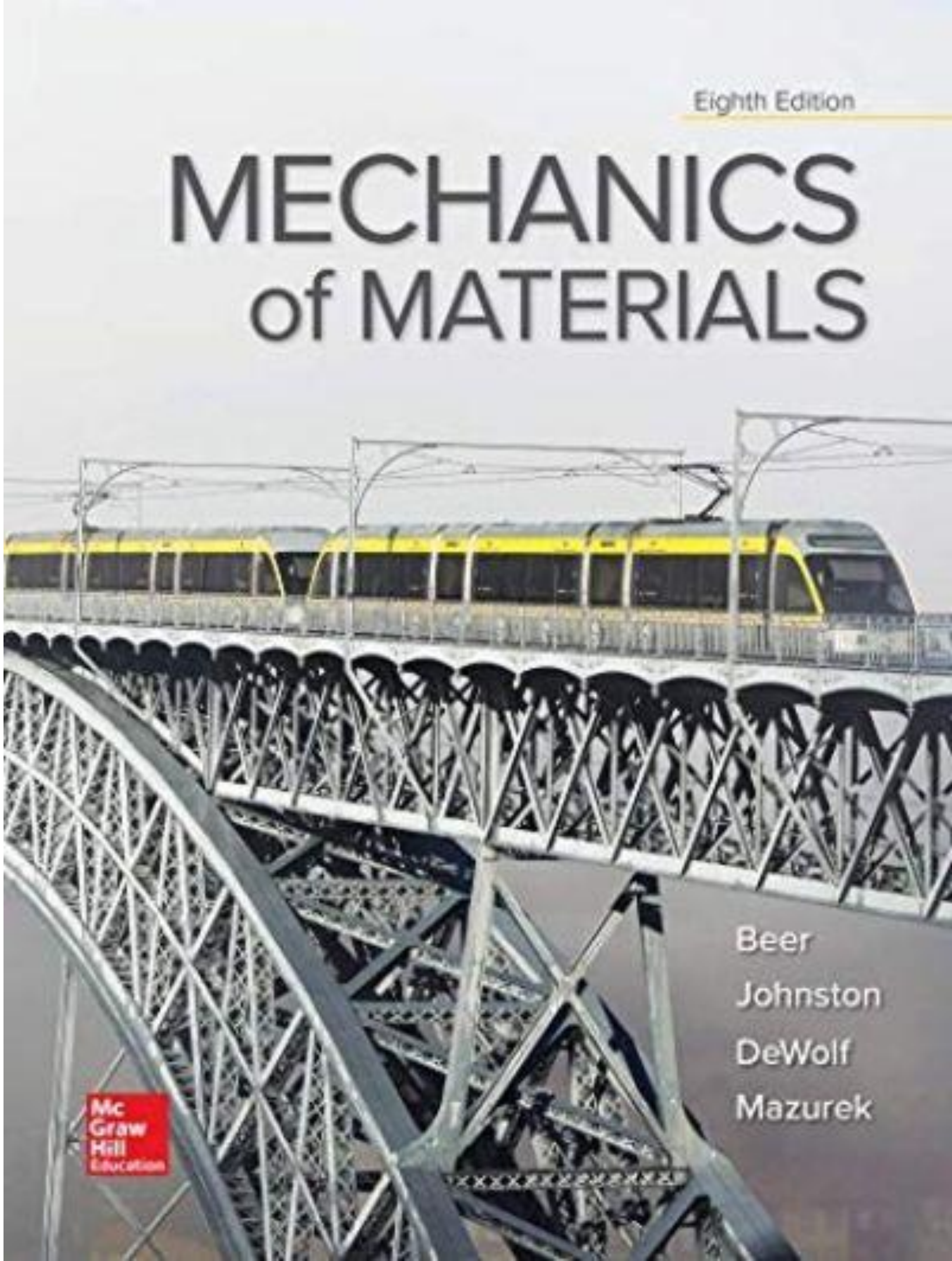
Pure Bending

Chapter 4



Contents

Pure Bending

Pure Bending:
Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane.

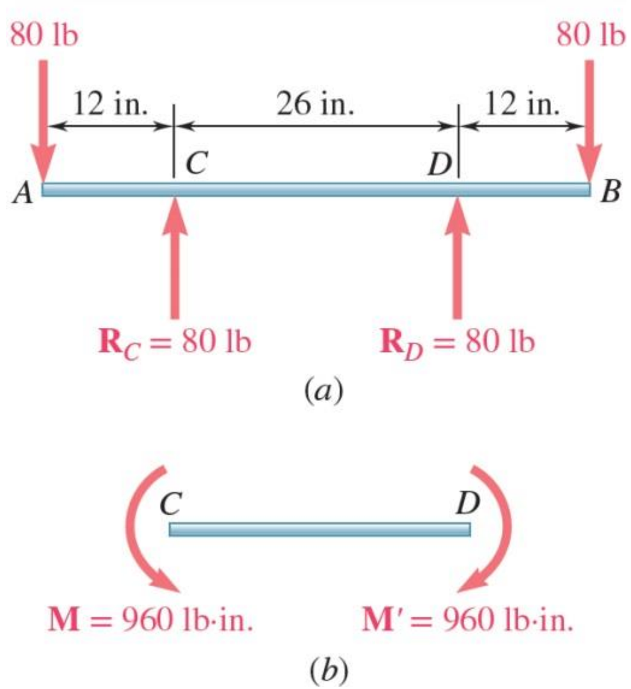


Figure 4.2 (a) Free-body diagram of the barbell pictured in the chapter opening photo and (b) Free-body diagram of the center bar portion showing pure bending.

Other Loading Types

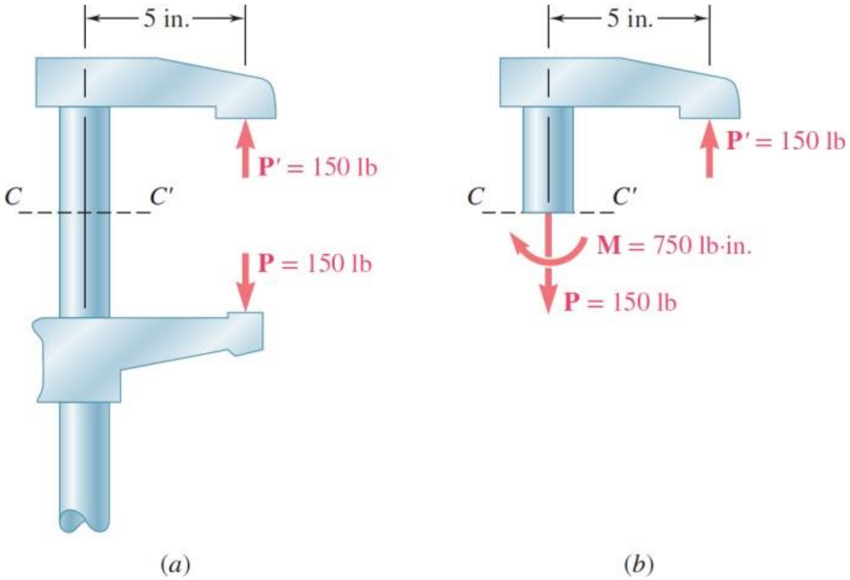


Figure 4.3 (a) Free-body diagram of a clamp, (b) free-body diagram of the upper portion of the clamp.

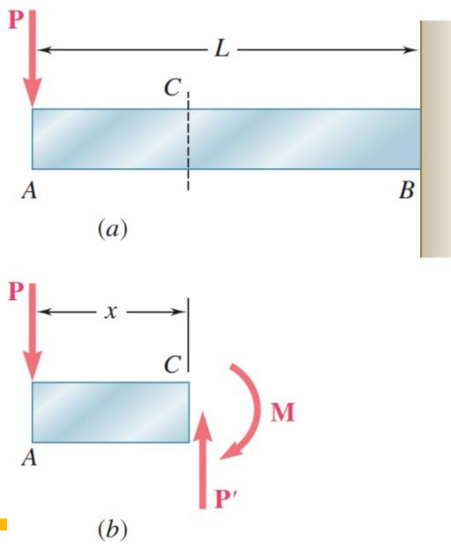


Figure 4.4 (a) Cantilevered beam with end loading. (b) As portion *AC* shows, beam is not in pure bending.

- *Eccentric Loading*: Axial loading that does not pass through the section centroid produces internal forces equivalent to an axial force and a couple.
- *Transverse Loading*: Concentrated or distributed transverse load produces internal forces equivalent to a shear force and a couple.
- *Principle of Superposition:* The normal stress due to pure bending may be combined with the normal stress due to axial loading and shear stress due to shear loading to find the complete state of stress.

Symmetric Member in Pure Bending

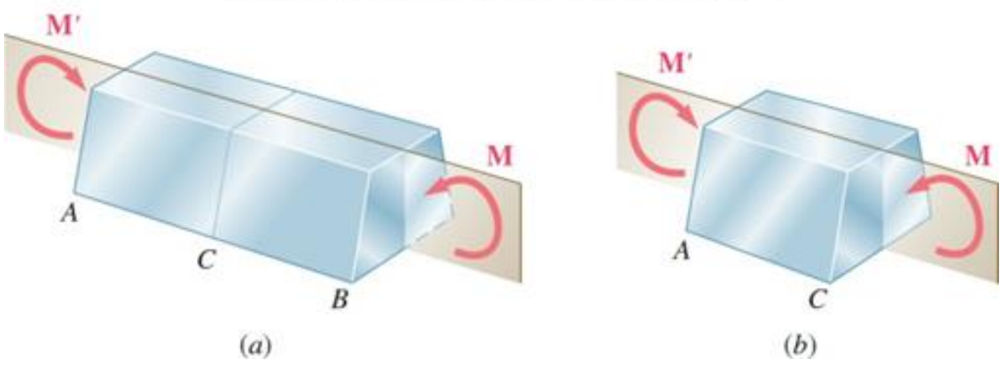


Figure 4.5 (a) A member in a state of pure bending. (b) Any intermediate portion of *AB* will also be in pure bending.

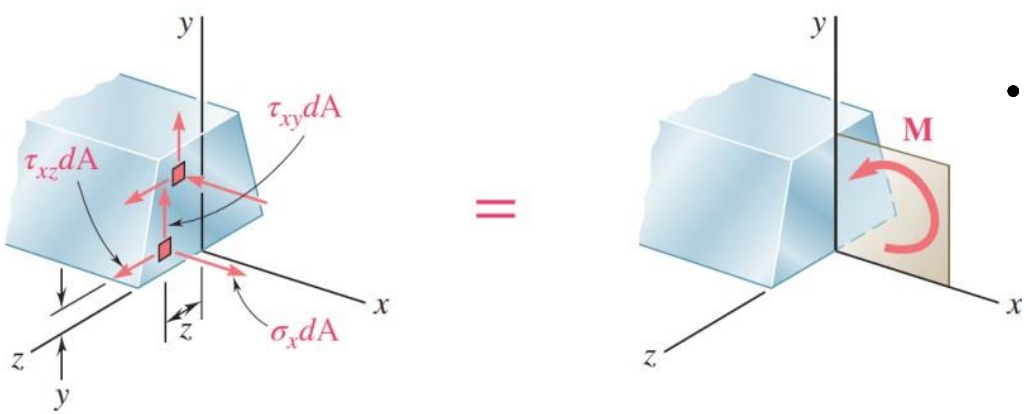


Figure 4.6 Summation of the infinitesimal stress elements must produce the equivalent pure-bending moment.

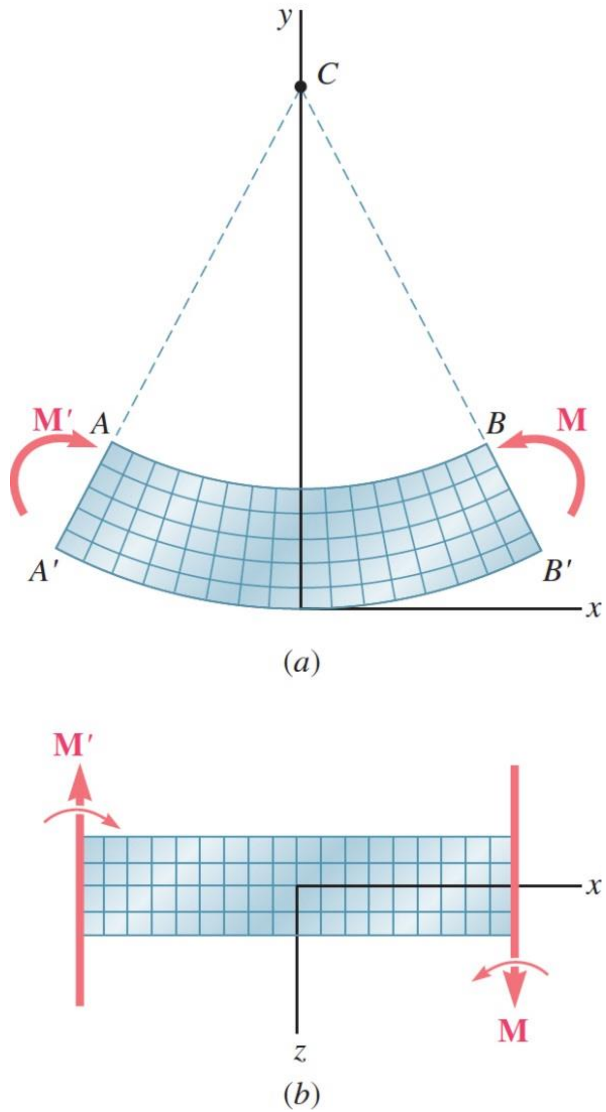
- Internal forces in any cross section are equivalent to a couple. The moment of the couple is the section *bending moment*.
- From statics, a couple **M** consists of two equal and opposite forces.
- The sum of the components of the forces in any direction is zero.
- The moment is the same about *any* axis perpendicular to the plane of the couple and zero about any axis contained in the plane.
- These requirements may be applied to the sums of the components and moments of the statically indeterminate elementary internal forces.

$$F_x = \int \sigma_x \, dA = 0$$

$$M_y = \int z\sigma_x \, dA = 0$$

$$M_z = \int -y\sigma_x \, dA = M$$

Bending Deformations



Beams with a plane of symmetry in pure bending.

- Member remains symmetric.
- Member bends uniformly to form a circular arc.
- Cross-sectional plane passes through arc center and remains planar.
- Length of top decreases and length of bottom increases.
- A *neutral surface* that is parallel to the upper and lower surfaces and for which the length does not change must exist.
- Stresses and strains are negative (compressive) above the neutral plane and positive (tension) below it.

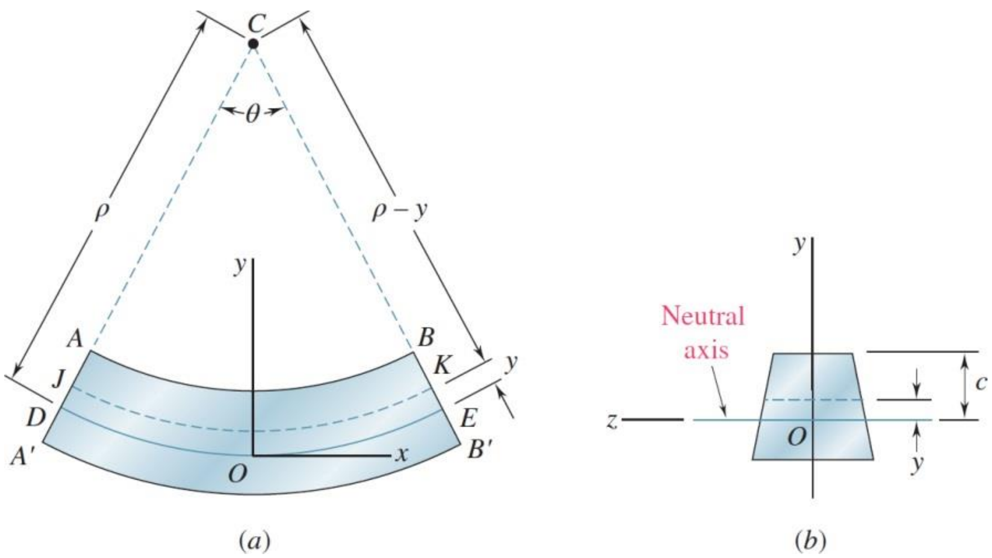
Figure 4.9 Member subject to pure bending shown in two views. (a) Longitudinal, vertical view (plane of symmetry) and (b) Longitudinal, horizontal view.

Strain Due to Bending

Consider a beam segment of length L.

$$L = \rho\theta$$

After deformation, the length of the neutral surface remains L. At other sections:



$$L' = (\rho - y)\theta$$

$$\delta = L' - L = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho}$$ (strain varies linearly)

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m$$

Figure 4.10 Kinematic definitions for pure bending. (a) Longitudinal-vertical view and (b) Transverse section at origin.

Stress Due to Bending

- For a linearly elastic and homogeneous material:

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

$$= -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$

- For static equilibrium:

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c}\int y \, dA$$

First moment with respect to neutral axis is zero. Therefore, the neutral axis must pass through the section centroid.

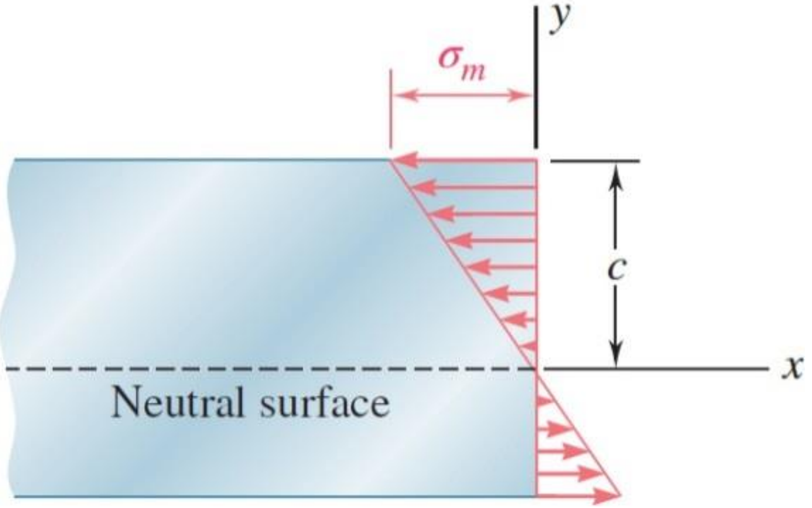


Figure 4.11 Bending stresses vary linearly with distance from the neutral axis.

- For static equilibrium:

$$M = \int(-y\sigma_x \, dA) = \int(-y)\left(-\frac{y}{c}\sigma_m\right) dA$$

$$M = \frac{\sigma_m}{c}\int y^2 \, dA = \frac{\sigma_m I}{c}$$

$$\boxed{\sigma_m = \frac{Mc}{I} = \frac{M}{S}}$$

Substituting $\sigma_x = -\frac{y}{c}\sigma_m$

$$\boxed{\sigma_x = -\frac{My}{I}}$$

Beam Section Properties

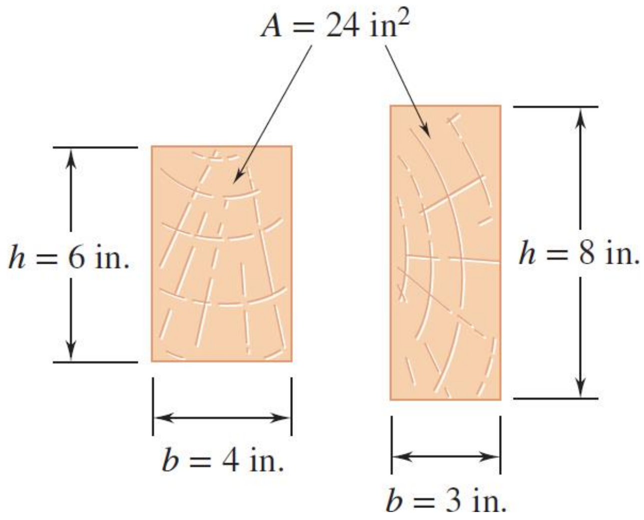


Figure 4.12 Wood beam cross sections.

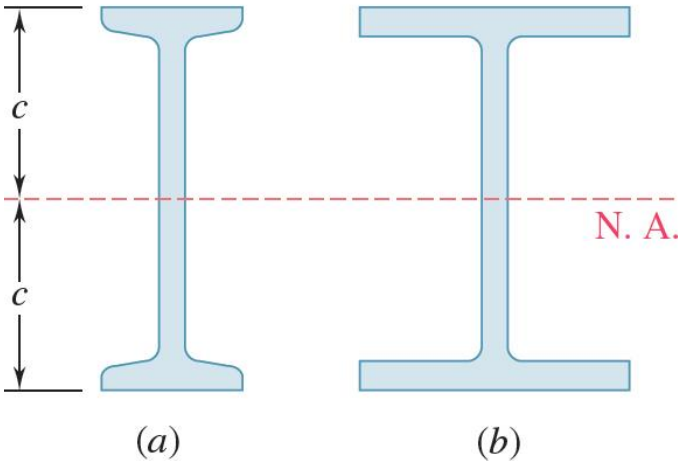


Figure 4.13 Two type of steel beam cross sections. (a) S-beam and (b) W-beam

- The maximum normal stress due to bending:

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

$I =$ section moment of inertia

$$S = \frac{I}{c} = \text{section modulus}$$

A beam section with a larger section modulus will have a lower maximum stress.

- Consider a rectangular beam cross section:

$$S = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \tfrac{1}{6}bh^2 = \tfrac{1}{6}Ah$$

Between two beams with the same cross sectional area, the beam with the larger depth h will be more effective in resisting bending.

- Structural steel beams are designed to have a large section modulus.

Beam Section Properties

$$\sigma_m = \frac{Mc}{I} = \frac{M}{S}$$

Concept Application 4.1

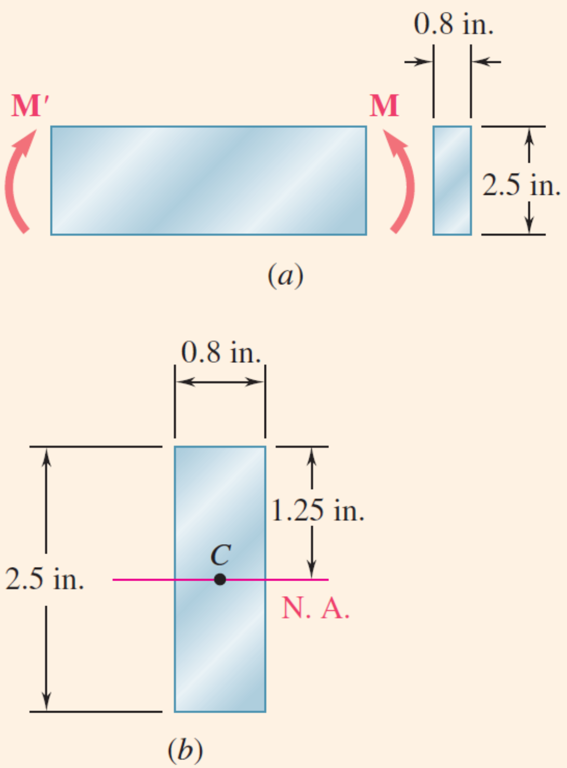


Fig. 4.14 (*a*) Bar of rectangular cross section in pure bending. (*b*) Centroid and dimensions of cross section.

A steel bar of 0.8 × 2.5-in. rectangular cross section is subjected to two equal and opposite couples acting in the vertical plane of symmetry of the bar (Fig. 4.14*a*). Determine the value of the bending moment M that causes the bar to yield. Assume $\sigma_Y = 36$ ksi.

Since the neutral axis must pass through the centroid C of the cross section, $c = 1.25$ in. (Fig. 4.14*b*). On the other hand, the centroidal moment of inertia of the rectangular cross section is

$$I = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(0.8\text{ in.})(2.5\text{ in.})^3 = 1.042\text{ in}^4$$

Solving Eq. (4.15) for M, and substituting the above data,

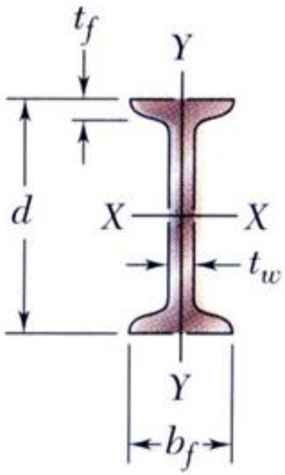
$$M = \frac{I}{c}\sigma_m = \frac{1.042\text{ in}^4}{1.25\text{ in.}}(36\text{ ksi})$$

$$M = 30\text{ kip}\cdot\text{in.}$$

Properties of American Standard Shapes

Appendix C. **Properties of Rolled-Steel Shapes**

(SI Units)



S Shapes

(American Standard Shapes)

Designation†	Area A, mm²	Depth d, mm	Flange Width b_f, mm	Flange Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X I_x 10^6 mm⁴	Axis X-X S_x 10^3 mm³	Axis X-X r_x mm	Axis Y-Y I_y 10^6 mm⁴	Axis Y-Y S_y 10^3 mm³	Axis Y-Y r_y mm
S610 × 180	22900	622	204	27.7	20.3	1320	4240	240	34.9	341	39.0
158	20100	622	200	27.7	15.7	1230	3950	247	32.5	321	39.9
149	19000	610	184	22.1	18.9	995	3260	229	20.2	215	32.3
134	17100	610	181	22.1	15.9	938	3080	234	19.0	206	33.0
119	15200	610	178	22.1	12.7	878	2880	240	17.9	198	34.0
S510 × 143	18200	516	183	23.4	20.3	700	2710	196	21.3	228	33.9
128	16400	516	179	23.4	16.8	658	2550	200	19.7	216	34.4
112	14200	508	162	20.2	16.1	530	2090	193	12.6	152	29.5
98.3	12500	508	159	20.2	12.8	495	1950	199	11.8	145	30.4
S460 × 104	13300	457	159	17.6	18.1	385	1685	170	10.4	127	27.5
81.4	10400	457	152	17.6	11.7	333	1460	179	8.83	113	28.8
S380 × 74	9500	381	143	15.6	14.0	201	1060	145	6.65	90.8	26.1
64	8150	381	140	15.8	10.4	185	971	151	6.15	85.7	27.1

Deformations in a Transverse Cross Section

$$\varepsilon_x = -\frac{y}{\rho}$$

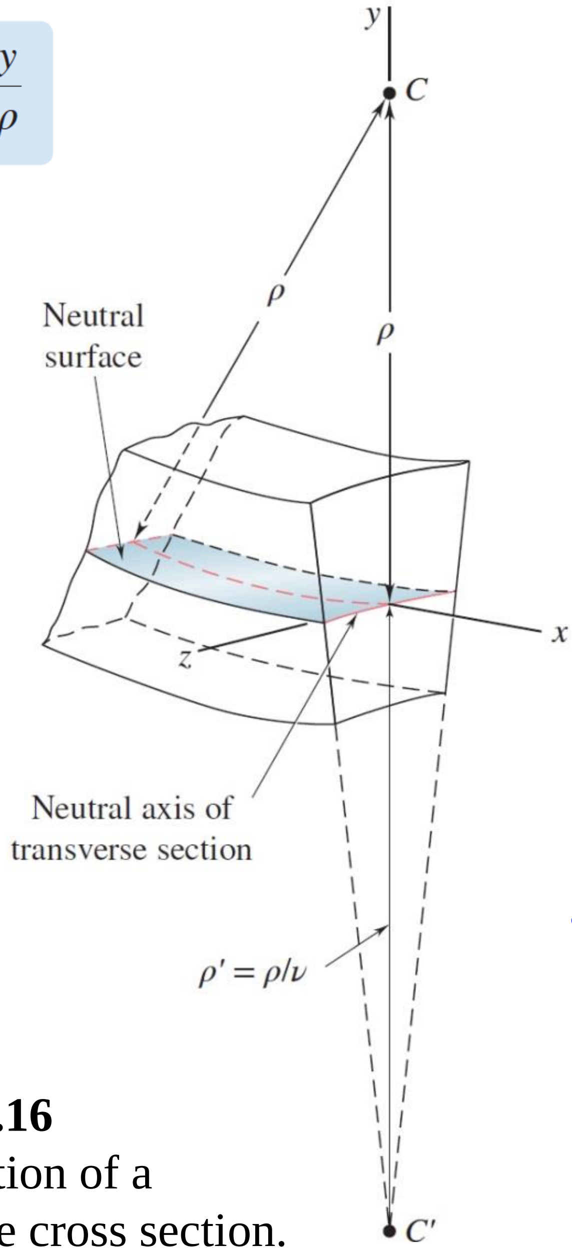


Figure 4.16
Deformation of a transverse cross section.

- Deformation due to bending moment M is quantified by the curvature of the neutral surface:

$$\frac{1}{\rho} = \frac{\varepsilon_m}{c} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$

$$= \frac{M}{EI}$$

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

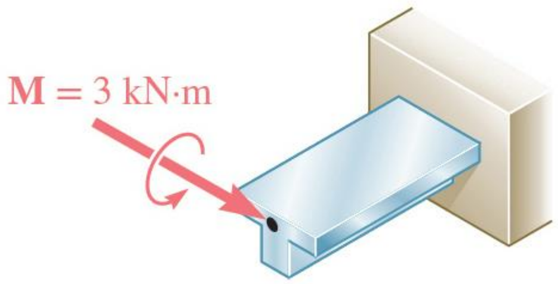
- Although transverse cross sectional planes remain planar when subjected to bending moments, in-plane deformations are nonzero:

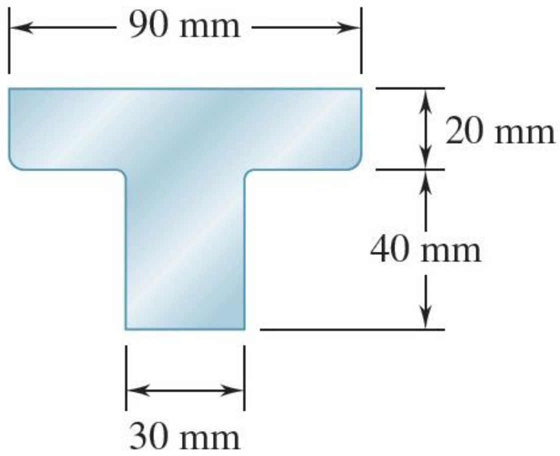
$$\varepsilon_y = -\nu\varepsilon_x = \frac{\nu y}{\rho} \qquad \varepsilon_z = -\nu\varepsilon_x = \frac{\nu y}{\rho}$$

- Expansion above the neutral surface and contraction below it cause an in-plane curvature:

$$\frac{1}{\rho'} = \frac{\nu}{\rho} = \text{anticlastic curvature}$$

Problem 4.2 [1]





A cast-iron machine part is acted upon by a 3 kN-m couple. Knowing E = 165 GPa and neglecting the effects of fillets, determine (a) the maximum tensile and compressive stresses, (b) the radius of curvature.

SOLUTION:

- Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} \qquad I_{x'} = \sum\left(\bar{I} + Ad^2\right)$$

- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

- Calculate the curvature:

$$\frac{1}{\rho} = \frac{M}{EI}$$

Problem 4.2 2

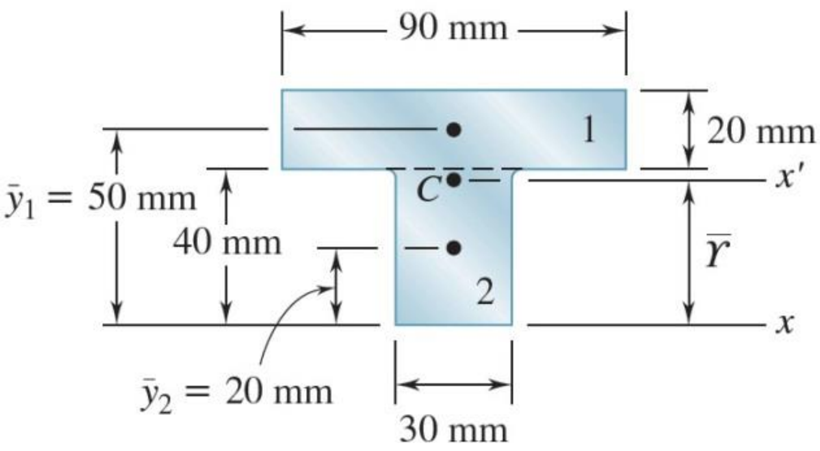


Figure 1 Composite areas for calculating centroid.

SOLUTION:

Based on the cross section geometry, calculate the location of the section centroid and moment of inertia.

	Area, mm^2	$\bar{y}$, mm	$\bar{y}A$, mm^3
1	$20 \times 90 = 1800$	50	90×10^3
2	$40 \times 30 = 1200$	20	24×10^3
	$\sum A = 3000$		$\sum \bar{y}A = 114 \times 10^3$

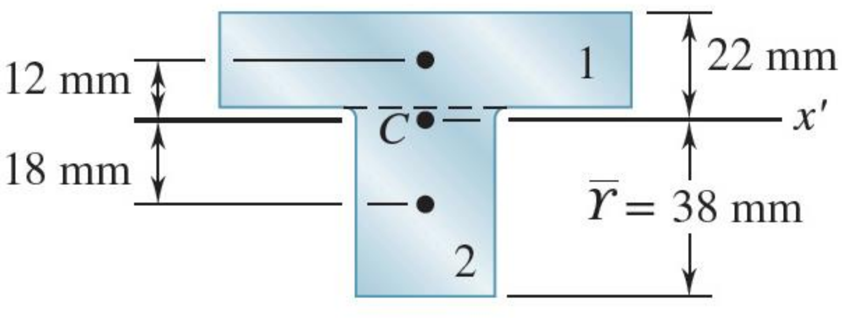


Figure 2 Composite sections for calculating moment of inertia.

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{114 \times 10^3}{3000} = 38 \text{ mm}$$

$$I_{x'} = \sum\left(\bar{I} + Ad^2\right) = \sum\left(\tfrac{1}{12}bh^3 + Ad^2\right)$$

$$= \left(\tfrac{1}{12}90 \times 20^3 + 1800 \times 12^2\right) + \left(\tfrac{1}{12}30 \times 40^3 + 1200 \times 18^2\right)$$

$$I = 868 \times 10^3 \text{mm}^4 \;=\; 868 \times 10^{-9} \text{m}^4$$

Problem 4.2 3

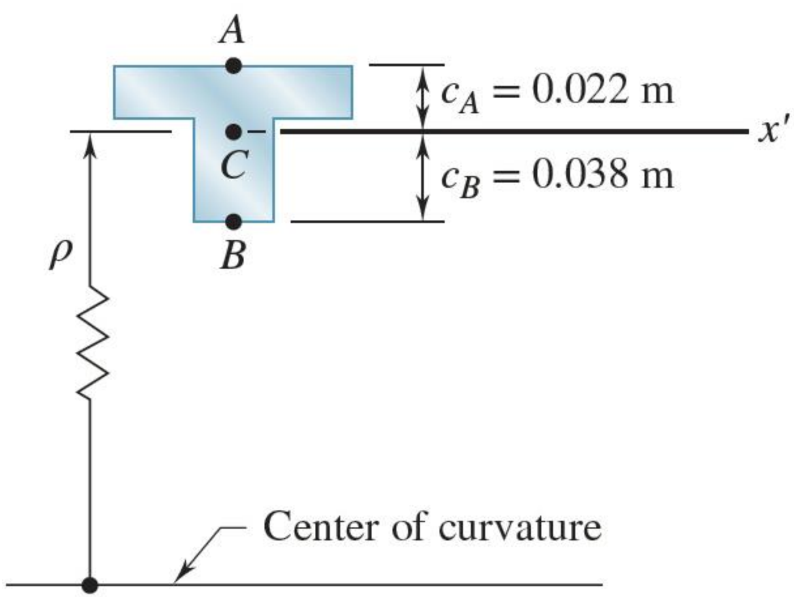


Figure 3 Deformed radius of curvature is measured to the centroid of the cross sections.

- Apply the elastic flexural formula to find the maximum tensile and compressive stresses.

$$\sigma_m = \frac{Mc}{I}$$

$$\sigma_A = \frac{M\,c_A}{I} = \frac{3\text{ kN}\cdot\text{m}\times 0.022\text{ m}}{868\times 10^{-9}\text{ m}^4} \qquad \boxed{\sigma_A = +76.0\text{ MPa}}$$

$$\sigma_B = -\frac{M\,c_B}{I} = -\frac{3\text{ kN}\cdot\text{m}\times 0.038\text{ m}}{868\times 10^{-9}\text{ m}^4} \qquad \boxed{\sigma_B = -131.3\text{ MPa}}$$

- Calculate the curvature:

$$\frac{1}{\rho} = \frac{M}{EI}$$

$$= \frac{3\text{ kN}\cdot\text{m}}{(165\text{ GPa})(868\times 10^{-9}\text{m}^4)}$$

$$\boxed{\begin{aligned}\frac{1}{\rho} &= 20.95\times 10^{-3}\text{m}^{-1}\\ \rho &= 47.7\text{ m}\end{aligned}}$$

Bending of Members Made of Several Materials

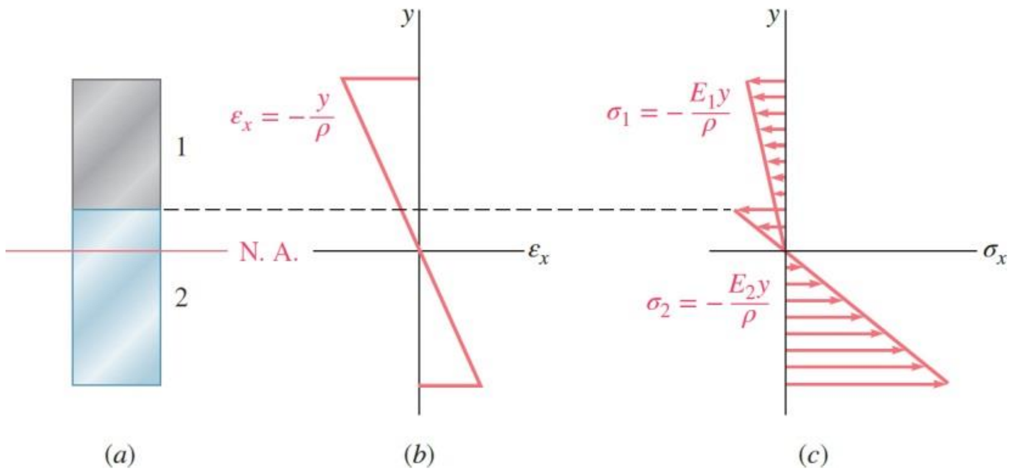


Figure 4.19 Stress and strain distributions in bar made of two materials. (a) Neutral axis shifted from centroid. (b) Strain distribution. (c) Corresponding stress distribution.

- Consider a bar consisting of two different materials with E_1 and E_2.
- Normal strain varies linearly with distance y

$$\varepsilon_x = -\frac{y}{\rho}$$

- Piecewise linear normal stress variation.

$$\sigma_1 = E_1\varepsilon_x = -\frac{E_1 y}{\rho} \qquad \sigma_2 = E_2\varepsilon_x = -\frac{E_2 y}{\rho}$$

Neutral axis **does not** pass through section centroid of composite section.

- Elemental forces on the section are

$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA \quad dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

- Define a transformed section such that

$$dF_2 = -\frac{(nE_1)y}{\rho} dA = -\frac{E_1 y}{\rho}(n\,dA) \qquad n = \frac{E_2}{E_1}$$

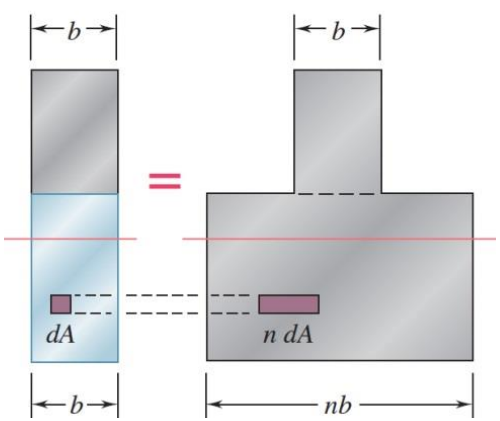


Figure 4.20 Transformed section based on stiffness is used to locate neutral axis.

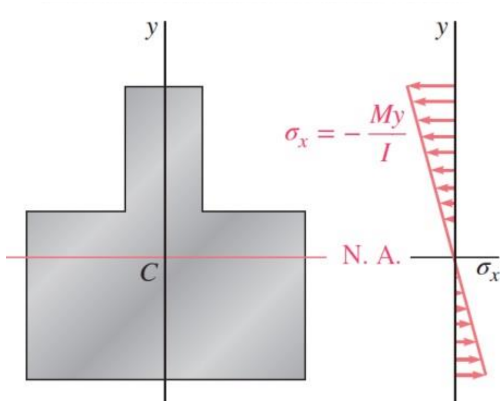


Figure 4.21 Distribution of stresses in transformed section.

Problem 4.3 1

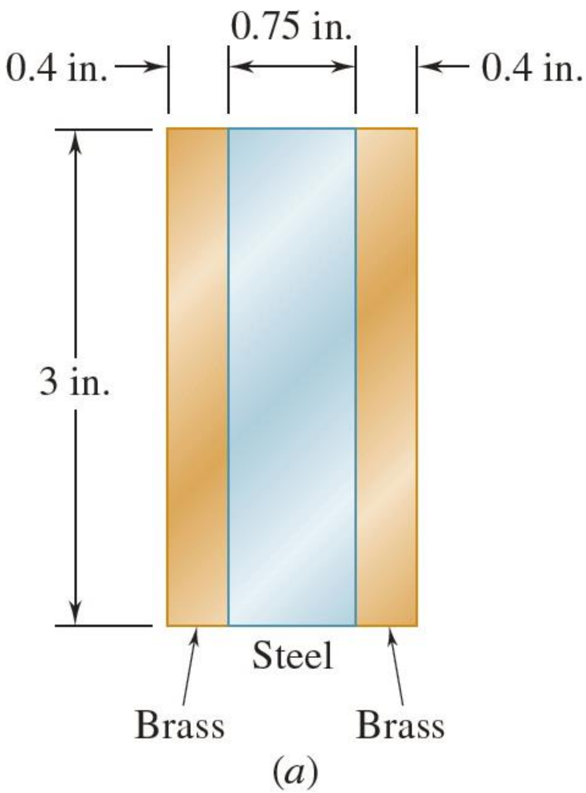


Figure 4.22a Composite, sandwich structure cross section.

Bar is made from bonded pieces of steel $\left(E_s = 29\times10^6\,\text{psi}\right)$ and brass $\left(E_b = 15\times10^6\,\text{psi}\right)$. Determine the maximum stress in the steel and brass when a moment of 40 kip·in is applied.

SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass.
- Evaluate the cross sectional properties of the transformed section.
- Calculate the maximum stress in the transformed section. This is the correct maximum stress for the brass pieces of the bar.
- Determine the maximum stress in the steel portion of the bar by multiplying the maximum stress for the transformed section by the ratio of the moduli of elasticity.

Problem 4.3 2

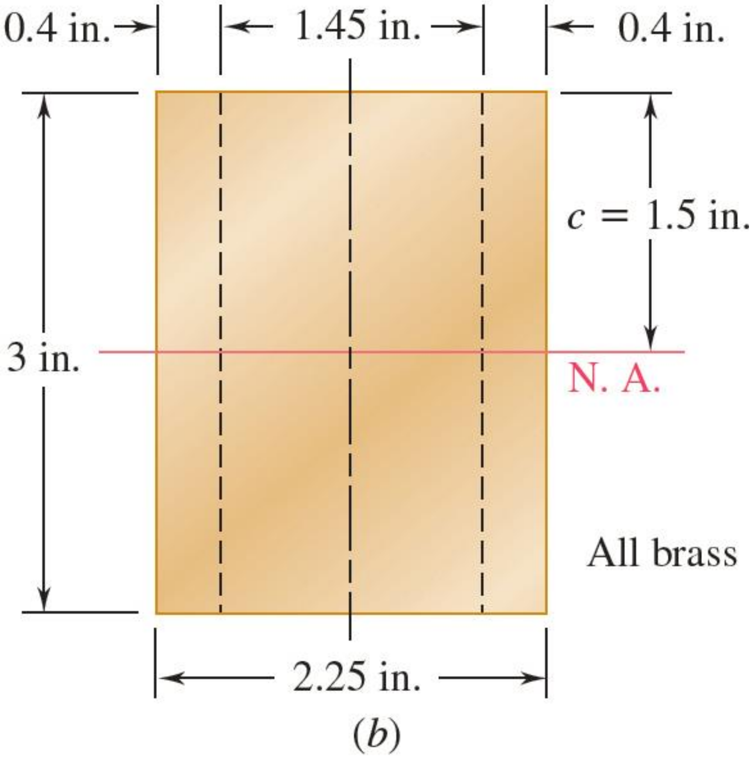


Figure 4.22b Bar length and height dimensions.

SOLUTION:

- Transform the bar to an equivalent cross section made entirely of brass.

$$n = \frac{E_s}{E_b} = \frac{29\times10^6\,\text{psi}}{15\times10^6\,\text{psi}} = 1.933$$

$$b_T = 0.4\text{ in} + 1.933\times0.75\text{ in} + 0.4\text{ in} = 2.25\text{ in}$$

- Evaluate the transformed cross sectional properties:

$$I = \tfrac{1}{12} b_T h^3 = \tfrac{1}{12}(2.25\text{ in.})(3\text{ in.})^3$$

$$= 5.063\text{ in.}^4$$

- Calculate the maximum stresses:

$$\sigma_m = \frac{Mc}{I} = \frac{(40\text{ kip}\cdot\text{in.})(1.5\text{ in.})}{5.063\text{ in.}^4} = 11.85\text{ ksi}$$

$$(\sigma_b)_{\max} = \sigma_m$$

$$(\sigma_s)_{\max} = n\sigma_m = 1.933\times11.85\text{ ksi}$$

$$\boxed{\begin{aligned}(\sigma_b)_{\max} &= 11.85\text{ ksi}\\ (\sigma_s)_{\max} &= 22.9\text{ ksi}\end{aligned}}$$

Reinforced Concrete Beams

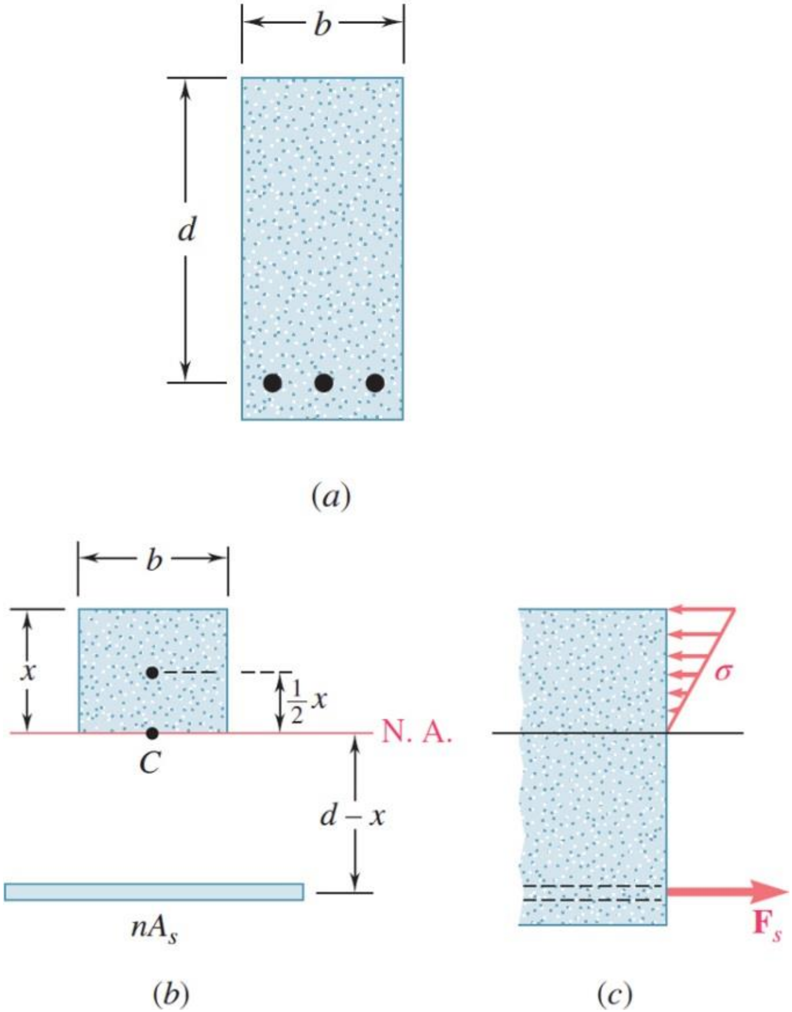


Figure 4.23 Reinforced concrete beam: (a) Cross section showing location of reinforcing steel. (b) Transformed section of all concrete. (c) Concrete stresses and resulting steel force.

- Concrete beams subjected to bending moments are reinforced by steel rods.
- The steel rods carry the entire tensile load below the neutral surface. The upper part of the concrete beam carries the compressive load.
- In the transformed section, the cross sectional area of the steel, A_s, is replaced by the equivalent area nA_s where $n = E_s / E_c$.
- To determine the location of the neutral axis:

$$(bx)\frac{x}{2} - n A_s (d - x) = 0$$

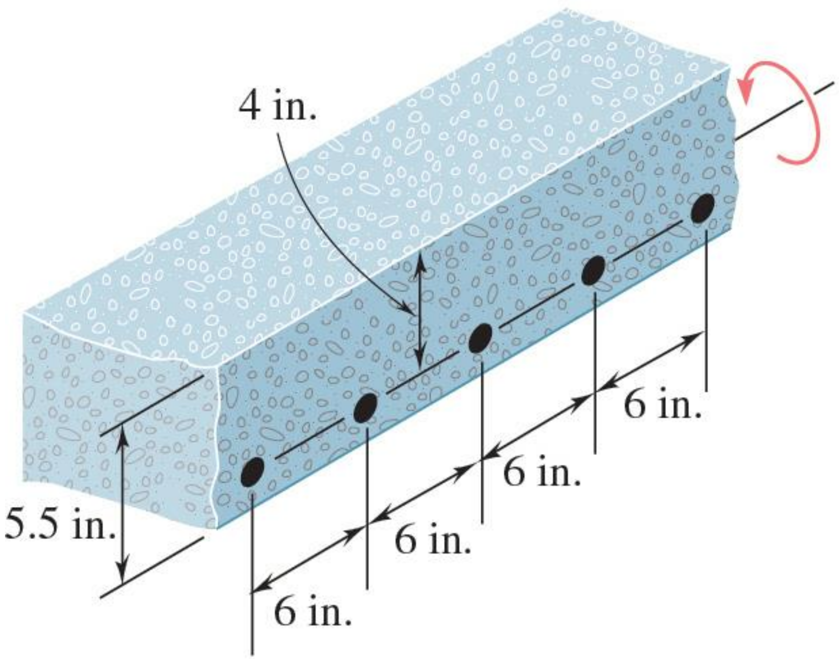
$$\frac{1}{2} b x^2 + n A_s x - n A_s d = 0$$

- The normal stress in the concrete and steel:

$$\sigma_x = -\frac{My}{I}$$

$$\sigma_c = \sigma_x \qquad \sigma_s = n\sigma_x$$

Problem 4.4 [1]



A concrete floor slab is reinforced with 5/8-inch-diameter steel rods placed 1.5 in. above the lower face of the slab and spaced 6 in. on centers. The modulus of elasticity is 29×10^{6} psi for steel and 3.6×10^{6} psi for concrete. With an applied bending moment of 40 kip·in for 1-ft width of the slab, determine the maximum stress in the concrete and steel.

SOLUTION:

- Transform to a section made entirely of concrete.
- Evaluate geometric properties of transformed section.
- Calculate the maximum stresses in the concrete and steel.

Problem 4.4 [2]

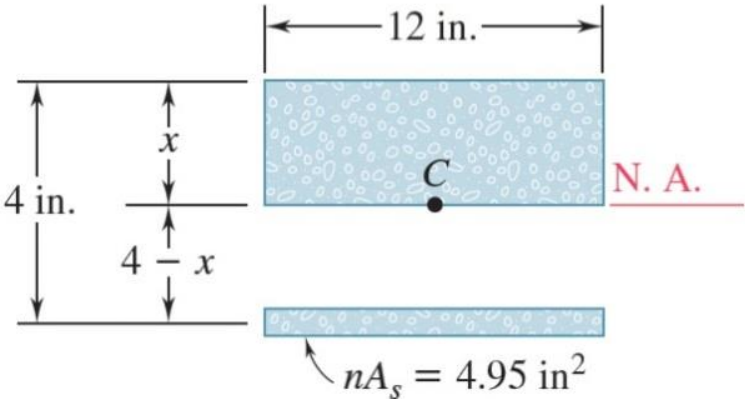


Figure 1 Transformed section to calculate neutral axis.

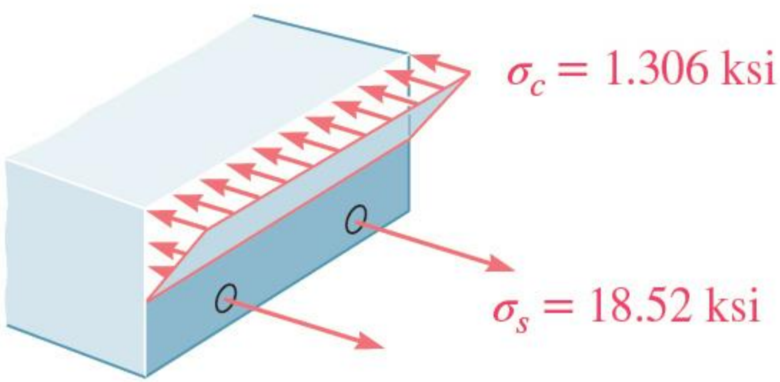


Figure 3 Force diagram at a cross section to calculate stresses.

SOLUTION:

- Transform to a section made entirely of concrete.

$$n = \frac{E_s}{E_c} = \frac{29\times10^6\ \text{psi}}{3.6\times10^6\ \text{psi}} = 8.06$$

$$nA_s = 8.06\times 2\left[\frac{\pi}{4}\left(\frac{5}{8}\text{in}\right)^2\right] = 4.95\,\text{in}^2$$

- Evaluate the geometric properties of the transformed section:

$$12x\left(\frac{x}{2}\right) - 4.95(4-x) = 0 \qquad x = 1.450\,\text{in}$$

$$I = \tfrac{1}{3}(12\,\text{in})(1.45\,\text{in})^3 + \left(4.95\,\text{in}^2\right)(2.55\,\text{in})^2 = 44.4\,\text{in}^4$$

- Calculate the maximum stresses:

$$\sigma_c = \frac{Mc_1}{I} = \frac{40\,\text{kip}\cdot\text{in}\times 1.45\,\text{in}}{44.4\,\text{in}^4} \qquad \boxed{\sigma_c = 1.306\,\text{ksi}}$$

$$\sigma_s = n\frac{Mc_2}{I} = 8.06\frac{40\,\text{kip}\cdot\text{in}\times 2.55\,\text{in}}{44.4\,\text{in}^4} \qquad \boxed{\sigma_s = 18.52\,\text{ksi}}$$

Stress Concentrations

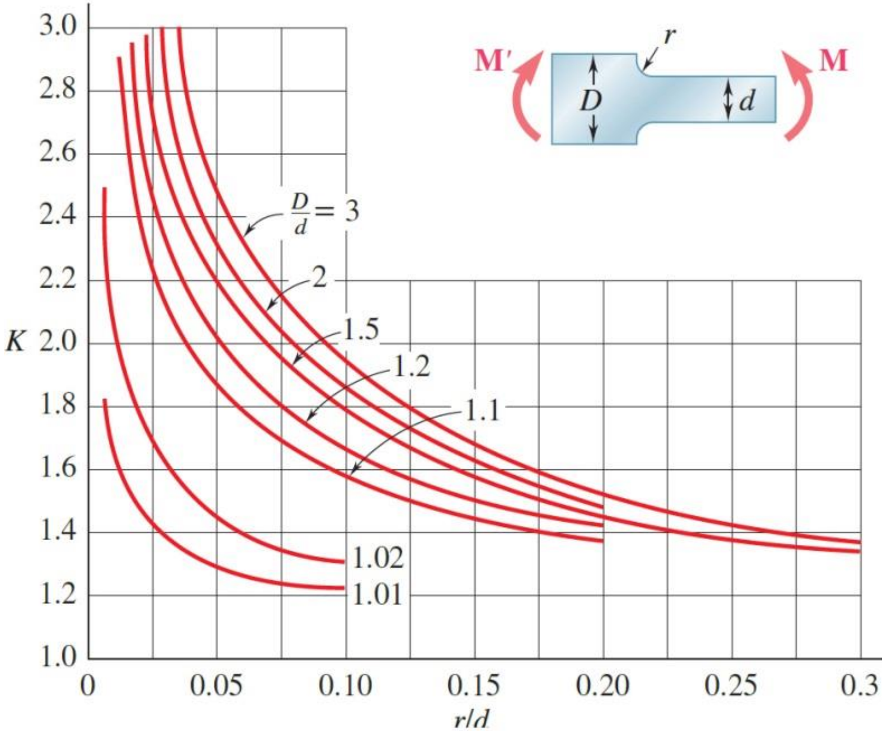


Figure 4.24 Stress-concentration factors for flat bars with fillets under pure bending.

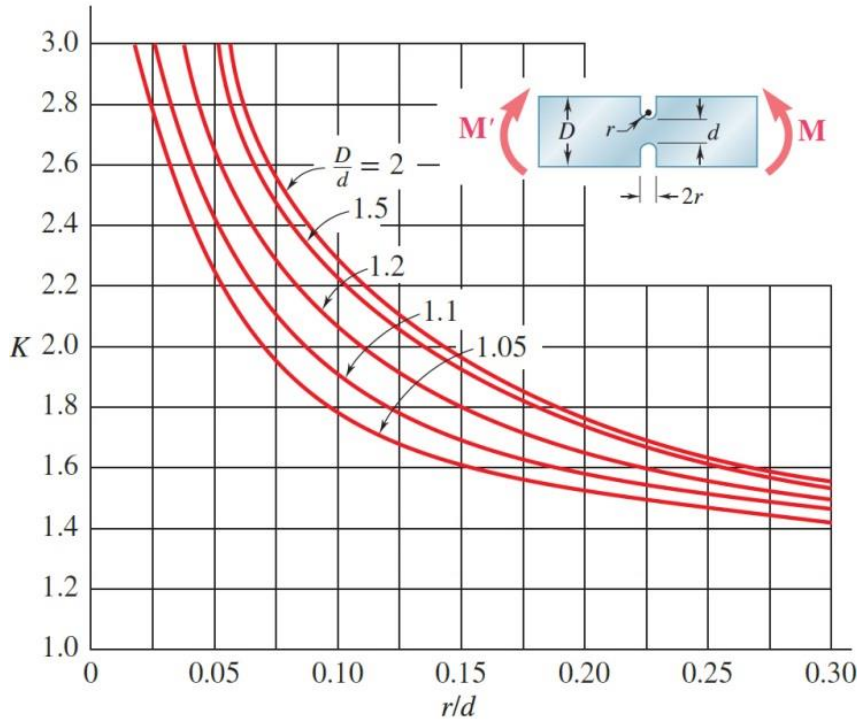


Figure 4.25 Stress-concentration factors for flat bars with grooves (notches) under pure bending.

Stress concentrations may occur

- in the vicinity of points where the loads are applied.
- in the vicinity of abrupt changes in cross section.

Maximum stress: $\sigma_m = K \dfrac{Mc}{I}$

Plastic Deformations 1

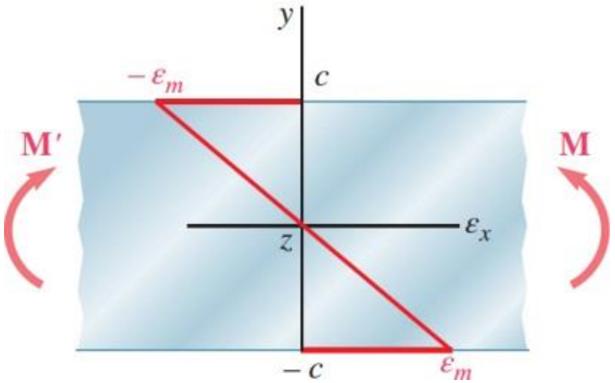


Figure 4.27 Linear strain distribution in beam under pure bending.

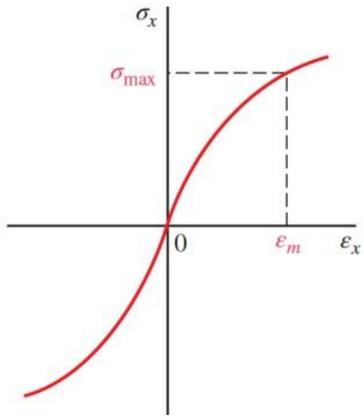


Figure 4.28 Material with nonlinear stress-strain diagram.

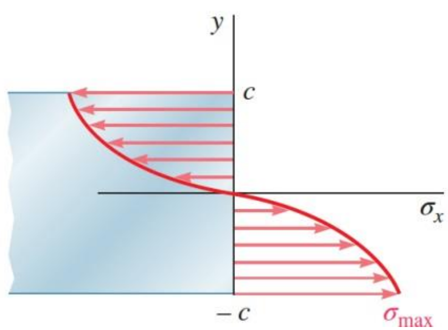


Figure 4.29 Nonlinear stress distribution in member under pure bending.

- For any member subjected to pure bending $\varepsilon_x = -\frac{y}{c}\varepsilon_m$ strain varies linearly across the section.

- <u>If</u> the member is made of a *linearly elastic material*, the neutral axis passes through the section centroid:

$$\text{and} \quad \sigma_x = -\frac{My}{I}$$

- For a member with vertical and horizontal planes of symmetry and a material with the same tensile and compressive stress-strain relationship, the neutral axis is located at the section centroid and the stress-strain relationship may be used to map the strain distribution from the stress distribution.

Plastic Deformations 2

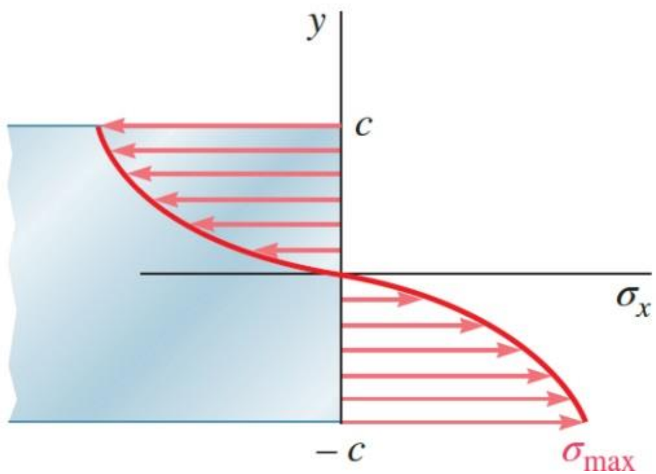


Figure 4.29 Nonlinear stress distribution in member under pure bending.

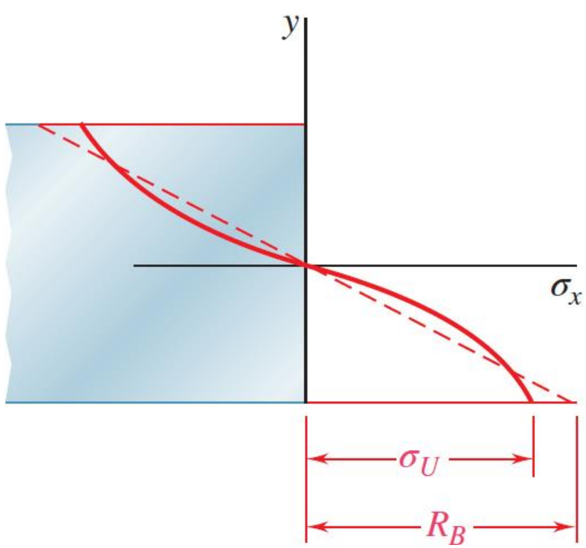


Figure 4.30 Member stress distribution at ultimate moment M_U.

- When the maximum stress is equal to the ultimate strength of the material, failure occurs and the corresponding moment M_U is referred to as the *ultimate bending moment*.
- The *modulus of rupture in bending*, R_B, is found from an experimentally determined value of M_U and a fictitious linear stress distribution:

$$R_B = \frac{M_U c}{I}$$

- R_B may be used to determine M_U of any member made of the same material and with the same cross sectional shape but different dimensions.

Members Made of an Elastoplastic Material

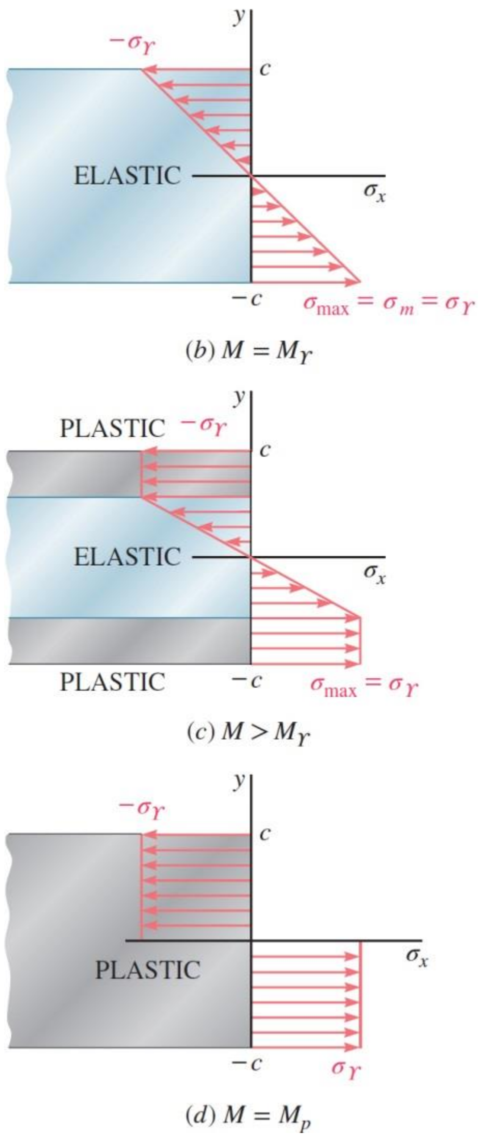


Figure 4.33 Bending stress distribution in a beam for: (b) yield impending, $M = M_Y$, (c) partially yielded, $M > M_Y$, and (d) fully plastic, $M = M_p$.

- Rectangular beam made of an elastoplastic material

$$\sigma_x \le \sigma_Y \qquad \sigma_m = \frac{Mc}{I}$$

$$\sigma_m = \sigma_Y \qquad M_Y = \frac{I}{c}\sigma_Y = \text{ maximum elastic moment}$$

- If the moment is increased beyond the maximum elastic moment, plastic zones develop around an elastic core.

rectangular member

$$M = \tfrac{3}{2} M_Y \left(1 - \tfrac{1}{3}\frac{y_Y^2}{c^2}\right) \qquad y_Y = \text{ elastic core (half-thickness)}$$

- As the moment increase, the plastic zones expand, and at the limit, the deformation is fully plastic.

$$M_p = \tfrac{3}{2} M_Y = \text{ plastic moment}$$

$$k = \frac{M_p}{M_Y} = \text{ shape factor (depends only on cross section shape)}$$

Plastic Deformations of Members With a Single Plane of Symmetry

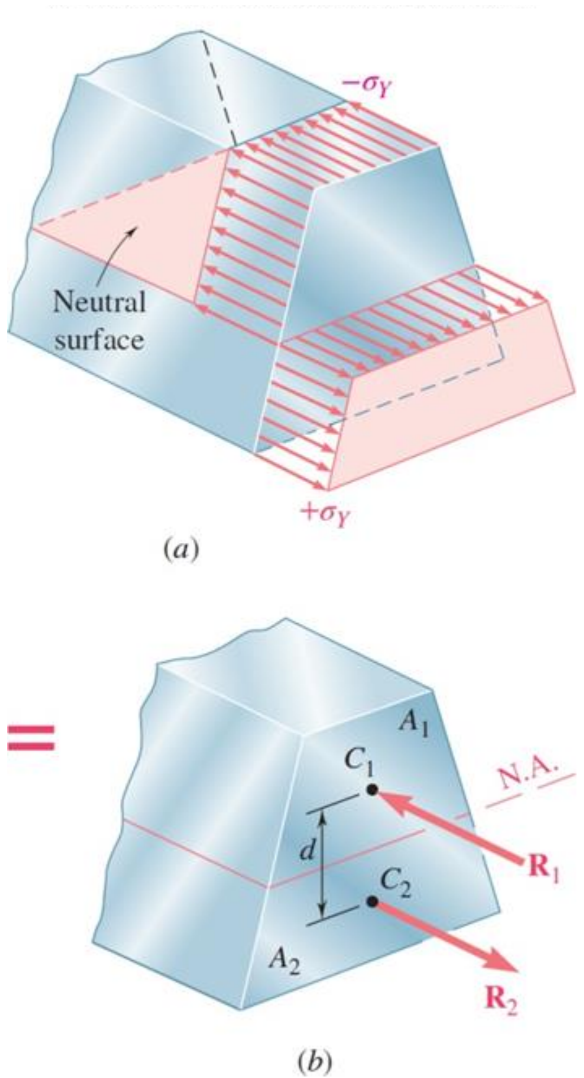


Figure 4.36 Nonsymmetrical beam subject to plastic moment. (a) Stress distributions and (b) resultant moments acting at tension/ compression centroids.

- Fully plastic deformation of a beam with only a vertical plane of symmetry.
- The neutral axis cannot be assumed to coincided with the centroidal axis of the cross section.
- Resultants R_1 and R_2 of the elementary compressive and tensile forces form a couple.

$$R_1 = R_2$$
$$A_1\sigma_Y = A_2\sigma_Y$$

The neutral axis divides the cross section into portions of equal areas.

- The plastic moment for the member:

$$M_p = \left(\tfrac{1}{2} A\sigma_Y\right) d$$

Residual Stresses

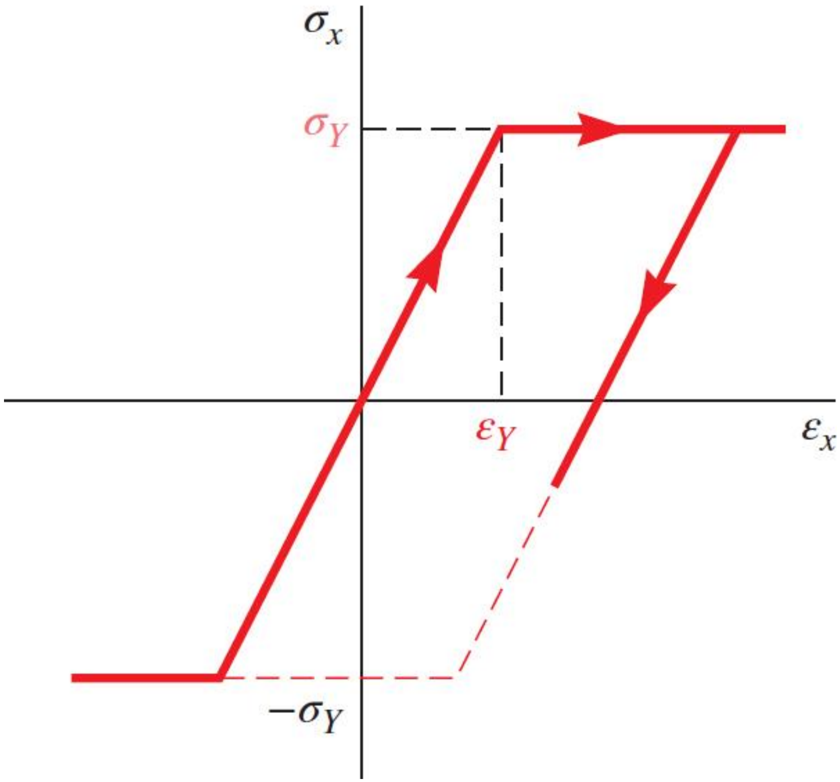


Figure 4.37 Elastoplastic material stress-strain diagram with load reversal.

- Plastic zones develop in a member made of an elastoplastic material if the bending moment is large enough.
- Since the linear relation between normal stress and strain applies at all points during the unloading phase, the unloading phase can be handled by assuming the member to be fully elastic.
- Residual stresses are obtained by applying the principle of superposition to combine the stresses due to loading with a moment *M* (elastoplastic deformation) and unloading with a moment -*M* (elastic deformation).
- The final value of stress at a point will not, in general, be zero.

Concept Application 4.5, 4.6 [1]

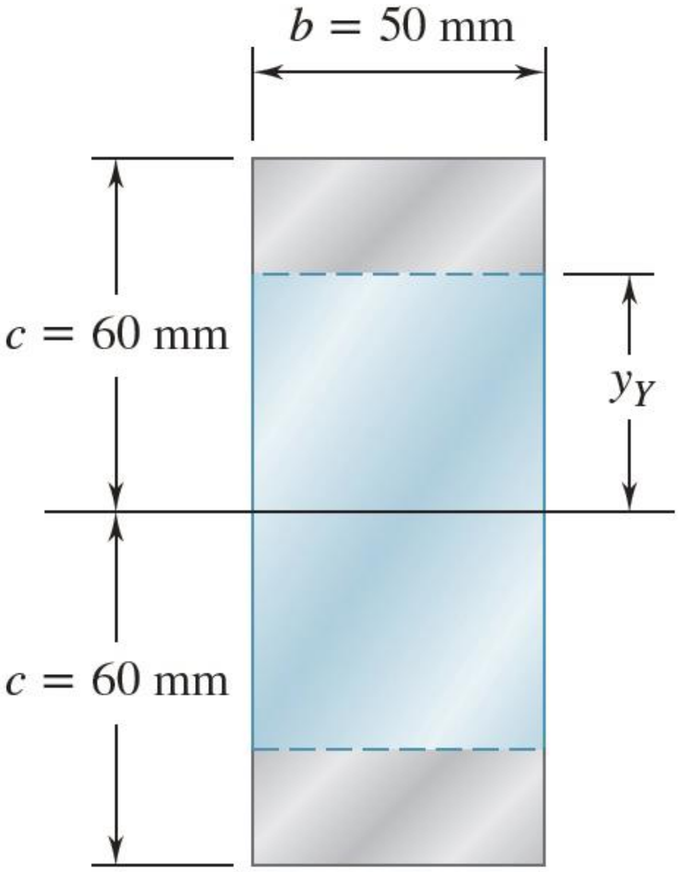


Figure 4.35 Rectangular cross section with load $M_y < M < M_p$.

A member of uniform rectangular cross section is subjected to a bending moment M = 36.8 kN·m. The member is made of an elastoplastic material with a yield strength of 240 MPa and a modulus of elasticity of 200 GPa.

Determine (a) the thickness of the elastic core, (b) the radius of curvature of the neutral surface.

After the loading has been reduced back to zero, determine (c) the distribution of residual stresses, and (d) the radius of curvature.

Concept Application 4.5, 4.6

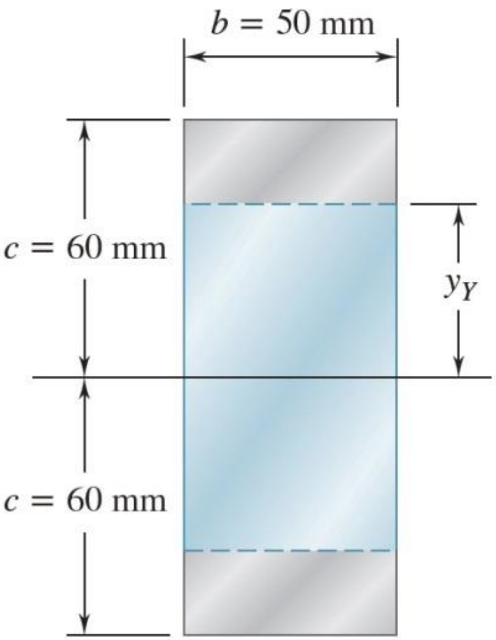


Figure 4.35 Rectangular cross section with load $M_y < M < M_p$.

- Maximum elastic moment:

$$\frac{I}{c} = \tfrac{2}{3} bc^2 = \tfrac{2}{3}\left(50\times10^{-3}\,\text{m}\right)\left(60\times10^{-3}\,\text{m}\right)^2$$

$$= 120\times10^{-6}\,\text{m}^3$$

$$M_Y = \frac{I}{c}\sigma_Y = \left(120\times10^{-6}\,\text{m}^3\right)\left(240\,\text{MPa}\right)$$

$$= 28.8\ \text{kN}\cdot\text{m}$$

- Thickness of elastic core:

$$M = \tfrac{3}{2} M_Y \left(1 - \tfrac{1}{3}\frac{y_Y^2}{c^2}\right)$$

$$36.8\,\text{kN}\cdot\text{m} = \tfrac{3}{2}\left(28.8\,\text{kN}\cdot\text{m}\right)\left(1 - \tfrac{1}{3}\frac{y_Y^2}{c^2}\right)$$

$$\frac{y_Y}{c} = \frac{y_Y}{60\,\text{mm}} = 0.666$$

$$\boxed{2y_Y = 80\,\text{mm}}$$

- Radius of curvature:

$$\varepsilon_Y = \frac{\sigma_Y}{E} = \frac{240\times10^6\ \text{Pa}}{200\times10^9\ \text{Pa}}$$

$$= 1.2\times10^{-3}$$

$$\varepsilon_Y = \frac{y_Y}{\rho}$$

$$\rho = \frac{y_Y}{\varepsilon_Y} = \frac{40\times10^{-3}\,\text{m}}{1.2\times10^{-3}}$$

$$\boxed{\rho = 33.3\,\text{m}}$$

Concept Application 4.5, 4.6 [3]

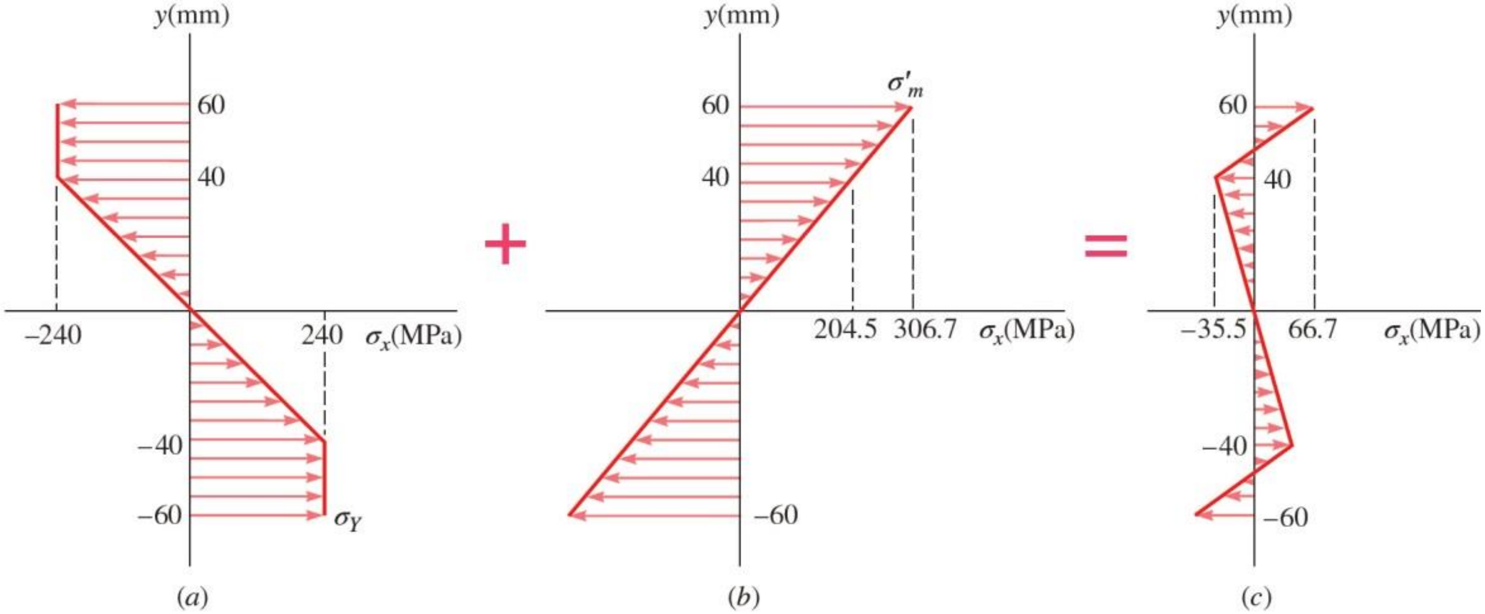


Figure 4.38 Determination of residual stress: (a) Stresses at maximum moment. (b) Unloading. (c) Residual stresses.

- $M = 36.8$ kN-m

$$y_Y = 40\,\text{mm}$$
$$\sigma_Y = 240\,\text{MPa}$$

- $M = -36.8$ kN-m

$$\sigma'_m = \frac{Mc}{I} = \frac{36.8\,\text{kN}\cdot\text{m}}{120\times10^6\,\text{m}^3}$$
$$= 306.7\,\text{MPa} < 2\sigma_Y$$

- $M = 0$

At the edge of the elastic core,

$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{-35.5\times10^6\ \text{Pa}}{200\times10^9\ \text{Pa}}$$
$$= -177.5\times10^{-6}$$

$$\rho = -\frac{y_Y}{\varepsilon_x} = \frac{40\times10^{-3}\,\text{m}}{177.5\times10^{-6}}$$

$$\boxed{\rho = 225\,\text{m}}$$

Eccentric Axial Loading in a Plane of Symmetry

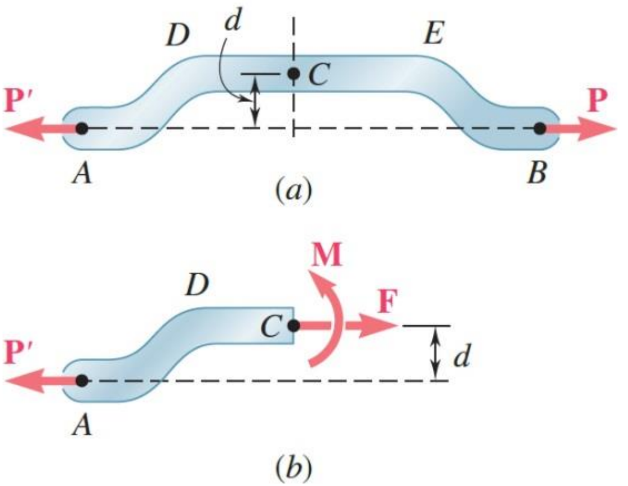


Figure 4.39 (a) Member with eccentric loading. (b) Free-body diagram of a member with internal loads at section C.

- Stress due to eccentric loading found by superposing the uniform stress due to a centric load and the linear stress distribution due to a pure bending moment.

$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$

$$= \frac{P}{A} - \frac{My}{I}$$

- Results are valid if stresses do not exceed the proportional limit, deformations have negligible effect on geometry, and stresses are not evaluated near points of load application.

- Eccentric loading.

$$F = P$$

$$M = Pd$$

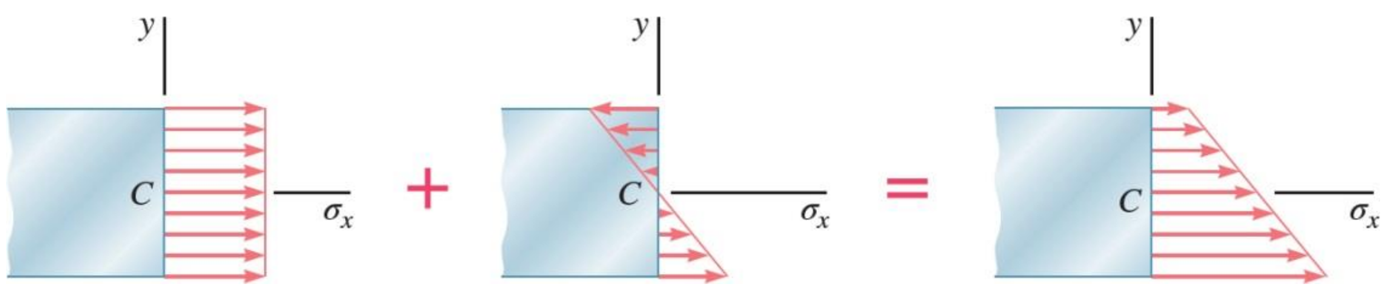


Figure 4.41 Stress distribution for eccentric loading is obtained by superposing the axial and pure bending distributions.

Concept Application 4.7 [1]

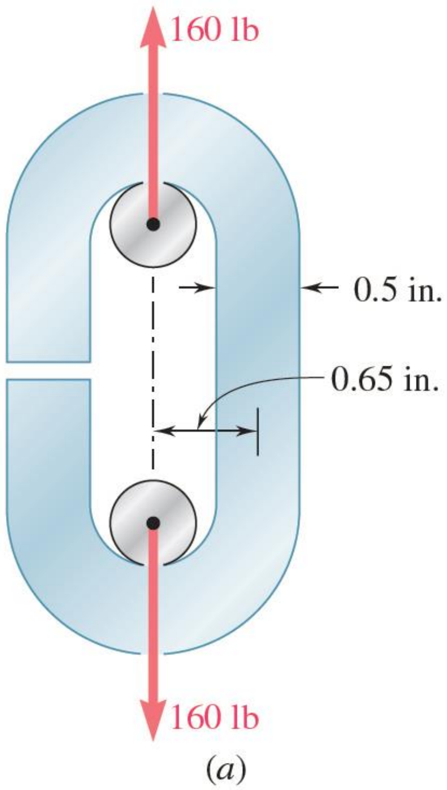


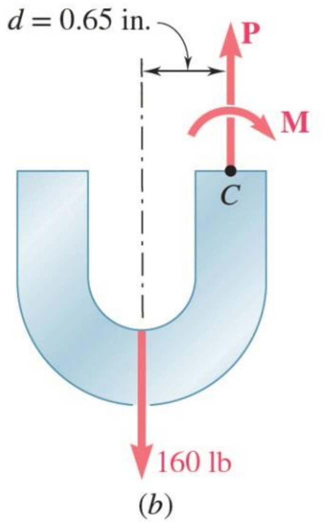
Figure 4.43 Open chain link under loading.

An open-link chain is obtained by bending low-carbon steel rods into the shape shown. For a 160 lb load, determine **(a)** maximum tensile and compressive stresses, **(b)** distance between section centroid and neutral axis.

SOLUTION:

- Find the equivalent centric load and bending moment.
- Superpose the uniform stress due to the centric load and the linear stress due to the bending moment.
- Evaluate the maximum tensile and compressive stresses at the inner and outer edges, respectively, of the superposed stress distribution.
- Evaluate the maximum tensile and compressive stresses at the inner and outer edges, respectively, of the superposed stress distribution.

Concept Application 4.7 [2]



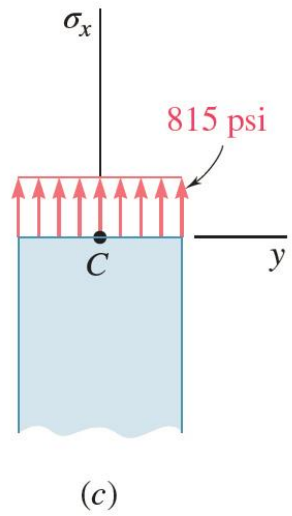
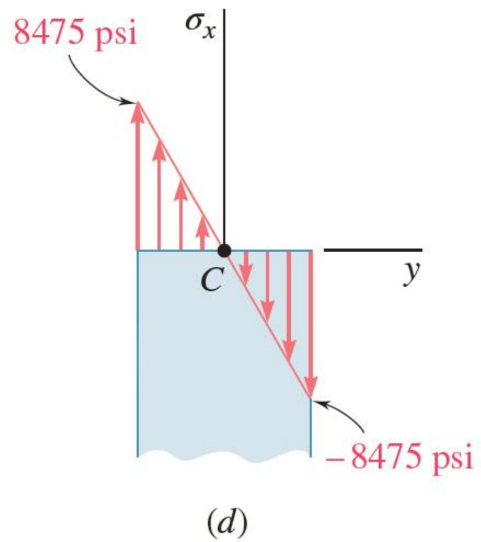


Figure 4.43 Free-body diagram for section at C to find axial force and moment. Stress at section C is superposed axial and bending stresses.

- Equivalent centric load and bending moment.

$$P = 160\,\text{lb}$$

$$M = Pd = (160\,\text{lb})(0.65\,\text{in}) = 104\,\text{lb}\cdot\text{in}$$



- Normal stress due to a centric load:

$$A = \pi c^2 = \pi(0.25\,\text{in})^2 = 0.1963\,\text{in}^2$$

$$\sigma_0 = \frac{P}{A} = \frac{160\,\text{lb}}{0.1963\,\text{in}^2} = 815\,\text{psi}$$

- Normal stress due to bending moment:

$$I = \tfrac{1}{4}\pi c^4 = \tfrac{1}{4}\pi(0.25)^4 = 3.068\times10^{-3}\,\text{in}^4$$

$$\sigma_m = \frac{Mc}{I} = \frac{(104\,\text{lb}\cdot\text{in})(0.25\,\text{in})}{3.068\times10^{-3}\,\text{in}^4} = 8475\,\text{psi}$$

Concept Application 4.7 3

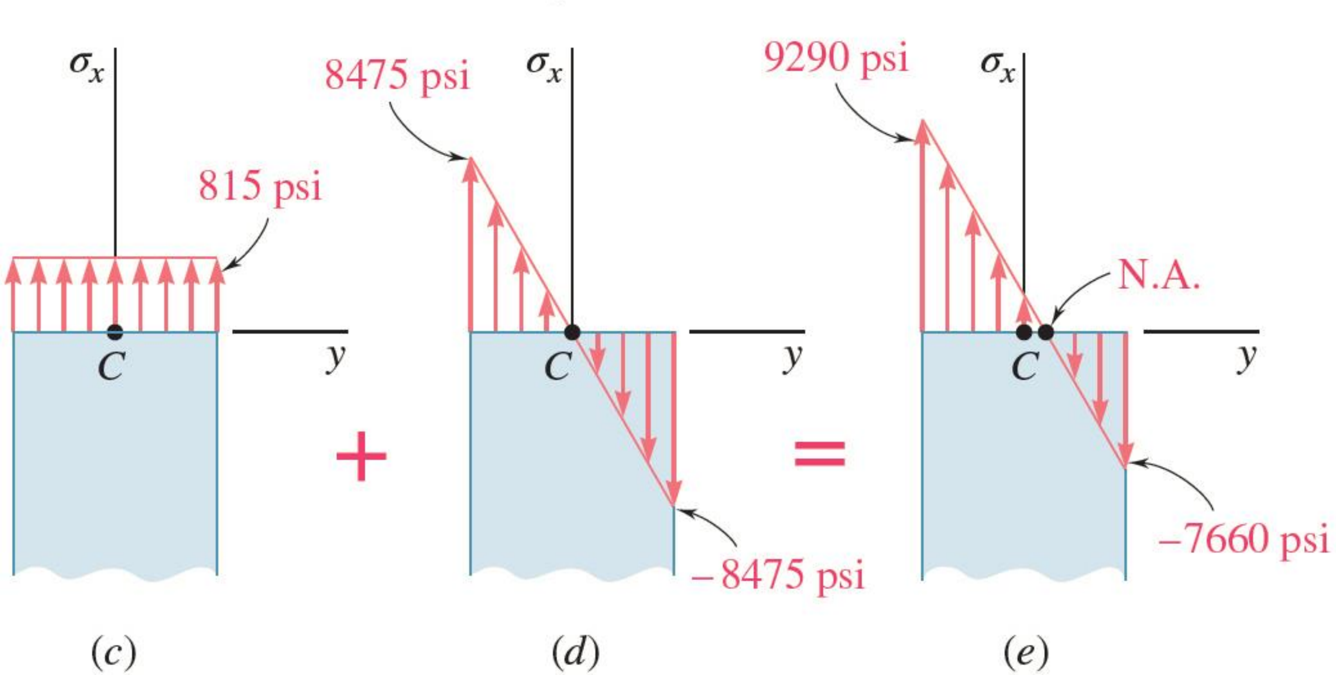


Figure 4.43 (c) Axial stress at section C. (d) Bending stress at C. (e) Superposition of stresses.

- Maximum tensile and compressive stresses:

$$\sigma_t = \sigma_0 + \sigma_m$$
$$= 815 + 8475 \qquad \boxed{\sigma_t = 9260\,\text{psi}}$$

$$\sigma_c = \sigma_0 - \sigma_m$$
$$= 815 - 8475 \qquad \boxed{\sigma_c = -7660\,\text{psi}}$$

- Neutral axis location:

$$0 = \frac{P}{A} - \frac{My_0}{I}$$

$$y_0 = \frac{P}{A}\frac{I}{M} = (815\,\text{psi})\frac{3.068\times10^{-3}\,\text{in}^4}{105\,\text{lb}\cdot\text{in}}$$

$$\boxed{y_0 = 0.0240\,\text{in}}$$

Problem 4.8 [1]

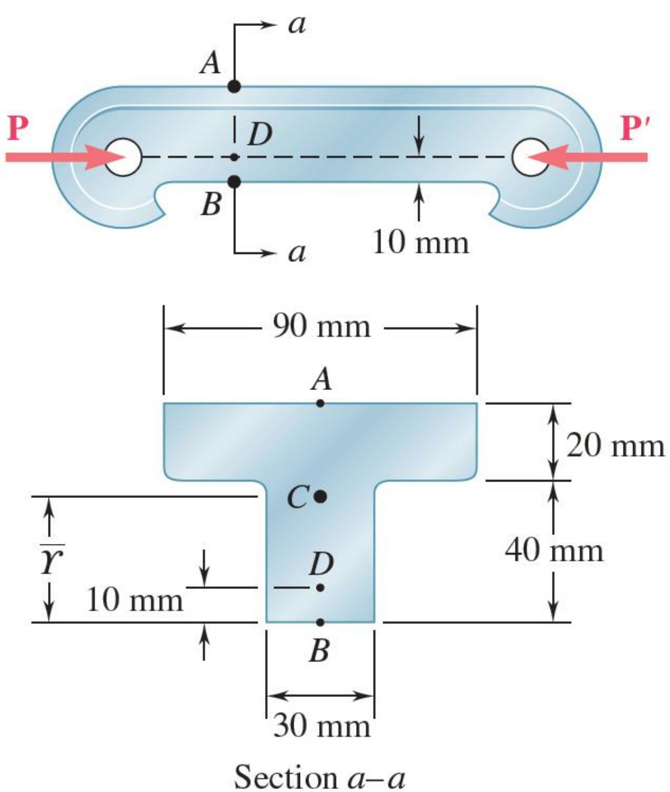


Figure 1 Section geometry to find centroid location.

From Problem 4.2

$$A = 3\times10^{-3}\ \mathrm{m}^2$$

$$\bar{Y} = 0.038\,\mathrm{m}$$

$$I = 868\times10^{-9}\ \mathrm{m}^4$$

The largest allowable stresses for the cast iron link are 30 MPa in tension and 120 MPa in compression. Determine the largest force *P* that can be applied to the link.

SOLUTION:

- Determine equivalent centric load and bending moment.
- Superpose the stress due to a centric load and the stress due to bending.
- Evaluate the critical loads for the allowable tensile and compressive stresses.
- The largest allowable load is the smallest of the two critical loads.

Problem 4.8 [2]

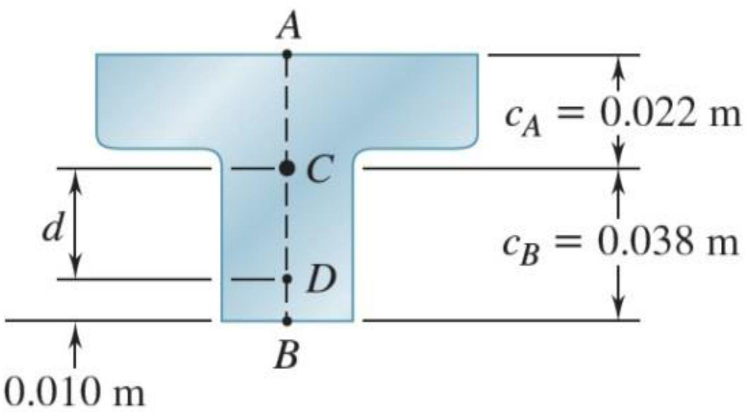


Figure 2 Section dimensions for finding location of point D.

- Determine equivalent centric and bending loads.

$$d = 0.038 - 0.010 = 0.028\,\text{m}$$

$$P = \text{centric load}$$

$$M = Pd = 0.028\,P = \text{ bending moment}$$

- Superpose stresses due to centric and bending loads:

$$\sigma_A = -\frac{P}{A} + \frac{Mc_A}{I} = -\frac{P}{3\times10^{-3}} + \frac{(0.028\,P)(0.022)}{868\times10^{-9}} = +377\,P$$

$$\sigma_B = -\frac{P}{A} - \frac{Mc_A}{I} = -\frac{P}{3\times10^{-3}} - \frac{(0.028\,P)(0.022)}{868\times10^{-9}} = -1559\,P$$

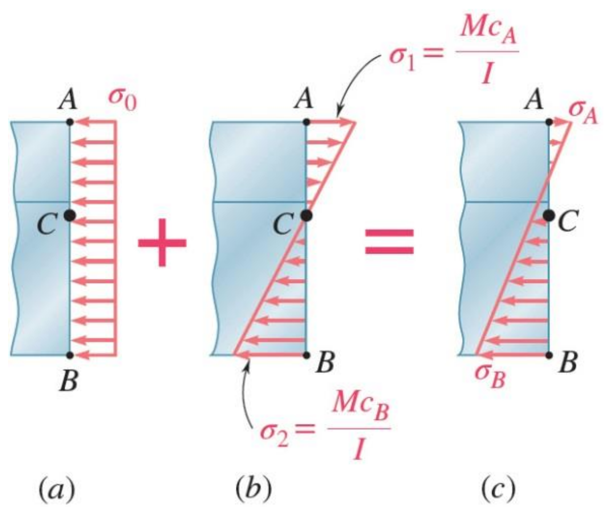


Figure 4 Stress distribution at section C is superposition of axial and bending distributions acting at centroid.

- Evaluate critical loads for allowable stresses:

$$\sigma_A = +377\,P = 30\,\text{MPa} \qquad P = 79.6\,\text{kN}$$

$$\sigma_B = -1559\,P = -120\,\text{MPa} \qquad P = 77.0\,\text{kN}$$

- The largest allowable load: $\boxed{P = 77.0\ \text{kN}}$

Unsymmetric Bending 1

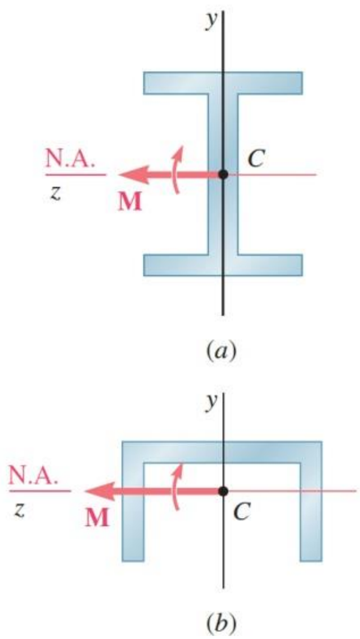


Figure 4.44 Moment in plane of symmetry.

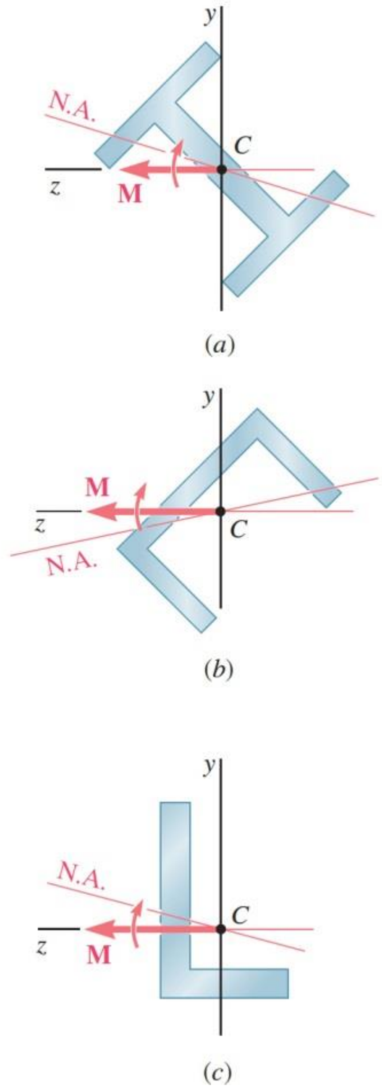


Figure 4.45 Moment not in plane of symmetry.

- Analysis of pure bending has been limited to members subjected to bending couples acting in a plane of symmetry.
- Members remain symmetric and bend in the plane of symmetry.
- The neutral axis of the cross section coincides with the axis of the couple.
- Will now consider situations in which the bending couples do *not* act in a plane of symmetry.
- Cannot assume that the member will bend in the plane of the couples.
- In general, the neutral axis of the section will not coincide with the axis of the couple.

Unsymmetric Bending 2

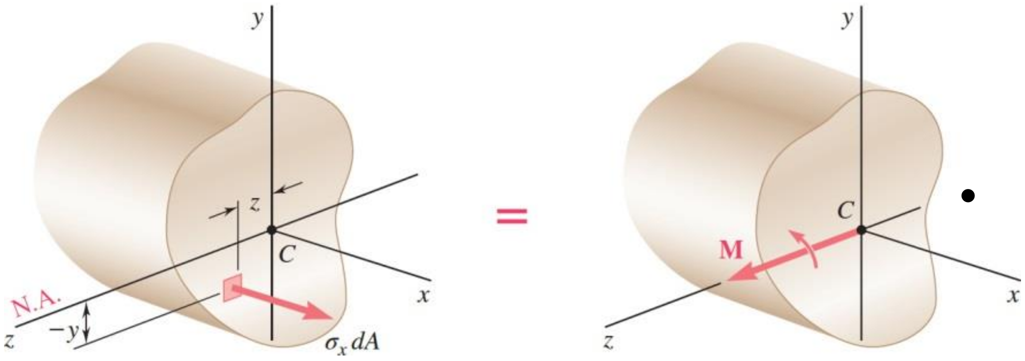


Figure 4.46 Section of arbitrary shape where the neutral axis coincides with the axis of couple **M**.

Determine the conditions under which the neutral axis of a cross section of arbitrary shape coincides with the axis of the couple as shown.

- The resultant force and moment from the distribution of elementary forces in the section must satisfy.

$$F_x = 0 = M_y \quad M_z = M = applied \quad couple$$

- $$0 = F_x = \int \sigma_x dA = \int \left(-\frac{y}{c}\sigma_m \right) dA$$
  or $0 = \int y\, dA$

  Neutral axis passes through centroid.

- $$M = M_z = -\int y \left(-\frac{y}{c}\sigma_m \right) dA$$
  or $M = \dfrac{\sigma_m I}{c}$ $\quad I = I_z = moment\ of\ inertia$

  defines stress distribution.

- $$0 = M_y = \int z\sigma_x dA = \int z \left(-\frac{y}{c}\sigma_m \right) dA$$
  or $0 = \int yz\, dA = I_{yz} = \ product\ of\ \ inertia$

  couple vector must be directed along a principal centroidal axis.

Unsymmetric Bending 3

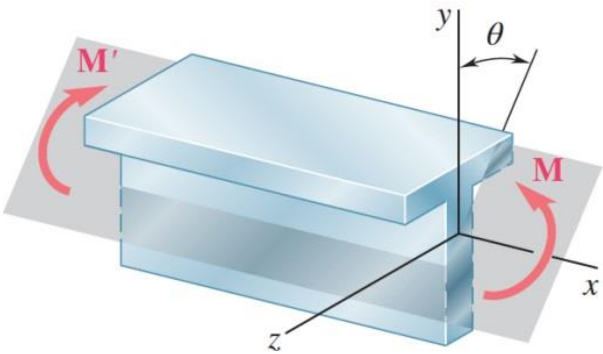


Figure 4.49 Unsymmetric bending, with bending moment not in a plane of symmetry.

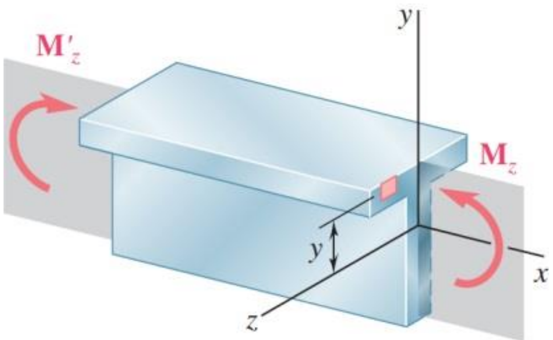


Figure 4.51 $\mathbf{M}_Z$ acts in a plane that includes a principal centroidal axis, bending the member in the vertical plane.

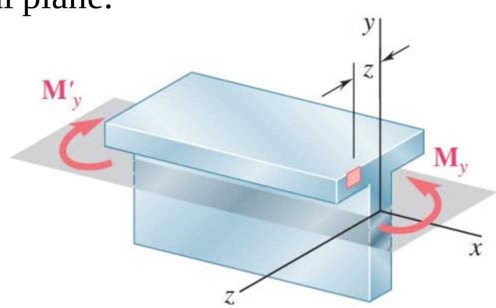


Figure 4.52 My acts in a plane that includes a principal centroidal axis, bending the member in the horizontal plane.

Superposition is applied to determine stresses in the most general case of unsymmetric bending.

- Resolve the couple vector into components along the principle centroidal axes.

$$M_z = M\cos\theta \qquad M_y = M\sin\theta$$

- Superpose the component stress distributions.

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- Along the neutral axis:

$$\sigma_x = 0 = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = -\frac{(M\cos\theta)y}{I_z} + \frac{(M\sin\theta)z}{I_y}$$

$$\tan\varphi = \frac{y}{z} = \frac{I_z}{I_y}\tan\theta$$

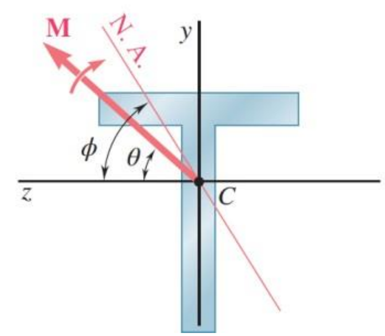
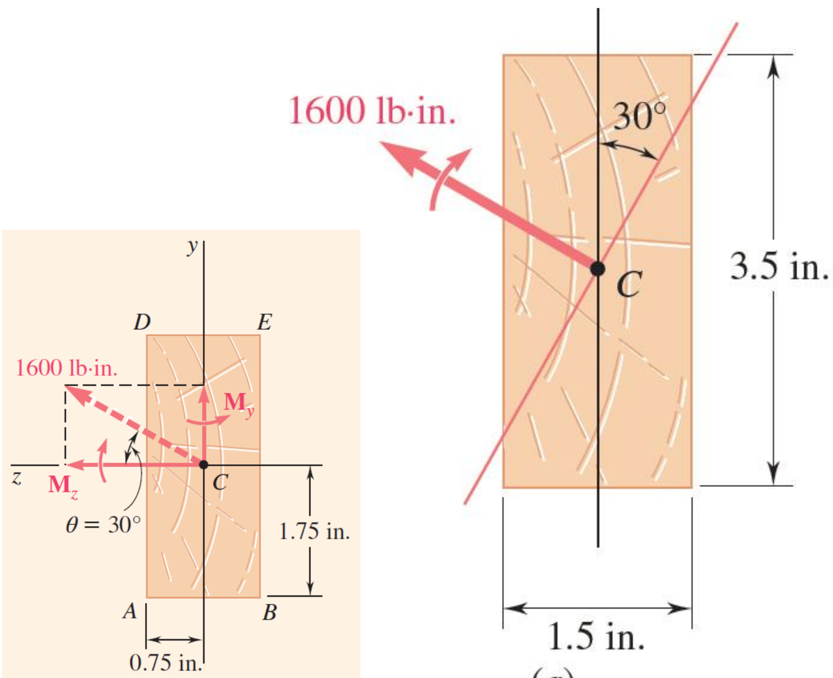


Figure 4.54 Neutral axis for unsymmetric bending.

Concept Application 4.8 [1]



(a)

A 1600 lb-in couple is applied to a rectangular wooden beam in a plane forming an angle of 30° with the vertical. Determine (a) the maximum stress in the beam, (b) the angle that the neutral axis forms with the horizontal plane.

SOLUTION:

- Resolve the couple vector into components along the principle centroidal axes and calculate the corresponding maximum stresses.

$$M_z = M\cos\theta \qquad M_y = M\sin\theta$$

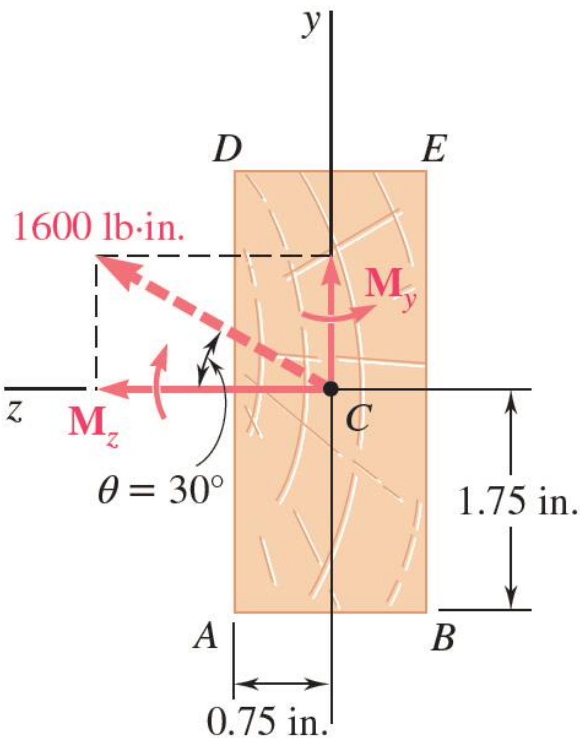
- Combine the stresses from the component stress distributions.

$$\sigma_x = \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- Determine the angle of the neutral axis.

$$\tan\varphi = \frac{y}{z} = \frac{I_z}{I_y}\tan\theta$$

Concept Application 4.8 2



(b)

- Resolve the couple vector into components and calculate the corresponding maximum stresses:

$$M_z = (1600\,\text{lb}\cdot\text{in})\cos 30 = 1386\,\text{lb}\cdot\text{in}$$

$$M_y = (1600\,\text{lb}\cdot\text{in})\sin 30 = 800\,\text{lb}\cdot\text{in}$$

$$I_z = \tfrac{1}{12}(1.5\,\text{in})(3.5\,\text{in})^3 = 5.359\,\text{in}^4$$

$$I_y = \tfrac{1}{12}(3.5\,\text{in})(1.5\,\text{in})^3 = 0.9844\,\text{in}^4$$

The largest tensile stress due to M_z occurs along *AB*

$$\sigma_1 = \frac{M_z y}{I_z} = \frac{(1386\,\text{lb}\cdot\text{in})(1.75\,\text{in})}{5.359\,\text{in}^4} = 452.6\,\text{psi}$$

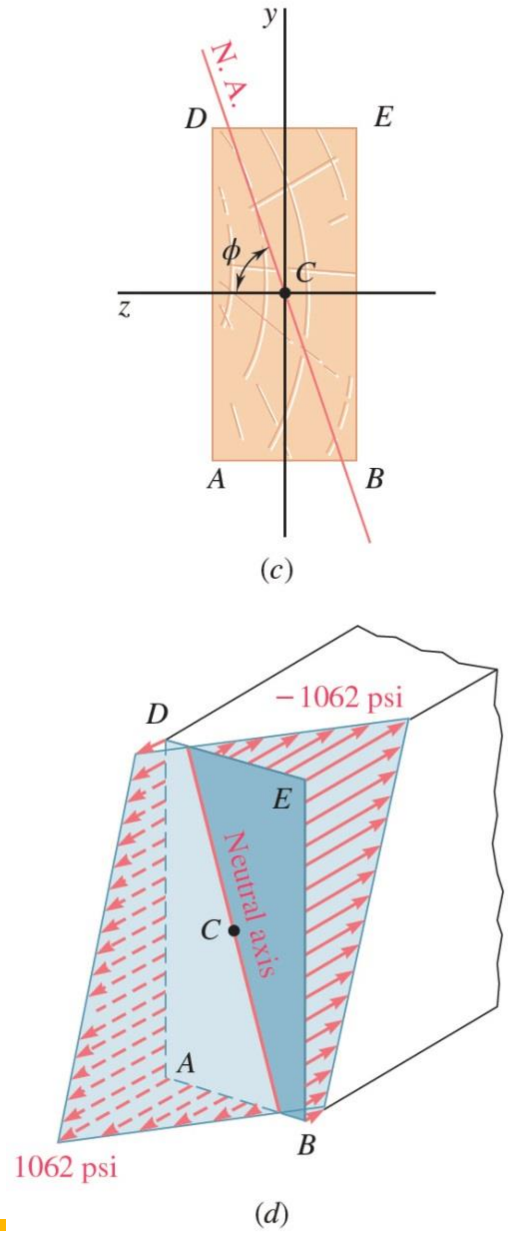
The largest tensile stress due to M_z occurs along *AD*

$$\sigma_2 = \frac{M_y z}{I_y} = \frac{(800\,\text{lb}\cdot\text{in})(0.75\,\text{in})}{0.9844\,\text{in}^4} = 609.5\,\text{psi}$$

- The largest tensile stress due to the combined loading occurs at *A:*

$$\sigma_{max} = \sigma_1 + \sigma_2 = 452.6 + 609.5 \qquad \boxed{\sigma_{max} = 1062\,\text{psi}}$$

Concept Application 4.8 3



- Determine the angle of the neutral axis.

$$\tan\varphi = \frac{I_z}{I_y}\tan\theta = \frac{5.359\,\text{in}^4}{0.9844\,\text{in}^4}\tan 30$$

$$= 3.143$$

$$\boxed{\phi = 72.4^o}$$

Figure 4.55 Cross section with neutral axis and stress distribution.

General Case of Eccentric Axial Loading

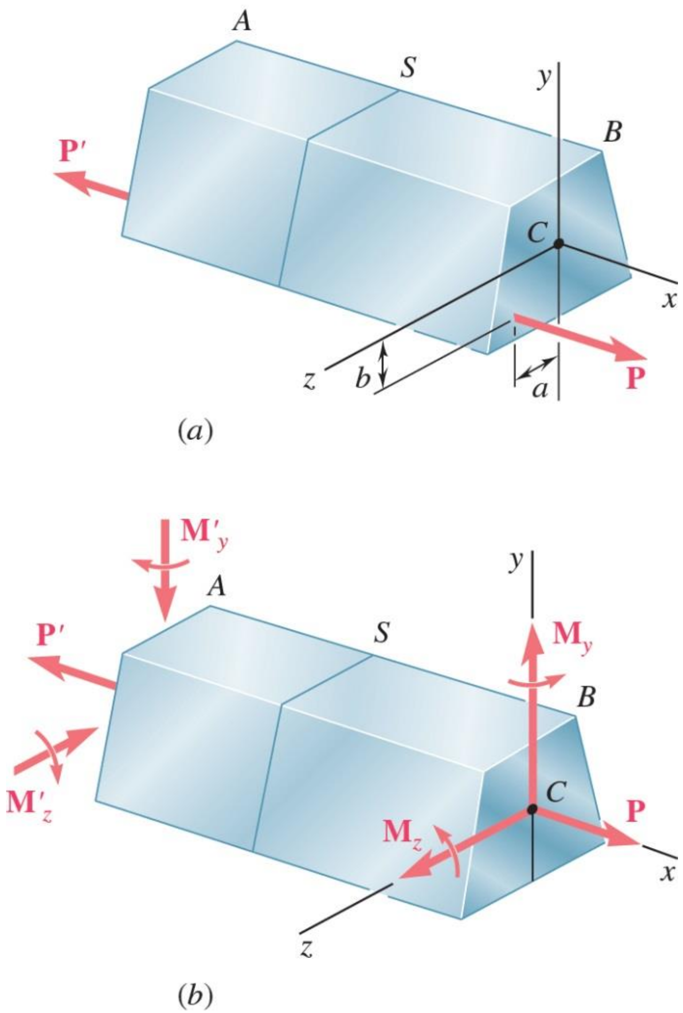


Figure 4.56 Eccentric axial loading. (a) Axial force applied away from section centroid. (b) Equivalent force-couple system acting at centroid.

- Consider a straight member subject to equal and opposite eccentric forces.
- The eccentric force is equivalent to the system of a centric force (***P***) and two couples (***M$_x$*** and ***M$_y$***).

$$P = \text{ centric force}$$

$$M_y = Pa \qquad M_z = Pb$$

- By the principle of superposition, the combined stress distribution is

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- If the neutral axis lies on the section, it may be found from

$$\frac{M_z}{I_z} y - \frac{M_y}{I_y} z = \frac{P}{A}$$