Ejercicios del capítulo 9 de *Classical Mechanics* de H. Goldstein

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Ejercicio 9.9

Sea u una función de $r^2, p^2, \mathbf{r}\cdot\mathbf{p}$. Para mostar que $[u, \mathbf{L}] = 0$ es suficiente ver que $[r^2, \mathbf{L}] = 0$, $[p^2, \mathbf{L}] = 0$ y $[\mathbf{r}\cdot\mathbf{p}, \mathbf{L}] = 0$ ya que:

$$
\begin{aligned}
[u, \mathbf{L}] =& \frac{\partial u}{\partial x_i}\frac{\partial \mathbf{L}}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial \mathbf{L}}{\partial x_i} = & (1)\\
=& \left(\frac{\partial u}{\partial (r^2)}\frac{\partial r^2}{\partial x_i} + \frac{\partial u}{\partial (p^2)}\frac{\partial p^2}{\partial x_i} + \frac{\partial u}{\partial (\mathbf{r}\cdot\mathbf{p})}\frac{\partial \mathbf{r}\cdot\mathbf{p}}{\partial x_i}\right)\frac{\partial \mathbf{L}}{\partial p_i} & (2)\\
&- \left(\frac{\partial u}{\partial (r^2)}\frac{\partial r^2}{\partial p_i} + \frac{\partial u}{\partial (p^2)}\frac{\partial p^2}{\partial p_i} + \frac{\partial u}{\partial (\mathbf{r}\cdot\mathbf{p})}\frac{\partial \mathbf{r}\cdot\mathbf{p}}{\partial p_i}\right)\frac{\partial \mathbf{L}}{\partial x_i} & \\
=& \left(\frac{\partial u}{\partial (r^2)}\left(\frac{\partial r^2}{\partial x_i}\frac{\partial \mathbf{L}}{\partial p_i} - \frac{\partial r^2}{\partial p_i}\frac{\partial \mathbf{L}}{\partial x_i}\right)\right) + \left(\frac{\partial u}{\partial (p^2)}\left(\frac{\partial p^2}{\partial x_i}\frac{\partial \mathbf{L}}{\partial p_i} - \frac{\partial p^2}{\partial p_i}\frac{\partial \mathbf{L}}{\partial x_i}\right)\right) & (3)\\
&+ \left(\frac{\partial u}{\partial (\mathbf{r}\cdot\mathbf{p})}\left(\frac{\partial \mathbf{r}\cdot\mathbf{p}}{\partial x_i}\frac{\partial \mathbf{L}}{\partial p_i} - \frac{\partial \mathbf{r}\cdot\mathbf{p}}{\partial p_i}\frac{\partial \mathbf{L}}{\partial x_i}\right)\right) & \\
=& \frac{\partial u}{\partial (r^2)}[r^2, \mathbf{L}] + \frac{\partial u}{\partial (p^2)}[p^2, \mathbf{L}] + \frac{\partial u}{\partial (\mathbf{r}\cdot\mathbf{p})}[\mathbf{r}\cdot\mathbf{p}, \mathbf{L}] & (4)
\end{aligned}
$$

Ahora calculemos los corchetes de Poisson de la anterior expresión:

$$
\begin{aligned}
[x_l x_l, \epsilon_{ijk} x_i p_j] =& \epsilon_{ijk}([x_l x_l, x_i] p_j + x_i [x_l x_l, p_j]) & (5)\\
=& -\epsilon_{ijk}(p_j([x_i, x_l] x_l + x_l [x_i, x_l]) + x_i([p_j, x_l] x_l + x_l [p_i, x_l])) &
\end{aligned}
$$

Pero $[x_i, x_j] = [p_i, p_j] = 0$ y $[x_i, p_j] = -[p_j, x_i] = \delta_{ij}$. Asi entonces:

$$
[x_l x_l, \epsilon_{ijk} x_i p_j] = -\epsilon_{ijk}(2 x_i x_l [p_j, x_l]) = 2\epsilon_{ijk} x_i x_j = \epsilon_{ijk} x_i x_j + \epsilon_{jik} x_j x_i = 0 \qquad (6)
$$

Para el segundo término tenemos:

$$
\begin{aligned}
[p_l p_l, \epsilon_{ijk} x_i p_j] =& \epsilon_{ijk}(p_j [p_l p_l, x_i] + x_i [p_l p_l, p_j]) = -\epsilon_{ijk}(p_j [x_i, p_l p_l]) & \\
=& -\epsilon_{ijk} 2 p_j p_l [x_i, p_l] = -2\epsilon_{ijk} p_j p_i = 0 & (7)
\end{aligned}
$$

Finalmente:

$$
\begin{aligned}
[x_l p_l, \epsilon_{ijk} x_i p_j] =& \epsilon_{ijk}([x_l p_l, x_i]p_j + x_i[x_l p_l, p_j]) \\
=& -\epsilon_{ijk}(p_j([x_i, x_l]p_l + x_l[x_i, p_l]) + x_i([p_j, x_l]p_l + x_l[p_j, p_l])) \\
=& -\epsilon_{ijk} p_j x_i + \epsilon_{ijk} x_i p_j = 0
\end{aligned} \tag{8}
$$

Si $\mathbf{F} = u\mathbf{r} + v\mathbf{p} + w\mathbf{r} \times \mathbf{p}$ donde u, v, w son funciones del mismo tipo que las del literal anterior, se pide mostrar que $[F_i, L_j] = \epsilon_{ijk} F_k$. Para ver lo anterior calculemos por separado los siguientes expresiones:

$$[ux_i, L_j] = u[x_i, L_j] \tag{9}$$

$$[vp_i, L_j] = v[p_i, L_j] \tag{10}$$

$$[wL_i, L_j] = w[L_i, L_j] \tag{11}$$

ya que $[u, L_j] = [v, L_j] = [w, L_j] = 0$. Ahora basta calcular las anteriores expresiones

$$[x_l, \epsilon_{ijk} x_i p_j] = \epsilon_{ijk}(p_j[x_l, x_i] + x_i[x_l, p_j]) = \epsilon_{ilk} x_i \tag{12}$$

$$[p_l, \epsilon_{ijk} x_i p_j] = \epsilon_{ijk}(x_i[p_l, p_j] + p_j[p_l, x_i]) = -\epsilon_{ljk} p_j = \epsilon_{jlk} p_j \tag{13}$$

$$
\begin{aligned}
[\epsilon_{lmi} x_l p_m, \epsilon_{abj} x_a p_b] &= \epsilon_{lmi}\epsilon_{abj}(x_a[x_l p_m, p_b] + p_b[x_l p_m, x_a]) \qquad (14) \\
&= -\epsilon_{lmi}\epsilon_{abj}(x_a([p_b, x_l]p_m) + p_b([x_a, p_m]x_l)) \\
&= -\epsilon_{lmi}\epsilon_{abj}(-x_a p_m \delta_{bl} + \delta_{am} p_b x_l)
\end{aligned}
$$

Finalmente

$$[L_i, L_j] = \epsilon_{lmi}\epsilon_{abj}\delta_{bl} x_a p_m - \epsilon_{lmi}\epsilon_{abj}\delta_{am} p_b x_l = \epsilon_{lmi}\epsilon_{alj} x_a p_m - \epsilon_{lmi}\epsilon_{mbj} p_b x_l = \epsilon_{ijk} L_k \tag{15}$$

Reuniendo todo lo anterior

$$[F_i, L_j] = u\epsilon_{ijk} x_k + v\epsilon_{ijk} p_k + \epsilon_{ijk} w L_k \tag{16}$$

Ejercicio 9.28

Para este problema tenemos que los momentos canonicos estan dados por

$$p_k = m\dot{x}_k + \frac{q}{2}\epsilon_{ijk} B_i x_j \tag{17}$$

De estos se obtiene facilmente que

$$v_k = \dot{x}_k = \frac{p_k - \frac{q}{2}\epsilon_{ijk} B_i x_j}{m} \tag{18}$$

Así entonces

$$[v_k, v_l] = \left[\frac{p_k - \frac{q}{2}\epsilon_{ijk}B_i x_j}{m}, \frac{p_l - \frac{q}{2}\epsilon_{abl}B_a x_b}{m}\right] \tag{19}$$

$$= \frac{1}{m^2}\left(\left[p_k, -\frac{q}{2}\epsilon_{abl}B_a x_b\right] - \left[\frac{q}{2}\epsilon_{ijk}B_i x_j, p_l\right]\right) \tag{20}$$

$$= \frac{1}{m^2}\left(-\frac{q}{2}\epsilon_{abl}B_a\left[p_k, x_b\right] - \frac{q}{2}\epsilon_{ijk}B_i\left[x_j, p_l\right]\right) \tag{21}$$

$$= \frac{1}{m^2}\left(\frac{q}{2}\epsilon_{abl}B_a\delta_{kb} - \frac{q}{2}\epsilon_{ijk}B_i\delta_{jl}\right) = \frac{1}{m^2}\left(\frac{q}{2}\epsilon_{akl}B_a - \frac{q}{2}\epsilon_{ilk}B_i\right) \tag{22}$$

$$= \frac{1}{m^2}\frac{q}{2}\left(\epsilon_{akl}B_a + \epsilon_{ikl}B_i\right) = \frac{q}{m^2}\epsilon_{akl}B_a \tag{23}$$

Por otro lado

$$[x_l, v_k] = \left[x_l, \frac{p_k - \frac{q}{2}\epsilon_{ijk}B_i x_j}{m}\right] = \frac{\delta_{lk}}{m} \tag{24}$$

$$[p_l, \dot{x}_k] = \frac{p_k - \frac{q}{2}\epsilon_{ijk}B_i x_j}{m} = \frac{-q\epsilon_{ijk}B_i}{2m}\left[p_l, x_j\right] = \frac{q\epsilon_{ijk}B_i}{2m}\delta_{lj} = \frac{q\epsilon_{ilk}B_i}{2m} \tag{25}$$

$$[x_k, \dot{p}_l] = [x_k, [p_l, H]] = -\left[H, [x_k, p_l]\right] - \left[p_l, [H, x_k]\right] = -\left[p_l, -\dot{x}_k\right] = \left[p_l, \dot{x}_k\right] = \frac{q\epsilon_{ilk}B_i}{2m} \tag{26}$$

Para el último cálculo es necesario encontrar una expresión para $\dot{p}_a$, para esto usamos las ecuaciones de Hamilton:

$$H = \frac{\left(p_i - \frac{q}{2}B_j x_k\epsilon_{ijk}\right)\left(p_i - \frac{q}{2}B_l x_n\epsilon_{iln}\right)}{2m} \tag{27}$$

Con estas se encuentran facilmente los $\dot{p}_a$:

$$\dot{p}_a = -\frac{\partial H}{\partial x_a} = \frac{q}{2}B_l\epsilon_{ila}\left(p_i - \frac{q}{2}B_j x_k\epsilon_{ijk}\right) + \frac{q}{2}B_j\epsilon_{ija}\left(p_i - \frac{q}{2}B_l x_n\epsilon_{iln}\right) \tag{28}$$

y se calcula el corchete de Poisson:

$$[p_b, \dot{p}_a] = \frac{1}{2m}\left(-\frac{q^2}{4}B_j B_l\epsilon_{ijk}\epsilon_{ila}\left[p_b, x_k\right] - \frac{q^2}{4}B_l B_j\epsilon_{iln}\epsilon_{ija}\left[p_b, x_n\right]\right) \tag{29}$$

$$= \frac{q^2}{8m}B_l B_j(\epsilon_{ijb}\epsilon_{ila} + \epsilon_{ilb}\epsilon_{ija}) = \frac{q^2}{4m}(B_l B_l\delta_{ba} - B_a B_b) \tag{30}$$

Ejercicio 9.30

Supongamos que Q, R son constantes de movimiento entonces::

$$[Q, H] = -\frac{\partial Q}{\partial t} \tag{31}$$

$$[R, H] = -\frac{\partial R}{\partial t} \tag{32}$$

Evaluemos las siguientes cantidades:

$$[[Q,R],H] = [R,[H,Q]] + [Q,[R,H]] = [R, \frac{\partial Q}{\partial t}] + [Q, -\frac{\partial R}{\partial t}] = \tag{33}$$
$$\frac{\partial R}{\partial x_i}\frac{\partial}{\partial p_i}\left(\frac{\partial Q}{\partial t}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial Q}{\partial t}\right)\frac{\partial R}{\partial p_i} - \frac{\partial Q}{\partial x_i}\frac{\partial}{\partial p_i}\left(\frac{\partial R}{\partial t}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial R}{\partial t}\right)\frac{\partial Q}{\partial p_i}$$

Por otro lado:

$$-\frac{\partial [Q,R]}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\partial Q}{\partial x_i}\frac{\partial R}{\partial p_i} - \frac{\partial Q}{\partial p_i}\frac{\partial R}{\partial x_i}\right) \tag{34}$$
$$= \frac{\partial R}{\partial x_i}\frac{\partial}{\partial t}\left(\frac{\partial Q}{\partial p_i}\right) - \frac{\partial}{\partial t}\left(\frac{\partial Q}{\partial x_i}\right)\frac{\partial R}{\partial p_i} - \frac{\partial Q}{\partial x_i}\frac{\partial}{\partial t}\left(\frac{\partial R}{\partial p_i}\right) + \frac{\partial}{\partial t}\left(\frac{\partial R}{\partial x_i}\right)\frac{\partial Q}{\partial p_i} \tag{35}$$

Para ver que $[Q,R]$ es constante de movimiento es suficiente comparar las 2 últimas expresiones y notar que las derivadas parciales conmutan.

Para mostrar que si F y H son constantes de movimiento entonces $\frac{\partial^n F}{\partial t^n}$ es constante de movimiento basta notar que $[F,H] = -\frac{\partial F}{\partial t}$ es constante de movimiento *i.e*

$$\frac{d[F,H]}{dt} = \frac{d}{dt}\left(-\frac{\partial F}{\partial t}\right) = 0 \tag{36}$$

Para probar que esto cierto para la n-esima derivada se usa inducción:
Se probo para $n=1$ se cumple i.e. : $\frac{d}{dt}\left(-\frac{\partial F}{\partial t}\right) = 0$
Ahora supongamos que se cumple para n es decir: $\frac{d}{dt}\left(\frac{\partial^n F}{\partial t^n}\right) = [\frac{\partial^n F}{\partial t^n}, H] + \frac{\partial^{n+1} F}{\partial t^{n+1}} = 0$ pero como por hipotesis $\frac{\partial^n F}{\partial t^n}$ es constante de movimiento entonces $[\frac{\partial^n F}{\partial t^n}, H]$ tambien lo es y por lo tanto $\frac{\partial^{n+1} F}{\partial t^{n+1}} = -[\frac{\partial^n F}{\partial t^n}, H]$ tambien es constante de movimiento.

Finalmente tomando $F = x - \frac{pt}{m}$ y $H = \frac{p^2}{2m}$ se verifica facilmente que:

$$\left[x - \frac{pt}{m}, \frac{p^2}{2m}\right] = \left[x, \frac{p^2}{2m}\right] = \frac{2p}{2m}[x,p] = \frac{2p}{2m} = -\frac{\partial F}{\partial t} = -\left(-\frac{p}{m}\right)\frac{\partial t}{\partial t} = \frac{p}{m} \tag{37}$$

Ejercicio 9.31

Sea $u = \ln(p + im\omega q) - i\omega t$ y $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$ Entonces:

$$[u,H] = \frac{\partial u}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial u}{\partial p} = \left(\frac{im\omega}{p + imq\omega}\right)\frac{p}{m} - mq\omega^2\left(\frac{1}{p + imq\omega}\right) = i\omega \tag{38}$$

Por otro lado claramente $\frac{\partial u}{\partial t} = -i\omega$, por lo tanto u es una constante de movimiento.
Como sabemos las soluciones explicitas del problema podemos ver que es exactamente u. Si $q = A\sin(\omega t + \phi)$ entonces $p = m\omega A\cos(\omega t + \phi)$ y la cantidad dentro del logaritmo es: $m\omega A e^{i(\omega t + \phi)}$. De acuerdo a lo anterior $u = \ln m\omega A + i(\omega t + \phi) - i\omega t = i\phi + \ln\sqrt{2Em}$, donde se ha usado que $E = \frac{1}{2}m\omega^2 A^2$. u no es mas que el logaritmo de una función de la energía mas i veces la fase inicial del oscilador.