Consider the linear transformation $D: P_2 \longrightarrow P_2$ defined as $Dp(x) = p'(x)$ i.e., the differentiation operator. Let $X = (1, x, x^2)$ be an ordered basis of P_2.

(i) Find ${}_X D_X$.

(ii) Consider the ordered set $Y = (1, 1+x+x^2, 2+4x+14x^2)$. Show Y is a basis of P_2.

(iii) Find ${}_X D_Y$.

(iv) Find a basis for the kernel of D $(\ker(D))$.

(v) Find a basis for the image of D $(\operatorname{im}(D))$.

(vi) Find $\ker(D)$

(vii) Find $\operatorname{im}(D)$

SOL:

(i) By Thm. 4.8.3 (II) (also see Def. 81 on p 186) we have:

$$ {}_X D_X = \left[K_X D(1) \quad K_X D(x) \quad K_X D(x^2) \right] $$

well

$$ D(1) = \frac{d}{dx}(1) = 0 $$

$$ D(x) = \frac{d}{dx}(x) = 1 $$

$$ D(x^2) = \frac{d}{dx}(x^2) = 2x $$

so we have

$$ {}_X D_X = \left[K_X(0) \quad K_X(1) \quad K_X(2x) \right] $$

now we need to find $K_X(0)$, $K_X(1)$ and $K_X(2x)$. See section 4.6 for details on how this works in general.

well we need to write 0 as a linear combination of the elements in the basis X:

$$ 0 = 0(1) + 0(x) + 0(x^2) $$

Remember: the order in which you write the linear combination matters! it must match the order in $X = (1, x, x^2)$.

which means

$$ K_X(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

first column of ${}_X D_X$

similarly we can find

$$ K_X(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} $$

second column of ${}_X D_X$

$$ K_X(2x) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} $$

third column of ${}_X D_X$

So finally we have

$$ \boxed{ {}_X D_X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} } $$

(ii) consider the basis $X = (1, x, x^2)$ provided.

Now consider the matrix A defined as :

$$A = \begin{bmatrix} K_X(p_1(x)) & K_X(p_2(x)) & K_X(p_3(x)) \end{bmatrix}$$

where $p_1(x)$, $p_2(x)$ and $p_3(x)$ are the polynomials in the ordered set Y.
(see how the general construction would work.)

so

$$A = \begin{bmatrix} K_X(1) & K_X(1+x+x^2) & K_X(2+4x+14x^2) \end{bmatrix}$$

now we need to find :

$$K_X(1), \quad K_X(1+x+x^2) \quad \text{and} \quad K_X(2+4x+14x^2)$$

well

$$1 = 1(1) + 0(x) + 0(x^2)$$

so $K_X(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$1 + x + x^2 = 1(1) + 1(x) + 1(x^2)$$

so $K_X(1+x+x^2) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

and

$$2 + 4x + 14x^2 = 2(1) + 4(x) + 14(x^2)$$

so $K_X(2+4x+14x^2) = \begin{bmatrix} 2 \\ 4 \\ 14 \end{bmatrix}$

So now we have

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 1 & 14 \end{bmatrix}$$

Now we want to find the rank of A:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 1 & 14 \end{bmatrix} \xrightarrow{R3 \to R3 - R2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 10 \end{bmatrix}$$ ← REF of A which has rank = 3

Since the rank of A is 3 and 3 is the number of columns of A, the set of vectors $K_X(Y) = \{ K_X(1), K_X(1+x+x^2), K_X(2+4x+14x^2) \} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 14 \end{bmatrix} \right\}$

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is linearly independent in $\mathbb{R}^3$. Moreover the rank(A) is the number of rows of A so the set $K_X(Y)$ also spans $\mathbb{R}^3$ (this means $\text{span}(K_X(Y)) = \mathbb{R}^3$) so since the set

$K_X(Y) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 14 \end{bmatrix} \right\}$ is both linearly independent and spans $\mathbb{R}^3$ it must be a basis of $\mathbb{R}^3$. (Note: you can shorten the argument by just saying $\text{rank}(A) = 3 \Rightarrow A$ is invertible $\Rightarrow K_X(Y)$ is a basis of $\mathbb{R}^3$ by lemma 3.5.5)

Therefore since K_X is an isomorphism from P_2 to $\mathbb{R}^3$ the set $Y = (1, 1 + x + x^2, 2 + 4x + 14x^2)$ is a basis of P_2.

(iii) 2 solutions:

SOL 1: By definition

$$
\begin{aligned}
{}_X D_Y &= \left[K_X(D(1)) \quad K_X(D(1+x+x^2)) \quad K_X(D(2+4x+14x^2)) \right] \\
&= \left[K_X(0) \quad K_X(1+2x) \quad K_X(4+28x) \right] \\
&= \boxed{\begin{bmatrix} 0 & 1 & 4 \\ 0 & 2 & 28 \\ 0 & 0 & 0 \end{bmatrix}}
\end{aligned}
$$

SOL 2: Change of basis

we want to find ${}_X D_Y$

well notice ${}_X D_Y = ({}_X D_X)({}_X I_Y)$

so we need to find ${}_X I_Y$

$$
\begin{aligned}
{}_X I_Y &= \left[K_X(1) \quad K_X(1+x+x^2) \quad K_X(2+4x+14x^2) \right] \\
&= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 1 & 14 \end{bmatrix}
\end{aligned}
$$

and now

$$
{}_X D_Y = ({}_X D_X)({}_X I_Y) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 1 & 14 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 & 4 \\ 0 & 2 & 28 \\ 0 & 0 & 0 \end{bmatrix}}
$$

(iv) Consider the matrix ${}_X D_X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and lemma 4.8.4.

we know how to find a basis for the nullspace of ${}_X D_X$:
solving the system $({}_X D_X)\vec{v} = \vec{0}$ where $\vec{v} \in \mathbb{R}^3$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftarrow \text{already in RREF.}$$

we have
$v_1 = \text{anything}$
$v_2 = 0$
$2v_3 = 0$

writing in vector parametric form:

$v_1 = v_1$
$v_2 = 0$
$v_3 = 0$

so $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

we have that a basis for the nullspace of ${}_X D_X$ is the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

to find a basis for the kernel of D we just need to use lemma 4.8.4 and realize $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the coordinate vector of some polynomial i.e. $K_X(p(x)) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (with respect to the basis X)

says $p(x) = 1(1) + 0(x) + 0(x^2)$
$= 1$

so a basis for the kernel of D is just $\boxed{\{1\}}$ (the set containing the polynomial $p(x) = 1$)

(v) Consider the matrix ${}_X D_X$ and lemma 4.8.4 again.

find a basis for the column space of ${}_X D_X$. This is just the set of pivot columns of ${}_X D_X$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$

to find a basis for the image of D we just need to use lemma 4.8.4 $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the coord. vector of the polynomial $p_1(x) = 1(1) + 0(x) + 0(x^2) = 1$ with respect to the basis X and $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ is the coord. vector of the poly $p_2(x) = 0(1) + 2(x) + 0(x^2) = 2x$ with respect to the basis X $\longrightarrow$

So a basis of the image of D is just

$\boxed{\{1, 2x\}}$ (the set containing the polynomials $p_1(x) = 1$ and $p_2(x) = 2x$.)

(vi) use (iv) you have a basis so

$$\ker(D) = \text{span}(\{1\}) = \{\alpha(1) \mid \alpha \in \mathbb{R}\}$$
$$= \underbrace{\boxed{\{\alpha \mid \alpha \in \mathbb{R}\}}}_{\text{the set of all constant polynomials}}$$

This should make a lot of sense since all constant polynomials have 0 for their derivative (which is our transformation)

(vii) use (x) you have a basis so

$$\text{im}(D) = \text{span}(\{1, 2x\})$$
$$= \boxed{\{\alpha_1(1) + \alpha_2(2x) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}}$$

this is an acceptable final answer, although you can simplify this a little more.