$\boxed{(21)\ 2}$

T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ i.e.)

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

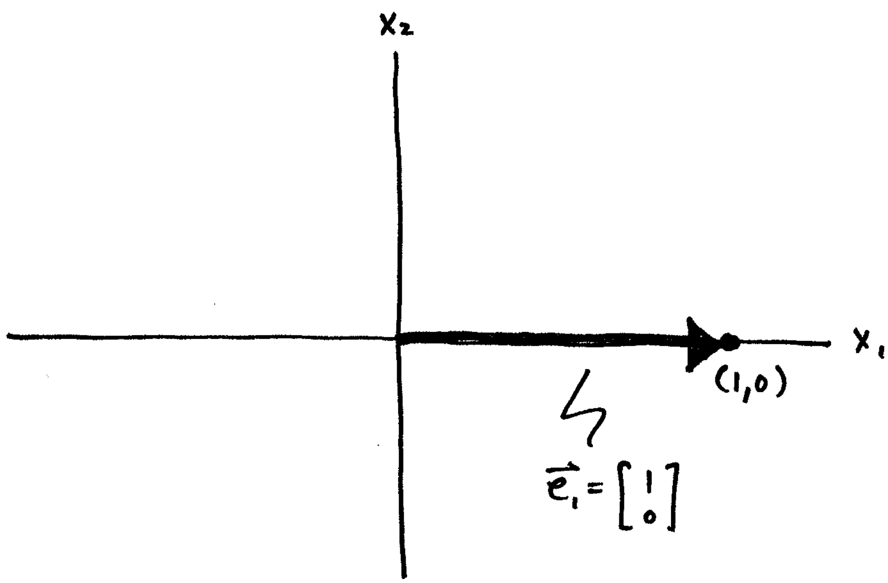
So this means A will be a $2\times 2$ matrix.

and

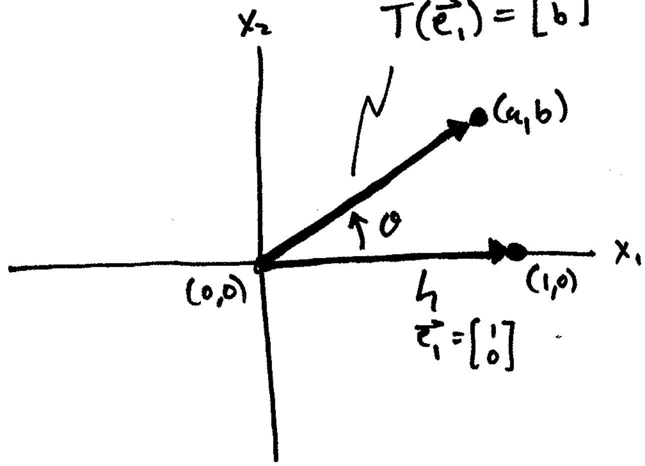
$$A = [\, T(\vec{e}_1) \quad T(\vec{e}_2)\, ]$$

So we need to do <u>two calculations</u> since there are two standard basis vectors in $\mathbb{R}^2$.

Let's find $T(\vec{e}_1)$:

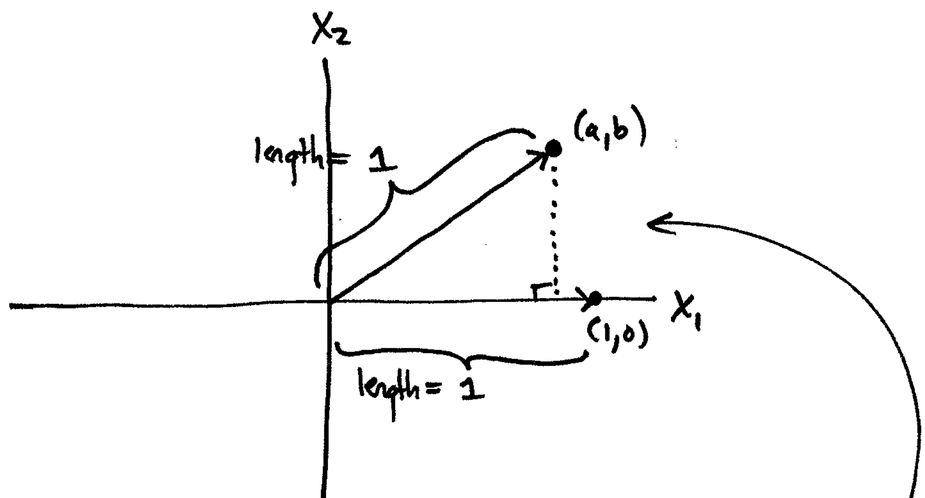



T will rotate $\vec{e}_1$ an angle of $\theta$ counterclockwise around the origin. So we have the picture:



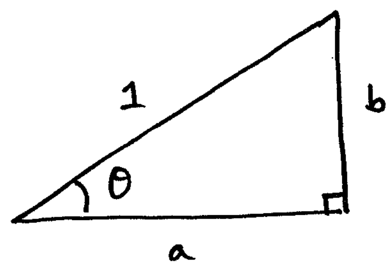

So we need to find $T(\vec{e}_1)$. Do you see the geometry here? What is the distance from the origin $(0,0)$ to $(1,0)$?

It is 1. So when we rotate the vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to the vector $T(\vec{e}_1) = \begin{bmatrix} a \\ b \end{bmatrix}$ will the length from $(0,0)$ to $(a,b)$ change? NO! It should stay 1. So we have the following picture:




Now form a right triangle as seen here

So we have the following triangle.



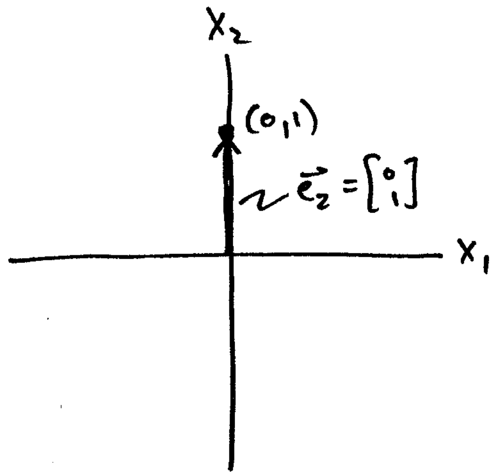
Can we solve for a and b? Sure! Use trig...

$$\sin\theta = \frac{b}{1} \quad \underline{\text{and}} \quad \cos\theta = \frac{a}{1}$$

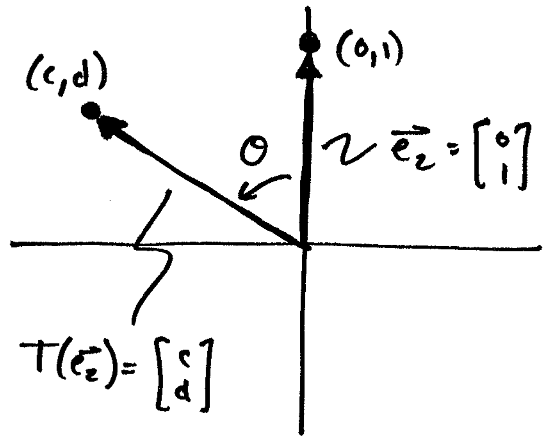
So $b = \sin\theta$ and $a = \cos\theta$

So $T(\vec{e_1}) = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

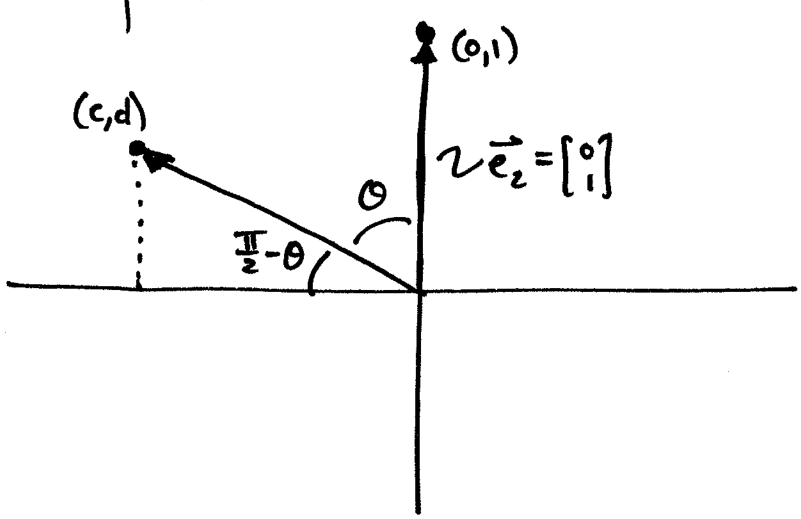
Now find $T(\vec{e_2})$:



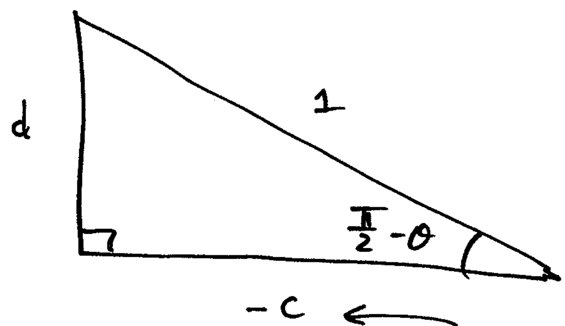

T will rotate $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ an angle of $\theta$ counterclockwise around the origin. So we have the picture




So we need to find $T(\vec{e}_2)$. Again we use trig and the picture:



To give us the triangle:




why do I need the minus here??
because you are in quadrant II
(remember your trig!)

using trig:

$$\sin\left(\frac{\pi}{2}-\theta\right)=\frac{d}{1}$$

$$\underbrace{\sin\left(\frac{\pi}{2}-\theta\right)}=d$$

$$\cos(\theta)=d$$

to see this you can either remember the formula OR use the formula

$$\sin(a-b)=\sin(a)\cos(b)-\cos(a)\sin(b)$$

$$\sin\left(\frac{\pi}{2}-\theta\right)=\sin\left(\frac{\pi}{2}\right)\cos(\theta)-\cos\left(\frac{\pi}{2}\right)\sin(\theta)$$

$$=\cos\theta-0$$

$$=\cos\theta$$

need this here

$$\cos\left(\frac{\pi}{2}-\theta\right)=\frac{-c}{1}$$

$$\underbrace{\cos\left(\frac{\pi}{2}-\theta\right)}=-c$$

$$\sin(\theta)=-c$$

$$c=-\sin(\theta)$$

to see this you can either remember the formula OR use the formula

$$\cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b)$$

$$\cos\left(\frac{\pi}{2}-\theta\right)=\cos\left(\frac{\pi}{2}\right)\cos(\theta)+\sin\left(\frac{\pi}{2}\right)\sin\theta$$

$$=0+\sin\theta$$

$$=\sin\theta$$

phew... so $T(\vec{e}_2)=\begin{bmatrix} c \\ d \end{bmatrix}=\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

so $A=\left[T(\vec{e}_1)\quad T(\vec{e}_2)\right]=\boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$