TRIGONOMETRIC INTEGRALS

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1. Trigonometric Integrals

Example 1.1.

$$\int \cos^5(x)dx$$

$$\begin{aligned}\int \cos^5(x)dx &= \int \cos(x)\cos^2(x)\cos^2(x)dx \\ &= \int \cos(x)(1-\sin^2(x))(1-\sin^2(x))dx \\ &= \int \cos(x)(1-\sin^2(x))^2dx.\end{aligned}$$

let $u = \sin(x) \implies \frac{du}{dx} = \cos(x) \implies dx = \frac{du}{\cos(x)}$.
Therefore,

$$\begin{aligned}\int \cos(x)(1-\sin^2(x))^2dx &= \int \cancel{\cos(x)}(1-u^2)^2\frac{du}{\cancel{\cos(x)}} \\ &= \int (1-u^2)^2du \\ &= \int (1-2u^2+u^4)du \\ &= u - \frac{2u^3}{3} + \frac{u^5}{5} + C \\ &= \sin(x) - \frac{2\sin^3(x)}{3} + \frac{\sin^5(x)}{5} + C.\end{aligned}$$

Example 1.2.

$$\int \sin^3(x)\cos^2(x)dx$$

$$\begin{aligned}\int \sin^3(x)\cos^2(x)dx &= \int \sin(x)\sin^2(x)\cos^2(x)dx \\ &= \int \sin(x)(1-\cos(x)^2)\cos^2(x)dx\end{aligned}$$

let $u = \cos(x) \implies \frac{du}{dx} = -\sin(x) \implies dx = \frac{du}{-\sin(x)}$.

$$\begin{aligned}\int \sin(x)(1-\cos(x)^2)\cos^2(x)dx &= \int \cancel{\sin(x)}(1-u^2)u^2\frac{du}{-\cancel{\sin(x)}}\\ &= \int -(1-u^2)u^2du\\ &= -\int (u^2-u^4)du\\ &= -\frac{u^3}{3}+\frac{u^5}{5}+C\\ &= -\frac{\cos^3(x)}{3}+\frac{\cos^5(x)}{5}+C.\end{aligned}$$

$$\int \sin^2(x)dx \text{ and } \int \cos^2(x)dx$$

are <u>very</u> common integrals that show in physics and engineering. To solve these we use the **half angle identities**:

$$\boxed{\sin^2(x) = \frac{1}{2}(1-\cos(2x)) = \frac{1}{2}-\frac{1}{2}\cos(2x).}$$

$$\boxed{\cos^2(x) = \frac{1}{2}(1+\cos(2x)) = \frac{1}{2}+\frac{1}{2}\cos(2x).}$$

Alternatively, you could also derive them from the identities:

$$\boxed{\begin{aligned}\sin(\alpha\pm\beta) = \sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta)\\ \cos(\alpha\pm\beta) = \cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)\end{aligned}}$$

Example 1.3.

$$\int_0^{\pi/4}\cos^2(x)dx$$

$$\begin{aligned}\int_0^{\pi/4}\cos^2(x)dx &= \int_0^{\pi/4}\frac{1}{2}(1+\cos(2x))dx\\ &= \frac{1}{2}\left[x+\frac{1}{2}\sin(2x)\right]_0^{\pi/4}\\ &= \frac{1}{2}\left[\left(\frac{\pi}{4}+\frac{1}{2}\cdot 1\right)-(0+0)\right]\\ &= \frac{\pi+2}{8}.\end{aligned}$$

STRATEGY for evaluating $\int \sin^m(x)\cos^n(x)dx$.

(1) If the power of COSINE is ODD (n=2k+1), save one cosine factor and use $\cos^2(x) = 1 - \sin^2(x)$:

$$\begin{aligned}\int \sin^m(x)\cos^{2k+1}(x)dx &= \int \sin^m(x)(\cos^2(x))^k\cos(x)dx \\ &= \int \sin^m(x)(1-\sin^2(x))^k\cos(x)dx\end{aligned}$$

Then substitute $u = \sin(x)$.

(2) If the power of SINE is ODD (m=2k+1), save one sine factor and use $\sin^2(x) = 1 - \cos^2(x)$:

$$\begin{aligned}\int \sin^{2k+1}(x)\cos^n(x)dx &= \int (\sin^2(x))^k\sin(x)\cos^n(x)dx \\ &= \int (1-\cos^2(x))^k\sin(x)\cos^n(x)dx\end{aligned}$$

Then substitute $u = \cos(x)$. [NOTE: if the pwers of both sine and coside are odd, either (1) or (2) can be used.]

(3) If the powers of BOTH SINE AND COSINE are EVEN, use the half angle formulas.

STRATEGY for evaluating $\int \tan^m(x)\sec^n(x)dx$.

(1) If the power of SECANT is EVEN, save a factor of $\sec^2(x)$ and use $\sec^2(x) = 1 + \tan^2(x)$.

(2) If the power of TAN is ODD, save a factor of $\sec(x)\tan(x)$ and use $\tan^2(x) = \sec^2(x) - 1$.

Example 1.4.

$$\int \sec^4(\theta)\tan^5(\theta)d\theta$$

SOLUTION 1:

$$\begin{aligned}\int \sec^4(\theta)\tan^5(\theta)d\theta &= \int \sec^2(\theta)\sec^2(\theta)\tan^5(\theta)d\theta \\ &= \int \sec^2(\theta)(1+\tan^2(\theta))\tan^5(\theta)d\theta\end{aligned}$$

$$\textit{let } u = \tan(\theta) \implies \frac{du}{d\theta} = \sec^2(\theta) \implies d\theta = \frac{du}{\sec^2(\theta)}.$$

$$\begin{aligned}
\int \sec^2(\theta)(1+\tan^2(\theta))\tan^5(\theta)d\theta &= \int \cancel{\sec^2(\theta)}(1+u^2)u^5 d\frac{du}{\cancel{\sec^2(\theta)}} \\
&= \int (u^5+u^7)du \\
&= \frac{u^6}{6}+\frac{u^8}{8}+C \\
&= \frac{\tan^6(\theta)}{6}+\frac{\tan^8(\theta)}{8}+C
\end{aligned}$$

<u>SOLUTION 2:</u>

$$\begin{aligned}
\int \sec^4(\theta)\tan^5(\theta)d\theta &= \int \sec^3(\theta)\sec(\theta)\tan(\theta)\tan^4(\theta)d\theta \\
&= \int \sec^3(\theta)\sec(\theta)\tan(\theta)(\sec^2(\theta)-1)^2 d\theta
\end{aligned}$$

$$\textit{let } u=\sec(\theta) \implies \frac{du}{d\theta}=\sec(\theta)\tan(\theta) \implies d\theta=\frac{du}{\sec(\theta)\tan(\theta)}.$$

$$\begin{aligned}
\int \sec^3(\theta)\sec(\theta)\tan(\theta)(\sec^2(\theta)-1)^2 d\theta &= \int u^3\cancel{\sec(\theta)\tan(\theta)}(u^2-1)^2\frac{du}{\cancel{\sec(\theta)\tan(\theta)}} \\
&= \int (u^3(1-2u^2+u^4))du \\
&= \int (u^3-2u^5+u^7)du \\
&= \frac{u^4}{4}-2\frac{u^6}{6}+\frac{u^8}{8}+C \\
&= \frac{\sec^4(\theta)}{4}-\frac{\sec^6(\theta)}{3}+\frac{\sec^8(\theta)}{8}+C
\end{aligned}$$

(Subtitute $\tan^2(x)=\sec^2(x)-1$ *into solution 1 to see that it matches solution 2.)*

For other cases we do not always have such a clear guideline. Sometimes you will need to use integration by parts, identities, or something else altogether.

Recall,

$$\boxed{\int \tan(x)dx = \ln|\sec(x)|+C.}$$

We will also need

$$\boxed{\int \sec(x)dx = \ln|\sec(x)+\tan(x)|+C.}$$

To evaluate the integrals

(1) $\int \sin(mx)\cos(nx)dx$.
(2) $\int \sin(mx)\sin(nx)dx$.
(3) $\int \cos(mx)\cos(nx)dx$.

Use the corresponding identities:

(1) $\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$.
(2) $\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$.
(3) $\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$.

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