Comments for Lecture 24
3.12.2010

Examples!.

Example 1: Consider the (ordered) set

$$X = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix} \right)$$

1. Is X linearly indepdent? If not find a depdence relation on X.
2. Does X span $\mathbb{R}^3$ (i.e., is $\mathrm{Span}(X)=\mathbb{R}^3$)?
3. Is X a (ordered) basis of $\mathbb{R}^3$?

Solution:

1. Using the "test for linear independece" (p129):

   Let $A = \begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & 5 \\ 0 & 1 & 4 \end{bmatrix}$

   Putting A into reduced row echelon form we obtain:

   $$A = \begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & 5 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\text{putting } A \text{ in RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

   So the rank(A)=2 (number of pivot columns). So rank(A)=2≠3 (the number of columns). Hence the answer is no, and X must be linearly depdendent.

   To find a depdence relation on X we need to find a nontrivial solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Well we did all of the work actually, but if you did this from scratch you would set up an augmented matrix for the system and reduce:

   $$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 1 & 1 & 5 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \xrightarrow{\text{putting into RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

   This has a parametric solution of the form:

$$
\begin{aligned}
x_1 &= -t \\
x_2 &= -4t \\
x_3 &= t
\end{aligned}
$$

So to get an example of a nontrival solution to the homogenous system $A\mathbf{x} = \mathbf{0}$ would could set $t = 1$ and so $x = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$ is a solution. So we have a depdence relation on X:

$$
(-1)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-4)\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}
$$

2. Using 1. We have rank(A)$= 2 \neq 3 =$ (the number of rows). Hence the answer is no, and X does not span all of $\mathbb{R}^3$.
3. Using 1. or 2. the answer is no. Don't forget the definition of a basis!

Example 2: Let $A = \begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & 5 \\ 0 & 1 & 4 \end{bmatrix}$

1. Find a basis X for the column space of A.
2. What is the dimension of the column space of A?
3. Find a basis Y for the null space of A.
4. What is the dimension of the null space of A?
5. Find a basis Z for the row space of A.
6. What is the dimension of the row space of A?

Solution:

1. We put A into reduced row ecehelon form (RREF):

$$
A = \begin{bmatrix} 1 & -1 & -3 \\ 1 & 1 & 5 \\ 0 & 1 & 4 \end{bmatrix} \xrightarrow{\text{putting } A \text{ in RREF}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} = A'
$$

So this means that columns 1 and 2 are the pivot columns. The set of pivot columns of A form a basis of the column space of A. Hence,

$$\boxed{X = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)}$$

2. The dimension of the column space is the number of elements in X hence $\boxed{2}$. (Remember we also have that the dimension of the column space $= \text{rank}(A) = 2$)

3. We find the vector parametric form of the solution set to the homogenous system $A\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 1 & 1 & 5 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \xrightarrow{\text{putting into RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This has a parametric solution of the form:

$$\begin{aligned} x_1 &= -t \\ x_2 &= -4t \\ x_3 &= t \end{aligned}$$

or we could write:

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= -4x_3 \\ x_3 &= x_3 \end{aligned}$$

So the vector parametric form would be:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -4x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$$

The set of vectors in the vector parametric form of the solution set of $A\mathbf{x} = \mathbf{0}$ form a basis for the null space of A. Hence,

$$\boxed{Y = \left(\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} \right)}$$

4. The dimension of the null space is the number of elements in Y hence $\boxed{1}$. (remember we also have that the dimension of the null space = number of non pivot columns = 3 - rank(A) = 3 - 2 = 1)

5. Using the work from 1. we have that a basis for the row space can be made from the nonzero rows of A' (the reduced row echelon form of A). Hence,

$$\boxed{Z = \left(\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 4 \end{bmatrix} \right)}$$

6. The dimension of the row space is the number of elements in Z hence $\boxed{2}$. (Remember we also have that the dimension of the row space = rank(A) = 2)