1

(a) We need to show three properties about X.

(i) Show $\vec{0}$ is in X: (what is $\vec{0}$ here? it is the polynomial $p(x)=0$ in P_2)

we can make a_1 and a_2 also equal to zero, ie. now $a_0=0, a_1=0, a_2=0$

so 0 is in X. ✓

(ii) Let $p(x), q(x) \in X$, show $p(x)+q(x) \in X$

Let $p(x) \in X \Rightarrow$ so $p(x) = a_0 + a_1 x + a_2 x^2$ and $a_0 = 0$, $a_1, a_2 \in \mathbb{R}$

Let $q(x) \in X \Rightarrow$ so $q(x) = b_0 + b_1 x + b_2 x^2$ and $b_0 = 0$, $b_1, b_2 \in \mathbb{R}$

now $p(x) + q(x) = (a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2)$

$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$

and since $a_0 + b_0 = 0 + 0 = 0$

$a_1 + b_1 \in \mathbb{R}$

$a_2 + b_2 \in \mathbb{R}$

we have $p(x) + q(x) \in X$ ✓

(iii) Let $p(x) \in X$ and $\alpha \in \mathbb{R}$, show $\alpha p(x) \in X$

Let $p(x) \in X$ and $\alpha \in \mathbb{R}$

so $p(x) = a_0 + a_1 x + a_2 x^2$ and $a_0 = 0$, $a_1, a_2 \in \mathbb{R}$

now $\alpha p(x) = \alpha (a_0 + a_1 x + a_2 x^2)$

$= \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2$

and since $\alpha a_0 = \alpha(0) = 0$

$\alpha a_1 \in \mathbb{R}$

$\alpha a_2 \in \mathbb{R}$

we have $\alpha p(x) \in X$ ✓

Conclusion: $\boxed{X \text{ is indeed a subspace of } P_2}$

(see Thm 4.1.2 on p164)

① (b) This is also a subspace of P_2 (solution is very similar to ①(a))

① (c) This is NOT a subspace. why?

well properties (i) and (ii) will hold (as in ① (a) and ①(b))

BUT property (iii) fails.

for example if $p(x) \in Z$ and $\alpha \in \mathbb{R}$

in general $\alpha p(x)$ might not be in Z

Consider $p(x) = 1 + x + x^2$ for example, this is in Z

since $a_0 = 1$, $a_1 = 1$, $a_2 = 1$ (which are all integers)

but if $\alpha = 1/2$ for example then

$$\alpha p(x) = \frac{1}{2} p(x) = \frac{1}{2}(1 + x + x^2)$$

$$= \frac{1}{2} + \frac{1}{2}x + \frac{1}{2}x^2$$

and this is NOT in Z since

$\frac{1}{2}$ is not an integer

(so none of the coefficients are integers)

(2) Again we use the same template as before

(i) Show $\vec{0}$ is in V: (what is $\vec{0}$ here? it is the function $f(x)=0$ in $C[0,1]$)

This is indeed in V since $\int_0^1 0\,dx = 0$ ✓

(ii) Let $f(x), g(x) \in V$, show $f(x)+g(x) \in V$:

since $f(x) \in V \Rightarrow f(x) \in C[0,1]$ and $\int_0^1 f(x)\,dx = 0$
since $g(x) \in V \Rightarrow g(x) \in C[0,1]$ and $\int_0^1 g(x)\,dx = 0$

so $f(x)+g(x) \in C[0,1]$ since the sum of two continuous functions is continuous so $f(x)+g(x)$ is continuous also on $[0,1]$

AND

$$\int_0^1 (f(x)+g(x))\,dx = \underbrace{\int_0^1 f(x)\,dx}_{} + \underbrace{\int_0^1 g(x)\,dx}_{}$$
$$= 0 + 0$$
$$= 0$$

so $f(x)+g(x) \in V$ ✓

(iii) Let $f(x) \in V$ and $\alpha \in \mathbb{R}$, show $\alpha f(x) \in V$:

since $f(x) \in V \Rightarrow f(x) \in C[0,1]$ and $\int_0^1 f(x)\,dx = 0$

so $\alpha f(x) \in C[0,1]$ since a constant multiple of a continuous function is continuous so $\alpha f(x)$ is continuous also on $[0,1]$

AND

$$\int_0^1 (\alpha f(x))\,dx = \alpha \int_0^1 f(x)\,dx = \alpha \cdot 0 = 0$$

so $\alpha f(x) \in V$

Conclusion: $\boxed{V \text{ is a subspace of } C[0,1]}$

(see Thm 4.1.2 on p164)

(3) This NOT a subspace of $C[0,1]$. why?

For instance property (i) fails. The function $f(x) = 0$ is NOT in W since $\int_0^1 0\, dx = 0 \neq 1$

(you could also show W is not a subspace via properties (ii) and (iii) failing. but (i) seems simple enough.)