ME4010 Computational Methods for Mechanical Engineering

Chapter 5 Curve Fitting and Interpolation

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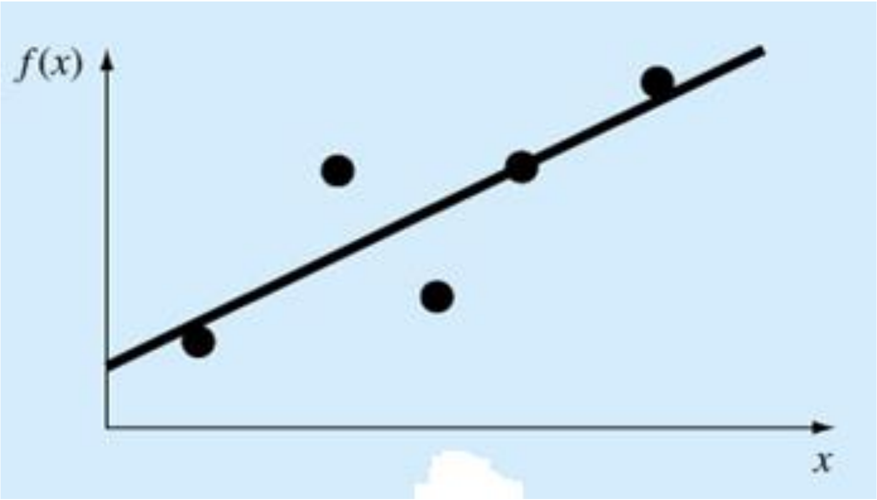
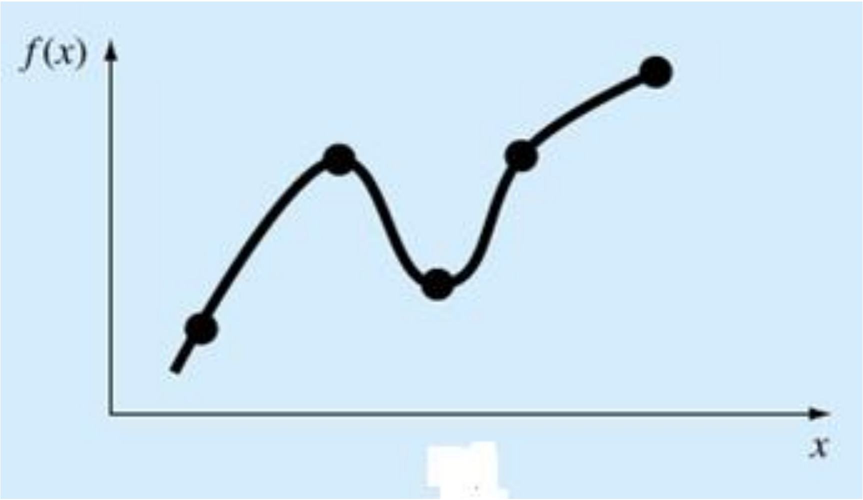
Curve Fitting and Interpolation





Regression

- Antoine equation: $P_v = 10^{\left(A + \frac{B}{T+C}\right)}$
- Heat capacity for a gas: $c_p = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + \ldots$
- Thermal conductivity: $k = cT^n$
- Viscosity: $\mu = \dfrac{C_1 T^n}{T + C_2}$
- Drag law: $C_D = c\,\mathrm{Re}^n$
- Heat transfer correlations: $Nu = c\,\mathrm{Re}^{n_1}\,\mathrm{Pr}^{n_2}\,(\mu / \mu_w)^{n_3}$

Nusselt number

Linear Regression – Least Squares Fit

- Fitting a straight line, $y=a_0+a_1x$, to a set of paired observations: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.
- Minimize the sum of the residual errors for all available data:

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_{i,\text{measured}} - y_{i,\text{model}})^2 = \sum_{i=1}^{n} \left[y_i - \left(a_0 + a_1 x_i \right) \right]^2$$

Least-Squares Fit of a Straight Line

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_{i,\text{measured}} - y_{i,\text{model}})^2 = \sum_{i=1}^{n} \left[y_i - \left(a_0 + a_1 x_i \right) \right]^2$$

$$\frac{\partial S_r}{\partial a_o} = -2 \sum (y_i - a_o - a_1 x_i) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2 \sum \left[(y_i - a_o - a_1 x_i) x_i \right] = 0$$

$$0 = \sum y_i - \sum a_0 - \sum a_1 x_i \Rightarrow \sum y_i = n a_0 + \left(\sum x_i \right) a_1$$

$$0 = \sum y_i x_i - \sum a_0 x_i - \sum a_1 x_i^2 \Rightarrow \sum y_i x_i = \left(\sum x_i \right) a_0 + \left(\sum x_i^2 \right) a_1$$

$$\therefore$$

$$a_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \left(\sum x_i \right)^2}$$

$$a_0 = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{n \sum x_i^2 - \left(\sum x_i \right)^2}$$

Example

x	0	1	2	3	4	5	6	7	8	9
y	4.00	6.10	8.30	9.90	12.40	14.30	15.70	17.40	19.80	22.30

$$\sum_{k=1}^{10} x_k = 0+1+2+3+4+5+6+7+8+9 = 45$$

$$\sum_{k=1}^{10} y_k = 4+6.1+8.3+9.9+12.4+14.3+15.7+17.4+19.8+22.3 = 130.2$$

$$\sum_{k=1}^{10} x_k^2 = (0)^2+(1)^2+(2)^2+(3)^2+(4)^2+(5)^2+(6)^2+(7)^2+(8)^2+(9)^2 = 285$$

$$\sum_{k=1}^{10} x_k y_k = 0+6.1+16.6+29.7+49.6+71.5+94.2+121.8+158.4+200.7 = 748.6$$

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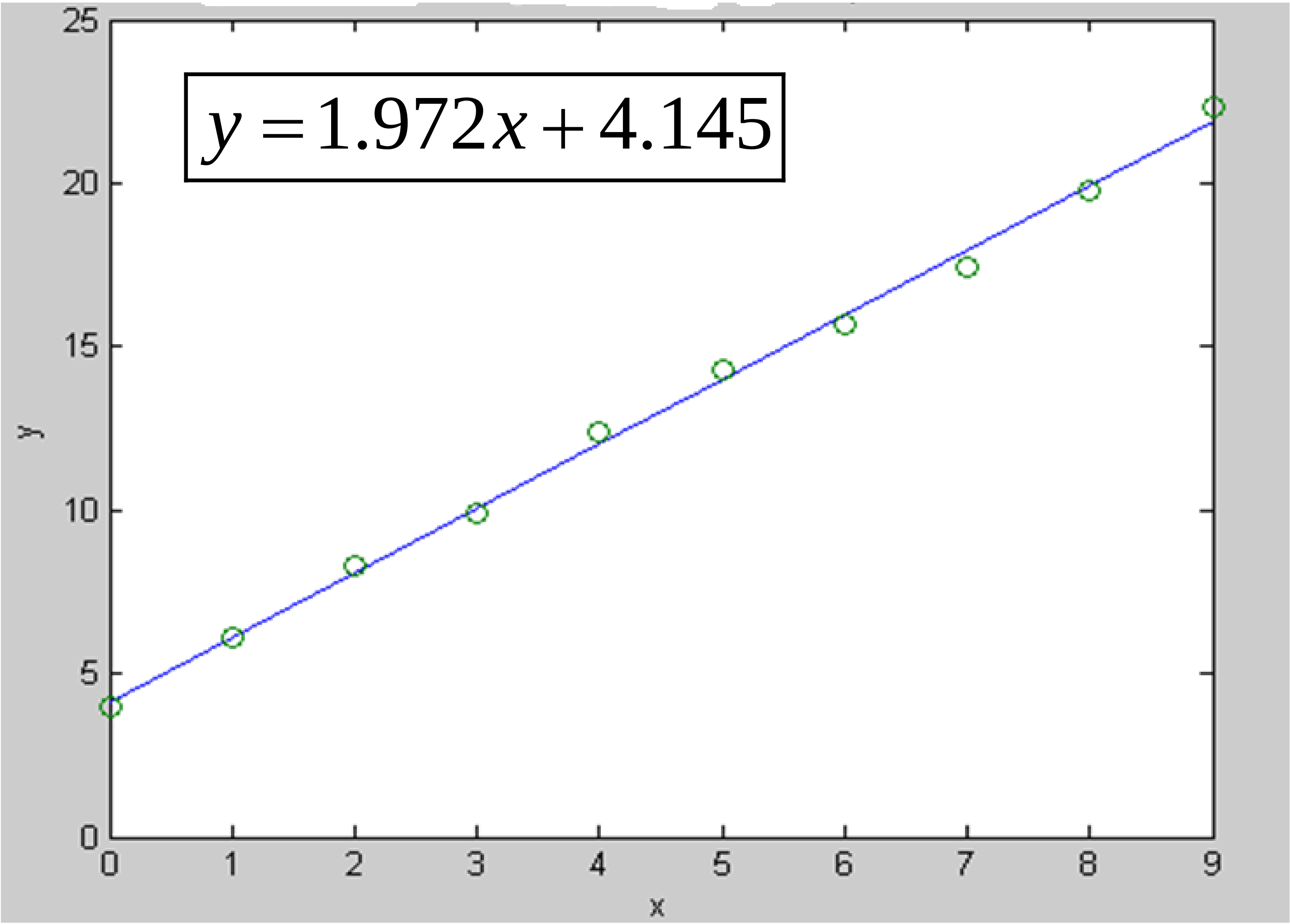
$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \frac{10\times 748.6 - 45\times 130.2}{10\times 285 - 45^2} = 1.972$$

$$a_0 = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \frac{285\times 130.2 - 45\times 748.6}{10\times 285 - 45^2} = 4.145$$

$$\boxed{y = 1.972x + 4.145}$$



$y = 1.972x + 4.145$
y
x
0
5
10
15
20
25
0
1
2
3
4
5
6
7
8
9

Other Functions

$$y = bx^m$$

$$\ln(y) = \ln(bx^m) = m\ln(x) + \ln(b)$$

$$Y \qquad\qquad\qquad a_1 \quad X \qquad\quad a_0$$

Once a_1 and a_0 are known, the constants,

$b = e^{a_0}$ and $m = a_1$

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Example

$$y = bx^m$$

$$\ln(y) = m\ln(x) + \ln(b)$$

$$Y \qquad a_1 \quad X \qquad a_0$$

	x_i	y_i	$X_i=\ln(x_i)$	$Y_i=\ln(y_i)$	X_iY_i	X_i^2
	1	0.5	0.0000	-0.6931	0.0000	0.0000
	2	1.7	0.6931	0.5306	0.3678	0.4805
	3	3.4	1.0986	1.2238	1.3445	1.2069
	4	5.7	1.3863	1.7404	2.4128	1.9218
	5	8.4	1.6094	2.2182	3.4253	2.5903
Sum	**15**	**19.700**	**4.7875**	**4.9300**	**7.5503**	**6.1995**

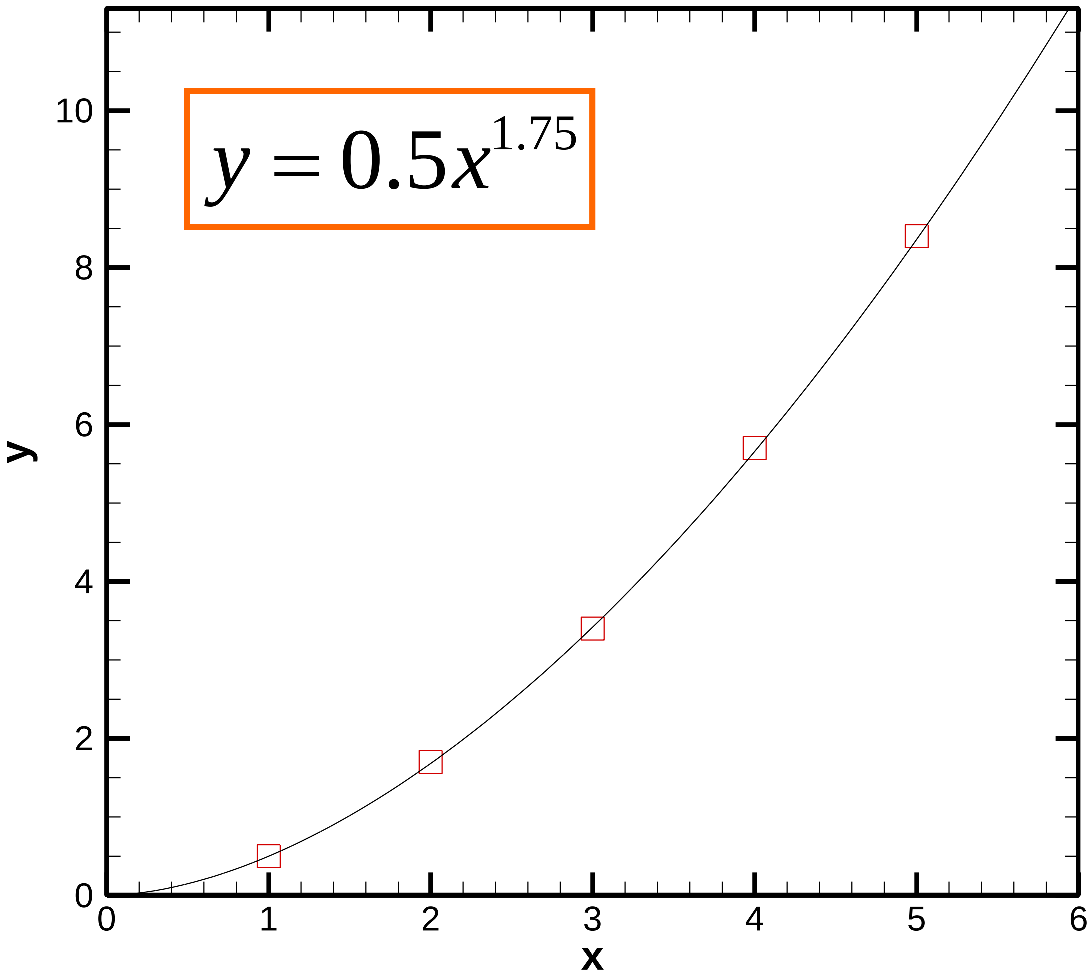
$$\sum_{k=1}^{5} X_k = 4.7875, \sum_{k=1}^{5} Y_k = 4.9300,$$

$$\sum_{k=1}^{5} X_k^2 = 6.1995, \sum_{k=1}^{5} X_k Y_k = 7.5503$$

$$a_1 = \frac{n\sum X_iY_i - \sum X_i \sum Y_i}{n\sum X_i^2 - \left(\sum X_i\right)^2} = 1.75 \Rightarrow m = 1.75$$

$$a_0 = \frac{\sum X_i^2 \sum Y_i - \sum X_i \sum X_iY_i}{n\sum X_i^2 - \left(\sum X_i\right)^2} = -0.69 \Rightarrow b = e^{-0.69} = 0.5$$

$$\boxed{y = 0.5x^{1.75}}$$



$y = 0.5x^{1.75}$
y
x
0
2
4
6
8
10
0
1
2
3
4
5
6

Other Functions

$$y = be^{mx}$$

$$\ln(y) = \ln(be^{mx}) = m\,x + \ln(b)$$

$$Y \qquad\qquad\qquad a_1 \qquad\qquad a_0$$

Once a_1 and a_0 are known, the constants,

$b = e^{a_0}$ and $m = a_1$

Other Functions

$$y = \frac{1}{mx+b}$$

$$\underbrace{1/y}_{Y} = \underbrace{m}_{a_1}x + \underbrace{b}_{a_0}$$

Once a_1 and a_0 are known, the constants,

$b = a_0$ and $m = a_1$

Other Functions

$$y = \frac{mx}{b + x}$$

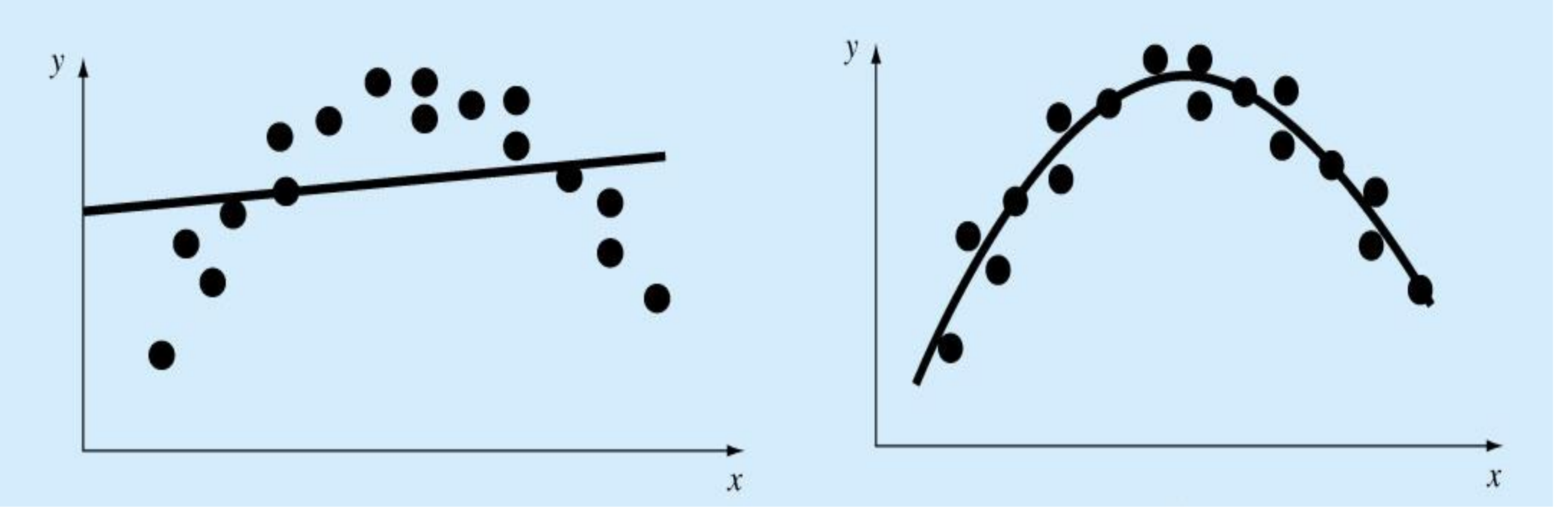
$$1/y = b/m \, 1/x + 1/m$$

$$Y \qquad a_1 \qquad X \qquad a_0$$

Once a_1 and a_0 are known, the constants,

$m = 1/a_0$ and $b = ma_1 = a_1/a_0$

Polynomial Regression



A parabola is preferable

2nd Order Polynomial

$$y = a_o + a_1 x + a_2 x^2$$

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_{i,\text{measured}} - y_{i,\text{model}})^2 = \sum_{i=1}^{n} \left[y_i - \left(a_0 + a_1 x_i + a_2 x_i^2 \right) \right]^2$$

$$\frac{\partial S_r}{\partial a_o} = -2 \sum (y_i - a_o - a_1 x_i - a_2 x_i^2) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2 \sum (y_i - a_o - a_1 x_i - a_2 x_i^2) x_i = 0$$

$$\frac{\partial S_r}{\partial a_2} = -2 \sum (y_i - a_o - a_1 x_i - a_2 x_i^2) x_i^2 = 0$$

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$$\frac{\partial S_r}{\partial a_o} = -2\sum (y_i - a_o - a_1 x_i - a_2 x_i^2) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum (y_i - a_o - a_1 x_i - a_2 x_i^2) x_i = 0$$

$$\frac{\partial S_r}{\partial a_2} = -2\sum (y_i - a_o - a_1 x_i - a_2 x_i^2) x_i^2 = 0$$

$$\sum y_i = n \cdot a_o + a_1 \sum x_i + a_2 \sum x_i^2$$

$$\sum x_i y_i = a_o \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3$$

$$\sum x_i^2 y_i = a_o \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4$$

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{Bmatrix}$$

A system of 3x3 equations needs to be solved to determine the coefficients of the polynomial.

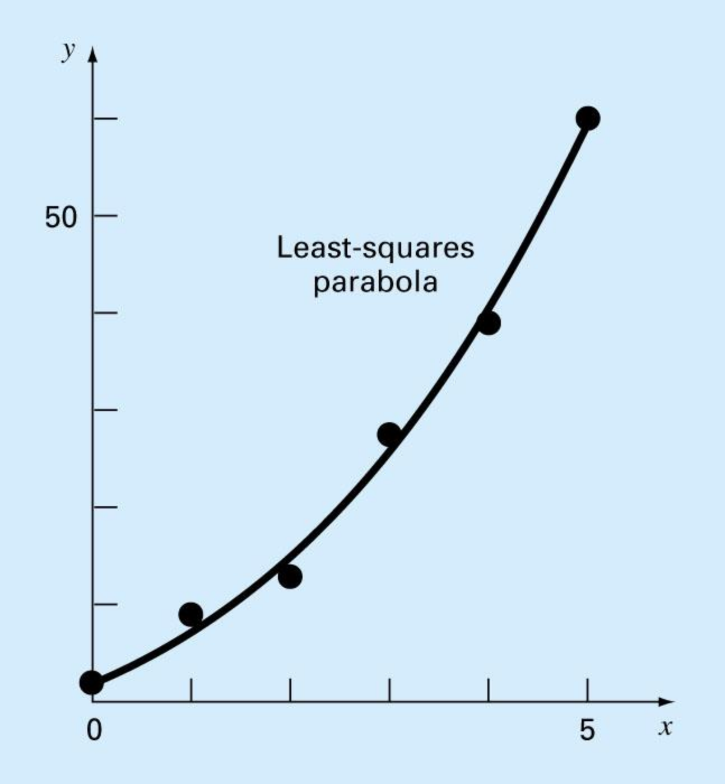
Example:

	x_i	y_i	x_i^2	x_i^3	x_i^4	x_iy_i	$x_i^2y_i$
	0	**2.1**	0	0	0	0	0
	1	**7.7**	1	1	1	7.7	7.7
	2	**13.6**	4	8	16	27.2	54.4
	3	**27.2**	9	27	81	81.6	244.8
	4	**40.9**	16	64	256	163.6	654.4
	5	**61.1**	25	125	625	305.5	1527.5
Sum	**15**	**152.6**	**55**	**225**	**979**	**585.6**	**2489**

$$\sum x_i = 15, \sum y_i = 152.6, \sum x_i^2 = 55, \sum x_i^3 = 225$$

$$\sum x_i^4 = 979, \sum x_i y_i = 585.6, \sum x_i^2 y_i = 2488.8$$

$$\begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 152.6 \\ 585.6 \\ 2488.8 \end{Bmatrix}$$



$$a_0 = 2.47857,\ a_1 = 2.35929,\ a_2 = 1.86071$$

$$y = 2.47857 + 2.35929\,x + 1.86071\,x^2$$

Multiple Linear Regression - example

Heat Transfer Data External to 3/4-inch OD Tubes

Point	**Re**	**Pr**	μ/μ_w	Nu
1	49000	2.3	0.947	277
2	68600	2.28	0.954	348
3	84800	2.27	0.959	421
4	34200	2.32	0.943	223
5	22900	2.36	0.936	177
6	1321	246	0.592	114.8
7	931	247	0.583	95.9
8	518	251	0.579	68.3
9	346	273	0.29	49.1
10	122.9	1518	0.294	56
11	54.0	1590	0.279	39.9
12	84.6	1521	0.267	47
13	1249	107.4	0.724	94.2
14	1021	186	0.612	99.9
15	465	414	0.512	83.1
16	54.8	1302	0.273	35.9

Nusselt number

$$Nu = a\,\mathrm{Re}^{b}\,\mathrm{Pr}^{c}\,(\mu/\mu_w)^{d}$$

$$\ln(Nu) = \ln(a) + b\ln(\mathrm{Re}) + c\ln(\mathrm{Pr}) + d\ln(Mu)$$

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n}\left[\ln(Nu_{i,\text{measured}}) - \ln(Nu_{i,\text{model}})\right]^2$$

$$= \sum_{i=1}^{n}\left[\ln\mathrm{Nu}_i - \left(\ln a + b\,\ln\mathrm{Re}_i + c\,\ln\mathrm{Pr}_i + d\,\ln\mathrm{Mu}_i\right)\right]^2$$

$$\frac{\partial S_r}{\partial a} = 2\left[n\ln a + b\sum_{i=1}^{n}\ln\mathrm{Re}_i + c\sum_{i=1}^{n}\ln\mathrm{Pr}_i + d\sum_{i=1}^{n}\ln\mathrm{Mu}_i - \sum_{i=1}^{n}\ln\mathrm{Nu}_i\right]/a = 0$$

$$\frac{\partial S_r}{\partial b} = 2\left[\ln a\sum_{i=1}^{n}\ln\mathrm{Re}_i + b\sum_{i=1}^{n}\left(\ln\mathrm{Re}_i\right)^2 + c\sum_{i=1}^{n}\ln\mathrm{Re}_i\cdot\ln\mathrm{Pr}_i + d\sum_{i=1}^{n}\ln\mathrm{Re}_i\cdot\ln\mathrm{Mu}_i - \sum_{i=1}^{n}\ln\mathrm{Re}_i\cdot\ln\mathrm{Nu}_i\right] = 0$$

$$\frac{\partial S_r}{\partial c} = 2\left[\ln a\sum_{i=1}^{n}\ln\mathrm{Pr}_i + b\sum_{i=1}^{n}\ln\mathrm{Re}_i\cdot\ln\mathrm{Pr}_i + c\sum_{i=1}^{n}\left(\ln\mathrm{Pr}_i\right)^2 + d\sum_{i=1}^{n}\ln\mathrm{Pr}_i\cdot\ln\mathrm{Mu}_i - \sum_{i=1}^{n}\ln\mathrm{Pr}_i\cdot\ln\mathrm{Nu}_i\right] = 0$$

$$\frac{\partial S_r}{\partial d} = 2\left[\ln a\sum_{i=1}^{n}\ln\mathrm{Mu}_i + b\sum_{i=1}^{n}\ln\mathrm{Re}_i\cdot\ln\mathrm{Mu}_i + c\sum_{i=1}^{n}\ln\mathrm{Pr}_i\cdot\ln\mathrm{Mu}_i + d\sum_{i=1}^{n}\left(\ln\mathrm{Mu}_i\right)^2 - \sum_{i=1}^{n}\ln\mathrm{Mu}_i\cdot\ln\mathrm{Nu}_i\right] = 0$$

$$\begin{bmatrix} n & \sum_{i=1}^{n} \ln \mathrm{Re}_i & \sum_{i=1}^{n} \ln \mathrm{Pr}_i & \sum_{i=1}^{n} \ln \mathrm{Mu}_i \\ \sum_{i=1}^{n} \ln \mathrm{Re}_i & \sum_{i=1}^{n} (\ln \mathrm{Re}_i)^2 & \sum_{i=1}^{n} \ln \mathrm{Re}_i \cdot \ln \mathrm{Pr}_i & \sum_{i=1}^{n} \ln \mathrm{Re}_i \cdot \ln \mathrm{Mu}_i \\ \sum_{i=1}^{n} \ln \mathrm{Pr}_i & \sum_{i=1}^{n} \ln \mathrm{Re}_i \cdot \ln \mathrm{Pr}_i & \sum_{i=1}^{n} (\ln \mathrm{Pr}_i)^2 & \sum_{i=1}^{n} \ln \mathrm{Pr}_i \cdot \ln \mathrm{Mu}_i \\ \sum_{i=1}^{n} \ln \mathrm{Mu}_i & \sum_{i=1}^{n} \ln \mathrm{Re}_i \cdot \ln \mathrm{Mu}_i & \sum_{i=1}^{n} \ln \mathrm{Pr}_i \cdot \ln \mathrm{Mu}_i & \sum_{i=1}^{n} (\ln \mathrm{Mu}_i)^2 \end{bmatrix} \begin{bmatrix} \ln a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \ln \mathrm{Nu}_i \\ \sum_{i=1}^{n} \ln \mathrm{Re}_i \cdot \ln \mathrm{Nu}_i \\ \sum_{i=1}^{n} \ln \mathrm{Pr}_i \cdot \ln \mathrm{Nu}_i \\ \sum_{i=1}^{n} \ln \mathrm{Mu}_i \cdot \ln \mathrm{Nu}_i \end{bmatrix}$$

$$\begin{bmatrix} 16 & 117.32401 & 71.450325 & -9.7193108 \\ 117.32401 & 962.16498 & 421.87845 & -52.829532 \\ 71.450325 & 421.87845 & 424.75516 & -61.5206146 \\ -9.7193108 & -52.8295326 & -61.5206146 & 9.7626619 \end{bmatrix} \begin{bmatrix} \ln a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 74.2386856 \\ 574.29163 \\ 302.39407 \\ -39.6257973 \end{bmatrix}$$

$$\ln a = -0.626014 \Rightarrow a = e^{-0.626014} = 0.534719$$

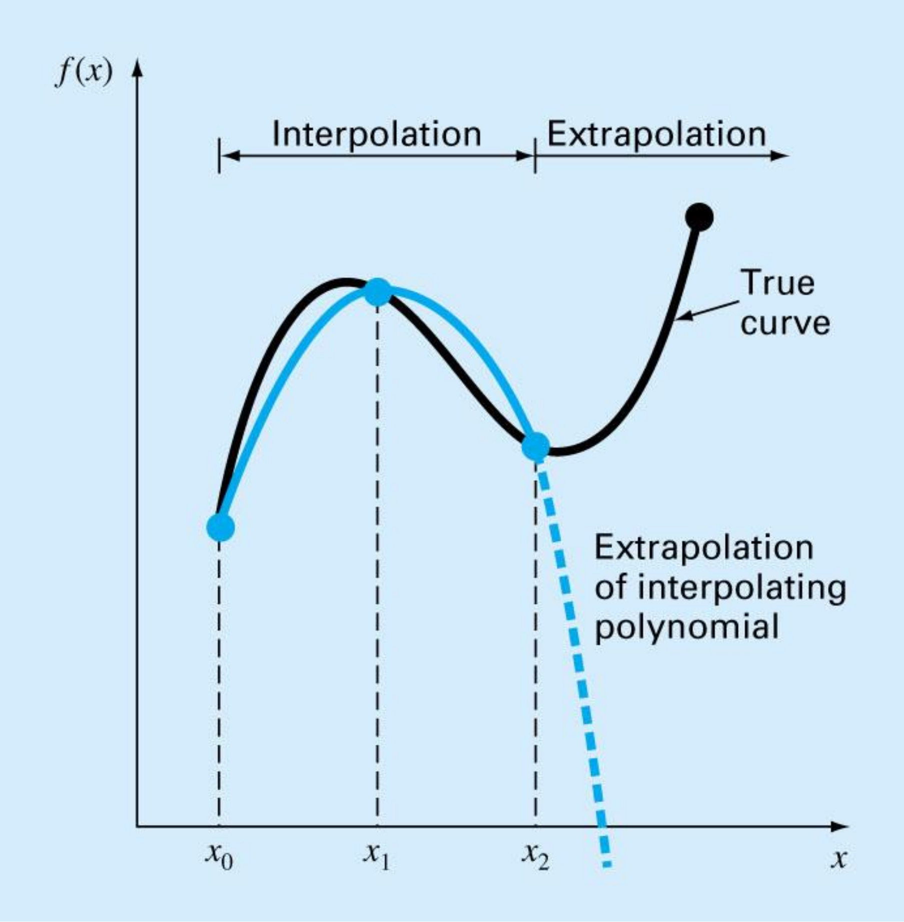
$$b = 0.558832,\ c = 0.252376,\ d = -0.0677135$$

$$\boxed{Nu = 0.66\,\mathrm{Re}^{0.54}\,\mathrm{Pr}^{0.25}}$$

$$\boxed{Nu = 0.53\,\mathrm{Re}^{0.56}\,\mathrm{Pr}^{0.25}(\mu/\mu_w)^{-0.068}}$$

Interpolation

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$



Construction of Polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

- Since $n+1$ data points are required to determine $n+1$ coefficients, simultaneous linear systems of equations can be used to calculate "*a*"s.

$$f(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 \cdots + a_n x_0^n$$
$$f(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 \cdots + a_n x_1^n$$
$$\vdots$$
$$f(x_n) = a_0 + a_1 x_n + a_2 x_n^2 \cdots + a_n x_n^n$$

where "x"s are the knowns and "a"s are the unknowns.

- However, this requires the solution of a large matrix.
- There are a variety of mathematical formats in which this polynomial can be expressed:
  - The Newton polynomial
  - The Lagrange polynomial

Newton's Divided-Difference Interpolating Polynomials

Linear Interpolation

- $P_1(x)$ is a first-order interpolating polynomial passing through $\left(x_0, f(x_o)\right)$ and $\left(x_1, f(x_1)\right)$.

$$\frac{P_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\boxed{P_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)}$$

Quadratic Interpolation

- If three data points are available, 0,1 and 2, the estimate is improved by introducing some curvature into the line connecting the points.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

- A simple procedure can be used to determine the values of the coefficients.

$$x = x_0 \qquad b_0 = f(x_0)$$

$$x = x_1 \qquad b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$x = x_2 \qquad b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

General Form of Newton's Interpolating Polynomials/

$$f_n(x) = f(x_0) + (x - x_0) f[x_1, x_0] + (x - x_0)(x - x_1) f[x_2, x_1, x_0]$$
$$+ \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1}) f[x_n, x_{n-1}, \cdots, x_0]$$

$$b_0 = f(x_0)$$
$$b_1 = f[x_1, x_0]$$
$$b_2 = f[x_2, x_1, x_0]$$
$$\vdots$$
$$b_n = f[x_n, x_{n-1}, \cdots, x_1, x_0]$$

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

$$\vdots$$

$$f[x_n, x_{n-1}, \ldots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_0]}{x_n - x_0}$$

$f(x)$	First divided differences	Second divided differences	Third divided differences
$f[x_0]$			
	$f[x_0,x_1]=\dfrac{f[x_1]-f[x_0]}{x_1-x_0}$		
$f[x_1]$		$f[x_0,x_1,x_2]=\dfrac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	
	$f[x_1,x_2]=\dfrac{f[x_2]-f[x_1]}{x_2-x_1}$		$f[x_0,x_1,x_2,x_3]=\dfrac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$
$f[x_2]$		$f[x_1,x_2,x_3]=\dfrac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	
	$f[x_2,x_3]=\dfrac{f[x_3]-f[x_2]}{x_3-x_2}$		$f[x_1,x_2,x_3,x_4]=\dfrac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}$
$f[x_3]$		$f[x_2,x_3,x_4]=\dfrac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$	
	$f[x_3,x_4]=\dfrac{f[x_4]-f[x_3]}{x_4-x_3}$		$f[x_2,x_3,x_4,x_5]=\dfrac{f[x_3,x_4,x_5]-f[x_2,x_3,x_4]}{x_5-x_2}$
$f[x_4]$		$f[x_3,x_4,x_5]=\dfrac{f[x_4,x_5]-f[x_3,x_4]}{x_5-x_3}$	
	$f[x_4,x_5]=\dfrac{f[x_5]-f[x_4]}{x_5-x_4}$		
$f[x_5]$			

Example

x	$f(x)$
2.0	0.85467
2.3	0.75682
2.6	0.43126
2.9	0.22364
3.2	0.08567

i	x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, \cdots, x_i]$	$f[x_{i-4}, \cdots, x_i]$
0	**2.0**	**0.85467**				
			-0.32617			
1	**2.3**	**0.75682**		**-1.26505**		
			-1.08520		**2.13363**	
2	**2.6**	**0.43126**		**0.65522**		**-2.02642**
			-0.69207		**-0.29808**	
3	**2.9**	**0.22364**		**0.38695**		
			-0.45990			
4	**3.2**	**0.08567**				

The 5 coefficients of the Newton's interpolating polynomial are:

$$b_0 = f[x_0] = 0.85467$$

$$b_1 = f[x_0, x_1] = -0.32617$$

$$b_2 = f[x_0, x_1, x_2] = -1.26505$$

$$b_3 = f[x_0, x_1, x_2, x_3] = 2.13363$$

$$b_4 = f[x_0, x_1, x_2, x_3, x_4] = -2.02642$$

$$P_4(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3)$$
$$= 0.85467 - 0.32617(x - 2.0) - 1.26505(x - 2.0)(x - 2.3)$$
$$+ 2.13363(x - 2.0)(x - 2.3)(x - 2.6)$$
$$- 2.02642(x - 2.0)(x - 2.3)(x - 2.6)(x - 2.9)$$

$P_4(x)$ can now be used to estimate the value of the function f(x) say at x = 2.8.

$$P_4(2.8) = 0.85467 - 0.32617(2.8 - 2.0)$$
$$-1.26505(2.8 - 2.0)(2.8 - 2.3)$$
$$+ 2.13363(2.8 - 2.0)(2.8 - 2.3)(2.8 - 2.6)$$
$$-2.02642(2.8 - 2.0)(2.8 - 2.3)(2.8 - 2.6)(2.8 - 2.9) = 0.275$$

Lagrange Interpolating Polynomials

Linear Interpolation

- $P_1(x)$ is a first-order interpolating polynomial passing through $\left(x_0, f(x_o)\right)$ and $\left(x_1, f(x_1)\right)$.

$$P_1(x) = L_0(x)\, f(x_0) + L_1(x)\, f(x_1)$$

at $x = x_0$, $L_0(x) = 1$ and $L_1(x) = 0$ and

at $x = x_1$, $L_0(x) = 0$ and $L_1(x) = 1$

The conditions can be satisfied if $L_0(x)$ and $L_1(x)$ are defined in the following way.

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

Given data points:

at $x_0 = 2, y_0 = 3$ and at $x_1 = 5, y_1 = 8$

$P(x)$ should satisfy the following conditions:

$P(x = 2) = 3$ and $P(x = 5) = 8$

$$P(x) = \mathbf{3}L_{\mathbf{0}}(x) + \mathbf{8}L_{\mathbf{1}}(x)$$

$P(x)$ can satisfy the above conditions if

at $x = x_0 = 2$, $L_0(x) = 1$ and $L_1(x) = 0$ and

at $x = x_1 = 5$, $L_0(x) = 0$ and $L_1(x) = 1$

$$L_{\mathbf{0}}(x) = \frac{x - \mathbf{5}}{\mathbf{2} - \mathbf{5}} \quad \textbf{and} \quad L_{\mathbf{1}}(x) = \frac{x - \mathbf{2}}{\mathbf{5} - \mathbf{2}}$$

$$P(x) = L_0(x) f(x_0) + L_1(x) f(x_1)$$

$$P(x) = \left(\frac{x-5}{2-5}\right)(3) + \left(\frac{x-2}{5-2}\right)(8)$$

$$P(x) = \frac{5x-1}{3}$$

$$\text{at } x = 4, P(4) = \frac{5\times 4-1}{3} = 6.333$$

The Lagrange interpolating polynomial passing through three given points; (x_0, y_0), (x_1, y_1) and (x_2, y_2) is:

$$P_2(x) = L_0(x)\,y_0 + L_1(x)\,y_1 + L_2(x)\,y_2$$

$$P_2(x) = L_0(x)\,f(x_0) + L_1(x)\,f(x_1) + L_2(x)\,f(x_2)$$

At x_0, $L_0(x)$ becomes 1.

At all other given data points $L_0(x)$ is 0.

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

At x_1, $L_1(x)$ becomes 1.

At all other given data points $L_1(x)$ is 0.

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

At x_2, $L_2(x)$ becomes 1.

At all other given data points $L_2(x)$ is 0.

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

General form of the Lagrange Interpolating Polynomial

$$P_n(x) = L_0(x)\,y_0 + L_1(x)\,y_1 + L_2(x)\,y_2 + \ldots\ldots\ldots\ldots + L_n(x)\,y_n$$

$$P_n(x) = L_0(x)\,f(x_0) + L_1(x)\,f(x_1) + L_2(x)\,f(x_2) + \ldots\ldots\ldots\ldots + L_n(x)\,f(x_n)$$

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

$$(x-x_0)(x-x_1)(x-x_2)\times\cdots\cdots$$
$$\times(x-x_{k-1})(x-x_{k+1})\times\cdots\cdots$$
$$\times(x-x_{n-1})(x-x_n)$$

$$(x_k-x_0)(x_k-x_1)(x_k-x_2)\times\cdots\cdots$$
$$\times(x_k-x_{k-1})(x_k-x_{k+1})\times\cdots\cdots$$
$$\times(x_k-x_{n-1})(x_k-x_n)$$

$$L_k(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$$

Data Set	Polynomial
$\{(x_1, y_1)\}$	$p(x) = y_1$
$\{(x_1, y_1), (x_2, y_2)\}$	$p(x) = y_1 \frac{(x-x_2)}{(x_1-x_2)} + y_2 \frac{(x-x_1)}{(x_2-x_1)}$
$\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$	$p(x) = y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$
$\{(x_1, y_1), (x_2, y_2), \cdots, (x_m, y_m)\}$	$p(x) = y_1 \frac{(x-x_2)(x-x_3)\cdots(x-x_m)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_m)} + y_2 \frac{(x-x_1)(x-x_3)\cdots(x-x_m)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_m)} + \cdots + y_m \frac{(x-x_1)(x-x_2)\cdots(x-x_{m-1})}{(x_m-x_1)(x_m-x_2)\cdots(x_m-x_{m-1})}$ $p(x) = \sum_{i=1}^{m} y_i \prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{(x-x_j)}{(x_i-x_j)}$

Example

$$\boxed{f(x) = \frac{1}{x}}$$

Find the Lagrange Interpolating Polynomial using the three given points.

$$(x_0, y_0) = (2,\ 0.5)$$

$$(x_1, y_1) = (2.5,\ 0.4)$$

$$(x_2, y_2) = (4,\ 0.25)$$

$$P_2(x) = L_0(x)\,y_0 + L_1(x)\,y_1 + L_2(x)\,y_2$$

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 2.5)(x - 4)}{(2 - 2.5)(2 - 4)}$$

$$= x^2 - 6.5x + 10$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 2)(x - 4)}{(2.5 - 2)(2.5 - 4)}$$

$$= \frac{-x^2 + 6x - 8}{0.75}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 2)(x - 2.5)}{(4 - 2)(4 - 2.5)}$$

$$= \frac{x^2 - 4.5x + 5}{3}$$

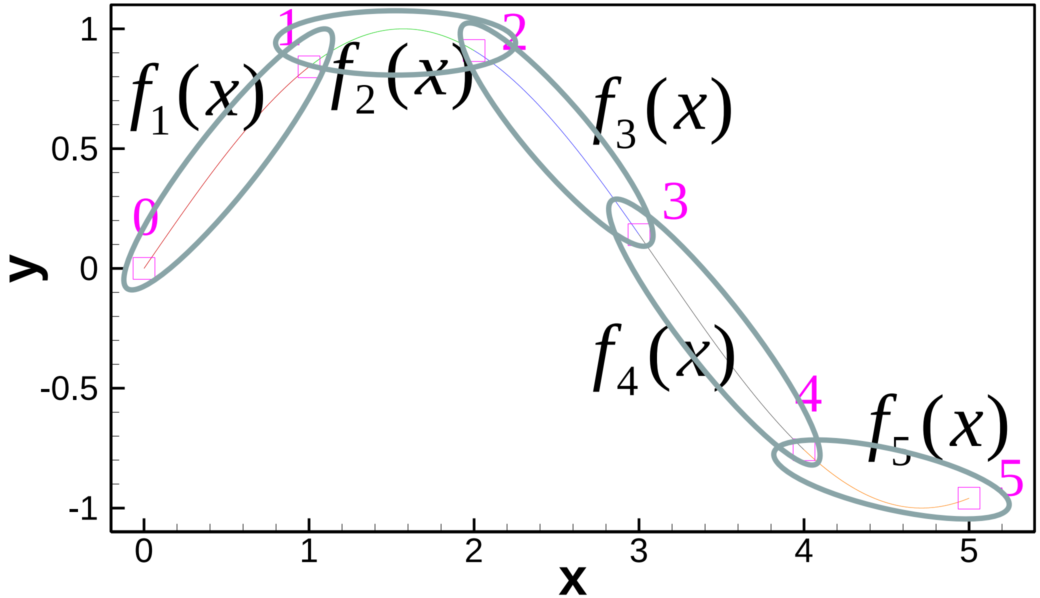
$$
\begin{aligned}
P(x) &= L_0(x)\,f(x_0) + L_1(x)\,f(x_1) + L_2(x)\,f(x_2) \\
&= \left(x^2 - 6.5x + 10\right)(0.5) \\
&\quad + \left(\frac{-x^2 + 6x - 8}{0.75}\right)(0.4) \\
&\quad + \left(\frac{x^2 - 4.5x + 5}{3}\right)(0.25) \\
&= 0.05x^2 - 0.425x + 1.15
\end{aligned}
$$

$$
\begin{aligned}
P(3) &= 0.05(3)^2 - 0.425(3) + 1.15 \\
&= 0.325 \approx f(3) = 1/3 = 0.333
\end{aligned}
$$

Cubic Spline

$$f_i(x) = a_i\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)^3 + b_i\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)^2 + c_i\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right) + d_i$$

x	y
0	0
1	0.841471
2	0.909297
3	0.14112
4	-0.756802
5	-0.958924



There will be 20 unknowns: a_i, b_i, c_i and d_i where $i = 1, 2, 3, 4$ and 5.

$n \times 4$ unknowns where n+1 is the number of points

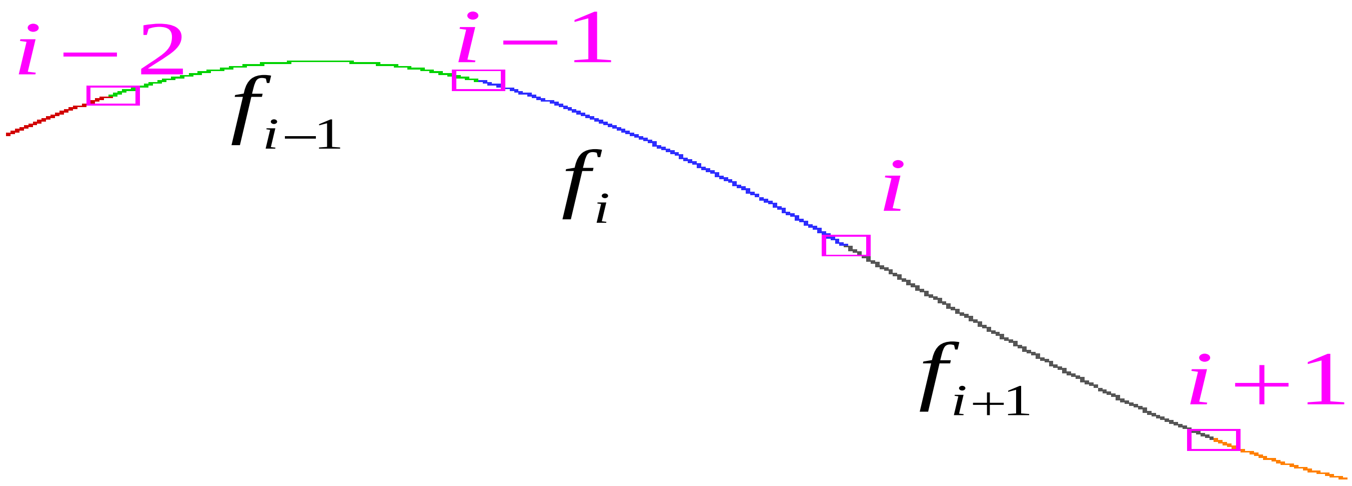
Cubic Spline

1. Each polynomial, $f_i(x)$, must pass through the endpoints of the interval.
2. At the interior knots, the slopes of the polynomials from the adjacent intervals are equal.
3. At the interior knots, the curvature of the polynomials from the adjacent intervals are equal.

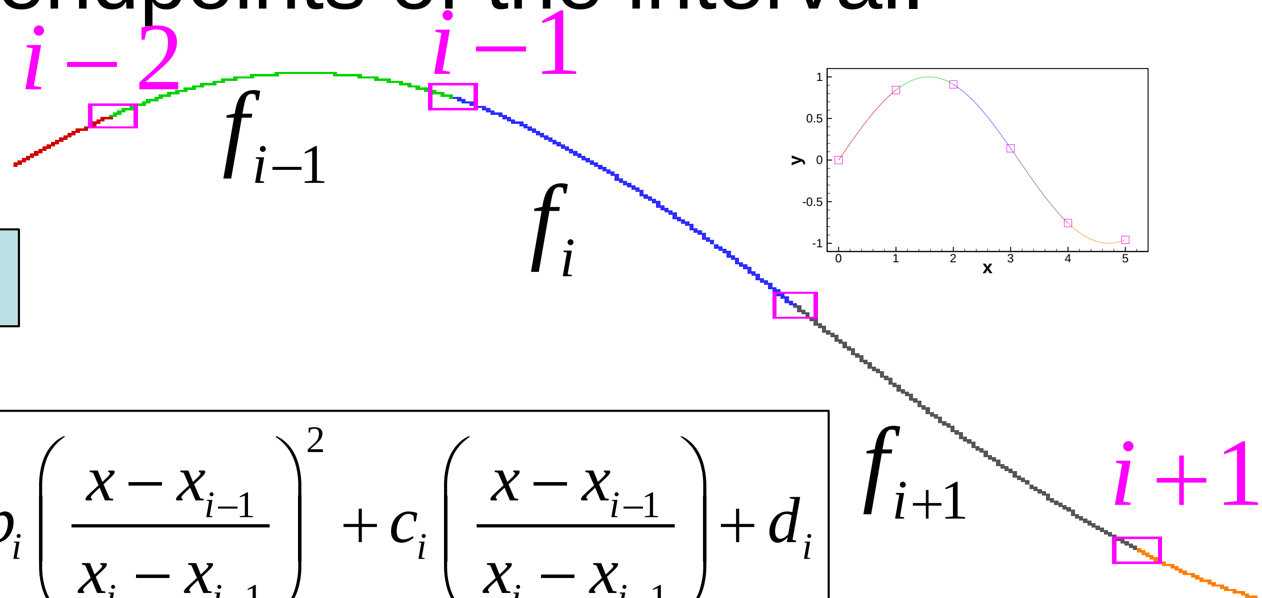
$$f_i(x) = a_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^3 + b_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2 + c_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) + d_i$$

$$f_i'(x) = \left[3a_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2 + 2b_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) + c_i\right] / (x_i - x_{i-1})$$

$$f_i''(x) = \left[6a_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) + 2b_i\right] / (x_i - x_{i-1})^2$$



1. Each polynomial, $y_i(x)$, must pass through the endpoints of the interval.



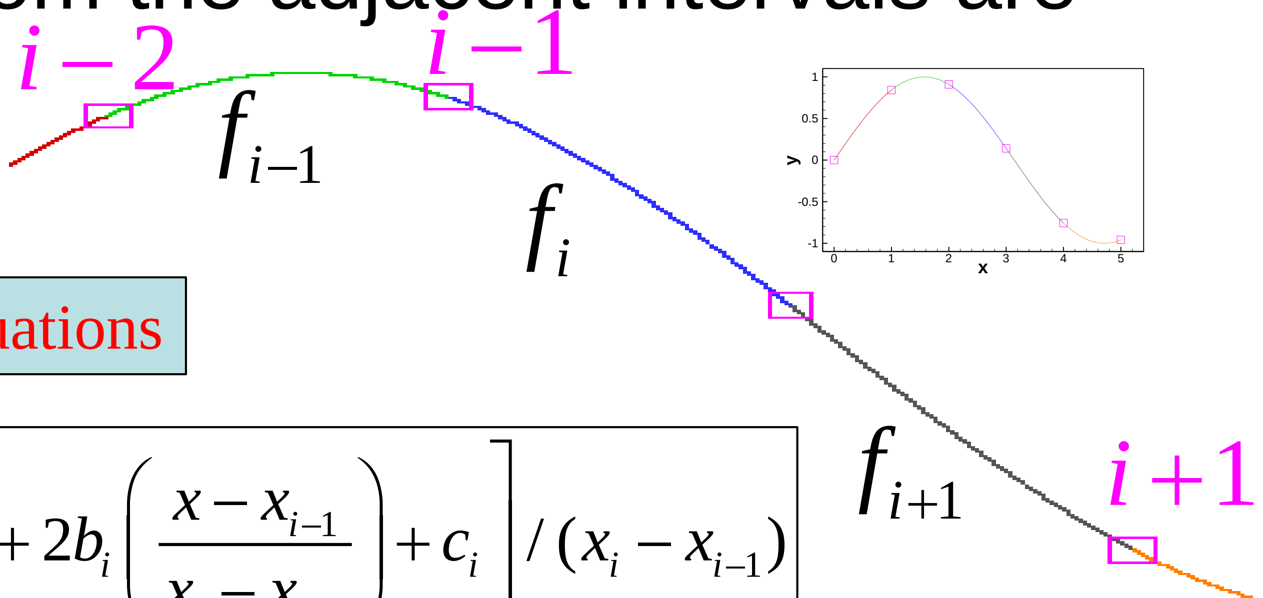
$2 \times n$ equations

$$f_i(x) = a_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^3 + b_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2 + c_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) + d_i$$

$$f_i(x_{i-1}) \Rightarrow d_i = y_{i-1}$$

$$f_i(x_i) \Rightarrow a_i + b_i + c_i + d_i = y_i$$

2. At the interior knots, the slopes of the polynomials from the adjacent intervals are equal.



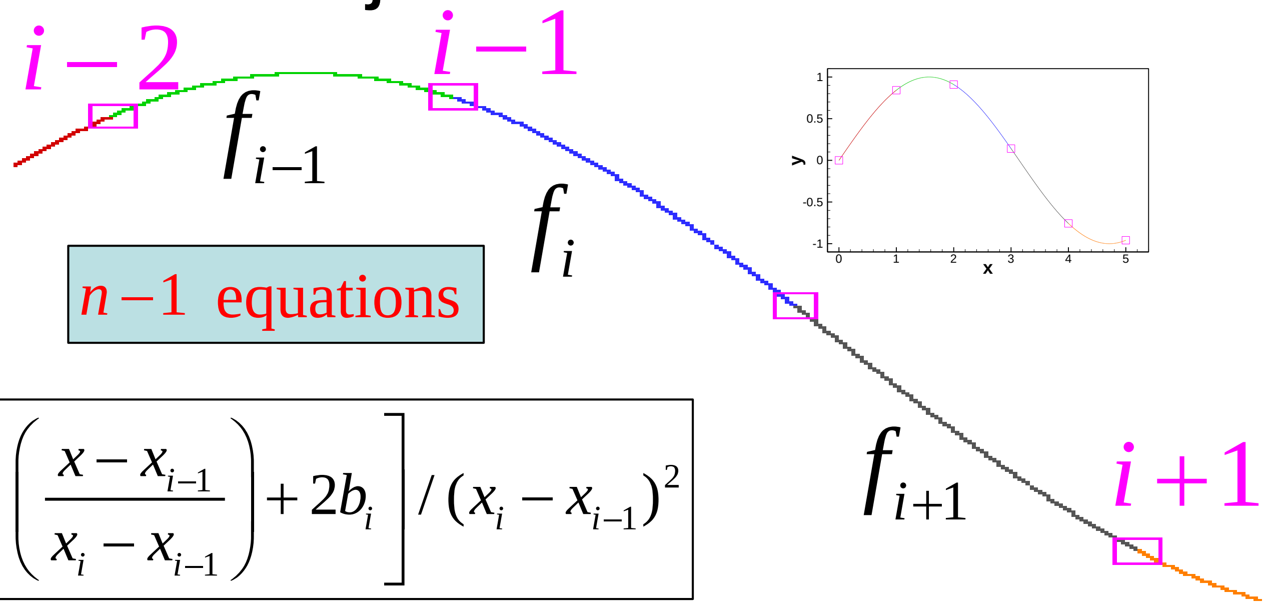
$n-1$ equations

$$f_i'(x) = \left[3a_i \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right)^2 + 2b_i \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right) + c_i \right] / (x_i - x_{i-1})$$

$$f_{i-1}'(x_{i-1}) = f_i'(x_{i-1}) \Rightarrow 3a_{i-1} + 2b_{i-1} + c_{i-1} = c_i$$

$$f_i'(x_i) = f_{i+1}'(x_i) \Rightarrow 3a_i + 2b_i + c_i = c_{i+1}$$

3. At the interior knots, the curvature of the polynomials from the adjacent intervals are equal.



$n-1$ equations

$$f_i''(x) = \left[6a_i \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right) + 2b_i \right] / (x_i - x_{i-1})^2$$

$$f_{i-1}''(x_{i-1}) = f_i''(x_{i-1}) \Rightarrow 3a_{i-1} + b_{i-1} = b_i$$

$$f_i''(x_i) = f_{i+1}''(x_i) \Rightarrow 3a_i + b_i = b_{i+1}$$

- The additional 2 equations can be obtained by assuming the second derivatives at the endpoints be zero, so

$$f_1''(x_o) = 0 \Rightarrow b_1 = 0$$

$$f_n''(x_n) = 0 \Rightarrow 3a_n + b_n = 0$$

OK, how do we solve it?

Make unknowns to be y_i', where $f_{i-1}'(x_i) = f_i'(x_i) = y'$

at $x = x_{i-1}$ $\begin{cases} d_i = y_{i-1} \\ c_i = y_{i-1}' \end{cases}$

at $x = x_i$ $\begin{cases} a_i + b_i + c_i + d_i = y_i \\ 3a_i + 2b_i + c_i = y_i' \end{cases}$

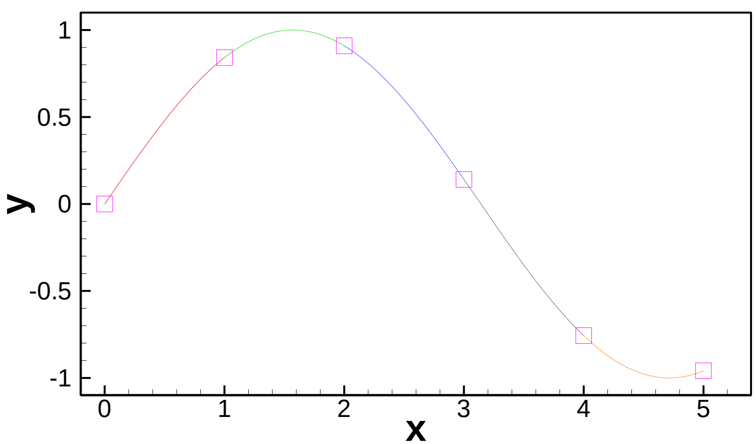
$$a_i + b_i = y_i - c_i - d_i = y_i - y_{i-1} - y_{i-1}'$$
$$3a_i + 2b_i = y_i' - c_i = y_i' - y_{i-1}'$$

$$a_i = y_i' + y_{i-1}' - 2(y_i - y_{i-1})$$

$$b_i = -y_i' - 2y_{i-1}' + 3(y_i - y_{i-1})$$

$$c_i = y_{i-1}'$$

$$d_i = y_{i-1}$$



$$f_i''(x_i) = f_{i+1}''(x_i) \Rightarrow 3a_i + b_i = b_{i+1}$$

$$3\underbrace{\left[y_i' + y_{i-1}' - 2(y_i - y_{i-1}) \right]}_{a_i} + \underbrace{\left[-y_i' - 2y_{i-1}' + 3(y_i - y_{i-1}) \right]}_{b_i}$$

$$= \underbrace{\left[-y_{i+1}' - 2y_i' + 3(y_{i+1} - y_i) \right]}_{b_{i+1}}$$

$$\boxed{y_{i-1}' + 4y_i' + y_{i+1}' = 3(y_{i+1} - y_{i-1})}$$

Point 0

$$f_1''(x_o) = 0 \Rightarrow b_1 = 0$$

$$-y_1' - 2y_0' + 3(y_1 - y_0) = 0$$

$$\boxed{2y_0' + y_1' = 3(y_1 - y_0)}$$

Point 1 to 4

$$y_{i-1}' + 4y_i' + y_{i+1}' = 3(y_{i+1} - y_{i-1})$$

$$\text{at } x_1 : \boxed{y_0' + 4y_1' + y_2' = 3(y_2 - y_0)}$$

$$\text{at } x_2 : \boxed{y_1' + 4y_2' + y_3' = 3(y_3 - y_1)}$$

$$\text{at } x_3 : \boxed{y_2' + 4y_3' + y_4' = 3(y_4 - y_2)}$$

$$\text{at } x_4 : \boxed{y_3' + 4y_4' + y_5' = 3(y_5 - y_3)}$$

Point 5

$$f_5''(x_5) = 0 \Rightarrow 3a_5 + b_5 = 0$$

$$3\underbrace{\left[y_5' + y_4' - 2(y_5 - y_4)\right]}_{a_5}$$

$$+\underbrace{\left[-y_5' - 2y_4' + 3(y_5 - y_4)\right]}_{b_5} = 0$$

$$\boxed{y_4' + 2y_5' = 3(y_5 - y_4)}$$

In Matrix Form

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \end{bmatrix} = \begin{bmatrix} 3(y_1 - y_0) \\ 3(y_2 - y_0) \\ 3(y_3 - y_1) \\ 3(y_4 - y_2) \\ 3(y_5 - y_3) \\ 3(y_5 - y_4) \end{bmatrix} = \begin{bmatrix} 2.524413 \\ 2.727891 \\ -2.101053 \\ -4.998297 \\ -3.300132 \\ -0.606366 \end{bmatrix}$$

The solution is: $$\begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \end{bmatrix} = \begin{bmatrix} 0.994576 \\ 0.535262 \\ -0.407732 \\ -1.00539 \\ -0.569018 \\ -0.0186741 \end{bmatrix}$$

Coefficients for equation i

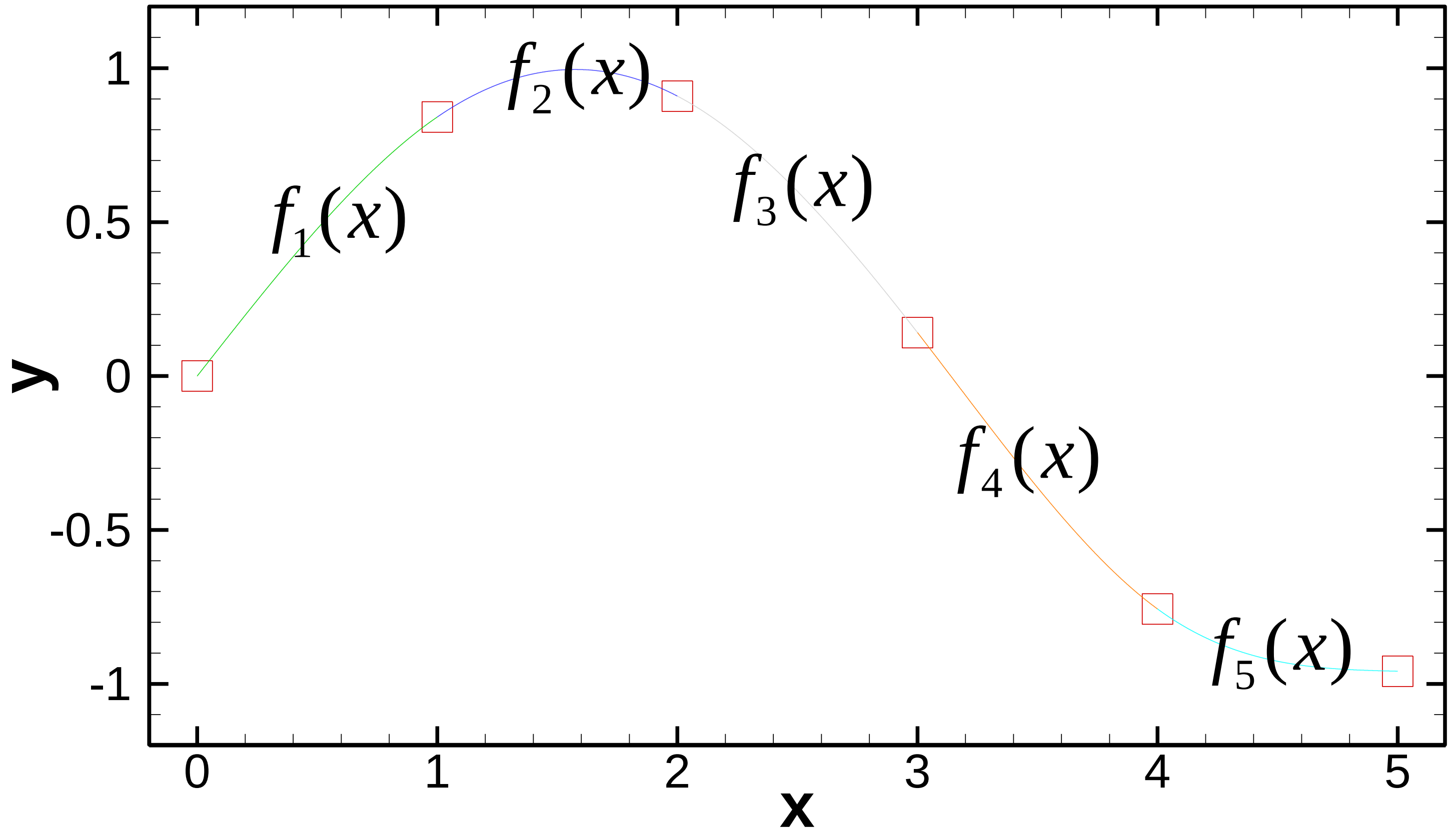
$$a_i = y_i' + y_{i-1}' - 2(y_i - y_{i-1}), \quad b_i = -y_i' - 2y_{i-1}' + 3(y_i - y_{i-1})$$

$$c_i = y_{i-1}', \quad d_i = y_{i-1}$$

i	a_i	b_i	c_i	d_i
1	-0.15310	0.00000	0.99458	0.00000
2	-0.00812	-0.45931	0.53526	0.84147
3	0.12323	-0.48368	-0.40773	0.90930
4	0.22144	-0.11397	-1.00539	0.14112
5	-0.18345	0.55034	-0.56902	-0.75680

Cubic Spline Interpolation

$$f_i(x) = a_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^3 + b_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2 + c_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) + d_i$$



In-class example

x	y
1	1
1.5	0.5
2	2
2.5	0.25
3	4

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} 3(y_1 - y_0) \\ 3(y_2 - y_0) \\ 3(y_3 - y_1) \\ 3(y_4 - y_2) \\ 3(y_5 - y_4) \end{bmatrix} = \begin{bmatrix} -1.5 \\ 3. \\ -0.75 \\ 6. \\ 11.25 \end{bmatrix}$$

$$\begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = \begin{bmatrix} -1.3660714285714284 \\ 1.2321428571428572 \\ -0.5625 \\ 0.267857142857143 \\ 5.49107142857142 8 \end{bmatrix}$$

What is the value at x=2.2?

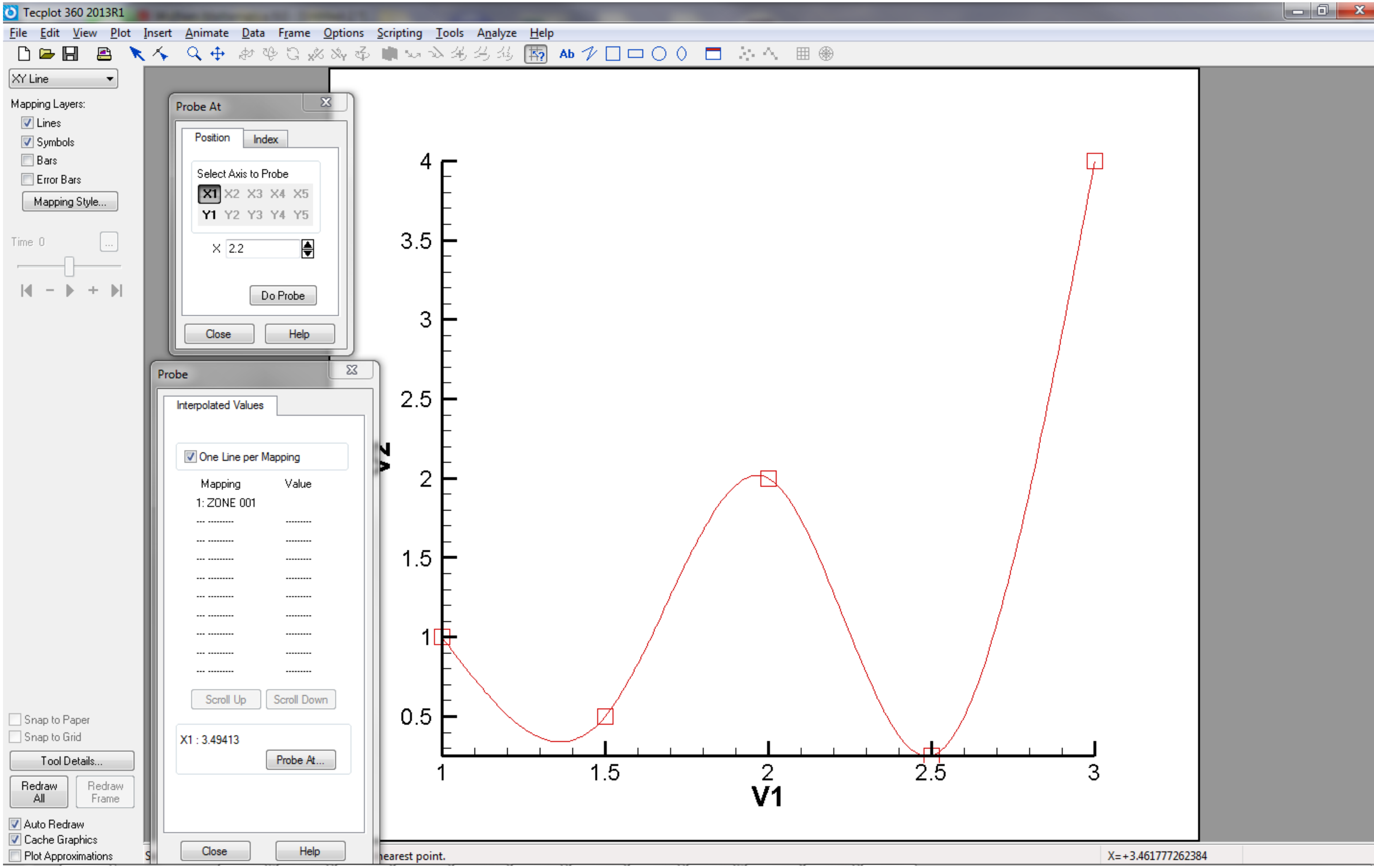
$$a_i = y_i' + y_{i-1}' - 2(y_i - y_{i-1}),\ \ b_i = -y_i' - 2y_{i-1}' + 3(y_i - y_{i-1})$$

$$c_i = y_{i-1}',\ \ d_i = y_{i-1}$$

i	a_i	b_i	c_i	d_i
1				
2				
3	3.20536	-4.39286	-0.5625	2
4				

$$f_i(x) = a_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^3 + b_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2 + c_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) + d_i$$

$$\frac{x - x_{i-1}}{x_i - x_{i-1}} = 0.4;\ f(2.2) = 1.27729$$



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