Vector Mechanics for Engineers: Statics

Moments of Forces

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Textbook: *Vector Mechanics for Engineers: Dynamics,* Beer, Johnston, Mazurek and Cornwell, McGraw-Hill, 10th edition, 2012.

Brief Review: Moment of a Force About a Point

- A force vector $\boldsymbol{F}$ is defined by its magnitude and direction. Its effect on the rigid body also depends on it point of application.

- The *moment* of $\boldsymbol{F}$ about point O is defined as

$$\boldsymbol{M}_O = \boldsymbol{r} \times \boldsymbol{F}$$

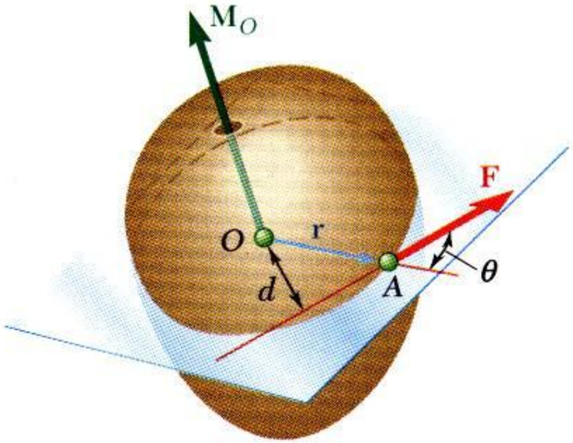
- The moment vector $\boldsymbol{M}_O$ is perpendicular to the plane containing O and the force $\boldsymbol{F}$.

- Magnitude of $\boldsymbol{M}_O$ measures the tendency of the force to cause rotation of the body about an axis along $\boldsymbol{M}_O$.

$$M_O = rF\sin\theta = Fd$$

  The sense **of the moment may be** determined by the right-hand rule.

- Any force $\boldsymbol{F'}$ that has the same magnitude and direction as $\boldsymbol{F}$, is *equivalent* if it also has the same line of action and therefore, produces the same moment.



(a)

(b)

Principle of Transmissibility!

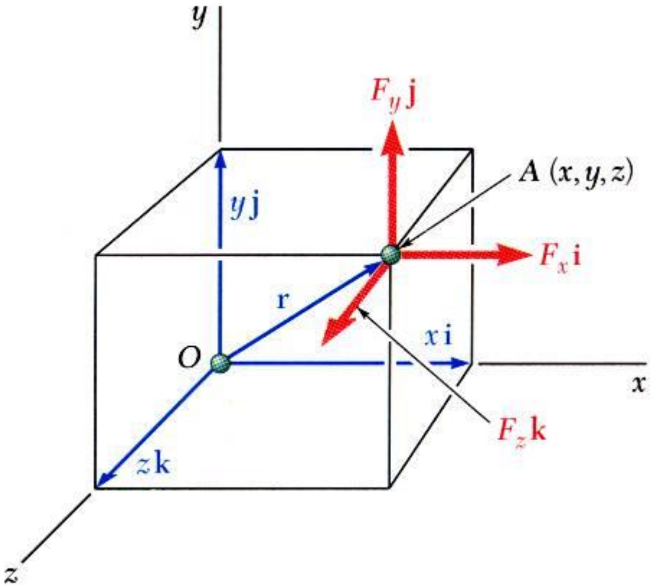
3.8 Rectangular Components of the Moment of a Force

The moment of ***F*** about *O*,

$$\vec{M}_O = \vec{r} \times \vec{F}, \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

$$\vec{M}_O = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$$



$$\vec{M}_O = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\boxed{M_O = rF\sin\theta = Fd}$$

$$= \left(yF_z - zF_y\right)\vec{i} + \underbrace{\left(zF_x - xF_z\right)}\vec{j} + \left(xF_y - yF_x\right)\vec{k}$$

Remember the (–) sign for j.

For 2D ($z = 0$ and $F_z = 0$)

$$\vec{M}_O = \left[xF_y - yF_x\right]\vec{k}$$

$$M_O = M_z$$

$$= xF_y - yF_x$$

$$M_x = M_y = 0$$

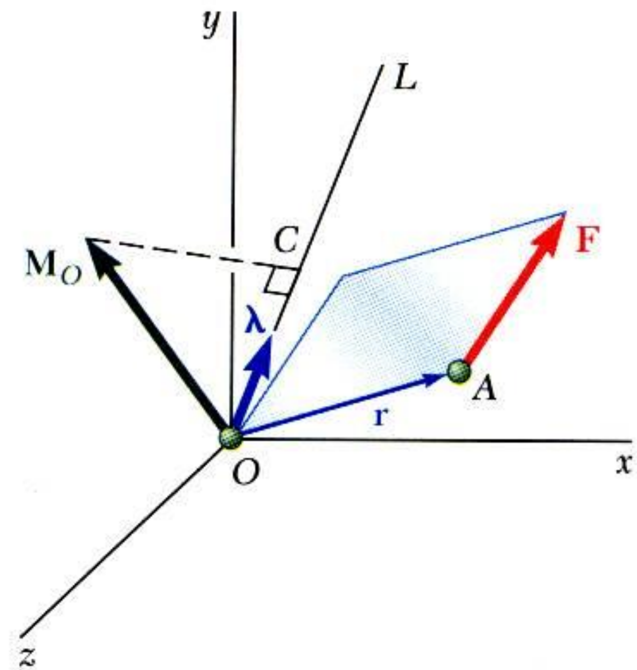
Review: Moment of a Force About a Given Axis

- Moment $\boldsymbol{M_O}$ of a force $\boldsymbol{F}$ applied at the point $\boldsymbol{A}$ about a point $\boldsymbol{O}$,

$$\vec{M}_O = \vec{r} \times \vec{F}$$

- Scalar moment M_{OL} about an axis $\boldsymbol{OL}$ is the projection of the moment vector $\boldsymbol{M_O}$ onto the axis:

$$M_{OL} = \vec{\lambda} \bullet \vec{M}_O = \vec{\lambda} \bullet \left(\vec{r} \times \vec{F}\right) = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$



- Moments of $\boldsymbol{F}$ about the coordinate axes:

$$M_x = yF_z - zF_y$$

$$M_y = zF_x - xF_z$$

$$M_z = xF_y - yF_x$$

Unit vector: $\vec{\lambda} = (\lambda_x, \lambda_y, \lambda_z)$

$$\vec{\lambda} = \cos\theta_x \vec{i} + \cos\theta_y \vec{j} + \cos\theta_z \vec{k}$$

In x-direction: $\vec{\lambda} = (1,0,0)$

In y-direction: $\vec{\lambda} = (0,1,0)$

In z-direction: $\vec{\lambda} = (0,0,1)$

3.11 Moment of a Force About a Given Axis

- Moment $\boldsymbol{M_O}$ of a force $\boldsymbol{F}$ applied at the point $\boldsymbol{A}$ about a point $\boldsymbol{O}$,

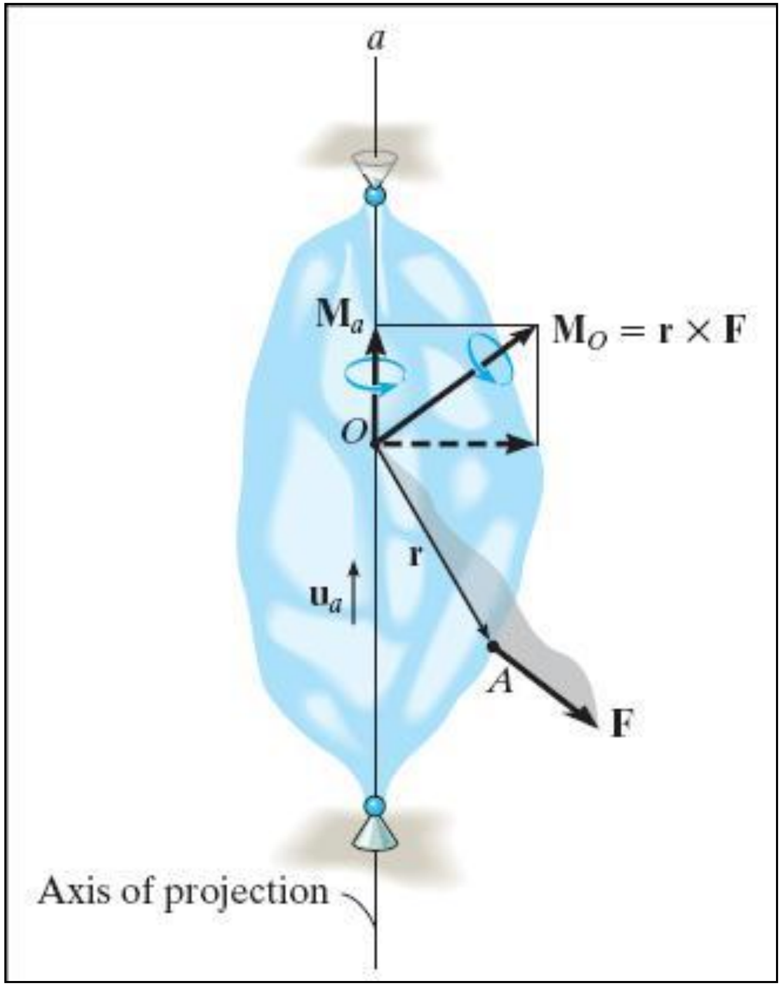
$$\vec{M}_O = \vec{r} \times \vec{F}$$

- Scalar moment M_{OL} about an axis $\boldsymbol{OL}$ is the projection of the moment vector $\boldsymbol{M_O}$ onto the axis,

$$M_{OL} = \vec{\lambda} \bullet \vec{M}_O = \vec{\lambda} \bullet \left(\vec{r} \times \vec{F}\right)$$

The tendency to rotate the body about the fixed axis.

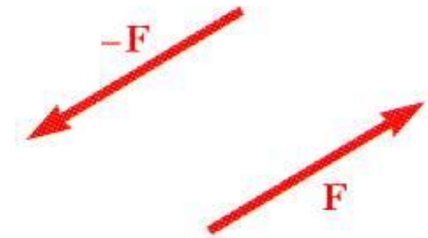
Only the force component perpendicular to the axis is important!



3.12 Moment of a Couple

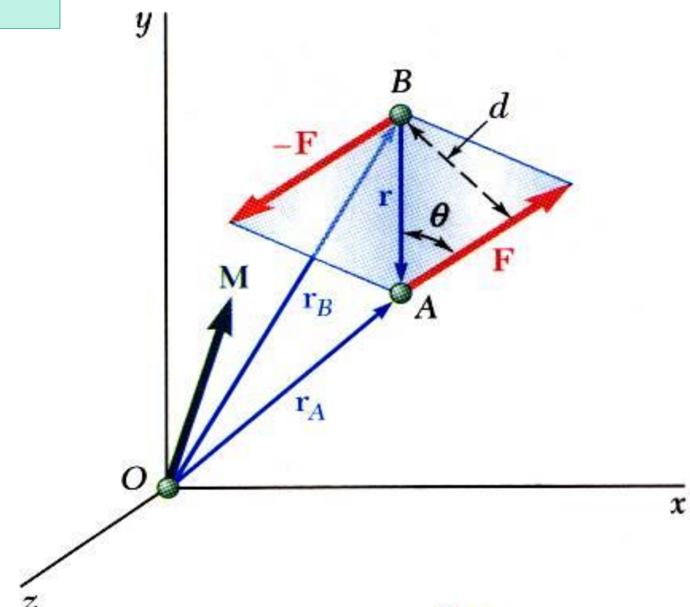
- **Two forces *F* and -*F* having**

1. the same magnitude,
2. parallel lines of action, and
3. opposite sense are said to form a ***couple.***

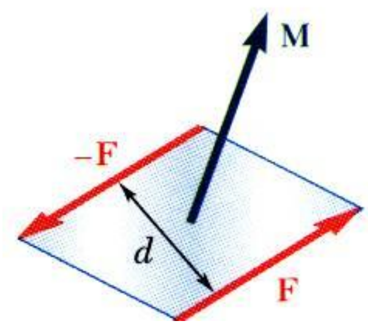


- **Moment of the couple**:

$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right)$$
$$= \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$
$$M = rF \sin\theta = Fd$$



- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.



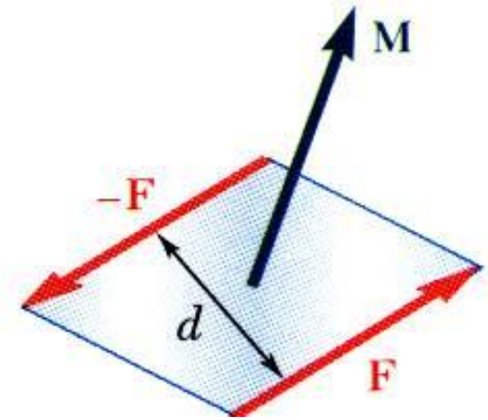
Example: Moment of a Couple

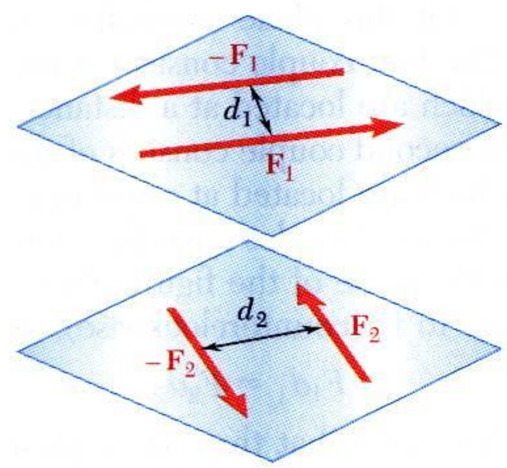
Photo 3.1 The parallel upward and downward forces of equal magnitude exerted on the arms of the lug nut wrench are an example of a couple.

Moment of a Couple (continued)

Two couples will have equal moments if

- $F_1 d_1 = F_2 d_2$
- the two couples lie in parallel planes, and
- the two couples have the same sense or the tendency to cause rotation in the same direction.
- Will be useful for drawing Free Body Diagram!



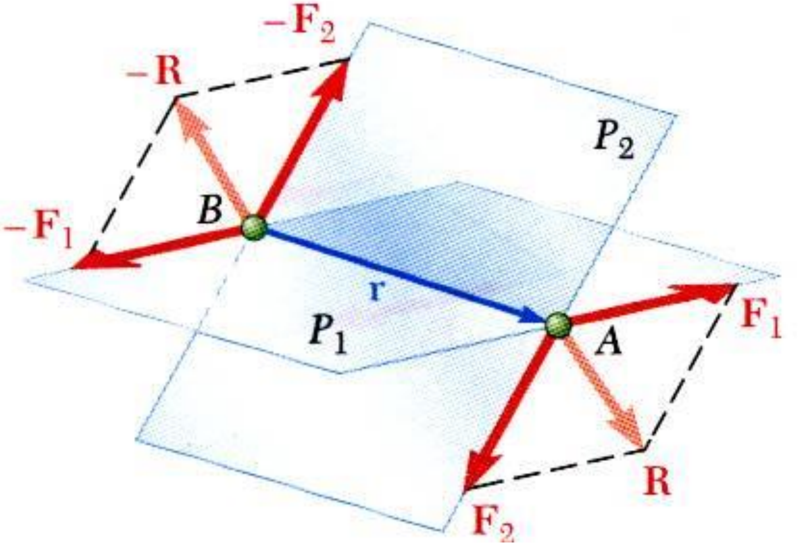


Addition of Couples

- Consider two intersecting planes P_1 and P_2 with each containing a couple

$$\vec{M}_1 = \vec{r} \times \vec{F}_1 \text{ in plane } P_1$$

$$\vec{M}_2 = \vec{r} \times \vec{F}_2 \text{ in plane } P_2$$



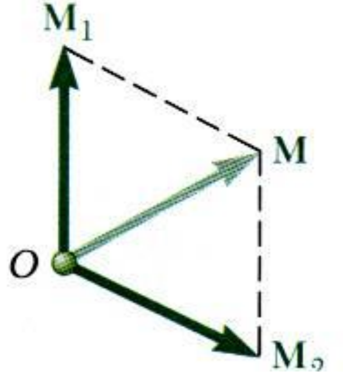
- Resultants of the vectors also form a couple

$$\vec{M} = \vec{r} \times \vec{R} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2\right)$$

- By Varigon's theorem

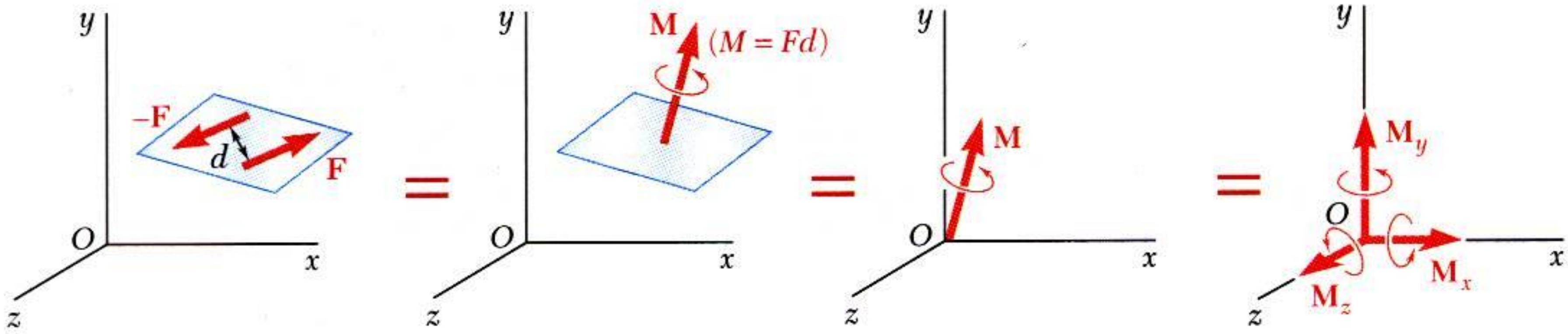
$$\vec{M} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2$$

$$= \vec{M}_1 + \vec{M}_2$$



- Sum of two couples is also a couple that is equal to the vector sum of the two couples.

Couples Can Be Represented by Vectors



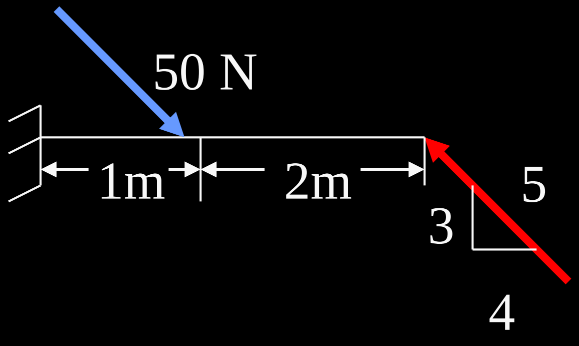
- A couple can be represented by a vector with magnitude and direction equal to the moment of the couple.
- *Couple vectors* obey the law of addition of vectors.
- Couple vectors are free vectors, i.e., the point of application is not significant.
- Couple vectors may be resolved into component vectors.

ATTENTION QUIZ

1. A couple is applied to the beam as shown. Its moment equals _____ N·m.

A) 50 B) 60

C) 80 D) 100

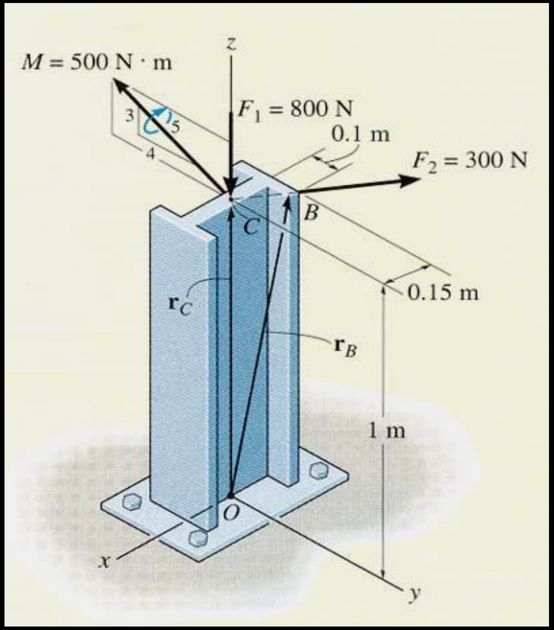


2. What is the direction of the moment vector of the couple ?

A) pointing towards us B) parallel to the red vector

C) impossible to tell D) pointing away from us

APPLICATIONS



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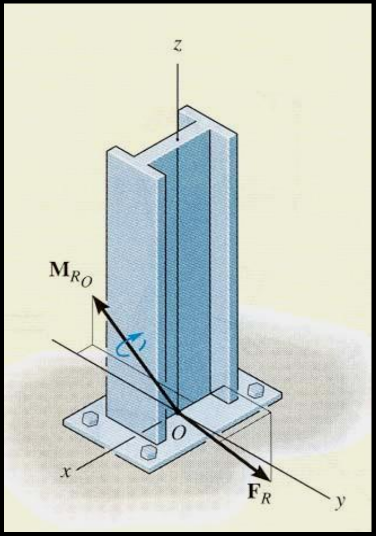


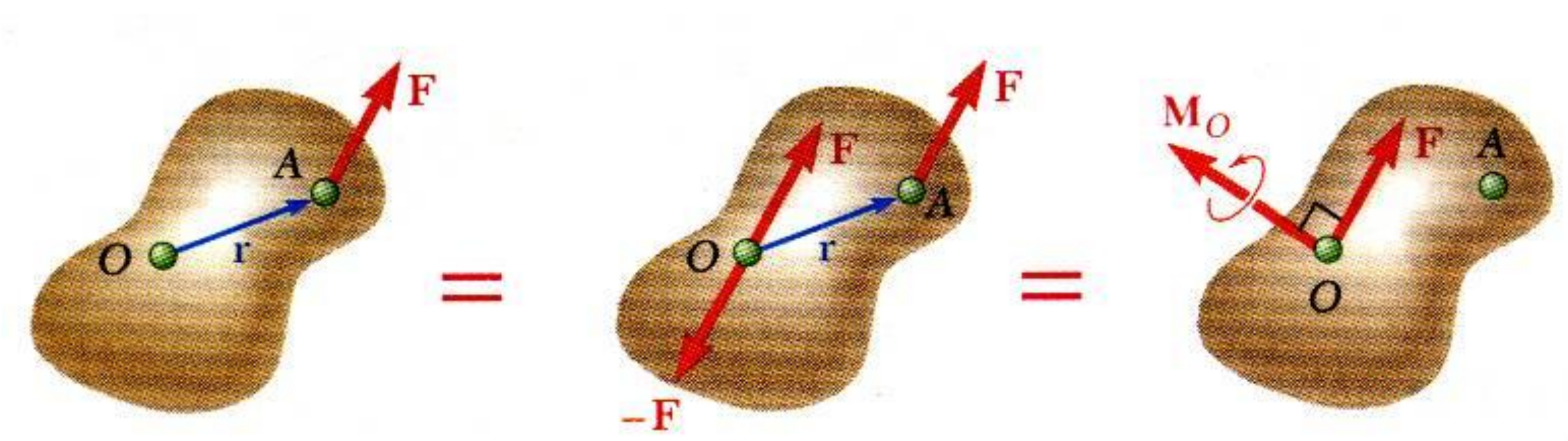
<u>Free Body Diagram</u>:

Several forces and a couple moment are acting on this vertical section of an I-beam.

Can you replace them with just one force and one couple moment at point O that will have the same external effect?

If yes, how will you do that?

Resolution of a Force Into a Force at *O* and a Couple



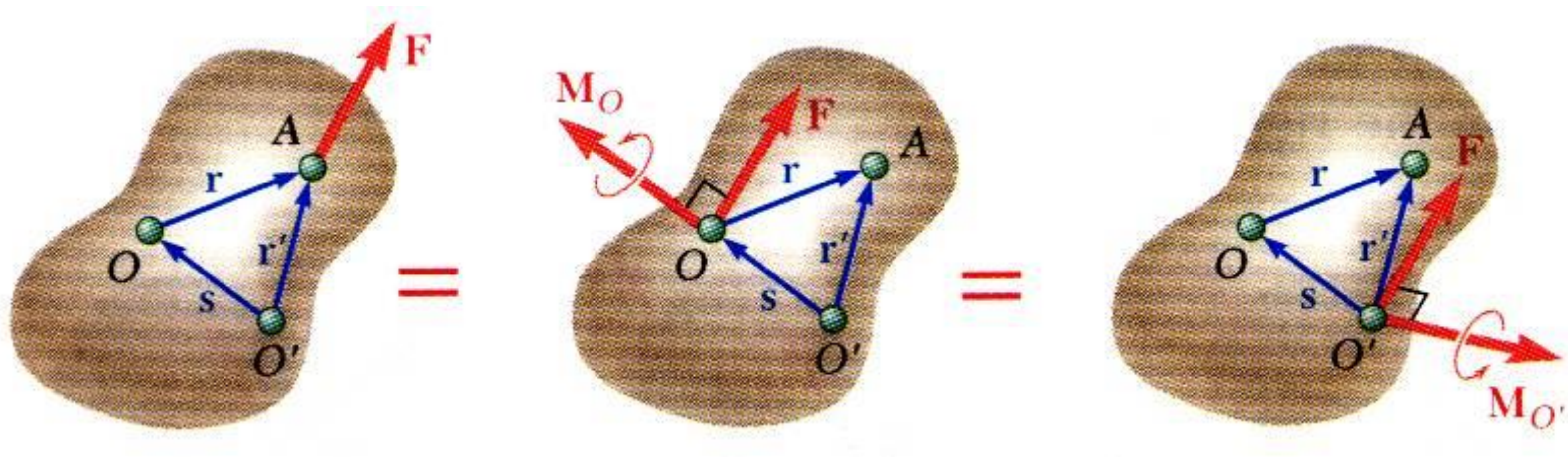
- Force vector ***F*** can not be simply moved to *O* without modifying its action on the body. Why?

- Attaching equal and opposite force vectors at *O* produces no net effect on the body.

- The three forces may be replaced by an equivalent force vector and couple vector, i.e., a *force-couple system*. Going backwards?

$$\vec{M}_O = \vec{r} \times \vec{F}$$

3.16 Resolution of a Force Into a Force at *O* and a Couple



- Moving ***F*** from *A* to a different point *O'* requires the addition of a different couple vector $\boldsymbol{M_{O'}}$

$$\vec{M}_{O'} = \vec{r}\,' \times \vec{F}$$

- The moments of ***F*** about O and *O'* are related,

$$\vec{M}_{O'} = \vec{r}\,'\times\vec{F} = (\vec{r} + \vec{s})\times\vec{F} = \vec{r}\times\vec{F} + \vec{s}\times\vec{F}$$

$$= \vec{M}_O + \vec{s}\times\vec{F}$$

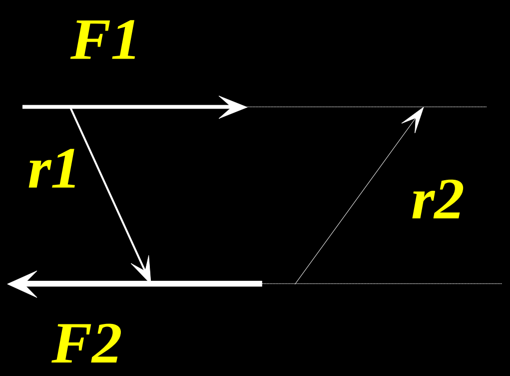
- Moving the force-couple system from *O* to *O'* requires the addition of the moment of the force at *O* about *O'*.

CONCEPT QUIZ

1. F_1 and F_2 form a couple. The moment of the couple is given by ____ .

   A) $r_1 \times F_1$ B) $r_2 \times F_1$

   C) $F_2 \times r_1$ D) $r_2 \times F_2$



2. If three couples act on a body, the overall result is that

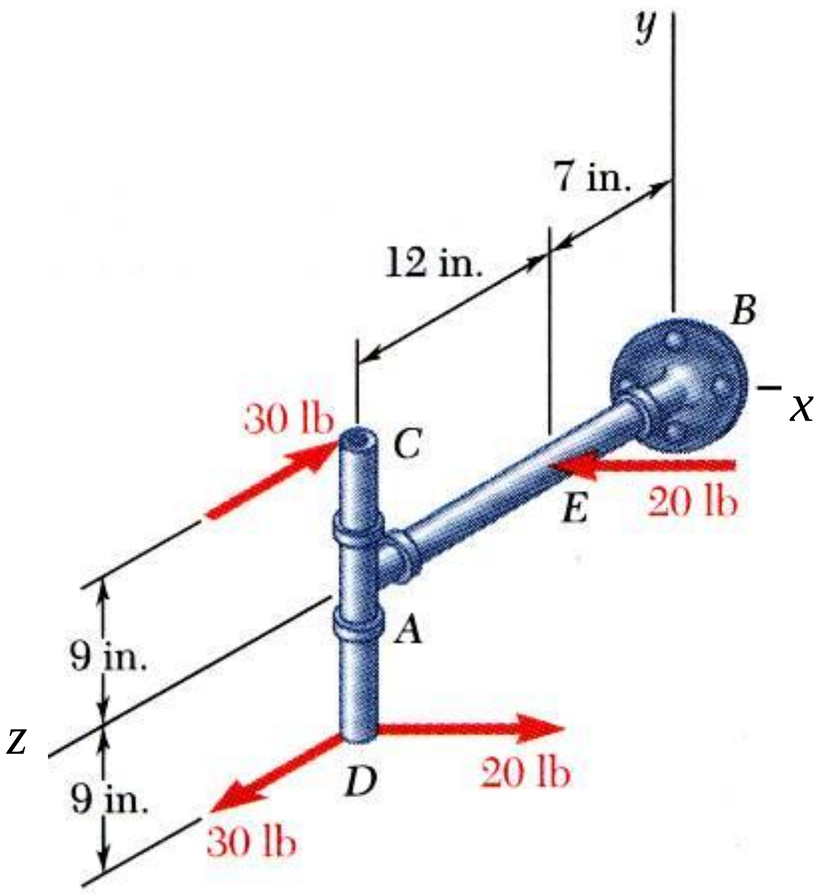
   A) the net force is not equal to 0.

   B) the net force and net moment are equal to 0.

   C) the net moment equals 0 but the net force is not necessarily equal to 0.

   D) the net force equals 0 but the net moment is not necessarily equal to 0 .

Sample Problem 3.6



Determine the components of the single couple equivalent to the couples shown.

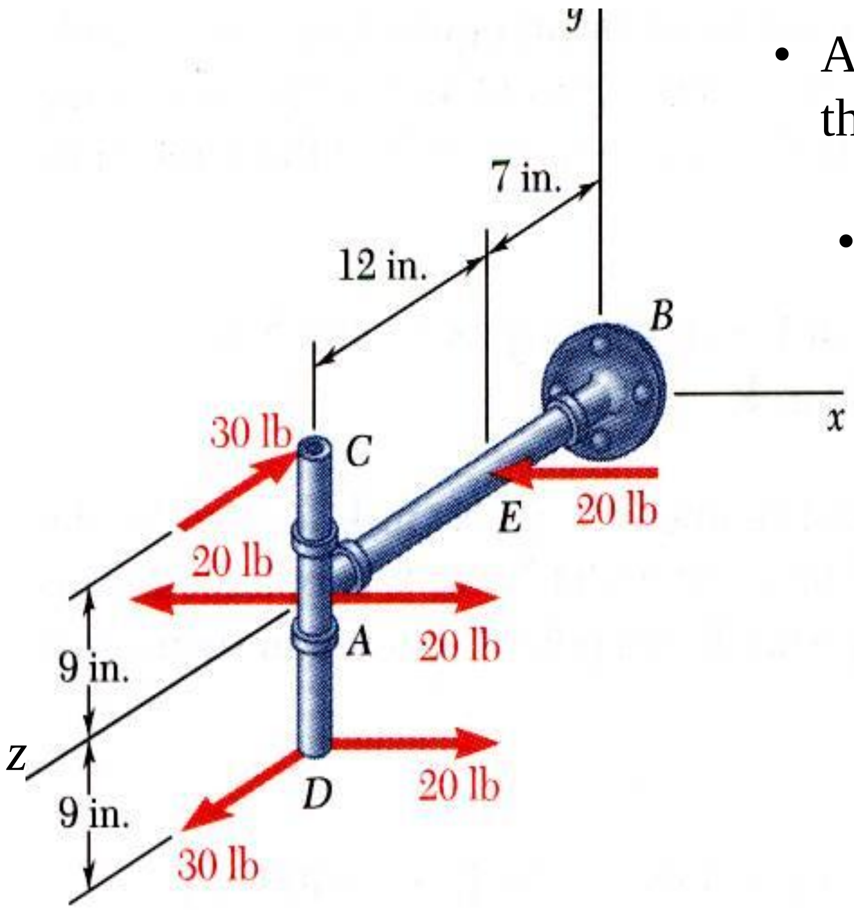
SOLUTION:

- Attach equal and opposite 20 lb forces in the $\pm x$ direction at *A*, thereby producing 3 couples for which the moment components are easily computed.

- Alternatively, compute the sum of the moments of the four forces about an arbitrary single point. The point *D* is a good choice as only two of the forces will produce non-zero moment contributions..

$$\vec{M}_D = \sum \vec{r} \times \vec{F}$$

Sample Problem 3.6



- Attach equal and opposite 20 lb forces in the $\pm x$ direction at A

- The three couples may be represented by three couple vectors,

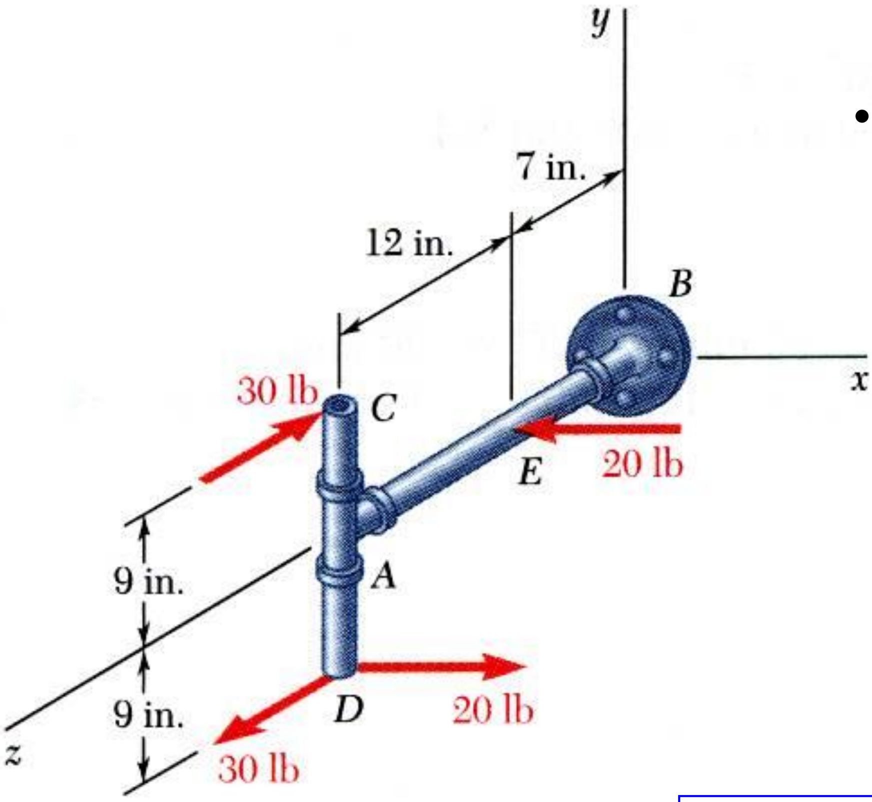
$$M_x = -(30\,\text{lb})(18\,\text{in.}) = -540\,\text{lb}\cdot\text{in.}$$
$$M_y = +(20\,\text{lb})(12\,\text{in.}) = +240\text{lb}\cdot\text{in.}$$
$$M_z = +(20\,\text{lb})(9\,\text{in.}) = +180\,\text{lb}\cdot\text{in.}$$

$$\boxed{\begin{aligned}\vec{M} &= -(540\,\text{lb}\cdot\text{in.})\vec{i} + (240\text{lb}\cdot\text{in.})\vec{j}\\ &\quad + (180\,\text{lb}\cdot\text{in.})\vec{k}\end{aligned}}$$

Moment of the couple:

$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right)$$
$$= \vec{r} \times \vec{F}$$
$$M = rF\sin\theta = Fd$$

Sample Problem 3.6



- Alternatively, compute the sum of the moments of the four forces about *D*.

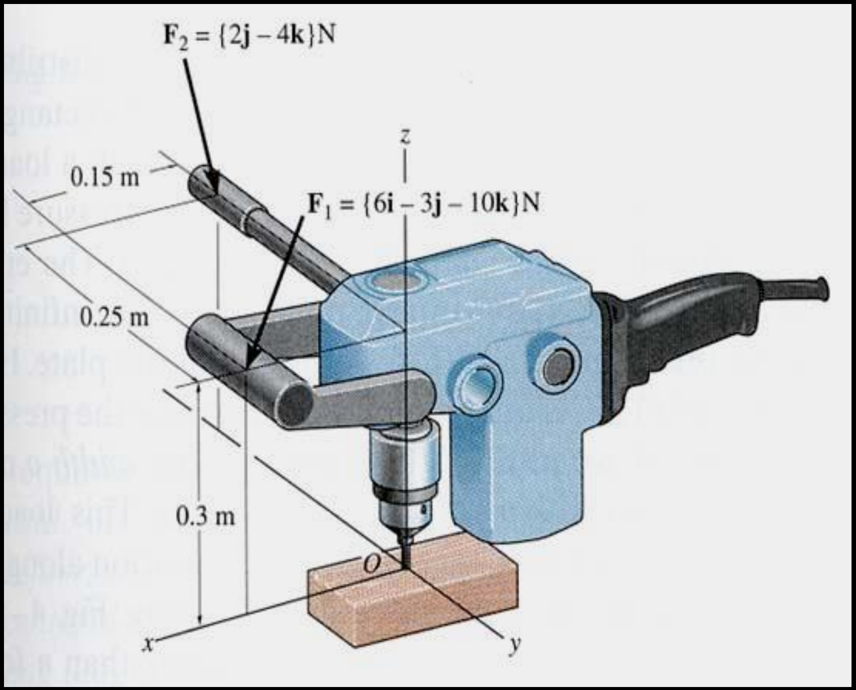
- Only the forces at *C* and *E* contribute to the moment about *D*.

$$\vec{M} = \vec{M}_D = (18\,\text{in.})\vec{j} \times (-30\,\text{lb})\vec{k} + \left[(9\,\text{in.})\vec{j} - (12\,\text{in.})\vec{k}\right] \times (-20\,\text{lb})\vec{i}$$

$$\boxed{\vec{M} = -(540\,\text{lb}\cdot\text{in.})\vec{i} + (240\text{lb}\cdot\text{in.})\vec{j} + (180\,\text{lb}\cdot\text{in.})\vec{k}}$$

- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.

PROBLEM



Given: Handle forces F_1 and F_2 are applied to the electric drill.

Find: An equivalent resultant force and couple moment at point O.

Plan:

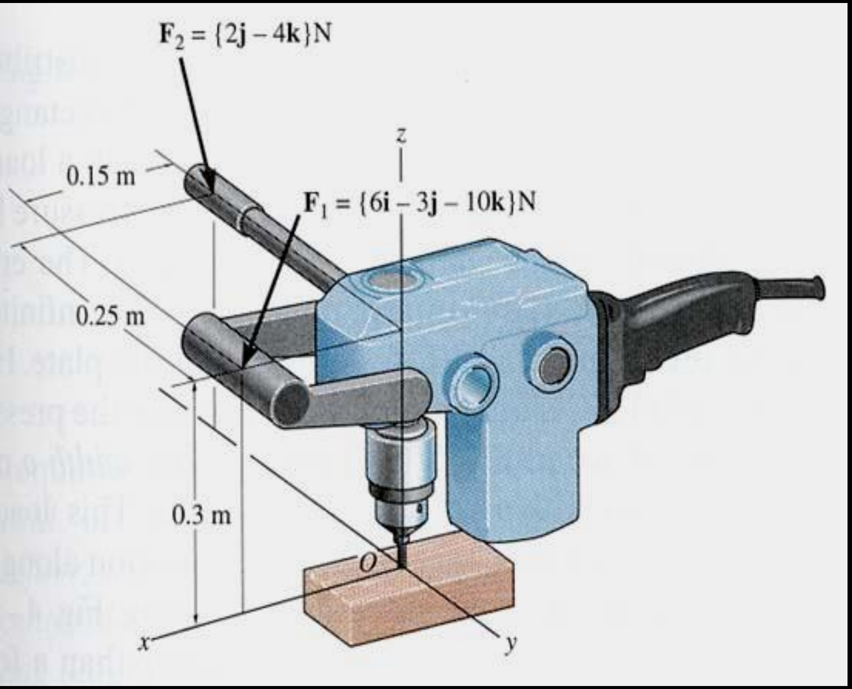
a) Find $\boldsymbol{F_{RO}} = \Sigma\, \boldsymbol{F_i}$

b) Find $\boldsymbol{M_{RO}} = \Sigma\, (\boldsymbol{r_i} \times \boldsymbol{F_i})$

where,

$\boldsymbol{F_i}$ are the individual forces in Cartesian vector notation.

$\boldsymbol{r_i}$ are the position vectors from the point O to any point on the line of action of $\boldsymbol{F_i}$.



SOLUTION

$$\boldsymbol{F_1} = \{6\,\boldsymbol{i} - 3\,\boldsymbol{j} - 10\,\boldsymbol{k}\}\ \text{N}$$

$$\boldsymbol{F_2} = \{0\,\boldsymbol{i} + 2\,\boldsymbol{j} - 4\,\boldsymbol{k}\}\ \text{N}$$

$$\boldsymbol{F_{RO}} = \{6\,\boldsymbol{i} - 1\,\boldsymbol{j} - 14\,\boldsymbol{k}\}\ \text{N}$$

$$\boldsymbol{r_1} = \{0.15\,\boldsymbol{i} + 0.3\,\boldsymbol{k}\}\ \text{m}$$

$$\boldsymbol{r_2} = \{-0.25\,\boldsymbol{j} + 0.3\,\boldsymbol{k}\}\ \text{m}$$

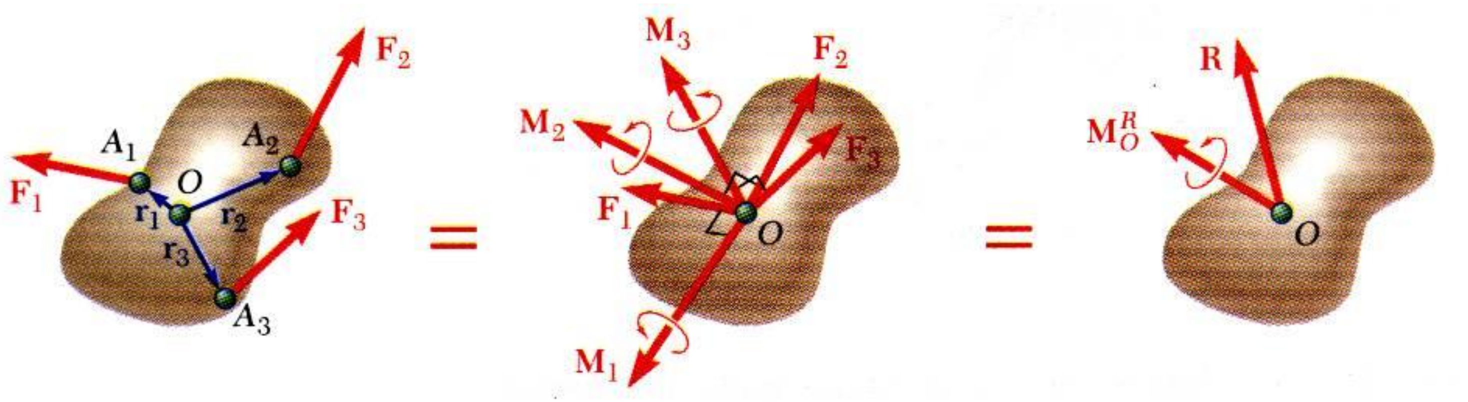
$$\boldsymbol{M_{RO}} = \boldsymbol{r_1} \times \boldsymbol{F_1} + \boldsymbol{r_2} \times \boldsymbol{F_2}$$

$$\boldsymbol{M_{RO}} = \left\{ \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0.15 & 0 & 0.3 \\ 6 & -3 & -10 \end{vmatrix} + \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & -0.25 & 0.3 \\ 0 & 2 & -4 \end{vmatrix} \right\}\ \text{N}\cdot\text{m}$$

$$= \{0.9\,\boldsymbol{i} + 3.3\,\boldsymbol{j} - 0.45\,\boldsymbol{k} + 0.4\,\boldsymbol{i} + 0\,\boldsymbol{j} + 0\,\boldsymbol{k}\}\ \text{N}\cdot\text{m}$$

$$= \{1.3\,\boldsymbol{i} + 3.3\,\boldsymbol{j} - 0.45\,\boldsymbol{k}\}\ \text{N}\cdot\text{m}$$

System of Forces: Reduction to a Force and Couple



- A system of forces may be replaced by a collection of force-couple systems acting a given point *O*

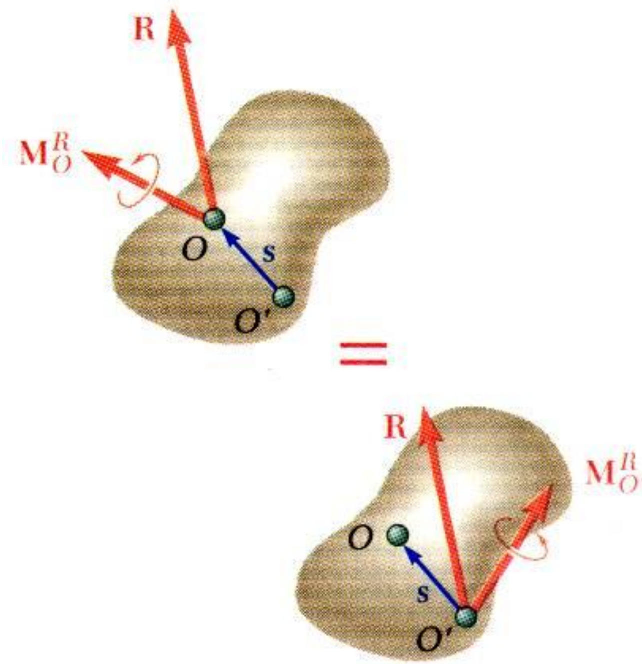
- The force and couple vectors may be combined into a resultant force vector and a resultant couple vector,

$$\vec{R} = \sum \vec{F} \qquad \vec{M}_O^R = \sum \left(\vec{r} \times \vec{F} \right)$$

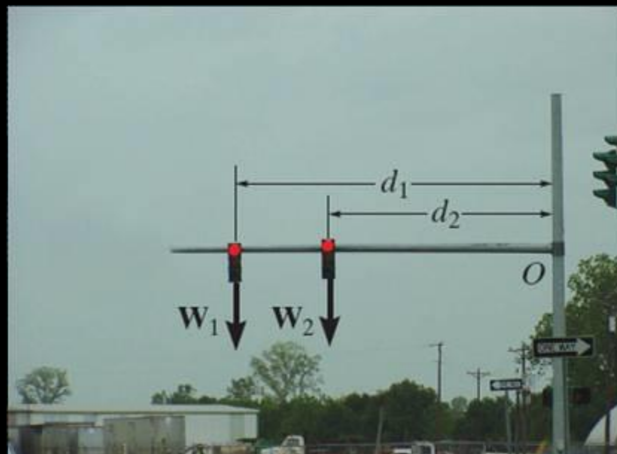
- The force-couple system at *O* may be moved to *O'* with the addition of the moment of ***R*** about *O'* ,

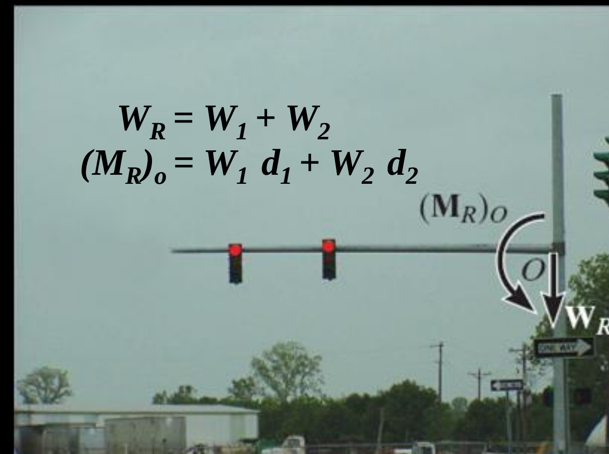
$$\vec{M}_{O'}^R = \vec{M}_O^R + \vec{s} \times \vec{R}$$

- Two systems of forces are <u>equivalent</u> if they can be reduced to the same force-couple system.



SIMPLIFICATION OF A FORCE AND COUPLE SYSTEM





If the force system lies in the x-y plane (a 2-D case), then the reduced equivalent system can be obtained using the following three scalar equations.

$$F_{R_x} = \Sigma F_x$$
$$F_{R_y} = \Sigma F_y$$
$$M_{R_O} = \Sigma M_c + \Sigma M_O$$

ATTENTION QUIZ

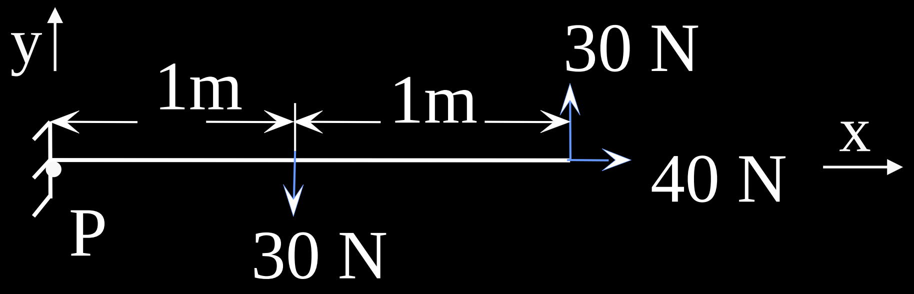
1. For this force system, the equivalent system at P is _____________ .

A) F_P = 40 N (along +x-dir.) and M_P = +60 N·m

B) F_P = 0 N and M_P = +30 N·m

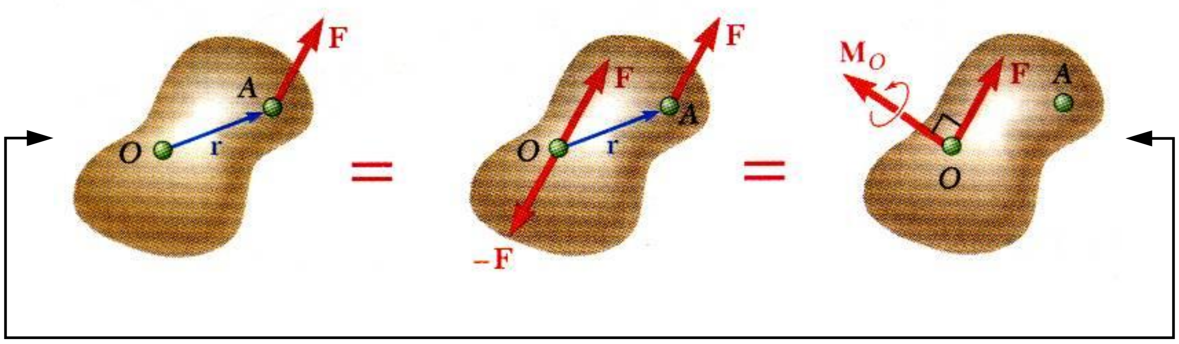
C) F_P = 30 N (along +y-dir.) and M_P = -30 N·m

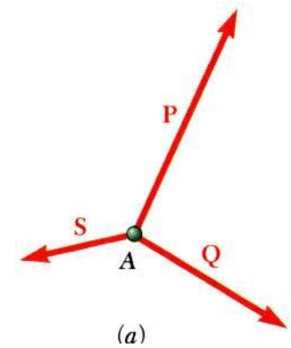
D) F_P = 40 N (along +x-dir.) and M_P = +30 N·m



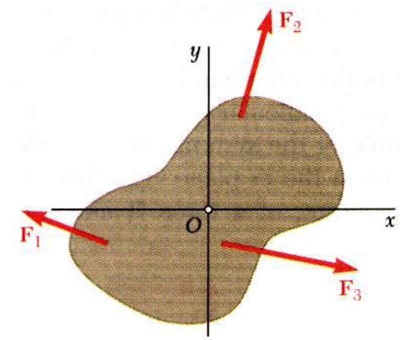
Further Reduction of a System of Forces (Special Cases)

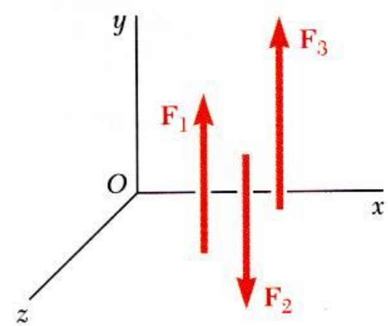
- <u>If the resultant force and couple at *O* are mutually perpendicular, they can be replaced by a single force acting along a new line of action</u>.



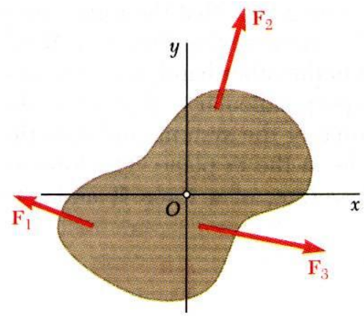


- The resultant force-couple system for a system of forces will be mutually perpendicular if:
  1) the forces are concurrent (just add the forces)
  2) the forces are coplanar (all components are ⊥ at O)
  3) the forces are parallel (moment is in *xz* plane).

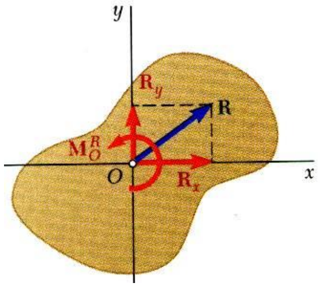


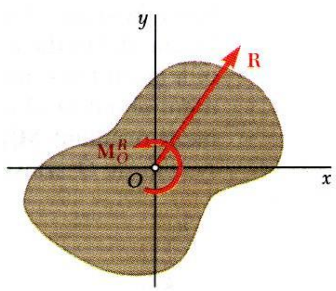


Further Reduction of a System of Forces

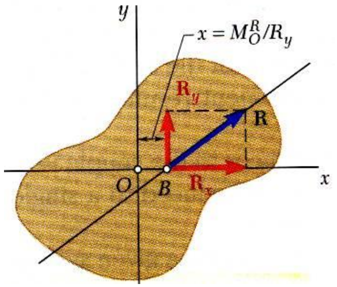


- System of coplanar forces is reduced to a force-couple system $\vec{R}$ and $\vec{M}_O^R$ that is mutually perpendicular.



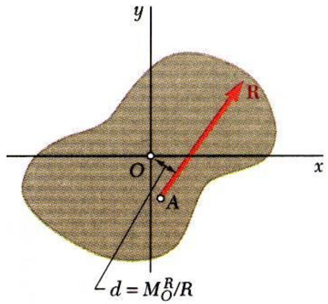


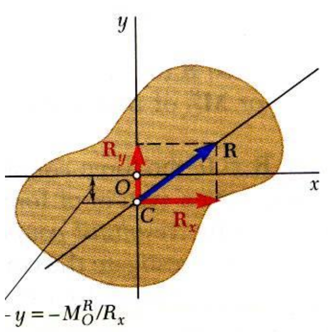
- System can be reduced to a single force by moving the line of action of $\vec{R}$ until its moment about O becomes $\vec{M}_O^R$



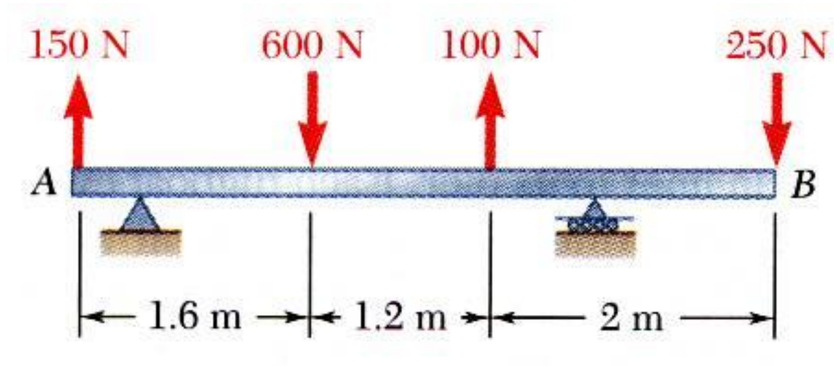
- In terms of rectangular coordinates,

$$xR_y - yR_x = M_O^R$$





Sample Problem 3.8



For the beam, reduce the system of forces shown to (a) an equivalent force-couple system at *A*, (b) an equivalent force couple system at *B*.

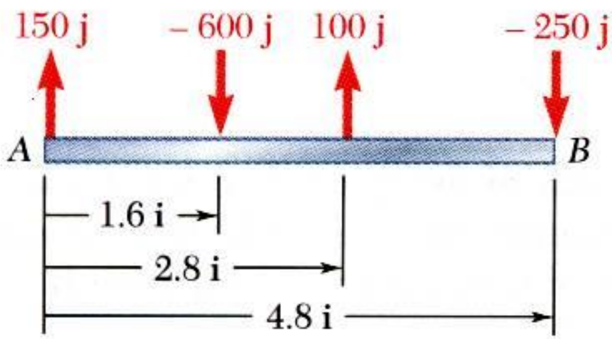
Note: Since the support reactions are not included, the given system will not maintain the beam in equilibrium.

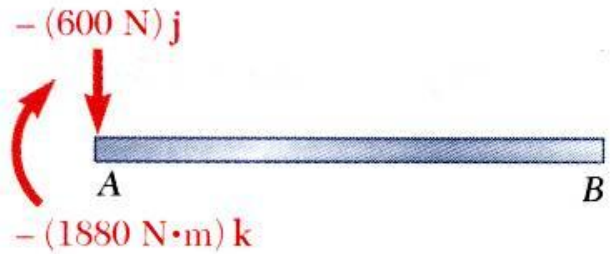
SOLUTION:

a) Compute the resultant force for the forces shown and the resultant couple for the moments of the forces about *A*.

b) Find an equivalent force-couple system at *B* based on the force-couple system at *A*.

Sample Problem 3.8





<u>SOLUTION:</u>

a) Compute the resultant force and the resultant couple at *A*.

$$\vec{R} = \sum \vec{F}$$
$$= (150\ \mathrm{N})\vec{j} - (600\ \mathrm{N})\vec{j} + (100\ \mathrm{N})\vec{j} - (250\ \mathrm{N})\vec{j}$$

$$\boxed{\vec{R} = -(600\ \mathrm{N})\vec{j}}$$

$$\vec{M}_A^R = \sum (\vec{r} \times \vec{F})$$
$$= (1.6\vec{i}) \times (-600\vec{j}) + (2.8\vec{i}) \times (100\vec{j}) + (4.8\vec{i}) \times (-250\vec{j})$$

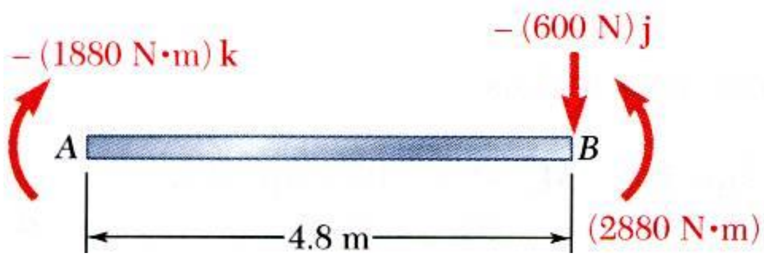
$$\boxed{\vec{M}_A^R = -(1880\ \mathrm{N \cdot m})\vec{k}}$$

Sample Problem 3.8

b) Find an equivalent force-couple system at *B* based on the force-couple system at *A*.

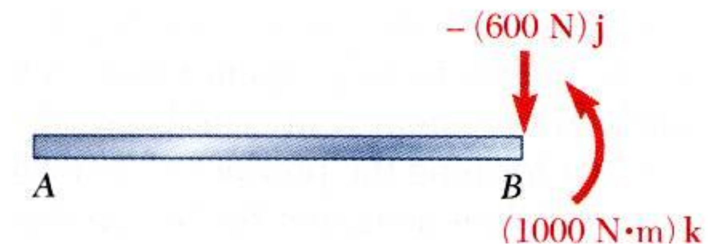
The force is unchanged by the movement of the force-couple system from *A* to *B*.

$$\boxed{\vec{R} = -(600\ \text{N})\vec{j}}$$



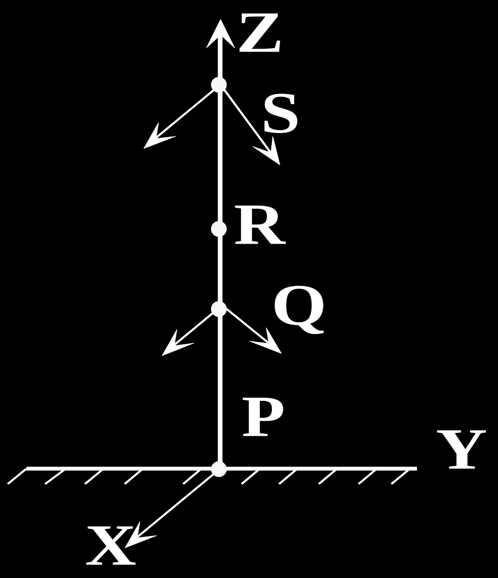
The couple at *B* is equal to the moment about *B* of the force-couple system found at *A*.

$$\begin{aligned}
\vec{M}_B^R &= \vec{M}_A^R + \vec{r}_{A/B} \times \vec{R} \\
&= -(1880\ \text{N}\cdot\text{m})\vec{k} + (-4.8\ \text{m})\vec{i} \times (-600\ \text{N})\vec{j} \\
&= -(1880\ \text{N}\cdot\text{m})\vec{k} + (2880\ \text{N}\cdot\text{m})\vec{k}
\end{aligned}$$



$$\boxed{\vec{M}_B^R = +(1000\ \text{N}\cdot\text{m})\vec{k}}$$

CONCEPT QUIZ



1. The forces on the pole can be reduced to a single force and a single moment at point _____ .

   A) P   B) Q   C) R

   D) S   E) Any of these points.

2. Consider two couples acting on a body. The simplest possible equivalent system at any arbitrary point on the body will have

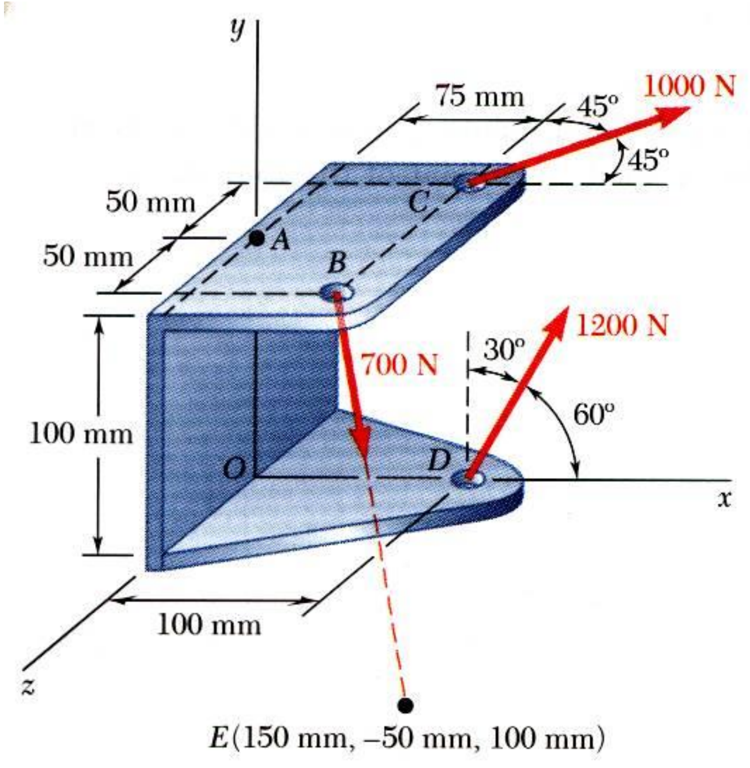
   A) One force and one couple moment.

   B) One force.

   C) One couple moment.

   D) Two couple moments.

Sample Problem 3.10



Three cables are attached to the bracket as shown. Replace the forces with an equivalent force-couple system at *A*.

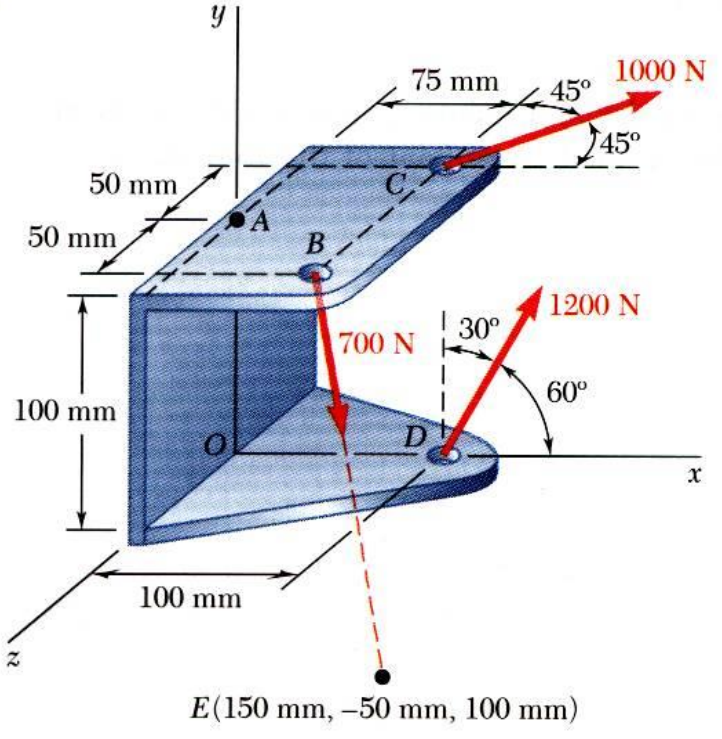
SOLUTION:

- Determine the relative position vectors for the points of application of the cable forces with respect to *A*.
- Resolve the forces into rectangular components.
- Compute the equivalent force,

$$\vec{R} = \sum \vec{F}$$

- Compute the equivalent couple,

$$\vec{M}_A^R = \sum \left(\vec{r} \times \vec{F} \right)$$



Sample Problem 3.10

SOLUTION:

- Determine the relative position vectors with respect to *A*.

$$\vec{r}_{B/A} = 0.075\vec{i} + 0.050\vec{k}\ (\mathrm{m})$$

$$\vec{r}_{C/A} = 0.075\vec{i} - 0.050\vec{k}\ (\mathrm{m})$$

$$\vec{r}_{D/A} = 0.100\vec{i} - 0.100\vec{j}\ (\mathrm{m})$$

- Resolve the forces into rectangular components.

$$\vec{F}_B = (700\ \mathrm{N})\vec{\lambda}$$

$$\vec{\lambda} = \frac{\vec{r}_{E/B}}{r_{E/B}} = \frac{75\vec{i} - 150\vec{j} + 50\vec{k}}{175}$$

$$= 0.429\vec{i} - 0.857\vec{j} + 0.289\vec{k}$$

$$\vec{F}_B = 300\vec{i} - 600\vec{j} + 200\vec{k}\ (\mathrm{N})$$

$$\vec{F}_C = (1000\ \mathrm{N})\left(\cos 45\vec{i} - \cos 45\vec{k}\right)$$

$$= 707\vec{i} - 707\vec{k}\ (\mathrm{N})$$

$$\vec{F}_D = (1200\ \mathrm{N})\left(\cos 60\vec{i} + \cos 30\vec{j}\right)$$

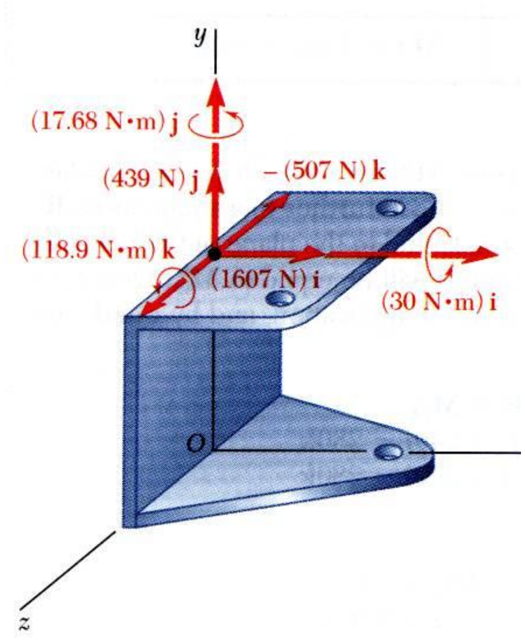
$$= 600\vec{i} + 1039\vec{j}\ (\mathrm{N})$$

Sample Problem 3.10

- Compute the equivalent force,

$$\vec{R} = \sum \vec{F}$$
$$= (300 + 707 + 600)\vec{i}$$
$$+ (-600 + 1039)\vec{j}$$
$$+ (200 - 707)\vec{k}$$

$$\boxed{\vec{R} = 1607\vec{i} + 439\vec{j} - 507\vec{k} \ \ (\mathrm{N})}$$



- Compute the equivalent couple,

$$\vec{M}_A^R = \sum (\vec{r} \times \vec{F})$$

$$\vec{r}_{B/A} \times \vec{F}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & 0.050 \\ 300 & -600 & 200 \end{vmatrix} = 30\vec{i} - 45\vec{k}$$

$$\vec{r}_{C/A} \times \vec{F}_c = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.075 & 0 & -0.050 \\ 707 & 0 & -707 \end{vmatrix} = 17.68\vec{j}$$

$$\vec{r}_{D/A} \times \vec{F}_D = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.100 & -0.100 & 0 \\ 600 & 1039 & 0 \end{vmatrix} = 163.9\vec{k}$$

$$\boxed{\vec{M}_A^R = 30\vec{i} + 17.68\vec{j} + 118.9\vec{k}}$$