Math 304 notes and problems, Spring, 2010

X1. Let $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Calculate several powers F^k (at least through $k = 6$). The numbers that occur in the $1,1$ entry of F^k form a famous sequence of numbers. Do you know what this sequence is called?

X2. Decide whether each of the following is a *linear transformation.* In each case, either give a specific violation of the definition of linear transformation or prove that the definition is satisfied.
(a) $F(x) = x^3$
(b) $F(x_1,\, x_2) = (x_1 x_2,\, x_1 + x_2)$
(c) $F(x_1,\, x_2) = 2(x_1,\, x_2) - (3,\, x_1)$
(d) $F(x_1,\, x_2) = (x_1 + x_2,\, x_1 - x_2,\, x_2 - |x_1|)$
(e) $F(x_1,\, x_2) = (x_1 + 2)^2 - (x_1 - 2)^2$

X3. Let $A = \begin{bmatrix} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 2 & 0 & 4 \\ 2 & -4 & 1 & 3 & -1 \end{bmatrix}$.

(a) Find a basis for the column space of A.
(b) What is the dimension of the column space of A?
(c) Find a basis for the null space of A.
(d) What is the dimension of the null space of A?

X4. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$.

(a) Show that $X = \{\, \mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3 \,\}$ is a linearly independent set.
(b) Find a vector $\mathbf{v}_4$ which is **not** in the span of X.
(c) Explain why $(\, \mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4 \,)$ is a basis for $\mathbb{R}^4$.

X5. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$, and $X = \{\, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \,\}$. Let $V = \mathrm{Span}(X)$.

(a) Why is $Z = (\, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \,)$ a basis for V? You might want to refer to problem **X4**.

(b) Let K be the coordinate transformation defined by the basis Z. The vector $\mathbf{w} = \begin{bmatrix} 5 \\ 5 \\ 3 \\ 10 \end{bmatrix}$ is in V. Calculate $K(\mathbf{w})$.

(c) Let M be the matrix with columns given by the vectors in the basis Z. Find a non-trivial solution $\mathbf{x}$ of the equation $M^T\mathbf{x} = \mathbf{0}$.

(d) Let $A = \mathbf{x}^T$, where $\mathbf{x}$ is your answer to part (c). Show that $AM = \mathbf{0}$. Explain why $A\mathbf{v} = \mathbf{0}$ for all vectors $\mathbf{v}$ in V.

A note from Quincy:

To receive any credit you must SHOW ALL OF YOUR WORK. Quoting a theorem or definition ONLY counts as "enough work" DEPENDING on the previous WORK YOU HAVE SHOWN.

X6. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -2 & 4 & -8 \\ 2 & 4 & -6 & -8 \\ 2 & 0 & 7 & 4 \end{bmatrix}$

(a) Find a basis for $\mathrm{Col}(A)$
(b) Find a basis for $\mathrm{Row}(A)$
(c) Find a basis for $\mathrm{Nul}(A)$
(d) Find a basis for $\mathrm{Col}(A^T)$
(e) Find a basis for $\mathrm{Row}(A^T)$
(f) Find a basis for $\mathrm{Nul}(A^T)$

X7. Let Let T be the transformation given by the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$

(a) Find a basis for $\mathrm{Dom}(T)$
(b) Find a basis for $\mathrm{CoDom}(T)$
(c) Find a basis for $\mathrm{Image}(T)$
(d) Find a basis for $\mathrm{Ker}(T)$

Hint: (a) and (b) require no work if you first say what $\mathrm{Dom}(T)$ and $\mathrm{CoDom}(T)$ are.

X8. This problem refers to the affine subsets described in section 3.6.4. Please use the notation of that section when answering the following.

(a) Using the matrix A from problem **X6** and the vector $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$, describe the solution set of the equation $A\mathbf{x} = \mathbf{b}$.

(b) Using the matrix A from problem **X7** and the vector $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$, describe the solution set of the equation $A\mathbf{x} = \mathbf{b}$.

(c) Using the matrix A from problem **X7** and the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, describe the solution set of the equation $A\mathbf{x} = \mathbf{b}$.

(d) Is the answer you got from part (c) a subspace of $\mathbb{R}^4$? If yes, then prove it using the definition or by quoting a theorem. If no, then show a **specific** counter example with **specific** vectors which disproves it.

X9. Let $f(x) = \sin^2(x)$, $g(x) = \cos^2(x)$, $h(x) = 1$. Then f, g and h are in $C(-\infty, \infty)$, the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Let $X = \{ f, g, h \}$.

(a) Use your knowledge of pre-calculus to show that X is a linearly dependent set.
(b) Find a subset Y of X which has the same span as X. (Hint: If you found a dependency relation in part (a) this should be straightforward.)
(c) Show that Y is now a basis for $\mathrm{Span}(X)$. (Hint: 0 and $\pi/4$ are good inputs to think about.)

X10. Let $X = \left\{ \sin^2(x), \cos^2(x), 1 \right\}$ be the set of functions defined in **X9** and let $V = \mathrm{Span}(X)$. Then $(\sin^2(x), \cos^2(x))$ is a basis for V. Let $K\colon V \to \mathbb{R}^2$ be the coordinate transformation defined by this basis.
For each of the following, either find the coordinate transform $K(u)$ or explain why $K(u)$ is not defined. You may need to remember some trig identities.
(a) $u = 3 - 4\cos^2(x) + 5\sin^2(x)$
(b) $u = \cos(2x)$
(c) $u = \sin(2x)$

X11. (This is a modification of problem S4-8.) Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 2x_1 - x_2 \\ 3x_1 + x_2 \end{bmatrix}$$

(a) Find the matrix of F with respect to the standard bases on $\mathbb{R}^3$ and $\mathbb{R}^2$.
(b) Find the matrix of F with respect to the bases $X = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$ and $Y = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{w}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

X12. Let $F\colon P_3 \to P_2$ be defined by the rule $F(p(x)) = p'(x) - p(1)$. This is a linear transformation.
(a) Find the matrix representation ${}_WF_Z$ where Z is the basis $(1, x, x^2, x^3)$ for P_3 and W is the basis $(1, x, x^2)$ for P_2.
(b) Use ${}_WF_Z$ to decide whether F is onto.
(c) Use ${}_WF_Z$ to find a basis for the kernel of F. Remember, elements of the kernel of F are polynomials, not column vectors, so write your answer appropriately.

X13. (This is a modification of problem S4-10.) Let $D\colon P_1 \to P_1$ be the linear transformation defined by

$$D(p(x)) = (x+2)p'(x) + 3p(x),$$

and let $X = \{p_1(x), p_2(x)\}$ and $Y = \{q_1(x), q_2(x)\}$, where

$$p_1(x) = 1 - x, \quad p_2(x) = 2x, \quad q_1(x) = 1 + x, \quad q_2(x) = -3.$$

(a) Find the matrix ${}_XD_X$.
(b) Find the change of basis matrix ${}_YI_X$.
(c) Find the change of basis matrix ${}_XI_Y$.
(d) Use parts (a), (b), and (c) to find the matrix ${}_YD_Y$.
(e) Check your answer to part (c) by directly computing the matrix of D with respect to the basis Y.

X14. (This is a modification of problem S4-13.) Consider the bases $X = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_2\}$ and $Y = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 6 \\ -3 \\ -1 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}.$$

(a) Find the change of basis matrix ${}_Y I_X$.
(b) Find the change of basis matrix ${}_X I_Y$.
(c) Compute the coordinate vector of

$$\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

with respect to the basis Y, i.e., $K_Y(\mathbf{w})$.
(d) Use parts (b) and (c) to find the coordinate vector of $\mathbf{w}$ with respect to the basis X, i.e., $K_X(\mathbf{w})$.
(e) Check your work by computing $K_X(\mathbf{w})$ directly.

Remember that the vector space M_{22} consists of all 2×2 matrices. The standard basis for M_{22} is (E_1, E_2, E_3, E_4), where $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
Let V be the subspace of M_{22} consisting of all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which satisfy the condition $a + d = 0$.

X15. (a) Let $H = E_1 - E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Show that $X = (H, E_2, E_3)$ is a basis for V.
(b) Define the linear transformation $L\colon V \to V$ by $L(A) = AH - HA$. Find ${}_X L_X$.

X16. There were several assertions without justification in problem **X15** and the discussion before it. Fix this by proving the following:
(a) V is a subspace of M_{22}.
(b) If A is in V then $L(A)$ is in V. (More generally, if A and B are in V then $AB - BA$ is in V.)
(c) L is a linear transformation from V to V.

Note: Quincy's doctoral research concerns *Lie algebras*. ("Lie" is pronounced "Lee".) The simplest type of Lie algebra is a vector space of square matrices which is closed under the *bracket operation*, which is defined by $[A, B] = AB - BA$. The vector space V in the last two problems is a well-known Lie algebra, with the special name $\mathfrak{sl}_2$.

To get any credit on these problems you MUST SHOW the appropriate WORK.

X17. Let $A = \begin{bmatrix} -1 & 5 \\ 2 & 7 \end{bmatrix}$. Calculate $\det(A)$ by the following method.
(a) Cross hatching.
(b) Row reducing A to a triangular matrix.
(c) Using an elementary matrix decomposition of A if one exists.
(d) Cofactor expansion

X18. Let $B = \begin{bmatrix} -1 & 5 & -1 \\ 2 & 7 & 5 \\ 1 & 12 & 4 \end{bmatrix}$. Calculate $\det(B)$ by the following method.
(a) Cross hatching.
(b) Row reducing B to a triangular matrix.
(c) Cofactor expansion

X19. Let $C = \begin{bmatrix} 0 & 2 & -2 \\ 2 & -2 & 0 \\ 2 & -2 & 2 \end{bmatrix}$. Calculate $\det(C)$ by the following method.
(a) Cross hatching.
(b) Row reducing C to a triangular matrix.
(c) Cofactor expansion

X20. Let $D = \begin{bmatrix} 0 & 0 & -1 & 2 \\ 0 & -3 & 0 & 4 \\ 5 & 0 & -6 & 7 \\ 8 & 0 & 0 & 9 \end{bmatrix}$. Calculate $\det(D)$ by the following method.
(a) Row reducing D to a triangular matrix.
(b) Cofactor expansion

X21. Let $E = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 2 & -3 & 2 & -3 \\ 4 & 4 & 4 & 4 \\ 5 & 6 & 3 & 2 \end{bmatrix}$. Calculate $\det(E)$ "As quickly as possible" using the method of your choice. HINT: You may want to familiarize yourself with the property list in section 5.2.

X22. Suppose that A is a square $n \times n$ matrix and λ is an eigenvalue of A with eigenvector $\mathbf{v}$. Give a short proof of each of the following statements.
(a) λ^2 is an eigenvalue of $M = A^2$ with corresponding eigenvector $\mathbf{v}$.
(b) $\lambda + 8$ is an eigenvalue of $M = A + 8I$ with corresponding eigenvector $\mathbf{v}$.
(c) If Q is an $n \times n$ invertible matrix then λ is an eigenvalue of $M = QAQ^{-1}$ with corresponding eigenvector $\mathbf{w} = Q\mathbf{v}$.
(d) If B is an $n \times n$ matrix and $AB = BA$ and $\mathbf{w} = B\mathbf{v} \neq \mathbf{0}$ then $\mathbf{w}$ is an eigenvector of A corresponnding to the eigenvalue λ.