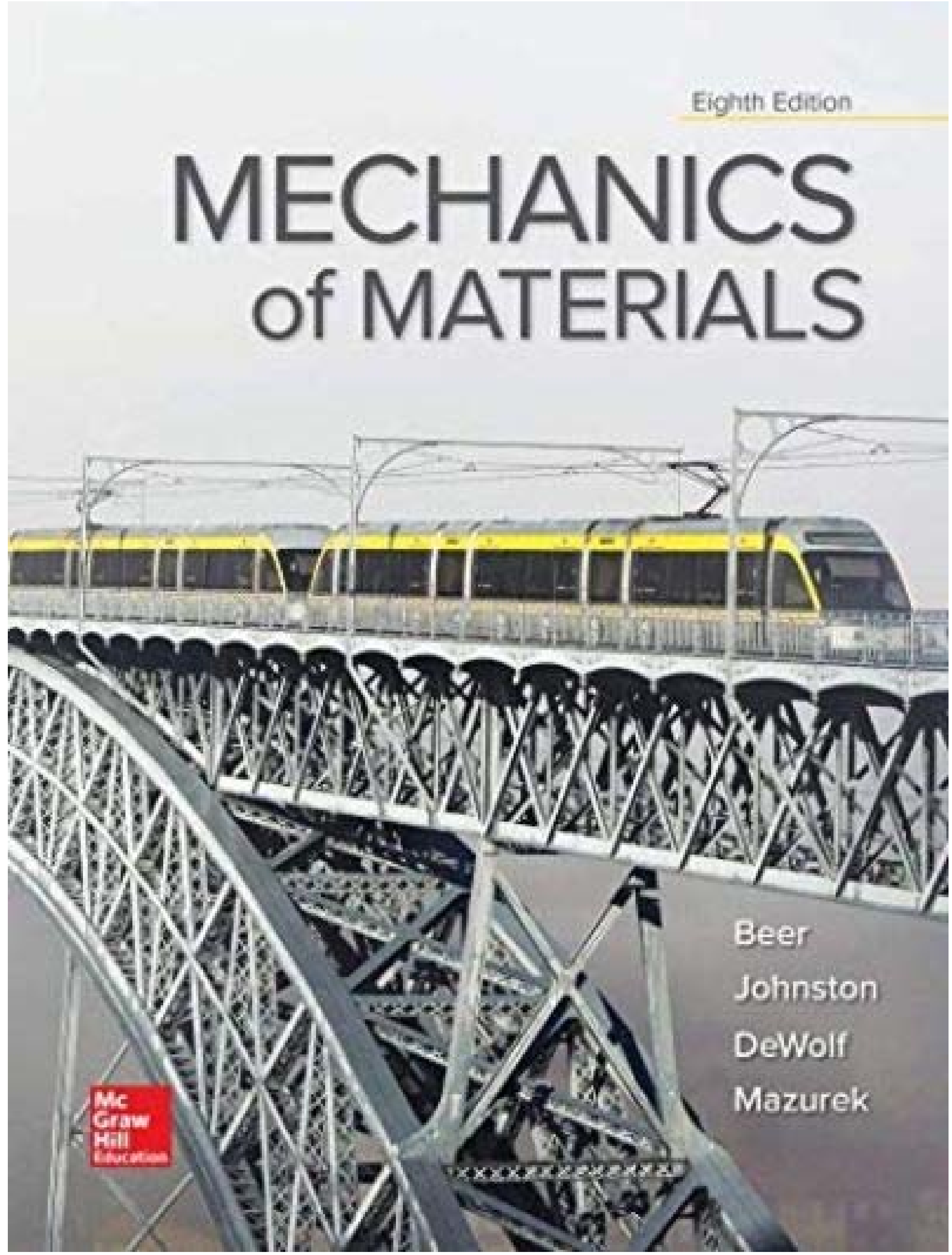
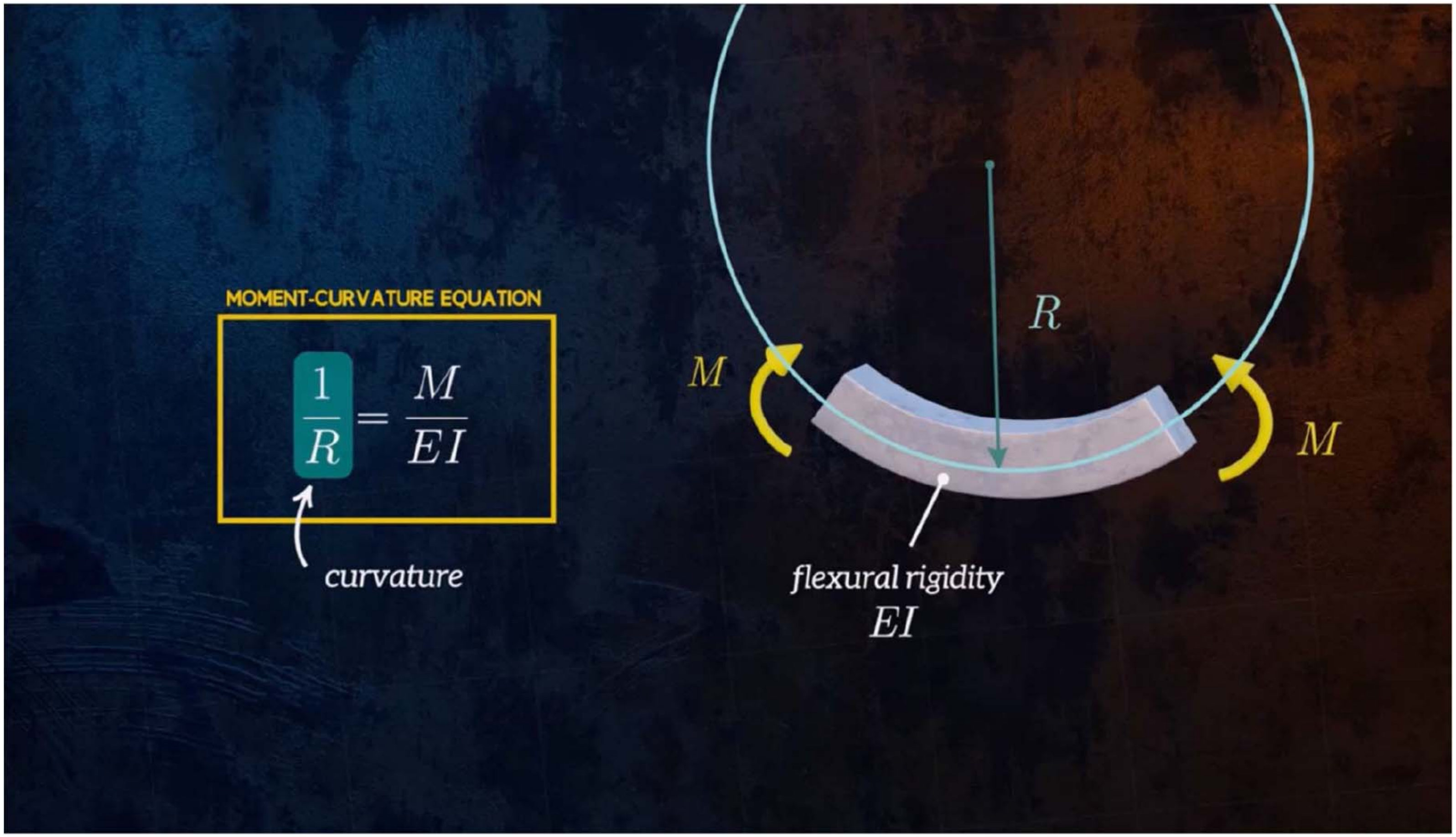
Deflection of Beams

Chapter 9

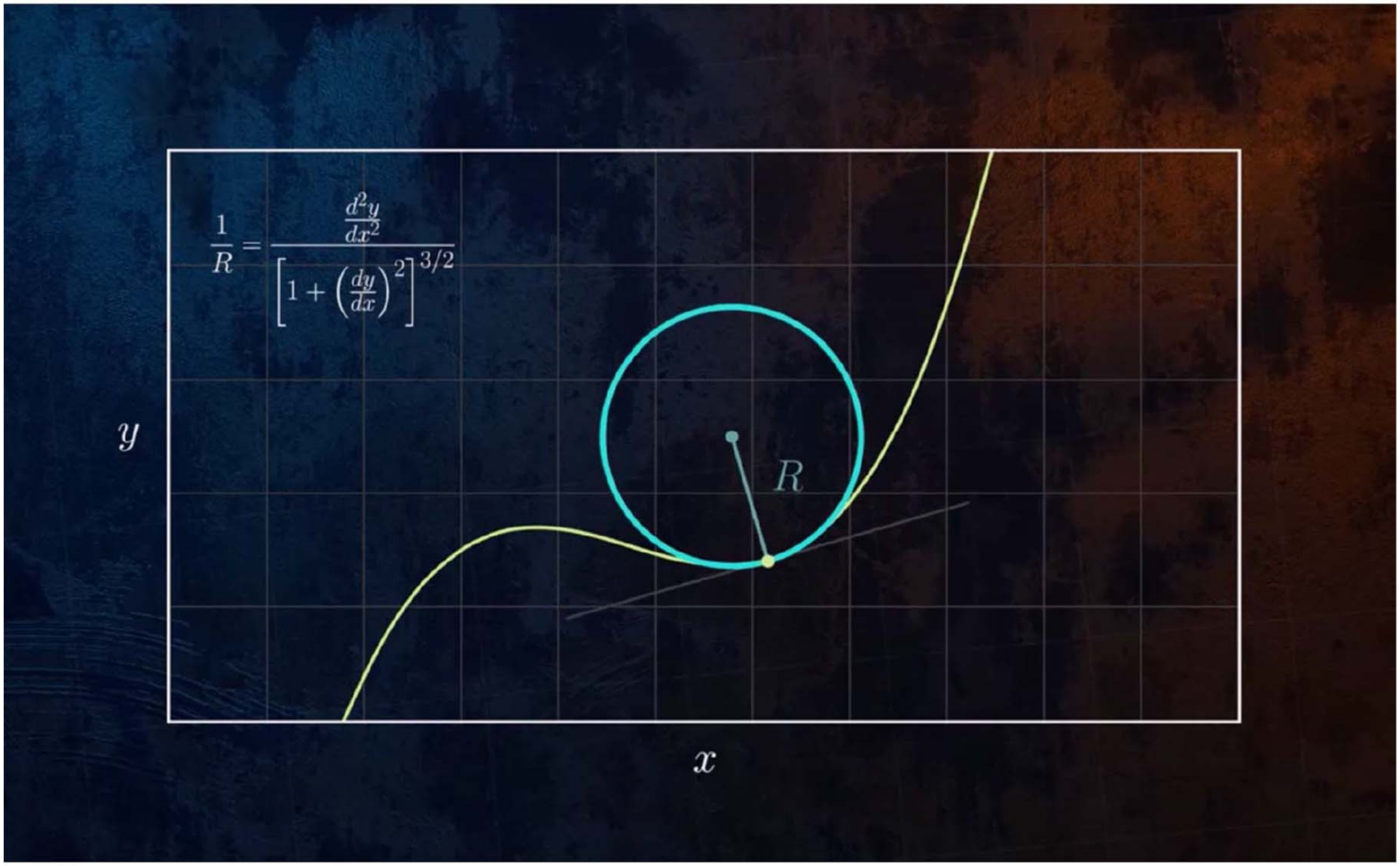


Contents

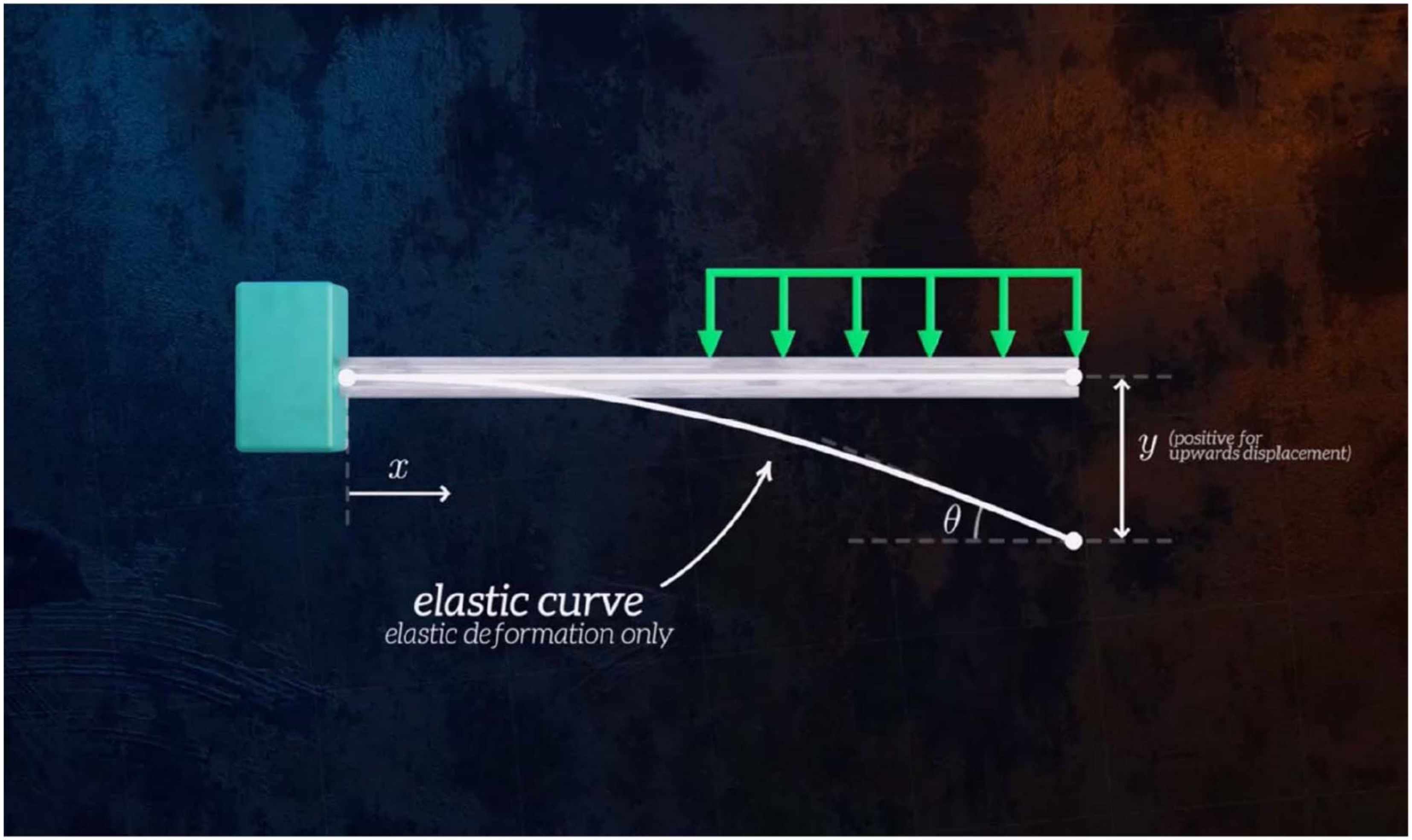
Pure Bending [1]



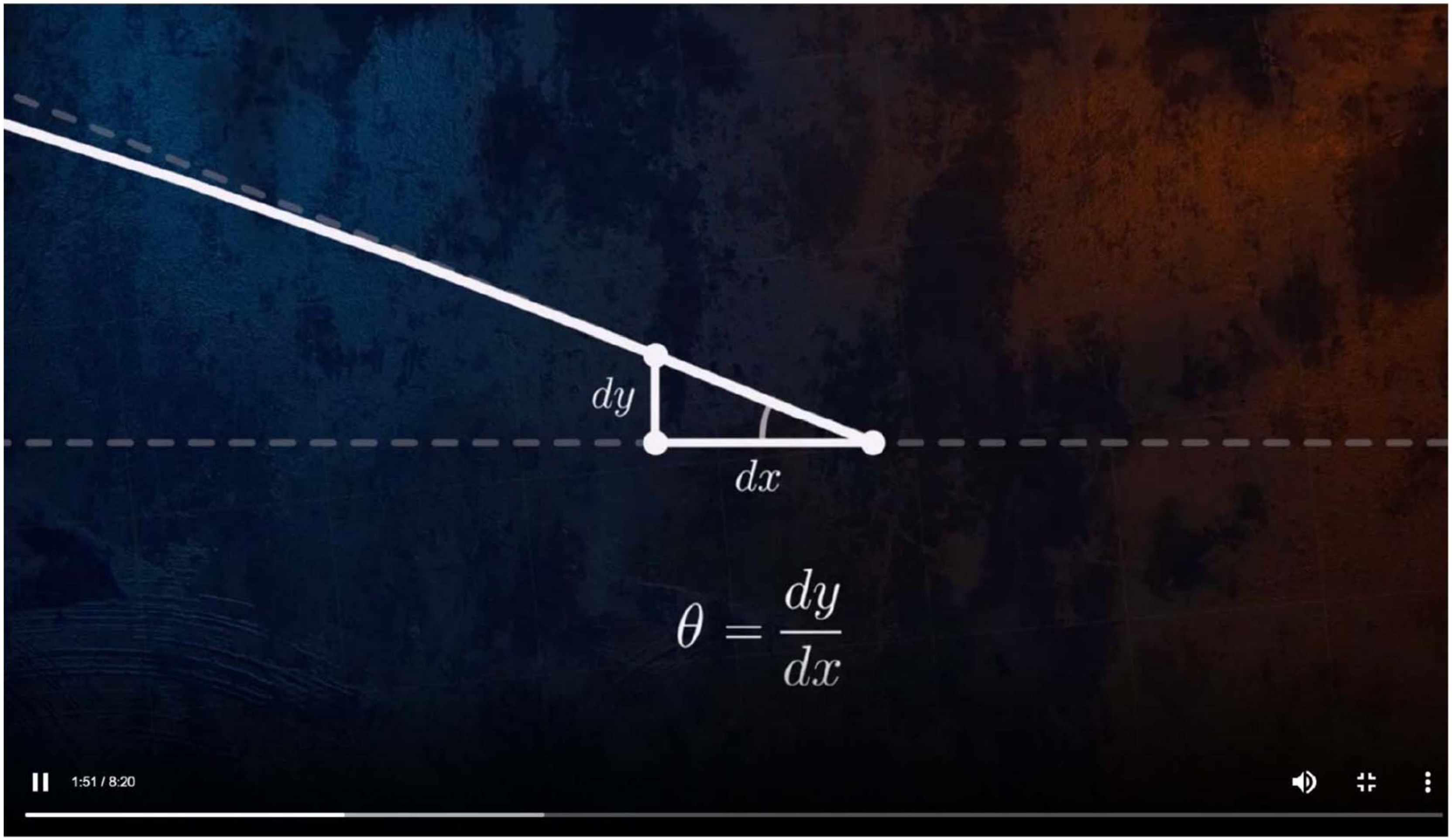
Pure Bending [1]



Beam Deflection [1]



Beam Deflection [1]



Deformation Under Transverse Loading 1

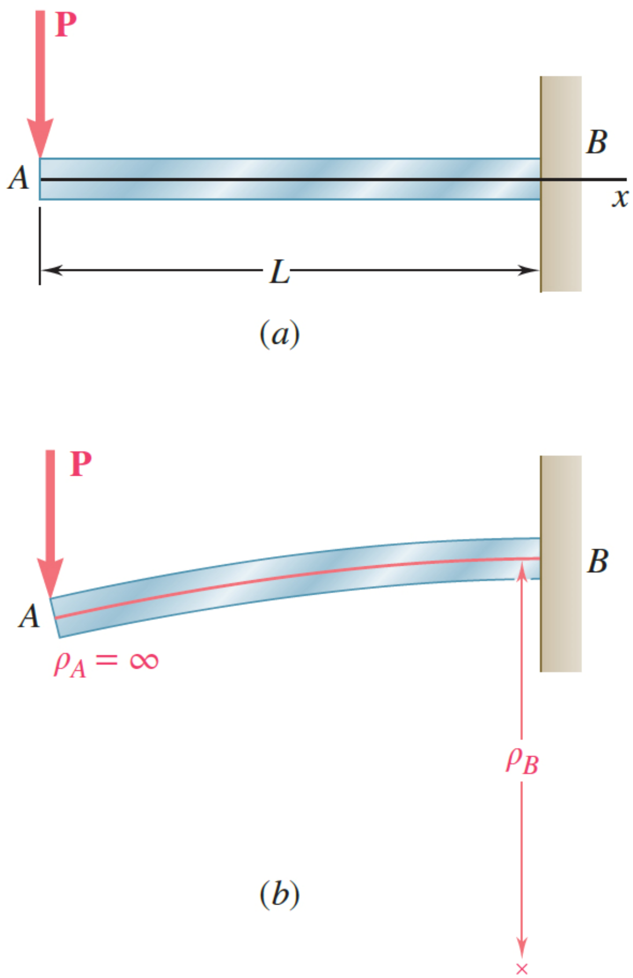


Figure 9.3 (a) Cantilever beam with concentrated load. (b) Deformed beam showing curvature at ends.

- Relationship between bending moment and curvature for pure bending remains valid for general transverse loadings.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Cantilever beam subjected to concentrated load **P** at the free end,

$$\frac{1}{\rho} = -\frac{Px}{EI}$$

- Curvature varies linearly with x
- At the free end A: $\frac{1}{\rho_A} = 0, \quad \rho_A = \infty$
- At the support B: $\frac{1}{\rho_B} \neq 0, \quad |\rho_B| = \frac{EI}{PL}$

Deformation Under Transverse Loading [2]

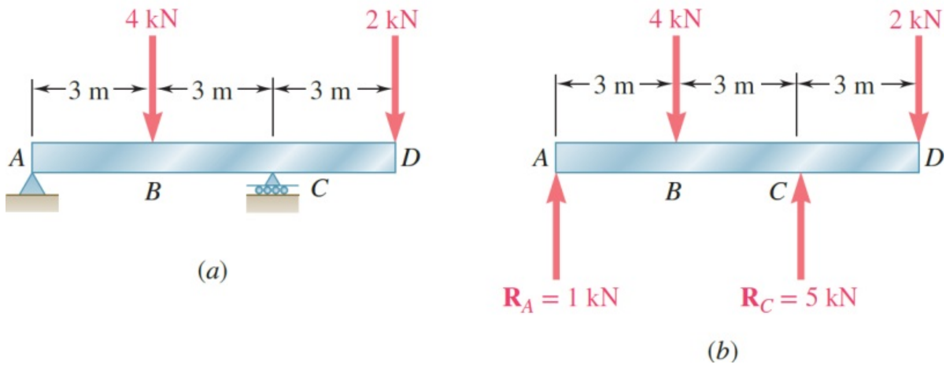


Figure 9.4 (a) Overhanging beam with two concentrated loads. (b) Free-body diagram showing reaction forces.

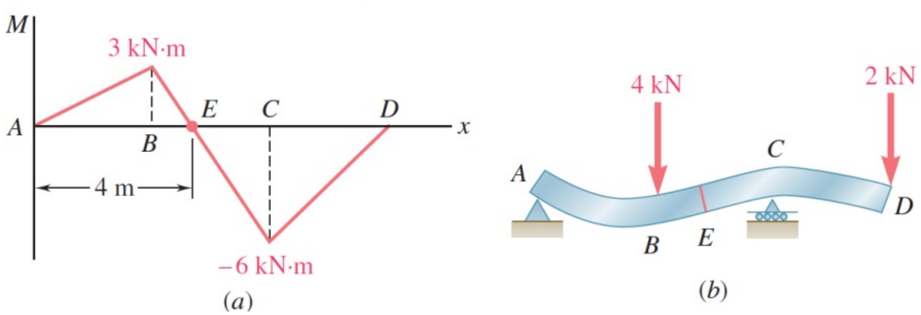


Figure 9.5 Beam of Fig. 9.4. (a) Bending-moment diagram. (b) Deformed shape.

- Overhanging beam.
- Reactions at A and C.
- Bending moment diagram.
- Curvature is zero at points where the bending moment is zero, i.e., at each end and at E.

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

- Beam is concave upwards where the bending moment is positive and concave downwards where it is negative.
- Maximum curvature occurs where the moment magnitude is a maximum.
- An equation for the beam shape or *elastic curve* is required to determine maximum deflection and slope.

Equation of the Elastic Curve 1

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

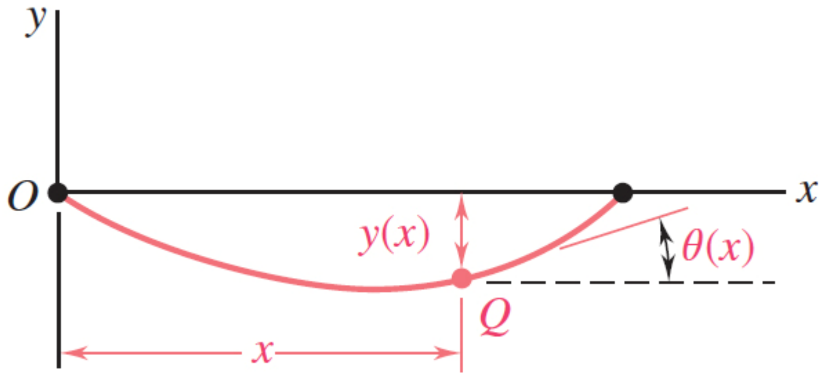


Figure 9.7 Slope *θ(x)* of tangent to the elastic curve.

$$\frac{dy}{dx} = \tan\theta \simeq \theta(x)$$

- From elementary calculus, simplified for beam parameters:

$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2}$$

- Substituting and integrating:

$$EI\frac{1}{\rho} = \boxed{EI\frac{d^2y}{dx^2} = M(x)}$$

$$EI\,\theta \approx EI\frac{dy}{dx} = \int_0^x M(x)\,dx + C_1$$

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1x + C_2$$

Equation of the Elastic Curve 2

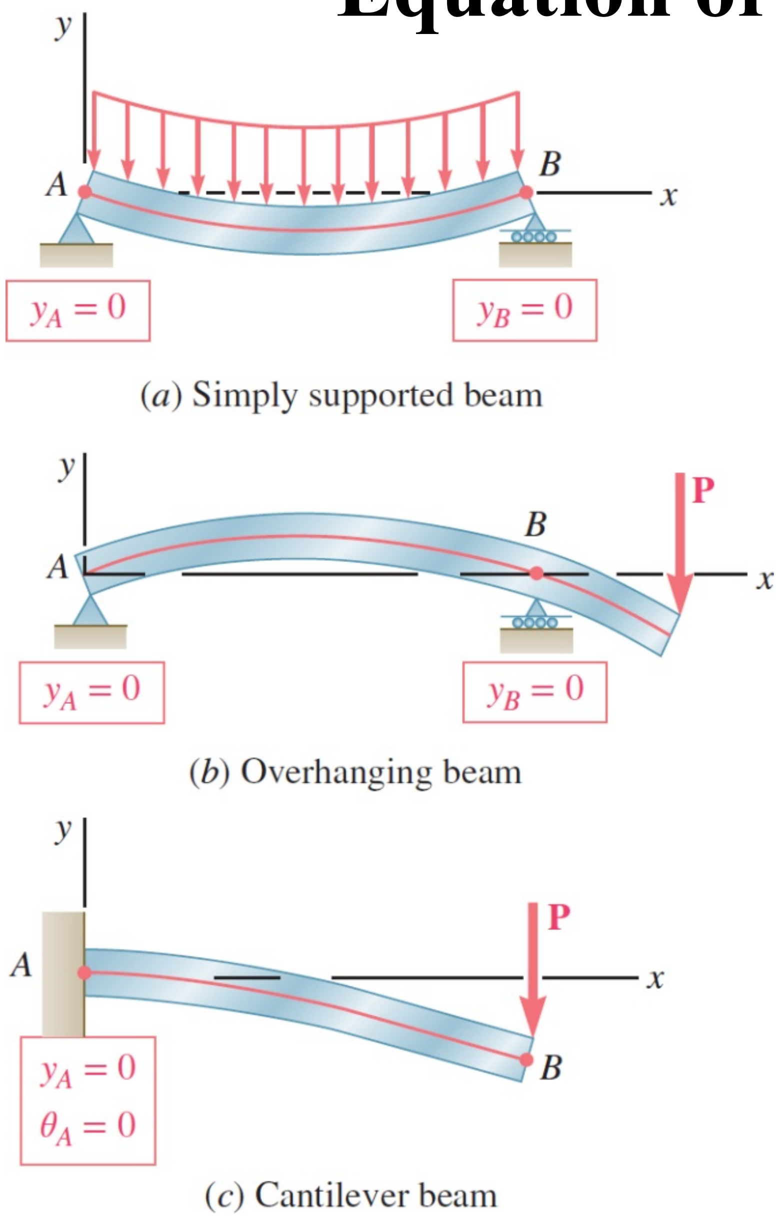


Figure 9.8 Known boundary conditions for statically determinate beams.

Constants are determined from boundary conditions:

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

Three cases for statically determinant beams,

- Simply supported beam

$$y_A = 0, \quad y_B = 0$$

- Overhanging beam

$$y_A = 0, \quad y_B = 0$$

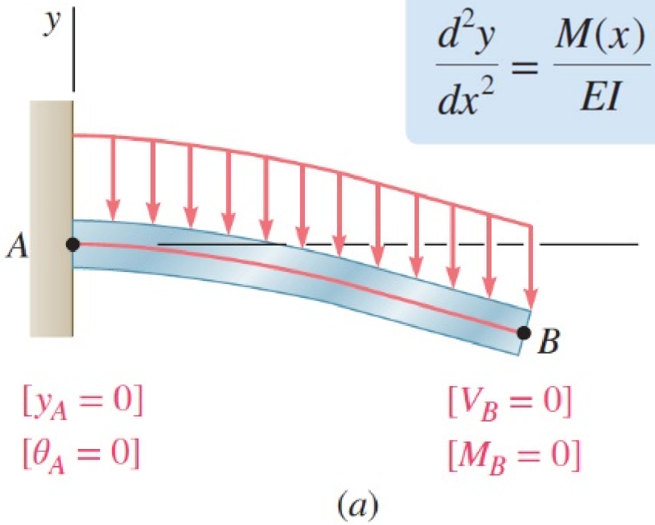
- Cantilever beam

$$y_A = 0, \quad \theta_A = 0$$

More complicated loadings require multiple integrals and application of requirement for continuity of displacement and slope.

Determination of the Elastic Curve from the Load Distribution

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$



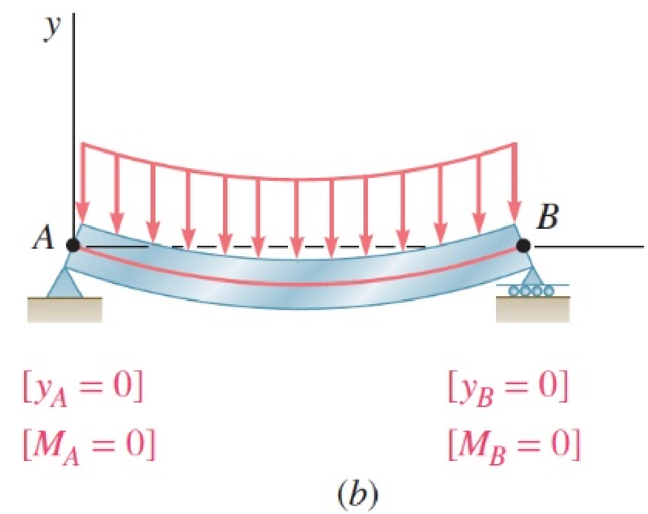


Figure 9.12 Boundary conditions for (a) cantilever beam (b) simply supported beam.

- For a beam subjected to a distributed load

$$\frac{dM}{dx} = V(x) \qquad \frac{d^2M}{dx^2} = \frac{dV}{dx} = -w(x)$$

- Equation for beam displacement becomes:

$$\frac{d^2M}{dx^2} = EI\frac{d^4y}{dx^4} = -w(x)$$

- Integrating four times yields:

$$EI\,y(x) = -\int dx\int dx\int dx\int w(x)\,dx$$
$$+\frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

- Constants are determined from boundary conditions.

Statically Indeterminate Beams 1

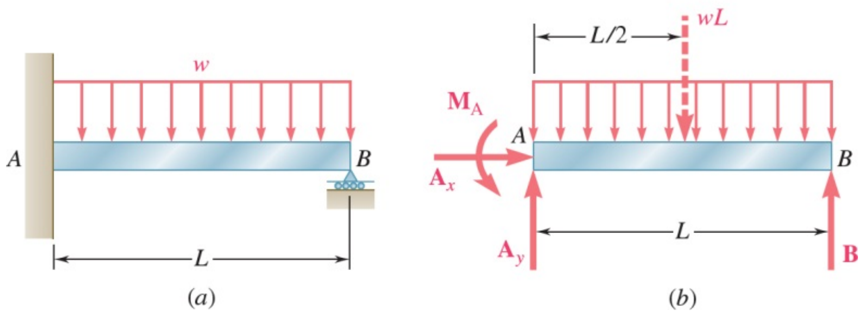


Figure 9.14 (a) Statically indeterminate beam with a uniformly distributed load. (b) Free-body diagram with four unknown

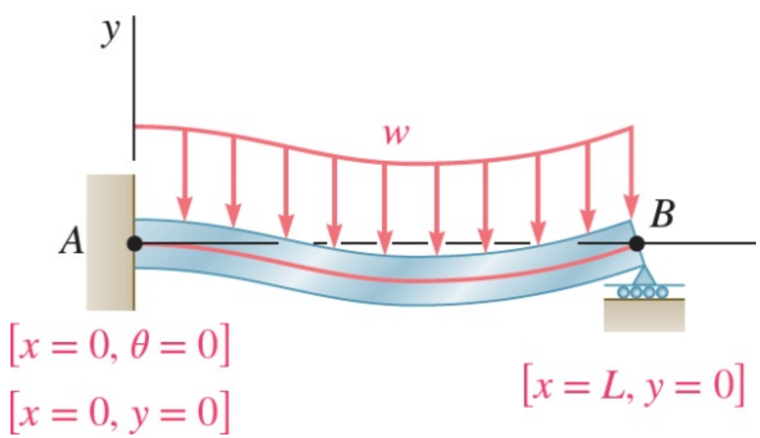


Figure 9.15 Boundary conditions for beam of Fig. 9.14.

- Consider beam with fixed support at *A* and roller support at *B*.
- From free-body diagram, note that there are four unknown reaction components.
- Conditions for static equilibrium yield:

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

The beam is statically indeterminate.

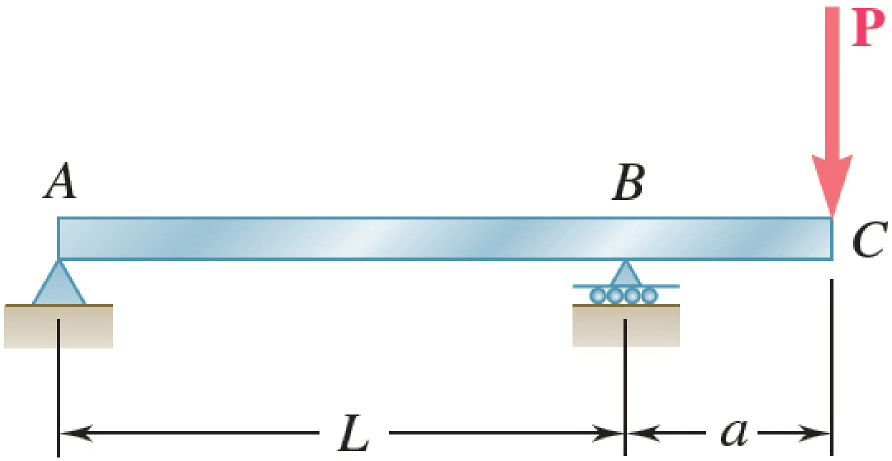
- Also have the beam deflection equation:

$$EI\, y = \int_0^x dx \int_0^x M(x)\, dx + C_1 x + C_2$$

which introduces two unknowns but provides three additional equations from the boundary conditions:

$$\text{At } x = 0,\ \theta = 0\ y = 0 \qquad \text{At } x = L,\ y = 0$$

Problem 9.1 [1]



$W14 \times 68 \quad I = 722\,\text{in}^4 \quad E = 29 \times 10^6\,\text{psi}$

$P = 50\,\text{kips} \quad L = 15\,\text{ft} \quad a = 4\ \text{ft}$

$L = 15\ \text{ft} = 180\ \text{in.} \qquad a = 4\ \text{ft} = 48\ \text{in.}$

For portion *AB* of the overhanging beam, (*a*) derive the equation for the elastic curve, (*b*) determine the maximum deflection, (*c*) evaluate y_{max}.

SOLUTION:

- Develop an expression for $M(x)$ and derive differential equation for elastic curve.
- Integrate differential equation twice and apply boundary conditions to obtain elastic curve.
- Locate point of zero slope or point of maximum deflection.
- Evaluate corresponding maximum deflection.

Problem 9.1 ₂

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

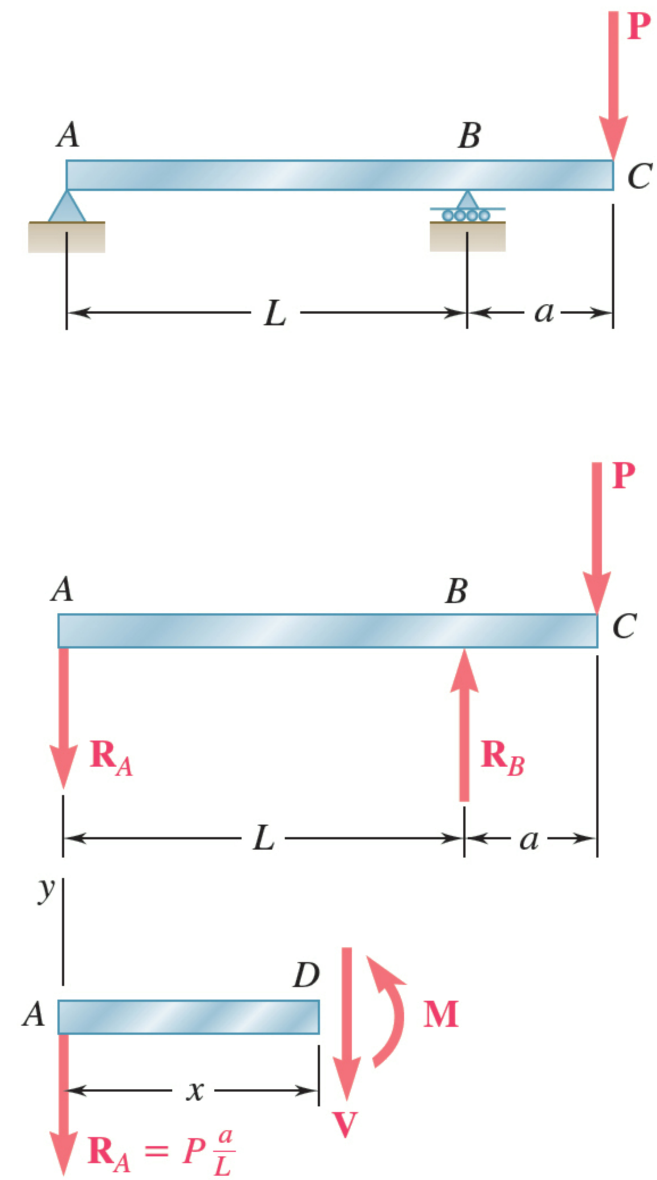


Figure 1 Free-body diagrams of beam and portion *AD*.

SOLUTION:

Develop an expression for $M(x)$ and derive differential equation for elastic curve.

- Reactions:

$$R_A = \frac{Pa}{L}\downarrow \quad R_B = P\left(1+\frac{a}{L}\right)\uparrow$$

- From the free-body diagram for section *AD*,

$$M = -P\frac{a}{L}x \quad (0 < x < L)$$

- The differential equation for the elastic curve,

$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

Problem 9.1 [3]

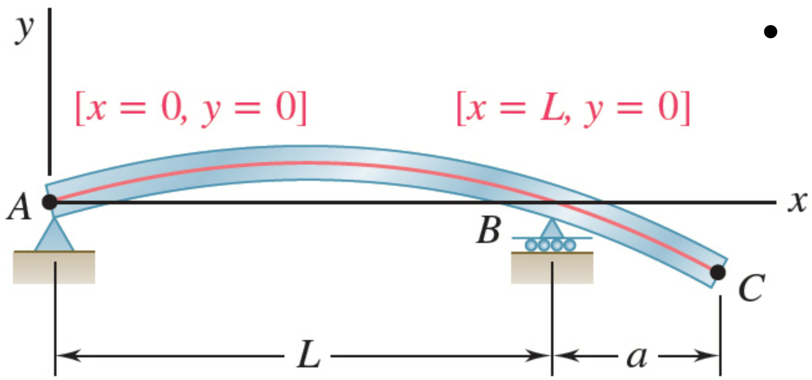


Figure 2 Boundary conditions.

$$\boxed{EI\frac{d^2y}{dx^2}=-P\frac{a}{L}x}$$

- Integrate differential equation twice and apply boundary conditions to obtain elastic curve.

$$EI\frac{dy}{dx}=-\frac{1}{2}P\frac{a}{L}x^2+C_1$$

$$EI\,y=-\frac{1}{6}P\frac{a}{L}x^3+C_1x+C_2$$

at $x=0,\ y=0:\quad C_2=0$

at $x=L,\ y=0:\quad EI(0)=-\frac{1}{6}P\frac{a}{L}L^3+C_1L \quad C_1=\frac{1}{6}PaL$

Substituting,

$$EI\frac{dy}{dx}=-\frac{1}{2}P\frac{a}{L}x^2+\frac{1}{6}PaL \qquad \frac{dy}{dx}=\frac{PaL}{6EI}\left[1-3\left(\frac{x}{L}\right)^2\right]$$

$$EI\,y=-\frac{1}{6}P\frac{a}{L}x^3+\frac{1}{6}PaLx$$

$$\boxed{y=\frac{PaL^2}{6EI}\left[\frac{x}{L}-\left(\frac{x}{L}\right)^3\right]}$$

Problem 9.1 [4]

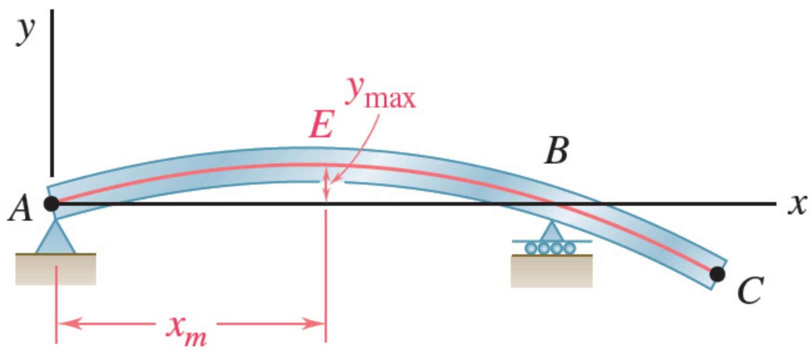


Figure 3 Deformed elastic curve with location of maximum deflection.

$$y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

- Locate point of zero slope or point of maximum deflection:

$$\frac{dy}{dx} = 0 = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x_m}{L}\right)^2\right] \qquad x_m = \frac{L}{\sqrt{3}} = 0.577L$$

- Evaluate corresponding maximum deflection.

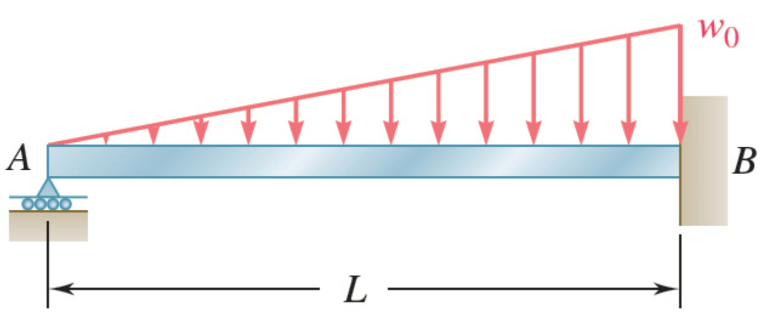
$$y_{max} = \frac{PaL^2}{6EI}\left[0.577 - (0.577)^3\right]$$

$$\boxed{y_{max} = 0.0642\frac{PaL^2}{6EI}}$$

$$y_{max} = 0.0642\frac{(50\,\text{kips})(48\,\text{in})(180\,\text{in})^2}{6\left(29\times10^6\,\text{psi}\right)\left(723\,\text{in}^4\right)}$$

$$\boxed{y_{max} = 0.238\,\text{in}}$$

Problem 9.3 1



For the uniform beam, determine the reaction at A, derive the equation for the elastic curve, and determine the slope at A. (Note that the beam is statically indeterminate to the first degree)

SOLUTION:

- Develop the differential equation for the elastic curve (will be functionally dependent on the reaction at A).
- Integrate twice and apply boundary conditions to solve for reaction at A and to obtain the elastic curve.
- Evaluate the slope at A.

Problem 9.3 $_2$

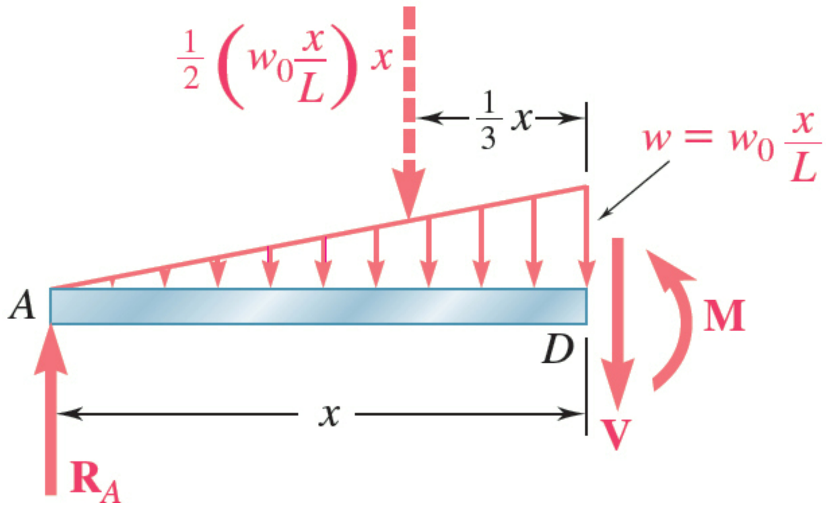
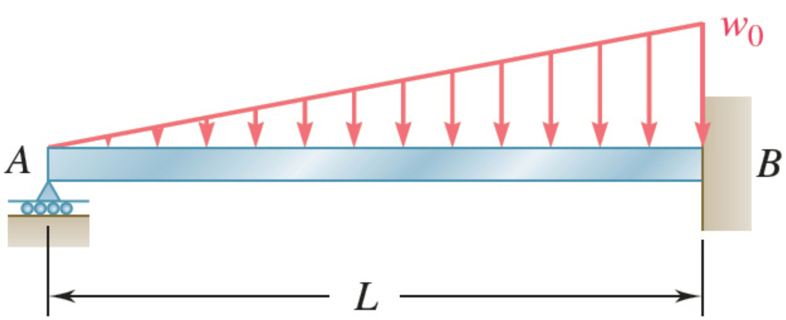


Figure 1 Free-body diagram of portion *AD* of beam.



- Consider moment acting at section *D*,

$$\sum M_D = 0$$

$$R_A x - \frac{1}{2}\left(\frac{w_0 x^2}{L}\right)\frac{x}{3} - M = 0$$

$$M = R_A x - \frac{w_0 x^3}{6L}$$

- The differential equation for the elastic curve

$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{w_0 x^3}{6L}$$

Problem 9.3 [3]

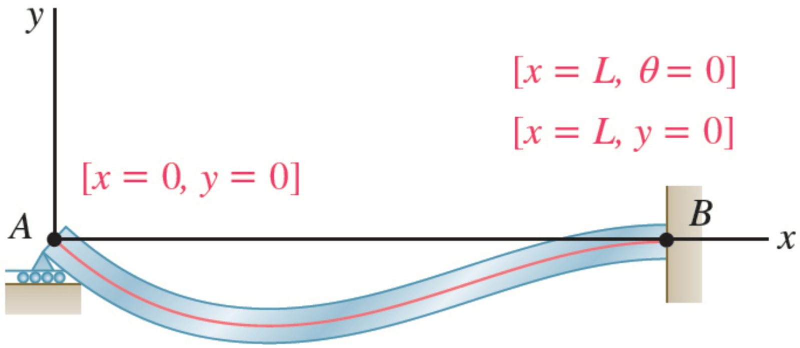


Figure 2 Boundary conditions.

$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{w_0 x^3}{6L}$$

$$C_1 = -\tfrac{1}{120} w_0 L^3$$

- Integrate twice:

$$EI\frac{dy}{dx} = EI\theta = \frac{1}{2}R_A x^2 - \frac{w_0 x^4}{24L} + C_1$$

$$EI\, y = \frac{1}{6}R_A x^3 - \frac{w_0 x^5}{120L} + C_1 x + C_2$$

- Apply boundary conditions:

at $x = 0,\ y = 0:\quad C_2 = 0$

at $x = L,\ \theta = 0:\quad \frac{1}{2}R_A L^2 - \frac{w_0 L^3}{24} + C_1 = 0 \qquad L$

at $x = L,\ y = 0:\quad \frac{1}{6}R_A L^3 - \frac{w_0 L^4}{120} + C_1 L + \cancel{C_2} = 0$

- Solve for reaction at A

$$\frac{1}{3}R_A L^3 - \frac{1}{30}w_0 L^4 = 0 \qquad \boxed{R_A = \frac{1}{10}w_0 L \uparrow}$$

Problem 9.3 [4]

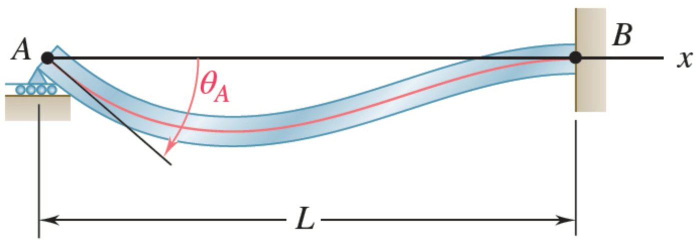


Figure 3 Deformed elastic curve showing slope at A.

- Substitute for C_1, C_2, and R_A in the elastic curve equation,

$$EI\,y = \frac{1}{6}\left(\frac{1}{10}w_0 L\right)x^3 - \frac{w_0 x^5}{120L} - \left(\frac{1}{120}w_0 L^3\right)x$$

$$\boxed{y = \frac{w_0}{120EIL}\left(-x^5 + 2L^2x^3 - L^4x\right)}$$

- Differentiate once to find the slope:

$$\theta = \frac{dy}{dx} = \frac{w_0}{120EIL}\left(-5x^4 + 6L^2x^2 - L^4\right)$$

at $x = 0$, $\theta_A = -\dfrac{w_0 L^3}{120EI}$

Method of Superposition

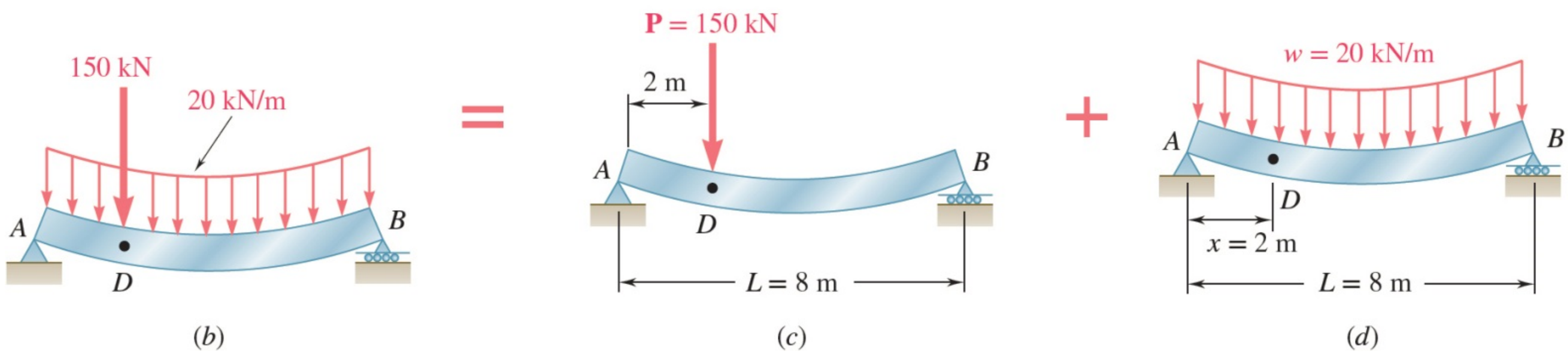
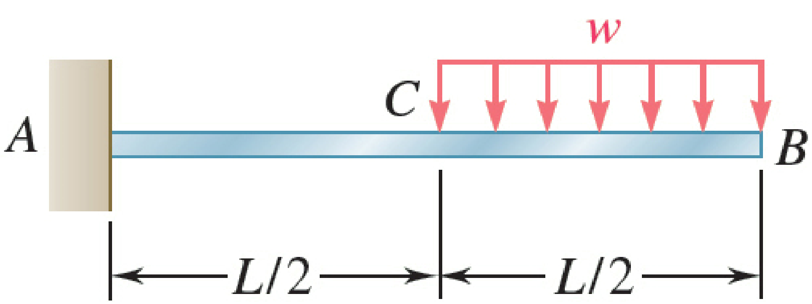


Figure 9.21b-d (b) The beam's loading can be obtained by superposing deflections due to (c) the concentrated load and (d) the distributed load.

Principle of Superposition:

- Deformations of beams subjected to combinations of loadings may be obtained as the linear combination of the deformations from the individual loadings
- Procedure is facilitated by tables of solutions for common types of loadings and supports.

Problem 9.7 [1]



For the beam and loading shown, determine the slope and deflection at point ***B***.

SOLUTION:

Superpose the deformations due to *Loading I* and *Loading II* as shown.

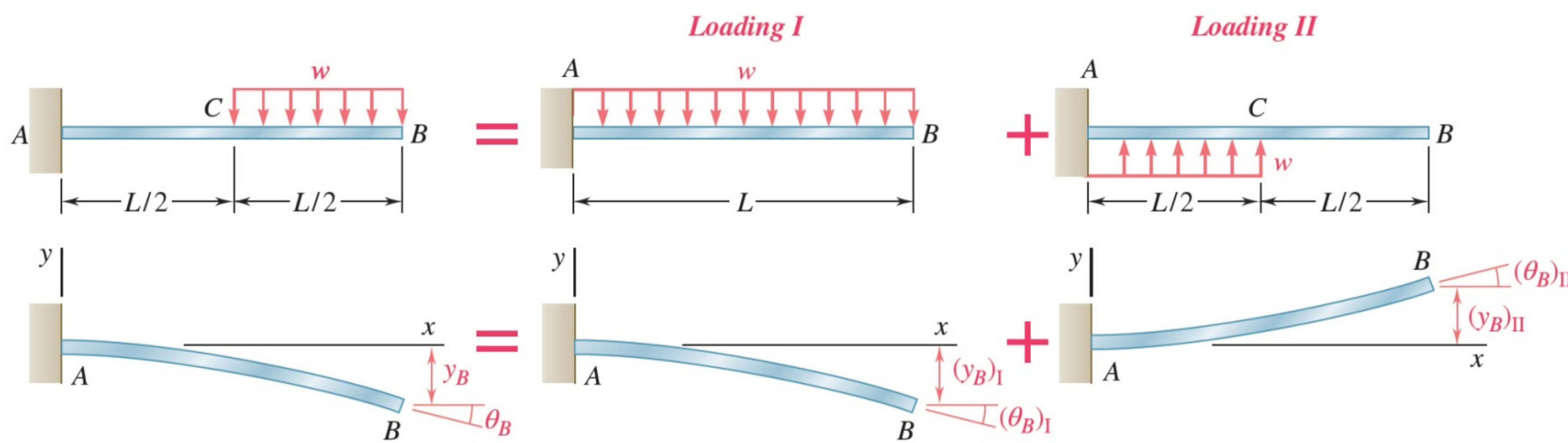


Figure 1 Actual loading is equivalent to the superposition of two distributed loads.

Problem 9.7

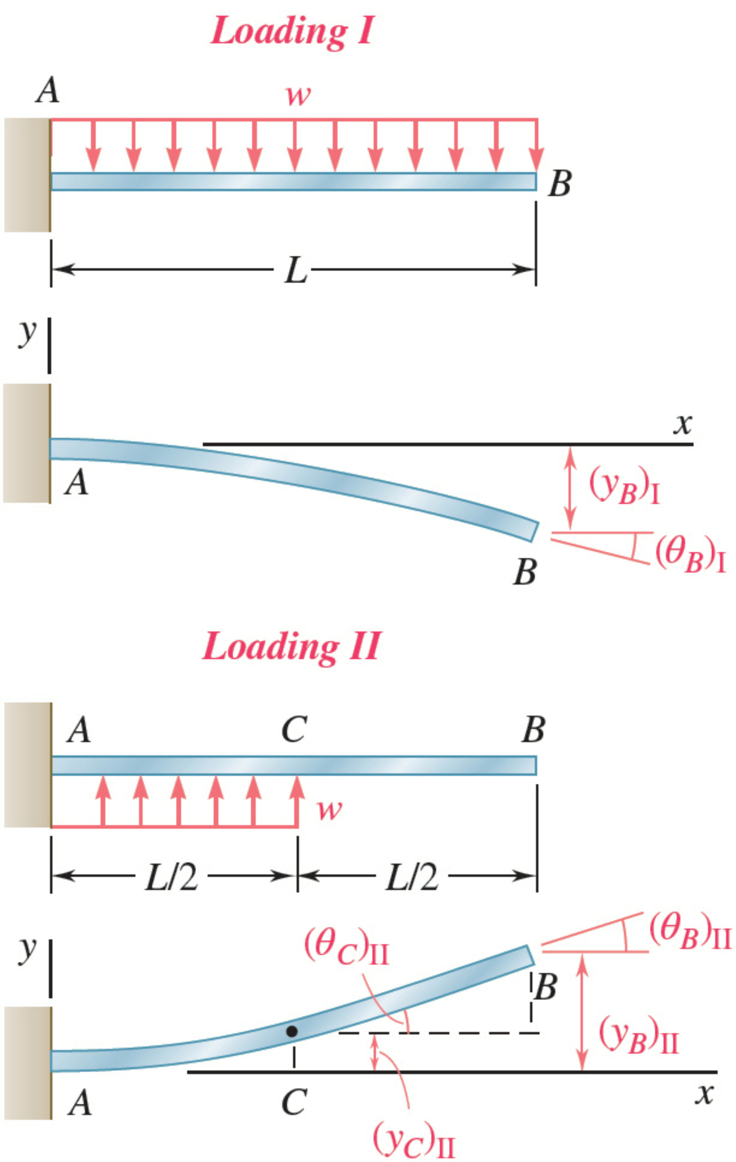


Figure 2 Deformation details of the superposed loadings I and II.

Loading I

$$\left(\theta_B\right)_I = -\frac{wL^3}{6EI} \qquad \left(y_B\right)_I = -\frac{wL^4}{8EI}$$

Loading II *(L => L/2)*

$$\left(\theta_C\right)_{II} = \frac{wL^3}{48EI} \qquad \left(y_C\right)_{II} = \frac{wL^4}{128EI}$$

In beam segment *CB*, the bending moment is zero and the elastic curve is a straight line.

$$\left(\theta_B\right)_{II} = \left(\theta_C\right)_{II} = \frac{wL^3}{48EI}$$

$$\left(y_B\right)_{II} = \frac{wL^4}{128EI} + \frac{wL^3}{48EI}\left(\frac{L}{2}\right) = \frac{7wL^4}{384EI}$$

Problem 9.7 [3]

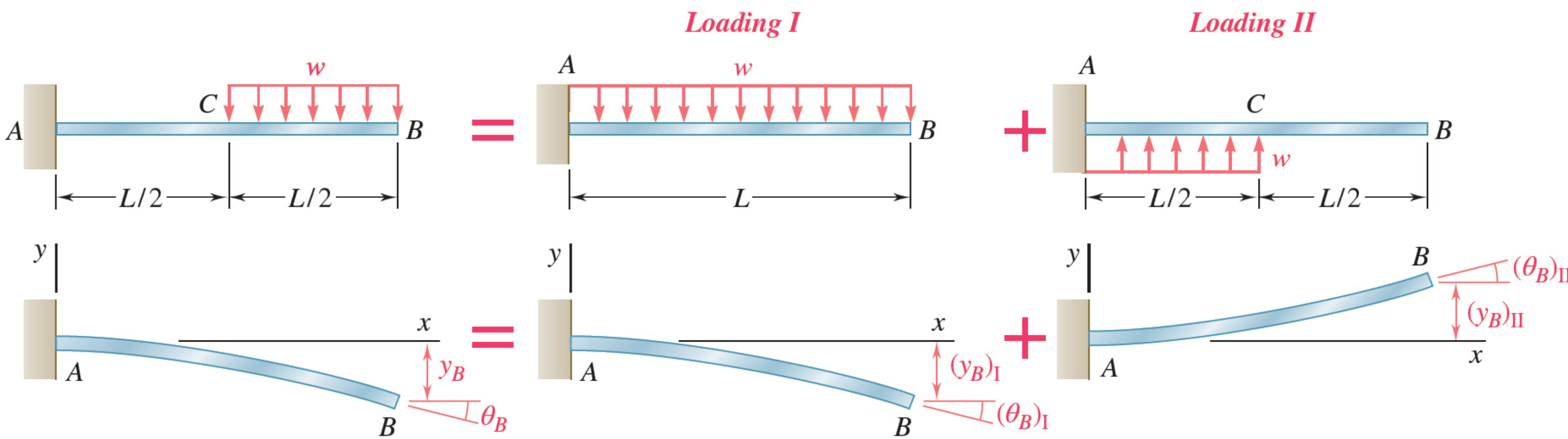


Figure 1 Actual loading is equivalent to the superposition of two distributed loads.

Combine the two solutions,

$$\theta_B = (\theta_B)_I + (\theta_B)_{II} = -\frac{wL^3}{6EI} + \frac{wL^3}{48EI} \qquad \boxed{\theta_B = -\frac{7wL^3}{48EI}}$$

$$y_B = (y_B)_I + (y_B)_{II} = -\frac{wL^4}{8EI} + \frac{7wL^4}{384EI} \qquad \boxed{y_B = -\frac{41wL^4}{384EI}}$$

Statically Indeterminate Beams 2

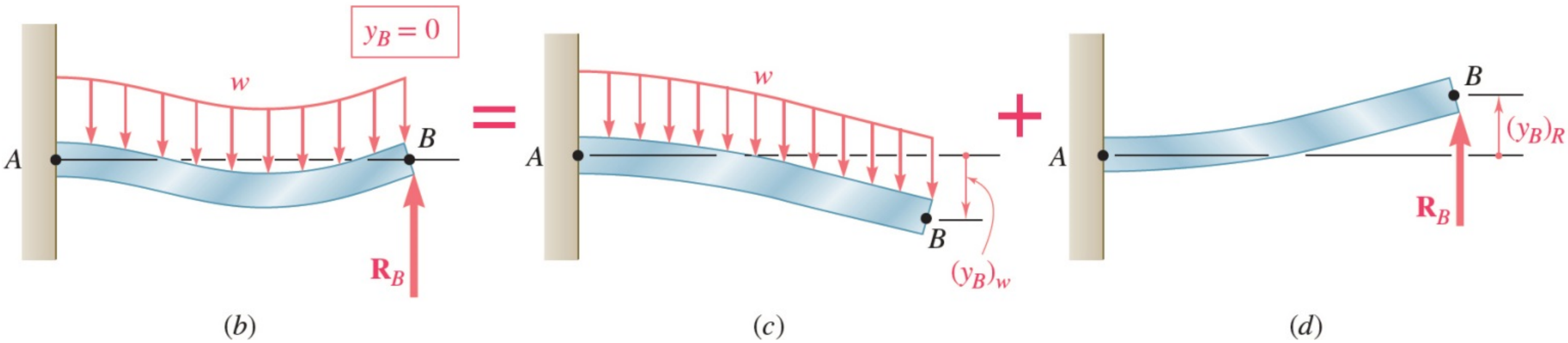
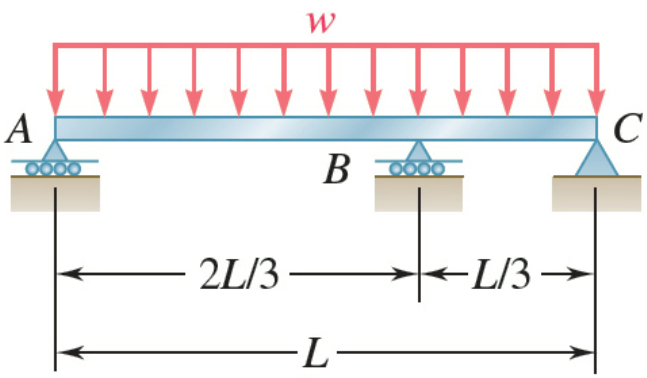


Figure 9.22 (b) Analyze the indeterminate beam by superposing two determinate cantilever beams, subjected to (c) a uniformly distributed load, (d) the redundant reaction.

- Method of superposition may be applied to determine the reactions at the supports of statically indeterminate beams.
- Designate one of the reactions as redundant and eliminate or modify the support.
- Determine the beam deformation without the redundant support.
- Treat the redundant reaction as an unknown load which, together with the other loads, must produce deformations compatible with the original supports.

Problem 9.8 [1]



For the uniform beam and loading shown, determine the reaction at each support and the slope at end A.

SOLUTION:

- Release the "redundant" support at B, and find deformation.
- Apply reaction at B as an unknown load to force zero displacement at B.

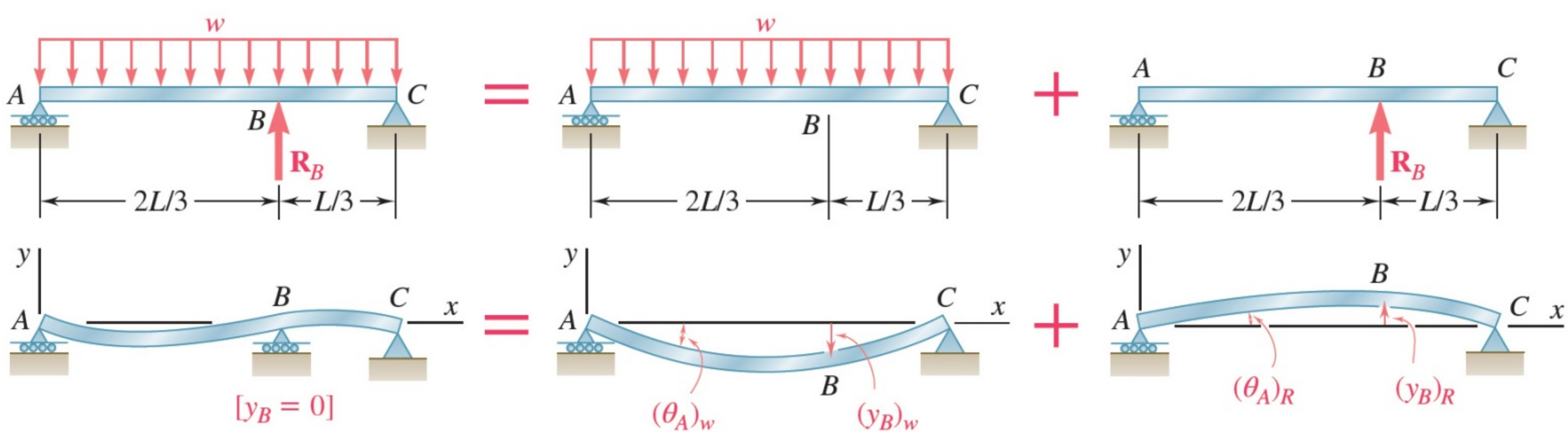
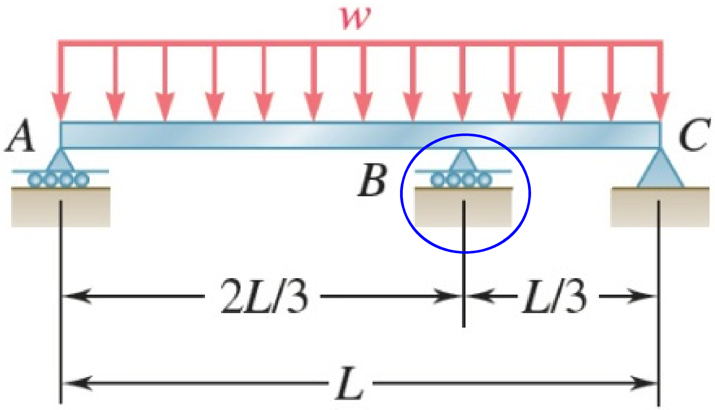
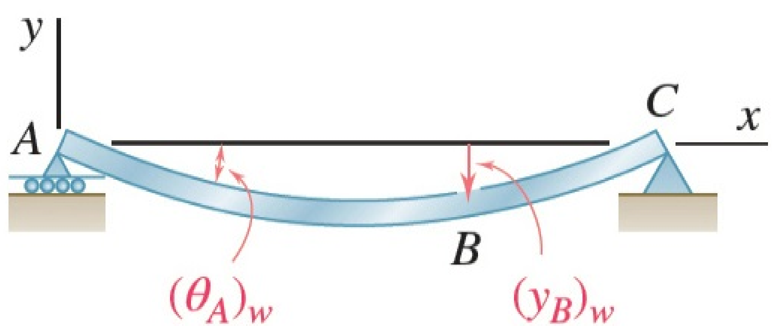


Figure 1 Indeterminate beam modeled as superposition of two determinate simply supported beams with reaction at B chosen redundant.

Problem 9.8 $_2$





- Distributed Loading:

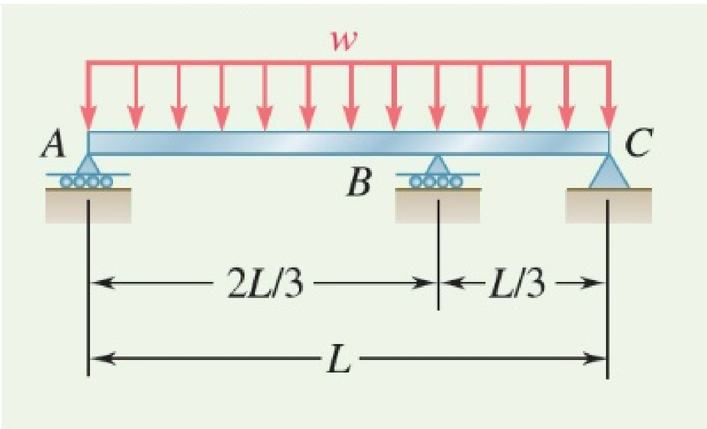
$$\left(y_B\right)_w = -\frac{w}{24EI}\left[x^4 - 2Lx^3 + L^3 x\right]$$

At point B, $x = \frac{2}{3}L$

$$\left(y_B\right)_w = -\frac{w}{24EI}\left[\left(\frac{2}{3}L\right)^4 - 2L\left(\frac{2}{3}L\right)^3 + L^3\left(\frac{2}{3}L\right)\right]$$

$$= -0.01132\frac{wL^4}{EI}$$

Problem 9.8 3



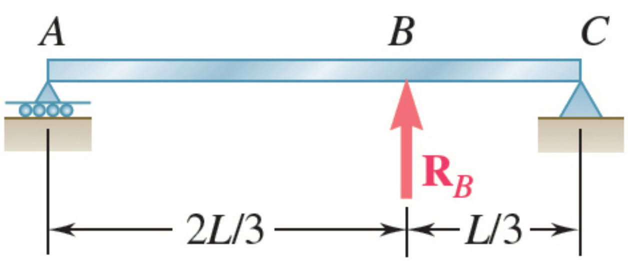
- Redundant Reaction Loading:

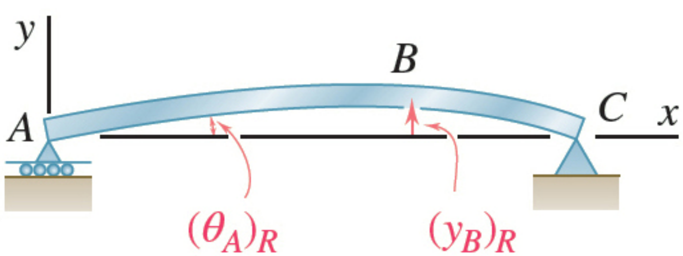
Case 6, Appendix F:

$$y = -\frac{w}{24EI}\left(x^4 - 2Lx^3 + L^3x\right)$$

$$\text{At } x = \frac{2}{3}L,\ \left(y_B\right)_w = \frac{w}{24EI}\left[\left(\frac{2}{3}L\right)^4 - 2L\left(\frac{2}{3L}\right)^3 + L^3\left(\frac{2}{3}L\right)\right]$$

$$= -0.01132\ \frac{wL^4}{EI}$$





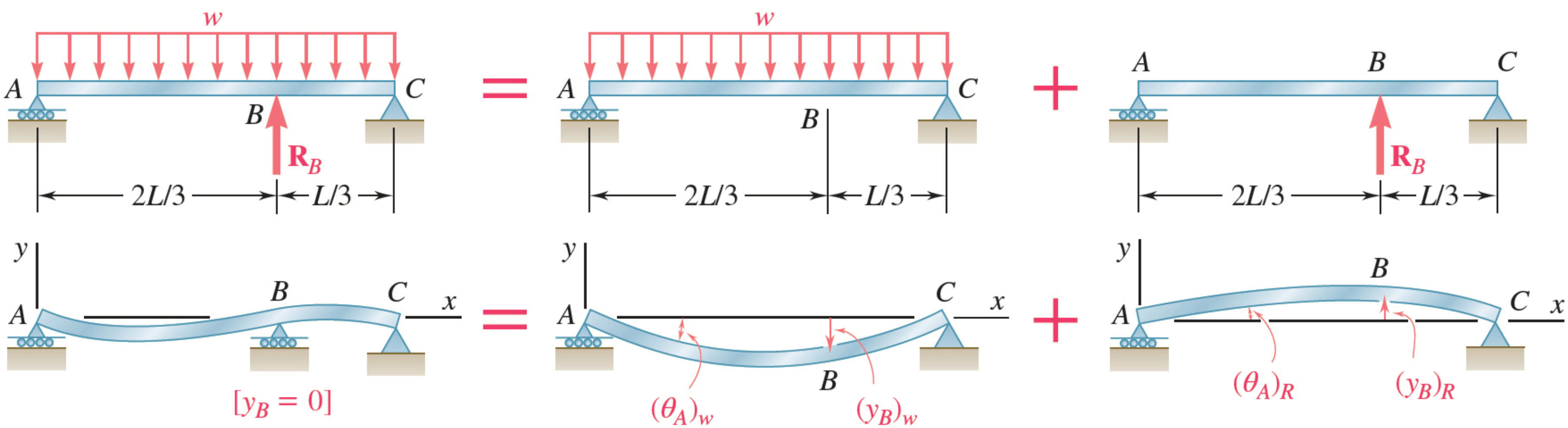
For $a = \frac{2}{3}L$ and $b = \frac{1}{3}L$

$$(y_B)_R = -\frac{Pa^2b^2}{3EIL}$$

$$\left(y_B\right)_R = \frac{R_B}{3EIL}\left(\frac{2}{3}L\right)^2\left(\frac{L}{3}\right)^2$$

$$= 0.01646\frac{R_BL^3}{EI}$$

Problem 9.8 4



- For compatibility with original supports, $y_B = 0$

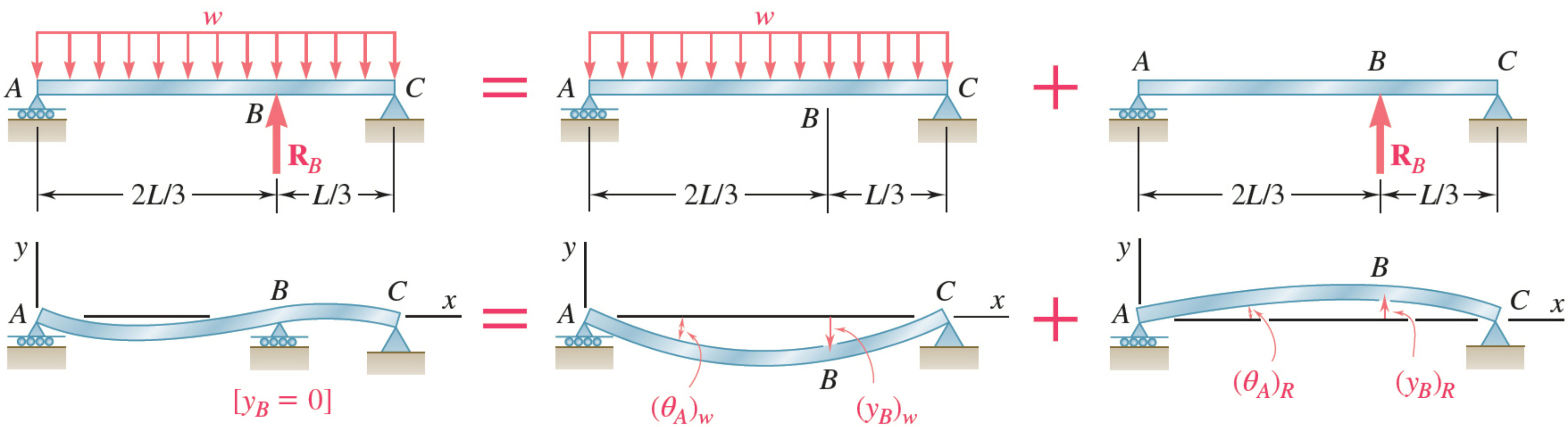
$$0 = (y_B)_w + (y_B)_R = -0.01132\frac{wL^4}{EI} + 0.01646\frac{R_B L^3}{EI}$$

$$\boxed{R_B = 0.688wL \uparrow}$$

- From statics,

$$\boxed{R_A = 0.271wL \uparrow \qquad R_C = 0.0413wL \uparrow}$$

Problem 9.8 5



Slope at end A (Appx F),

$$\left(\theta_A\right)_w = -\frac{wL^3}{24EI} = -0.04167\frac{wL^3}{EI}$$

$$\mathbf{R}_B = 0.688wL \uparrow$$

$$\left(\theta_A\right)_R = -\frac{Pb\left(L^2 - b^2\right)}{6EIL} = \frac{0.0688wL}{6EIL}\left(\frac{L}{3}\right)\left[L^2 - \left(\frac{L}{3}\right)^2\right] = 0.03398\frac{wL^3}{EI}$$

$$\theta_A = \left(\theta_A\right)_w + \left(\theta_A\right)_R = -0.04167\frac{wL^3}{EI} + 0.03398\frac{wL^3}{EI}$$

$$\boxed{\theta_A = -0.00769\frac{wL^3}{EI}}$$