Math 222 Spring 2011
2/25/2011
Quiz #5

Name: Key

Please box your answers. Show all work clearly and in order.

$\boxed{5}$ 1. Evaluate

$$\int \frac{x}{\sqrt{25-x^2}}\,dx.$$

$\boxed{\text{SOLUTION 1}}$: Let $u = 25 - x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

so,

$$\int \frac{x}{\sqrt{25-x^2}}\,dx = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{-2x}$$

$$= \frac{-1}{2}\int u^{-1/2}\,du$$

$$= \frac{-1}{2}\,\frac{u^{1/2}}{1/2} + C$$

$$= \boxed{-\sqrt{25-x^2} + C}$$

$\boxed{\text{SOLUTION 2}}$: Let $x = 5\sin\theta$, $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

$\frac{dx}{d\theta} = 5\cos\theta \Rightarrow dx = 5\cos\theta\,d\theta$

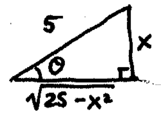
$$\sqrt{25-x^2} = \sqrt{25-(5\sin\theta)^2} = \sqrt{25-25\sin^2\theta}$$
$$= \sqrt{25(1-\sin^2\theta)}$$
$$= \sqrt{25\cos^2\theta}$$
$$= 5|\cos\theta|$$
$$= 5\cos\theta, \text{ since } \theta\in[\tfrac{-\pi}{2},\tfrac{\pi}{2}]$$

so, $$\int \frac{x}{\sqrt{25-x^2}}\,dx = \int \frac{5\sin\theta}{\cancel{5\cos\theta}} \cdot \cancel{5\cos\theta}\,d\theta$$

$$= -5\cos\theta + C = -5\left(\frac{\sqrt{25-x^2}}{5}\right) + C$$

$$= \boxed{-\sqrt{25-x^2} + C}$$

$x = 5\sin\theta$
$\Rightarrow \sin\theta = \frac{x}{5}$
so



$\Rightarrow \cos\theta = \frac{\sqrt{25-x^2}}{5}$

$\boxed{5}$ 2. Pick ONE of the following. Cross out the problem you do not want graded. Otherwise I will grade the first problem worked on.

(a) Evaluate: $\displaystyle\int \frac{\cos(x)}{\sqrt{1+\sin^2(x)}}\,dx.$

(b) Evaluate: $\displaystyle\int \frac{x}{\sqrt{4x^2-4}}\,dx.$

$\boxed{\text{SOLUTION}}$ Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x) \Rightarrow dx = \frac{du}{\cos(x)}$

So $$\int \frac{\cos(x)}{\sqrt{1+\sin^2(x)}}\,dx = \int \frac{\cancel{\cos(x)}}{\sqrt{1+u^2}} \cdot \frac{du}{\cancel{\cos(x)}}$$

$$= \int \frac{1}{\sqrt{1+u^2}}\,du$$

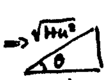
Let $u = \tan\theta \Rightarrow \frac{du}{d\theta} = \sec^2\theta$
$(\theta \in (\tfrac{-\pi}{2},\tfrac{\pi}{2}))$
$\Rightarrow du = \sec^2\theta\,d\theta$

$$\sqrt{1+u^2} = \sqrt{1+\tan^2\theta}$$
$$= \sqrt{\sec^2\theta}$$
$$= |\sec\theta|$$
$$= \sec\theta, \text{ since } \theta\in(\tfrac{-\pi}{2},\tfrac{\pi}{2})$$

So $$\int \frac{1}{\sqrt{1+u^2}}\,du = \int \frac{1}{\sec\theta} \cdot \sec^2\theta\,d\theta = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$u = \tan\theta \Rightarrow$



$\Rightarrow \sec\theta = \sqrt{1+u^2}$ so $= \ln|\sqrt{1+u^2} + u| + C$

$$= \boxed{\ln\left|\sqrt{1+\sin^2(x)} + \sin(x)\right| + C}$$

$\boxed{\text{SOLUTION 1}}$ Let $u = 4x^2 - 4 \Rightarrow \frac{du}{dx} = 8x \Rightarrow dx = \frac{du}{8x}$

so $$\int \frac{x}{\sqrt{4x^2-4}}\,dx = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{8x} = \frac{1}{8}\int u^{-1/2}\,du$$

$$= \frac{1}{8}\left(\frac{u^{1/2}}{1/2}\right) + C$$

$$= \frac{1}{4}\sqrt{4x^2-4} + C$$

$$= \boxed{\frac{\sqrt{x^2-1}}{2} + C}$$

$\boxed{\text{SOLUTION 2}}$ Let's write the integral as: (where $\theta\in[0,\tfrac{\pi}{2})\cup[\pi,\tfrac{3\pi}{2})$)

$$\int \frac{x}{\sqrt{(2x)^2-4}}\,dx.$$ Now let $2x = 2\sec\theta \Rightarrow x = \sec\theta$
$\Rightarrow dx = \sec\theta\tan\theta\,d\theta$

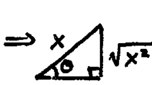
so $$\sqrt{(2x)^2-4} = \sqrt{(2\sec\theta)^2-4} = \sqrt{4\sec^2\theta-4}$$
$$= \sqrt{4(\sec^2\theta-1)}$$
$$= \sqrt{4\tan^2\theta}$$
$$= 2|\tan\theta|$$
$$= 2\tan\theta \text{ (since our assumption on } \theta)$$

so

$$\int \frac{\sec\theta}{2\cancel{\tan\theta}} \cdot \sec\theta\cancel{\tan\theta}\,d\theta = \frac{1}{2}\int \sec^2\theta\,d\theta = \frac{1}{2}\tan\theta + C$$

$$= \boxed{\frac{1}{2}\sqrt{x^2-1} + C}$$

$x = \sec\theta \Rightarrow$



so $\tan\theta = \sqrt{x^2-1}$