$\boxed{X10}$ (<u>NOTE</u>: This problem is similar to the examples in lecture 41 comments. Keep in mind that here the manipulations are simple and do not require the method involving matrices as in the general case and more complicated situations like example 2.2 in lecture 41 comments.)

(a) $u = 3 - 4\cos^2(x) + 5\sin^2(x)$

(<u>Recall</u>: $1 = \sin^2(x) + \cos^2(x)$)

so

$$u = 3(1) - 4\cos^2(x) + 5\sin^2(x)$$
$$= 3(\sin^2(x) + \cos^2(x)) - 4\cos^2(x) + 5\sin^2(x)$$
$$= 3\sin^2(x) + 3\cos^2(x) - 4\cos^2(x) + 5\sin^2(x)$$
$$= 8\sin^2(x) - \cos^2(x)$$

(in the correct order with respect to the basis $(\sin^2(x), \cos^2(x))$)

so $\boxed{K(u) = \begin{bmatrix} 8 \\ -1 \end{bmatrix}}$

(b) $u = \cos(2x)$ (<u>Recall</u>: $\cos(2x) = \cos^2(x) - \sin^2(x)$)

so $u = \cos^2(x) - \sin^2(x)$

$$= -1\sin^2(x) + 1\cos^2(x)$$

(in the correct order with respect to the basis $(\sin^2(x), \cos^2(x))$)

so $\boxed{K(u) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

(c) $u = \sin(2x)$ (<u>Recall</u>: $\sin(2x) = 2\sin(x)\cos(x)$)

$\underline{u = 2\sin(x)\cos(x)}$

so we cannot write u as a linear combination of $\sin^2(x)$ and $\cos^2(x)$ hence $K(u)$ is not defined.

$\boxed{X12}$

(a) by definition

$$_W F_Z = \left[K_W(F(1)) \quad K_W(F(x)) \quad K_W(F(x^2)) \quad K_W(F(x^3)) \right]$$

first find:

$$F(1) = \frac{d}{dx} 1 - 1 = 0 - 1 = -1$$

$$F(x) = \frac{d}{dx} x - (1) = 1 - 1 = 0$$

$$F(x^2) = \frac{d}{dx} x^2 - (1)^2 = 2x - 1$$

$$F(x^3) = \frac{d}{dx} x^3 - (1)^3 = 3x^2 - 1$$

so now

$$_W F_Z = \left[K_W(-1) \quad K_W(0) \quad K_W(2x-1) \quad K_W(3x^2-1) \right]$$

since W is such a nice basis of P_2 it is simple to calculate the rest of this. (see lecture comments 41 example 2.2 for a more complicated situation)

we have

$$K_W(-1) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$K_W(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$K_W(2x-1) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$K_W(3x^2-1) = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

so

$$
{}_wF_z = \begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}
$$

(b) ${}_wF_z$ is already in REF

$$
\begin{bmatrix} \boxed{-1} & 0 & -1 & -1 \\ 0 & 0 & \boxed{2} & 0 \\ 0 & 0 & 0 & \boxed{3} \end{bmatrix}
$$

and $\operatorname{rank}({}_wF_z) = 3 = \#$ of rows so by Thm 1.6.2 ${}_wF_z$ is onto (regarded as a function from $\mathbb{R}^4$ to $\mathbb{R}^3$) meaning F is onto.

(c) Essentially we use lemma 4.8.4 on p192 and just find a basis for the nullspace of ${}_wF_z$ and then use our coordinate isomorphism to "translate" this to have a basis for the kernel of F.

Find a basis for $\mathrm{Nul}({}_wF_z)$:

$$
\left[\begin{array}{cccc|c} -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]
$$

$$
\begin{aligned}
-x_1 \quad - x_3 - x_4 &= 0 \\
2x_3 &= 0 \Rightarrow x_3 = 0 \\
3x_4 &= 0 \Rightarrow x_4 = 0
\end{aligned}
$$

so

$x_1 = 0$

find vector parametric form of solution.

so
$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

so this gives us what?

the set $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis of $Nul(_wF_z)$

but this ~~vector~~ is not a basis for $\ker(F)$, why?

because $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is in $\mathbb{R}^4$ and not P_3 (the domain of F)

so use the coordinate transformation $K_z : P_3 \longrightarrow \mathbb{R}^4$

to translate to give us a "vector"/polynomial in P_3.

i.e. find $p(x)$ if $K_z(p(x)) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

well this is actually very quick, by

definition

$$p(x) = 0(1) + 1(x) + 0(x^2) + 0(x^3)$$

$$p(x) = x$$

so the set $\boxed{\{x\}}$ is a basis of $\ker(F)$