COMMENTS FOR LECTURE 4 - 1.29.2010

NATHAN REFF

Today we started to discuss 1.3.3 and unfortunately I did not get to tell you the end of the story. I hope that the following is not too difficult to follow so we can do the homework for Monday.

Writing the solution set to a system of Linear Equations

We start with a system of linear equations $S \longleftrightarrow$ Augmented matrix $A \rightarrow$ Take off the last column of A to get the coefficient matrix C

Let A' be the matrix obtained by reducing A until C is in reduced row echelon form (RREF) and let C' be the matrix obtained by deleting the last column of A'

In some sense think of the following picture to help make sense of the notation:

$$A = \left[\begin{array}{c|c} C & b \end{array}\right] \xrightarrow{\text{Putting } C \text{ into RREF}} A' = \left[\begin{array}{c|c} C' & b' \end{array}\right]$$

Where b is the rightmost column of numbers in A and b' is the rightmost column of numbers in A'.

For example: If $A = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ -1 & -2 & 1 & 1 \\ -1 & -2 & 0 & 5 \end{array}\right]$

Then

$$C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

And after putting C into RREF we get:

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ -1 & -2 & 1 & 1 \\ -1 & -2 & 0 & 5 \end{array}\right] \xrightarrow{\text{Putting } C \text{ into RREF}} A' = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

Then

$$C' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$$

What can we say at this point?

1. If the right most column of A' has a pivot position THEN S has no solutions. (The system is *inconsistent*).

For example: See the above example!

2. If the right most column of A' has NO pivot position THEN S will have a solution to the system. The questions now is how many solutions and how do we represent them?

Well, are there are any *free variables*?

If the answer is **NO**: Then there is exactly one solution that can be read directly from A'.

For example: If $A' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Then $x_1 = 2$, $x_2 = -1$ and $x_3 = 3$.

If the answer is **YES**: Then there are infinitely many solutions. We just need to know how to write all of the solutions. They key here is to solve for the *basic variables* in terms of the *free variables*.

For example: If $A' = \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Here the free variables are x_2 and x_4 (these can take on any value), and the basic variables are x_1 and x_3.

Row 2 says $x_3 = 2$ and row 1 says $x_1 - x_2 = 3$ so in other words $x_1 = 3 + x_2$.

So all solutions to the system are of the form:

$$\begin{aligned} x_1 &= 3 + x_2 \\ x_2 &= \text{Anything} \\ x_3 &= 2 \\ x_4 &= \text{Anything} \end{aligned}$$

Here are some practice examples with RREF. Please try to do them on your own and then check your answers!

Simpler Example:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ -1 & 2 & -1 & 1 \\ 2 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R2 \to R2+R1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R3 \to R3-2R1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 3 & 1 & -4 \end{array}\right] \xrightarrow{R3 \to R3-3R2}$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -13 \end{array}\right] \xrightarrow{R1 \to R1+R2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -13 \end{array}\right] \xrightarrow{R1 \to R1-R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -13 \end{array}\right]$$

So we have exactly one solution to the system: $x_1 = 18$, $x_2 = 3$ and $x_3 = 13$. Now just to make sure that we did not make a mistake we should check our answer...

Good Practice with fractions:

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 10 \\ 2 & 1 & 2 & 5 \\ -2 & 2 & 3 & 1 \end{array}\right] \xrightarrow{R2 \to R2-\frac{2}{3}R1} \left[\begin{array}{ccc|c} 3 & 1 & -1 & 10 \\ 0 & 1/3 & 8/3 & -5/3 \\ -2 & 2 & 3 & 1 \end{array}\right] \xrightarrow{R3 \to R3+\frac{2}{3}R1}$$
$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 10 \\ 0 & 1/3 & 8/3 & -5/3 \\ 0 & 8/3 & 7/3 & \frac{23}{3} \end{array}\right] \xrightarrow{R3 \to R3-8R2} \left[\begin{array}{ccc|c} 3 & 1 & -1 & 10 \\ 0 & 1/3 & 8/3 & -5/3 \\ 0 & 0 & -19 & 21 \end{array}\right] \xrightarrow{R1 \to \frac{1}{3}R1}$$
$$\left[\begin{array}{ccc|c} 1 & 1/3 & -1/3 & 10/3 \\ 0 & 1/3 & 8/3 & -5/3 \\ 0 & 0 & -19 & 21 \end{array}\right] \xrightarrow{R2 \to 3R2} \left[\begin{array}{ccc|c} 1 & 1/3 & -1/3 & 10/3 \\ 0 & 1 & 8 & -5 \\ 0 & 0 & -19 & 21 \end{array}\right] \xrightarrow{R1 \to R1-\frac{1}{3}R2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 5 \\ 0 & 1 & 8 & -5 \\ 0 & 0 & -19 & 21 \end{array}\right] \xrightarrow{R3 \to \frac{-1}{19}R3}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 5 \\ 0 & 1 & 8 & -5 \\ 0 & 0 & 1 & -\frac{21}{19} \end{array}\right] \xrightarrow{R1 \to R1+3R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{32}{19} \\ 0 & 1 & 8 & -5 \\ 0 & 0 & 1 & -\frac{21}{19} \end{array}\right] \xrightarrow{R2 \to R2-8R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{32}{19} \\ 0 & 1 & 0 & \frac{73}{19} \\ 0 & 0 & 1 & -\frac{21}{19} \end{array}\right]$$

So we have exactly one solution to the system: $x_1 = 32/19$, $x_2 = 73/19$ and $x_3 = -21/19$. Now just to make sure that we did not make a mistake we should check our answer...

Department of Mathematical Sciences, Binghamton University (SUNY), Binghamton, NY 13902-6000, U.S.A.

E-mail address: reff@math.binghamton.edu