EXAM 1

Math 221 - 09 - Calculus I
2/26/2009

Name: Key

When you are finished please sign the following:

Signature: Key

By signing my name I pledge that I have not broken the Student Academic Honesty Code at any point during this examination.

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 6 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Part A. Fill the blank (30% of the total points).

1. (*30 points*)

1. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ $\boxed{\text{exist}}$, then

a. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

b. $\lim_{x\to a}[cf(x)g(x)] = c\lim_{x\to a} f(x) \lim_{x\to a} g(x)$

c. $\lim_{x\to a}[f(x)/g(x)] = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$ provided $\lim_{x\to a} g(x)$ $\boxed{\neq 0}$

d. $\lim_{x\to a}(f(x))^n = [\lim_{x\to a} f(x)]^n$ if n is a positive integer.

e. $\lim_{x\to a}(f(x))^{1/n} = [\lim_{x\to a} f(x)]^{1/n}$ if n is a positive integer and if n is even then $\lim_{x\to a} f(x) > 0$.

f. $\lim_{x\to a} c = \boxed{c}$

g. $\lim_{x\to a} x^n = \boxed{a^n}$ where n is a positive integer.

h. $\lim_{x\to 0} \frac{\sin x}{x} = \boxed{1}$

2. (a) If $f(x) \boxed{\leq} g(x)$ for x $\boxed{\text{near } a}$, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$, provided the limits exist.

(b) if $f(x) \leq g(x) \leq h(x)$ for x $\boxed{\text{near } a}$, and $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = \boxed{L}$

3. $f(x)$ is continuous at a if $\lim_{x\to a^-} f(x) \boxed{= f(a) =} \lim_{x\to a^+} f(x)$.
If f and g are $\boxed{\text{continuous at } a}$, then so are $f \pm g$, fg and f/g (provided $g(a)$ $\boxed{\neq 0}$)
polynomial, rational, root and trigonometric functions are continuous $\boxed{\text{on their domain}}$

4. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

5. $(cf(x) \pm g(x))' = cf'(x) \pm g'(x)$

6. $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

7. $(f(x)/g(x))' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

8. $(x^n)' = nx^{n-1}$
$(\sin x)' = \cos(x)$
$(\cos x)' = -\sin(x)$
$(\tan x)' = \sec^2(x)$
$(\sec x)' = \sec(x)\tan(x)$
$(\cot x)' = -\csc^2(x)$
$(\csc x)' = -\csc(x)\cot(x)$

9. $\frac{d}{dx} f(u(x)) = \frac{d}{du} f \frac{d}{dx} u = f'(u(x))\, u'(x)$

Part B. Problems solving. (70% of the total points) **You need to show your work!!**

2. (*15 points*) Evaluate the limit, if it exists, and justify each step by appropriate limit law(s) with respect to Part A.

a. (*5 pts*) $\lim\limits_{x\to 3}\dfrac{x-3}{\sqrt{x}-\sqrt{3}} = \lim\limits_{x\to 3}\dfrac{x-3}{\sqrt{x}-\sqrt{3}}\cdot 1$

$= \lim\limits_{x\to 3}\dfrac{(x-3)(\sqrt{x}+\sqrt{3})}{(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}$ +2 points

$= \lim\limits_{x\to 3}\dfrac{\cancel{(x-3)}(\sqrt{x}+\sqrt{3})}{\cancel{(x-3)}}$ +2 points

$= \lim\limits_{x\to 3}\sqrt{x}+\sqrt{3} = \sqrt{3}+\sqrt{3} = \boxed{2\sqrt{3}}$ + 1 point

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b. (*5 pts*) $\lim\limits_{x\to 1}\dfrac{x^2-1}{x^2+x-2} = \lim\limits_{x\to 1}\dfrac{\cancel{(x-1)}(x+1)}{(x+2)\cancel{(x-1)}}$ factorization +2 points

$= \lim\limits_{x\to 1}\dfrac{x+1}{x+2}$ reduction + 2 points

$= \boxed{\dfrac{2}{3}}$ correct answer +1 point

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c. (*5 pts*) $\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{(x^2-1)} = \lim\limits_{x\to 1}\dfrac{\sin(x-1)}{(x-1)(x+1)}$ + 2 points

+2 points

$= \underbrace{\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{(x-1)}}\ \lim\limits_{x\to 1}\dfrac{1}{(x+1)} = 1\cdot\lim\limits_{x\to 1}\dfrac{1}{x+1} = \boxed{\dfrac{1}{2}}$ +1 point

The following was not needed but this may be nice to refer to when studying:

Notice this limit is of the form: $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$ (from PART A #1h)

if $x\to 1$ and $\theta = x-1$ then $\theta\to 0$

so $\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x-1} = 1 = \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}$ ✓

3. (*15 points*) Show that the function

$$f(x) = \begin{cases} x^4 \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$. *Hint:* Show why f is continuous when $x \neq 0$, and then show f is continuous at $x = 0$.

$\boxed{\text{if } x \neq 0}$ then $f(x) = x^4 \cos\left(\frac{1}{x}\right)$

x^4 is continuous everywhere (it is a polynomial)

$\frac{1}{x}$ is continuous everywhere except $x=0$ (rational function)

$\cos\left(\frac{1}{x}\right)$ is continuous everywhere except $x=0$ since it is a composition of $\cos(x)$ which is continuous everywhere and $\frac{1}{x}$ which is continuous everywhere except $x=0$.

so, $x^4 \cos\left(\frac{1}{x}\right)$ is continuous everywhere except $x=0$ since it is a product of two continuous functions (everywhere except at $x=0$)

<u>So</u>, $f(x)$ is continuous for $x \neq 0$. +5 points

$\boxed{\text{if } x = 0}$ check to see if $f(x)$ is continuous at $x=0$: +3 points

we want to show: $\lim\limits_{x \to 0} x^4 \cos\left(\frac{1}{x}\right) = f(0)$

well, $f(0) = 0$.

how do we show $\lim\limits_{x \to 0} x^4 \cos\left(\frac{1}{x}\right) = 0$?

notice: $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$ +3 points

So, $-x^4 \leq x^4 \cos\left(\frac{1}{x}\right) \leq x^4$

$\lim\limits_{x \to 0} (-x^4) = 0 = \lim\limits_{x \to 0} x^4$ so by the squeeze theorem, $\lim\limits_{x \to 0} x^4 \cos\left(\frac{1}{x}\right) = 0$, hence $f(x)$ is continuous at $x=0$ +4 points

SO, f is continuous on $(-\infty, \infty)$ ✓

4. (*15 points*)
The displacement (in meters) of a τ-particle moving in a straight line is given by the equation of motion $s(t) = \sqrt{2t-1}$, where t is measured in seconds. Find the velocity of the τ-particle when $t = 1$ using the definition of the derivative. **You are not allowed to use the derivative formulas from Part A.**

$$v(1) = s'(1) = \lim_{h\to 0} \frac{s(1+h) - s(1)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2(1+h)-1} - \sqrt{2(1)-1}}{h}$$

5 points

$$= \lim_{h\to 0} \frac{\sqrt{2+2h-1} - \sqrt{1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+2h} - 1}{h} \cdot 1$$

5 points

$$= \lim_{h\to 0} \frac{(\sqrt{1+2h}-1)(\sqrt{1+2h}+1)}{h \, (\sqrt{1+2h}+1)}$$

$$= \lim_{h\to 0} \frac{(1+2h) - 1}{h(\sqrt{1+2h}+1)}$$

5 points.

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{1+2h}+1)}$$

$$= \lim_{h\to 0} \frac{2}{\sqrt{1+2h}+1}$$

$$= \frac{2}{\sqrt{1+0}+1} = \boxed{1}$$

NOTE: Many of you found $v(t)$ dy definition first and then found $v(1)$. This is fine, but I'm sure most of you would agree finding $v(1)$ by definition is <u>much</u> quicker. Keep this in mind for the future.

5. (*15 points*) Differentiate the following functions. **You may use the derivative formulas from Part A.**

a. (*5 pts*) $f(x) = \dfrac{x}{1-x^2}$

$$f'(x) = \frac{(1-x^2)\frac{d}{dx}x - x\frac{d}{dx}(1-x^2)}{(1-x^2)^2}$$

$$= \frac{(1-x^2)(1) - x(-2x)}{(1-x^2)^2}$$

$$= \frac{1-x^2+2x^2}{(1-x^2)^2} = \boxed{\frac{1+x^2}{(1-x^2)^2}}$$

b. (*5 pts*) $f(x) = 2009 \cdot 5^{2009}$

↑ this is a constant! so,

$$\boxed{f'(x) = 0}$$

c. (*5 pts*) $f(x) = \tan^2(4x)$

$$f'(x) = 2\tan(4x)\frac{d}{dx}\tan(4x)$$

$$= 2\tan(4x)\sec^2(4x)\frac{d}{dx}4x$$

$$= 2\tan(4x)\sec^2(4x)4$$

$$= \boxed{8\tan(4x)\sec^2(4x)}$$

PICK ONE OF THE FOLLOWING:

Please indicate which one you do NOT want me to grade by putting an X through it, otherwise I will grade the first one worked on:

6. (*10 points*)

a. (*10 pts*) Use the Intermediate Value Theorem to show there is some point where the function $f(x) = \frac{1}{4}x^4 - x^3 - \frac{1}{2}x^2 + 3x + 2$ has zero slope on the interval $(0, 2)$.

$f'(x) = x^3 - 3x^2 - x + 3$, looking at the closed interval $[0,2]$

$f'(0) = 0 - 0 - 0 + 3 = 3$

$f'(2) = 8 - 3 \cdot 4 - 2 + 3 = -3$

and $0 \in (-3,3)$ in other words 0 is between -3 and 3 on the real number line.

So by the Intermediate Value Theorem there exists some point $c \in (0,2)$ such that $f'(c) = 0$ ✓

b. (*10 pts*) Find an equation of the tangent line to the function:

$$f(x) = \begin{cases} 6 & \text{if } x \leq -1 \\ x & \text{if } -1 < x < 0 \\ \sin(\sin(x)) & \text{if } x \geq 0 \end{cases}$$

at the point $(\pi, 0)$.

well at $x = \pi$ since $\pi \geq 0$ we have $f(x) = \sin(\sin(x))$ there.

So, for x at π

$$\left.\begin{aligned} f'(x) &= \cos(\sin(x)) \frac{d}{dx} \sin(x) \\ &= \cos(\sin(x)) \cos(x) \end{aligned}\right\} \text{5 points.}$$

so $f'(\pi) = \cos(\sin(\pi)) \cos(\pi) = \cos(0)(-1) = -1$ } 2 points

so equation of tangent line at the point $(\pi, 0)$ is

$y - 0 = -1(x - \pi)$ } 2 points

$\boxed{y = -x + \pi}$ ✓ 1 point.