Supplementary homework problems for week 6.

1. Let $A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & -1 & 1 & 0 \\ 1 & -1 & 0 & -1 \end{bmatrix}$

   (a) Find a basis X for the column space (image) of A.

   (b) What is the dimension of the column space of A?

   (c) Find a basis Y for the null space of A.

   (d) What is the dimension of the null space (kernel) of A?

   (e) Find a basis Z for the row space of A.

   (f) What is the dimension of the row space of A?

2. Let $V = \{\mathbf{a}, \mathbf{b}\}$ be a collection of vectors in $\mathbb{R}^n$. Show that Span(V) is a subspace of $\mathbb{R}^n$.

① Please see lecture 26 comments for an example of this type.
You should get the following solutions:

(a) $X = \left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right)$

(b) 2

(c) $Y = \left(\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$

(d) 2

(e) $Z = \left([1 \ 0 \ 1 \ 1], [0 \ 1 \ 1 \ 2] \right)$

(f) 2

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② $\text{Span}(V) = \{ c_1\vec{a} + c_2\vec{b} \mid c_1, c_2 \in \mathbb{R} \}$.

we want to show Span (V) is a subspace of $\mathbb{R}^n$.

Use theorem 3.3.2 so we need to show span (V) satisfies the three properties

(i) If $c_1 = 0$ and $c_2 = 0$ then $0\vec{a} + 0\vec{b} = \vec{0}$ so $\vec{0} \in \text{span}(V)$ ✓

(ii) Let $\vec{x} \in \text{span}(V)$ so $\vec{x} = \alpha_1\vec{a} + \alpha_2\vec{b}$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$

Let $\vec{y} \in \text{span}(V)$ so $\vec{y} = \beta_1\vec{a} + \beta_2\vec{b}$ for some $\beta_1, \beta_2 \in \mathbb{R}$

then
$$\begin{aligned}\vec{x} + \vec{y} &= (\alpha_1\vec{a} + \alpha_2\vec{b}) + (\beta_1\vec{a} + \beta_2\vec{b}) \\ &= \alpha_1\vec{a} + \alpha_2\vec{b} + \beta_1\vec{a} + \beta_2\vec{b} \\ &= \alpha_1\vec{a} + \beta_1\vec{a} + \alpha_2\vec{b} + \beta_2\vec{b} \\ &= \underbrace{(\alpha_1 + \beta_1)}_{\text{real number}}\vec{a} + \underbrace{(\alpha_2 + \beta_2)}_{\text{real number}}\vec{b}\end{aligned}$$
so $\vec{x} + \vec{y} \in \text{span}(V)$ ✓

(iii) Let $\vec{x} \in \text{span}(V)$ so $\vec{x} = \alpha_1 \vec{a} + \alpha_2 \vec{b}$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$

Let $c \in \mathbb{R}$

then $c\vec{x} = c(\alpha_1 \vec{a} + \alpha_2 \vec{b})$

$$= \underbrace{c\alpha_1}_{\text{real number}} \vec{a} + \underbrace{c\alpha_2}_{\text{real number}} \vec{b}$$

so $c\vec{x} \in \text{span}(V)$ ✓

so by (i), (ii) and (iii)

and thm 3.3.2

we have $\text{span}(V)$ is a subspace of $\mathbb{R}^n$