

COMMENTS FOR LECTURE 26 - 3.12.2010

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Examples!

Example 1: Consider the (ordered) set

$$X = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix} \right)$$

- (1) Is X linearly independent? If not find a dependence relation on X .
- (2) Does X span \mathbb{R}^3 (i.e., is $\text{Span}(X) = \mathbb{R}^3$)?
- (3) Is X a (ordered) basis of \mathbb{R}^3 ?

Solution:

- (1) Using the “test for linear independence” (p129):

$$\text{Let } A = \begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & 5 \\ 0 & 1 & 4 \end{bmatrix}$$

Putting A into reduced row echelon form we obtain:

$$A = \begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & 5 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\text{putting } A \text{ in RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

So the $\text{rank}(A)=2$ (number of pivot columns). So $\text{rank}(A)=2 \neq 3$ (the number of columns). Hence the answer is **no**, and X must be linearly dependent.

To find a dependence relation on X we need to find a nontrivial solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Well we did all of the work actually, but if you did this from scratch you would set up an augmented matrix for the system and reduce:

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 1 & 1 & 5 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] \xrightarrow{\text{putting into RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This has a parametric solution of the form:

$$x_1 = -t$$

$$x_2 = -4t$$

$$x_3 = t$$

So to get an example of a nontrivial solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ would could set $t = 1$ and so $x = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$ is a solution. So we have a dependence relation on X :

$$(-1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-4) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

- (2) Using 1. We have $\text{rank}(A) = 2 \neq 3 = (\text{the number of rows})$. Hence the answer is **no**, and X does not span all of \mathbb{R}^3 .
 (3) Using 1. or 2. the answer is **no**. Don't forget the definition of a basis!

Example 2: Let $A = \begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & 5 \\ 0 & 1 & 4 \end{bmatrix}$

- (1) Find a basis X for the column space of A .
- (2) What is the dimension of the column space of A ?
- (3) Find a basis Y for the null space of A .
- (4) What is the dimension of the null space of A ?
- (5) Find a basis Z for the row space of A .
- (6) What is the dimension of the row space of A ?

Solution:

- (1) We put A into reduced row echelon form (RREF):

$$A = \begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & 5 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\text{putting } A \text{ in RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = A'$$

So this means that columns 1 and 2 are the pivot columns. The set of pivot columns of A form a basis of the column space of A . Hence,

$$X = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)$$

- (2) The dimension of the column space is the number of elements in X hence **2**. (Remember we also have that the dimension of the column space = $\text{rank}(A) = 2$)
 (3) We find the vector parametric form of the solution set to the homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 1 & 1 & 5 & 0 \\ 0 & 1 & 4 & 0 \end{array} \right] \xrightarrow{\text{putting into RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This has a parametric solution of the form:

$$\begin{aligned}x_1 &= -t \\x_2 &= -4t \\x_3 &= t\end{aligned}$$

or we could write:

$$\begin{aligned}x_1 &= -x_3 \\x_2 &= -4x_3 \\x_3 &= x_3\end{aligned}$$

So the vector parametric form would be:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -4x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$$

The set of vectors in the vector parametric form of the solution set of $A\mathbf{x} = \mathbf{0}$ form a basis for the null space of A . Hence,

$$Y = \left(\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} \right)$$

- (4) The dimension of the null space is the number of elements in Y hence 1. (remember we also have that the dimension of the null space = number of non pivot columns = $3 - \text{rank}(A) = 3 - 2 = 1$)
- (5) Using the work from 1. we have that a basis for the row space can be made from the nonzero rows of A' (the reduced row echelon form of A). Hence,

$$Z = ([1 \ 0 \ 1], [0 \ 1 \ 4])$$

- (6) The dimension of the row space is the number of elements in Z hence 2. (Remember we also have that the dimension of the row space = $\text{rank}(A) = 2$)

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