Uncertainty Quantification in Synthetic Controls with Staggered Treatment Adoption*

Matias D. Cattaneo† Yingjie Feng‡ Filippo Palomba§ Rocio Titiunik¶

September 13, 2022

Abstract

We propose principled prediction intervals to quantify the uncertainty of a large class of synthetic control estimators in settings with staggered treatment adoption, offering precise non-asymptotic coverage probability guarantees. From a methodological perspective, we provide a detailed discussion of different treatment effects of interest, allowing for multiple treated units with treatment adoption at possibly different points in time. From a theoretical perspective, our proposed uncertainty quantification methods improve on prior literature by (i) covering a large class of estimands in staggered adoption settings, (ii) allowing for synthetic control methods with possibly nonlinear constraints, (iii) proposing principled and feasible tuning parameter selection, and (iv) offering valid uniform inference across post-treatment periods. We illustrate our proposed inference methodology with a substantive empirical application studying the effects of economic liberalization in the 1990s on GDP for emerging European countries. Companion general purpose software packages in `Python`, `R` and `Stata` are provided.

Keywords: causal inference, synthetic controls, staggered treatment adoption, prediction intervals, non-asymptotic inference.

PRELIMINARY AND INCOMPLETE -- COMMENTS WELCOME

*We thank Alberto Abadie, Simon Freyaldenhoven, and Bartolomeo Stellato for many insightful discussions. Cattaneo and Titiunik gratefully acknowledge financial support from the National Science Foundation (SES-2019432), and Cattaneo gratefully acknowledges financial support from the National Institute of Health (R01 GM072611-16).

†Department of Operations Research and Financial Engineering, Princeton University.

‡School of Economics and Management, Tsinghua University.

§Department of Economics, Princeton University.

¶Department of Politics, Princeton University.

1 Introduction

The synthetic control method was introduced by Abadie and Gardeazabal (2003), and since then many extensions and generalizations have been proposed in the literature (Abadie, 2021, and references therein). This methodology is now part of the standard toolkit for program evaluation and treatment effect analysis, offering a complement to traditional difference-in-differences, event study and other panel data approaches for causal inference employing longitudinal aggregated data with a few treated units. Most of the synthetic control literature concentrates on identification, as well as on prediction or point estimation of treatment effects, under different causal inference frameworks and algorithmic implementations. However, despite its importance in empirical work, principled uncertainty quantification of synthetic control predictions or estimators remains largely unexplored, particularly in more complex data settings such as those spanning beyond the canonical single treated unit setup where asymptotic approximations may be unreliable.

This paper proposes prediction intervals to quantify the uncertainty of a large class of synthetic control predictions or estimators in settings with staggered treatment adoption, offering precise non-asymptotic coverage probability guarantees and principled tuning parameter selection. Employing a causal inference framework where potential outcomes are assumed to be random, we propose inferential procedures with non-asymptotic probability guarantees because asymptotic approximations are often suspect in synthetic control applications due to small sample sizes. Conceptually, our proposed prediction intervals capture two sources of uncertainty: one coming from the construction of the synthetic control weights using pre-treatment data, and the other generated by the irreducible sampling variability introduced by the post-treatment outcomes. The proposed prediction intervals also take into account potential misspecification error explicitly, and enjoy other robustness properties due to their non-asymptotic construction.

Our first contribution is methodological in nature due to the complexity added by the staggered treatment adoption setup, which allows for (but does not require) the existence of multiple treatment units changing from control to treatment status at possibly different points in time. More specifically, in Section 3, we introduce a fairly general causal inference framework that is not only specifically tailored to synthetic control methods, but also incorporates staggered treatment adoption. Using this framework, we discuss a large class of treatment effects in the context of synthetic

controls, and explain how our inferential methods can be used in each case. Furthermore, our proposed causal framework explicitly allows for misspecification error, multiple covariate features, and cross-equation re-weighting when constructing the synthetic control weights based on pre-treatment data.

Building on our general causal inference framework, we present two main theoretical contributions in Section 4. First, we give high-level sufficient conditions leading to valid prediction intervals with precise non-asymptotic guarantees, allowing for stationary and non-stationary data and for synthetic control estimators constructed using multiple re-weighted features and nonlinear constraints. We also provide easy-to-verify primitive conditions, which cover all common synthetic control methods in the literature (e.g., Ridge or elastic net regression). Second, we extend our methods to provide not only pointwise but also joint inference validity across post-treatment time periods. This result allows for the construction of joint "prediction bands" to complement the prediction intervals for different treatment effects at each point in the post-treatment period.

To complement our methodological and theoretical work, Section 5 discusses principled tuning parameter selection based on our theoretical results. Our proposed methods employ explicitly the non-asymptotic characterizations of the coverage errors associated with the proposed prediction intervals to calibrate the underlying tuning parameters. In particular, we discuss in detail relaxation methods for possibly nonlinear constraints.

Our proposed synthetic control methods are illustrated with a substantive empirical application investigating the effect of economic liberalization in the 1990s on GDP for emerging European countries. This empirical work is motivated by Billmeier and Nannicini (2013) who, also employing synthetic control methods, studied the same substantive question for countries in other regions of the world. Our main empirical findings suggest that economic liberalization in the 1990s did not have a major economic impact for for emerging European countries. This finding is in line with prior empirical results in the literature. The supplemental appendix provides additional empirical evidence supporting our main findings, including an empirical re-analysis using alternative synthetic control estimators for different countries in the world and a discussion of specific cases where the synthetic control method does not appear to be well-suited for the analysis.

Finally, we also provide general purpose software implementing all our results in `Python`, `R`, and `Stata`, accompanied with detailed documentation and additional replication materials. This

software is introduced in detail in our companion article Cattaneo et al. (2022), where we also discuss several numerical and implementation issues related to numerical optimization and tuning parameter selection. See also the supplemental appendix (Section S.3) for more details on how to prepare the data to analyze staggered treatment adoption using synthetic control methods using our companion software.

1.1 Related Literature

This paper contributes to two strands of the synthetic control literature. First, it develops estimation/prediction and inference methods for staggered treatment adoption settings. Putting aside generic linear factor model methods, Ben-Michael et al. (2021) and Shaikh and Toulis (2021) appear to be the only papers that have studied staggered treatment adoption for synthetic controls before. The first paper focuses on estimation/prediction in settings where the pre-treatment fit is poor, and develops penalization methods for improving the performance of the canonical synthetic control estimator. Ben-Michael et al. (2021) also suggest employing a bootstrap method for assessing uncertainty, but no formalization is provided guaranteeing its (asymptotic) validity. Shaikh and Toulis (2021) focus on uncertainty quantification employing a parametric duration model and propose a permutation-based inferential method under a symmetry assumption. Our paper complements prior work by offering an alternative nonparametric inference method with demonstrable non-asymptotic coverage guarantees and allowing for misspecification in the construction of the synthetic control weights, which can be applied directly to a large class of synthetic control (possibly penalized) estimators and causal quantities of interest.

From a general methodological perspective, our proposed inference methods are motivated by Vovk (2012), and are most closely related to prior work by Chernozhukov et al. (2021a,b) on conformal prediction intervals and by Cattaneo et al. (2021) on non-asymptotic prediction intervals (see also Wainwright (2019) for a modern introduction to non-asymptotic statistical learning). Our paper contributes to this strand of the literature by developing new prediction interval methods. First, we allow for a large class of estimands in staggered adoption settings (prior work covered only the canonical single treated unit case). Second, we cover a large class of synthetic control estimators with possibly nonlinear constrains (prior work allowed only for estimators with linear constraints). Third, we propose principled and feasible tuning parameter selection (prior work did not provide

guidance on this regard). Fourth, we also develop valid uniform inference across post-treatment periods (absent in prior work).

There are several other recent proposals in the literature to quantify uncertainty and conduct inference for synthetic controls. For example, Li (2020) study correctly specified linear factor models, Masini and Medeiros (2021) study high-dimensional penalization methods, Agarwal et al. (2021) investigate matrix completion methods, and Shen et al. (2022) explore panel data methods, among many others. All these methods rely on asymptotic approximations, in most cases employing Gaussian critical values that assume away misspecification errors and other small sample issues. Our work complements these contributions by providing prediction intervals with non-asymptotic coverage guarantees. Finally, all the inferential methods mentioned so far contrast with the original method proposed by Abadie et al. (2010), which relies on design-based permutation of treatment assignment assuming the potential outcomes are non-random.

2 The Effect of Liberalization on GDP for Emerging European Countries

During the second half of the twentieth century, a considerable number of countries all over the world launched programs of (external) economic liberalization, booming from 22% in 1960 to 73% in the early 2000s (Wacziarg and Welch, 2008). In the last thirty years, political scientists and economists investigated the social and economic consequences of such liberalization programs, often ending up with conflicting answers (see among many Levine and Renelt, 1992; Sachs, Warner, Åslund and Fischer, 1995; DeJong and Ripoll, 2006).

The main reason behind the lack of agreement among researchers can be attributed to the usage of different econometric techniques. On the one hand, cross-country econometric studies are beset by the well-known endogeneity or measurement problems (Rodriguez and Rodrik, 2000). On the other hand, individual case studies (Bhagwati and Srinivasan, 2002) are frequently difficult to generalize, lack external validity, and often rely on far-fetched country comparisons. The synthetic control, on the other end, represents a third alternative to the existing cross-country studies (e.g. Sachs et al., 1995) and single-country episodes (e.g. Bhagwati and Srinivasan, 2001) that aim at unveiling the relationship between economic liberalization and economic welfare. Synthetic controls provide

a transparent methodology for addressing endogeneity from omitted variable bias by taking into account the presence of time-varying unobservable confounders.

With this caveat in mind, Billmeier and Nannicini (2013) analyze the effects of liberalization in four continents: Africa, Asia, North America, and South America. To do so, they use a pre-existing dataset of economic variables (Giavazzi and Tabellini, 2005) which includes 180 countries, covers the period 1963-2000, and contains an indicator for economic liberalization as defined in Sachs et al. (1995) and updated in Wacziarg and Welch (2008) (hereafter, Sachs-Werner indicator). (More details on the data and the definition of economic liberalization can be found in the Supplemental Appendix Section S.3.) Armed with these data, the authors use the dummy variable for economic liberalization as a treatment indicator in the synthetic control framework originally developed in Abadie and Gardeazabal (2003).

The main finding of in Billmeier and Nannicini (2013) is that economic liberalization—in the sense of the Sachs-Werner indicator—has a non-negative effect on GDP per capita. Moreover, the authors document the presence of substantial heterogeneity in the effects depending on the period of time in which the liberalization took place. On the one hand, the estimated effect on real income per capita is positive in those countries that embarked on programs of economic liberalization before the 1980s. On the other hand, in the MENA (Middle East and North Africa) region and in sub-Saharian Africa, where many liberalization episodes occurred after 1985, the magnitude of the estimated effect becomes either statistically insignificant or economically irrelevant.

Our work builds on Billmeier and Nannicini (2013) in three ways. *First*, we form prediction intervals around treatment effect estimates using recent advancements in uncertainty quantification for synthetic control methods through finite-sample probability concentration methods. *Second*, rather than estimating *independently* a synthetic control for each treated unit, we leverage both the presence of multiple treated units and the staggered nature of treatment adoption to *jointly* estimate synthetic controls for each treated unit. *Third*, by borrowing donors from other continents, we also analyze the impact of economic liberalization events in Europe (see Table 1 below).

As reported in Table 1, we select six episodes out of the nine available ones. In particular, North Macedonia is not included in the final analysis due to the lack of other variables in the dataset besides GDP per capita before 1994, whereas Czech Republic and Poland are excluded in virtue of

Table 1: *Economic liberalization episodes in the Billmeier and Nannicini (2013) dataset.*

Country Name	Event Date	Years Closed	Years Liberalized	Analyzed in this work
Albania	1992	29	9	✓
Bulgaria	1991	28	10	✓
Czech Republic	1991	28	10	✗
Hungary	1990	27	11	✓
North Macedonia	1994	31	7	✗
Poland	1990	27	11	✗
Romania	1992	29	9	✓
Slovak Republic	1991	28	10	✓
Slovenia	1991	28	10	✓

Notes: a comprehensive description of the events that led the Sachs-Werner indicator to switch on for these countries is contained in the Supplemental Appendix Section S.3 together with the full set of donors.

the poor pre-treatment fits yielded by the synthetic controls for the two countries.[1] We discover that the real income per capita trajectory following the liberalization is lower after 10 years than it would have been in the absence of the liberalization in all the countries we analyze. Similarly, the average treatment effect (see Section 3 for a precise definition) is negative for all the treated units. However, when individual and simultaneous prediction intervals are taken into account, the trajectory of the synthetic control becomes indistinguishable from the realized time series for real income with high probability. Results in other continents - fully presented in the Supplemental Appendix Section S.5 - confirm in magnitude the ones found in Billmeier and Nannicini (2013) whenever it has been possible to compare them. However, the data do not allow us to draw the conclusion that liberalization events changed the trajectory of GDP per capita in either a favorable or negative way.

3 Causal Inference Framework and Quantities of Interest

We consider the synthetic control framework with a fixed number of units that may adopt treatment at different times. Specifically, a researcher observes N units for T time periods. Units are indexed by $i = 1, \ldots, N$, and time periods are indexed by $t = 1, \ldots, T$. Let T_i represent the time when unit i receives the treatment, with $T_i = \infty$ denoting that unit i is never treated. Each unit i remains

[1] As detailed in Supplemental Appendix Section S.3 and Supplemental Appendix Section S.7, we still use Czech Republic and Poland as donors whenever possible. Supplemental Appendix Section S.7 presents the results for Czech Republic, North Macedonia, and Poland.

untreated in $t = 1, \ldots, T_i - 1$ and remains treated since T_i. We assume there is a (non-empty) set of units that are never treated, i.e., $N_0 := \sum_{i=1}^{N} \mathbb{1}(T_i = \infty) > 0$, and let $N_1 := N - N_0$ be the number of units that are eventually treated by time period T. Without loss of generality, assume units are ordered in the adoption time: $T_1 \leq T_2 \leq \cdots \leq T_{N_1}$. In our empirical application, the treatment of interest is economic liberalization, the adoption time of which is heterogeneous across different countries.

The staggered adoption problem can be analyzed in a multi-valued treatment effects framework. Let $Y_{it}(s)$ denote the potential outcome of unit i in period t that would be observed if unit i had adopted the treatment in period s, for $s = 1, \ldots, T, \infty$, and we set $Y_{it}(s) = Y_{it}(\infty)$ for $t < s$. Implicitly, these simplifications impose two standard assumptions: no spillovers (the potential outcomes of unit i depend only on i's adoption time) and no anticipation (a unit's potential outcomes prior to the treatment are equal to that had it never been treated). Then, the observed outcome can be written as

$$Y_{it} = Y_{it}(\infty)\mathbb{1}(t < T_i) + Y_{it}(T_i)\mathbb{1}(t \geq T_i).$$

A large set of causal quantities can be defined in this context. In particular, for $k \geq 0$, let τ_{ik} be the (individual) treatment effect of unit i in $T_i + k$ (k periods after the treatment):

$$\tau_{ik} := Y_{i(T_i+k)}(T_i) - Y_{i(T_i+k)}(\infty).$$

This is the treatment effect of interest in the classical synthetic control analysis with only one treated unit. When multiple treated units or multiple post-treatment periods are available, a researcher might be interested in a variety of other causal quantities. The following are some typical examples:

(i) Average post-treatment effect on unit i:

$$\tau_{i\cdot} := \frac{1}{T - T_i + 1} \sum_{k=0}^{T-T_i} \tau_{ik} \ ;$$

(ii) Average treatment effect on the treated k periods after the treatment:

$$\tau_{\cdot k} := \frac{1}{N_1} \sum_{i=1}^{N_1} \tau_{ik} \ ;$$

(iii) Average treatment effect on units treated at time s_0 after k periods:

$$\tau_{\cdot k,s_0} := \frac{1}{|\{j : T_j = s_0\}|} \sum_{j:T_j=s_0} \tau_{ik},$$

where $|\{j : T_j = s_0\}|$ denotes the number of units that get treated at s_0.

(iv) Average treatment effects on the treated across $K+1$ post-treatment periods:

$$\tau_{\cdot\cdot} := \frac{1}{K+1} \sum_{k=0}^{K} \tau_{\cdot k}.$$

Since the observation ends at time T, the number of treated units included in the definition of the average treatment effect $\tau_{\cdot k}$ could vary across k. To avoid this complication, we assume that all treated units are observed at least K periods after the treatment for some $K \geq 1$, i.e., $T_{N_1} \leq T-K$, and attention is restricted to $\tau_{\cdot k}$ for $k \leq K$.

We emphasize that as in the classical treatment effects framework, the potential outcomes, the adoption times and the individual treatment effects are viewed as *random* quantities. Also, we assume that there are only a fixed (possibly small) number of treated units and post-treatment periods, which is very likely in many synthetic control applications. Thus, the various average treatment effects defined above are also viewed as random quantities. By contrast, in classical large-sample-based causal analysis, the target parameters are *fixed* limits of these average effects as $N_1 \to \infty$, $T - T_i \to \infty$, and/or $K \to \infty$. Accordingly, we develop in this paper statistical inference methods based on prediction intervals rather than confidence intervals for the treatment effects defined above.

The canonical synthetic control analysis for one single treated unit can be viewed as a special case of the more general setup described above. Specifically, suppose that unit 1 is the only treated unit who receives the treatment at T_1, and all other units are *never treated*, i.e., $T_i = \infty$ for all

$i \geq 2$. Then, the observed outcome is

$$Y_{it} = \begin{cases} Y_{it}(\infty) & i = 2, \ldots, N \\ Y_{1t}(\infty) & i = 1 \text{ and } t = 1, \ldots, T_1 - 1 \\ Y_{1t}(T_1) & i = 1 \text{ and } t = T_1, \ldots, T \end{cases}.$$

The target causal quantity in this canonical case is usually the individual treatment effect on the treated, i.e., τ_{1k} defined previously.

3.1 Synthetic Control Method

We consider the case where multiple treated units are available, and one would like to find a vector of SC weights possibly for each treated unit. From now on, we use a superscript $i = 1, \ldots, N_1$ to index the treated units that enter the construction of the desired causal quantity, and a subscript $l = 1, \ldots, M$ to denote different features of the treated on which one would like to match.

Let $\mathbf{A}_l^{[i]} = (a_{1,l}^{[i]}, \cdots, a_{T_0,l}^{[i]})' \in \mathbb{R}^{T_0}$ be the lth feature of the treated unit i measured in T_0 (user-specified) pre-treatment periods. For each feature l and each treated unit i, there exist $J + K$ variables that are used to predict or match the T_0-dimensional vector $\mathbf{A}_l^{[i]}$. These $J + K$ variables are separated into two groups denoted by $\mathbf{B}_l^{[i]} = (\mathbf{B}_{1,l}^{[i]}, \mathbf{B}_{2,l}^{[i]}, \cdots, \mathbf{B}_{J,l}^{[i]}) \in \mathbb{R}^{T_0 \times J}$ and $\mathbf{C}_l^{[i]} = (\mathbf{C}_{1,l}^{[i]}, \cdots, \mathbf{C}_{K,l}^{[i]}) \in \mathbb{R}^{T_0 \times K}$, respectively. More precisely, for each $j = 1, \cdots, J$, $\mathbf{B}_{j,l}^{[i]} = (b_{j1,l}^{[i]}, \cdots, b_{jT_0,l}^{[i]})'$ corresponds to the lth feature of the jth unit in the donor pool measured in T_0 pre-treatment periods, and for each $k = 1, \cdots, K$, $\mathbf{C}_{k,l}^{[i]} = (c_{k1,l}^{[i]}, \cdots, c_{kT_0,l}^{[i]})'$ is another vector of control variables used to predict $\mathbf{A}_l$ over the same pre-intervention time span. For ease of notation, we let $d = J + KM$. Stacking the M equations (corresponding to M features) for each treated unit, we define

$$\underbrace{\mathbf{A}^{[i]}}_{T_0 \cdot M \times 1} = \begin{bmatrix} \mathbf{A}_1^{[i]} \\ \vdots \\ \mathbf{A}_M^{[i]} \end{bmatrix}, \quad \underbrace{\mathbf{B}^{[i]}}_{T_0 \cdot M \times J} = \begin{bmatrix} \mathbf{B}_1^{[i]} \\ \vdots \\ \mathbf{B}_M^{[i]} \end{bmatrix}, \quad \underbrace{\mathbf{C}^{[i]}}_{T_0 \cdot M \times K \cdot M} = \begin{bmatrix} \mathbf{C}_1^{[i]} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2^{[i]} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}_M^{[i]} \end{bmatrix}.$$

The goal of the SC method is to search for a vector of weights $\mathbf{w} = (\mathbf{w}^{[1]\prime}, \cdots, \mathbf{w}^{[N_1]\prime})' \in \mathcal{W} \subseteq \mathbb{R}^{JN_1}$ which is common across the M features and a vector of coefficients $\mathbf{r} = (\mathbf{r}^{[1]\prime}, \cdots, \mathbf{r}^{[N_1]\prime})' \in$

$\mathcal{R} \subseteq \mathbb{R}^{KMN_1}$, such that the linear combination of $\mathbf{B}^{[i]}$ and $\mathbf{C}^{[i]}$ matches $\mathbf{A}^{[i]}$ as close as possible, for all $1 \leq i \leq N_1$. The feasibility sets $\mathcal{W}$ and $\mathcal{R}$ capture the restrictions imposed. Typical examples include simplex-type, lasso-type and ridge-type constraints (see more details in Cattaneo et al., 2022).

Such SC weights are typically obtained via the following optimization problem: for some $T_0 \cdot M \cdot N_1 \times T_0 \cdot M \cdot N_1$ symmetric weighting matrix $\mathbf{V}$,

$$\widehat{\boldsymbol{\beta}} := (\widehat{\mathbf{w}}',\ \widehat{\mathbf{r}}')' \in \underset{\mathbf{w}\in\mathcal{W},\, \mathbf{r}\in\mathcal{R}}{\arg\min}\ (\mathbf{A} - \mathbf{B}\mathbf{w} - \mathbf{C}\mathbf{r})'\mathbf{V}(\mathbf{A} - \mathbf{B}\mathbf{w} - \mathbf{C}\mathbf{r}) \tag{3.1}$$

where

$$\underbrace{\mathbf{A}}_{T_0\cdot M\cdot N_1\times 1} = \begin{bmatrix} \mathbf{A}^{[1]} \\ \vdots \\ \mathbf{A}^{[N_1]} \end{bmatrix}, \quad \underbrace{\mathbf{B}}_{T_0\cdot M\cdot N_1\times J\cdot N_1} = \begin{bmatrix} \mathbf{B}^{[1]} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{[2]} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}^{[N_1]} \end{bmatrix}, \quad \underbrace{\mathbf{C}}_{T_0\cdot M\cdot N_1\times K\cdot M\cdot N_1} = \begin{bmatrix} \mathbf{C}^{[1]} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{[2]} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}^{[N_1]} \end{bmatrix}.$$

Accordingly, we write $\widehat{\mathbf{w}} = (\widehat{\mathbf{w}}^{[1]\prime}, \cdots, \widehat{\mathbf{w}}^{[N_1]\prime})'$ where each $\widehat{\mathbf{w}}^{[i]} = (\widehat{w}_1^{[i]}, \cdots, \widehat{w}_J^{[i]})'$ is the SC weights on J control units that are used to predict the counterfactual of the treated unit i. Similarly, write $\widehat{\mathbf{r}} = (\widehat{\mathbf{r}}^{[1]\prime}, \cdots, \widehat{\mathbf{r}}^{[N_1]\prime})'$ and $\widehat{\boldsymbol{\beta}} = (\widehat{\boldsymbol{\beta}}^{[1]\prime}, \cdots, \widehat{\boldsymbol{\beta}}^{[N_1]\prime})'$.

Then, the predicted counterfactual outcome of each treated unit is given by

$$\widehat{Y}_{it}(\infty) := \mathbf{x}_t^{[i]\prime}\widehat{\mathbf{w}}^{[i]} + \mathbf{g}_t^{[i]\prime}\widehat{\mathbf{r}}^{[i]} = \mathbf{p}_t^{[i]\prime}\widehat{\boldsymbol{\beta}}^{[i]}, \quad \mathbf{p}_t^{[i]} = (\mathbf{x}_t^{[i]\prime},\ \mathbf{g}_t^{[i]\prime})', \quad i = 1, \cdots, N_1, \quad t > T_i,$$

where $\mathbf{x}_t^{[i]}$ is a vector of predictors of the control units measured in time t used to predict the counterfactual of the treated unit i, and $\mathbf{g}_t^{[i]}$ is a vector of predictors that correspond to the additional control variables specified in $\mathbf{C}^{[i]}$. For convenience of later exposition, we write $\mathbf{p}_t = (\mathbf{p}_t^{[1]\prime}, \cdots, \mathbf{p}_t^{[N_1]\prime})'$. Variables included in $\mathbf{x}_t^{[i]}$ and $\mathbf{g}_t^{[i]}$ need not be the same as those in $\mathbf{B}^{[i]}$ and $\mathbf{C}^{[i]}$.

Note that any causal quantity τ discussed before can be written as the difference between an observed outcome (possibly a linear combination of outcomes of a few treated units or in different periods) and the corresponding counterfactual outcome. To construct an estimate $\widehat{\tau}$ of τ, one only needs to substitute the "SC prediction" for the unobserved counterfactual:

$$\mathsf{SC\ prediction} := \mathbf{p}_\tau'\widehat{\boldsymbol{\beta}},$$

where the predictor vector $\mathbf{p}_\tau$ needs to be defined in context. See details in the following examples.

Example 3.1 (Individual Treatment Effect τ_{ik})**.** *Suppose that the individual treatment effect τ_{ik} of the treated unit i with $T_i < \infty$ is of interest for some $0 \le k \le T - T_i$. Let the set of pre-treatment periods be $\mathcal{T}_0 := \{t : t \le T_i - 1\}$. The donor pool consists of units that receive treatment later than $T_i + k$: $\mathcal{D}_{ik} := \{j : T_j > T_i + k\}$. The SC weights are constructed using the data of the treated unit i and the control units in $\mathcal{D}_{ik}$ from the pre-treatment periods in $\mathcal{T}_0$. Given the prediction of the counterfactual outcome $\widehat{Y}_{i(T_i+k)}$, the estimated treatment effect is*

$$\widehat{\tau}_{ik} := Y_{i(T_i+k)} - \widehat{Y}_{i(T_i+k)}(\infty) = Y_{i(T_i+k)} - \mathbf{p}'_{\tau_{ik}}\widehat{\boldsymbol{\beta}}.$$

The predictor vector $\mathbf{p}_{\tau_{ik}}$ in this case is given by

$$\mathbf{p}_{\tau_{ik}} := (\underbrace{\mathbf{0}'_{J+KM}, \cdots, \mathbf{0}'_{J+KM}}_{(i-1)\ vectors},\ \mathbf{p}^{[i]\prime}_{T_i+k},\ \underbrace{\mathbf{0}'_{J+KM}, \cdots, \mathbf{0}'_{J+KM}}_{(N_1-i)\ vectors})'.$$

where $\mathbf{0}_{J+KM}$ is a $(J + KM)$-vector of zeros. See Figure 1 for a graphical representation of $\widehat{\tau}_{ik}$.

Example 3.2 (Average Post-Treatment Effect $\tau_{i\cdot}$)**.** *Suppose that the quantity of interest is the treatment effect on the treated unit i averaged across all the post-treatment periods, i.e., $\tau_{i\cdot}$ defined previously. Let the set of pre-treatment periods be $\mathcal{T}_0 = \{t : t \le T_i - 1\}$, and take the set of all units that are never treated as the donor pool: $\mathcal{D}_{\cdot\infty} := \{j : T_j = \infty\}$. The SC weights in this scenario can be constructed the same way as those for individual treatment effects. The estimate of the average post-treatment effect then is given by*

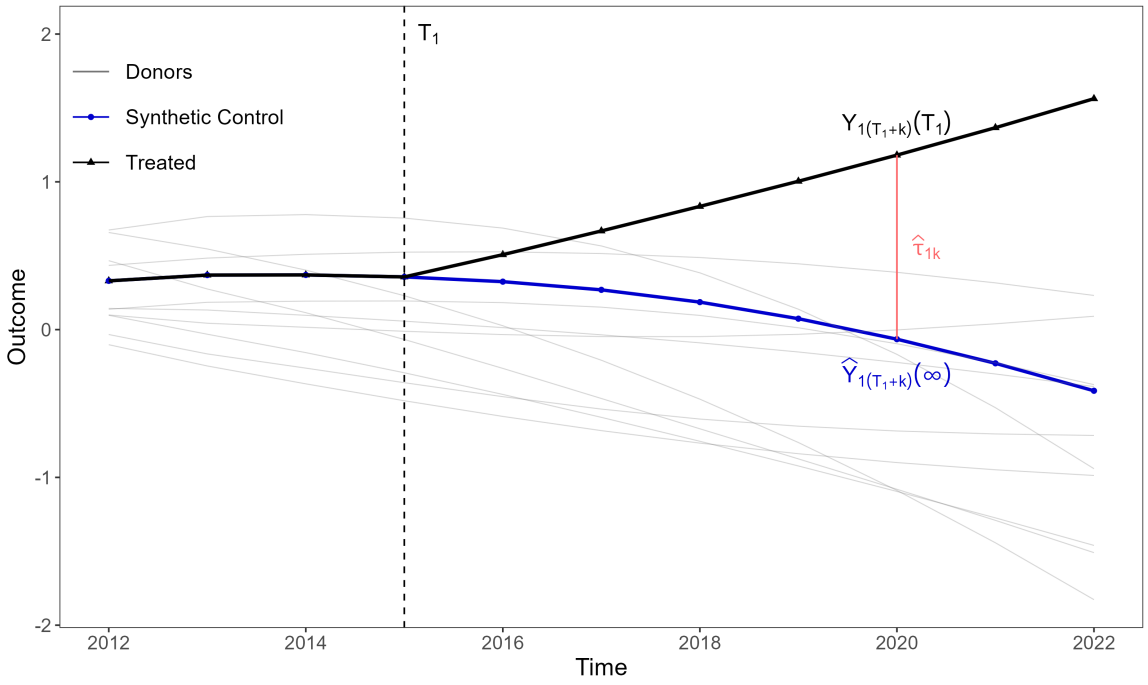
$$\widehat{\tau}_{i\cdot} := \frac{1}{T - T_i + 1}\sum_{t=T_i}^{T}\Big(Y_{it} - \widehat{Y}_{it}(\infty)\Big) = \frac{1}{T - T_i + 1}\sum_{t=T_i}^{T} Y_{it} \ - \ \mathbf{p}'_{\tau_{i\cdot}}\widehat{\boldsymbol{\beta}},$$

where the predictor vector in this case is given by

$$\mathbf{p}_{\tau_{i\cdot}} := \Big(\underbrace{\mathbf{0}'_{J+KM}, \cdots, \mathbf{0}'_{J+KM}}_{(i-1)\ vectors},\ \frac{1}{T - T_i + 1}\sum_{t\ge T_i}\mathbf{p}^{[i]\prime}_t,\ \underbrace{\mathbf{0}'_{J+KM}, \cdots, \mathbf{0}'_{J+KM}}_{(N_1-i)\ vectors}\Big)'.$$

See Figure 2 for a graphical representation of $\widehat{\tau}_{i\cdot}$.

Figure 1: *Graphical illustration of the individual treatment effect* $\widehat{\tau}_{ik}$.



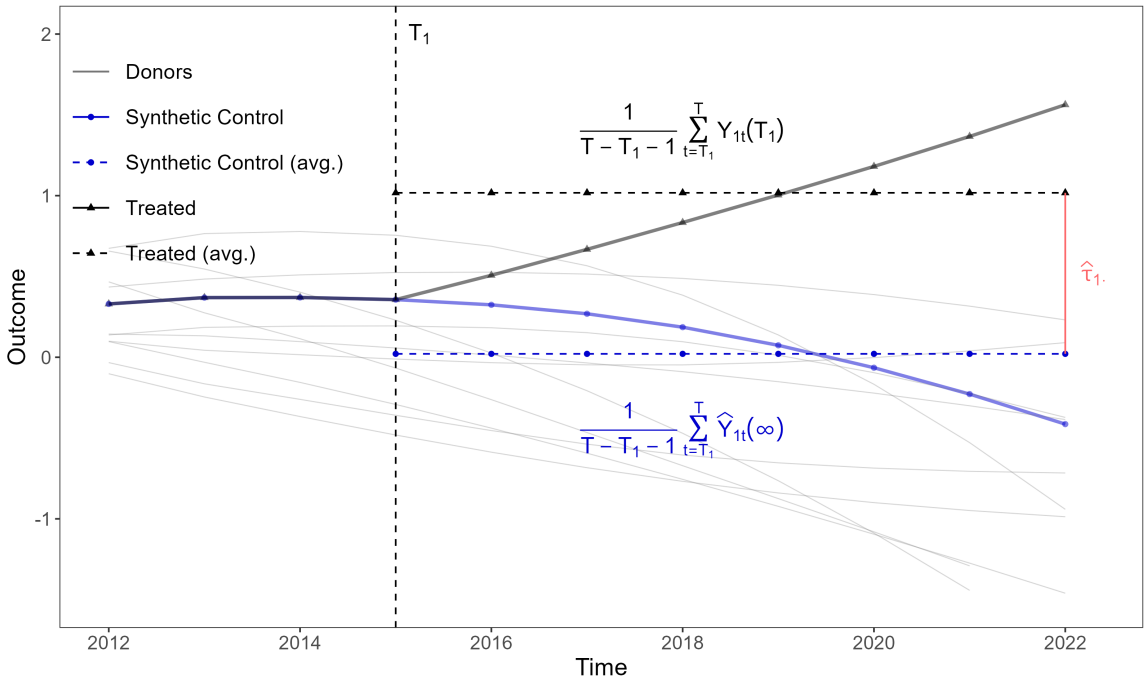
Notes: The black line displays the times series of the treated unit's outcome, whereas the blurred gray lines portray the same variable for the donor units. The blue line is the synthetic control constructed out of the donor units using a simplex-type constraint. The pink vertical line represents the quantity $\widehat{\tau}_{ik}$ described in Example 3.1, where k is set to 5 in this example.

Example 3.3 (Average Treatment Effect on the Treated $\tau_{\cdot k,s_0}$)**.** *Suppose that the causal quantity of interest is the average treatment effect on units that adopt the treatment at time* s_0*, which is measured* k *periods after the adoption, i.e.,* $\tau_{\cdot k,s_0}$ *defined above. Let the set of pre-treatment periods be* $\mathcal{T}_0 = \{t : t \leq s_0 - 1\}$*. Denote by* $\mathcal{D}_{\cdot k,s_0}$ *the donor pool consisting of units that adopt the treatment later than* $s_0 + k$*. We have two strategies to conduct the SC analysis:*

- *Implement the procedure described above, which allows for different SC weights on different treated units. Suppose* $\{i : T_i = s_0\} = \{i_1, \cdots, i_{N_{s_0}}\}$*. Then, the estimated effect is given by*

$$\widehat{\tau}_{\cdot k,s_0} := \frac{1}{N_{s_0}} \sum_{i:T_i=s_0} \left(Y_{i(s_0+k)} - \widehat{Y}_{i(s_0+k)}(\infty) \right) = \frac{1}{N_{s_0}} \sum_{i:T_i=s_0} Y_{i(s_0+k)} \; - \; \mathbf{p}'_{\tau_{\cdot k,s_0}} \widehat{\boldsymbol{\beta}},$$

Figure 2: *Graphical illustration of the post-treatment effect $\widehat{\tau}_{i\cdot}$.*



Notes: The black line displays the times series of the treated unit's outcome, whereas the blurred gray lines portray the same variable for the donor units. The blue line is the synthetic control constructed out of the donor units using a simplex-type constraint. The dashed lines represent the post-treatment average of the treated time series (black) and the synthetic control time series (blue). The pink vertical line represents the quantity $\widehat{\tau}_{i\cdot}$ described in Example 3.2.

where the predictor vector in this case is given by

$$\mathbf{p}_{\tau_{\cdot k,s_0}} := \Big(\underbrace{\mathbf{0}'_{J+KM}, \cdots, \mathbf{0}'_{J+KM}}_{(i_1-1)\ vectors}, \ \frac{1}{N_{s_0}} \mathbf{p}_{s_0+k}^{[i_1]\prime}, \cdots, \frac{1}{N_{s_0}} \mathbf{p}_{s_0+k}^{[i_{N_{s_0}}]\prime}, \ \underbrace{\mathbf{0}'_{J+KM}, \cdots, \mathbf{0}'_{J+KM}}_{(N_1 - i_{N_{s_0}})\ vectors} \Big)'.$$

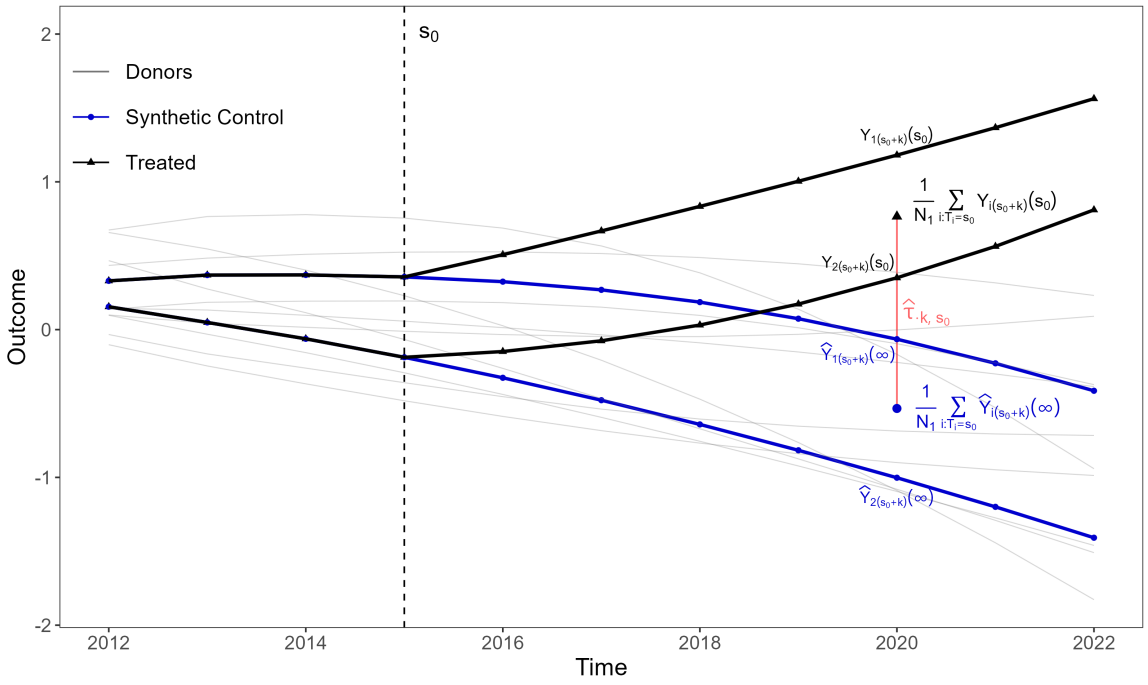
- *Aggregate different treated units into one single unit, denoted by "*`ave`*", whose potential outcomes are given by the average of all units treated at time s_0:*

$$Y_t^{\texttt{ave}}(s; s_0) := \frac{1}{N_{s_0}} \sum_{i:T_i = s_0} Y_{it}(s), \quad t = 1, \cdots, T, \quad s = 1, \cdots, T, \infty.$$

Other features of this aggregate unit can be constructed similarly as the average of multiple units treated at s_0. The SC weights can be obtained using the data of the aggregate unit "`ave`*" and control units in $\mathcal{D}_{\cdot k,s_0}$ from pre-treatment periods in $\mathcal{T}_0$. Then, the SC analysis proceeds*

exactly the same way as in Example 3.1. See Figure 3 for a graphical representation of $\widehat{\tau}_{\cdot k,s_0}$.

Figure 3: *Graphical illustration of the average treatment effect on the treated* $\widehat{\tau}_{\cdot k,s_0}$.



Notes: The black lines display the times series of the treated units' outcome, whereas the blurred gray lines portray the same variable for the donor units. The blue lines are the synthetic controls constructed out of the donor units using a simplex-type constraint. The largest black triangle represents the average of the treated units' outcomes at $s_0 + k$ where s_0 and k are set to 2015 and 5 in this example. Similarly, the largest blue circle is the average of the synthetic controls at $s_0 + k$. The pink vertical line represents the quantity $\widehat{\tau}_{\cdot k,s_0}$ described in Example 3.3.

Example 3.4 (Average Treatment Effect on the Treated after k Periods $\tau_{\cdot k}$)**.** *Suppose that the quantity of interest is the average treatment effect on all treated units measured k periods after the adoption of the treatment, i.e.,* $\tau_{\cdot k}$ *defined above. Let the set of pre-treatment periods be* $\mathcal{T} = \{t : t \leq T_1 - 1\}$ *and the donor pool be* $\mathcal{D}_{\cdot\infty} = \{j : T_j = \infty\}$*. Then, the estimated effect is given by*

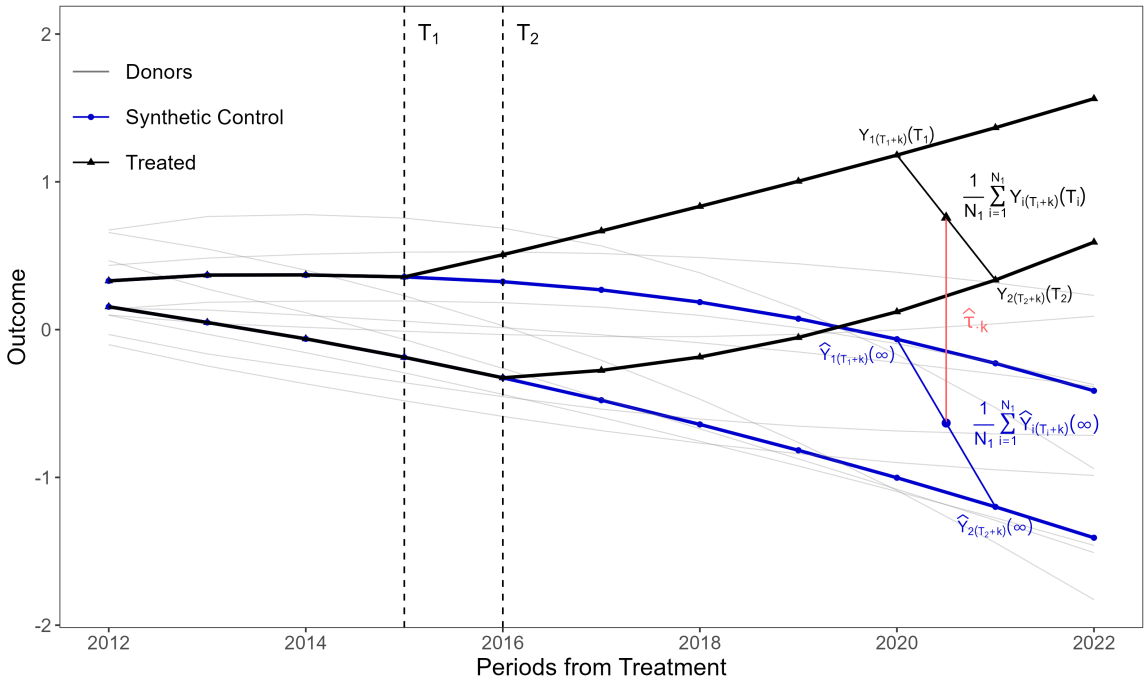
$$\widehat{\tau}_{\cdot k} := \frac{1}{N_1}\sum_{i=1}^{N_1}\left(Y_{i(T_i+k)} - \widehat{Y}_{i(T_i+k)}\right) = \frac{1}{N_1}\sum_{i=1}^{N_1} Y_{it} \;-\; \mathbf{p}'_{\tau_{\cdot k}}\widehat{\boldsymbol{\beta}},$$

where the predictor vector in this case is given by

$$\mathbf{p}_{\tau_{\cdot k}} := \left(\frac{1}{N_1}\mathbf{p}^{[1]\prime}_{T_1+k}, \cdots, \frac{1}{N_1}\mathbf{p}^{[N_1]\prime}_{T_{N_1}+k}\right)'.$$

See Figure 4 for a graphical representation of $\widehat{\tau}_{\cdot k}$.

Figure 4: *Graphical illustration of the average treatment effect on the treated after k periods $\widehat{\tau}_{\cdot k}$.*



Notes: The black lines display the times series of the treated units' outcome, whereas the blurred gray lines portray the same variable for the donor units. The blue lines are the synthetic controls constructed out of the donor units using a simplex-type constraint. The largest black triangle represents the average of the treated units' outcomes k periods after treatment where k is set to 5 in this example. Similarly, the largest blue circle is the average of the synthetic controls k periods after treatment. The pink vertical line represents the quantity $\widehat{\tau}_{k,s_0}$ described in Example 3.4.

As pointed out in Ben-Michael et al. (2021), with multiple treated units, the SC weights could be constructed in two ways: (i) optimize the separate fit for each treated unit; (ii) optimize the pooled fit for the average of the treated units. These ideas can be accommodated by choosing a proper weighting matrix $\mathbf{V}$. For example, taking $\mathbf{V} = \mathbf{I}_{T_0 M N_1}$ yields

$$\widehat{\boldsymbol{\beta}} = \underset{\mathbf{w}\in\mathcal{W},\mathbf{r}\in\mathcal{R}}{\arg\min} \sum_{i=1}^{N_1}\sum_{l=1}^{M}\sum_{t=1}^{T_0}\left(a_{t,l}^{[i]} - \mathbf{b}_{t,l}^{[i]\prime}\mathbf{w}_l^{[i]} - \mathbf{c}_{t,l}^{[i]\prime}\mathbf{r}_l^{[i]}\right)^2,$$

where $\mathbf{B}_l^{[i]} := (\mathbf{b}_{1,l}^{[i]}, \cdots, \mathbf{b}_{T_0,l}^{[i]})'$ is the lth feature of the J control units in the donor pool, and $\mathbf{C}_l^{[i]} := (\mathbf{c}_{1,l}^{[i]}, \cdots, \mathbf{c}_{T_0,l}^{[i]})'$ is the additional K variables used to predict $\mathbf{A}_l^{[i]}$. The objective above is equivalent to minimizing the sum of squared errors of the pre-treatment fit for *each* treated unit and thus is termed "separate fit".

By contrast, consider the following weighting matrix:

$$\mathbf{V} = \frac{1}{N_1^2}\mathbf{1}_{N_1}\mathbf{1}_{N_1}' \otimes \mathbf{I}_{T_0 M}$$

where $\otimes$ denotes the Kronecker product operator. Then,

$$\widehat{\boldsymbol{\beta}} = \underset{\mathbf{w}\in\mathcal{W},\mathbf{r}\in\mathcal{R}}{\arg\min} \sum_{l=1}^{M}\sum_{t=1}^{T_0}\left[\frac{1}{N_1}\sum_{i=1}^{N_1}\left(a_{t,l}^{[i]} - \mathbf{b}_{t,l}^{[i]\prime}\mathbf{w}^{[i]} - \mathbf{c}_{t,l}^{[i]\prime}\mathbf{r}_l^{[i]}\right)\right]^2.$$

That is, the goal is to minimize the sum of squared *averaged* errors across all treated units, which is usually termed "pooled fit".

4 Prediction Intervals

Given the generic framework introduced before, we aim to construct prediction intervals for various treatment effects defined in Section 3. See Vovk (2012), Chernozhukov et al. (2021a,b); Cattaneo et al. (2021), and references therein, for recent papers on (conditional) prediction intervals and related methods.

Generally, suppose that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ introduced previously are random quantities defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and $\mathscr{H} \subseteq \mathscr{F}$ is a sub-σ-field. For some $\alpha, \pi \in (0,1)$, we say a random interval $\mathcal{I}$ is an (α,π)-valid $\mathscr{H}$-conditional prediction interval for a causal quantity τ (any of the treatment effects defined in Section 3) if

$$\mathbb{P}\Big\{\mathbb{P}\big[\tau \in \mathcal{I} \,\big|\, \mathscr{H}\big] \geq 1-\alpha\Big\} \geq 1-\pi. \tag{4.1}$$

If $\mathscr{H}$ is the trivial σ-field over Ω, then $\mathcal{I}$ reduces to an unconditional prediction interval for τ. In the general case, the prediction interval $\mathcal{I}$ is $\mathscr{H}$-conditionally (α,π)-valid: the conditional coverage probability of $\mathcal{I}$ for τ is at least $(1-\alpha)$, which holds with probability over $\mathscr{H}$ at least $(1-\pi)$. In practice, $(1-\alpha)$ is a desired coverage level chosen by users, say 95%, and π is a "small" number that depends on the sample size and typically goes to zero in some asymptotic sense. In this paper, all results are valid for all T_0 large enough, with the associated probability loss π characterized precisely. Thus, we say that the conditional coverage of the prediction interval $\mathcal{I}$ is at least $(1-\alpha)$

with high probability, or that the conditional prediction interval offers finite-sample probability guarantees. Our results imply $\pi \to 0$ as $T_0 \to \infty$, but no limits or asymptotic arguments are used in this paper.

Generally, the choice of the conditioning set $\mathscr{H}$ determines the uncertainty that would be or would not be taken into account by prediction intervals. In this paper, we consider prediction intervals conditional on all control units as well as other predictors used to construct the SC prediction. That is, $\mathscr{H} = \{\mathbf{B}, \mathbf{C}, \mathbf{p}_\tau\}$. Therefore, the uncertainty to be characterized arises from the treated units only. Define the population-based estimand of the SC weights that is analogous to (3.1):

$$\boldsymbol{\beta}_0 := (\mathbf{w}_0', \mathbf{r}_0')' = \underset{\mathbf{w}\in\mathcal{W},\, \mathbf{r}\in\mathcal{R}}{\arg\min}\ \mathbb{E}\Big[(\mathbf{A} - \mathbf{B}\mathbf{w} - \mathbf{C}\mathbf{r})'\mathbf{V}(\mathbf{A} - \mathbf{B}\mathbf{w} - \mathbf{C}\mathbf{r})\Big|\mathscr{H}\Big]. \tag{4.2}$$

Thus, we can write

$$\mathbf{A} = \mathbf{B}\mathbf{w}_0 + \mathbf{C}\mathbf{r}_0 + \mathbf{U}, \qquad \mathbf{w}_0 \in \mathcal{W}, \qquad \mathbf{r}_0 \in \mathcal{R}, \tag{4.3}$$

where $\mathbf{U} = (\mathbf{u}^{[1]\prime}, \cdots, \mathbf{u}^{[N_1]\prime})' \in \mathbb{R}^{T_0 M N_1}$ is the corresponding pseudo-true residual relative to a σ-field $\mathscr{H}$. Note that each $\mathbf{u}^{[i]} = (u^{[i]}_{1,1}, \cdots, u^{[i]}_{T_0,1}, \cdots, u^{[i]}_{1,M}, \cdots, u^{[i]}_{T_0,M})' \in \mathbb{R}^{T_0 M}$ is the pseudo-true residual from the M equations (corresponding to the M features) of the treated unit i. Given the pseudo-true value $\boldsymbol{\beta}_0$ and a desired causal quantity τ, we generally have the following decomposition of the estimated effect $\widehat{\tau}$:

$$\widehat{\tau} - \tau \equiv -\mathbf{p}_\tau'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) + e_\tau,$$

where $\mathbf{p}_\tau'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)$ captures the *in-sample uncertainty* from the SC weights construction using pre-treatment information, and e_τ captures the *out-of-sample uncertainty* from the stochastic error in one or a few post-treatment periods. Notice that in-sample uncertainty quantification is necessary in this scenario since the conditioning set $\mathscr{H} \supseteq \{\mathbf{B}, \mathbf{C}\}$ but $\mathscr{H} \not\supseteq \mathbf{A}$.

To construct prediction intervals for a causal quantity τ, we propose to find constants $M_{1,\mathtt{L}}$, $M_{1,\mathtt{U}}$, $M_{2,\mathtt{L}}$ and $M_{2,\mathtt{U}}$, possibly depending on $\alpha_1, \alpha_2, \pi_1, \pi_2 \in (0,1)$ such that

$$\begin{aligned}
&\mathbb{P}\Big\{\mathbb{P}\big[M_{1,\mathtt{L}} \le \mathbf{p}_\tau'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \le M_{1,\mathtt{U}} \,\big|\, \mathscr{H}\big] \ge 1 - \alpha_1\Big\} \ge 1 - \pi_1, \qquad \text{and} \\
&\mathbb{P}\Big\{\mathbb{P}\big[M_{2,\mathtt{L}} \le \quad e_\tau \quad \le M_{2,\mathtt{U}} \,\big|\, \mathscr{H}\big] \ge 1 - \alpha_2\Big\} \ge 1 - \pi_2,
\end{aligned}$$

which suffices to guarantee

$$\mathbb{P}\Big\{\mathbb{P}\big[\widehat{\tau}+M_{1,\mathtt{L}}-M_{2,\mathtt{U}}\leq\tau\leq\widehat{\tau}+M_{1,\mathtt{U}}-M_{2,\mathtt{L}}\big|\mathscr{H}\big]\geq 1-\alpha_1-\alpha_2\Big\}\geq 1-\pi_1-\pi_2,$$

that is, the prediction interval $\mathcal{I}=[\widehat{\tau}+M_{1,\mathtt{L}}-M_{2,\mathtt{U}},\,\widehat{\tau}+M_{1,\mathtt{U}}-M_{2,\mathtt{L}}]$ achieves $(1-\alpha_1-\alpha_2)$ $\mathscr{H}$-conditional coverage probability, which holds with probability at least $1-\pi_1-\pi_2$, as defined in (4.1).

In-sample uncertainty. We propose a simulation-based strategy to bound the in-sample error $\mathbf{p}_\tau'(\boldsymbol{\beta}_0-\widehat{\boldsymbol{\beta}})$. Let $\mathbf{Z}=(\mathbf{B},\mathbf{C})$ and $d=(J+KM)N_1$. Using (3.1) and (4.2), we obtain the following optimization problem characterizing the centered synthetic control weights estimator:

$$\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0=\underset{\boldsymbol{\beta}-\boldsymbol{\beta}_0\in\Delta}{\arg\min}\Big\{(\boldsymbol{\beta}-\boldsymbol{\beta}_0)'\widehat{\mathbf{Q}}(\boldsymbol{\beta}-\boldsymbol{\beta}_0)-2\widehat{\boldsymbol{\gamma}}'(\boldsymbol{\beta}-\boldsymbol{\beta}_0)\Big\},$$

where $\widehat{\mathbf{Q}}=\mathbf{Z}'\mathbf{V}\mathbf{Z}$, $\widehat{\boldsymbol{\gamma}}'=\mathbf{U}'\mathbf{V}\mathbf{Z}$, and $\Delta=\{\boldsymbol{\beta}-\boldsymbol{\beta}_0\in\mathbb{R}^d:\boldsymbol{\beta}\in\mathcal{W}\times\mathcal{R}\}$. Throughout the paper, we assume the constraint sets $\mathcal{W}$ and $\mathcal{R}$ are convex.

Let $\boldsymbol{\gamma}:=\mathbb{E}[\widehat{\boldsymbol{\gamma}}|\mathscr{H}]$, which is not necessarily equal to $\mathbf{0}$. By optimality of $\widehat{\boldsymbol{\beta}}$ and the convexity of $\mathcal{W}$ and $\mathcal{R}$, it can be shown that $\widehat{\boldsymbol{\beta}}$ has to satisfy $\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0\in\Delta$ and $(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)'\widehat{\mathbf{Q}}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)-2(\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma})'(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)\leq 0$. Thus, the minimum and the maximum of $\mathbf{p}_\tau'(\boldsymbol{\beta}-\boldsymbol{\beta}_0)$ over the set for $\boldsymbol{\beta}$ satisfying these restrictions are natural lower and upper bounds on the in-sample error $\mathbf{p}_\tau'(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)$. Conditional on $\mathscr{H}$, the uncertainty of these (stochastic) bounds come from $\widehat{\boldsymbol{\gamma}}$ only. Conceptually, we can employ a normal distributional approximation of $\widehat{\boldsymbol{\gamma}}$ and set $M_{1,\mathtt{L}}=\mathfrak{c}_{\mathtt{L}}(\alpha_1/2)$ and $M_{1,\mathtt{U}}=\mathfrak{c}_{\mathtt{U}}(1-\alpha_1/2)$ where

$$\begin{aligned}&\mathfrak{c}_{\mathtt{L}}(\alpha_1/2):=(\alpha_1/2)\text{-quantile of }\inf\{\mathbf{p}_\tau'\boldsymbol{\delta}:\boldsymbol{\delta}\in\mathcal{M}_{\mathbf{G}}\}\quad\text{and}\\&\mathfrak{c}_{\mathtt{U}}(1-\alpha_1/2):=(1-\alpha_1/2)\text{-quantile of }\sup\{\mathbf{p}_\tau'\boldsymbol{\delta}:\boldsymbol{\delta}\in\mathcal{M}_{\mathbf{G}}\}\end{aligned}$$

conditional on $\mathscr{H}$, where $\mathcal{M}_{\mathbf{G}}=\{\boldsymbol{\delta}\in\Delta:\boldsymbol{\delta}'\widehat{\mathbf{Q}}\boldsymbol{\delta}-2\mathbf{G}'\boldsymbol{\delta}\leq 0\}$, $\mathbf{G}|\mathscr{H}\sim\mathsf{N}(\mathbf{0},\boldsymbol{\Sigma})$ and $\boldsymbol{\Sigma}=\mathbb{V}[\widehat{\boldsymbol{\gamma}}|\mathscr{H}]$.

However, $\mathfrak{c}_{\mathtt{L}}(\alpha_1/2)$ and $\mathfrak{c}_{\mathtt{U}}(1-\alpha_1/2)$ cannot be directly used since they still rely on the infeasible normalized constraint set Δ and the unknown covariance matrix $\boldsymbol{\Sigma}$. In the following we propose a feasible simulation-based strategy, which generalizes available results in the literature by allowing for *possibly nonlinear* constraints. For example, our main empirical results in Section 6 employ a

L1-L2 constrained synthetic control estimator, which exhibits good performance in finite samples (but was not covered by prior results on prediction intervals until this paper).

First, we need a feasible constraint set $\Delta^\star$ used in simulation. Specifically, define the distance between a point $\mathbf{a} \in \mathbb{R}^d$ and a set $\Lambda \subseteq \mathbb{R}^d$ by $\text{dist}(\mathbf{a}, \Lambda) = \inf_{\lambda \in \Lambda} \|\mathbf{a} - \boldsymbol{\lambda}\|$, where $\|\cdot\|$ is a generic ℓ_p vector norm on $\mathbb{R}^d$ with $p \geq 1$ (e.g., Euclidean norm or ℓ_1 norm). Intuitively, we require that every point in the original infeasible constraint set Δ be *close* to the feasible constraint set $\Delta^\star$ in simulation. Consequently, searching for an upper (or lower) bound within the infeasible set Δ can be replaced with doing so within the feasible set $\Delta^\star$. This requirement will be formalized as condition (iii) in Theorem 1 below.

Second, we need an estimator $\widehat{\mathbf{\Sigma}}$ of the covariance matrix $\mathbf{\Sigma}$. A variety of heteroskedasticity/serial-correlation-robust estimators in the literature can be used. We require $\widehat{\mathbf{\Sigma}}$ be a "good" approximation of $\mathbf{\Sigma}$ in the sense of condition (iv) in Theorem 1 below. It would allow us to approximate the infeasible normal distribution $\mathsf{N}(\mathbf{0}, \mathbf{\Sigma})$ by $\mathsf{N}(\mathbf{0}, \widehat{\mathbf{\Sigma}})$ which can simulated using the data.

Once $\Delta^\star$ and $\widehat{\mathbf{\Sigma}}$ are available, we can simply draw random vectors from $\mathsf{N}(\mathbf{0}, \widehat{\mathbf{\Sigma}})$ conditional on the data, and then set

$$M_{1,\mathtt{L}} = \mathfrak{c}^\star_{\mathtt{L}}(\alpha_1/2) \quad \text{and} \quad M_{1,\mathtt{U}} = \mathfrak{c}^\star_{\mathtt{U}}(1-\alpha_1/2) \tag{4.4}$$

where

$$\begin{aligned}
&\mathfrak{c}^\star_{\mathtt{L}}(\alpha_1/2) := (\alpha_1/2)\text{-quantile of } \inf\{\mathbf{p}_\tau' \boldsymbol{\delta} : \boldsymbol{\delta} \in \mathcal{M}^\star_{\mathbf{G}}\} \quad \text{and} \\
&\mathfrak{c}^\star_{\mathtt{U}}(1-\alpha_1/2) := (1-\alpha_1/2)\text{-quantile of } \sup\{\mathbf{p}_\tau' \boldsymbol{\delta} : \boldsymbol{\delta} \in \mathcal{M}^\star_{\mathbf{G}}\}
\end{aligned}$$

conditional on the data, $\mathcal{M}^\star_{\mathbf{G}} = \{\boldsymbol{\delta} \in \Delta^\star : \ \boldsymbol{\delta}' \widehat{\mathbf{Q}} \boldsymbol{\delta} - 2\mathbf{G}'\boldsymbol{\delta} \leq 0\}$, and $\mathbf{G}|\mathsf{Data} \sim \mathsf{N}(\mathbf{0}, \widehat{\mathbf{\Sigma}})$.

Out-of-sample uncertainty. To bound the out-of-sample error e_τ, we propose an easy-to-implement approach based on non-asymptotic concentration inequalities. Specifically, assume that e_τ is condition-on-$\mathscr{H}$ sub-Gaussian with parameter $\sigma_{\mathscr{H}}$. Then for any $\varepsilon > 0$,

$$\mathbb{P}\Big(|e_\tau - \mathbb{E}[e_\tau|\mathscr{H}]| \geq \varepsilon \Big| \mathscr{H}\Big) \leq 2\exp\Big(-\frac{\varepsilon^2}{2\sigma^2_{\mathscr{H}}}\Big).$$

Consequently, we set

$$M_{2,\mathtt{L}} = \mathbb{E}[e_\tau|\mathscr{H}] - \sqrt{2\sigma^2_{\mathscr{H}}\log(2/\alpha_2)} \qquad \text{and} \qquad M_{2,\mathtt{U}} = \mathbb{E}[e_\tau|\mathscr{H}] + \sqrt{2\sigma^2_{\mathscr{H}}\log(2/\alpha_2)}, \tag{4.5}$$

which yields a prediction interval $[M_{2,\mathtt{L}}, M_{2,\mathtt{U}}]$ that covers $e^{[i]}_{T_i+k}$ with at least $(1-\alpha_2)$ conditional coverage probability. We emphasize that the sub-Gaussianity assumption imposed here is only one illustrative example. The above strategy can still applied using other concentration inequalities that only require weaker moment conditions, though the resulting prediction intervals may be wider.

As discussed in examples below, the generic out-of-sample error e_τ associated with some treatment effect τ is typically a linear combination of individual error terms $e^{[i]}_t = Y_{it}(\infty) - \mathbf{p}^{[i]\prime}_t\boldsymbol{\beta}_0$ in the post-treatment period, and the sub-Gaussianity of e_τ is implied by assuming $e^{[i]}_t$ is sub-Gaussian. In practice, one could first construct pre-treatment residuals $\widehat{e}^{[i]}_t = Y_{it}(\infty) - \mathbf{p}^{[i]\prime}_t\widehat{\boldsymbol{\beta}}^{[i]}$, $t = 1, \cdots, T_0$, and estimate the conditional moments of $e^{[i]}_t$ employing various parametric or nonparametric regression of $\widehat{e}^{[i]}_t$. Such estimates then can be translated into the necessary estimates of $\mathbb{E}[e_\tau|\mathscr{H}]$ and $\sigma^2_{\mathscr{H}}$ for constructing $M_{2,\mathtt{L}}$ and $M_{2,\mathtt{U}}$. Note that the necessary conditional moments could also be set using external information, or tabulated across different values to assess the sensitivity of the resulting prediction intervals. See Cattaneo et al. (2022) for more implementation details.

Example 4.1 (Individual Treatment Effect τ_{ik}, continued)**.** *For the causal quantity τ_{ik}, the out-of-sample error is given by*

$$e_{\tau_{ik}} = Y_{i(T_i+k)}(\infty) - \mathbf{p}^{[i]\prime}_{T_i+k}\boldsymbol{\beta}_0 = e^{[i]}_{T_i+k},$$

If we assume $e^{[i]}_t$ is sub-Gaussian conditional on $\mathscr{H}$, then the strategy outlined above can be applied.

Example 4.2 (Average Post-Treatment Effect $\tau_{i\cdot}$, continued)**.** *For the causal quantity $\tau_{i\cdot}$, the out-of-sample error is given by*

$$e_{\tau_{i\cdot}} := \frac{1}{T-T_i+1}\sum_{t=T_i}^{T}\left(Y_{it}(\infty) - \mathbf{p}^{[i]\prime}_t\boldsymbol{\beta}^{[i]}_0\right) = \frac{1}{T-T_i+1}\sum_{t=T_i}^{T} e^{[i]}_t.$$

The approach outlined above can still be applied to $e_{\tau_{i\cdot}}$. For example, if $e^{[i]}_t$ is condition-on-$\mathscr{H}$ sub-Gaussian with parameter $\sigma_{\mathscr{H},t} > 0$ for $t = T_i, \cdots, T$, it can be shown that $e_{\tau_{i\cdot}}$, as the average

of $e_t^{[i]}$ *across time, satisfies that for any* $\varepsilon > 0$,

$$\mathbb{P}(|e_{\tau_i\cdot} - \mathbb{E}[e_{\tau_i\cdot}|\mathscr{H}]| \geq \varepsilon|\mathscr{H}) \leq 2\exp\Big(-\varepsilon^2/(2\bar{\sigma}^2_{\mathscr{H}})\Big), \quad \bar{\sigma}_{\mathscr{H}} = \frac{1}{T-T_i+1}\sum_{t=T_i}^{T}\sigma_{\mathscr{H},t}.$$

Note that this inequality holds regardless of the dependence structure of $e_t^{[i]}$. *If* $e_t^{[i]}$ *is independent over* t, *the above result can be improved:*

$$\mathbb{P}(|e_{\tau_i\cdot} - \mathbb{E}[e_{\tau_i\cdot}|\mathscr{H}]| \geq \varepsilon|\mathscr{H}) \leq 2\exp\Big(-\varepsilon^2/(2\tilde{\sigma}^2_{\mathscr{H}})\Big), \quad \tilde{\sigma}_{\mathscr{H}} = \frac{1}{T-T_i+1}\Big(\sum_{t=T_i}^{T}\sigma^2_{\mathscr{H},t}\Big)^{1/2}.$$

Then, one can characterize each $\sigma_{\mathscr{H},t}$, $t = T_i, \cdots, T$, *and use the idea outlined above to construct bounds on* $e_{\tau_i\cdot}$.

Alternatively, one could construct a pre-treatment sequence of errors that is analogous to $e_{\tau_i\cdot}$:

$$\tilde{e}_t^{[i]} := \frac{1}{T-T_i+1}\sum_{t=\ell}^{\ell+T-T_i} e_t^{[i]}, \quad 1 \leq \ell \leq 2T_i - T - 1.$$

Then, apply the strategy outlined before to this new sequence of errors, which requires characterizing the (conditional) moments of $\tilde{e}_t^{[i]}$. *One caveat is that by construction,* $\tilde{e}_t^{[i]}$ *could have a different dependence structure than the original sequence* $e_t^{[i]}$. *For example, even if* $e_t^{[i]}$ *is independent over* t *conditional on* $\mathscr{H}$, $\tilde{e}_t^{[i]}$ *would be* $(T-T_i+1)$*-dependent conditional on* $\mathscr{H}$ *in general, that is,* $\tilde{e}_t^{[i]}$ *is independent of* $\tilde{e}_{t+\ell}^{[i]}$ *conditional on* $\mathscr{H}$ *if* $\ell \geq T - T_i + 1$.

Example 4.3 (Average Treatment Effect on the Treated $\tau_{\cdot k,s_0}$, continued)**.** *In this scenario, the out-of-sample error is given by*

$$e_{\tau_{\cdot k,s_0}} := \frac{1}{N_{s_0}}\sum_{i:T_i=s_0}\Big(Y_{i(s_0+k)}(\infty) - \mathbf{p}^{[i]\prime}_{s_0+k}\boldsymbol{\beta}^{[i]}_0\Big) = \frac{1}{N_{s_0}}\sum_{i:T_i=s_0} e^{[i]}_{s_0+k}.$$

The out-of-sample error above is similar to that defined in Example 4.2 except that $e_{\tau_{\cdot k,s_0}}$ *is a cross-sectional average of* $e_t^{[i]}$ *rather than a time-series average. The uncertainty quantification strategy outlined in Example 4.2 can still be applied, with the caveat that it is uncommon in SC analysis to assume* $e_t^{[i]}$ *is stationary and/or independent over* i. *By contrast, it is reasonable to assume* $e_t^{[i]}$ *is stationary and/or independent (at least weakly dependent) over time.*

Example 4.4 (Average Treatment Effect on the Treated $\tau_{\cdot k}$, continued)**.** *In this scenario, the out-of-sample error is given by*

$$e_{\tau_{\cdot k}} := \frac{1}{N_1}\sum_{i=1}^{N_1}\left(Y_{i(T_i+k)}(\infty) - \mathbf{p}_{T_i+k}^{[i]\prime}\boldsymbol{\beta}_0^{[i]}\right) = \frac{1}{N_1}\sum_{i=1}^{N_1} e_{T_i+k}^{[i]}.$$

Since the adoption time T_i may be heterogeneous across i, $e_{\tau_{\cdot k}}$ is an average of the individual errors $e_t^{[i]}$ of different units in different periods. Again, one can characterize the (conditional) moments of each $e_t^{[i]}$ and use the concentration-inequality-based approach to bound $e_{\tau_{\cdot k}}$. By contrast, it would be difficult to implement the other strategy outlined in Example 4.2 that relies on constructed pre-treatment averaged errors analogous to $e_{\tau_{\cdot k}}$, since such averages may include errors that are far away in time and their dependence on $\mathscr{H}$ may be very complex.

In addition to the concentration-based approach described above, other strategies, including location-scale models and quantile regression, were proposed in CFT for out-of-sample uncertainty quantification. We briefly review them in Appendix S.1.1.

Main theorem. Now, we are ready to present our main theorem, which shows that the prediction interval constructed above achieves approximately $(1-\alpha_1-\alpha_2)$ conditional coverage probability, which holds with high probability on $\mathscr{H}$. In the following we use $\|\cdot\|_*$ to denote the dual norm of $\|\cdot\|$ on $\mathbb{R}^d$ and use $\|\cdot\|_{\mathtt{F}}$ to denote the Frobenius matrix norm.

Theorem 1. *Assume $\mathcal{W}$ and $\mathcal{R}$ are convex, $\widehat{\boldsymbol{\beta}}$ in (3.1) and $\boldsymbol{\beta}_0$ in (4.2) exist, $\mathscr{H} = \sigma(\mathbf{B}, \mathbf{C}, \mathbf{p}_\tau)$, and $M_{1,\mathtt{L}}$, $M_{1,\mathtt{U}}$, $M_{2,\mathtt{L}}$ and $M_{2,\mathtt{U}}$ are specified as in (4.4) and (4.5). In addition, for some finite non-negative constants ϵ_γ, π_γ, $\varpi_\delta^\star$, $\epsilon_\delta^\star$, $\pi_\delta^\star$, $\varpi_\Delta^\star$, $\epsilon_\Delta^\star$, $\pi_\Delta^\star$, $\epsilon_{\gamma,1}^\star$, $\epsilon_{\gamma,2}^\star$, $\pi_\gamma^\star$ and $\sigma_{\mathscr{H}}$, the following conditions hold:*

(i) $\mathbb{P}[\mathbb{P}(\mathbf{p}_\tau'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \in [\mathfrak{c}_{\mathtt{L}}(\alpha_1/2), \mathfrak{c}_{\mathtt{U}}(1-\alpha_1/2)]|\mathscr{H}) \geq 1-\alpha_1-\epsilon_\gamma] \geq 1-\pi_\gamma$;

(ii) $\mathbb{P}[\mathbb{P}(\sup\{\|\boldsymbol{\delta}\| : \boldsymbol{\delta} \in \mathcal{M}_{\mathbf{G}}\} \leq \varpi_\delta^\star|\mathscr{H}) \geq 1-\epsilon_\delta^\star] \geq 1-\pi_\delta^\star$;

(iii) $\mathbb{P}[\mathbb{P}(\sup_{\mathbf{a}\in\Delta\cap\mathcal{B}(0,\varpi_\delta^\star)} \mathrm{dist}(\mathbf{a}, \Delta^\star \cap \mathcal{B}(0,\varepsilon)) \leq \varpi_\Delta^\star\ |\mathscr{H}) \geq 1-\epsilon_\Delta^\star] \geq 1-\pi_\Delta^\star$;

(iv) $\mathbb{P}[\mathbb{P}(\|\boldsymbol{\Sigma}^{-1/2}\widehat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1/2} - \mathbf{I}_d\|_{\mathtt{F}} \leq 2\epsilon_{\gamma,1}^\star|\mathscr{H}) \geq 1-\epsilon_{\gamma,2}^\star] \geq 1-\pi_\gamma^\star$;

(v) *e_τ is sub-Gaussian conditional on $\mathscr{H}$ with parameter $\sigma_{\mathscr{H}}$.*

Then, for $\epsilon^\star_{\gamma,1} \in [0, 1/4]$,

$$\mathbb{P}\Big\{\mathbb{P}\big(\tau \in [\widehat{\tau} + M_{1,\mathtt{L}} - M_{2,\mathtt{U}} - \varepsilon_\Delta,\ \widehat{\tau} + M_{1,\mathtt{U}} - M_{2,\mathtt{L}} + \varepsilon_\Delta]\big|\mathscr{H}\big) \geq 1 - \alpha_1 - \alpha_2 - \epsilon\Big\} \geq 1 - \pi,$$

where $\epsilon = \epsilon_\gamma + \epsilon^\star_{\gamma,1} + \epsilon^\star_{\gamma,2} + \epsilon^\star_\delta + \epsilon^\star_\Delta$, $\pi = \pi_\gamma + \pi^\star_\gamma + \pi^\star_\delta + \pi^\star_\Delta$ *and* $\varepsilon_\Delta = \|\mathbf{p}_\tau\|_* \varpi^\star_\Delta$.

Among the five conditions imposed in this theorem, (i)-(iv) are high-level conditions used for in-sample uncertainty quantification, which can be verified in many practically relevant scenarios. See more detailed discussion in Section 4.1. Condition (v), as we emphasized before, is a moment condition on e_τ that is used to showcase our out-of-sample uncertainty quantification strategy and can be relaxed by utilizing other appropriate concentration inequalities. Moreover, the constant ε_Δ in the theorem is used to adjust the prediction interval for nonlinear constraints. In many SC applications with linear constraints only (e.g., simplex or lasso constraint), such adjustment is *unnecessary* and we can set $\varepsilon_\Delta = 0$. See Section 5.2 for more discussion.

4.1 Discussion of Conditions (i)-(iv)

Theorem 1 relies on high-level conditions (i)-(iv). In this section we discuss each of them in more detail.

- **Condition (i)**. This condition formalizes the idea of distributional approximation of $\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}$ by a Gaussian vector $\mathbf{G}$, which is an immediate extension of the conclusion given in Theorem 1 and Theorem A of Cattaneo et al. (2021) to the setup with multiple treated units. In general, condition (i) can be verified if the error term $(u^{[1]}_{t,1}, \cdots, u^{[1]}_{t,M}, \cdots, u^{[N_1]}_{t,1}, \cdots, u^{[N_1]}_{t,M})'$ is independent or weakly dependent (e.g., β-mixing) over t conditional on $\mathscr{H}$, in addition to other regularity conditions. Importantly, the features included in $\mathbf{A}$ and $\mathbf{B}$ could be non-stationary, thus covering the cointegration case which is common in SC applications.

- **Condition (ii)**. This is a mild condition on the concentration of $\boldsymbol{\delta} \in \mathcal{M}_\mathbf{G}$. The requirement $\boldsymbol{\delta}'\widehat{\mathbf{Q}}\boldsymbol{\delta} - 2\mathbf{G}'\boldsymbol{\delta} \leq 0$ is usually known as the *basic inequality* in regression analysis (see Chapter 7 of Wainwright (2019) for the example of Lasso). Note that $\mathbf{G}$ is (conditional) Gaussian by construction, making condition (ii) easy to verify based on well-known bounds for Gaussian distributions. Section 4.1 of Cattaneo et al. (2021) verifies this condition in three concrete

SC examples: outcomes-only regression with i.i.d. data, multi-equation regression with weakly dependent data, and cointegration.

- **Condition (iii)**. This is a high-level requirement on the "closeness" between Δ and $\Delta^\star$. Here we propose a strategy for constructing $\Delta^\star$, which can be shown to satisfy (iii) if the constraints specified in $\mathcal{W}$ and $\mathcal{R}$ are formed by smooth functions. To be specific, suppose that

$$\mathcal{W} \times \mathcal{R} = \Big\{ \boldsymbol{\beta} \in \mathbb{R}^d : \boldsymbol{m}_{\mathtt{eq}}(\boldsymbol{\beta}) = \mathbf{0}, \boldsymbol{m}_{\mathtt{in}}(\boldsymbol{\beta}) \leq \mathbf{0} \Big\},$$

where $\boldsymbol{m}_{\mathtt{eq}}(\cdot) \in \mathbb{R}^{d_{\mathtt{eq}}}$ and $\boldsymbol{m}_{\mathtt{in}}(\cdot) \in \mathbb{R}^{d_{\mathtt{in}}}$. Let the jth constraint in $\boldsymbol{m}_{\mathtt{in}}(\cdot)$ be $m_{\mathtt{in},j}(\cdot)$. Given tuning parameters $\varrho_j > 0$, $j = 1, \cdots, d_{\mathtt{in}}$, let $\mathcal{A} = \{j_1, \cdots, j_k\}$ denote the set of indices for the inequality constraints such that $m_{\mathtt{in},j}(\widehat{\boldsymbol{\beta}}) > -\varrho_j$. Then define

$$\Delta^\star = \Big\{ \boldsymbol{\beta} - \widehat{\boldsymbol{\beta}} : \boldsymbol{m}_{\mathtt{eq}}(\boldsymbol{\beta}) = \mathbf{0}, m_{\mathtt{in},j}(\boldsymbol{\beta}) \leq m_{\mathtt{in},j}(\widehat{\boldsymbol{\beta}}) \text{ for } j \in \mathcal{A}, m_{\mathtt{in},l}(\boldsymbol{\beta}) \leq \mathbf{0} \text{ for } l \notin \mathcal{A} \Big\}. \tag{4.6}$$

The following lemma verifies condition (iii) for this $\Delta^\star$. We use $s_{\min}(\mathbf{M})$ to denote the minimum singular value of a matrix $\mathbf{M}$.

Lemma 1. *Let $\|\cdot\|$ be the Euclidean norm for vectors and the spectral norm for matrices. Assume condition (ii) in Theorem 1 holds. In addition, assume that with probability at least $1 - \pi^\star_\beta$, the following conditions hold: (i) $\boldsymbol{m}(\cdot) = (\boldsymbol{m}_{\mathtt{eq}}(\cdot)', \boldsymbol{m}_{\mathtt{in}}(\cdot)')'$ is twice continuously differentiable on $\mathcal{B}(\boldsymbol{\beta}_0, \varpi^\star_\delta)$ with $\inf_{\boldsymbol{\beta} \in \mathcal{B}(\boldsymbol{\beta}_0, \varpi^\star_\delta)} s_{\min}(\frac{\partial}{\partial \boldsymbol{\beta}'} \boldsymbol{m}(\boldsymbol{\beta})) \geq c_{\min}$ for some constant $c_{\min} > 0$; (ii) for all $1 \leq j \leq d_{\mathtt{in}}$, $\varrho_j \in (\mathfrak{c}\varpi^\star_\delta, |m_{\mathtt{in},j}(\boldsymbol{\beta}_0)| - \mathfrak{c}\varpi^\star_\delta)$ for some constant $\mathfrak{c} > 0$ specified in the proof. Then, for $\Delta^\star$ defined in (4.6), condition (iii) holds with $\varpi^\star_\Delta = \mathfrak{C}(\varpi^\star_\delta)^2$, $\epsilon^\star_\Delta = \epsilon^\star_\delta$ and $\pi^\star_\Delta = \pi^\star_\delta + \pi^\star_\beta$ for some constant $\mathfrak{C} > 0$.*

In this lemma the tuning parameters ϱ_j are introduced to guarantee that with high probability, we can correctly differentiate the binding inequality constraints from the other non-binding ones. In Section 5 below, we provide more practical details about choosing ϱ_j.

- **Condition (iv)**. This is a requirement that $\widehat{\boldsymbol{\Sigma}}$ is a "good" approximation of the unknown covariance matrix $\boldsymbol{\Sigma}$. Many standard covariance estimation strategies such as the family of well-known heteroskedasticity-consistent estimators can be utilized.

4.2 Simultaneous Prediction Intervals

So far we have focused on constructing prediction intervals that have high coverage of the desired treatment effects, in particular, the individual treatment effect in *each* post-treatment period. In some applications, it might be appealing to construct prediction intervals that have high *simultaneous* coverage in multiple post-treatment periods, usually termed *simultaneous prediction intervals* in the literature. Specifically, for a particular treated unit $1 \leq i \leq N_1$, we aim to construct a sequence of intervals $\mathcal{I}_k$ for $0 \leq k \leq L$ for some $L \leq T - T_i$ such that

$$\mathbb{P}\Big\{\mathbb{P}\big[\tau_{ik} \in \mathcal{I}_k, \text{ for all } 0 \leq k \leq L \,\big|\, \mathscr{H}\big] \geq 1 - \alpha\Big\} \geq 1 - \pi.$$

As described before, the uncertainty of the estimated individual treatment effect $\widehat{\tau}_{ik}$ comes from the in-sample error $\mathbf{p}'_{\tau_{ik}}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)$ and the out-of-sample error $e^{[i]}_{T_i+k}$.

Regarding the in-sample error, the following is an immediate generalization of the prediction interval described in (4.4), which enjoys simultaneous coverage in multiple periods: If constraints imposed in Δ are linear (such as simplex and lasso constraints), set

$$\begin{aligned}
M_{1,\mathtt{L}} &:= (\alpha_1/2)\text{-quantile of } \inf\Big\{\mathbf{p}'_{\tau_{ik}}\boldsymbol{\delta} :\ \boldsymbol{\delta} \in \Delta^\star,\ \ell^\star(\boldsymbol{\delta}) \leq 0,\ 0 \leq k \leq L\Big\} \text{ and} \\
M_{1,\mathtt{U}} &:= (1 - \alpha_1/2)\text{-quantile of } \sup\Big\{\mathbf{p}'_{\tau_{ik}}\boldsymbol{\delta} :\ \boldsymbol{\delta} \in \Delta^\star,\ \ell^\star(\boldsymbol{\delta}) \leq 0,\ 0 \leq k \leq L\Big\},
\end{aligned}$$

which guarantees that with high probability,

$$\mathbb{P}\big[M_{1,\mathtt{L}} \leq \mathbf{p}'_{\tau_{ik}}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \leq M_{1,\mathtt{U}} \text{ for all } 0 \leq k \leq L \,\big|\, \mathscr{H}\big] \geq 1 - \alpha_1;$$

if constraints imposed in Δ are nonlinear (such as the ridge-type constraint), further decrease the lower bound $M_{1,\mathtt{L}}$ and increase the upper bound $M_{1,\mathtt{U}}$ defined above by some $\overline{\varepsilon}_\Delta$. We regard $\overline{\varepsilon}_\Delta$ as a small tuning parameter used to adjust for nonlinear constraints. See more discussion about selecting this parameter in Section 5.2.

Regarding the out-of-sample error, the goal is to find $M_{2,\mathtt{L}}$ and $M_{2,\mathtt{U}}$ such that with high probability,

$$\mathbb{P}\big[M_{2,\mathtt{L}} \leq e^{[i]}_{T_i+k} \leq M_{2,\mathtt{U}} \text{ for all } 0 \leq k \leq L \,\big|\, \mathscr{H}\big] \geq 1 - \alpha_2.$$

An easy-to-implement strategy analogous to that described in (4.5) is to adjust the intervals based on maximal inequalities. For example, suppose that each $e^{[i]}_{T_i+k}$, $0 \leq k \leq L$, is conditional sub-Gaussian with parameter $\sigma_{\mathscr{H},k}$ (not necessarily independent over k). Then,

$$\mathbb{P}\Big(\max_{0\leq k\leq L} |e^{[i]}_{T_i+k} - \mathbb{E}[e^{[i]}_{T_i+k}|\mathscr{H}]| \geq \varepsilon \Big| \mathscr{H} \Big) \leq 2\sum_{k=0}^{L} \exp\Big(-\frac{\varepsilon^2}{2\sigma^2_{\mathscr{H},k}} \Big).$$

If $\sigma_{\mathscr{H},k} \leq \sigma_{\mathscr{H}}$ for all $0 \leq k \leq T - T_i$, then one can take $M_{2,\mathtt{L}} = \mathbb{E}[e_{T_i+k}|\mathscr{H}] - \varepsilon$ and $M_{2,\mathtt{U}} = \mathbb{E}[e_{T_i+k}|\mathscr{H}] + \varepsilon$ with $\varepsilon = \sqrt{2\sigma^2_{\mathscr{H}} \log(2(L+1)/\alpha_2)}$. Compared with prediction intervals with validity for each period constructed the same way, these simultaneous prediction intervals are slightly wider due to the additional factor $\sqrt{\log(L+1)}$. In practice, one only needs to estimate the conditional mean and variance of $e^{[i]}_t$ using the pre-treatment residuals. Flexible parametric or non-parametric estimation methods can be used.

Again, the sub-Gaussianity assumption can be relaxed by using other concentration inequalities that only require weaker moment conditions, though resulting simultaneous prediction intervals may be wider. Also, there are other strategies to construct prediction intervals that simultaneously cover multiple out-of-sample errors, though they are computationally more cumbersome and usually require more stringent conditions. See Appendix S.1.2 for a brief discussion.

5 Tuning Parameter Selection

To implement the uncertainty quantification procedure described before, one needs to select two kinds of tuning parameters: $\{\varrho_j : 1 \leq j \leq d_{\mathtt{in}}\}$ and ε_Δ. ϱ_js are used to define the constraint set $\Delta^\star$ in simulation that would approximately preserve the local geometry of the original constraint set Δ, while ε_Δ is an adjustment of the bounds on the in-sample error that take into account the "distance" between Δ and $\Delta^\star$. In most SC applications, each constraint is imposed on parameters associated with one treated unit rather than multiple treated units, which will be the focus of the following discussion.

5.1 Defining Constraints in Simulation

In the proposed procedure, a sequence of tuning parameters ϱ_j, $j = 1, \cdots, d_{\mathtt{in}}$, are introduced to determine which inequality constraints are binding. In this section we propose a feasible strategy to select ϱ_j. Consider a (generic) treated unit i, and suppose that it is associated with inequality constraints with indices in $\mathcal{S}^{[i]}$.

We start with the idea of condition (ii) in Theorem 1 and use a parameter $\varrho^{[i]}$ to bound the deviation of $\widehat{\boldsymbol{\beta}}^{[i]}$ from $\boldsymbol{\beta}_0^{[i]}$. The following formula is used:

$$\varrho^{[i]} = \mathcal{C}\frac{\log(T_0)^c}{T_0^{1/2}},$$

where $c = 1/2$ if the data are i.i.d. or weakly dependent, and $c = 1$ if $\mathbf{A}^{[i]}$ and $\mathbf{B}^{[i]}$ form a cointegrated system, while $\mathcal{C}$ is one of the following

$$\mathcal{C}_1 = \frac{\widehat{\sigma}_u}{\min_{1\leq j\leq J}\widehat{\sigma}_{b_j}}, \qquad \mathcal{C}_2 = \frac{\max_{1\leq j\leq J}\widehat{\sigma}_{b_j}\widehat{\sigma}_u}{\min_{1\leq j\leq J}\widehat{\sigma}_{b_j}^2}, \qquad \mathcal{C}_3 = \frac{\max_{1\leq j\leq J}\widehat{\sigma}_{b_j u}}{\min_{1\leq j\leq J}\widehat{\sigma}_{b_j}^2},$$

where $\widehat{\sigma}_{b_j,u}$ is the estimated (unconditional) covariance between the pseudo-true residual $\mathbf{u}^{[i]}$ and the jth column of $\mathbf{B}^{[i]}$ (the features of the jth control unit), and $\widehat{\sigma}_u$ and $\widehat{\sigma}_{b_j}$ are the estimated (unconditional) standard deviation of $\mathbf{u}^{[i]}$ and the jth conlumn of $\mathbf{B}^{[i]}$ respectively. Notice that if the synthetic control weights are constructed based on both stationary features and non-stationary features, the non-stationary components would govern the precision of the estimation. In such cases, one could ignore the stationary components and take $c = 1$ in the above calculation.

Next, we define possibly heterogeneous parameters ϱ_j, $j \in \mathcal{S}^{[i]}$, for different inequality constraints associated with unit i. Note that by the first-order Taylor expansion, if the jth inequality constraint is binding, i.e., $m_{\mathtt{in},j}(\boldsymbol{\beta}_0^{[i]}) = 0$, then

$$m_{\mathtt{in},j}(\widehat{\boldsymbol{\beta}}^{[i]}) \approx \frac{\partial}{\partial \boldsymbol{\beta}'} m_{\mathtt{in},j}(\boldsymbol{\beta}_0^{[i]})(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0).$$

Then, an intuitive choice of ϱ_j would be

$$\varrho_j := \left\| \frac{\partial}{\partial \boldsymbol{\beta}} m_{\mathtt{in},j}(\widehat{\boldsymbol{\beta}}^{[i]}) \right\|_1 \times \varrho^{[i]}, \quad j \in \mathcal{S}^{[i]},$$

where $\|\cdot\|_1$ denotes the ℓ_1 norm. If $m_{\mathtt{in},j}(\widehat{\boldsymbol{\beta}}^{[i]}) > -\varrho_j$, let the jth constraint be binding in the simulation-based uncertainty quantification.

5.2 Adjustment for Nonlinear Constraints

When some constraints imposed in Δ are nonlinear (e.g., ridge-type constraints), we introduce a constant ε_Δ to adjust the bounds on the in-sample error (see Theorem 1), which depends on the distance $\varpi_\Delta^\star$ between the localized constraint sets Δ and $\Delta^\star$ specified in condition (iii). We emphasize that this adjustment is *unnecessary* if constraints are linear in parameters (e.g., simplex or lasso constraints).

The distance between Δ and $\Delta^\star$ typically depends on the the first and second derivatives of the constraint functions $\boldsymbol{m}_{\mathtt{in}}(\cdot)$. Again, we first focus on the inequality constraints related to one particular treated unit i. Denote by $\boldsymbol{m}_{\mathtt{in},\mathcal{S}^{[i]}}(\cdot)$ the vector of constraint functions $m_{\mathtt{in},j}(\cdot)$ with $j \in \mathcal{S}^{[i]}$. We propose to set

$$\varepsilon_\Delta^{[i]} = \|\mathbf{p}_\tau^{[i]}\|_1 \times \frac{\sqrt{|\mathcal{S}^{[i]}|}}{2} s_{\min}^{-1}\Big(\frac{\partial}{\partial \boldsymbol{\beta}} \boldsymbol{m}_{\mathtt{in},\mathcal{S}^{[i]}}(\widehat{\boldsymbol{\beta}}^{[i]})\Big) \times \max_{j\in\mathcal{S}^{[i]}} s_{\max}\Big(\frac{\partial}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}'} m_{\mathtt{in},j}(\widehat{\boldsymbol{\beta}}^{[i]})\Big) \times (\varrho^{[i]})^2,$$

where $\mathbf{p}_\tau^{[i]}$ denotes the subvector of $\mathbf{p}_\tau$ that corresponds to $\boldsymbol{\beta}_0^{[i]}$, and $|\mathcal{S}^{[i]}|$ denotes the cardinality of $\mathcal{S}^{[i]}$. Denote by $\mathcal{N}_\tau$ the set of treated units to which the causal quantity τ are related. Then set

$$\varepsilon_\Delta = \sum_{i\in\mathcal{N}_\tau} \varepsilon_\Delta^{[i]}.$$

Notice that when constraints are linear, the second derivative of $m_{\mathtt{in},j}(\cdot)$ is zero, and then the above choice of ε_Δ would be exactly 0.

One typical example of nonlinear constraints that is commonly used in practice is the ridge-type restriction. Specifically, suppose that there is one treated unit i and one constraint

$$\|\boldsymbol{\beta}^{[i]}\|_2^2 = 1.$$

The above choice of ε_Δ simplifies to

$$\varepsilon_\Delta^{[i]} = \|\mathbf{p}_\tau^{[i]}\|_1 \times (2\|\widehat{\boldsymbol{\beta}}^{[i]}\|_2)^{-1} \times (\varrho^{[i]})^2.$$

Remark 1 (Simultaneous Prediction Intervals)**.** In Section 4.2 we discussed how to construct simultaneous prediction intervals. If nonlinear constraints are imposed, a tuning parameter $\overline{\varepsilon}_\Delta$ was introduced to adjust the bounds on in-sample errors. In this context, apply the procedure described above to each period $T_i + k$, $0 \leq k \leq L$, and thus we can obtain a sequence of constants, denoted by $\varepsilon_{\Delta,k}$, $0 \leq k \leq L$. We let $\overline{\varepsilon}_\Delta = \max_{0\leq k\leq L} \varepsilon_{\Delta,k}$. ⌟

6 Empirical Application

This section presents the results of applying the methodology thoroughly described above to the setting studied in Billmeier and Nannicini (2013). For brevity, here we only report results for Europe, whereas the interested reader can find complete results for Africa, Asia, North America, and South America in Section S.5 of the Supplemental Appendix. Specifically, we estimate three of the causal quantities introduced along Section 3: (i) country-year treatment effects (Example 3.1) up to 10 periods after the liberalization occurred; (ii) average treatment effects for each country (Example 3.2); (iii) the average treatment effect on countries liberalized in 1991 (Example 3.3).

6.1 Empirical Strategy

In our main specification, we match on two features ($M = 2$), logarithm of GDP per capita and percentage of complete secondary schooling attained in population. We obtain the SC weights under the L1-L2 constraint, i.e.,

$$\mathcal{W} = \bigtimes_{i=1}^{N_1} \left\{ \mathbf{w}^{[i]} \in \mathbb{R}^J : ||\mathbf{w}^{[i]}||_1 = 1, ||\mathbf{w}^{[i]}||_2 \leq Q_2^{[i]} \right\},$$

and conduct covariate adjustment by including a constant term that is common across features and a constant term and a linear time trend for each feature. We impose no constraints on these additional parameters, i.e., $\mathcal{R} = \mathbb{R}^{KMN_1}$.[2] We set the weighting matrix $\mathbf{V} = \mathbf{I}$. The vector of predictors $\mathbf{p}_t$ contains the (log) GDP per capita of countries in the donor pool in each post-treatment period, a constant term, and a linear trend. As described in Section 3, given a particular

[2] In the Supplemental Appendix Section S.4, we replicates the whole analysis using the popular "simplex" constraint, i.e., $\mathcal{W} = \times_{i=1}^{N_1} \{\mathbf{w}^{[i]} \in \mathbb{R}_+^J : ||\mathbf{w}^{[i]}||_1 = 1\}$.

treatment effect τ, the predictor $\mathbf{p}_\tau$ is defined accordingly. Table 2 describes all the matrices in greater detail.

Table 2: *Description of main quantities used for estimation/inference.*

Matrix	Description
$\mathbf{A}^{[i]} = \begin{bmatrix} \mathbf{A}_1^{[i]} \\ \mathbf{A}_2^{[i]} \end{bmatrix}$	matrix of pre-treatment features for the ith treated unit
$\mathbf{B}^{[i]} = \begin{bmatrix} \mathbf{B}_1^{[i]} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2^{[i]} \end{bmatrix}$	matrix of pre-treatment features for the donors of the ith treated unit
$\mathbf{C}^{[i]} = \begin{bmatrix} \mathbf{1} & \mathbf{C}_1^{[i]} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{C}_2^{[i]} \end{bmatrix}$	matrix of adjustment covariates used for the ith treated unit's equation
$\mathbf{p}_t^{[i]} = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{g}_t \end{bmatrix}$	vector of predictors for the ith treated unit

Notes: all quantities in the table are defined for a generic treated unit $i = 1, \ldots, N_1$. Feature 1 is GDP per capita and feature 2 is the percentage of complete secondary schooling attained in population. The vectors $\mathbf{0}$ and $\mathbf{1}$ are conformable vectors of zeros and ones, respectively.

In addition, since political and economic reforms do not happen overnight, we take into account the possibility of anticipation effects up to 1 year before the treatment. To further clarify, take the example of Albania, which underwent a process of economic liberalization in 1992, according to the Sachs-Werner indicator. To control for the presence of plausible anticipation effects, we simply define the pre-treatment period for Albania as 1963-1990 rather than 1963-1991.

Recall that to quantify the in-sample uncertainty from estimating the SC weights, we need to construct the bounds $M_{1,\mathtt{L}}$ and $M_{1,\mathtt{U}}$ on $\mathbf{p}_\tau'(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)$. The following strategy is adopted. First, we treat the synthetic control weights as possibly misspecified, thus estimating both the first and second conditional moments of the pseudo-true residuals $\mathbf{u}$. The conditional first moment $\mathbb{E}[\mathbf{u} \,|\, \mathscr{H}]$ is estimated feature-by-feature using a linear-in-parameters regression of $\widehat{\mathbf{u}} = \mathbf{A} - \mathbf{B}\widehat{\mathbf{w}} - \mathbf{C}\widehat{\mathbf{r}}$ on $\mathbf{B}$ and the first lag of $\mathbf{B}$, whereas the conditional second moment $\mathbb{V}[\mathbf{u} \,|\, \mathscr{H}]$ is estimated with an HC1-type estimator. We then draw $S = 200$ i.i.d. random vectors from the Gaussian distribution $\mathsf{N}(0, \widehat{\boldsymbol{\Sigma}})$, conditional on the data, to simulate the criterion function $\ell_{(s)}^{\dagger}(\boldsymbol{\delta}) := \boldsymbol{\delta}'\widehat{\mathbf{Q}}\boldsymbol{\delta} - 2\mathbf{G}_{(s)}'\boldsymbol{\delta}$, $s = 1, \ldots, 200$,

and solve the following optimization problems

$$l_{(s)} := \inf_{\boldsymbol{\delta}\in\Delta^\star,\, \ell^\star_{(s)}(\boldsymbol{\delta})\leq 0} \mathbf{p}_\tau'\boldsymbol{\delta} \qquad \text{and} \qquad u_{(s)} := \sup_{\boldsymbol{\delta}\in\Delta^\star,\, \ell^\star_{(s)}(\boldsymbol{\delta})\leq 0} \mathbf{p}_\tau'\boldsymbol{\delta},$$

where $\Delta^\star$ is constructed as explained in Section 4.1. Finally, $M_{1,\mathtt{L}}$ is the $(\alpha_1/2)-$quantile of $\{l_{(s)}\}_{s=1}^S$ and $M_{1,\mathtt{U}}$ is the $(1-\alpha_1/2)-$quantile of $\{u_{(s)}\}_{s=1}^S$, where α_1 is set to 0.05.

Recall that to quantify the out-of-sample uncertainty from the stochastic error in the post-treatment period, we need to construct the bounds $M_{2,\mathtt{L}}$ and $M_{2,\mathtt{U}}$ on e_τ. We employ the non-asymptotic bounds described in (4.5), assuming that e_τ is sub-Gaussian conditional on $\mathscr{H}$. We set $\alpha_2 = 0.05$, and the conditional mean $\mathbb{E}[e_\tau|\mathscr{H}]$ and the sub-Gaussian parameter $\sigma_{\mathscr{H}}$ are parametrized and estimated by a linear-in-parameters regression of the pre-treatment residuals on $\mathbf{B}$.

Finally, the prediction intervals for the counterfactual outcome and the treatment effect of interest are given by

$$\left[\mathbf{p}_\tau'\widehat{\boldsymbol{\beta}} - M_{1,\mathtt{U}} + M_{2,\mathtt{L}};\ \mathbf{p}_\tau'\widehat{\boldsymbol{\beta}} - M_{1,\mathtt{L}} + M_{2,\mathtt{U}}\right] \quad \text{and} \quad [\widehat{\tau} + M_{1,\mathtt{L}} - M_{2,\mathtt{U}};\ \widehat{\tau} + M_{1,\mathtt{U}} - M_{2,\mathtt{L}}]$$

respectively.

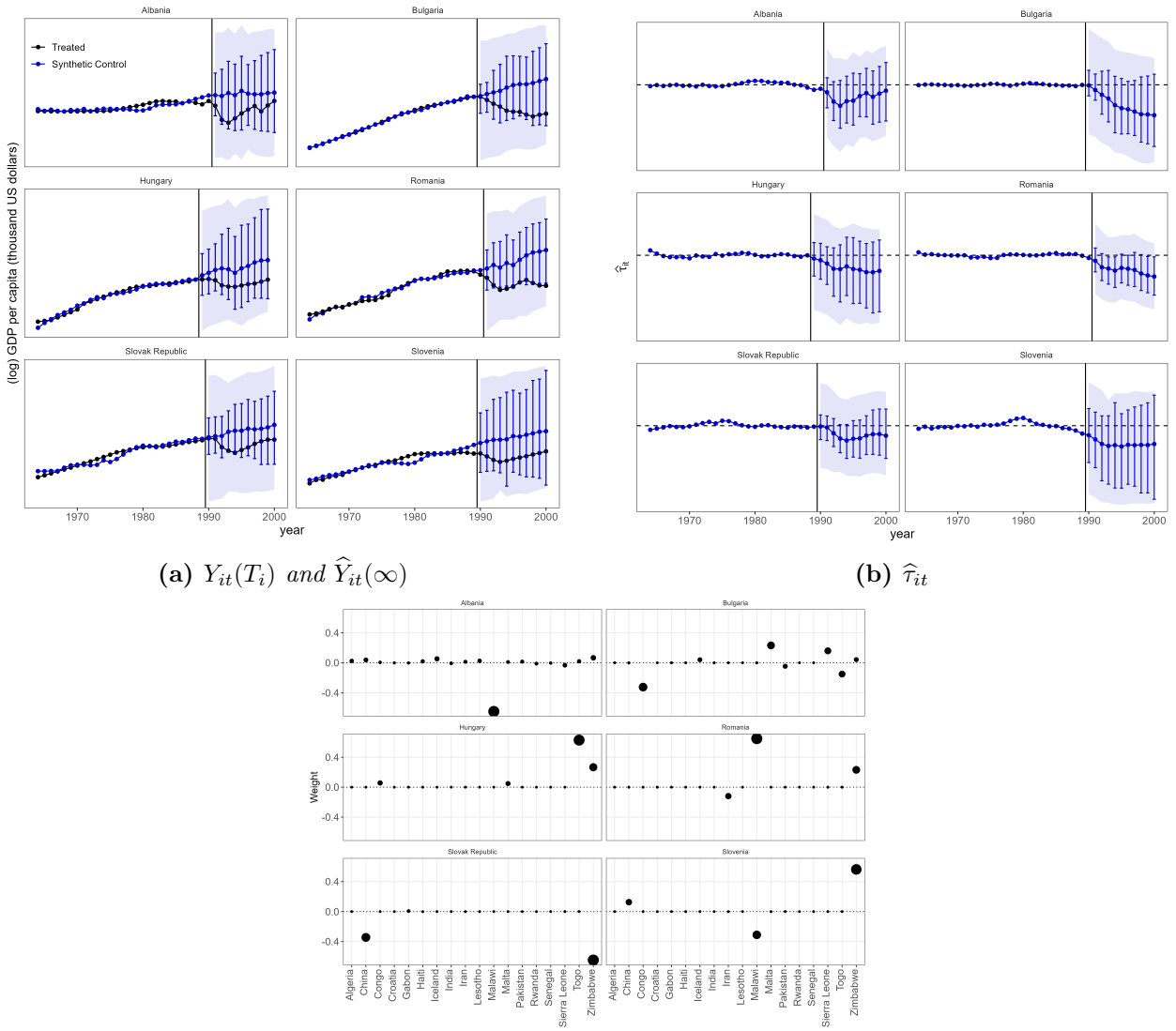
6.2 Results

Overall, our point estimates suggest that (external) liberalization episodes in Europe had a negative impact on GDP per capita. These findings might be explained by the nation's net trade position, among other things. For example, a negative effect on GDP can be justified if, following the liberalization event, imports rise more (or decrease less) than exports do, and this difference more than offsets the likely rise in consumption and investment. However, once we take uncertainty into account, the prediction intervals for the synthetic control always contain the realized GDP per capita series, implying that there is no strong evidence that the real income trajectory has been altered.

Country-year treatment effect. In Figure 5 we show the estimated synthetic controls (panel (a)) and the estimated treatment effects (panel (b)) with the corresponding 90% prediction intervals, whilst the estimated weights $\widehat{\mathbf{w}}$ are depicted in the bottom panel. We clearly see that in all six countries

the realized trajectory of GDP per capita (black lines) lies below the synthetic one (blue lines), implying that in the absence of the liberalization event, real income per capita would have been higher. Looking at the 90% prediction intervals (blue vertical bars), we can see that in most cases the distance between the actual GDP series and the counterfactual one is not different from zero with high probability for almost all units and periods. If we consider 90% simultaneous prediction intervals for each unit (blue shaded areas), it is clear that the realized and synthetic trajectories do not simultaneously differ with high probability.

Figure 5: *Estimated Individual Treatment Effects* $\widehat{\tau}_{it}$.



(a) $Y_{it}(T_i)$ *and* $\widehat{Y}_{it}(\infty)$ **(b)** $\widehat{\tau}_{it}$

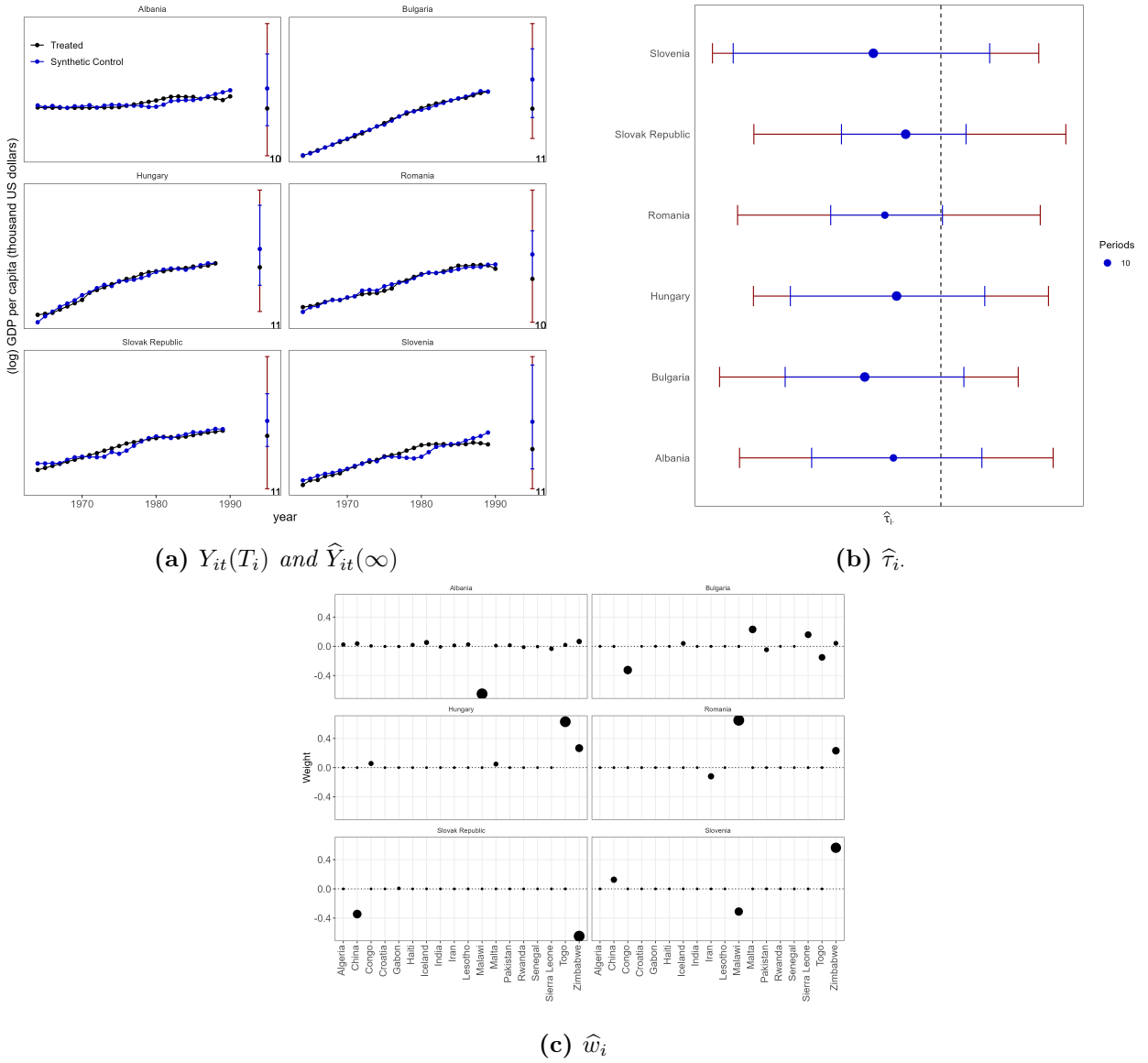
(c) $\widehat{w}_i$

Notes: Blue bars report 90% prediction intervals, whereas blue shaded areas report 90% simultaneous prediction intervals. In-sample uncertainty is quantified using 200 simulations, whereas out-of-sample uncertainty is quantified using sub-Gaussian bounds.

Average post-treatment effect for each country. The second causal quantity we estimate is the average post-treatment effect for each of the six European countries we study. Specifically, we target the average effect over the period following the liberalization up to the year 2000. Figure 6 shows that in all countries the liberalization episode depressed the real income per capita. However, if we consider individual and simultaneous prediction intervals that possess high simultaneous coverage

across treated units, we can see that no treated unit shows a negative average post-treatment effect with high probability.

Figure 6: *Estimated Average Post-Treatment Effects* $\widehat{\tau}_{i.}$.



(a) $Y_{it}(T_i)$ *and* $\widehat{Y}_{it}(\infty)$ **(b)** $\widehat{\tau}_{i.}$

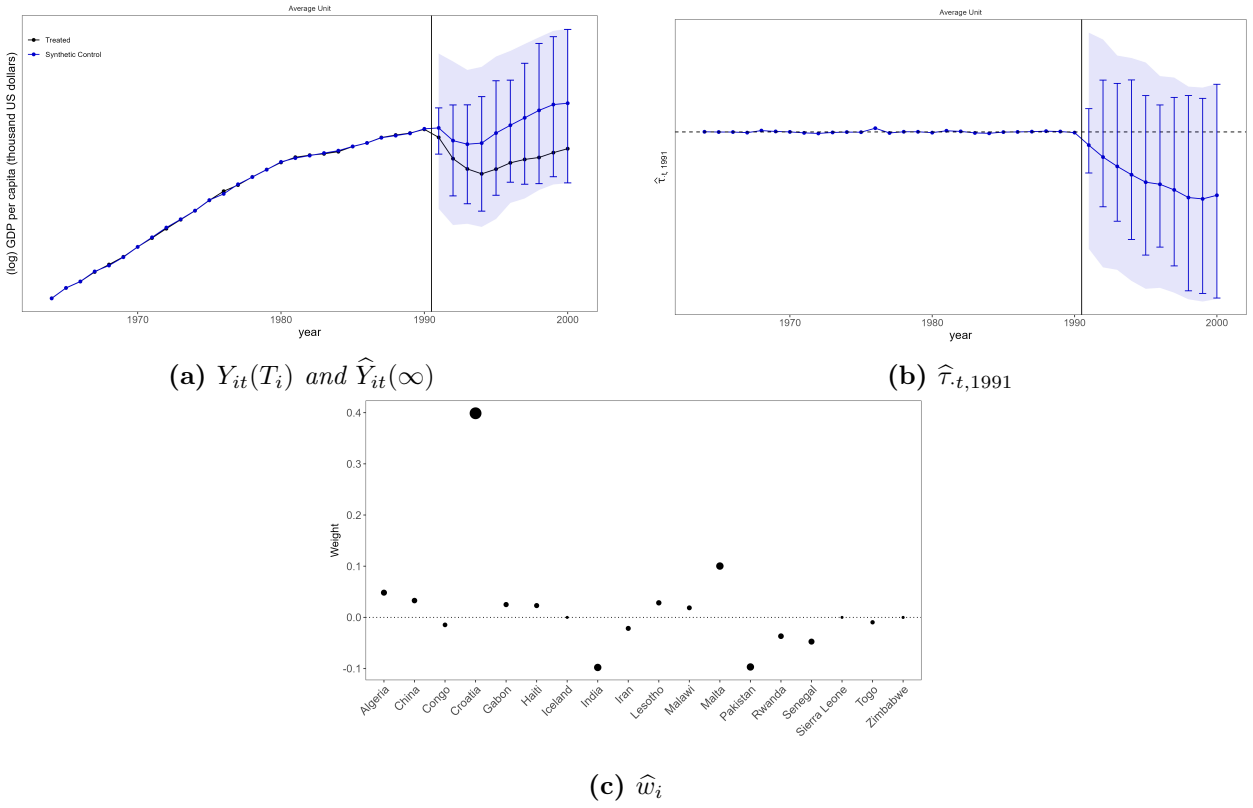
(c) $\widehat{w}_i$

Notes: Blue bars report 90% prediction intervals, whereas red bars report 90% simultaneous prediction intervals. In-sample uncertainty is quantified using 200 simulations, whereas out-of-sample uncertainty is quantified using sub-Gaussian bounds. The small number at the bottom-right corner of panel (a) represents the number of periods over which the post-treatment average is computed.

Average treatment effect on countries liberalized in 1991. In this third and last exercise, we estimate the average treatment effect on countries liberalized in 1991, which are Bulgaria, Czech Republic,

Slovak Republic, and Slovenia. To study this causal quantity, we are forced to match on the real GDP per capita series for the "average treated unit" only. Indeed, matching on two features in this case yields a poor pre-treatment fit (see Supplemental Appendix Section S.6 for the results using a simplex-type constraint and the results with $M = 2$). Even in this case, the trajectory for the synthetic real income lies above the realized one, suggesting a negative impact of the liberalization event. Again, quantifying uncertainty shows that the two series are not different with high probability.

Figure 7: *Estimated Average Treatment Effect on the Treated in 1991.*



(a) $Y_{it}(T_i)$ *and* $\widehat{Y}_{it}(\infty)$ **(b)** $\widehat{\tau}_{\cdot t,1991}$

(c) $\widehat{w}_i$

Notes: Blue bars report 90% prediction intervals, whereas blue shaded areas report 90% simultaneous prediction intervals. In-sample uncertainty is quantified using 200 simulations, whereas out-of-sample uncertainty using sub-Gaussian bounds.

7 Conclusion

References

Abadie, A. (2021), "Using Synthetic Controls: Feasibility, Data Requirements, and Methodological Aspects," *Journal of Economic Literature*, 59, 391–425.

Abadie, A., Diamond, A., and Hainmueller, J. (2010), "Synthetic control methods for comparative case studies: Estimating the effect of California's tobacco control program," *Journal of the American statistical Association*, 105, 493–505.

Abadie, A., and Gardeazabal, J. (2003), "The Economic Costs of Conflict: A Case Study of the Basque Country," *American Economic Review*, 93, 113–132.

Agarwal, A., Shah, D., Shen, D., and Song, D. (2021), "On Robustness of Principal Component Regression," *Journal of the American Statistical Association*.

Ben-Michael, E., Feller, A., Rothstein, J. et al. (2021), "Synthetic Controls with Staggered Adoption," *Journal of the Royal Statistical Society, Series B*, forthcoming.

Bhagwati, J., and Srinivasan, T. N. (2001), "Outward-orientation and development: are revisionists right?" in *Trade, development and political economy*, Springer, pp. 3–26.

——— (2002), "Trade and poverty in the poor countries," *American Economic Review*, 92, 180–183.

Billmeier, A., and Nannicini, T. (2013), "Assessing economic liberalization episodes: A synthetic control approach," *Review of Economics and Statistics*, 95, 983–1001.

Cattaneo, M. D., Feng, Y., Palomba, F., and Titiunik, R. (2022), "`scpi`: Uncertainty Quantification for Synthetic Control Estimators," working paper, Princeton University.

Cattaneo, M. D., Feng, Y., and Titiunik, R. (2021), "Prediction Intervals for Synthetic Control Methods," *Journal of the American Statistical Association*, 116, 1865–1880.

Chernozhukov, V., Wüthrich, K., and Zhu, Y. (2021a), "Distributional Conformal Prediction," *Proceedings of the National Academy of Sciences*, 118, e2107794118.

Chernozhukov, V., Wüthrich, K., and Zhu, Y. (2021b), "An Exact and Robust Conformal Inference Method for Counterfactual and Synthetic Controls," *Journal of the American Statistical Association*.

DeJong, D. N., and Ripoll, M. (2006), "Tariffs and growth: an empirical exploration of contingent relationships," *The Review of Economics and Statistics*, 88, 625–640.

Giavazzi, F., and Tabellini, G. (2005), "Economic and political liberalizations," *Journal of monetary economics*, 52, 1297–1330.

Levine, R., and Renelt, D. (1992), "A sensitivity analysis of cross-country growth regressions," *The American Economic Review*, 942–963.

Li, K. T. (2020), "Statistical Inference for Average Treatment Effects Estimated by Synthetic Control Methods," *Journal of the American Statistical Association*, 115, 2068–2083.

Masini, R., and Medeiros, M. C. (2021), "Counterfactual Analysis with Artificial Controls: Inference, High Dimensions and Nonstationarity," *Journal of the American Statistical Association*.

Rodriguez, F., and Rodrik, D. (2000), "Trade policy and economic growth: a skeptic's guide to the cross-national evidence," *NBER macroeconomics annual*, 15, 261–325.

Sachs, J. D., Warner, A., Åslund, A., and Fischer, S. (1995), "Economic reform and the process of global integration," *Brookings papers on economic activity*, 1995, 1–118.

Shaikh, A. M., and Toulis, P. (2021), "Randomization Tests in Observational Studies with Staggered Adoption of Treatment," *Journal of the American Statistical Association*.

Shen, D., Ding, P., Sekhon, J., and Yu, B. (2022), "A Tale of Two Panel Data Regressions," *arXiv preprint arXiv:2207.14481*.

Vovk, V. (2012), "Conditional Validity of Inductive Conformal Predictors," in *JMLR: Workshop and Conference Proceedings 25, Asian Conference on Machine Learning*, pp. 475–490.

Wacziarg, R., and Welch, K. H. (2008), "Trade liberalization and growth: New evidence," *The World Bank Economic Review*, 22, 187–231.

Wainwright, M. J. (2019), *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*, Cambridge University Press.