examples involving coordinate transformations.

e.g. There is a vector (polynomial) $p(x)$ in P_2 which has the coordinate vector $K_\gamma(p(x)) = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

(i) with respect to the basis $B_1 = \{1, x, x^2\}$, find $p(x)$. (so here $\gamma = B_1$)

(ii) with respect to the basis $B_2 = \{1-x, x, x^2\}$, find $p(x)$. (so here $\gamma = B_2$)

SOL: (i) $p(x) = (1)(1) + (-1)(x) + (3)(x^2)$

$= 1 - x + 3x^2$

(ii) $p(x) = 1(1-x) + (-1)(x) + (3)x^2$

$= 1 - x - x + 3x^2$

$= 1 - 2x + 3x^2$

exercise: show B_2 is a basis of P_2 (used in this example)

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e.g. (i) find the coordinate vector $K_{B_1}(p(x))$ ~~with res~~

if $p(x) = 5 + 6x + 6x^2$ (already as a lin. comb. of elements from $B_1 = \{1, x, x^2\}$)

SOL: $K_{B_1}(p(x)) = \begin{bmatrix} 5 \\ 6 \\ 6 \end{bmatrix}$ ← $\boxed{SO}$

(ii) find the coordinate vector $K_{B_2}(p(x))$

if $p(x) = 5 + 6x + 6x^2$

we need to write $p(x)$ as a linear combination of elements in $B_2 = \{1-x, x, x^2\}$

SOL: $p(x) = 5 + (-5x + 11x) + 6x^2$

$= 5 - 5x + 11x + 6x^2$

$= 5(1-x) + 11x + 6x^2$

so $K_{B_2}(p(x)) = \begin{bmatrix} 5 \\ 11 \\ 6 \end{bmatrix}$

Showing a set is linearly independent in P_2

e.g. Let $X = \{1, 1-x\}$ (this is a set of vectors/polynomials in P_2) Show that X is linearly independent in P_2.

SOL ① to show X is linearly independent we show that the ONLY solution to the equation

$$c_1(1) + c_2(1-x) = 0 \quad (*)$$

is when $c_1 = 0$ and $c_2 = 0$ (see definition of linearly independent on p 165)

well let's manipulate this equation $(*)$:

$$c_1(1) + c_2(1-x) = 0$$

$$\iff c_1 + c_2 - c_2 x = 0$$

$$\iff (c_1 + c_2)1 + (-c_2)x = 0 \quad (**)$$

but we know the set $\{1, x\}$ is linearly independent (see prop 4.4.1 on p174)

So the only way for $(**)$ to be true is for both $c_1 + c_2 = 0$ AND $-c_2 = 0$ by definition of linear independence.

SO →

just solve this linear system:

$$c_1 + c_2 = 0$$
$$-c_2 = 0$$

This is solving the equation:

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is done by row reduction:

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow{R1 \to R1+R2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow{R2 \to -R2}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

so the only solution is

$$c_1 = 0$$
$$c_2 = 0$$

This is what we wanted to show originally with $(*)$. Hence

$$X = \{1, 1-x\}$$ is

linearly independent in P_2.

$\boxed{\text{SOL } ②}$ There is another way to solve this problem via isomorphisms.

Let $S = \{1, x, x^2\}$ be a basis of P_2
(we know this is a basis of P_2: see p174)
we want to show $X = \{p_1(x) = 1,\ p_2(x) = 1 - x\}$ is linearly independent.
consider the matrix:

$$A = \left[\, K_s(p_1(x)) \quad K_s(p_2(x)) \,\right]$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

putting A into RREF :

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 + R2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \xrightarrow{R2 \to -R2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

since the rank of A is 2 and 2 is the number of columns of A, the set of vectors

$$K_s(X) = \Big\{ \underset{K_s(p_1(x))}{\underset{\shortparallel}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}},\ \underset{K_s(p_2(x))}{\underset{\shortparallel}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}} \Big\}$$ is linearly independent in $\mathbb{R}^3$

and since K_S is an isomorphism, the set of vectors
$X = \{\underset{p_1(x)}{1},\ \underset{p_2(x)}{1 - x}\}$ is linearly independent in P_2.