Correction for paper *Cooper, 1967*

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1 Introduction

This serves as an appendix of paper [2]. It aims at pointing out some formulation errors in paper [1] by Cooper in 1967, as reported in [2]. Proof will be given in the following. Equations from [1] with the same sequence number are first listed and the corrected ones are given and highlighted alongside.

2 Equations

Equation of motion:

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \rho\omega^2\mathbf{u} = 0 \tag{1}$$

Constitutive equation:

$$\tau_{ij} = \lambda\delta_{ij}\nabla\cdot\mathbf{u} + \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i) \tag{2}$$

Displacement components by eight potential functions Φ_{lmn}:

$$\begin{aligned} u_{lm} &= \partial\Phi_{lm1}/\partial x + \partial\Phi_{lm2}/\partial y \\ v_{lm} &= \partial\Phi_{lm1}/\partial y - \partial\Phi_{lm2}/\partial x \end{aligned} \tag{3}$$

Helmholtz equation:

$$\Delta\Phi_{lmn} + k_{mn}^2\Phi_{lmn} = 0, \quad l,m,n = 1,2 \tag{4}$$

where

$$k_{mn} \equiv \omega/S_{mn}; \quad S_{m1}^2 = (\lambda_m + 2\mu_m)/\rho_m; \quad S_{m2}^2 = \mu_m/\rho_m \tag{5}$$

Complex wave number:

$$\begin{aligned} k_{mn} &= (\omega/C_{mn})(1 + i\,\tan\Omega_{mn}) \\ &= (\omega/C_{mn})(1 - i\,\tan\Omega_{mn}) \end{aligned} \tag{6}$$

where

$$\begin{aligned} \tan 2\Omega_{mn} &\equiv -ImS_{mn}^2/ReS_{mn}^2, \quad 0 \le \Omega_{mn} < \pi/2 \\ &\equiv ImS_{mn}^2/ReS_{mn}^2, \quad 0 \le \Omega_{mn} < \pi/2 \\ C_{mn} &\equiv |S_{mn}|\sec\Omega_{mn} \end{aligned} \tag{7}$$

Combining Equations 2, 3 and 4:

$$\begin{aligned} \sigma_{xlm} &= 2\mu_m(\partial^2\Phi_{lm1}/\partial x^2 + \partial^2\Phi_{lm2}/\partial x\partial y) - \lambda_m k_{m1}^2\Phi_{lm1} \\ \sigma_{ylm} &= 2\mu_m(\partial^2\Phi_{lm1}/\partial y^2 - \partial^2\Phi_{lm2}/\partial x\partial y) - \lambda_m k_{m1}^2\Phi_{lm1} \\ \tau_{lm} &= \mu_m(2\partial^2\Phi_{lm1}/\partial x\partial y + \partial^2\Phi_{lm2}/\partial y^2 - \partial^2\Phi_{lm2}/\partial x^2) \end{aligned} \tag{8}$$

Boundary conditions at interface $y = 0$:

$$\begin{aligned} u_{l1}(x,0) &= u_{l2}(x,0), \qquad v_{l1}(x,0) = v_{l2}(x,0) \\ \sigma_{yl1}(x,0) &= \sigma_{yl2}(x,0), \qquad \tau_{l1}(x,0) = \tau_{l2}(x,0) \end{aligned} \tag{9}$$

Total potential functions:

$$\Phi_{l1n} = \delta_{ln}\Psi_l + \psi_{l1n}; \quad \Phi_{l2n} = \psi_{l2n} \tag{10}$$

Potential functions:

$$\begin{aligned} \Psi_l &= (I_l S_{l1}/\omega)\exp\{ik_{1l}(\mathbf{r}_l\cdot\mathbf{x})\} \\ &= (I_l S_{1l}/\omega)\exp\{ik_{1l}(\mathbf{r}_l\cdot\mathbf{x})\} \\ \psi_{lmn} &= (I_l R_{lmn} S_{mn}/\omega)\exp\{ik_{mn}(\mathbf{r}_{lmn}\cdot\mathbf{x})\} \end{aligned} \tag{11}$$

where

$$\mathbf{r}_l \equiv \mathbf{i}\,\sin\theta_l - \mathbf{j}\,\cos\theta_l, \qquad \mathbf{x} \equiv \mathbf{i}\,x + \mathbf{j}\,y,$$
$$\mathbf{r}_{lmn} \equiv \mathbf{i}\,\sin\zeta_{lmn} + \mathbf{j}\,\epsilon_m\cos\zeta_{lmn}, \qquad \epsilon_m \equiv (-1)^{m+1} \tag{12}$$

Combing Equations 3, 8, 10, 11 and 12:

$$\begin{Bmatrix} u_{lm} \\ v_{lm} \\ \sigma_{xlm} \\ \sigma_{ylm} \\ \tau_{lm} \end{Bmatrix} = \begin{bmatrix} ik_{m1}\sin\zeta_{lm1} & ik_{m2}\epsilon_m\cos\zeta_{lm2} \\ \epsilon_m ik_{m1}\cos\zeta_{lm1} & -ik_{m2}\sin\zeta_{lm2} \\ -k_{m1}^2(\lambda_m + 2\mu_m\sin^2\zeta_{lm1}) & -\mu_m k_{m2}^2\epsilon_m\sin 2\zeta_{lm2} \\ -k_{m1}^2(\lambda_m + 2\mu_m\cos^2\zeta_{lm1}) & \mu_m k_{m2}^2\epsilon_m\sin 2\zeta_{lm2} \\ -\epsilon_m\mu_m k_{m1}^2\sin 2\zeta_{lm1} & -\mu_m k_{m2}^2\cos 2\zeta_{lm2} \end{bmatrix} \begin{Bmatrix} \psi_{lm1} \\ \psi_{lm2} \end{Bmatrix}$$
$$+\,\delta_{m1}\Psi_l \begin{Bmatrix} i(\delta_{l1}k_{11}\sin\theta_1 - \delta_{l2}k_{12}\cos\theta_2) \\ -i(\delta_{l1}k_{11}\cos\theta_1 + \delta_{l2}k_{12}\sin\theta_2) \\ -\delta_{l1}k_{11}^2(\lambda_1 + 2\mu_1\sin^2\theta_1) + \delta_{l2}\mu_1 k_{12}^2\sin 2\theta_2 \\ -\delta_{l1}k_{11}^2(\lambda_1 + 2\mu_1\cos^2\theta_1) - \delta_{l2}\mu_1 k_{12}^2\sin 2\theta_2 \\ \delta_{l1}\mu_1 k_{11}^2\sin 2\theta_1 - \delta_{l2}\mu_1 k_{12}^2\cos 2\theta_2 \end{Bmatrix} \tag{13}$$

Complex Snell's law:

$$\sin\zeta_{lmn} = S_{mn}\sin\theta_l / S_{1l} \tag{14}$$

The linear set of equations for coefficients:

$$\mathbf{A}_l\mathbf{R}_l = \mathbf{B}_l, \quad l = 1, 2 \tag{15}$$

$$\mathbf{A}_l = \begin{bmatrix} \sin\zeta_{l11} & \cos\zeta_{l12} & -\sin\zeta_{l21} & \cos\zeta_{l22} \\ \cos\zeta_{l11} & -\sin\zeta_{l12} & \cos\zeta_{l21} & \sin\zeta_{l22} \\ -\rho_1 S_{11}\cos\zeta_{l12} & \rho_1 S_{12}\sin 2\zeta_{l12} & \rho_2 S_{21}\cos 2\zeta_{l22} & \rho_2 S_{22}\sin 2\zeta_{l22} \\ (\rho_1 S_{12}^2/S_{11})\sin 2\zeta_{l11} & \rho_1 S_{12}\cos 2\zeta_{l12} & (\rho_2 S_{22}^2/S_{21})\sin 2\zeta_{l21} & -\rho_2 S_{22}\cos 2\zeta_{l22} \end{bmatrix}$$
$$= \begin{bmatrix} \sin\zeta_{l11} & \cos\zeta_{l12} & -\sin\zeta_{l21} & \cos\zeta_{l22} \\ \cos\zeta_{l11} & -\sin\zeta_{l12} & \cos\zeta_{l21} & \sin\zeta_{l22} \\ -\rho_1 S_{11}\cos 2\zeta_{l12} & \rho_1 S_{12}\sin 2\zeta_{l12} & \rho_2 S_{21}\cos 2\zeta_{l22} & \rho_2 S_{22}\sin 2\zeta_{l22} \\ (\rho_1 S_{12}^2/S_{11})\sin 2\zeta_{l11} & \rho_1 S_{12}\cos 2\zeta_{l12} & (\rho_2 S_{22}^2/S_{21})\sin 2\zeta_{l21} & -\rho_2 S_{22}\cos 2\zeta_{l22} \end{bmatrix} \tag{16}$$

$$\mathbf{B}_1 = \begin{Bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ \rho_1 S_{11}\cos 2\zeta_{112} \\ \rho_1 S_{12}^2\sin 2\theta_1 \end{Bmatrix}, \quad \mathbf{B}_2 = \begin{Bmatrix} \cos\theta_2 \\ \sin\theta_2 \\ -\rho_1 S_{12}\sin 2\theta_2 \\ -\rho_1 S_{12}\cos 2\theta_2 \end{Bmatrix}, \quad \mathbf{R}_l = \begin{Bmatrix} R_{l11} \\ R_{l12} \\ R_{l21} \\ R_{l22} \end{Bmatrix}$$
$$= \begin{Bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ \rho_1 S_{11}\cos 2\zeta_{112} \\ (\rho_1 S_{12}^2/S_{11})\sin 2\theta_1 \end{Bmatrix} \qquad = \begin{Bmatrix} \cos\theta_2 \\ \sin\theta_2 \\ +\rho_1 S_{12}\sin 2\theta_2 \\ -\rho_1 S_{12}\cos 2\theta_2 \end{Bmatrix} \tag{17}$$

3 Proofs

3.1 Eqs. 6 & 7

For Eqs. 6 & 7, it's hard to find the original source from which the expression in Cooper's paper comes. However, it can be shown by numeric implementation that the sign taken will give rise to improper and unreasonable results. We give here a simple explanation for our corrections.

For the sake of simplicity, we use a complex modulus noted by E^* instead of $\lambda(\omega)$ or $\mu(\omega)$. Generally, the complex dynamic modulus of viscoelastic material can be expressed by $E^* = E' + iE''$ where E' is called storage modulus representing stored energy and E'' is loss modulus representing dissipated energy. The ratio of loss modulus and storage modulus is defined as $\tan\delta$ where δ is actually the phase lag between stress and strain in viscoelastic material. Thus, we have $\tan\delta = E''/E'$.

Using Euler's formula, E^* can be alternatively expressed by $E^* = |E^*|e^{i\delta}$. Similar to Eq. 5, we have $S^2 = E^*/\rho = |E^*|e^{i\delta}/\rho$. Thus, $S = \sqrt{|E^*|e^{i\delta}/\rho} = \sqrt{|E^*|/\rho}\,e^{i\frac{\delta}{2}}$ and the complex wave number reads:

$$
\begin{aligned}
k = \frac{\omega}{S} &= \frac{\omega}{\sqrt{|E^*|/\rho}}\, e^{-i\frac{\delta}{2}} \\
&= \frac{\omega}{\sqrt{|E^*|/\rho}}\, (\cos\frac{\delta}{2} - i\sin\frac{\delta}{2}) \\
&= \frac{\omega}{\sqrt{|E^*|/\rho}\sec\frac{\delta}{2}}\, (1 - i\tan\frac{\delta}{2}) \\
&= \frac{\omega}{C}\, (1 - i\tan\frac{\delta}{2})
\end{aligned}
\tag{18}
$$

where $\frac{\delta}{2}$ correspond to Ω_{mn} in Eq.6 and C correspond to C_{mn} in Eq.7.

Note that the corrections made are just flipping the sign from plus to minus and from minus to plus. It's thus possible that these two versions are equivalent. We demonstrate here that with the condition $0 \le \frac{\delta}{2} < \pi/2$, they are no more the same.

Generally, storage and loss modulus are all non-negative, namely $E' \ge 0$ and $E'' \ge 0$. Assuming hereby $\arctan(E''/E') = \theta$ where $0 \le \theta < \pi/2$. Our corrected formulas work as:

$$
\begin{cases} k = \frac{\omega}{C}\,(1 - i\tan\frac{\delta}{2}) \\ \tan\delta = E''/E', 0 \le \frac{\delta}{2} < \pi/2 \end{cases}
\Rightarrow
\begin{cases} \delta = \theta \\ \tan\frac{\delta}{2} = \tan\frac{\theta}{2} \\ k = \frac{\omega}{C}\,(1 - i\tan\frac{\theta}{2}) \end{cases}
\tag{19}
$$

And the original formulas work as:

$$
\begin{cases} k = \frac{\omega}{C}\,(1 + i\tan\frac{\delta}{2}) \\ \tan\delta = -E''/E', 0 \le \frac{\delta}{2} < \pi/2 \end{cases}
\Rightarrow
\begin{cases} \delta = \pi - \theta \\ \tan\frac{\delta}{2} = \frac{1}{\tan\frac{\theta}{2}} \\ k = \frac{\omega}{C}\,(1 + i\frac{1}{\tan\frac{\theta}{2}}) \end{cases}
\tag{20}
$$

3.2 Eq. 8

Eq. 2 can be alternatively expressed using the strain ε_{ij} as $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$ where $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$. Combining with Eqs. 3 & 4, we have the three stress components in Eq. 8:

$$
\begin{aligned}
\sigma_{xlm} &= \lambda_m(\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu_m\varepsilon_{xx} \\
&= \lambda_m(\frac{\partial u_{lm}}{\partial x} + \frac{\partial v_{lm}}{\partial y}) + 2\mu_m\frac{\partial u_{lm}}{\partial x} \\
&= \lambda_m(\frac{\partial^2\Phi_{lm1}}{\partial x^2} + \frac{\partial^2\Phi_{lm2}}{\partial x\partial y} + \frac{\partial^2\Phi_{lm1}}{\partial y^2} - \frac{\partial^2\Phi_{lm2}}{\partial x\partial y}) + 2\mu_m(\frac{\partial^2\Phi_{lm1}}{\partial x^2} + \frac{\partial^2\Phi_{lm2}}{\partial x\partial y}) \\
&= 2\mu_m(\frac{\partial^2\Phi_{lm1}}{\partial x^2} + \frac{\partial^2\Phi_{lm2}}{\partial x\partial y}) - \lambda_m k_{m1}^2\Phi_{lm1}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\sigma_{ylm} &= \lambda_m(\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu_m\varepsilon_{yy} \\
&= \lambda_m(\frac{\partial u_{lm}}{\partial x} + \frac{\partial v_{lm}}{\partial y}) + 2\mu_m\frac{\partial v_{lm}}{\partial y} \\
&= \lambda_m(\frac{\partial^2\Phi_{lm1}}{\partial x^2} + \frac{\partial^2\Phi_{lm2}}{\partial x\partial y} + \frac{\partial^2\Phi_{lm1}}{\partial y^2} - \frac{\partial^2\Phi_{lm2}}{\partial x\partial y}) + 2\mu_m(\frac{\partial^2\Phi_{lm1}}{\partial y^2} - \frac{\partial^2\Phi_{lm2}}{\partial x\partial y}) \\
&= 2\mu_m(\frac{\partial^2\Phi_{lm1}}{\partial y^2} - \frac{\partial^2\Phi_{lm2}}{\partial x\partial y}) - \lambda_m k_{m1}^2\Phi_{lm1}
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\tau_{lm} = 2\mu_m\varepsilon_{xy} &= \mu_m(\frac{\partial u_{lm}}{\partial y} + \frac{\partial v_{lm}}{\partial x}) \\
&= \mu_m(\frac{\partial^2\Phi_{lm1}}{\partial x\partial y} + \frac{\partial^2\Phi_{lm2}}{\partial y^2} + \frac{\partial^2\Phi_{lm1}}{\partial x\partial y} - \frac{\partial^2\Phi_{lm2}}{\partial x^2}) \\
&= \mu_m(2\frac{\partial^2\Phi_{lm1}}{\partial x\partial y} + \frac{\partial^2\Phi_{lm2}}{\partial y^2} - \frac{\partial^2\Phi_{lm2}}{\partial x^2})
\end{aligned}
\tag{23}
$$

3.3 Eq. 11

S_{mn} is referred to as complex wave speed. The subscript m represents the medium, i.e. $m = 1$ implies incident side and $m = 2$ implies the other side. The subscript n is used to denote dilatational and shear waves, respectively. Therefore, in Eq. 11, it ought to be S_{1l} instead of S_{l1} to represent the incident wave Φ_l.

3.4 Eq. 13

- u_{lm}

  From Eq. 3, $u_{lm} = \partial\Phi_{lm1}/\partial x + \partial\Phi_{lm2}/\partial y$. Using Eqs. 10 & 11 & 12 and discussing different cases in terms of m, we have:

  when $m = 1$

$$\begin{aligned} u_{l1} &= \frac{\partial\Phi_{l11}}{\partial x} + \frac{\partial\Phi_{l12}}{\partial y} \\ &= \frac{\partial(\delta_{l1}\Psi_l + \psi_{l11})}{\partial x} + \frac{\partial(\delta_{l2}\Psi_l + \psi_{l12})}{\partial y} \end{aligned} \tag{24}$$

  when $m = 2$

$$\begin{aligned} u_{l2} &= \frac{\partial\Phi_{l21}}{\partial x} + \frac{\partial\Phi_{l22}}{\partial y} \\ &= \frac{\partial\psi_{l21}}{\partial x} + \frac{\partial\psi_{l22}}{\partial y} \end{aligned} \tag{25}$$

  It concludes that

$$\begin{aligned} u_{lm} &= \frac{\partial\psi_{lm1}}{\partial x} + \frac{\partial\psi_{lm2}}{\partial y} + \delta_{m1}(\delta_{l1}\frac{\partial\Psi_l}{\partial x} + \delta_{l2}\frac{\partial\Psi_l}{\partial y}) \\ &= ik_{m1}\sin\zeta_{lm1}\psi_{lm1} + ik_{m2}\epsilon_m\cos\zeta_{lm2}\psi_{lm2} \\ &\quad + \delta_{m1}\Psi_l\, i\,(\delta_{l1}k_{1l}\sin\theta_l - \delta_{l2}k_{1l}\cos\theta_l) \\ &= ik_{m1}\sin\zeta_{lm1}\psi_{lm1} + ik_{m2}\epsilon_m\cos\zeta_{lm2}\psi_{lm2} \\ &\quad + \delta_{m1}\Psi_l\, i\,(\delta_{l1}k_{11}\sin\theta_1 - \delta_{l2}k_{12}\cos\theta_2) \\ &= \begin{bmatrix} ik_{m1}\sin\zeta_{lm1} & ik_{m2}\epsilon_m\cos\zeta_{lm2} \end{bmatrix} \begin{Bmatrix} \psi_{lm1} \\ \psi_{lm2} \end{Bmatrix} \\ &\quad + \delta_{m1}\Psi_l\left[i(\delta_{l1}k_{11}\sin\theta_1 - \delta_{l2}k_{12}\cos\theta_2)\right] \end{aligned} \tag{26}$$

- v_{lm}

  From Eq. 3, $v_{lm} = \partial\Phi_{lm1}/\partial y - \partial\Phi_{lm2}/\partial x$. Using Eqs. 10 & 11 & 12 and discussing different cases in terms of m, we have:

  when $m = 1$

$$\begin{aligned} v_{l1} &= \frac{\partial\Phi_{l11}}{\partial y} - \frac{\partial\Phi_{l12}}{\partial x} \\ &= \frac{\partial(\delta_{l1}\Psi_l + \psi_{l11})}{\partial y} - \frac{\partial(\delta_{l2}\Psi_l + \psi_{l12})}{\partial x} \end{aligned} \tag{27}$$

  when $m = 2$

$$\begin{aligned} v_{l2} &= \frac{\partial\Phi_{l21}}{\partial y} - \frac{\partial\Phi_{l22}}{\partial x} \\ &= \frac{\partial\psi_{l21}}{\partial y} - \frac{\partial\psi_{l22}}{\partial x} \end{aligned} \tag{28}$$

  It concludes that

$$\begin{aligned} v_{lm} &= \frac{\partial\psi_{lm1}}{\partial y} - \frac{\partial\psi_{lm2}}{\partial x} + \delta_{m1}(\delta_{l1}\frac{\partial\Psi_l}{\partial y} - \delta_{l2}\frac{\partial\Psi_l}{\partial x}) \\ &= ik_{m1}\epsilon_m\cos\zeta_{lm1}\psi_{lm1} - ik_{m2}\sin\zeta_{lm2}\psi_{lm2} \\ &\quad + \delta_{m1}\Psi_l\, i\,(-\delta_{l1}k_{1l}\cos\theta_l - \delta_{l2}k_{1l}\sin\theta_l) \\ &= ik_{m1}\epsilon_m\cos\zeta_{lm1}\psi_{lm1} - ik_{m2}\sin\zeta_{lm2}\psi_{lm2} \\ &\quad + \delta_{m1}\Psi_l\, i\,(-\delta_{l1}k_{11}\cos\theta_1 - \delta_{l2}k_{12}\sin\theta_2) \\ &= \begin{bmatrix} ik_{m1}\epsilon_m\cos\zeta_{lm1} & -ik_{m2}\sin\zeta_{lm2} \end{bmatrix} \begin{Bmatrix} \psi_{lm1} \\ \psi_{lm2} \end{Bmatrix} \\ &\quad + \delta_{m1}\Psi_l\left[-i(\delta_{l1}k_{11}\cos\theta_1 + \delta_{l2}k_{12}\sin\theta_2)\right] \end{aligned} \tag{29}$$

- σ_{xlm}

  From Eq. 8, $\sigma_{xlm} = 2\mu_m(\partial^2\Phi_{lm1}/\partial x^2 + \partial^2\Phi_{lm2}/\partial x\partial y) - \lambda_m k_{m1}^2\Phi_{lm1}$. Using Eqs. 10 & 11 & 12 and discussing different cases in terms of m, we have:

when $m = 1$

$$\begin{aligned}\sigma_{xl1} &= 2\mu_1(\frac{\partial^2\Phi_{l11}}{\partial x^2} + \frac{\partial^2\Phi_{l12}}{\partial x\partial y}) - \lambda_1 k_{11}^2\Phi_{l11} \\ &= 2\mu_1[\frac{\partial^2(\delta_{l1}\Psi_l + \psi_{l11})}{\partial x^2} + \frac{\partial^2(\delta_{l2}\Psi_l + \psi_{l12})}{\partial x\partial y}] - \lambda_1 k_{11}^2(\delta_{l1}\Psi_l + \psi_{l11})\end{aligned} \tag{30}$$

when $m = 2$

$$\begin{aligned}\sigma_{xl2} &= 2\mu_2(\frac{\partial^2\Phi_{l21}}{\partial x^2} + \frac{\partial^2\Phi_{l22}}{\partial x\partial y}) - \lambda_2 k_{21}^2\Phi_{l21} \\ &= 2\mu_2(\frac{\partial^2\psi_{l21}}{\partial x^2} + \frac{\partial^2\psi_{l22}}{\partial x\partial y}) - \lambda_2 k_{21}^2\psi_{l21}\end{aligned} \tag{31}$$

It concludes that

$$\begin{aligned}\sigma_{xlm} &= 2\mu_m(\frac{\partial^2\psi_{lm1}}{\partial x^2} + \frac{\partial^2\psi_{lm2}}{\partial x\partial y}) - \lambda_m k_{m1}^2\psi_{lm1} \\ &\quad + \delta_{m1}[\,2\mu_1(\delta_{l1}\frac{\partial^2\Psi_l}{\partial x^2} + \delta_{l2}\frac{\partial^2\Psi_l}{\partial x\partial y}) - \lambda_1 k_{11}^2\delta_{l1}\Psi_l\,] \\ &= 2\mu_m(-k_{m1}^2\sin^2\zeta_{lm1}\psi_{lm1} - k_{m2}^2\sin\zeta_{lm2}\epsilon_m\cos\zeta_{lm2}\psi_{lm2}) - \lambda_m k_{m1}^2\psi_{lm1} \\ &\quad + \delta_{m1}[2\mu_1(-\delta_{l1}k_{1l}^2\sin^2\theta_l\Psi_l + \delta_{l2}k_{1l}^2\sin\theta_l\cos\theta_l\Psi_l) - \lambda_1 k_{11}^2\delta_{l1}\Psi_l] \\ &= -k_{m1}^2(\lambda_m + 2\mu_m\sin^2\zeta_{lm1})\psi_{lm1} - \mu_m k_{m2}^2\epsilon_m\sin 2\zeta_{lm2}\psi_{lm2} \\ &\quad + \delta_{m1}\Psi_l[-\delta_{l1}(\lambda_1 k_{11}^2 + 2\mu_1 k_{1l}^2\sin^2\theta_l) + \delta_{l2}\mu_1 k_{1l}^2\sin 2\theta_l] \\ &= -k_{m1}^2(\lambda_m + 2\mu_m\sin^2\zeta_{lm1})\psi_{lm1} - \mu_m k_{m2}^2\epsilon_m\sin 2\zeta_{lm2}\psi_{lm2} \\ &\quad + \delta_{m1}\Psi_l[-\delta_{l1}k_{11}^2(\lambda_1 + 2\mu_1\sin^2\theta_1) + \delta_{l2}\mu_1 k_{12}^2\sin 2\theta_2] \\ &= \begin{bmatrix}-k_{m1}^2(\lambda_m + 2\mu_m\sin^2\zeta_{lm1}) & -\mu_m k_{m2}^2\epsilon_m\sin 2\zeta_{lm2}\end{bmatrix}\begin{Bmatrix}\psi_{lm1} \\ \psi_{lm2}\end{Bmatrix} \\ &\quad + \delta_{m1}\Psi_l\left[-\delta_{l1}k_{11}^2(\lambda_1 + 2\mu_1\sin^2\theta_1) + \delta_{l2}\mu_1 k_{12}^2\sin 2\theta_2\right]\end{aligned} \tag{32}$$

- σ_{ylm}

  From Eq. 8, $\sigma_{ylm} = 2\mu_m(\partial^2\Phi_{lm1}/\partial y^2 - \partial^2\Phi_{lm2}/\partial x\partial y) - \lambda_m k_{m1}^2\Phi_{lm1}$. Using Eqs. 10 & 11 & 12 and discussing different cases in terms of m, we have:

  when $m = 1$

$$\begin{aligned}\sigma_{yl1} &= 2\mu_1(\frac{\partial^2\Phi_{l11}}{\partial y^2} - \frac{\partial^2\Phi_{l12}}{\partial x\partial y}) - \lambda_1 k_{11}^2\Phi_{l11} \\ &= 2\mu_1[\frac{\partial^2(\delta_{l1}\Psi_l + \psi_{l11})}{\partial y^2} - \frac{\partial^2(\delta_{l2}\Psi_l + \psi_{l12})}{\partial x\partial y}] - \lambda_1 k_{11}^2(\delta_{l1}\Psi_l + \psi_{l11})\end{aligned} \tag{33}$$

  when $m = 2$

$$\begin{aligned}\sigma_{yl2} &= 2\mu_2(\frac{\partial^2\Phi_{l21}}{\partial y^2} - \frac{\partial^2\Phi_{l22}}{\partial x\partial y}) - \lambda_2 k_{21}^2\Phi_{l21} \\ &= 2\mu_2(\frac{\partial^2\psi_{l21}}{\partial y^2} - \frac{\partial^2\psi_{l22}}{\partial x\partial y}) - \lambda_2 k_{21}^2\psi_{l21}\end{aligned} \tag{34}$$

It concludes that

$$
\begin{aligned}
\sigma_{ylm} &= 2\mu_m(\frac{\partial^2\psi_{lm1}}{\partial y^2} - \frac{\partial^2\psi_{lm2}}{\partial x\partial y}) - \lambda_m k_{m1}^2\psi_{lm1} \\
&\quad + \delta_{m1}[\,2\mu_1(\delta_{l1}\frac{\partial^2\Psi_l}{\partial y^2} - \delta_{l2}\frac{\partial^2\Psi_l}{\partial x\partial y}) - \lambda_1 k_{11}^2\delta_{l1}\Psi_l\,] \\
&= 2\mu_m(-k_{m1}^2\epsilon_m^2\cos^2\zeta_{lm1}\psi_{lm1} + k_{m2}^2\sin\zeta_{lm2}\epsilon_m\cos\zeta_{lm2}\psi_{lm2}) - \lambda_m k_{m1}^2\psi_{lm1} \\
&\quad + \delta_{m1}[2\mu_1(-\delta_{l1}k_{1l}^2\cos^2\theta_l\Psi_l - \delta_{l2}k_{1l}^2\sin\theta_l\cos\theta_l\Psi_l) - \lambda_1 k_{11}^2\delta_{l1}\Psi_l] \\
&= -k_{m1}^2(\lambda_m + 2\mu_m\cos^2\zeta_{lm1})\psi_{lm1} + \mu_m k_{m2}^2\epsilon_m\sin 2\zeta_{lm2}\psi_{lm2} \\
&\quad + \delta_{m1}\Psi_l[-\delta_{l1}(\lambda_1 k_{11}^2 + 2\mu_1 k_{1l}^2\cos^2\theta_l) - \delta_{l2}\mu_1 k_{1l}^2\sin 2\theta_l] \\
&= -k_{m1}^2(\lambda_m + 2\mu_m\cos^2\zeta_{lm1})\psi_{lm1} + \mu_m k_{m2}^2\epsilon_m\sin 2\zeta_{lm2}\psi_{lm2} \\
&\quad + \delta_{m1}\Psi_l[-\delta_{l1}k_{11}^2(\lambda_1 + 2\mu_1\cos^2\theta_1) - \delta_{l2}\mu_1 k_{12}^2\sin 2\theta_2] \\
&= \begin{bmatrix} -k_{m1}^2(\lambda_m + 2\mu_m\cos^2\zeta_{lm1}) & \mu_m k_{m2}^2\epsilon_m\sin 2\zeta_{lm2} \end{bmatrix} \begin{Bmatrix} \psi_{lm1} \\ \psi_{lm2} \end{Bmatrix} \\
&\quad + \delta_{m1}\Psi_l\left[-\delta_{l1}k_{11}^2(\lambda_1 + 2\mu_1\cos^2\theta_1) - \delta_{l2}\mu_1 k_{12}^2\sin 2\theta_2\right]
\end{aligned} \tag{35}
$$

- τ_{lm}

From Eq. 8, $\tau_{lm} = \mu_m(2\partial^2\Phi_{lm1}/\partial x\partial y + \partial^2\Phi_{lm2}/\partial y^2 - \partial^2\Phi_{lm2}/\partial x^2)$. Using Eqs. 10 & 11 & 12 and discussing different cases in terms of m, we have:

when $m = 1$

$$
\begin{aligned}
\tau_{l1} &= \mu_1(2\frac{\partial^2\Phi_{l11}}{\partial x\partial y} + \frac{\partial^2\Phi_{l12}}{\partial y^2} - \frac{\partial^2\Phi_{l12}}{\partial x^2}) \\
&= \mu_1[2\frac{\partial^2(\delta_{l1}\Psi_l + \psi_{l11})}{\partial x\partial y} + \frac{\partial^2(\delta_{l2}\Psi_l + \psi_{l12})}{\partial y^2} - \frac{\partial^2(\delta_{l2}\Psi_l + \psi_{l12})}{\partial x^2}]
\end{aligned} \tag{36}
$$

when $m = 2$

$$
\begin{aligned}
\tau_{l2} &= \mu_2(2\frac{\partial^2\Phi_{l21}}{\partial x\partial y} + \frac{\partial^2\Phi_{l22}}{\partial y^2} - \frac{\partial^2\Phi_{l22}}{\partial x^2}) \\
&= \mu_2(2\frac{\partial^2\psi_{l21}}{\partial x\partial y} + \frac{\partial^2\psi_{l22}}{\partial y^2} - \frac{\partial^2\psi_{l22}}{\partial x^2})
\end{aligned} \tag{37}
$$

It concludes that

$$
\begin{aligned}
\tau_{lm} &= \mu_m(2\frac{\partial^2\psi_{lm1}}{\partial x\partial y} + \frac{\partial^2\psi_{lm2}}{\partial y^2} - \frac{\partial^2\psi_{lm2}}{\partial x^2}) \\
&\quad + \delta_{m1}\mu_1(2\delta_{l1}\frac{\partial^2\Psi_l}{\partial x\partial y} + \delta_{l2}\frac{\partial^2\Psi_l}{\partial y^2} - \delta_{l2}\frac{\partial^2\Psi_l}{\partial x^2}) \\
&= \mu_m(-2k_{m1}^2\sin\zeta_{lm1}\epsilon_m\cos\zeta_{lm1}\psi_{lm1} - k_{m2}^2\epsilon_m^2\cos^2\zeta_{lm2}\psi_{lm2} + k_{m2}^2\sin^2\zeta_{lm2}\psi_{lm2}) \\
&\quad + \delta_{m1}\mu_1(2\delta_{l1}k_{1l}^2\sin\theta_l\cos\theta_l\Psi_l - \delta_{l2}k_{1l}^2\cos^2\theta_l\Psi_l + \delta_{l2}k_{1l}^2\sin^2\theta_l\Psi_l) \\
&= -\mu_m k_{m1}^2\epsilon_m\sin 2\zeta_{lm1}\psi_{lm1} - \mu_m k_{m2}^2\cos 2\zeta_{lm2}\psi_{lm2} \\
&\quad + \delta_{m1}\Psi_l(\delta_{l1}\mu_1 k_{1l}^2\sin 2\theta_l - \delta_{l2}\mu_1 k_{1l}^2\cos 2\theta_l) \\
&= -\mu_m k_{m1}^2\epsilon_m\sin 2\zeta_{lm1}\psi_{lm1} - \mu_m k_{m2}^2\cos 2\zeta_{lm2}\psi_{lm2} \\
&\quad + \delta_{m1}\Psi_l(\delta_{l1}\mu_1 k_{11}^2\sin 2\theta_1 - \delta_{l2}\mu_1 k_{12}^2\cos 2\theta_2) \\
&= \begin{bmatrix} -\mu_m k_{m1}^2\epsilon_m\sin 2\zeta_{lm1} & -\mu_m k_{m2}^2\cos 2\zeta_{lm2} \end{bmatrix} \begin{Bmatrix} \psi_{lm1} \\ \psi_{lm2} \end{Bmatrix} \\
&\quad + \delta_{m1}\Psi_l\left[\delta_{l1}\mu_1 k_{11}^2\sin 2\theta_1 - \delta_{l2}\mu_1 k_{12}^2\cos 2\theta_2\right]
\end{aligned} \tag{38}
$$

3.5 Eq. 14

Since the relation $\exp\{ik_{1l}(\mathbf{r}_l \cdot \mathbf{x})\} = \exp\{ik_{mn}(\mathbf{r}_{lmn} \cdot \mathbf{x})\}$ should be satisfied on $y = 0$, we have the following equations:

$$
\begin{aligned}
k_{mn}\sin\zeta_{lmn} &= k_{1l}\sin\theta_l \\
\frac{\omega}{S_{mn}}\sin\zeta_{lmn} &= \frac{\omega}{S_{1l}}\sin\theta_l
\end{aligned} \tag{39}
$$

which results in the complex Snell's law:

$$
\sin\zeta_{lmn} = \frac{S_{mn}}{S_{1l}}\sin\theta_l \tag{40}
$$

3.6 Eqs. 15, 16 & 17

In the following demonstrations, we assume all the equal exponents being noted as Exp.

- First row

  Using Eq. 26 and potential functions defined in Eq. 11, we can write $u_{lm}(x,0)$ for different cases of m:

  when $m=1$

$$\begin{aligned} u_{l1}(x,0) &= ik_{11}\sin\zeta_{l11}I_lR_{l11}\frac{S_{11}}{\omega}Exp + ik_{12}(-1)^2\cos\zeta_{l12}I_lR_{l12}\frac{S_{12}}{\omega}Exp \\ &+ i(\delta_{l1}k_{11}\sin\theta_1 - \delta_{l2}k_{12}\cos\theta_2)I_l\frac{S_{1l}}{\omega}Exp \end{aligned} \tag{41}$$

  when $m=2$

$$u_{l2}(x,0) = ik_{21}\sin\zeta_{l21}I_lR_{l21}\frac{S_{21}}{\omega}Exp + ik_{22}(-1)^3\cos\zeta_{l22}I_lR_{l22}\frac{S_{22}}{\omega}Exp \tag{42}$$

  Considering the definition of wave number k_{mn} in Eq. 5, the boundary condition $u_{l1}(x,0) = u_{l2}(x,0)$ results in:

$$\begin{aligned} \sin\zeta_{l11}R_{l11} + \cos\zeta_{l12}R_{l12} + (\delta_{l1}\sin\theta_1 - \delta_{l2}\cos\theta_2) &= \sin\zeta_{l21}R_{l21} - \cos\zeta_{l22}R_{l22} \\ \sin\zeta_{l11}R_{l11} + \cos\zeta_{l12}R_{l12} - \sin\zeta_{l21}R_{l21} + \cos\zeta_{l22}R_{l22} &= -\delta_{l1}\sin\theta_1 + \delta_{l2}\cos\theta_2 \end{aligned} \tag{43}$$

  Thus the first row of $\mathbf{A}_l$, $\mathbf{B}_1$ and $\mathbf{B}_2$ read:

$$\begin{aligned} \mathbf{A}_l(1,:) &= \begin{bmatrix}\sin\zeta_{l11} & \cos\zeta_{l12} & -\sin\zeta_{l21} & \cos\zeta_{l22}\end{bmatrix} \\ \mathbf{B}_1(1,:) &= \begin{bmatrix}-\sin\theta_1\end{bmatrix} \\ \mathbf{B}_2(1,:) &= \begin{bmatrix}\cos\theta_2\end{bmatrix} \end{aligned} \tag{44}$$

- Second row

  Using Eq. 29 and potential functions defined in Eq. 11, we can write $v_{lm}(x,0)$ for different cases of m:

  when $m=1$

$$\begin{aligned} v_{l1}(x,0) &= ik_{11}(-1)^2\cos\zeta_{l11}I_lR_{l11}\frac{S_{11}}{\omega}Exp - ik_{12}\sin\zeta_{l12}I_lR_{l12}\frac{S_{12}}{\omega}Exp \\ &- i(\delta_{l1}k_{11}\cos\theta_1 + \delta_{l2}k_{12}\sin\theta_2)I_l\frac{S_{1l}}{\omega}Exp \end{aligned} \tag{45}$$

  when $m=2$

$$v_{l2}(x,0) = ik_{21}(-1)^3\cos\zeta_{l21}I_lR_{l21}\frac{S_{21}}{\omega}Exp - ik_{22}\sin\zeta_{l22}I_lR_{l22}\frac{S_{22}}{\omega}Exp \tag{46}$$

  Considering the definition of wave number k_{mn} in Eq. 5, the boundary condition $v_{l1}(x,0) = v_{l2}(x,0)$ results in:

$$\begin{aligned} \cos\zeta_{l11}R_{l11} - \sin\zeta_{l12}R_{l12} - (\delta_{l1}\cos\theta_1 + \delta_{l2}\sin\theta_2) &= -\cos\zeta_{l21}R_{l21} - \sin\zeta_{l22}R_{l22} \\ \cos\zeta_{l11}R_{l11} - \sin\zeta_{l12}R_{l12} + \cos\zeta_{l21}R_{l21} + \sin\zeta_{l22}R_{l22} &= \delta_{l1}\cos\theta_1 + \delta_{l2}\sin\theta_2 \end{aligned} \tag{47}$$

  Thus the second row of $\mathbf{A}_l$, $\mathbf{B}_1$ and $\mathbf{B}_2$ read:

$$\begin{aligned} \mathbf{A}_l(2,:) &= \begin{bmatrix}\cos\zeta_{l11} & -\sin\zeta_{l12} & \cos\zeta_{l21} & \sin\zeta_{l22}\end{bmatrix} \\ \mathbf{B}_1(2,:) &= \begin{bmatrix}\cos\theta_1\end{bmatrix} \\ \mathbf{B}_2(2,:) &= \begin{bmatrix}\sin\theta_2\end{bmatrix} \end{aligned} \tag{48}$$

- Third row

  Using Eq. 35 and potential functions defined in Eq. 11, we can write $\sigma_{ylm}(x,0)$ for different cases of m:

  when $m=1$

$$\begin{aligned} \sigma_{yl1}(x,0) &= -k_{11}^2(\lambda_1 + 2\mu_1\cos^2\zeta_{l11})I_lR_{l11}\frac{S_{11}}{\omega}Exp + \mu_1k_{12}^2(-1)^2\sin 2\zeta_{l12}I_lR_{l12}\frac{S_{12}}{\omega}Exp \\ &+ [-\delta_{l1}k_{11}^2(\lambda_1 + 2\mu_1\cos^2\theta_1) - \delta_{l2}\mu_1k_{12}^2\sin 2\theta_2]I_l\frac{S_{1l}}{\omega}Exp \end{aligned} \tag{49}$$

  when $m=2$

$$\sigma_{yl2}(x,0) = -k_{21}^2(\lambda_2 + 2\mu_2\cos^2\zeta_{l21})I_lR_{l21}\frac{S_{21}}{\omega}Exp + \mu_2k_{22}^2(-1)^3\sin 2\zeta_{l22}I_lR_{l22}\frac{S_{22}}{\omega}Exp \tag{50}$$

Considering the definition of wave number k_{mn} in Eq. 5, the boundary condition $\sigma_{yl1}(x,0)=\sigma_{yl2}(x,0)$ results in:

$$\begin{aligned}
-k_{11}(\lambda_1+2\mu_1\cos^2\zeta_{l11})R_{l11}+\mu_1 k_{12}\sin 2\zeta_{l12}R_{l12}-[\delta_{l1}k_{11}(\lambda_1+2\mu_1\cos^2\theta_1)+\delta_{l2}\mu_1 k_{12}\sin 2\theta_2] \\
=-k_{21}(\lambda_2+2\mu_2\cos^2\zeta_{l21})R_{l21}-\mu_2 k_{22}\sin 2\zeta_{l22}R_{l22} \\
-\frac{1}{S_{11}}(\lambda_1+2\mu_1\cos^2\zeta_{l11})R_{l11}+\mu_1\frac{1}{S_{12}}\sin 2\zeta_{l12}R_{l12}+\frac{1}{S_{21}}(\lambda_2+2\mu_2\cos^2\zeta_{l21})R_{l21}+\mu_2\frac{1}{S_{22}}\sin 2\zeta_{l22}R_{l22} \\
=\delta_{l1}\frac{1}{S_{11}}(\lambda_1+2\mu_1\cos^2\theta_1)+\delta_{l2}\mu_1\frac{1}{S_{12}}\sin 2\theta_2
\end{aligned} \tag{51}$$

For the factors of R_{l11}, R_{l12}, R_{l21}, R_{l22}, δ_{l1} and δ_{l2}, we have the following deductions considering the definition of wave speed S_{mn} in Eq. 5 and Snell's law:

- R_{l11}

$$\begin{aligned}
-\frac{1}{S_{11}}(\lambda_1+2\mu_1\cos^2\zeta_{l11}) &= -\frac{1}{S_{11}}(\lambda_1+2\mu_1-2\mu_1\sin^2\zeta_{l11}) \\
&= -\frac{1}{S_{11}}(\rho_1 S_{11}^2-2\rho_1 S_{12}^2\frac{S_{11}^2}{S_{12}^2}\sin^2\zeta_{l12}) \\
&= -\frac{1}{S_{11}}\rho_1 S_{11}^2(1-2\sin^2\zeta_{l12}) \\
&= -\rho_1 S_{11}\cos 2\zeta_{l12}
\end{aligned} \tag{52}$$

- R_{l12}

$$\begin{aligned}
\mu_1\frac{1}{S_{12}}\sin 2\zeta_{l12} &= \rho_1 S_{12}^2\frac{1}{S_{12}}\sin 2\zeta_{l12} \\
&= \rho_1 S_{12}\sin 2\zeta_{l12}
\end{aligned} \tag{53}$$

- R_{l21}

$$\begin{aligned}
\frac{1}{S_{21}}(\lambda_2+2\mu_2\cos^2\zeta_{l21}) &= \frac{1}{S_{21}}(\lambda_2+2\mu_2-2\mu_2\sin^2\zeta_{l21}) \\
&= \frac{1}{S_{21}}(\rho_2 S_{21}^2-2\rho_2 S_{22}^2\frac{S_{21}^2}{S_{22}^2}\sin^2\zeta_{l22}) \\
&= \frac{1}{S_{21}}\rho_2 S_{21}^2(1-2\sin^2\zeta_{l22}) \\
&= \rho_2 S_{21}\cos 2\zeta_{l22}
\end{aligned} \tag{54}$$

- R_{l22}

$$\begin{aligned}
\mu_2\frac{1}{S_{22}}\sin 2\zeta_{l22} &= \rho_2 S_{22}^2\frac{1}{S_{22}}\sin 2\zeta_{l22} \\
&= \rho_2 S_{22}\sin 2\zeta_{l22}
\end{aligned} \tag{55}$$

- δ_{l1}

$$\begin{aligned}
\frac{1}{S_{11}}(\lambda_1+2\mu_1\cos^2\theta_1) &= \frac{1}{S_{11}}(\lambda_1+2\mu_1-2\mu_1\sin^2\theta_1) \\
&= \frac{1}{S_{11}}(\rho_1 S_{11}^2-2\rho_1 S_{12}^2\frac{S_{11}^2}{S_{12}^2}\sin^2\zeta_{112}) \\
&= \frac{1}{S_{11}}\rho_1 S_{11}^2(1-2\sin^2\zeta_{112}) \\
&= \rho_1 S_{11}\cos 2\zeta_{112}
\end{aligned} \tag{56}$$

- δ_{l2}

$$\begin{aligned}
\mu_1\frac{1}{S_{12}}\sin 2\theta_2 &= \rho_1 S_{12}^2\frac{1}{S_{12}}\sin 2\theta_2 \\
&= \rho_1 S_{12}\sin 2\theta_2
\end{aligned} \tag{57}$$

Thus the third row of $\mathbf{A}_l$, $\mathbf{B}_1$ and $\mathbf{B}_2$ read:

$$\begin{aligned}
\mathbf{A}_l(3,:) &= \begin{bmatrix}-\rho_1 S_{11}\cos 2\zeta_{l12} & \rho_1 S_{12}\sin 2\zeta_{l12} & \rho_2 S_{21}\cos 2\zeta_{l22} & \rho_2 S_{22}\sin 2\zeta_{l22}\end{bmatrix} \\
\mathbf{B}_1(3,:) &= \begin{bmatrix}\rho_1 S_{11}\cos 2\zeta_{112}\end{bmatrix} \\
\mathbf{B}_2(3,:) &= \begin{bmatrix}\rho_1 S_{12}\sin 2\theta_2\end{bmatrix}
\end{aligned} \tag{58}$$

- Fourth row

Using Eq. 38 and potential functions defined in Eq. 11, we can write $\tau_{lm}(x,0)$ for different cases of m: when $m=1$

$$\begin{aligned}\tau_{l1}(x,0) &= -(-1)^2\mu_1 k_{11}^2 \sin 2\zeta_{l11} I_l R_{l11}\frac{S_{11}}{\omega}Exp - \mu_1 k_{12}^2 \cos 2\zeta_{l12} I_l R_{l12}\frac{S_{12}}{\omega}Exp \\ &+ (\delta_{l1}\mu_1 k_{11}^2 \sin 2\theta_1 - \delta_{l2}\mu_1 k_{12}^2 \cos 2\theta_2) I_l \frac{S_{1l}}{\omega}Exp\end{aligned} \tag{59}$$

when $m=2$

$$\tau_{l2}(x,0) = -(-1)^3\mu_2 k_{21}^2 \sin 2\zeta_{l21} I_l R_{l21}\frac{S_{21}}{\omega}Exp - \mu_2 k_{22}^2 \cos 2\zeta_{l22} I_l R_{l22}\frac{S_{22}}{\omega}Exp \tag{60}$$

Considering the definition of wave number k_{mn} in Eq. 5, the boundary condition $\tau_{l1}(x,0) = \tau_{l2}(x,0)$ results in:

$$\begin{aligned} -\mu_1 k_{11}\sin 2\zeta_{l11}R_{l11} - \mu_1 k_{12}\cos 2\zeta_{l12}R_{l12} + (\delta_{l1}\mu_1 k_{11}\sin 2\theta_1 - \delta_{l2}\mu_1 k_{12}\cos 2\theta_2) \\ = \mu_2 k_{21}\sin 2\zeta_{l21}R_{l21} - \mu_2 k_{22}\cos 2\zeta_{l22}R_{l22} \\ \mu_1\frac{1}{S_{11}}\sin 2\zeta_{l11}R_{l11} + \mu_1\frac{1}{S_{12}}\cos 2\zeta_{l12}R_{l12} + \mu_2\frac{1}{S_{21}}\sin 2\zeta_{l21}R_{l21} - \mu_2\frac{1}{S_{22}}\cos 2\zeta_{l22}R_{l22} \\ = \delta_{l1}\mu_1\frac{1}{S_{11}}\sin 2\theta_1 - \delta_{l2}\mu_1\frac{1}{S_{12}}\cos 2\theta_2 \end{aligned} \tag{61}$$

For the factors of R_{l11}, R_{l12}, R_{l21}, R_{l22}, δ_{l1} and δ_{l2}, we have the following deductions considering the definition of wave speed S_{mn} in Eq. 5 and Snell's law:

  - R_{l11}

$$\mu_1\frac{1}{S_{11}}\sin 2\zeta_{l11} = \frac{\rho_1 S_{12}^2}{S_{11}}\sin 2\zeta_{l11} \tag{62}$$

  - R_{l12}

$$\begin{aligned}\mu_1\frac{1}{S_{12}}\cos 2\zeta_{l12} &= \frac{\rho_1 S_{12}^2}{S_{12}}\cos 2\zeta_{l12} \\ &= \rho_1 S_{12}\cos 2\zeta_{l12}\end{aligned} \tag{63}$$

  - R_{l21}

$$\mu_2\frac{1}{S_{21}}\sin 2\zeta_{l21} = \frac{\rho_2 S_{22}^2}{S_{21}}\sin 2\zeta_{l21} \tag{64}$$

  - R_{l22}

$$\begin{aligned}-\mu_2\frac{1}{S_{22}}\cos 2\zeta_{l22} &= -\frac{\rho_2 S_{22}^2}{S_{22}}\cos 2\zeta_{l22} \\ &= -\rho_2 S_{22}\cos 2\zeta_{l22}\end{aligned} \tag{65}$$

  - δ_{l1}

$$\mu_1\frac{1}{S_{11}}\sin 2\theta_1 = \frac{\rho_1 S_{12}^2}{S_{11}}\sin 2\theta_1 \tag{66}$$

  - δ_{l2}

$$\begin{aligned}-\mu_1\frac{1}{S_{12}}\cos 2\theta_2 &= -\frac{\rho_1 S_{12}^2}{S_{12}}\cos 2\theta_2 \\ &= -\rho_1 S_{12}\cos 2\theta_2\end{aligned} \tag{67}$$

Thus the fourth row of $\mathbf{A}_l$, $\mathbf{B}_1$ and $\mathbf{B}_2$ read:

$$\begin{aligned}\mathbf{A}_l(4,:) &= \left[(\rho_1 S_{12}^2/S_{11})\sin 2\zeta_{l11} \quad \rho_1 S_{12}\cos 2\zeta_{l12} \quad (\rho_2 S_{22}^2/S_{21})\sin 2\zeta_{l21} \quad -\rho_2 S_{22}\cos 2\zeta_{l22}\right] \\ \mathbf{B}_1(4,:) &= \left[(\rho_1 S_{12}^2/S_{11})\sin 2\theta_1\right] \\ \mathbf{B}_2(4,:) &= \left[-\rho_1 S_{12}\cos 2\theta_2\right]\end{aligned} \tag{68}$$

4 References

[1] H. F. Cooper Jr, "Reflection and transmission of oblique plane waves at a plane interface between viscoelastic media", The Journal of the Acoustical Society of America, vol. 42, no. 5, Art. no. 5, 1967.

[2] TO BE FILLED