Math 222 Spring 2011
4/29/2011
Quiz #11

Name: Key

Please box your answers. Show all work clearly and in order.

1. Let $f(x) = \cos(4x)$.

   (a) Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series.

$a=0$,

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos(4x)$	$\cos(0)=1$
1	$-\sin(4x)4$	$-\sin(4\cdot 0)4 = 0$
2	$-\cos(4x)4^2$	$-\cos(4\cdot 0)4^2 = -1\cdot 4^2 = -4^2$
3	$+\sin(4x)4^3$	$\sin(4\cdot 0)4^3 = 0$
4	$\cos(4x)4^4$	$\cos(4x)4^4 = 1\cdot 4^4$
$\vdots$	$\vdots$	$\vdots$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 1 + 0 + \frac{(-4^2)}{2!}x^2 + 0 + \frac{(4^4)}{4!}x^4 + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}}$$

   (b) Find the radius of convergence for the Maclaurin series in part (a).

using the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}4^{2(n+1)}x^{2(n+1)}}{(2(n+1))!}\cdot\frac{(2n)!}{(-1)^n 4^{2n}x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)4^{2n+2}x^{2n+2}\cdot(2n)(2n-1)\cdots(2)(1)}{(2n+2)(2n+1)(2n)(2n-1)\cdots(2)(1)4^{2n}x^{2n}}\right|$$

$$= \lim_{n\to\infty}\frac{4^2|x^2|}{(2n+2)(2n+1)} = 0 < 1$$

, so the power series converges for all real x.

Hence, $\boxed{R=\infty}$

Pick ONE of the following (Either 2 or 3). Cross out the problem you do not want graded. Otherwise I will grade the first problem worked on.

2. Let $f(x) = \sin(x)$.

   (a) Find the Taylor series for $f(x)$ centered at $a = \frac{\pi}{2}$ using the definition of a Taylor series.

$a=\frac{\pi}{2}$,

n	$f^{(n)}(x)$	$f^{(n)}(\frac{\pi}{2})$
0	$\sin(x)$	$\sin(\pi/2) = 1$
1	$\cos(x)$	$\cos(\pi/2) = 0$
2	$-\sin(x)$	$-\sin(\pi/2) = -1$
3	$-\cos(x)$	$-\cos(\pi/2) = 0$
4	$\sin(x)$	$\sin(\pi/2) = 1$
$\vdots$	$\vdots$	$\vdots$

$$f(\tfrac{\pi}{2}) + f'(\tfrac{\pi}{2})(x-\tfrac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2!}(x-\tfrac{\pi}{2})^2 + \frac{f'''(\frac{\pi}{2})}{3!}(x-\tfrac{\pi}{2})^3 + \frac{f^{(4)}(\frac{\pi}{2})}{4!}(x-\tfrac{\pi}{2})^4 + \cdots$$

$$= 1 + 0 + \frac{(-1)}{2!}(x-\tfrac{\pi}{2})^2 + 0 + \frac{1}{4!}(x-\tfrac{\pi}{2})^4 + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x-\tfrac{\pi}{2})^{2n}}$$

   (b) Show that the series obtained in part (a) represents $f(x) = \sin(x)$ for all x. (So you need to show the series really does equal the function).

we use Taylor's Inequality (see p773). Notice $f^{(n+1)}(x)$ is either $+\sin(x)$, $-\sin(x)$, $+\cos(x)$ or $-\cos(x)$.

and all of these are bounded above in absolute value by 1.

i.e., $|f^{(n+1)}(x)| \le 1$ for all x. so let $M=1$.

Therefore, by Taylor's Inequality:

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-\tfrac{\pi}{2}|^{n+1} = \frac{|x-\frac{\pi}{2}|^{n+1}}{(n+1)!} \text{ for all } x$$

It follows by the Squeeze Thm. that $\lim_{n\to\infty}|R_n(x)| = 0$ so $\lim_{n\to\infty} R_n(x) = 0$

therefore, by Thm. 8 (p773) the series in (a) represents $f(x)=\sin(x)$ for all x.

3. (a) Use the binomial series to expand $\dfrac{1}{\sqrt{1-x^2}}$.

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x^2)^n$$

$$= 1 + \left(-\tfrac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \cdots$$

$$= 1 + \frac{1 \cdot x^2}{2^1} + \frac{1 \cdot 3 \cdot x^4}{2^2 \cdot 2!} + \frac{1 \cdot 3 \cdot 5 \cdot x^6}{2^3 \cdot 3!} + \cdots$$

$$= \boxed{1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \cdot n!} x^{2n}}$$

(b) Use part (a) to find the Maclaurin series for $\sin^{-1}(x)$.

$$\sin^{-1}(x) = \int \frac{1}{\sqrt{1-x^2}}\,dx + C$$

$$= \int \left[1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!} x^{2n} \right] dx + C$$

$$= D + x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \cdot n!} \frac{x^{2n+1}}{2n+1}$$

Notice $\sin^{-1}(0) = 0 = D + 0 + 0 \implies D = 0$

so

$$\boxed{\sin^{-1}(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n+1)\, 2^n \cdot n!} x^{2n+1}}$$