Comments for Lecture 29
3.18.2010

A Zoo of Vector Spaces.

NOTE: In the following examples I do not verify all 10 axioms of a vector space are satisfied. This is left to the motivated reader.

Example 0: The most trivial vector space is the *zero vector space* $\{\mathbf{0}\}$. This is a vector space over $\mathbb{R}$ (but actually over any field).

Example 1: Our old friend *Euclidean space* (or *real coordinate space*) $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ with vector addition and scalar multiplication as defined on p38.

Example 2: The set of all polynomials with coefficients in $\mathbb{R}$ of degree less than or equal to n (where n is a nonnegative integer) is a vector space over $\mathbb{R}$ with vector addition defined as addition of polynomials and scalar multiplication defined as scalar multiplication of polynomials. In the book this set is denoted P_n.

Formally: The set

$$P_n = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}\}$$

is a vector space over $\mathbb{R}$ with vector addition and scalar multiplication defined as follows

Vector addition: let $p(x), q(x) \in P_n$. So we can write: $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and $q(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$ where each $a_j \in \mathbb{R}$ and each $b_k \in \mathbb{R}$ for $j, k = 0, 1, 2, \ldots, n$. Then

$$\begin{aligned} p(x) + q(x) &= (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) + (b_0 + b_1x + b_2x^2 + \cdots + b_nx^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n \end{aligned}$$

Scalar multiplication: let $p(x) \in P_n$ and let $\alpha \in \mathbb{R}$. So we can write: $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where each $a_j \in \mathbb{R}$ for $j = 0, 1, 2, \ldots, n$. Then

$$\begin{aligned} \alpha p(x) &= \alpha(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) \\ &= \alpha a_0 + \alpha a_1x + \alpha a_2x^2 + \cdots + \alpha a_nx^n \end{aligned}$$

Example 3: The set of polynomials with coefficients in $\mathbb{R}$ is a vector space over $\mathbb{R}$ with vector addition defined as addition of polynomials and scalar multiplication defined as scalar multiplication of polynomials. This vector space is often denoted $\mathbb{R}[x]$.

Example 4: The set of all $m \times n$ matrices with entries in $\mathbb{R}$ is a vector space over $\mathbb{R}$ with vector addition defined as matrix addition and scalar multiplication defined as scalar multiplication of matrices. This vector space is sometimes denoted $M(m,n)$ or $M_{n,m}(\mathbb{R})$ or $\mathbb{R}^{m\times n}$.

Example 5: The set of all functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space over $\mathbb{R}$ with vector addition defined as function addition and scalar multiplication defined as scalar multiplication of functions.

Formally: The set $F = \{f{:}\mathbb{R} \to \mathbb{R}\}$ is a vector space over $\mathbb{R}$ with vector addition and scalar multiplication defined as follows

Vector addition: let $f, g \in F$. Then

$$(f+g)(x) = f(x) + g(x) \text{ for all } x \in \mathbb{R}$$

Scalar multiplication: let $f \in F$ and let $\alpha \in \mathbb{R}$. Then

$$(\alpha f)(x) = \alpha f(x) \text{ for all } x \in \mathbb{R}$$

Example 6: The set of all continuous functions from $[0,1]$ to $\mathbb{R}$ is a vector space over $\mathbb{R}$ with vector addition defined as function addition and scalar multiplication defined as scalar multiplication of functions.

Formally: The set $C[0,1] = \{f{:}[0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ is a vector space over $\mathbb{R}$ with vector addition and scalar multiplication defined as in example 5.