Comments for Lecture 16
2.22.2010

Using **Theorem 1.6.2**.

Suppose that C is an $m \times n$ matrix regarded as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. We can use **Theorem 1.6.2** on page 33 to determine when C is *onto*, *one-to-one* or even when it is a *one-to-one correspondence* using **Corollary 1.6.3** simply by computing the rank.

You could also be asked the more challenging problems:

Problem 1. Suppose that C is not onto. Then find a $\mathbf{b}$ in $\mathbb{R}^m$ such that there is no $\mathbf{x}$ in $\mathbb{R}^n$ such that $C\mathbf{x} = \mathbf{b}$.

Problem 2. Suppose that C is not one-to-one. Then find two different vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ such that $C\mathbf{x} = C\mathbf{y}$ (note that $C\mathbf{x}$ and $C\mathbf{y}$ are in $\mathbb{R}^m$).

There are a few ways that you can solve these problems and I will show you a method that will work in general.

To solve **problem 1** keep in mind the idea of *onto* reminding you of "existence of solutions". In other words to solve problem 1 we are trying to find a vector $\mathbf{b}$ such that the system:

$$C\mathbf{x} = \mathbf{b}$$

is inconsistent. In other words you cannot find an $\mathbf{x}$ in $\mathbb{R}^n$ to solve it. We use the following procedure to solve this problem:

1. Create the augmented matrix $A = \left[\begin{array}{c|c} C & \mathbf{b} \end{array}\right]$ where $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$. Here we keep $b_1, b_2, \ldots, b_m$ as unknown variables. (remember we don't know what $\mathbf{b}$ is yet!)
2. Put A into row echelon form (you could of course put A into reduced row echelon form but row echelon form should be enough) to obtain the matrix $A' = \left[\begin{array}{c|c} C' & \mathbf{b}' \end{array}\right]$. In other words:

$$A = \left[\begin{array}{c|c} C & \mathbf{b} \end{array}\right] \xrightarrow{\text{Putting } C \text{ into REF}} A' = \left[\begin{array}{c|c} C' & \mathbf{b}' \end{array}\right]$$

3. Notice you must have at least one row of C' be a row of all zeros. Why? Well otherwise the rank of C would be equal to the number of rows which means that C would be onto by **Theorem 1.6.2** (which cannot happen since we are starting with a matrix C

which is not onto). So let's look at the very last row of A' which we agreed must look like the following:

$$\left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & b_m^{'} \end{array}\right]$$

Where $b_m^{'}$ will be a linear combination of the b_i's (since $b_m^{'}$ is obtained by elementary row operations of A with the rightmost column only involving the b_i's).

In other words $b_m^{'} = t_1b_1 + t_2b_2 + \ldots + t_mb_m$ where each of the t_i's are real numbers for $1 \le i \le m$. So how do we make our original system:

$$C\mathbf{x} = \mathbf{b}$$

An inconsistent system? Just make $b_m^{'} \neq 0$. So you just start to pick some values of $b_1, b_2, \ldots b_m$ which makes $b_m^{'} \neq 0$ and you have such a vector $\mathbf{b}$ that we were looking for.

4. Phew!

Let's try to use this method.
Example 1:

(a) Show that the matrix $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ does not represent an onto function.

(b) Find a vector $\mathbf{b}$ in $\mathbb{R}^2$ that is not in the image of the function (in other words find a $\mathbf{b}$ in $\mathbb{R}^2$ such that the equation $C\mathbf{x} = \mathbf{b}$ has no solution). (in other other words find a vector $\mathbf{b}$ in $\mathbb{R}^2$ so there is no $\mathbf{x}$ in $\mathbb{R}^2$ such that $C\mathbf{x} = \mathbf{b}$).

Solution to (a):

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The rank of the matrix C is 1 which is not equal to the number of rows which is 2. Therefore by Theorem 1.6.2 C cannot represent an onto function.
Solution to (b):
Using the method described above to solve Problem 1 we have:

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 2 & 2 & b_2 \end{array}\right] \xrightarrow{R2 \to R2 - 2R1} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 0 & b_2 - 2b_1 \end{array}\right]$$

So in order to make our system inconsistent we just need $b_2 - 2b_1 \neq 0$. So think of anything that makes this true. Well for example $b_2 = 5$ and $b_1 = 0$ works right? So we can say for sure that the vector $\mathbf{b} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$ is not in the image of the function described by C. In other words the system $C\mathbf{x} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$ has no solutions for the above C.

To solve **problem 2** keep in mind the idea of *one-to-one* reminding you of "uniqueness of solutions". In other words we are trying to find two different vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ such that $C\mathbf{x} = C\mathbf{y}$. We use the following procedure to solve this problem:

1. Find something in the image of the function described by C. For example consider where the zero vector $\mathbf{x} = \mathbf{0}$ in $\mathbb{R}^n$. Where does this go? Well $C\mathbf{0} = \mathbf{0}$.

2. Solve the system $C\mathbf{y} = \mathbf{0}$ and pick any solution $\mathbf{y} \neq \mathbf{0}$. Then you have two different vectors $\mathbf{x} = \mathbf{0}$ and $\mathbf{y}$ such that $C\mathbf{x} = C\mathbf{y}$.

Question: Will this method always work if we pick $\mathbf{x} = \mathbf{0}$ initially (for some $m \times n$ matrix C which is not one-to-one)? The answer is actually yes and we will see why later in the course.

Let's try to use this method.
Example 2:

(a) Show that the matrix $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ does not represent a one-to-one function.

(b) Find two different vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ such that $C\mathbf{x} = C\mathbf{y}$.

Solution to (a):
The rank of the matrix C is 1 (shown above) which is not equal to the number of columns which is 2. Therefore by Theorem 1.6.2 C cannot represent a one-to-one function.
Solution to (b):
Using the method described above to solve Problem 2 we have:
Let $\mathbf{x} = \mathbf{0}$ and notice $C\mathbf{x} = \mathbf{0}$.

Now let's solve the system $C\mathbf{y} = \mathbf{0}$. Well using our normal method of creating an augmented matrix and row reducing we have:

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{R2 \to R2 - 2R1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So the solution in parametric form would be:
$y_1 = -s$ and $y_2 = s$. Therefore let's pick a vector $\mathbf{y} \neq \mathbf{0}$ such that $y_1 = -s$ and $y_2 = s$. For example let $s = 1$ and we have $y = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Indeed we have $\mathbf{y} \neq \mathbf{x} = \mathbf{0}$ and $C\mathbf{x} = C\mathbf{y} = \mathbf{0}$.