Final Review MTH 201

1. An object whose position (in feet) at time t (where t is in seconds) is given by $s(t) = 2t^2 + 5t + 1$, (label your answers appropriately)

   (a) Find the average velocity of the object on the interval $[0, 2]$.

   $$A.V. = \frac{\Delta \text{displacement}}{\Delta \text{time}}$$

   $$A.V. = \frac{s(2) - s(0)}{2 - 0}$$

   $$A.V. = \frac{19 - 1}{2}$$

   $$A.V. = 9 \text{ ft/sec}$$

   $s(2) = 2(2)^2 + 5(2) + 1$
   $s(2) = 8 + 10 + 1$
   $s(2) = 19$
   $s(0) = 2(0)^2 + 5(0) + 1$
   $s(0) = 1$

   (b) Find the instantaneous velocity of the object at $t = 1$.

   $$v(t) = s'(t) = 4t + 5$$

   $$v(1) = 4(1) + 5 = 9 \text{ ft/sec}$$

2. Determine the following limits:

   (a) $$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 7x + 10} = \lim_{x \to 2} \frac{(x+3)\cancel{(x-2)}}{(x-5)\cancel{(x-2)}} = \lim_{x \to 2} \frac{(x+3)}{(x-5)} = \frac{(2)+3}{(2)-5} = \frac{5}{-3}$$

   This limit can also be solved using L'H Rule

   (b) $$\lim_{x \to \infty} \frac{5x^3 - 1}{x + x^2 - 2x^3} = \lim_{x \to \infty} \frac{\frac{5x^3}{x^3} - \frac{1}{x^3}}{\frac{x}{x^3} + \frac{x^2}{x^3} - \frac{2x^3}{x^3}} = \lim_{x \to \infty} \frac{5 + \overset{\to 0}{\frac{-1}{x^3}}}{\underset{\downarrow 0}{\frac{1}{x^2}} + \underset{\downarrow 0}{\frac{1}{x}} - 2} = \frac{5}{-2}$$

   "divide numerator and denominator by highest power of x present in denominator"

   (c) $$\lim_{x \to -\infty} \frac{500x^2 + 750x + 1000}{1 - 3x^3}$$

   $$= \lim_{x \to -\infty} \left(\frac{\overset{\to 0}{\frac{500}{x}} + \overset{\to 0}{\frac{750}{x^2}} + \overset{\to 0}{\frac{1000}{x^3}}}{\underset{\to 0}{\frac{1}{x^3}} - 3} \right) = 0$$

   These limits can also be solved using L'H Rule

   (d) $$\lim_{x \to -\infty} \frac{1 - x^3}{x^2 + x + 1000}$$

   $$= \lim_{x \to -\infty} \left(\frac{\frac{1}{x^2} - x}{1 + \underset{\downarrow 0}{\frac{1}{x}} + \underset{\downarrow 0}{\frac{1000}{x^2}}} \right) = \infty$$

(e) $\lim_{x \to \frac{\pi}{2}} \cos(2x + \cos x) = \cos\left(2\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right) = \cos(\pi + 0)$

$= \cos(\pi) = -1$

(f) $\lim_{x \to 2^-} \frac{x^2 - 4}{|x - 2|} = \lim_{x \to 2^-} \frac{\cancel{(x-2)}(x+2)}{-\cancel{(x-2)}} = \frac{2+2}{-1} = -4$

Note: negative here because as x approaches 2 from the left, $(x-2) < 0 \Rightarrow |x-2| = -(x-2) > 0$.

(g) $\lim_{x \to 0} \left(\frac{1 - \cos 2x}{x} + \frac{\sin 3x}{x}\right) = \lim_{x \to 0} \frac{1-\cos 2x}{x} + \lim_{x \to 0} \frac{\sin 3x}{x}$

Note: you could also use L'Hospital's Rule.

$= 2 \cdot \lim_{x \to 0} \frac{1 - \cos 2x}{2x} + 3 \cdot \lim_{x \to 0} \frac{\sin 3x}{3x}$

$= -2 \lim_{x \to 0} \frac{\cos 2x - 1}{2x} + 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = -2(0) + 3(1) = \boxed{3}$

(h) $\lim_{x \to 6^-} [x] = 5$

SEE EXAMPLE 10 (p105) for the definition of this function.
It is sometimes called the greatest integer function,
or floor function.

note

$[x] = \lfloor x \rfloor$

"Floor Function"

(i) $\lim_{x \to 6} [x] =$ DNE since $\lim_{x \to 6^+} \lfloor x \rfloor = 6 \neq \lim_{x \to 6^-} \lfloor x \rfloor = 5$

(j) $\lim_{x \to 6^+} ([x] + 2x) = 6 + 2(6) = 18$

(k) $\lim_{x \to 1} \frac{\ln x}{x - 1} \overset{LH}{=} \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$

(l) $\lim_{x\to\infty} 5x^2e^{-x}$ $= \lim_{x\to\infty} \frac{5x^2}{e^x}$ $\left(\frac{\infty}{\infty}\right) \overset{LH}{=} \lim_{x\to\infty} \frac{10x}{e^x} \overset{LH}{=} \lim_{x\to\infty} \frac{10}{e^x} = \boxed{0}$

(m) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$ $\left(\frac{0}{0}\right) \overset{LH}{=} \lim_{x\to 0} \frac{e^x - 1}{2x}$ $\left(\frac{0}{0}\right) \overset{LH}{=} \lim_{x\to 0} \frac{e^x}{2} = \boxed{\frac{1}{2}}$

(n) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$. $\left(\frac{\infty}{\infty}\right) \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{2x^{1/2}}{1} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = \boxed{0}$

3. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^6 \cos(\ln|x|) = 0$

$-1 \le \cos\theta \le 1$

$\Rightarrow \quad -1 \le \cos(\ln|x|) \le 1$

$\Rightarrow \quad -x^6 \le x^6\cos(\ln|x|) \le x^6.$

$\lim_{x\to 0} -x^6 = 0$ and $\lim_{x\to 0} x^6 = 0$ $\underset{\text{Squeeze Thm.}}{\Rightarrow}$ $\lim_{x\to 0} x^6\cos(\ln|x|) = 0$

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4. What are the vertical and horizontal asymptotes of the function $f(x) = \dfrac{2x^2-6x}{x^2-9}$? Label the asymptotes as to whether they are vertical or horizontal. Be sure to clearly and completely justify your answers.

$$f(x) = \frac{2x^2-6x}{x^2-9} = \frac{2x(\cancel{x-3})}{(x+3)(\cancel{x-3})} \qquad D\{x \mid x \neq \pm 3\}$$

$$f(x) = \frac{2x}{x+3} \quad \text{for } x \neq 3$$

$\boxed{\begin{array}{c} V.A. \\ x=-3 \end{array}}$

$\boxed{H.A. \mid y=2}$

$$\cancel{\lim_{x\to\infty} \frac{2x^2-6x}{x^2-9}} \quad \lim_{x\to\infty} \frac{2x}{x+3} = \lim_{x\to\infty} \frac{2}{1+\frac{3}{x}} = \boxed{2} \quad ; \quad \lim_{x\to-\infty} \frac{2x}{x+3} = \lim_{x\to-\infty} \frac{2}{1+\frac{3}{x}} = \boxed{2}$$

0

This repeats

5. What is the definition of the derivative of $f(x)$?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{or} \quad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

6. Find the derivative of $g(x) = \frac{4}{2x-3}$ using the definition of the derivative.

$$g'(x) = \lim_{h\to 0} \left(\frac{\frac{4}{2(x+h)-3} - \frac{4}{2x-3}}{h} \right) = \lim_{h\to 0} \left(\frac{4(2x-3) - 4(2(x+h)-3)}{(2(x+h)-3)(2x-3)} \right)\left(\frac{1}{h}\right)$$

$$= \lim_{h\to 0} \left(\frac{\cancel{8x}\cancel{-12} - (\cancel{8x}+8h\cancel{-12})}{(2(x+h)-3)(2x-3)} \right)\left(\frac{1}{h}\right)$$

$$= \lim_{h\to 0} \left(\frac{-8\cancel{h}}{(2(x+h)-3)(2x-3)} \right)\left(\frac{1}{\cancel{h}}\right) = \frac{-8}{(2(x+0)-3)(2x-3)} = \frac{-8}{(2x-3)^2}$$

$\square$

7. Find the derivative of $g(x) = x^2 + 3x$ using the definition of the derivative.

$$g'(x) = \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - (x^2+3x)}{h} = \lim_{h \to 0} \left(\frac{1}{h}\right)\left(\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h - \cancel{x^2} - \cancel{3x}\right)$$

$$= \lim_{h \to 0} \left(\frac{1}{\cancel{h}}\right)\left(\cancel{h}(2x + h + 3)\right) = 2x + 0 + 3 = 2x + 3$$

B

8. Differentiate.

(a) $f(x) = 3x^5 - 4x^2 + x - 2 + 6e^x + 5^e$

$f'(x) = 15x^4 - 8x + 1 + 6e^x$

(b) $h(\theta) = \tan(\sin(7\theta + 1))$

$h'(\theta) = \sec^2(\sin(7\theta + 1)) \cdot \frac{d}{d\theta}(\sin(7\theta + 1))$

$h'(\theta) = \sec^2(\sin(7\theta + 1)) \cdot \cos(7\theta + 1) \cdot 7$

(c) $y = x\tan^{-1}(3x) - \sin^{-1}(3x) + [\sin 3x]^{-1}$

$$y' = x\left(\frac{3}{1+(3x)^2}\right) + \tan^{-1}(3x) - \frac{1}{\sqrt{1-(3x)^2}}\cdot 3 - (\sin(3x))^{-2}\cdot\cos(3x)\cdot 3$$

(d) $y = [3\ln x + \cot x]^{\ln 4} - \tan^7[5x + \pi^x]^2$

$$y' = \ln 4\,[3\ln x + \cot x]^{\ln 4 - 1}\left(\frac{3}{x} - \csc^2 x\right) - 7\tan^6(5x+\pi^x)^2\cdot\sec^2(5x+\pi^x)^2\cdot 2(5x+\pi^x)\cdot$$

$$(5 + \pi^x \ln\pi)$$

(e) $h(x) = 1 - \dfrac{7}{\sqrt{x}} + \dfrac{2}{3x+5} - 3\sqrt{2x-1} + \dfrac{5}{6x}$

$$h'(x) = \frac{7}{2}x^{-3/2} - 2(3x+5)^{-2}(3) - 3\left(\frac{1}{2}(2x-1)^{-1/2}\cdot 2\right) + \frac{5}{6}(-1)x^{-2}$$

$$h'(x) = \frac{7}{2\sqrt{x^3}} - \frac{6}{(3x+5)^2} - \frac{3}{\sqrt{2x-1}} - \frac{5}{6x^2}$$

(f) $g(x) = \dfrac{\cos 4x}{3 - \sin 4x}$

$$g'(x) = \frac{(3-\sin 4x)(-\sin 4x)(4) - (\cos 4x)(-\cos 4x)4}{(3-\sin 4x)^2}$$

(g) $y = e^{x^3+x^2+x+1} + \ln(2 - 5x + 7x^3) - \pi^3$

$$y' = e^{x^3+x^2+x+1}\cdot(3x^2+2x+1) + \frac{-5+21x^2}{2-5x+7x^3}$$