Math 304 Spring 2010
4/22/2010
Quiz #12 Name: Key

Show all work clearly and in order. Please box your answers. 10 minutes.

1. (a) Let $S = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ be the standard basis of $\mathbb{R}^2$. Let $X = \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$ be a basis of $\mathbb{R}^2$ (you do not need to show this). Let $F{:}\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by the matrix (with respect to the basis S)

$$ {}_SF_S = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}. $$

Find the matrix ${}_XF_X$.

I want to write ${}_XF_X = ({}_XI_S)({}_SF_S)({}_SI_X)$ so we need to find ${}_XI_S$ and ${}_SI_X$

${}_SI_X = \left[k_S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \;\; k_S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \right] = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ (since S was the standard basis)

${}_XI_S = ({}_SI_X)^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$ to find this $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2 \to R2+R1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]$

$\xrightarrow{R1 \to R1-R2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right]$ so ${}_XI_S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

hence ${}_XF_S = ({}_XI_S)({}_SF_S)({}_SI_X) = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & 1 \\ -5 & 0 \end{bmatrix}}$

(b) Show that F is an isomorphism. (Hint: use either ${}_XF_X$ or ${}_SF_S$).

F is an isomorphism if F is a linear transformation (given)
AND F is also a one-to-one correspondence.

well this can be shown by considering a matrix representing F. consider ${}_SF_S$. This matrix is invertible

since $\det({}_SF_S) = (1)(3) - (-1)(2) = 3 + 2 = 5 \neq 0$

hence ${}_SF_S$ is a one-to-one correspondence thought of as a function. i.e. F is a one to one correspondence.
Together with "F is a linear transformation" from (a)
we have F is an isomorphism