Comments for Lecture 41
4.16.2010

Suppose V is some finite dimensional vector space and $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is an ordered basis of V. Recall this means we have an isomorphism $K_B{:}V \to \mathbb{R}^n$ (The coordinate transformation). In the following we consider $\mathbf{u} \in V$.

Read 4.5 and 4.6, especially **General Lemma 4.5.10**

$\boxed{\text{How to find the coordinate vector } K_B(\mathbf{u}) \text{ given } \mathbf{u}}$

Suppose you are given $\mathbf{u}$ and are asked to find $K_B(\mathbf{u})$ (the coordinate vector of $\mathbf{u}$ with respect to the basis B). To solve this problem you need to first write $\mathbf{u}$ as a linear combination of the elements of the basis B:

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Once you have done this we have

$$K_B(\mathbf{u}) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

.

i.e., we just peel the coefficients off the linear combination we found and create a vector in $\mathbb{R}^n$ with the correct ordering. Warning: The order does matter so be careful! NOTE: The real task is writing $\mathbf{u}$ as a linear combination of the elements in B (this can take some work!). See examples below.

$\boxed{\text{How to find } \mathbf{u} \text{ given the coordinate vector } K_B(\mathbf{u})}$ (EASY PROBLEM)

Suppose you are given the coordinate vector $K_B(\mathbf{u}) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ of some unknown vector $\mathbf{u}$ that you must find. Well by definition we have

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Example 1: Suppose $V = P_3$ and $B = (1, x, x^2, x^3)$. Find $K_B(1 - x + 2x^3)$.
Solution: Here we have $\mathbf{u} = 1 - x + 2x^3$. To solve this problem we need to write $\mathbf{u}$ as a linear combination of the basis elements in B. Since B is such a simple basis of P_3 we don't have

to do much work. We have $\mathbf{u} = 1 - x + 2x^3 = (1)(1) + (-1)(x) + (0)(x^2) + (2)(x^3)$. So this means $K_B(\mathbf{u}) = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$.

Example 2: Suppose $V = P_2$ and $S = (p_1(x) = 2 - 2x - x^2, p_2(x) = 1 + x - x^2, p_3(x) = 3 - x + 3x^2)$.

1. Show that S is a basis of P_2.
2. Find $K_S(3 + 4x - x^2)$.
3. Find $p(x)$ if $K_S(p(x)) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solution:

1. Consider the ordered basis $T = (1, x, x^2)$ of P_2. Why do I need this basis? Well, to solve this problem I will use General Lemma 4.5.10(g) on p179:

$$S \text{ is a finite basis for } P_2 \iff K_T(S) \text{ is a finite basis for } \mathbb{R}^3$$

So now we are just going to work in $\mathbb{R}^3$ and show $K_T(S) = (K_T(p_1(x)), K_T(p_2(x)), K_T(p_3(x)))$ is a basis of $\mathbb{R}^3$. First we need to find $K_T(p_1(x))$,$K_T(p_2(x))$ and $K_T(p_3(x))$. Now it should be clear why we chose the basis T to work with. These coordinate vectors are as follows:

$$K_S(p_1(x)) = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, K_S(p_2(x)) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } K_S(p_3(x)) = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

Now show these three coordinate vectors form a basis in $\mathbb{R}^3$. We work with the matrix $A = \begin{bmatrix} K_T(p_1(x)) & K_T(p_2(x)) & K_T(p_3(x)) \end{bmatrix}$. (STOP! Now it should be clear why we keep constructing this kind of matrix when solving this kind of problem. Make sure you understand what the goal is!).

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix} \xrightarrow{\text{Putting } A \text{ into RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence A is invertible. So this means the set $K_T(S)$ is a basis of $\mathbb{R}^3$, and since K_T is an isomorphism we have that S is a basis of P_2.

2. This problem is going to require much more work than what we did in example 1. Notice here it is not obvious how to write $3 + 4x - x^2$ in terms of the basis S. Again we use the isomorphism K_T to turn this into a problem in $\mathbb{R}^3$. Notice we can write $K_T(3 + 4x - x^2) = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$. Now we try to write this vector as a linear combination of the vectors in the basis $K_T(S)$ (This is the kind of problem you solved in chapter 3). In other words we want to solve this equation:

$$c_1 \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

This is just solving the system:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

Solving this system:

$$A = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ -2 & 1 & -1 & 4 \\ -1 & -1 & 3 & -1 \end{array}\right] \xrightarrow{\text{Putting } A \text{ into RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7/10 \\ 0 & 1 & 0 & 61/20 \\ 0 & 0 & 1 & 9/20 \end{array}\right]$$

So $c_1 = -7/10, c_2 = 61/20$, and $c_3 = 9/20$. This is gives us

$$K_S(3 + 4x - x^2) = \begin{bmatrix} -7/10 \\ 61/20 \\ 9/20 \end{bmatrix}.$$

You can see this is the correct answer since: $3 + 4x - x^2 = (-7/10)(2 - 2x - x^2) + (61/20)(1 + x - x^2) + (9/20)(3 - x + 3x^2)$

3. This problem is very quick, we have

$$\begin{aligned} p(x) &= 1p_1(x) + 2p_2(x) + 3p_3(x) \\ &= 1(2 - 2x - x^2) + 2(1 + x - x^2) + 3(3 - x + 3x^2) \\ &= 2 - 2x - x^2 + 2 + 2x - 2x^2 + 9 - 3x + 9x^2 \\ &= 13 - 3x + 6x^2 \end{aligned}$$

Make sure you see the difference between this problem and the previous one.