Example of Euclidean Algorithm

Example 1: Suppose we wanted to find gcd(50,35).

Using the Euclidean Algorithm (gcd reduction) we get the following:

$$\begin{aligned}\gcd(50,35) &= \gcd(35,15)\\ &= \gcd(15,5)\\ &= \gcd(5,0)\\ &= 5\end{aligned}$$

via the following equations

$$50 = 35 \cdot 1 + 15 \quad (1)$$
$$35 = 15 \cdot 2 + 5 \quad (2)$$
$$15 = 5 \cdot 3 + 0 \quad (3)$$

Suppose you are asked to find $x, y \in \mathbb{Z}$ such that $50x + 35y = 5$. We can use our work from (1) and (2) (we won't need (3)) to solve this.

First solve for the remainder of each equation and write them in reverse order.

So we have the following equations

$$\begin{aligned} 5 &= 35 - 15 \cdot 2 \quad \text{from (2)}\\ 15 &= 50 - 35 \cdot 1 \quad \text{from (1)}\end{aligned}$$

Let's introduce some color to make the next step clear:

$$\begin{aligned} 5 &= 35 - 15 \cdot 2\\ 15 &= 50 - 35 \cdot 1\end{aligned}$$

Now substitute equations starting from top to bottom

$$\begin{aligned} 5 &= 35 - 15 \cdot 2\\ &= 35 - (50 - 35 \cdot 1) \cdot 2\\ &= 35 - 50 \cdot 2 + 35 \cdot 2\\ &= 35 \cdot 3 - 50 \cdot 2\\ &= 50 \cdot (-2) + 35 \cdot (3)\end{aligned}$$

So $x = -2$ and $y = 3$.

Example 2:

(a) Show that 120 and 23 are relatively prime using the Euclidean algorithm (gcd reduction).

(b) Use (a) to find integers x and y such that $120x + 23y = 1$.

Solution to (a): Using gcd reduction we have:

$$\begin{aligned} \gcd(120, 23) &= \gcd(23, 5) \\ &= \gcd(5, 3) \\ &= \gcd(3, 2) \\ &= \gcd(2, 1) \\ &= \gcd(1, 0) \\ &= 1 \end{aligned}$$

via the following equations

$$\begin{aligned} 120 &= 23 \cdot 5 + 5 && (4) \\ 23 &= 5 \cdot 4 + 3 && (5) \\ 5 &= 3 \cdot 1 + 2 && (6) \\ 3 &= 2 \cdot 1 + 1 && (7) \\ 2 &= 1 \cdot 2 + 0. && (8) \end{aligned}$$

Solution to (b): Solving for the remainders in equations (4)-(7) and writing in reverse order we have:

$$\begin{aligned} 1 &= 3 - 2 \cdot 1 \\ 2 &= 5 - 3 \cdot 1 \\ 3 &= 23 - 5 \cdot 4 \\ 5 &= 120 - 23 \cdot 5 \end{aligned}$$

Substituting and simplifying at each step (in order to continue substituting the subsequent equation in the above) we have:

$$
\begin{aligned}
1 &= 3 - 2 \cdot 1 \\
&= 3 - (5 - 3 \cdot 1) \cdot 1 \\
&= 3 - 5 + 3 \\
&= 2 \cdot 3 - 5 \\
&= 2 \cdot (23 - 5 \cdot 4) - 5 \\
&= 2 \cdot 23 - 5 \cdot 8 - 5 \\
&= 2 \cdot 23 - 9 \cdot 5 \\
&= 2 \cdot 23 - 9 \cdot (120 - 23 \cdot 5) \\
&= 2 \cdot 23 - 9 \cdot 120 + 23 \cdot 45 \\
&= -9 \cdot 120 + 23 \cdot 47 \\
&= 120 \cdot (-9) + 23 \cdot (47)
\end{aligned}
$$

Therefore $x = -9$ and $y = 47$.