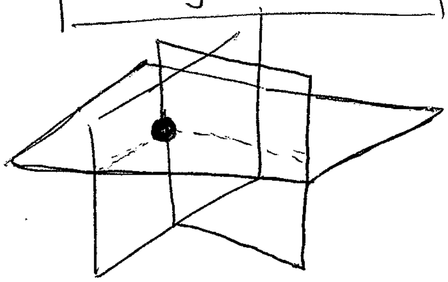
Linear systems with three unknowns (geometrically)

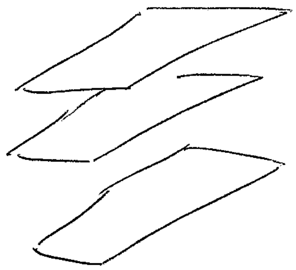
Each linear equation in three variables: $c_1x_1+c_2x_2+c_3x_3=k$ represents a plane in (x_1,x_2,x_3)-space.

The Good
Exactly One Solution

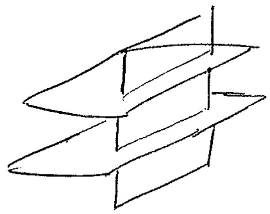


(intersection is a point)

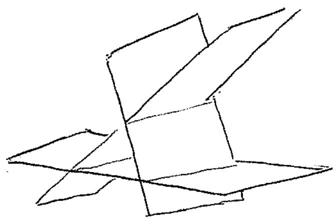
The Bad
No Solutions



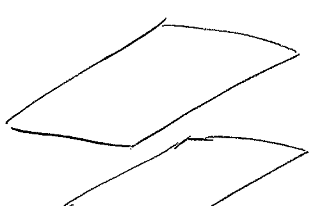
(3 parallel planes)
no common intersection



(2 parallel planes)
no common intersection

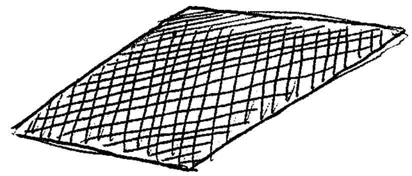


(no common intersection)

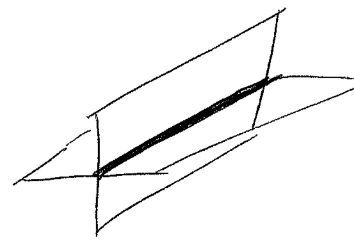


(two coincident planes
parallel to the third:
no common intersection)

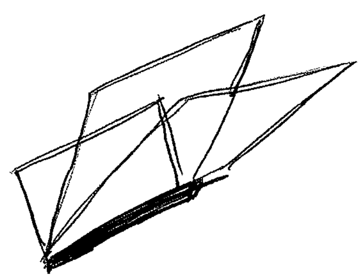
The Ugly
Infinitely many solutions



all planes are coincident
intersection is a plane



(two coincident planes;
intersection is a line)



(intersection is a line)
(think of pages in a book)

System of Linear Equations

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ -x_1 + 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 1 \end{aligned}$$

Add equations 1 to equation 2:

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 0x_1 + 4x_2 + 4x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 1 \end{aligned}$$

Add -2 times equation 1 to equation 3:

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 0x_1 + 4x_2 + 4x_3 &= 2 \\ 0x_1 - 3x_2 - 5x_3 &= -3 \end{aligned}$$

Add 3/4 times equation 2 to equation 3:

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 0x_1 + 4x_2 + 4x_3 &= 2 \\ 0x_1 + 0x_2 - 2x_3 &= -3/2 \end{aligned}$$

Multiply equation 2 by 1/4:

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 0x_1 + x_2 + x_3 &= 1/2 \\ 0x_1 + 0x_2 - 2x_3 &= -3/2 \end{aligned}$$

Multiply equation 3 by $-1/2$:

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 0x_1 + x_2 + x_3 &= 1/2 \\ 0x_1 + 0x_2 + x_3 &= 3/4 \end{aligned}$$

Add -1 times equation 3 to equation 2:

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2 \\ 0x_1 + x_2 + 0x_3 &= -1/4 \\ 0x_1 + 0x_2 + x_3 &= 3/4 \end{aligned}$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ -1 & 3 & 1 & 0 \\ 2 & -1 & 1 & 1 \end{array}\right]$$

Add row 1 to row 2:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 4 & 4 & 2 \\ 2 & -1 & 1 & 1 \end{array}\right]$$

Add -2 times row 1 to row 3:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 4 & 4 & 2 \\ 0 & -3 & -5 & -3 \end{array}\right]$$

Add 3/4 times row 2 to row 3:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 4 & 4 & 2 \\ 0 & 0 & -2 & -3/2 \end{array}\right]$$

Multiply row 2 by 1/4:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & -2 & -3/2 \end{array}\right]$$

Multiply row 3 by -1/2:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 3/4 \end{array}\right]$$

Add -1 times row 3 to row 2:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 3/4 \end{array}\right]$$

System of Linear Equations

Add -3 times equation 3 to equation 1:

$$\begin{aligned} x_1 + x_2 + 0x_3 &= -1/4 \\ 0x_1 + x_2 + 0x_3 &= -1/4 \\ 0x_1 + 0x_2 + x_3 &= 3/4 \end{aligned}$$

Add -1 times equation 2 to equation 1:

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + x_2 + 0x_3 &= -1/4 \\ 0x_1 + 0x_2 + x_3 &= 3/4 \end{aligned}$$

So $x_1 = 0$, $x_2 = -1/4$ and $x_3 = 3/4$.

Augmented Matrix

Add -3 times row 3 to row 1:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1/4 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 3/4 \end{array}\right]$$

Add -1 times row 2 to row 1:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 3/4 \end{array}\right]$$

So $x_1 = 0$, $x_2 = -1/4$ and $x_3 = 3/4$.

Elementary row operations (Definition 8, p10)

1. Switch two equations (rows).
2. Multiply one equation (row) by a non-zero constant.
3. Add a multiple of one equation (row) to a different equation (row). Specifically, replace a row by the sum of that row and a multiple of some different row.

IMPORTANT! On tests you will be required to identify the row operation that you are using in each step. The way to describe this is by annotating each step in the following style:

1. $R2 \leftrightarrow R5$ (switch rows 2 and 5).
2. $R4 \to \frac{1}{13}R4$ (divide row 4 by 13).
3. $R1 \to R1 - 3R2$ (replace row 1 with row 1 minus 3 times row 2).

WARNING! The operation of type 3 is *ALWAYS* 'replace a row by itself plus or minus a multiple of a different row'. So $R2 \to R1 - 3R2$ is <u>*NOT*</u> a legal row operation.

The previous example using the above convention:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ -1 & 3 & 1 & 0 \\ 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R2\to R2+R1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 4 & 4 & 2 \\ 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R3\to R3-2R1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 4 & 4 & 2 \\ 0 & -3 & -5 & -3 \end{array}\right]$$

$$\xrightarrow{R3\to R3+\frac{3}{4}R2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 4 & 4 & 2 \\ 0 & 0 & -2 & -3/2 \end{array}\right] \xrightarrow{R2\to \frac{1}{4}R2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & -2 & -3/2 \end{array}\right] \xrightarrow{R3\to \frac{-1}{2}R3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 3/4 \end{array}\right] \xrightarrow{R2\to R2-R3} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 3/4 \end{array}\right] \xrightarrow{R1\to R1-3R3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1/4 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 3/4 \end{array}\right]$$

$$\xrightarrow{R1\to R1-R2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & 3/4 \end{array}\right]$$

So $x_1 = 0$, $x_2 = -1/4$ and $x_3 = 3/4$.