Math 304 Spring 2010
3/11/2010
Quiz #7 Name: ____________________

Show all work clearly and in order. Please box your answers. 10 minutes.

1. Show that the set

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^3$.

Using Theorem 3.3.2 : (i) if $x_1 = 0$, $x_2 = 0$ then we have $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0} \in V$ (so $\vec{0}$ is in V) ✓

(ii) Let $\vec{x} \in V$ so $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ for some $x_1, x_2 \in \mathbb{R}$

Let $\vec{y} \in V$ so $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}$ for some $y_1, y_2 \in \mathbb{R}$

then $\vec{x}+\vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ 0 \end{bmatrix}$ ← real numbers

so $\vec{x}+\vec{y} \in V$ (so $\vec{x}+\vec{y}$ is in V) ✓

(iii) Let $\vec{x} \in V$ so $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$ for some $x_1, x_2 \in \mathbb{R}$

Let $c \in \mathbb{R}$

then $c\vec{x} = c\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ 0 \end{bmatrix}$ ← real numbers

so $c\vec{x} \in V$ (so $c\vec{x}$ is in V) ✓

Hence by (i), (ii) and (iii) and Thm 3.3.2 we have $\boxed{V \text{ is a subspace of } \mathbb{R}^3}$

2. Show that the set $X = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right)$ is an (ordered) basis of $\mathbb{R}^3$.

Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R3 \to R3+R2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ ← this matrix is in REF (row echelon form) and has 3 pivot cols

hence $\text{rank}(A) = 3$. There are two ways to write your conclusion of why X is a basis of $\mathbb{R}^3$:

Solution 1:
$= \#$ columns $\Rightarrow$ X is lin independent by Thm 3.4.4
$= \#$ rows $\Rightarrow$ X spans all of $\mathbb{R}^3$ (i.e., $\text{Span}(X) = \mathbb{R}^3$) by lemma 3.3.6
So $\boxed{X \text{ is a basis of } \mathbb{R}^3}$

Solution 2: $\text{rank}(A) = 3 \Rightarrow$ A is invertible so by lemma 3.5.5 $\boxed{X \text{ is a basis of } \mathbb{R}^3}$

3. Give an example of a set of vectors in $\mathbb{R}^2$ that is linearly dependent.

There are infinitely many examples. The most obvious one is a set containing just the zero vector:

$$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

(see lemma 3.4.3)

Other examples: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\}$ (see corollary 3.4.5)

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ (again see lemma 3.4.3)

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ (again see lemma 3.4.3)

MANY others....