Math 222 Spring 2011
4/15/2011
Quiz #10

Name: ____________________

Please box your answers. Show all work clearly and in order. Due on Wednesday 4/27/2011.

1. Find the radius of convergence and the interval of convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} n^{4n}x^n$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2x^n}{2\cdot 4\cdot 6\cdots(2n)}$ (Hint: simplify the sum first).

2. Find a power series representation for the given function and determine the interval of convergence.

(a) $f(x) = \dfrac{x}{1-x^2}$

(b) $f(x) = \dfrac{1}{x^6+4}$

(c) $f(x) = \ln(3-x)$

$\boxed{Q10}$

(1) use ratio test :

(a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{n+1}}{n+2}\cdot\frac{n+1}{(-1)^{n+1}x^n}\right|$

$= \lim_{n\to\infty}\frac{n+1}{n+2}|x| = |x|$

so if $|x|<1 \Rightarrow$ power series converges $\Rightarrow \boxed{R=1}$

if $|x|>1 \Rightarrow$ p.s. diverges.

if $|x|=1 \Rightarrow x=1$ or $x=-1$ check if the power series converges or diverges by ~~eva~~ pluging in!

$\underline{x=1}$: $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(1)^n}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$

alternating, let $b_n=\frac{1}{n+1}$, (i) $b_{n+1}=\frac{1}{n+2}<b_n=\frac{1}{n+1}$ ✓

(so decreasing)

(ii) $\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{1}{n+1}=0$ ✓

So by Alternating series test the power series is $\underline{\text{convergent for } x=1}$.

$\underline{x=-1}$: $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(-1)^n}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n(-1)(-1)^n}{n+1}$

$= \sum_{n=0}^{\infty}\frac{((-1)^n)^2(-1)}{n+1}$

$= \sum_{n=0}^{\infty}\frac{-1}{n+1} = -\left(\sum_{n=0}^{\infty}\frac{1}{n+1}\right)$

notice that this looks similar to the harmonic series

in fact

$= -\left(\sum_{n=1}^{\infty}\frac{1}{(n-1)+1}\right) = -\left(\sum_{n=1}^{\infty}\frac{1}{n}\right)$

harmonic series so $\underline{\text{divergent for } x=-1}$

So the interval of convergence is $\boxed{(-1,1]}$

(b) $\boxed{\text{SOL 1}}$ use ratio test : $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{4(n+1)}x^{n+1}}{n^{4n}x^n}\right| = \lim_{n\to\infty}\left(|x|\cdot\frac{(n+1)^{4n+4}}{n^{4n}}\right)$

$= |x|\lim_{n\to\infty}\frac{(n+1)^{4n}\cdot(n+1)^4}{n^{4n}}$

$= |x|\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{4n}(n+1)^4$

$= |x|\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^4(n+1)^4$

$= |x|\cdot e^4\cdot\lim_{n\to\infty}(n+1)^4 = \infty$

so this means the power series diverges for all x values other than the center $x=0$

So radius of conv. is $\boxed{R=0}$ and the interval of convergence is just the single value $x=0$ so we can write: $\boxed{\{0\}}$

(the set containing just one element: 0)

$\boxed{\text{SOL 2}}$ use root test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|n^{4n}x^n|} = \lim_{n\to\infty} \sqrt[n]{(n^4)^n |x|^n}$

$= \lim_{n\to\infty} n^4|x| = \infty$

(then the same conclusion as above: $\boxed{R=0}$, $\boxed{\text{Interval } \{0\}}$)

(c) <u>Note</u>: $2\cdot4\cdot6\cdots(2n)$ is <u>NOT</u> $(2n)! = (2n)(2n-1)(2n-2)\cdots(3)(2)(1)$

one way you can simplify is the following:

$2\cdot4\cdot6\cdots(2n) = (2\cdot1)\cdot(2\cdot2)\cdot(2\cdot3)\cdots(2\cdot n) = \underbrace{2\cdot2\cdot2\cdot2\cdots2}_{n\text{-times}}\cdot\underbrace{1\cdot2\cdot3\cdots n}_{n!}$

so we have n factors of 2 here:

$= 2^n n!$

but I will show you the solution without this!

$\boxed{\text{SOL}}$ use ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2x^{n+1}}{\cancel{2\cdot4\cdot6\cdots(2n)}(2(n+1))}\cdot\frac{\cancel{2\cdot4\cdot6\cdots(2n)}}{n^2\ x^n}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^2|x|}{2(n+1)\cdot n^2}$$

$$= \lim_{n\to\infty}\frac{(n^2+2n+1)|x|}{(2n+2)n^2}$$

$$= \lim_{n\to\infty}\frac{n^2+2n+1}{2n^3+2n^2}|x| = 0 < 1$$

so the power series converges for all x values. Hence, $\boxed{R=\infty}$ and the interval of convergence is all real numbers: $\boxed{(-\infty,\infty)}$

using: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 \leq x \leq 1$

② (a)

$$f(x) = \frac{x}{1-x^2} = x \cdot \frac{1}{1-x^2} = x \sum_{n=0}^{\infty} (x^2)^n \quad \text{for } -1 \leq x^2 \leq 1 \text{ (so } -1 \leq x \leq 1)$$

$$= x \sum_{n=0}^{\infty} x^{2n} \quad \text{for } -1 \leq x \leq 1 \quad (\text{so } |x|<1)$$

$$= \boxed{\sum_{n=0}^{\infty} x^{2n+1}} \quad \text{for } |x| < 1$$

Interval of convergence is $\boxed{(-1, 1)}$

(b)
$$f(x) = \frac{1}{x^6+4} = \frac{1}{4\left(\frac{x^6}{4}+1\right)} = \frac{1}{4}\cdot\left(\frac{1}{1+\frac{x^6}{4}}\right) = \frac{1}{4}\left(\frac{1}{1-\left(-\frac{x^6}{4}\right)}\right)$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x^6}{4}\right)^n \quad \text{for } \left|\frac{-x^6}{4}\right| < 1$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n}}{4^n} \quad \text{for } |x^6| < 4, \ |x| < 4^{1/6}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n}}{4^{n+1}}} \quad |x| < 4^{1/6}$$

Interval of convergence: $\boxed{(-4^{1/6}, 4^{1/6})}$

(c) Notice: $\frac{d}{dx}(\ln(3-x)) = \frac{1}{3-x}\cdot(-1) = \frac{-1}{3-x}$ so

$$f(x) = \ln(3-x) = -\int \frac{1}{3-x}\,dx + C$$

$$= -\int \frac{1}{3(1-\frac{x}{3})}\,dx + C$$

$$= \frac{-1}{3}\int \sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n dx + C$$

$$= \frac{-1}{3}\left(\sum_{n=0}^{\infty}\frac{1}{3^n}\cdot\frac{x^{n+1}}{(n+1)}\right) + D$$

$$= -\sum_{n=0}^{\infty}\frac{x^{n+1}}{3^{n+1}(n+1)} + D$$

and $\ln(3-0) = \ln(3) = 0 + D \Rightarrow D = \ln(3)$

Hence, $f(x) = \boxed{\ln(3) + \left(-\sum_{n=0}^{\infty}\frac{x^{n+1}}{3^{n+1}(n+1)}\right)}$

for $\left|\frac{x}{3}\right| < 1$

$|x| < 3$

so

Interval of convergence: $\boxed{(-3, 3)}$