Math 222 Spring 2011
4/8/2011
Quiz #9

Name: Key

Please box your answers. Show all work clearly and in order. Due on Monday 4/11/2011.

1. Determine whether each series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{\sqrt{n}+4}}$

SOL: consider $a_n = \dfrac{1}{\sqrt{\sqrt{n}+4}}$ and $b_n = \dfrac{1}{\sqrt{\sqrt{n}}}$, notice both a_n and b_n are positive for $n \geq 1$.

Notice $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{\sqrt{n}+4}}{1/\sqrt{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{\sqrt{n}}}{\sqrt{\sqrt{n}+4}} = \lim_{n\to\infty} \sqrt{\frac{\sqrt{n}}{\sqrt{n}+4}} =$

$\displaystyle = \lim_{n\to\infty} \sqrt{\frac{\sqrt{n}\left(\frac{1}{\sqrt{n}}\right)}{(\sqrt{n}+4)\left(\frac{1}{\sqrt{n}}\right)}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{4}{\sqrt{n}}}} = \sqrt{\frac{1}{1+0}} = \sqrt{1} = 1 > 0$

since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{\sqrt{n}}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/4}}$ diverges (p-series $p = 1/4 \leq 1$), By the limit comparison test $\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{\sqrt{n}+4}} \text{ diverges}}$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2-3n+2}$

SOL: consider $a_n = \dfrac{1}{n^2-3n+2}$ and $b_n = \dfrac{1}{n^2}$, notice both a_n and b_n are positive for $n \geq 3$ (how do we know this for a_n?)

Notice $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^2-3n+2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n^2}{n^2-3n+2} = 1 > 0.$

Since $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges (p-series $p = 2 > 1$), By the limit comparison test $\boxed{\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2-3n+2} \text{ converges}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3}$

SOL: consider the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{n^3}\right|$. If we show this series is convergent then the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3}$ is absolutely convergent, and so it is also convergent.

terms are positive for $n \geq 1$.

Notice that

$$\left|\frac{\cos(n)}{n^3}\right| \overset{\text{for } n\geq 1}{=} \frac{|\cos(n)|}{n^3} \leq \frac{1}{n^3}$$

since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent (p-series $p = 3 > 1$) By the comparison test $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos(n)}{n^3}\right|$ converges. Hence, from above

$\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3} \text{ converges}}$

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{5^n}$

SOL 1: consider using the ratio test with $a_n = (-1)^n \frac{n}{5^n}$

Notice that $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(n+1)}{5^{n+1}}}{(-1)^n\frac{n}{5^n}}\right| =$

$\displaystyle= \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}}\cdot\frac{5^n}{n}\right| = \lim_{n\to\infty}\frac{(n+1)\cdot\cancel{5^n}}{\cancel{5^n}\cdot 5\cdot n} =$

$\displaystyle= \lim_{n\to\infty}\left(\frac{1}{5}\cdot\frac{n+1}{n}\right) = \frac{1}{5} < 1$ so

by the ratio test the series is absolutely convergent and therefore $\boxed{\sum_{n=1}^{\infty}(-1)^n\frac{n}{5^n} \text{ converges}}$

SOL 2: consider using the alternating series test. with $b_n = \frac{n}{5^n}$ (positive terms)

(a) Show $b_{n+1} \le b_n$ for all n. [redacted]

consider $f(x) = \frac{x}{5^x}$

so $\displaystyle f'(x) = \frac{5^x\cdot 1 - x\cdot\ln(5)\cdot 5^x}{(5^x)^2} = \frac{5^x(1 - x\ln(5))}{(5^x)^2}$

so since $5^x > 0$ and $(5^x)^2 > 0$.

and $1 - x\ln(5) < 0$ if $\frac{1}{\ln(5)} < x$

we have $f'(x) < 0$ for $x \ge 1$.

Hence the b_n's are decreasing, so $b_{n+1} \le b_n$.

(b) Notice that $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n}{5^n}$

consider $f(x) = \frac{x}{5^x}$ and $\displaystyle\lim_{x\to\infty}\frac{x}{5^x} \overset{L'H}{=}$

$\displaystyle= \lim_{x\to\infty}\frac{(1)}{\ln(5)\cdot 5^x} = 0$

so by the alternating series test $\boxed{\sum_{n=1}^{\infty}(-1)^n\frac{n}{5^n} \text{ converges}}$

(e) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 5^n}{n!}$

SOL: consider using the ratio test with $a_n = (-1)^{n+1}\frac{n^2 5^n}{n!}$

Notice that

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)^2\cdot 5^{n+1}}{(n+1)!}\cdot\frac{(n!)}{(-1)^{n+1}n^2 5^n}\right| = \lim_{n\to\infty}\left|\frac{(n [redacted] +1)^{\cancel{2}}\cancel{5^n}\cdot 5\cdot\cancel{n!}}{\cancel{(n+1)}\cancel{n!}\cdot n^2\cdot\cancel{5^n}}\right|$

$\displaystyle= \lim_{n\to\infty}\frac{5(n+1)}{n^2} = 0 < 1$. Therefore, $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 5^n}{n!}$ is absolutely convergent by the ratio test and hence, $\boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 5^n}{n!} \text{ converges}}$

(f) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-3n}{n+2}\right)^{2n}$

SOL: consider using the root test with $a_n = \left(\frac{-3n}{n+2}\right)^{2n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-3n}{n+2}\right)^{2n}\right|}$$

$$= \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{9n^2}{(n+2)^2}\right)^n\right|}$$

$$= \lim_{n\to\infty}\frac{9n^2}{(n+2)^2} = \lim_{n\to\infty}\frac{9n^2}{n^2+4n+4} = 9 > 1$$

Therefore by the root test $\boxed{\sum_{n=1}^{\infty}\left(\frac{-3n}{n+2}\right)^{2n} \text{ is divergent}}$