Comments for Lecture 20
3.3.2010

When is a vector w in Span(X)?.

Please review the "membership test" described in section 3.3.2 starting on p122. This is the method you can use in order to answer the above question. Also look at quiz 6 for an example of this type of problem.

Take a look at **lemma 3.3.5** on p123. This gives a nice summary of the "membership test" (take a look at this again once you know what the column space of A is. How can we relate this lemma to Col(A)?).

Take a look at **lemma 3.3.6** on p123. This gives us a method for determining if a given set of vectors $X = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ spans all of $\mathbb{R}^m$. In other words we know when Span(X)=$\mathbb{R}^m$. This is a good foot in the door to answer the general question of when we having a spanning set of a vector space. Here we have the answer specifically for the vector space $\mathbb{R}^m$. But we need to know the answer also for subspaces and general vector spaces as well.

Lemma 3.3.7 gives a very quick method (only in certain situations, look closely!)to say when a set cannot span all of $\mathbb{R}^m$. For example consider the set:

$$X = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Notice there is only one vector in X and $1 < 2$. Hence by corollary 3.3.7 X cannot span all of $\mathbb{R}^2$. Notice however this does not help you in the general case. We still need to do some row reduction in the general setting.

Linear transformations and span.

Since linear transformations behave nicely on linear combinations of vectors it seems reasonable to think that it must behave nicely on the span of a set of vectors X (since the Span(X) is just the set of all linear combinations of vectors in X). This is what **lemma 3.3.8** essentially gives us. This can be useful in several situations. One example is given in exercise (36)1. The solution provided encourages you to use lemma 3.3.8 and the kernel of a linear transformation (see below). However don't forget! If you are ever asked to show a set is a subspace of $\mathbb{R}^n$ you can usually use the shortcut method from **theorem 3.3.2** on p121 unless you are told otherwise.

Column space, null space, image and kernel.

Suppose A is a $m \times n$ matrix. For the following definitions let's write A in terms of its columns, so we assume $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$ for some vectors $\mathbf{v}_i \in \mathbb{R}^m$ where $1 \le i \le n$.

Definitions involving the matrix A

The *column space* of A denoted Col A or Col(A) is defined as the set of all linear combinations of the columns of A. In other words:

$$\boxed{\text{Col } A = \text{Span }(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\})}$$

The *null space* of A denoted Nul A of Nul(A) is defined as the set of all $\mathbf{x} \in \mathbb{R}^n$ for which $A\mathbf{x} = \mathbf{0}$. In other words:

$$\boxed{\text{Nul } A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}}$$

Definitions involving the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$

Recall the function defined as $\mathbf{x} \mapsto A\mathbf{x}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

The *image* of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of all vectors $A\mathbf{x}$ where $\mathbf{x}$ is in $\mathbb{R}^n$. In other words:

$$\boxed{\text{image of } \mathbf{x} \mapsto A\mathbf{x} = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}}$$

The *kernel* of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of all vectors $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{0}$. In other words:

$$\boxed{\text{kernel of } \mathbf{x} \mapsto A\mathbf{x} = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}}$$

Exercise (36)5 shows that we actually have:

$$\text{Col } A = \text{image of } \mathbf{x} \mapsto A\mathbf{x}$$
$$\text{Nul } A = \text{kernel of } \mathbf{x} \mapsto A\mathbf{x}$$

Later we will look at linear transformations in a more general setting. We will have linear transformations $T : V \to W$ from a vector space V to a vector space W. Then we can define the image and kernel of T (denoted im(T) and ker(T) respectively) just as you would expect. We will have im(T)= $\{T(\mathbf{x}) \mid \mathbf{x} \in V\}$ and ker(T)= $\{\mathbf{x} \in V \mid T(\mathbf{x}) = \mathbf{0}\}$.

Theorem 3.3.9 gives us a nice result. If you read this theorem and proof carefully we actually do not need to have our vector spaces be $\mathbb{R}^n$ and $\mathbb{R}^m$. We will have a general result which I will restate for you here:

Theorem. *Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Then*

1. *im(T) is a subspace of W.*
2. *ker(T) is a subspace of V.*