Math 222 Spring 2011
4/1/2011
Quiz #8

Name: key

Please box your answers. Show all work clearly and in order.

$\boxed{6}$ 1. Determine whether each series is convergent or divergent. If it is convergent, find its sum.

(a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

$\boxed{\text{SOL}}$ the series is $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$. This is a geometric series with $r = 1/2$ and since $|r| = \frac{1}{2} < 1$ the series $\boxed{\text{converges}}$ and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1-1/2} = \frac{1}{1/2} = \boxed{2}$$

(b) $\sum_{n=1}^{\infty} \frac{4^{n-1}}{(-5)^n} = \sum_{n=1}^{\infty} \frac{4^{n-1}}{(-5)(-5)^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{-5}\right)\left(\frac{4}{-5}\right)^{n-1}$. This is a geometric series with $r = -4/5$ and since $|r| = 4/5 < 1$ the series $\boxed{\text{converges}}$ and

$$\sum_{n=1}^{\infty} \frac{4^{n-1}}{(-5)^n} = \frac{1/-5}{1-\left(\frac{-4}{5}\right)} = \boxed{\frac{-1}{9}}$$

(c) $\sum_{n=1}^{\infty} \tan^{-1}(n)$

Notice the limit of the sequence of terms does not tend toward 0.

$\lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2} \neq 0$. Therefore, by the Test for Divergence the series $\boxed{\text{diverges}}$

$\boxed{4}$ 2. Pick ONE of the following. Cross out the problem you do not want graded. Otherwise I will grade the first problem worked on.

Determine whether each the series is convergent or divergent.

(a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$\boxed{\text{SOL}}$ Consider the function $f(x) = \frac{1}{x\ln(x)}$ for $x \geq 2$

Notice this function is

(i) continuous on $x \geq 2$. This is because it is a ratio of cts. functions on $x \geq 2$ and $x\ln(x)$ is not equal to 0 for any $x \geq 2$.

(ii) positive on $x \geq 2$. This is because
$x > 0$ if $x \geq 2$
$\ln(x) > 0$ if $x \geq 2$
hence $\frac{1}{x\ln(x)} > 0$ for $x \geq 2$.

(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$\boxed{\text{SOL}}$ (short): $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$

notice $a_n = \frac{1}{n^2+n} > 0$ for all $n \geq 1$

notice $\frac{1}{n^2+n} < \frac{1}{n^2}$

and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (its a p-series with $p=2>1$)

the comparison test tells us $\sum \frac{1}{n(n+1)}$ also converges.

(iii) decreasing on $x \geq 2$. This can be checked by showing $f'(x) < 0$ for $x \geq 2$:

$$f'(x) = \frac{-(x \cdot (\frac{1}{x}) + \ln(x))}{(x\ln(x))^2} = \frac{-(1+\ln(x))}{(x\ln(x))^2}$$

notice $1+\ln(x) > 0$ if $x \geq 2$

so $-(1+\ln(x)) < 0$ iff $x \geq 2$ (the numerator)

and $(x\ln(x))^2 > 0$ if $x \geq 0$

hence $f'(x) < 0$ if $x \geq 2$.

Now we can try to use the integral test:

$$\int_2^\infty \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\ln(x)}\,dx$$

let $u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\Rightarrow dx = x\,du$

$u(2) = \ln(2)$

$u(t) = \ln(t)$

$$= \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{1}{\cancel{x} u} \cdot \cancel{x}\,du$$

$$= \lim_{t\to\infty} \Big[\ln|u|\Big]_{\ln(2)}^{\ln(t)}$$

$$= \lim_{t\to\infty} \Big[\ln|\ln(t)| - \ln|\ln(2)|\Big]$$

$= \infty$ so the integral diverges.

Hence, by the integral test, the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(2)}$ $\boxed{\text{diverges}}$

$\boxed{\text{SOL 2}}$ (A bit longer, but this method is needed if the question also asks to find the sum if it converges) so keep this in mind.

we can use partial fractions to write

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$= \frac{A(n+1) + Bn}{n(n+1)}$$

So

$$1 = A(n+1) + Bn$$

$$1 = An + A + Bn$$

$$1 = (A+B)n + A$$

So $\left.\begin{array}{l} A+B=0 \\ A=1 \end{array}\right\} \Rightarrow B=-A=-1$

So $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

so we can write:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

consider the n^{th} partial sum of the series:

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

now notice

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Since the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} S_n = 1$$

the series $\boxed{\text{converges}}$ (and it is equal to 1)