EXAM 2

Score: ________ out of 100

Math 324 - Linear Algebra   Name: ______________________________

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please $\boxed{\text{box/circle}}$ or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 7 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Circle your answer for each of the following:

(a) | True | False | Every vector space is a subspace of itself.

(b) | True | False | If V is a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a collection of vectors in V, then Span(S) is always a subspace of V.

(c) | True | False | The solution set of a consistent linear system $A\mathbf{x} = \mathbf{0}$ of m equations in n variables is a subspace of $\mathbb{R}^n$.

(d) | True | False | The solution set of a consistent linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n variables is a subspace of $\mathbb{R}^n$.

(e) | True | False | If V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a collection of linearly independent vectors in V, then for any nonzero scalar k, $\{k\mathbf{v}_1, k\mathbf{v}_2, \ldots, k\mathbf{v}_k\}$ is also a collection of linearly independent vectors in V.

(f) | True | False | A set containing a single vector is always linearly independent.

(g) | True | False | Every dependent set contains the zero vector.

(h) | True | False | $\text{Row}(A)^{\perp} = \text{Null}(A)$.

(i) | True | False | $\text{Span}(\{1, x, 1-x\}) = P_2$.

(j) | True | False | Let $T{:}V \to W$ be a linear transformation, then $\text{range}(T) = \text{im}(T) = \{T(\mathbf{x}) : \mathbf{x} \in V\}$.

2. Suppose a 5×9 matrix A has rank 4. Then
$m \times n$

(a) The dimension of the column space of A is $\boxed{4}$

(b) The dimension of the row space of A is $\boxed{4}$

(c) The dimension of the null space of A is $\boxed{5}$

(d) The dimension of the null space of A^T is $\boxed{1}$

3. Show that $W = \{a_1x + a_4x^4 : a_1, a_4 \in \mathbb{R}\}$ forms a subspace of P_4.

(i) $\vec{0} \in P_4$ is 0 (the constant polynomial.)
since $0 = 0x + 0x^4 \in W$. ✓

(ii) Let $a_1x + a_4x^4 \in W$, $k \in \mathbb{R}$.

$$k(a_1x + a_4x^4) = \underbrace{(ka_1)}_{\in \mathbb{R}}x + \underbrace{(ka_4)}_{\in \mathbb{R}}x^4 \in W. \checkmark$$

(iii) Let $a_1x + a_4x^4$, $b_1x + b_4x^4 \in W$

$$(a_1x + a_4x^4) + (b_1x + b_4x^4) = (a_1+b_1)x + (a_4+b_4)x^4 \in W$$ ✓

4. Determine whether or not $S = \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^3$.

$\boxed{\text{SOL 1}}$: Show $A = \begin{bmatrix} 3 & -1 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ has zero determinant.

$$\det(A) = \begin{vmatrix} 3 & -1 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 1 \end{vmatrix} = -12 \neq 0$$

hence S is a basis of $\mathbb{R}^3$ ∎

$\boxed{\text{SOL 2}}$: (i) S spans $\mathbb{R}^3$. i.e., $\text{span}(S) = \mathbb{R}^3$
That is, Show $\forall \vec{b} \in \mathbb{R}^3$ we can write $\vec{b} = c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -4 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

which means we want to show the linear system is consistent (always has a solution. That is, $[A \mid \vec{b}]$ is consistent.

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & b_1 \\ 0 & -4 & 3 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & -1/3 & -1/3 & b_1/3 \\ 0 & 1 & 3/-4 & b_2/-4 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

consistent! so $\vec{b}$ is always in $\text{span}(S)$. ✓

(ii) S is linearly independent. The REF work shows we have 3 pivots (3 leading 1's) and since we are working in $\mathbb{R}^3$ this shows S is linearly independent.

Hence, S is a basis of $\mathbb{R}^3$ ∎

$\boxed{\text{SOL 3}}$: use anything on the Equivalent Statements Thm to get an alternate proof.

5. Let $T : M_{22} \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 - x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

(a) Show that T is a linear transformation

$$\text{(i)}\quad T\left(k\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) = T\left(\begin{bmatrix} kx_1 & kx_2 \\ kx_3 & kx_4 \end{bmatrix}\right) = \begin{bmatrix} kx_1 - kx_2 \\ kx_3 \\ kx_4 \end{bmatrix} = k\begin{bmatrix} x_1 - x_2 \\ x_3 \\ x_4 \end{bmatrix} = kT\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right)$$

$$\begin{aligned}
\text{(ii)}\quad T\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} + \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}\right) &= T\left(\begin{bmatrix} x_1+y_1 & x_2+y_2 \\ x_3+y_3 & x_4+y_4 \end{bmatrix}\right) \\
&= \begin{bmatrix} (x_1+y_1) - (x_2+y_2) \\ x_3+y_3 \\ x_4+y_4 \end{bmatrix} \\
&= \begin{bmatrix} x_1 - x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} y_1 - y_2 \\ y_3 \\ y_4 \end{bmatrix} = T\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}\right)
\end{aligned}$$

(b) Find $\ker(T)$

$$\begin{aligned}
\ker(T) &= \{ \vec{x} \in M_{22} \mid T(\vec{x}) = \vec{0} \in \mathbb{R}^3 \} \\
&= \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in M_{22} \;\middle|\; T\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \;\middle|\; \begin{bmatrix} x_1 - x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \;\middle|\; \underbrace{x_1 - x_2 = 0}_{x_1 = x_2} \text{ AND } x_3 = 0 \text{ AND } x_4 = 0 \right\} \\
&= \left\{ \begin{bmatrix} x_1 & x_1 \\ 0 & 0 \end{bmatrix} \right\}
\end{aligned}$$

6. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$

(a) Find a basis for Null(A).

Solve: $A\vec{x} = \vec{0}$. $\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ -1 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R2 \to R2+R1} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$

free variables

$x_1 - x_2 - x_4 = 0$
$x_3 = 0$
so
$x_1 = s + t$
$x_2 = s$
$x_3 = 0$
$x_4 = t$
, so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

basis for Null(A): $\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$

(b) Find a basis for Row(A)

using the work in part (a) the RREF form of A: $\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

leading 1's. give the rows that form a basis of Row(A):

$\boxed{\{ [1\ -1\ 0\ -1], [0\ 0\ 1\ 0] \}}$

(c) Find a basis for Col(A)

The leading 1's in the RREF form of A point to the colums of A that form a basis of Col(A). i.e., ~~the~~ columns 1 and 3 form a basis of Col(A):

$\boxed{\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}}$

(d) Find a basis for Null(A^T)

Same idea as part (a). Solve $A^T\vec{x} = \vec{0}$. $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow[R4 \to R4+R1]{R2 \to R2+R1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 - x_2 = 0$
$x_2 = 0$
so
$x_1 = 0$
$x_2 = 0$

The only vector in Null(A^T) $= \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ so the basis is the empty set $\emptyset$.

(e) rank(A) = $\boxed{2}$

(f) nullity(A) = $\boxed{2}$

(g) nullity(A^T) = $\boxed{0}$

(h) The columns of A span a $\boxed{2}$ dimensional subspace of $\boxed{\mathbb{R}^2}$

(i) The rows of A span a $\boxed{2}$ dimensional subspace of $\boxed{\mathbb{R}^4}$

7. **Solve 2 of the following problems**. Please put an X through the problem that you do not want graded (otherwise I will grade the first two problems worked on).

(a) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ be a collection of vectors in some vector space V. Suppose $\mathbf{v}_1 + 2\mathbf{v}_2 = \mathbf{v}_3 - \mathbf{v}_4$. Show that S is a set of linearly dependent vectors.

$$v_1 + 2v_2 - v_3 + v_4 = 0$$

nontrivial combination of v_1, v_2, v_3, v_4, v_5 (coef. of v_5 is 0)

Hence, S is linearly dependent.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and suppose $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

Let $\mathbf{x} \in \ker(T)$.

Compute $T\left(2\mathbf{x} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T(2\vec{x}) - T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$$= 2T(\vec{x}) - \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

since $\vec{x} \in \ker(T)$

$$= 2\begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}$$

(c) Prove that for any matrix A, $\text{rank}(A^T) = \text{rank}(A)$.

SOL 1

$$\text{rank}(A) = \dim(\text{Col}(A))$$
$$= \dim(\text{Row}(A^T))$$
$$= \text{rank}(A^T)$$

SOL 2

$$\text{rank}(A) = \dim(\text{Row}(A))$$
$$= \dim(\text{Col}(A^T))$$
$$= \text{rank}(A^T)$$

N.B. we are using previous knowledge:
for a matrix B
$\text{rank}(B) = \dim(\text{Col}(B)) = \dim(\text{Row}(B))$