EXAM 2

Math 221 - 09 - Calculus I
3/26/2009

Name: Key

When you are finished please sign the following:

Signature: ______________________________

By signing my name I pledge that I have not broken the Student Academic Honesty Code at any point during this examination.

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 6 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Part B. Problems solving. (70% of the total points) **You need to show your work!!**

2. (*10 points*)

a. (*5 pts*) Find $\frac{dy}{dx}$ by implicit differentiation:

$$3x^3 + xy = 15$$

$$\frac{d}{dx}\left(3x^3 + xy\right) = \frac{d}{dx} 15$$

$$9x^2 + x\frac{dy}{dx} + y = 0$$

4 points

$$\boxed{\frac{dy}{dx} = \frac{-y - 9x^2}{x}}$$

1 point

b. (*5 pts*) Using the result from above find y''. Simplify!

use the quotient rule:

$$y'' = \frac{x\left[-\frac{dy}{dx} - 18x\right] - \left[-y - 9x^2\right](1)}{x^2}$$

2 points

$$= \frac{x\left[-\left(\frac{-y-9x^2}{x}\right) - 18x\right] + \left[y + 9x^2\right]}{x^2}$$

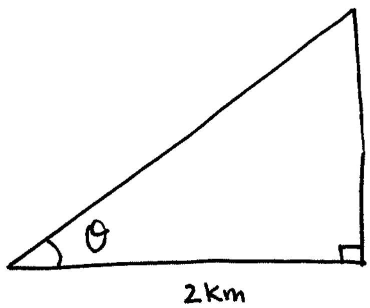
2 points

$$= \frac{-(-y-9x^2) - 18x^2 + y + 9x^2}{x^2}$$

$$= \frac{y + \cancel{9x^2} - \cancel{18x^2} + \cancel{9x^2} + y}{x^2} = \boxed{\frac{2y}{x^2}}$$

1 point

3. (*15 points*) You are watching a rocket launch exactly 2 km. away. As the rocket climbs in altitude you begin tilt your head upwards in order to see it. If the rocket were to move vertically upwards at a rate of 10 km/s, how fast is the angle at which you tilt your head changing when the rocket is 9 km above the launching station? (Note: Your picture should be something nice! Do not worry about meaningless things such as your height and the height of the rocket, etc.)



given : $\frac{dy}{dt} = 10\ \text{km}/\text{s}$

unknown : $\frac{d\theta}{dt}$ when $y = 9\ \text{km}.$

5 points

Equation : $\tan\theta = \frac{y}{2}$ 2 points

Differentiation

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{y}{2}\right)$$

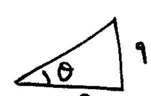
$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{2}\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{1}{2\sec^2\theta}\frac{dy}{dt}$$

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{2}\frac{dy}{dt}$$

5 points

Substitution : when $y=9$ we have this picture :



so $\cos\theta = \frac{2}{\sqrt{2^2+9^2}} = \frac{2}{\sqrt{85}}$

hence $\frac{d\theta}{dt} = \frac{\left(\frac{2}{\sqrt{85}}\right)^2}{2}(10) = \frac{\frac{4}{85}}{2} 10 = \frac{20}{85}$

$= 4/17$

3 points

Solution : $\boxed{\frac{d\theta}{dt} = 4/17\ \text{rad}/\text{s}}$

4. (*15 points*) Evaluate the limit, if it exists. You must show your work by using appropriate limit laws/methods .

a. (*5 pts*) $\displaystyle\lim_{x\to\infty}\frac{x^2-1}{x^2+x-2} = \lim_{x\to\infty}\frac{(x^2-1)\left(\frac{1}{x^2}\right)}{(x^2+x-2)\left(\frac{1}{x^2}\right)}$

$$= \lim_{x\to\infty}\frac{\frac{x^2}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{2}{x^2}} = \lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}} = \frac{1-0}{1+0-0} = \boxed{1}$$

b. (*5 pts*) $\displaystyle\lim_{x\to1}\frac{x^2-1}{x^2+x-2} = \lim_{x\to1}\frac{\cancel{(x-1)}(x+1)}{(x+2)\cancel{(x-1)}} = \lim_{x\to1}\frac{(x+1)}{(x+2)} = \boxed{\frac{2}{3}}$

c. (*5 pts*) $\displaystyle\lim_{x\to0}\underbrace{\frac{x^2-1}{x^2+x-2}} = \frac{0-1}{0+0-2} = \boxed{\frac{1}{2}}$

↑

this function is continuous everywhere except when $x = 1$ OR $x = -2$, neither of which is $x=0$! So the result follows.

5. (*20 points*) Let $f(x) = \dfrac{x^2}{x^2+4}$. Find

a. (*2 pts*) domain,

$\mathbb{R}$, this is because f is a rational function which is continuous everywhere except. when $x^2+4=0$ which there are no such x in the real numbers that solve this hence the domain is all real numbers

b. (*2 pts*) intercepts,

<u>y-intercept:</u> $f(0) = \frac{0}{0+4} = 0$ so the point $(0,0)$

<u>x-intercept(s)</u>: $f(x) = 0 = \frac{x^2}{x^2+4} = 0 \iff x^2 = 0 \iff x = 0$

so the point $(0,0)$

c. (*2 pts*) symmetry,

$$f(-x) = \frac{(-x)^2}{(-x)^2+4} = \frac{x^2}{x^2+4} = f(x)$$ <u>EVEN</u>

d. (*5 pts*) intervals of increase and decrease,

$$f'(x) = \frac{(x^2+4)2x \;-x^2(2x)}{(x^2+4)^2} = \frac{\cancel{2x^3} + 8x \cancel{-2x^3}}{(x^2+4)^2} = \frac{8x}{(x^2+4)^2}$$

so $f'(x) = 0$ when $8x = 0$ so $x = 0$ and $f'(x)$ is always defined so

f': $-$ | $+$; $x=0$

<u>intrvals</u>: $\underbrace{(-\infty, 0)}$ $\{0\}$ $\underbrace{(0, \infty)}$

f: decreasing increasing

e. (*2 pts*) maximum and minimum values,

local minimum at $f(0) = 0$

so at the point $(0,0)$

f. (*5 pts*) intervals of concavity and points of inflection,

$$f''(x) = \frac{(x^2+4)^2 8 - 8x(2(x^2+4)2x)}{((x^2+4)^2)^2} = \frac{8(x^2+4)[(x^2+4) - 4x^2]}{(x^2+4)^4}$$

$$= \frac{8(x^2+4)(-3x^2+4)}{(x^2+4)^4}$$

so that $f''(x) = 0$ when $8(x^2+4)(-3x^2+4) = 0$

so $-3x^2+4 = 0$ (only factor with roots)

$3x^2 = 4$

$x^2 = 4/3$

$x = \pm\sqrt{4/3}$

f'': $-$ $+$ $-$

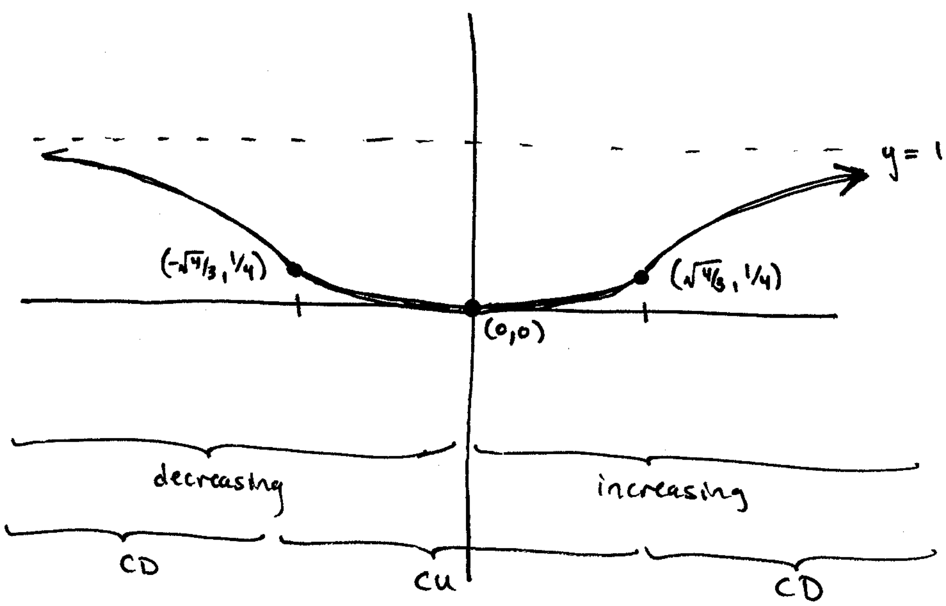
$-\sqrt{4/3}$ $\sqrt{4/3}$

intervals: $(0, -\sqrt{4/3})$ $\{-\sqrt{4/3}\}$ $(-\sqrt{4/3}, +\sqrt{4/3})$ $\{+\sqrt{4/3}\}$ $(\sqrt{4/3}, \infty)$

f: CD $\uparrow$ CU $\uparrow$ CD

inf point: $(-\sqrt{4/3}, 1/4)$ inf. point: $(+\sqrt{4/3}, 1/4)$

g. (*2 pts*) sketch its graph (it will be helpful to note any asymptote(s) present).



(at both ends)

<u>Notice</u>: there will be a horizontal asymptote at $y = 1$

because $\lim_{x\to\infty} f(x) = 1$

<u>and</u> $\lim_{x\to -\infty} f(x) = 1$

PICK ONE OF THE FOLLOWING:

Please indicate which one you do NOT want me to grade by putting an X through it, otherwise I will grade the first one worked on:

6. (*10 points*)

a. (*10 pts*) Find the point on the line $y = 2x + 1$ that is closest to the origin.

Suppose (x,y) is a point on the line $y=2x+1$. If D is the distance from (x,y) to the origin $(0,0)$ then:

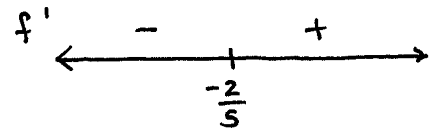
$$D = \sqrt{(x-0)^2 + (y-0)^2} \quad \text{(distance formula)}$$
$$= \sqrt{x^2 + (2x+1)^2}$$

consider $f(x) = D^2 = (x-0)^2 + (y-0)^2$

$$= x^2 + (2x+1)^2 = x^2 + 4x^2 + 4x + 1$$
$$= 5x^2 + 4x + 1$$

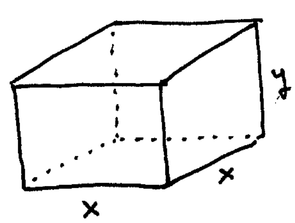
$$f'(x) = 10x + 4$$
$$f'(x) = 0 = 10x + 4 \implies x = \frac{-4}{10} = -\frac{2}{5}.$$

Check we have a minimum:



so the minimum value of f occurs when $x = -\frac{2}{5}$ which will be the same x-value that gives the minimum value for D. Hence, the point on the line $y=2x+1$ that is closest to the origin is $\left(-\frac{2}{5}, \frac{1}{5}\right)$

b. (*10 pts*) A closed box with a square base must have a volume of 8 ft^3. Find the dimensions of the box that will minimize the amount of material used. (Note: only the base is assumed to be square!)



square base!

we are given the volume : $8 = x^2 y$

we want to minimize surface area :

$$A = 2x^2 + 4xy$$

so using the given volume we see $y = \frac{8}{x^2}$ so

$$A(x) = 2x^2 + 4x\left(\frac{8}{x^2}\right) = 2x^2 + \frac{32}{x}, \quad x > 0$$ (since length must be nonnegative and volume was positive)

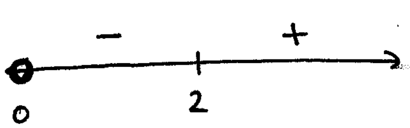
so

$$A'(x) = 4x - \frac{32}{x^2} = \frac{4x^3 - 32}{x^2}$$

$A'(x) = 0$ when $4x^3 - 32 = 0$

$$4x^3 = 32$$
$$x^3 = 8$$
$$x = \sqrt[3]{8}$$
$$x = 2$$

Check we have a minimum:



, so indeed $x=2$ corresponds to the minimum of $A(x)$

so $y = 8/4 = 2$

so the dimensions are 2ft × 2ft × 2ft. (a cube)