Additional homework problems for week 1.

1. Describe an elementary row operation that produces C from B, and then describe an elementary row operation that recovers B from C.

(a) $B = \begin{bmatrix} 2 & 0 & -4 \\ -3 & -2 & 6 \\ 2 & 5 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 & 1 \\ -3 & -2 & 6 \\ 2 & 0 & -4 \end{bmatrix}$

produces C from B: $R1 \longleftrightarrow R3$
recovers B from C: $R1 \longleftrightarrow R3$

(b) $B = \begin{bmatrix} 2 & 0 & -4 \\ -3 & -2 & 1 \\ 2 & 0 & 3 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 & -4 \\ -3 & -2 & 1 \\ 10 & 0 & 15 \end{bmatrix}$

produces C from B: $R3 \longrightarrow 5R3$
recovers B from C: $R3 \longrightarrow \frac{1}{5}R3$

2. Find the solution set to the given linear systems, and then check your solutions (by substituting your found values to verify they are correct):

(a)

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

(a) $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right] \xrightarrow{R2 \to R2 + R1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{array}\right] \xrightarrow{R3 \to R3 - 3R1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$

$\xrightarrow{R3 \to R3 - 10R2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & -52 & -104 \end{array}\right]$ so

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_2 + 5x_3 &= 9 \\ -52x_3 &= -104 \end{aligned}$$

$\Rightarrow x_3 = 2$
so $-x_2 + 5(2) = 9$
$\Rightarrow x_2 = 1$
so $x_1 + 1 + 2(2) = 8$
$\Rightarrow x_1 = 3$

Check: $3 + 1 + 2(2) = 8$ ✓
$-3 - 2 + 3(2) = 1$ ✓
$3(3) - 7(1) + 4(2) = 10$ ✓

solution set has exactly one solution:
$\boxed{\begin{matrix} x_1 = 3 \\ x_2 = 1 \\ x_3 = 2 \end{matrix}}$

solution to (b) on next page

(b)

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right] \xrightarrow{R2 \to R2 + R1} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - 4R1} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right] \xrightarrow{R3 \to R3 + R2} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Column 3 is not a pivot column $\Rightarrow$ x_3 is a free variable.
now represent the basic variables x_1 and x_2 in terms of x_3:

row 1 says: $2x_1 + 2x_2 + 2x_3 = 0$

$7x_2 + 4x_3 = 1$ so $7x_2 = 1 - 4x_3$

so $x_2 = \frac{1}{7} - \frac{4}{7}x_3$

now $2x_1 = -2x_2 - 2x_3$

so $x_1 = -x_2 - x_3$

so $x_1 = -\left(\frac{1}{7} - \frac{4}{7}x_3\right) - x_3$

so $x_1 = -\frac{1}{7} + \frac{4}{7}x_3 - x_3$

so $x_1 = -\frac{1}{7} - \frac{3}{7}x_3$

Really all this work is doing the extra work to obtain RREF so either way Hence.

$$\boxed{\begin{aligned} x_1 &= -\tfrac{1}{7} - \tfrac{3}{7}x_3 \\ x_2 &= \tfrac{1}{7} - \tfrac{4}{7}x_3 \\ x_3 &= \text{Anything} \end{aligned}}$$

We can actually still check these solutions algebraically but I will leave that to you to try.