Rings:

(Q1) Give an example for the following:

(i) A field:

$\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{Z}_p$, etc.

(ii) A commutative ring with unity but NOT a field.

$\mathbb{Z}$

(iii) A commutative ring without unity

$2\mathbb{Z}$, $3\mathbb{Z}$, ..., $n\mathbb{Z}$, ect.

(since $n\mathbb{Z} = \{ \ldots, -n, 0, n, \ldots \}$ missing 1

(iv) A NON-commutative ring with unity.

$M_2(\mathbb{R})$

$M_2(\mathbb{Z})$

(unity here is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in both examples.)

(v) A NON-commutative ring without unity.

$M_2(2\mathbb{Z})$

2x2 matrices with entries in $2\mathbb{Z}$ (I is not in here.).

Q2 Show $\mathbb{Z}[i] = \{a+bi \mid a,b \in \mathbb{Z}\}$ forms a subring of $\mathbb{C}$.

proof: using the subring test:

(i) $0+0i \in \mathbb{Z}[i]$ ✓

Let $a+bi,\ c+di \in \mathbb{Z}[i]$

(ii) $(a+bi)-(c+di) = (a-c)+(b-d)i \in \mathbb{Z}[i]$ ✓

(iii) $(a+bi)(c+di) = (ac-bd)+(bc+ad)i \in \mathbb{Z}[i]$ ✓

Hence $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$.

Q3 Show the set $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \ a,b \in \mathbb{Z} \right\}$ is a subring of $M_2(\mathbb{Z})$

2x2 matrices with $\mathbb{Z}$-entries.