ME4010 Computational Methods for Mechanical Engineering

Chapter 4 Solution of system of equations

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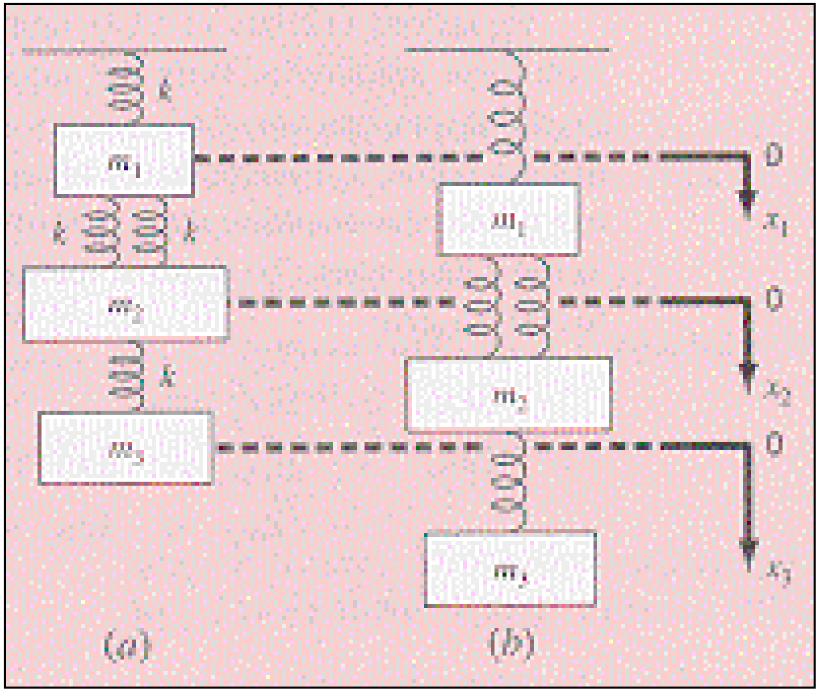
Dynamics

$$m_1 \frac{d^2}{dt^2} x_1 = 2k(x_2 - x_1) + m_1 g - kx_1$$

$$m_2 \frac{d^2}{dt^2} x_2 = k(x_3 - x_2) + m_2 g - 2k(x_2 - x_1)$$

$$m_3 \frac{d^2}{dt^2} x_3 = m_3 g - k(x_3 - x_2)$$



The steady state equations are:

$$\boxed{\begin{aligned} 3kx_1 - 2kx_2 + 0 &= m_1 g \\ -2kx_1 + 3kx_2 - kx_3 &= m_2 g \\ 0 - kx_2 + kx_3 &= m_3 g \end{aligned}}$$

In matrix form:

$$\begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} m_1 g \\ m_2 g \\ m_3 g \end{bmatrix}$$

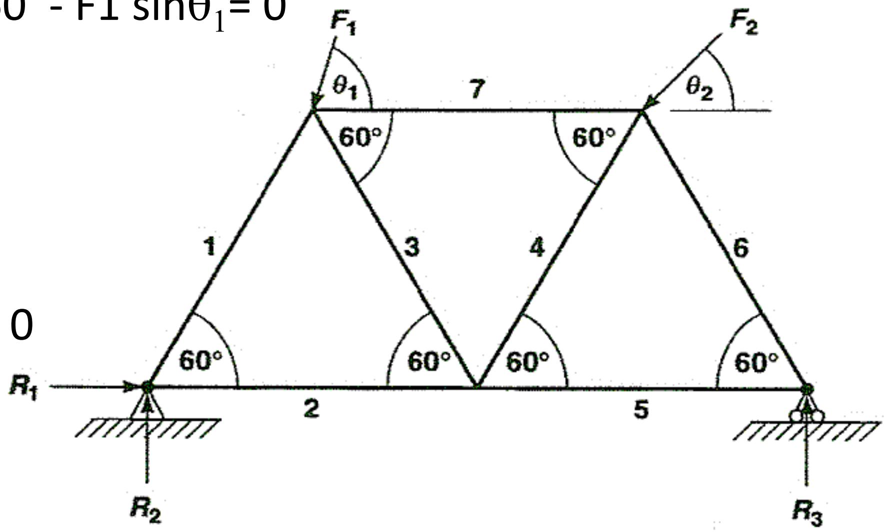
Statics

$-T1 \cos 60 + T3 \cos 60 + T7 - F1 \cos \theta_1 = 0$

$-T1 \sin 60 - T3 \sin 60 - F1 \sin\theta_1 = 0$

$-T4 \cos 60 - T7 + T6 \cos 60 - F2 \cos \theta_2 = 0$

$-T4 \sin 60 - T6 \sin 60 - F2 \sin\theta_2 = 0$



$R1 + T1 \cos 60 + T2 = 0$

$R2 + T1 \sin 60 = 0$

$-T5 - T6 \cos 60 = 0$

$T6 \sin 60 + R3 = 0$

$-T2 - T3 \cos 60 + T4 \cos 60 + T5 = 0$

$T3 \sin 60 + T4 \sin 60 = 0$

In matrix form

$$\begin{bmatrix} 1 & \cos 60 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sin 60 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -\cos 60 & \cos 60 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin 60 & \sin 60 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\cos 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin 60 & 1 & 0 \\ 0 & -\cos 60 & 0 & 0 & \cos 60 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\sin 60 & 0 & 0 & -\sin 60 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\cos 60 & 0 & \cos 60 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -\sin 60 & 0 & -\sin 60 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_1 \\ T_1 \\ T_2 \\ R_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ R_3 \\ T_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F_1 \cos \theta_1 \\ F_1 \sin \theta_1 \\ F_2 \cos \theta_2 \\ F_2 \sin \theta_2 \end{bmatrix}$$

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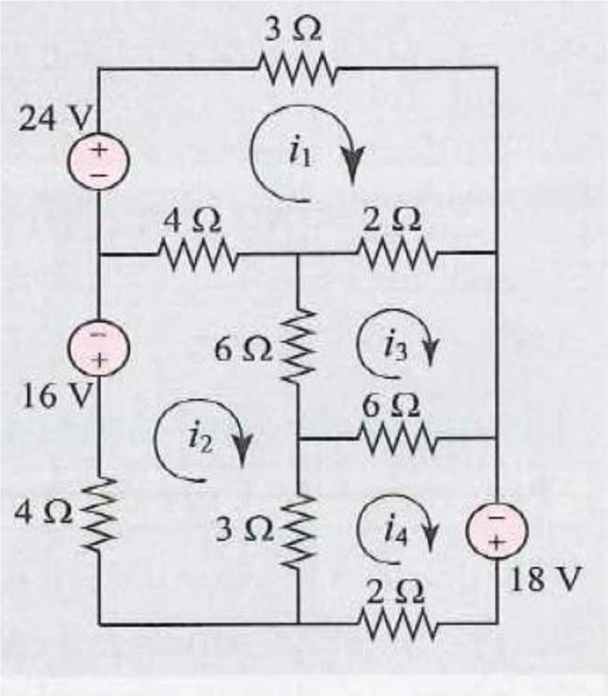
Circuit

$$V = I * R$$



1: $i_1*9 - i_2*4 - i_3*2 = 24$

$$\begin{bmatrix} 9 & -4 & -2 & 0 \\ -4 & 17 & -6 & -3 \\ -2 & -6 & 14 & -6 \\ 0 & -3 & -6 & 11 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 24 \\ -16 \\ 0 \\ 18 \end{bmatrix}$$

LU Decomposition

Method

$$[A] = [L][U]$$

where

$[L]$ = lower triangular matrix

$[U]$ = upper triangular matrix

Given $[A][X] = [B]$

1. Decompose $[A]$ into $[L]$ and $[U]$
2. Solve $[L][Z] = [B]$ for $[Z]$
3. Solve $[U][X] = [Z]$ for $[X]$

Method: [A] Decomposes to [L] and [U]

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

[*U*] is the same as the coefficient matrix at the end of the forward elimination step.

[*L*] is obtained using the *multipliers* that were used in the forward elimination process

Finding the [*U*] matrix

Using the Forward Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Step 1: $\frac{64}{25} = 2.56; \quad Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$

$$\frac{144}{25} = 5.76; \quad Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Finding the [U] Matrix

Matrix after Step 1: $\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$

Step 2: $\dfrac{-16.8}{-4.8} = 3.5; \quad Row3 - Row2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Finding the [L] matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

Using the multipliers used during the Forward Elimination Procedure

From the first step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

Finding the [L] Matrix

From the second step of forward elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \qquad \ell_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Does [L][U] = [A]?

$$[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}= \ ?$$

$$\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$$

Try this…

$$\begin{bmatrix} 4 & -2 & -3 & 6 \\ -6 & 7 & 6.5 & -6 \\ 1 & 7.5 & 6.25 & 5.5 \\ -12 & 22 & 15.5 & -1 \end{bmatrix}$$

Using LU Decomposition to solve SLEs

Solve the following set of linear equations using LU Decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the procedure for finding the [*L*] and [*U*] matrices

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Example

Set $[L][Z] = [B]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solve for $[Z]$

$$z_1 = 106.8$$
$$2.56 z_1 + z_2 = 177.2$$
$$5.76 z_1 + 3.5 z_2 + z_3 = 279.2$$

Example

Complete the forward substitution to solve for $[Z]$

$$z_1 = 106.8$$

$$\begin{aligned} z_2 &= 177.2 - 2.56 z_1 \\ &= 177.2 - 2.56(106.8) \\ &= -96.2 \end{aligned}$$

$$\begin{aligned} z_3 &= 279.2 - 5.76 z_1 - 3.5 z_2 \\ &= 279.2 - 5.76(106.8) - 3.5(-96.21) \\ &= 0.735 \end{aligned}$$

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example

Set $[U][X] = [Z]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Solve for $[X]$

The 3 equations become

$$25x_1 + 5x_2 + x_3 = 106.8$$

$$-4.8x_2 - 1.56x_3 = -96.21$$

$$0.7x_3 = 0.735$$

Example

From the 3rd equation

$$0.7x_3 = 0.735$$

$$x_3 = \frac{0.735}{0.7}$$

$$x_3 = 1.050$$

Substituting in a_3 and using the second equation

$$-4.8x_2 - 1.56x_3 = -96.21$$

$$x_2 = \frac{-96.21 + 1.56x_3}{-4.8}$$

$$x_2 = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$x_2 = 19.70$$

Example

Substituting in a_3 and a_2 using the first equation

$$25x_1 + 5x_2 + x_3 = 106.8$$

$$\begin{aligned} x_1 &= \frac{106.8 - 5x_2 - x_3}{25} \\ &= \frac{106.8 - 5(19.70) - 1.050}{25} \\ &= 0.2900 \end{aligned}$$

Hence the Solution Vector is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Try this…

$$\begin{bmatrix} 4 & -2 & -3 & 6 \\ -6 & 7 & 6.5 & -6 \\ 1 & 7.5 & 6.25 & 5.5 \\ -12 & 22 & 15.5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ -6.5 \\ 16 \\ 17 \end{bmatrix}$$

```
Module LU_solver
Contains
! SUBROUTINE TO PERFORM DOLITTLE FACTORIZATION: A=LU
 SUBROUTINE LU(N,A,B,X)
    integer::N,I,j,k
    double precision:: A(N,N),L(N,N),U(N,N),B(N),Z(N),X(N)
    double precision:: S,S1,S2,SS,S3,S4
    DO I = 1,N
     L(I,I)=1.
    ENDDO
    U(1,1)=A(1,1)
    IF(U(1,1).EQ.0.)THEN
     WRITE(1,*)'LU FACTORIZATION IS IMPOSSIBLE'
     RETURN
    ENDIF
    DO J =2,N
     U(1,J)=A(1,J)            ! DETERMINE THE FIRST ROW OF U
     L(J,1)=A(J,1)/U(1,1)     ! DETERMINE THE FIRST COLUMN OF XL
    ENDDO
    DO I = 2, N-1
     S=0.
     DO K = 1, I-1
      S=S+L(I,K)*U(K,I)
     ENDDO
     U(I,I)= A(I,I)-S
     IF(U(I,I).EQ.0.)THEN
      WRITE(1,*)'LU FACTORIZATION IS IMPOSSIBLE'
      RETURN
     ENDIF
     DO J = I+1, N
      S1=0.
      S2=0.
      DO K = 1, I-1
       S1=S1+L(I,K)*U(K,J)     ! DETERMINE THE (ith) ROWS OF U
       S2=S2+L(J,K)*U(K,I)     ! DETERMINE THE (ith) COLUMN OF XL
       U(I,J)=A(I,J)-S1
       L(J,I)=(A(J,I)-S2)/U(I,I)
      ENDDO
     ENDDO
    ENDDO
    SS=0
    DO K = 1,N-1
     SS=SS+L(N,K)*U(K,N)
     U(N,N)=A(N,N)-SS
    ENDDO
    IF(U(N,N).EQ.0) THEN
     WRITE(1,*)' THE MATRIX A IS SINQULAR'
    ENDIF
! SUBROUTINE TO SOLVE LZ=B USING FORWARD SUBSTITUTION
    Z(1)=B(1)
    DO I =2,N
     S3=0.
     DO J =1,I-1
      S3=S3+L(I,J)*Z(J)
     ENDDO
     Z(I)=B(I)-S3
    ENDDO
    X(N)=Z(N)/U(N,N)
    DO K = N-1,1,-1            ! SOLVING UX=Z
     S4=0.
     DO J=K+1,N
      S4=S4+U(K,J)*X(J)
     ENDDO
     X(K)=(Z(K)-S4)/U(K,K)
    ENDDO
    RETURN
    END subroutine LU
 end module solver
```

```
program main
   use LU_solver
   double precision, dimension(4,4)::A
   double precision, dimension (4)::B,X
   integer::i,j,N=4
   open (1, file='matrix.dat')
   do i=1,n
     read(1,*)(a(i,j),J=1,N)
   enddo
   read(1,*)B
   call LU(N,A,B,X)
   print *,x
   stop
end program main
```

$$\begin{bmatrix} 4 & -2 & -3 & 6 \\ -6 & 7 & 6.5 & -6 \\ 1 & 7.5 & 6.25 & 5.5 \\ -12 & 22 & 15.5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ -6.5 \\ 16 \\ 17 \end{bmatrix}$$

matrix.dat file

```
4.   -2. -3.  6.
-6.  7.  6.5  -6.
1.   7.5 6.25 5.5
-12. 22. 15.5 -1.
12. -6.5 16. 17.
```