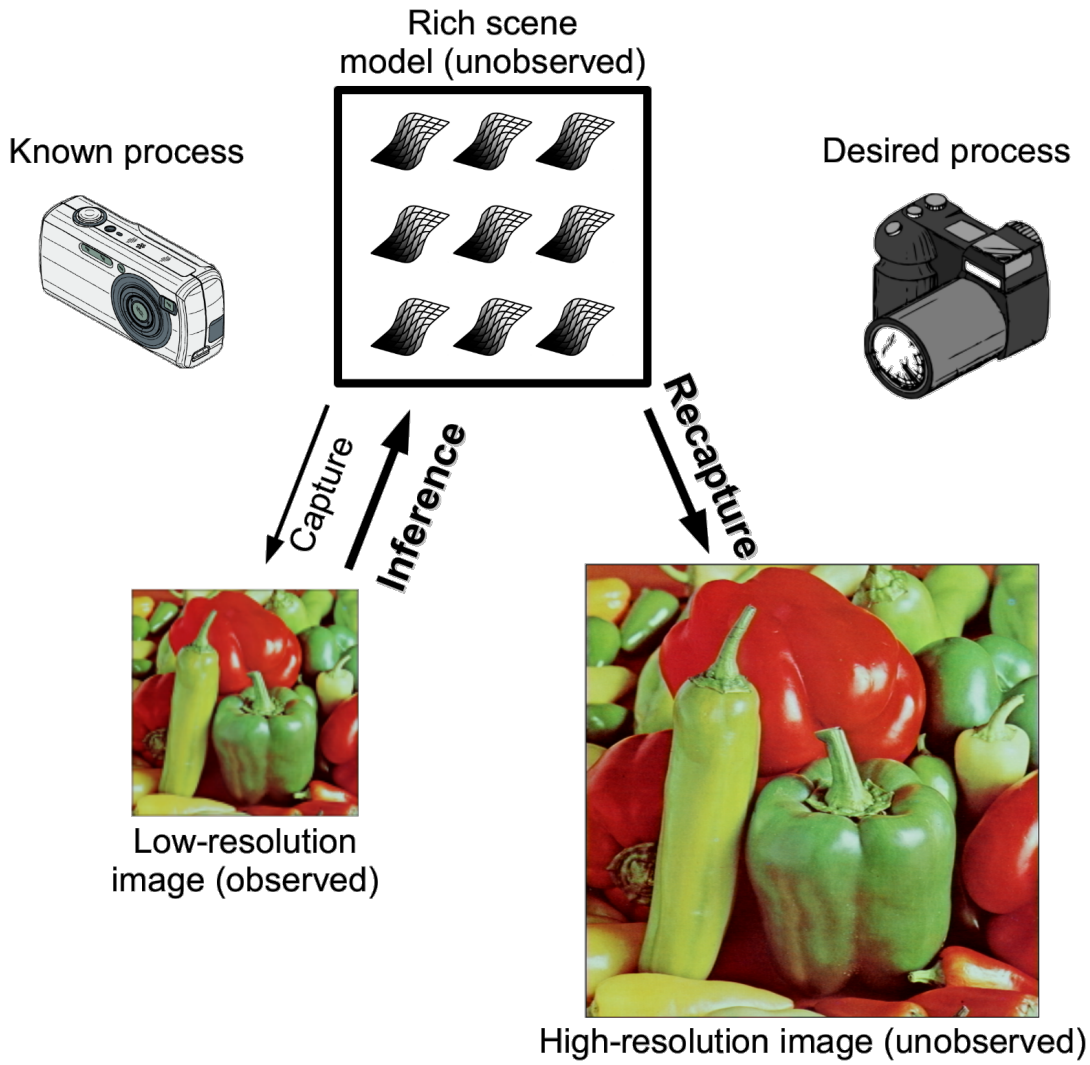
Running Probabilistic Programs Backward

Neil Toronto

PLT @ Brigham Young University

Master's Research: Super-Resolution



Toronto et al. Super-Resolution via Recapture and Bayesian Effect Modeling. CVPR 2009

Master's Research: Super-Resolution

- Model and query: Half a page of beautiful math

$$\mathbf{C}^x_{i,j} \equiv i + \tfrac{1}{2} \qquad i \in 0..m-1$$
$$\mathbf{C}^y_{i,j} \equiv j + \tfrac{1}{2} \qquad j \in 0..n-1$$

$$\mathrm{N9}(x,y) \equiv \{i \in \mathbb{Z} \mid -1 \le i - \lfloor x \rfloor \le 1\} \times \{j \in \mathbb{Z} \mid -1 \le j - \lfloor y \rfloor \le 1\}$$

$$\mathrm{dist}(x,y,\theta,d) \equiv x\cos\theta + y\sin\theta - d$$

$$\mathrm{prof}(d,\sigma,v^+,v^-) \equiv \tfrac{v^+ - v^-}{2}\,\mathrm{erf}\left(\tfrac{d}{\sqrt{2}\sigma}\right) + \tfrac{v^+ + v^-}{2}$$

$$\mathrm{edge}(x,y,\theta,d,v^+,v^-,\sigma) \equiv \mathrm{prof}(\mathrm{dist}(x,y,\theta,d),\sigma,v^+,v^-)$$

$$\mathbf{S}^{\mathrm{edge}}_{i,j}(x,y) \equiv \mathrm{edge}(x - \mathbf{C}^x_{i,j}, y - \mathbf{C}^y_{i,j}, \mathbf{S}^\theta_{i,j}, \mathbf{S}^d_{i,j}, \mathbf{S}^{v^+}_{i,j}, \mathbf{S}^{v^-}_{i,j}, \mathbf{S}^\sigma_{i,j})$$

$$\mathrm{E}[h(\mathbf{S}_{x,y})] \equiv \sum_{k,l \in \mathrm{N9}(x,y)} w(x - \mathbf{C}^x_{k,l}, y - \mathbf{C}^y_{k,l})\; h(\mathbf{S}^{\mathrm{edge}}_{k,l}(x,y))$$

$$\mathbf{S}^\theta_{i,j} \sim \mathrm{Uniform}(-\pi,\pi) \qquad \mathbf{S}^{v^+}_{i,j} \sim \mathrm{Uniform}(0,1)$$
$$\mathbf{S}^d_{i,j} \sim \mathrm{Uniform}(-3,3) \qquad \mathbf{S}^{v^-}_{i,j} \sim \mathrm{Uniform}(0,1)$$
$$\mathbf{S}^\sigma_{i,j} \sim \mathrm{Beta}(1.6,1)$$

$$\mathbf{I}_{i,j} | \mathbf{S}_{\mathrm{N9}(i,j)} \sim \mathrm{Normal}(\mathrm{E}[\mathbf{S}_{i,j}], \omega)$$

$$\Phi_{i,j}(\mathbf{S}_{\mathrm{N9}(i,j)}) \equiv \exp\left(-\frac{\mathrm{Var}[\mathbf{S}_{i,j}]}{2\gamma^2}\right)$$

Master's Research: Super-Resolution

- Query implementation: 600 lines of Python

Main Results: Super-Resolution

- Competitor and BEI on 4x super-resolution:

Resolution Synthesis

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Bayesian Edge Inference

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- Also beat state-of-the-art on “objective” measures

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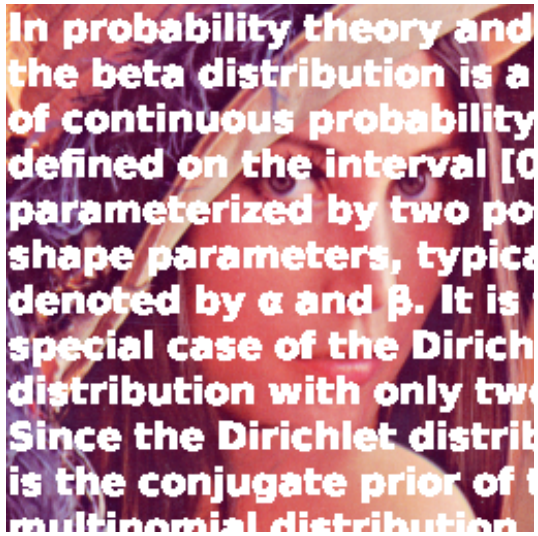

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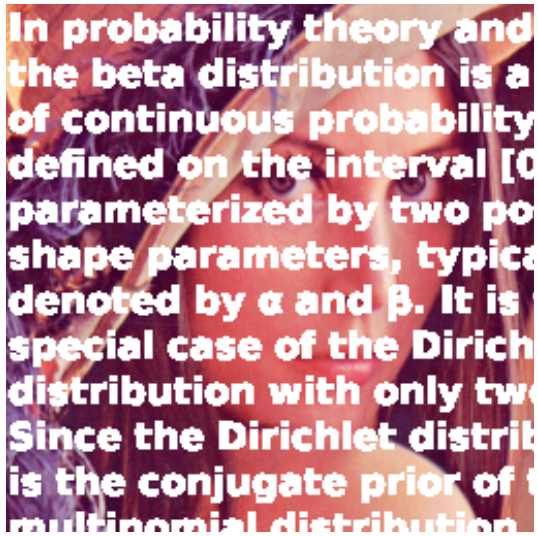

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Bayesian Edge Inference

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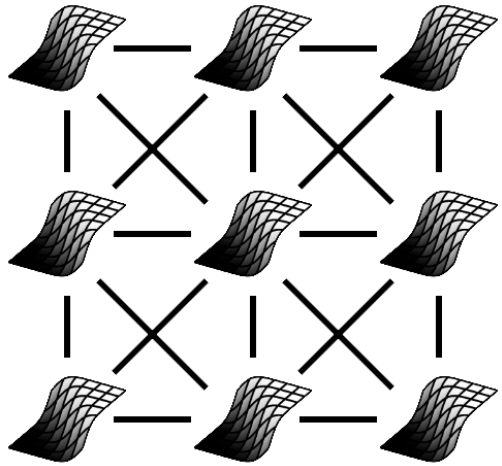

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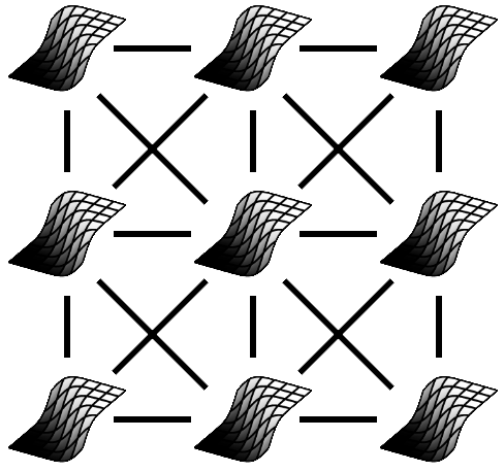

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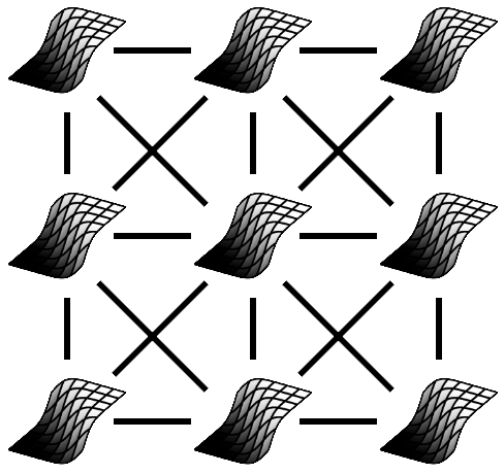

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i.e. “We can’t model it correctly so here’s a hack.”

Simple Example Process

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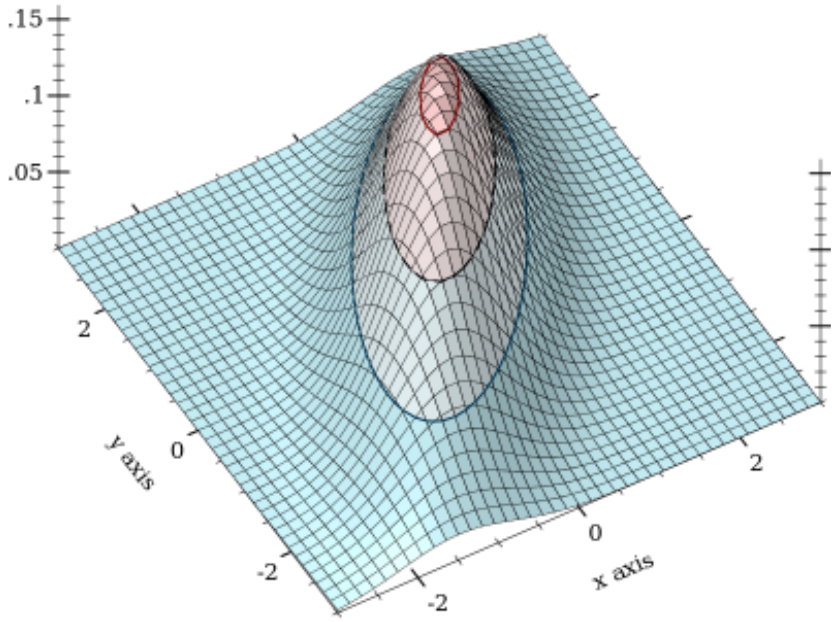
- Density model $f : \mathbb{R} \times \mathbb{R} \to [0, \infty)$:



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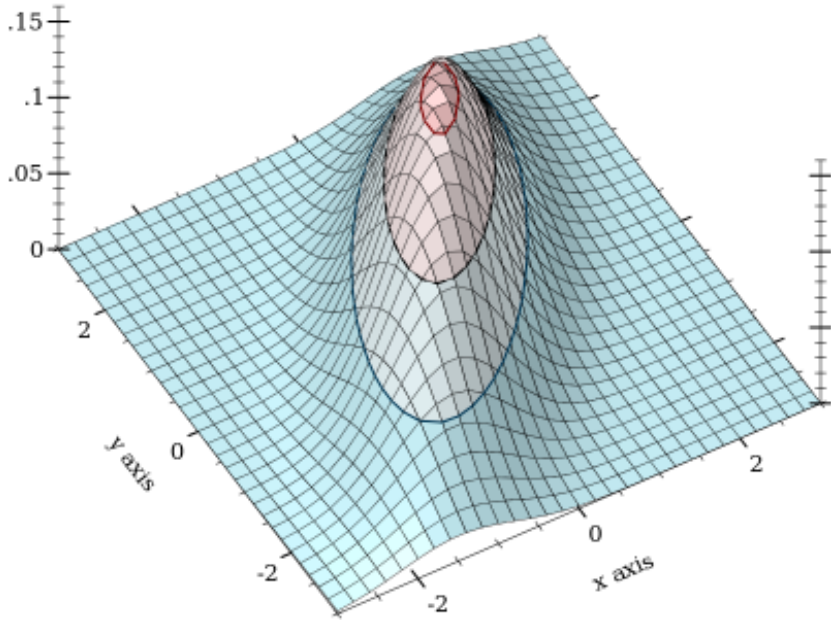

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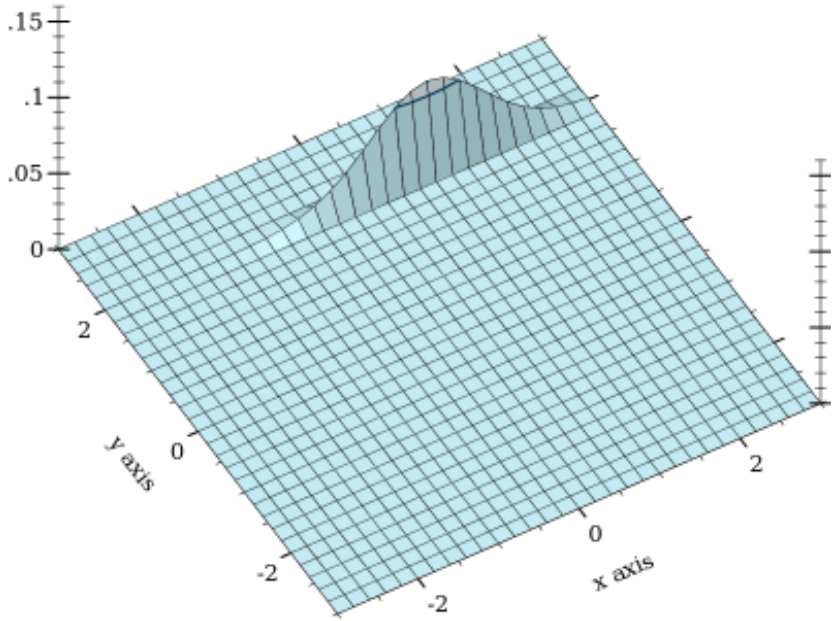

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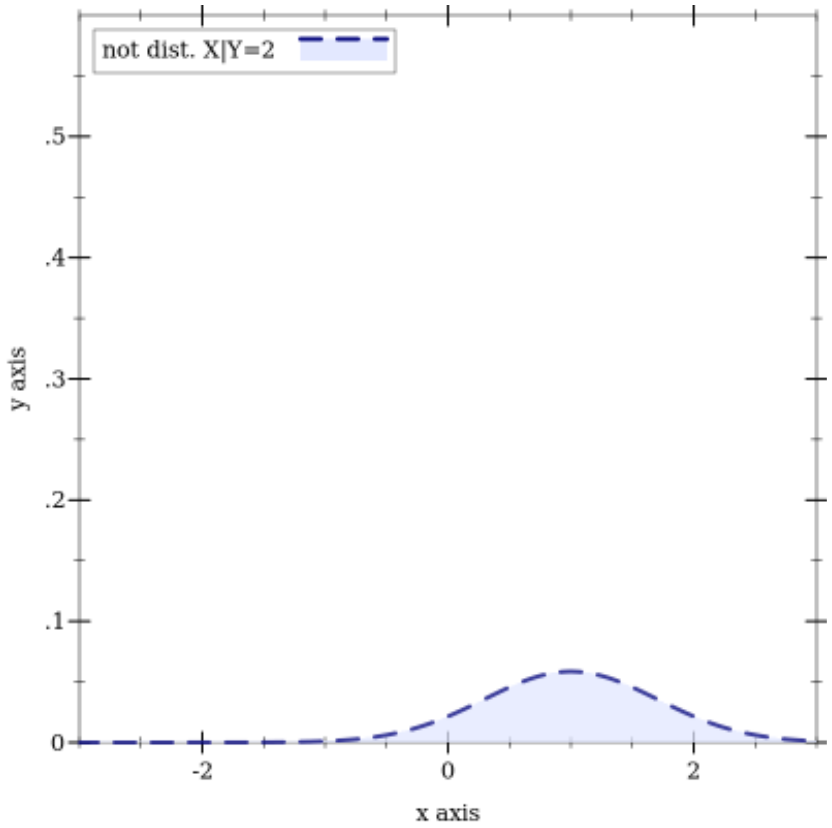

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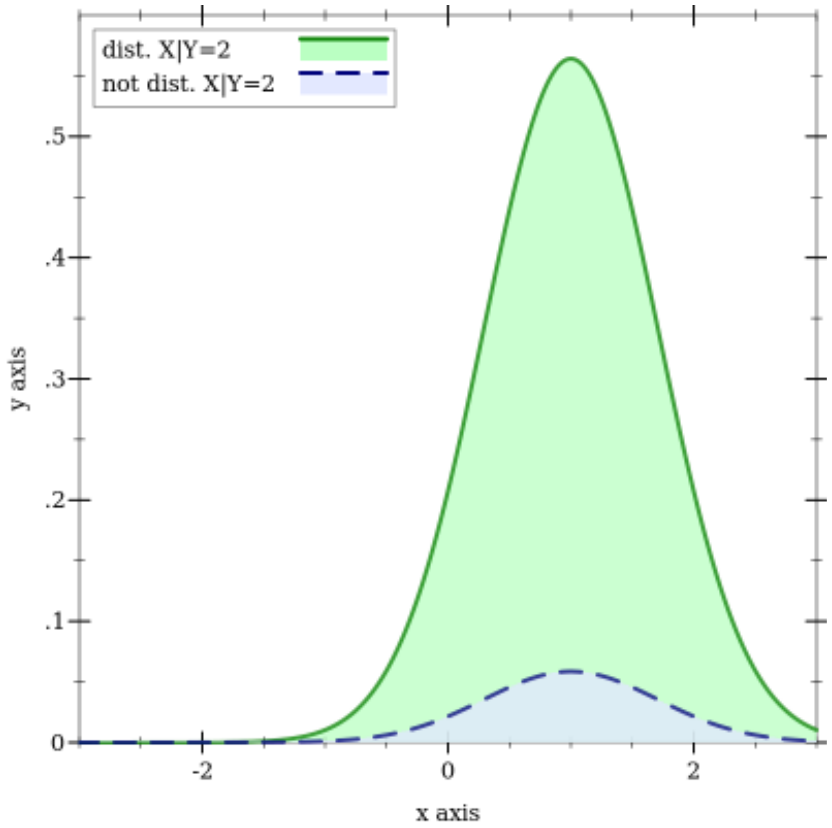

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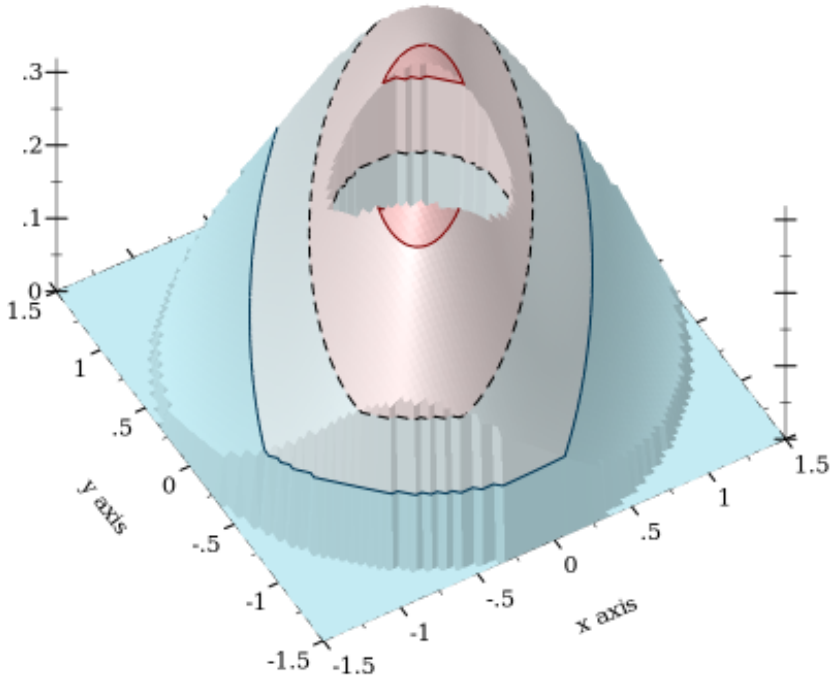

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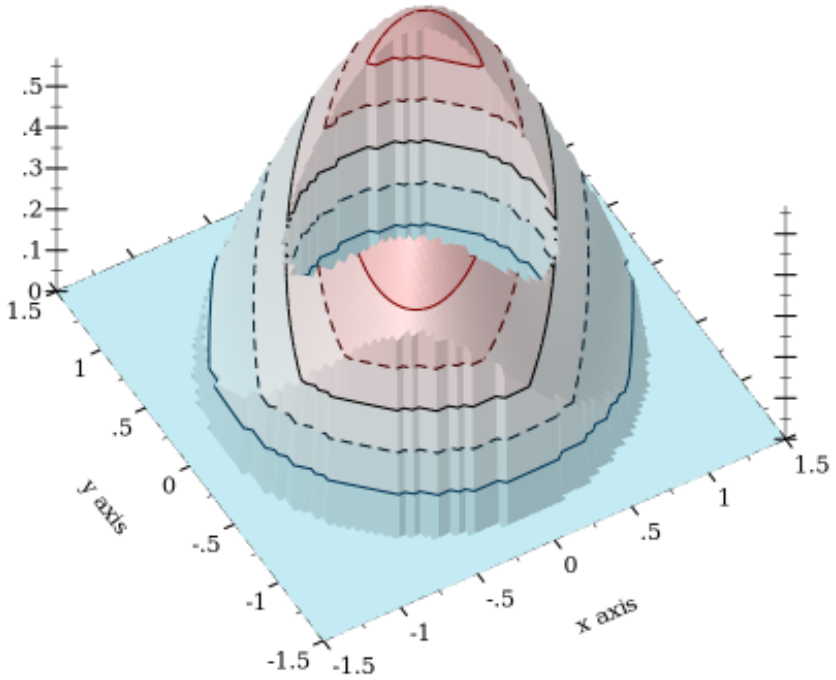

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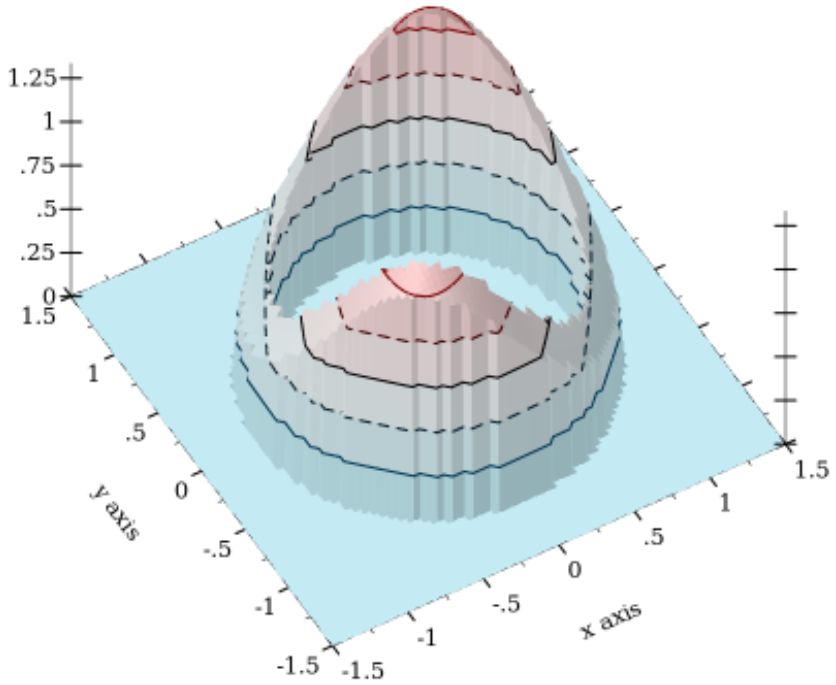

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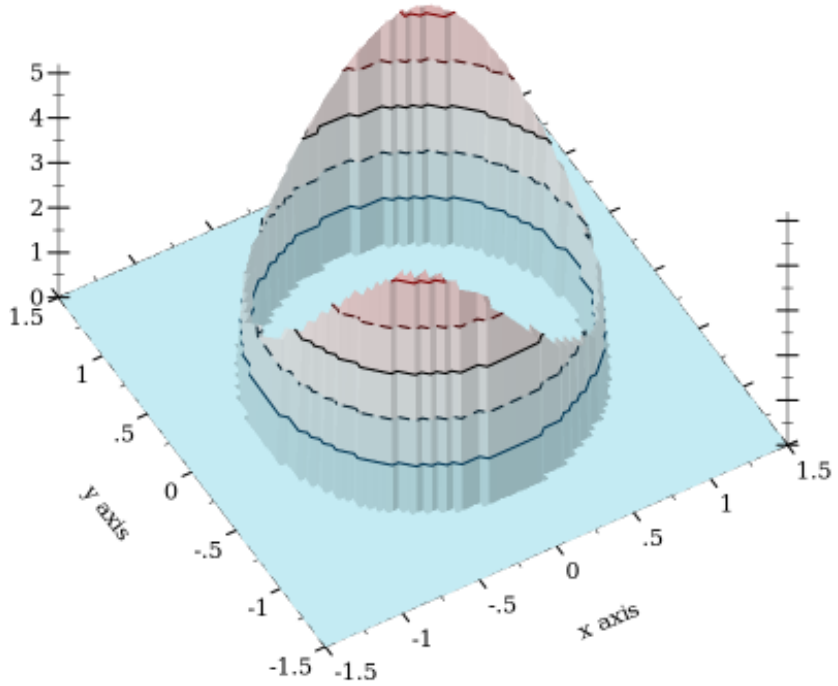

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There are tricks to get around limitations, but none are generally applicable...

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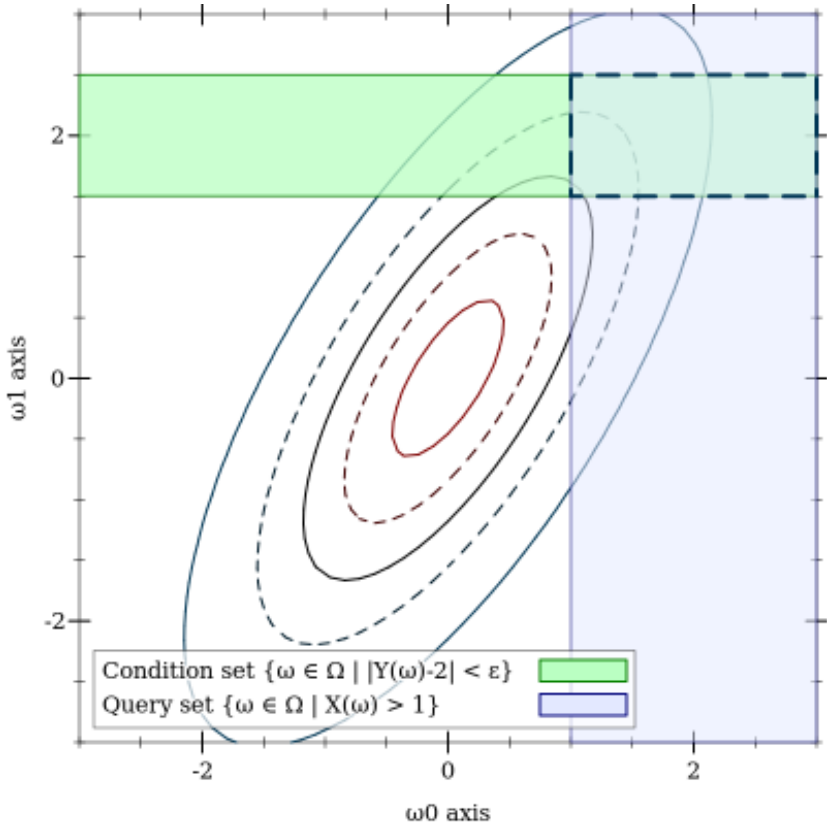

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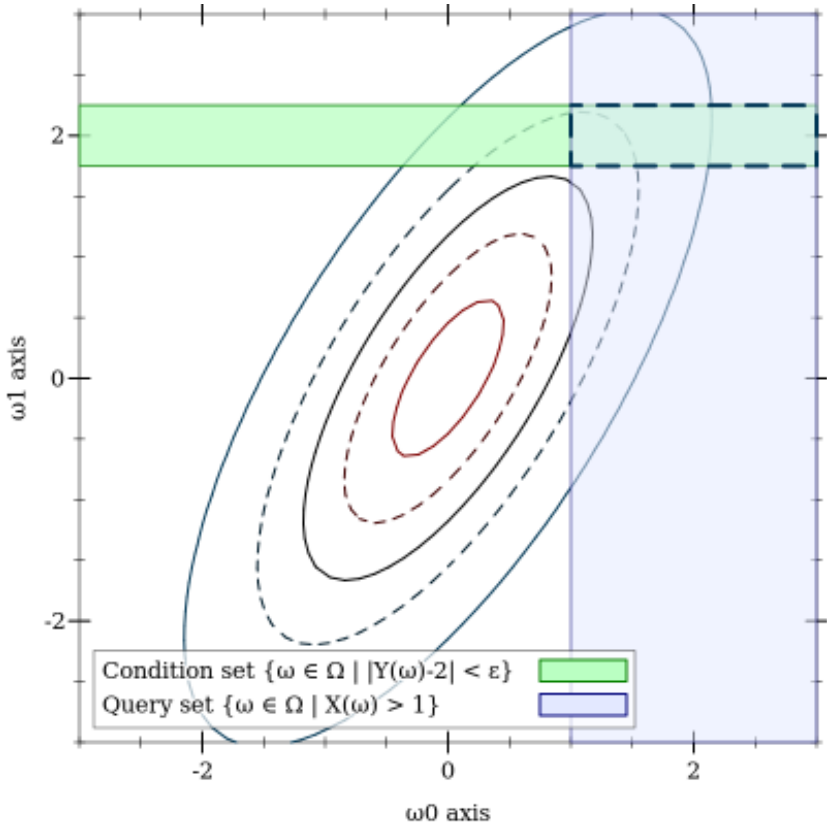

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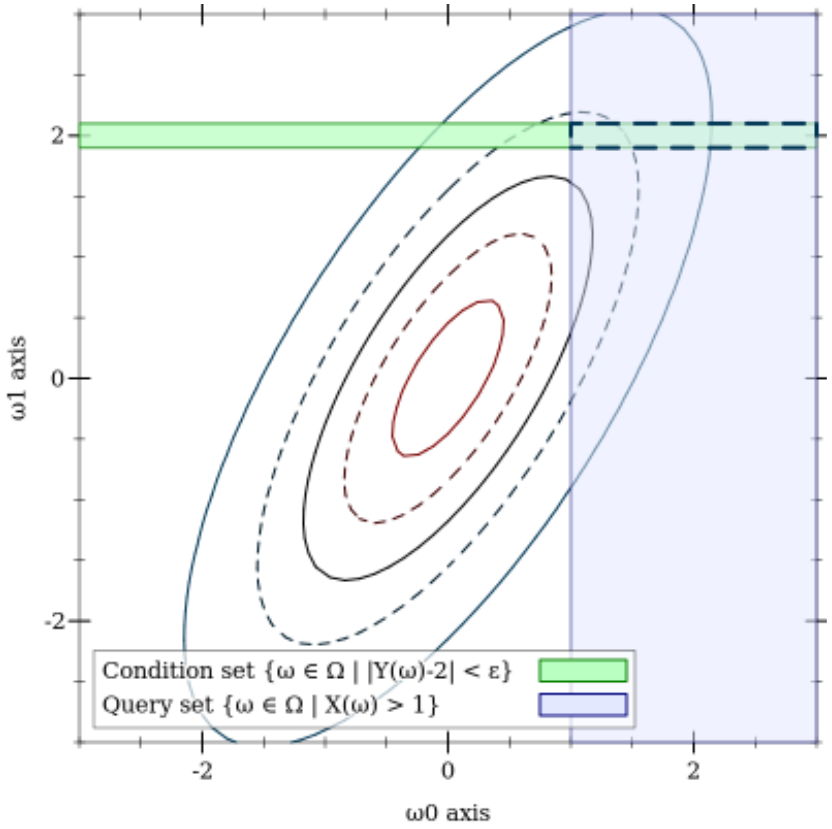

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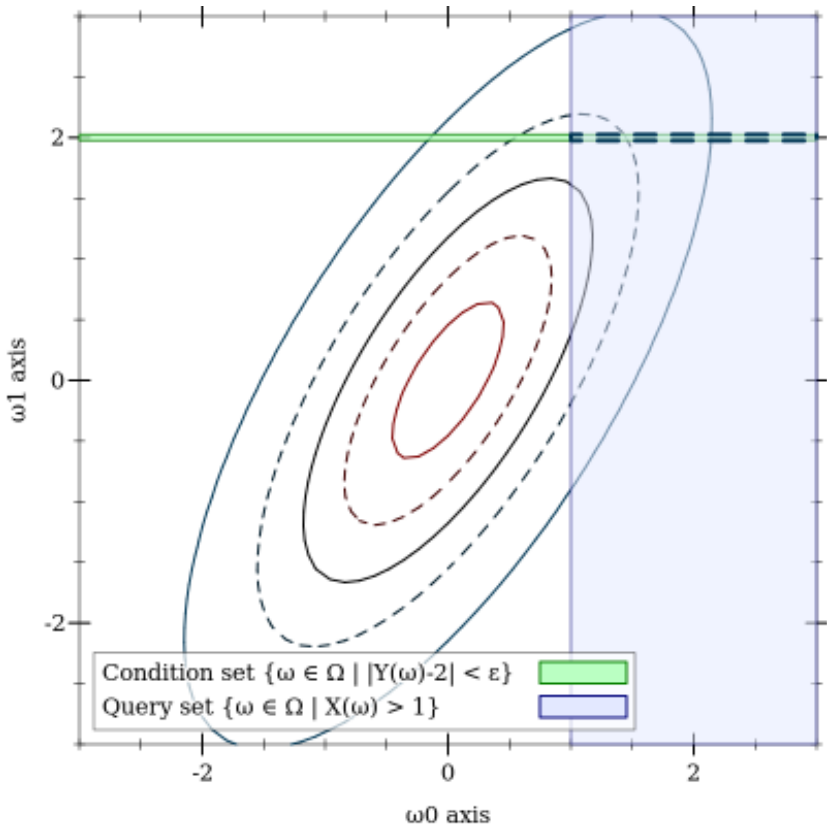

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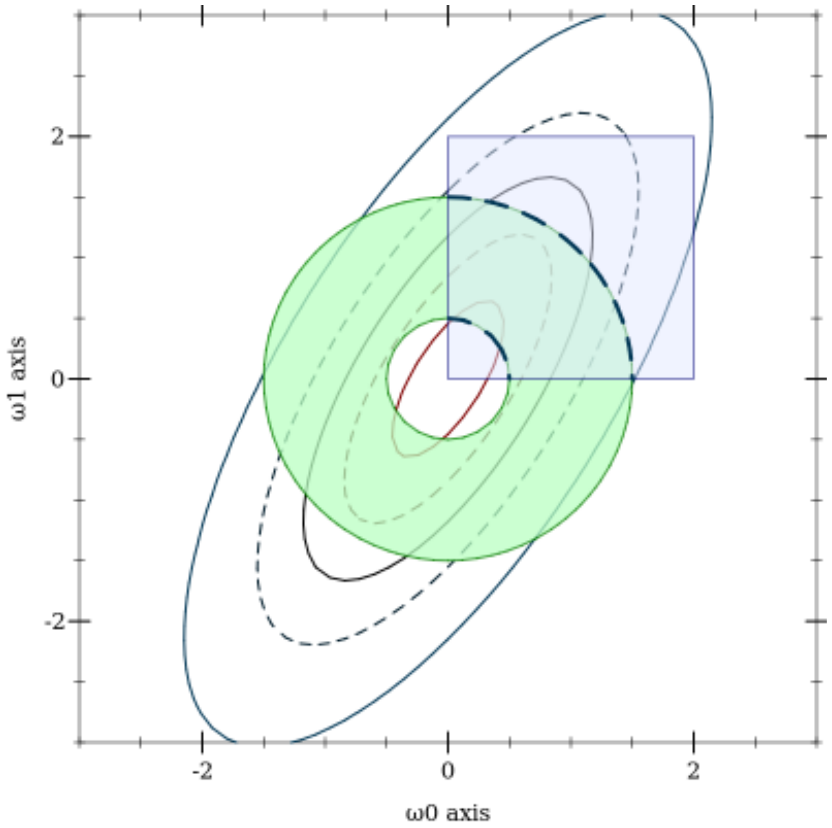

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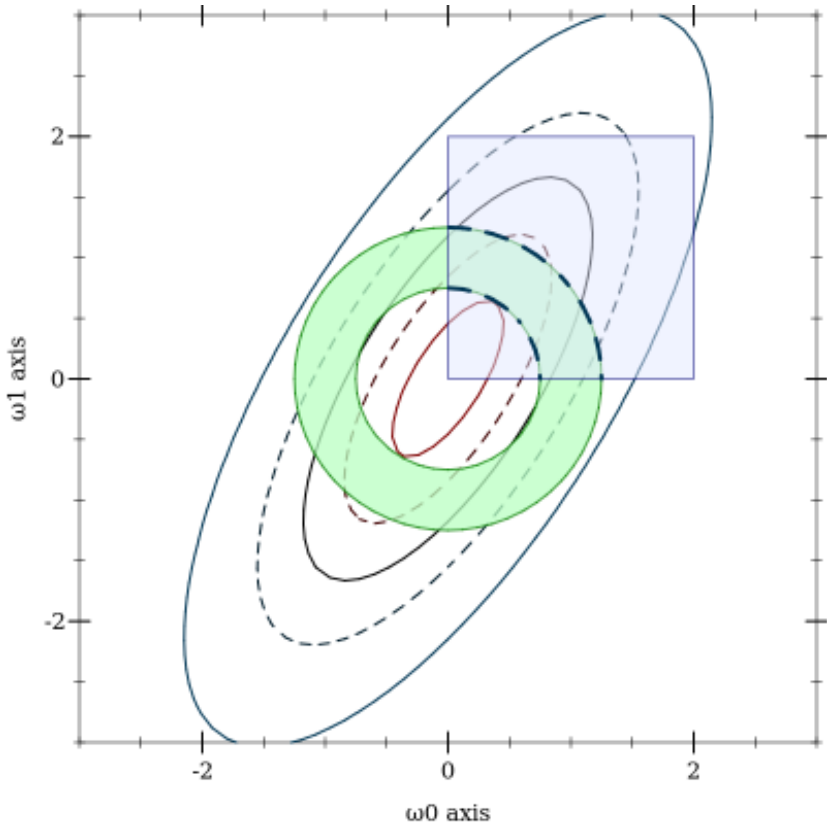

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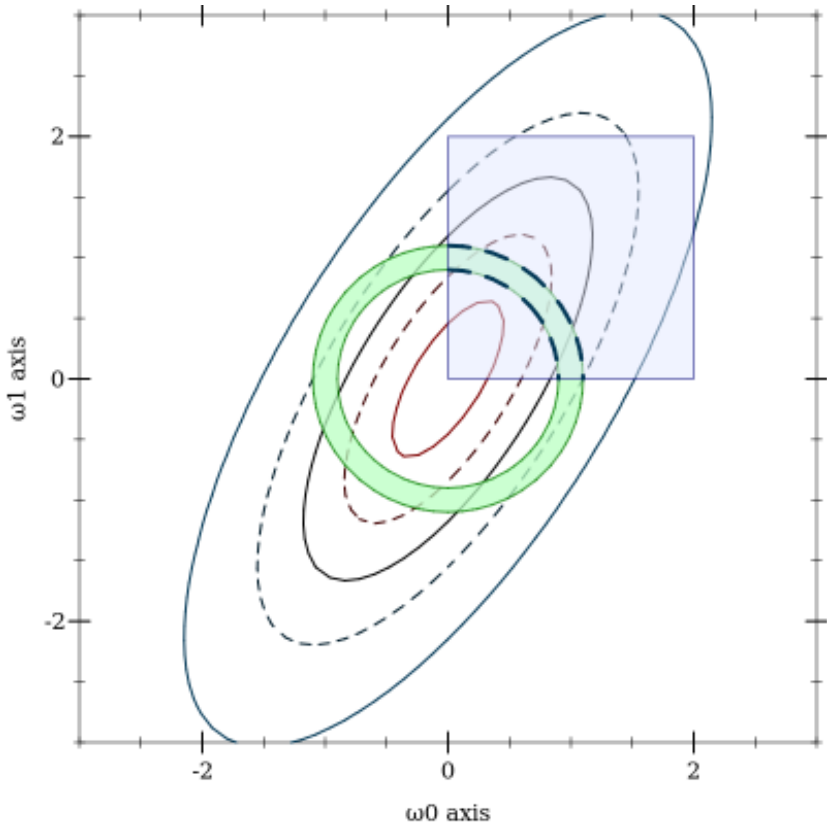

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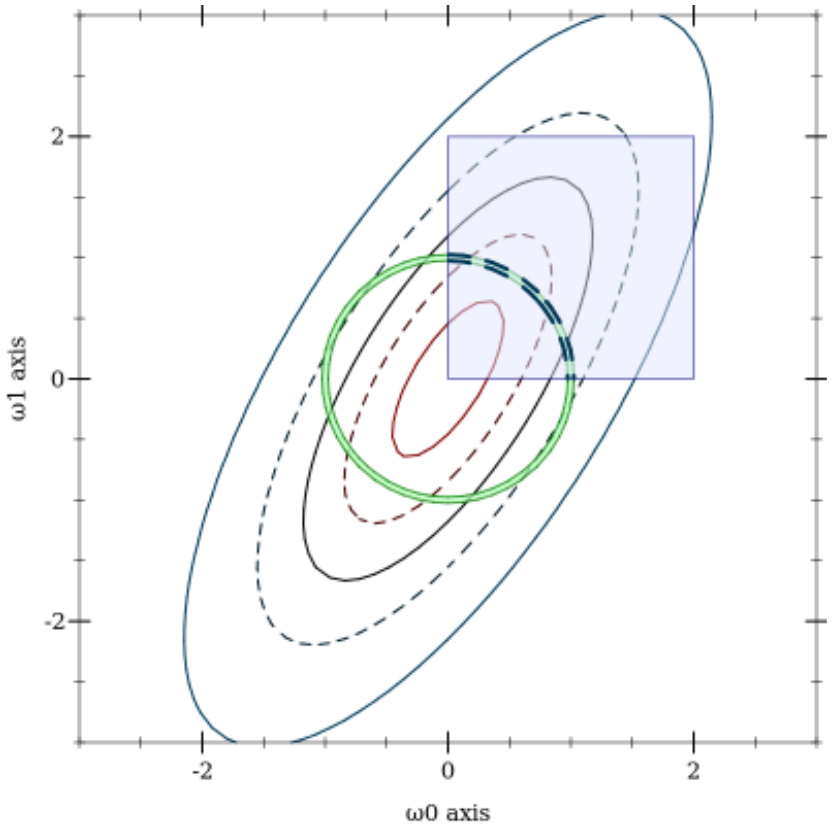

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Let $F : \mathbb{R} \to [0, 1]$ be the Normal CDF, and define a **uniform random source** model:

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$$X : \Omega \to \mathbb{R},\ X(\omega) = F^{-1}(\omega_0)$$
$$Y : \Omega \to \mathbb{R},\ Y(\omega) = F^{-1}(\omega_1) + X(\omega)$$

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- What do preimages look like?

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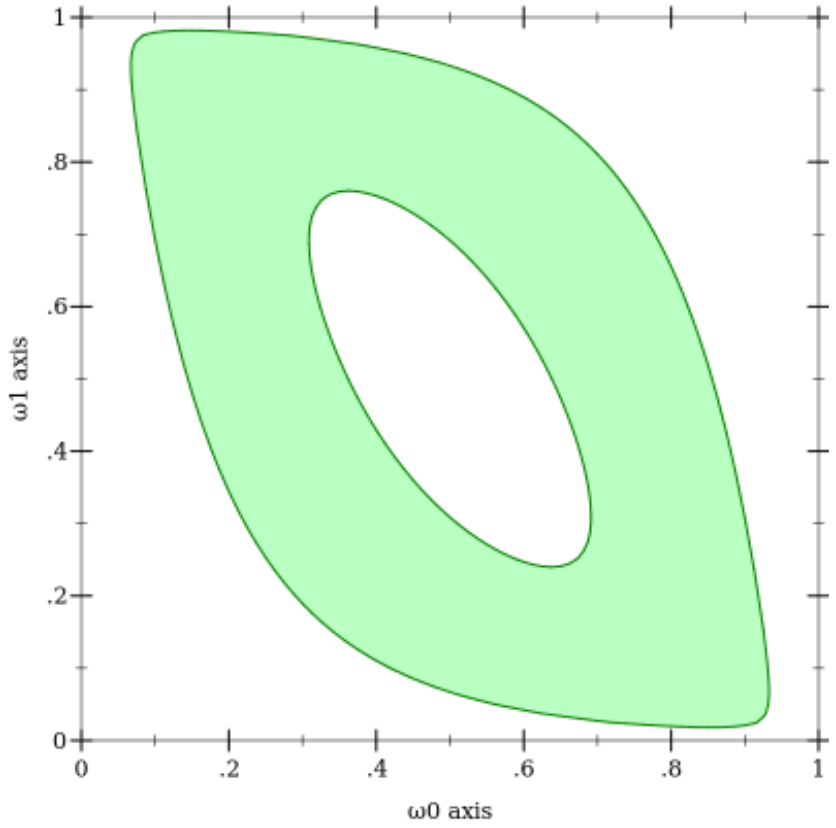

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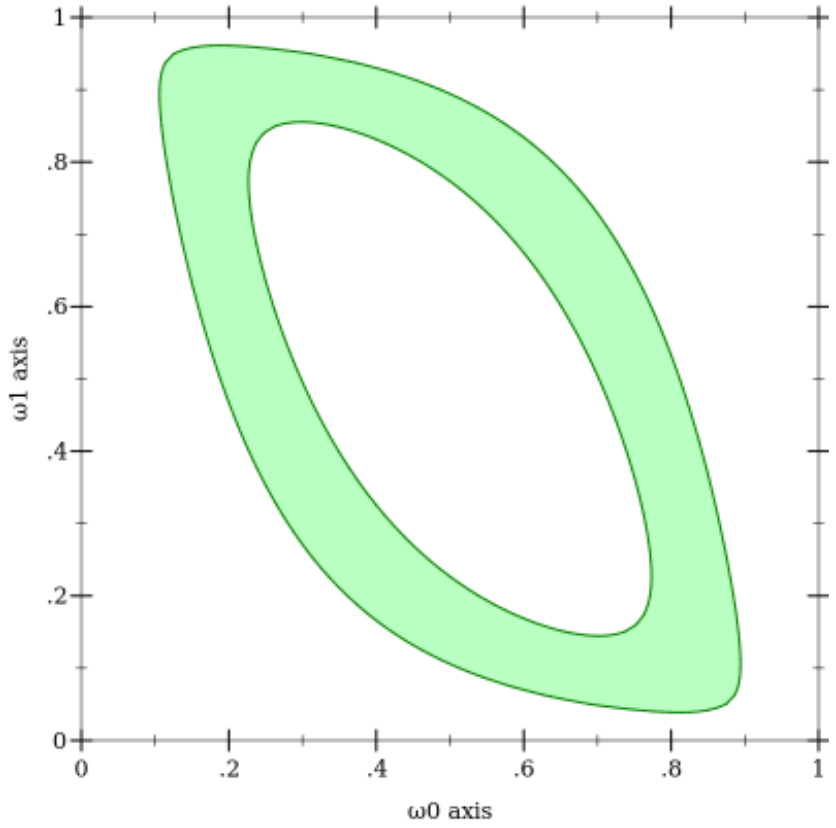

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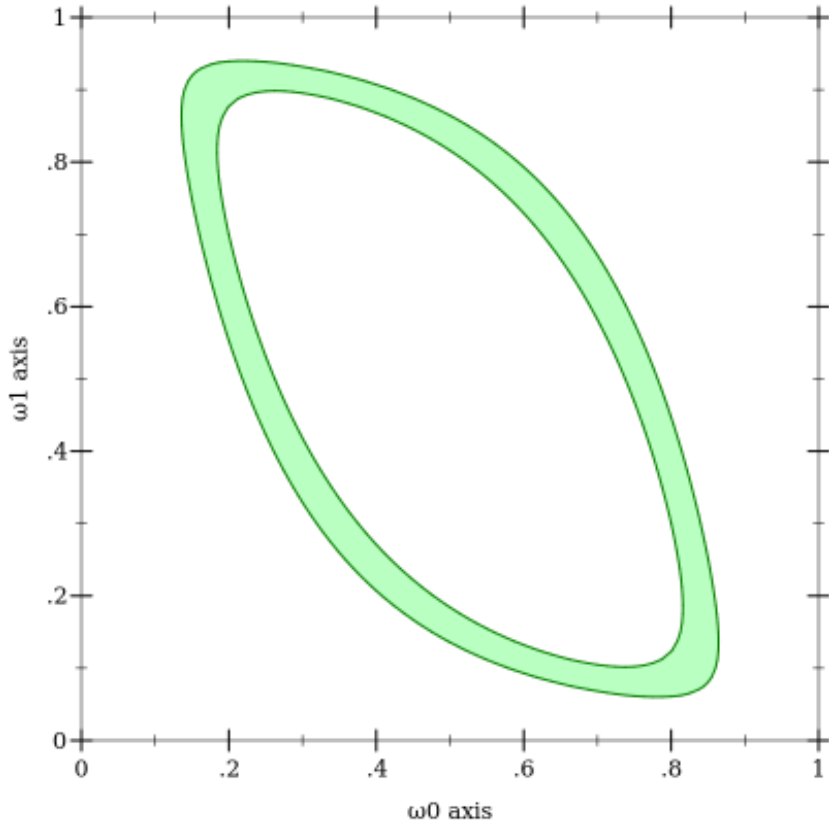

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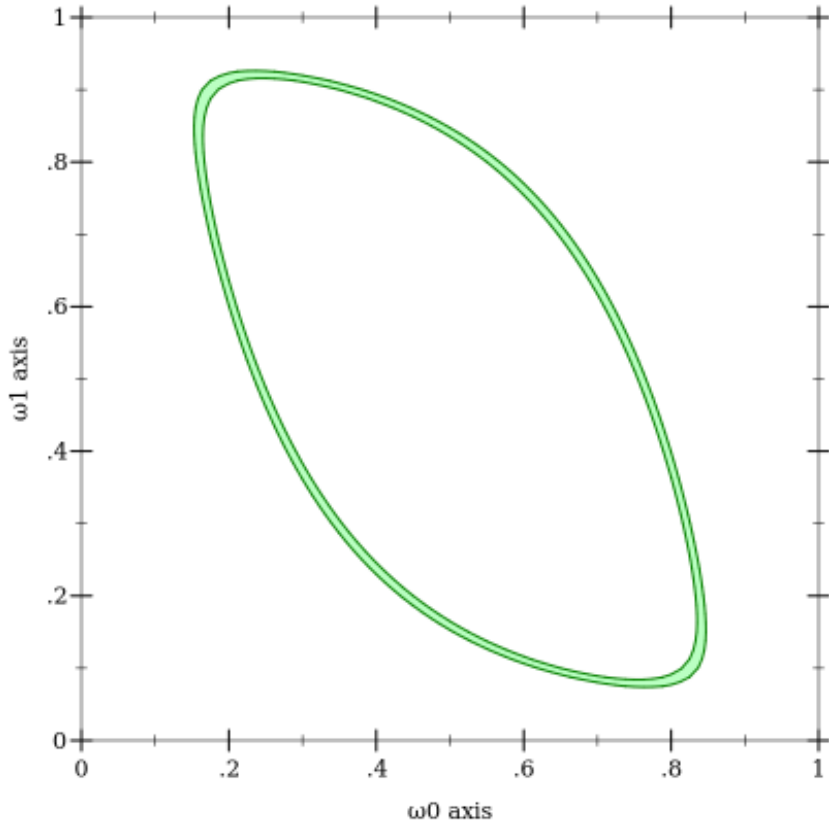
$Z^{-1}([0, \epsilon)), \epsilon = 0.025$:



Crazy Idea

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- Implementation should:
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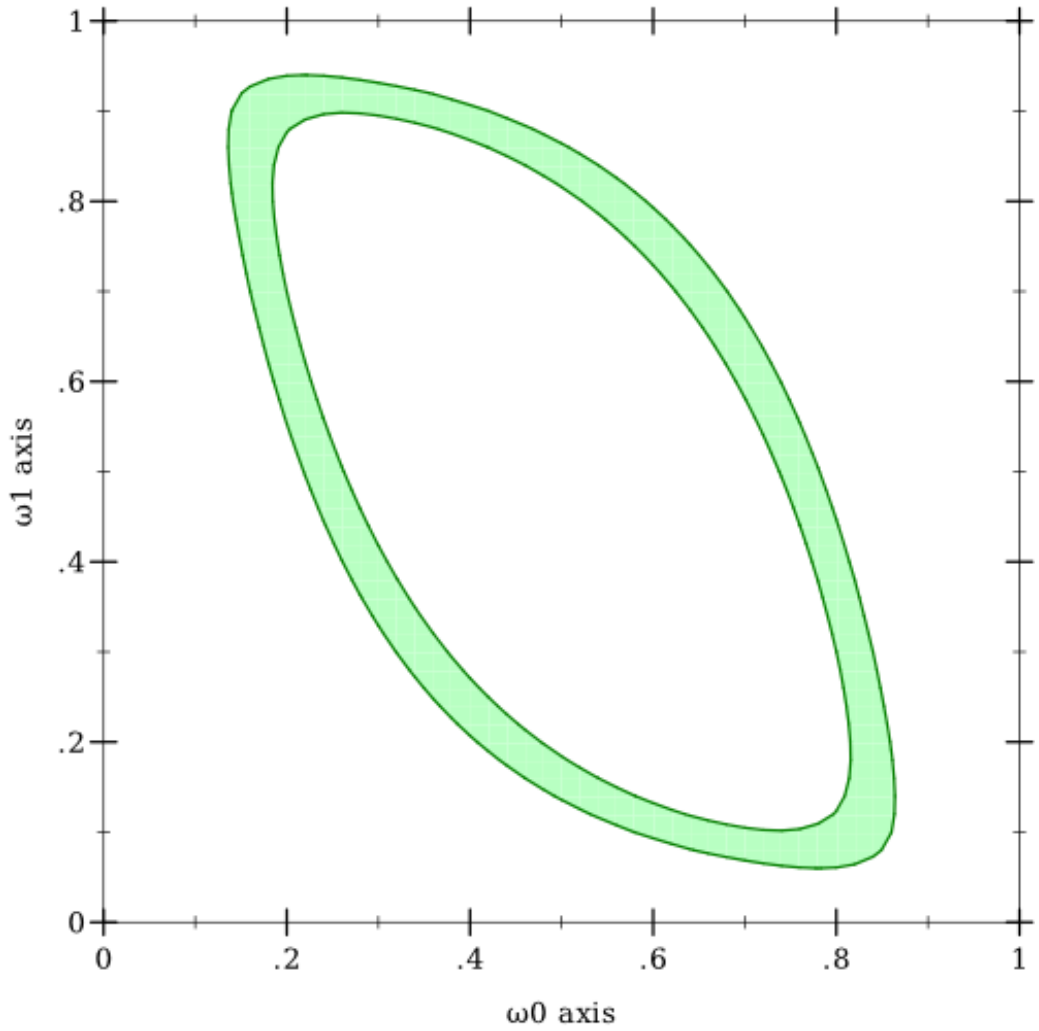
- Exact implementation is out of reach, but...

What About Approximating?

Conservative approximation with rectangles:



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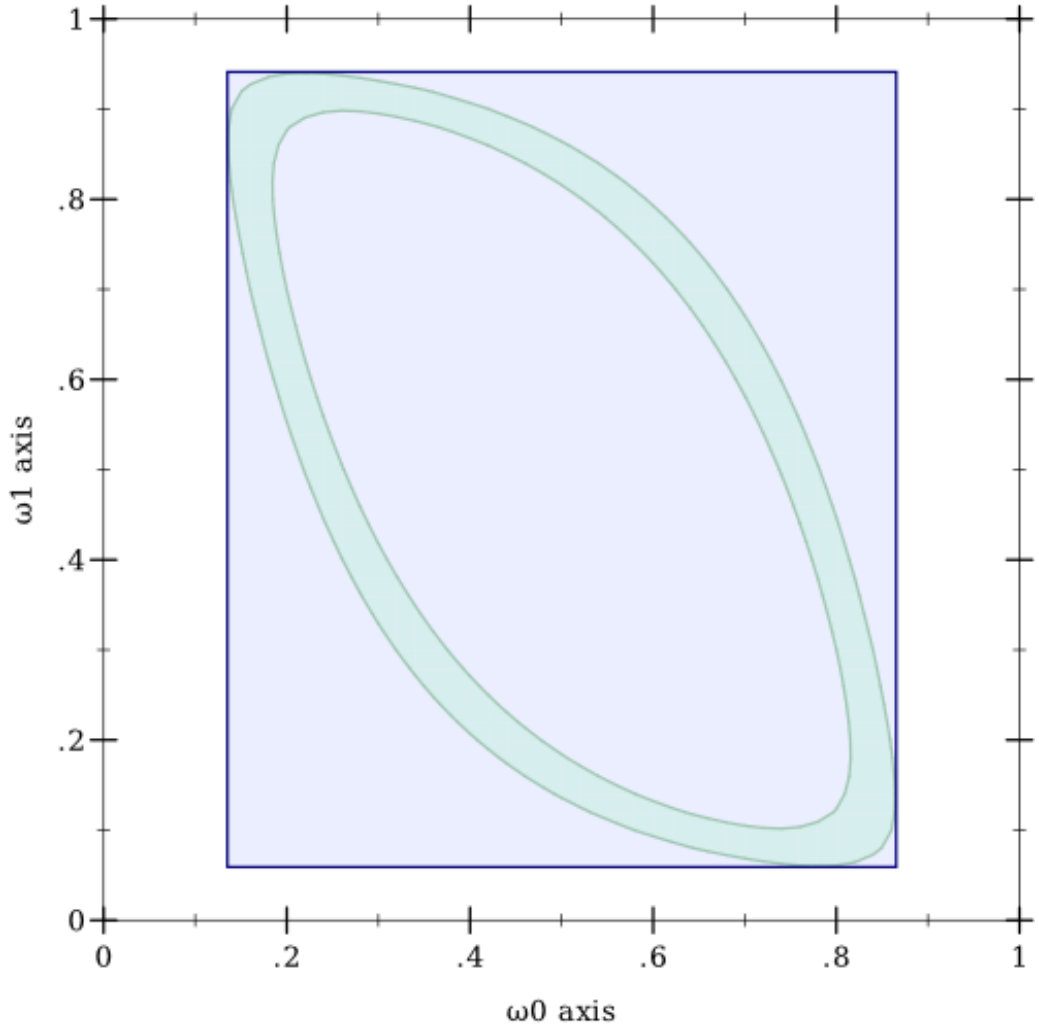

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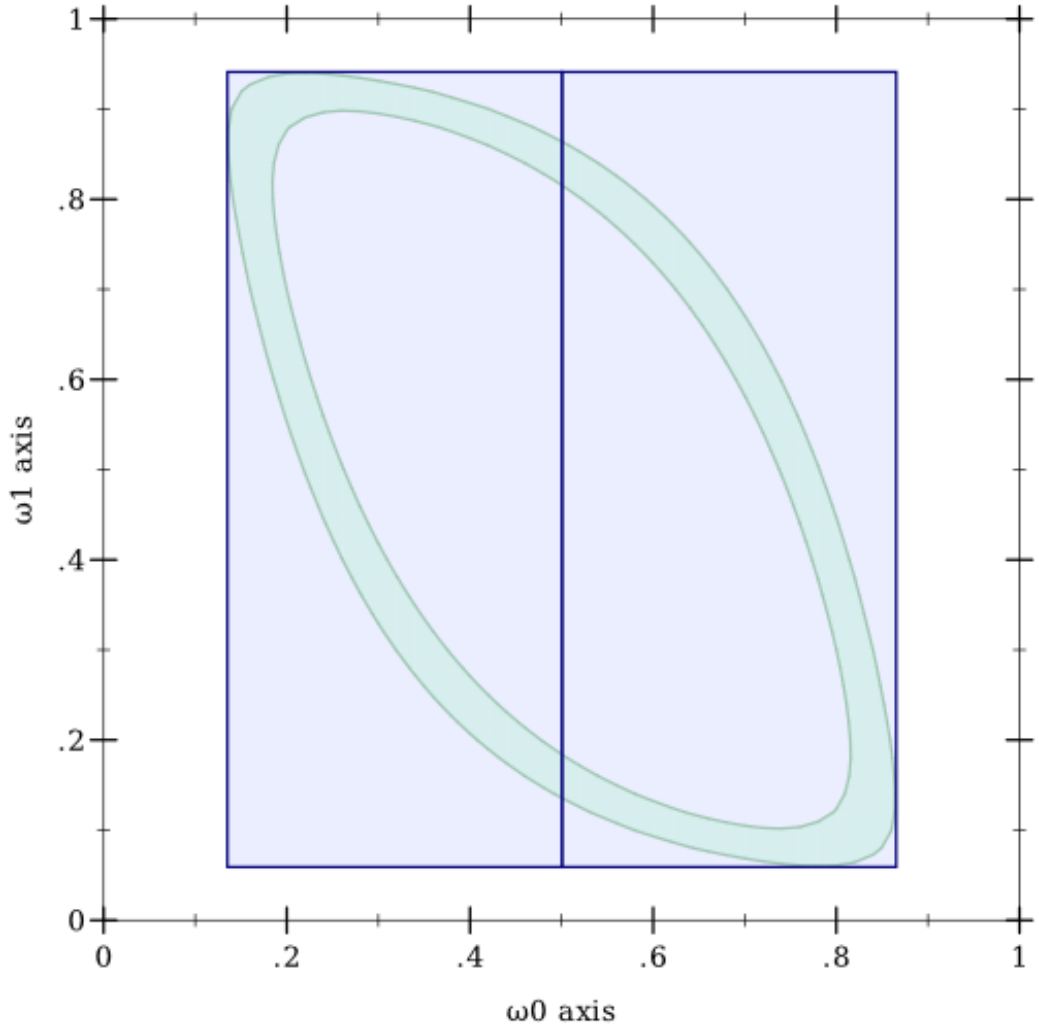
Restricting preimages to rectangular subdomains:



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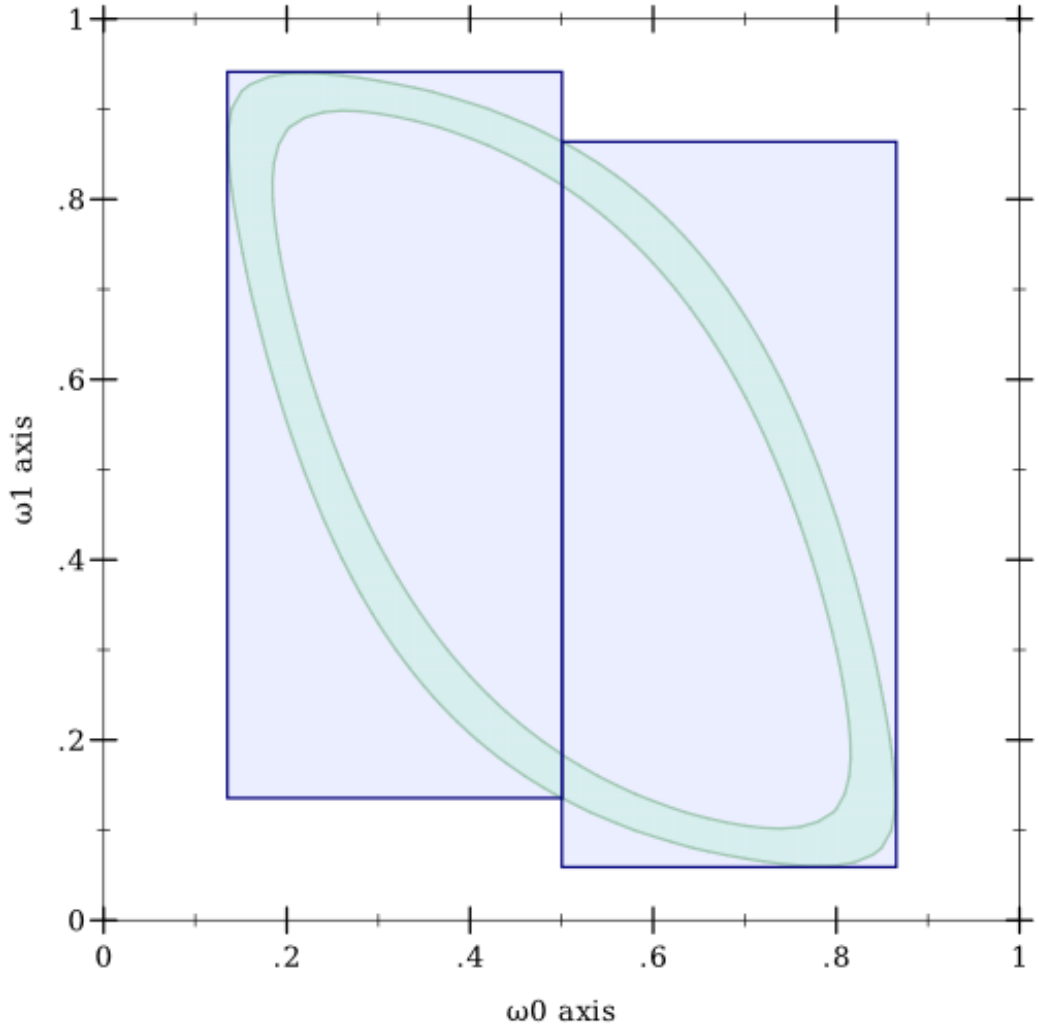

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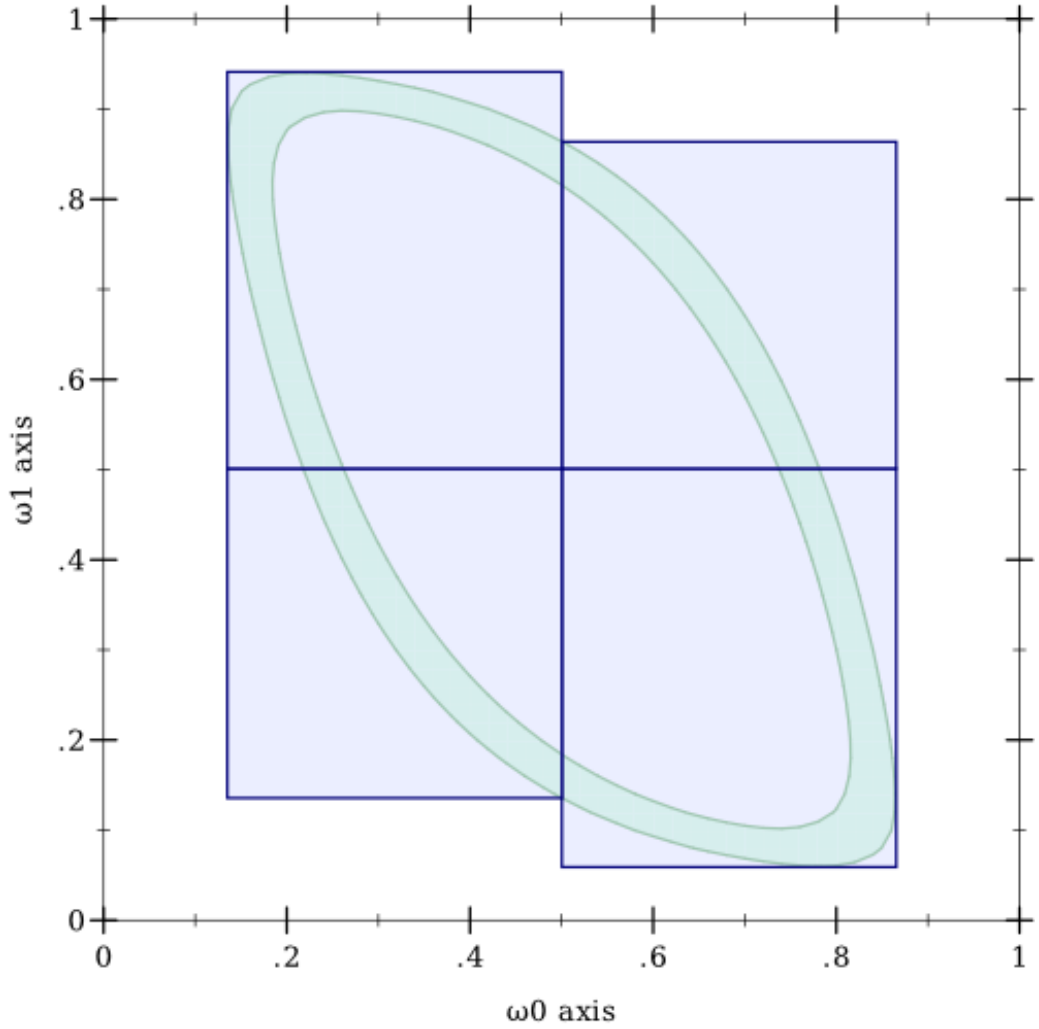

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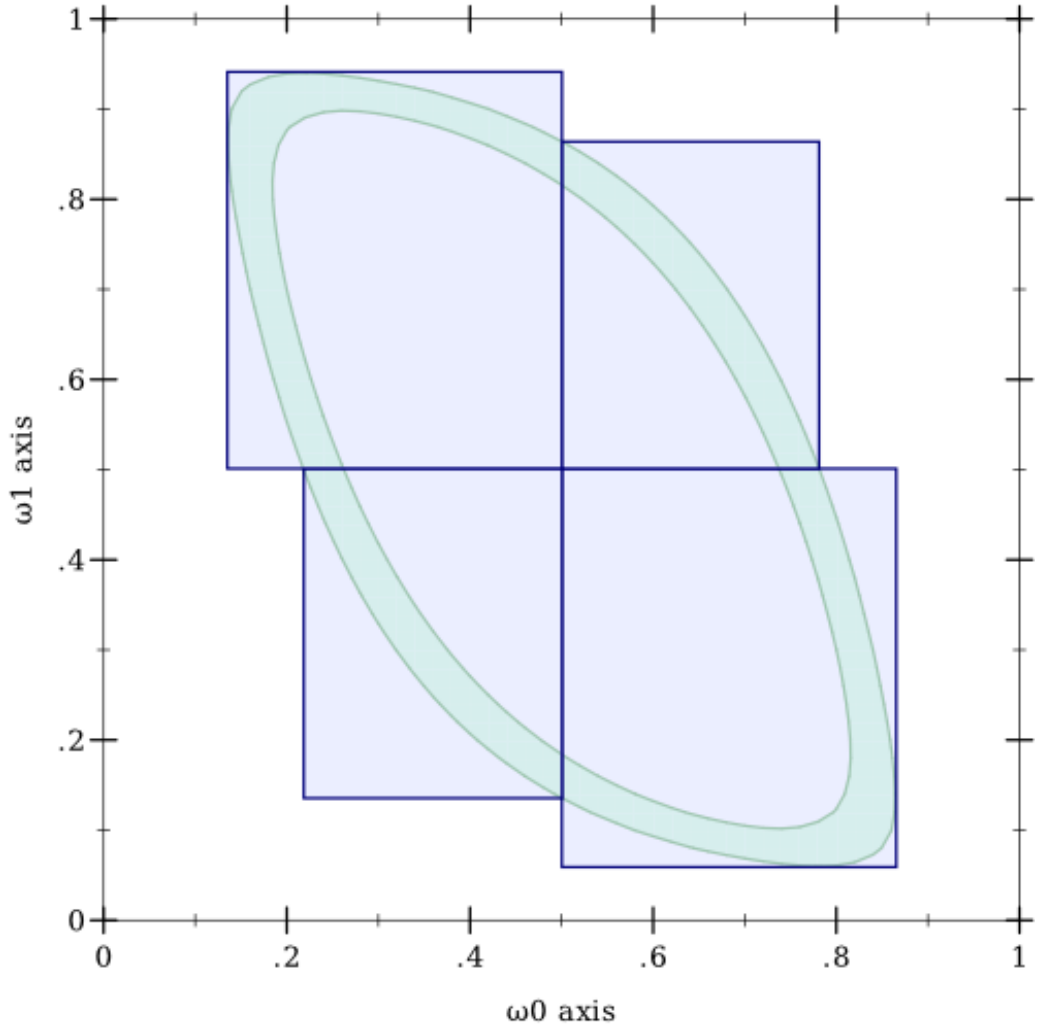

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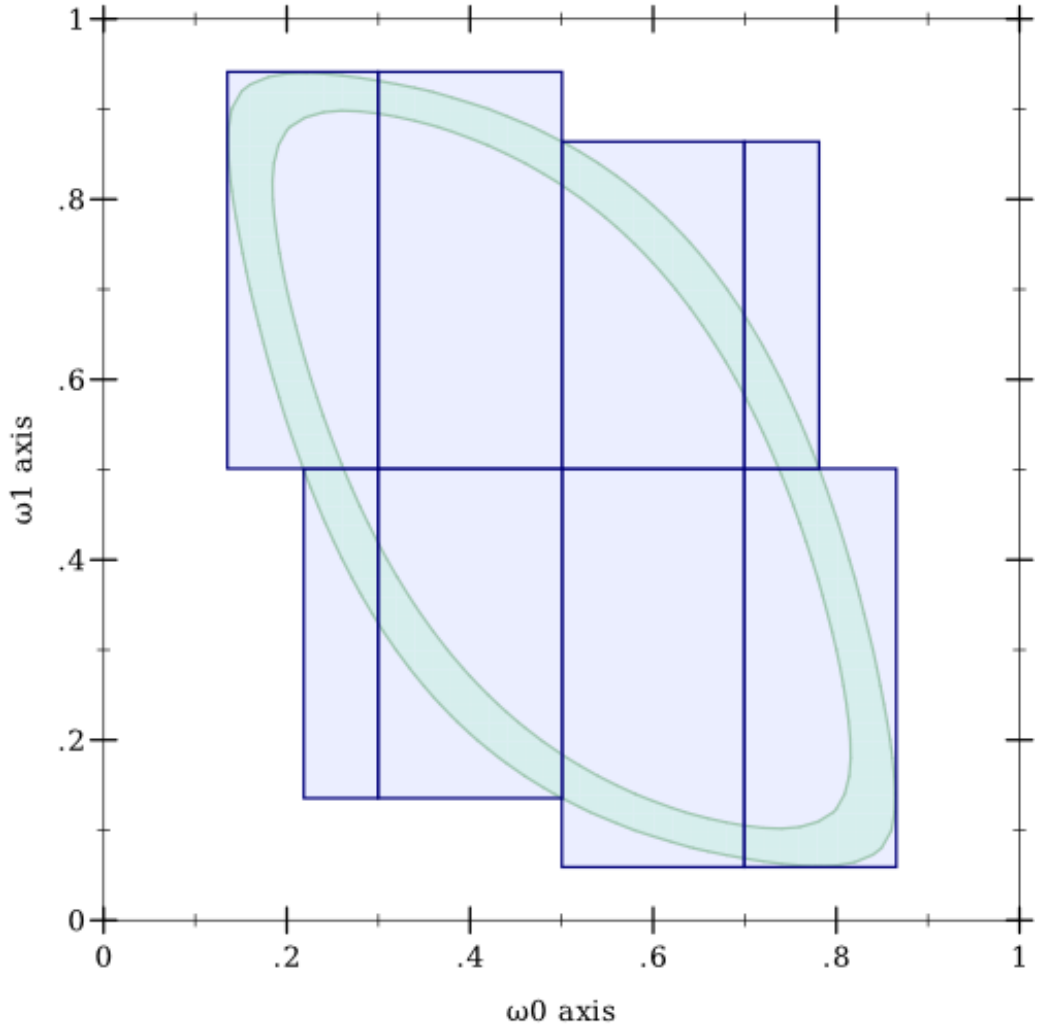

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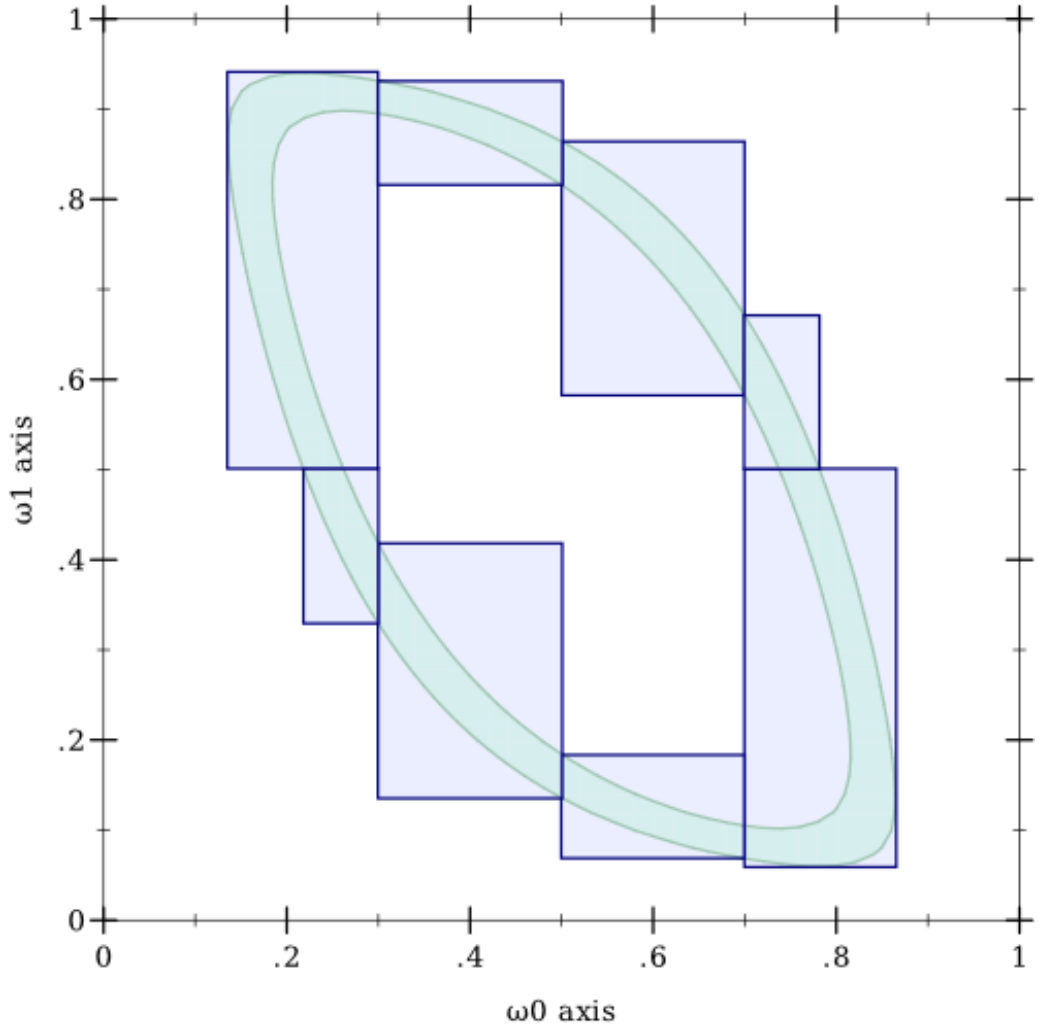

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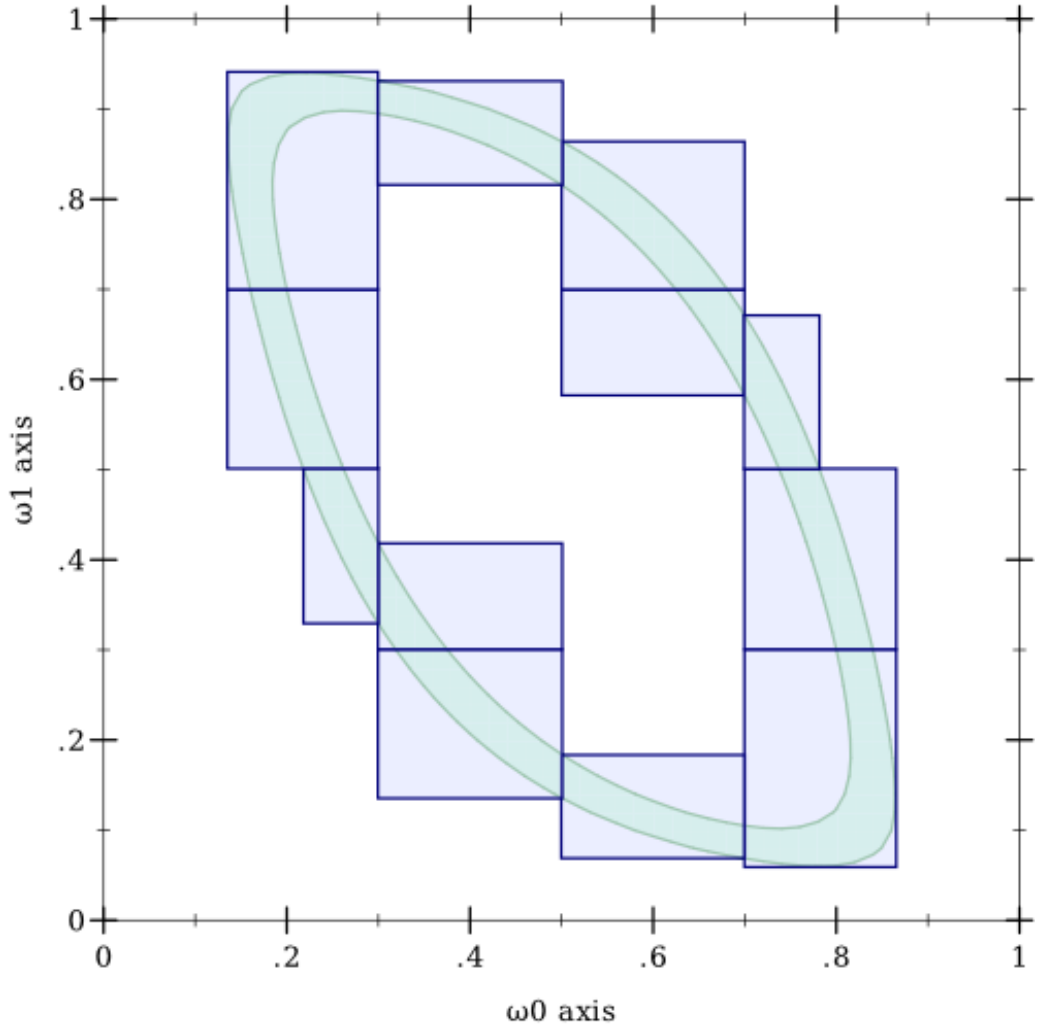

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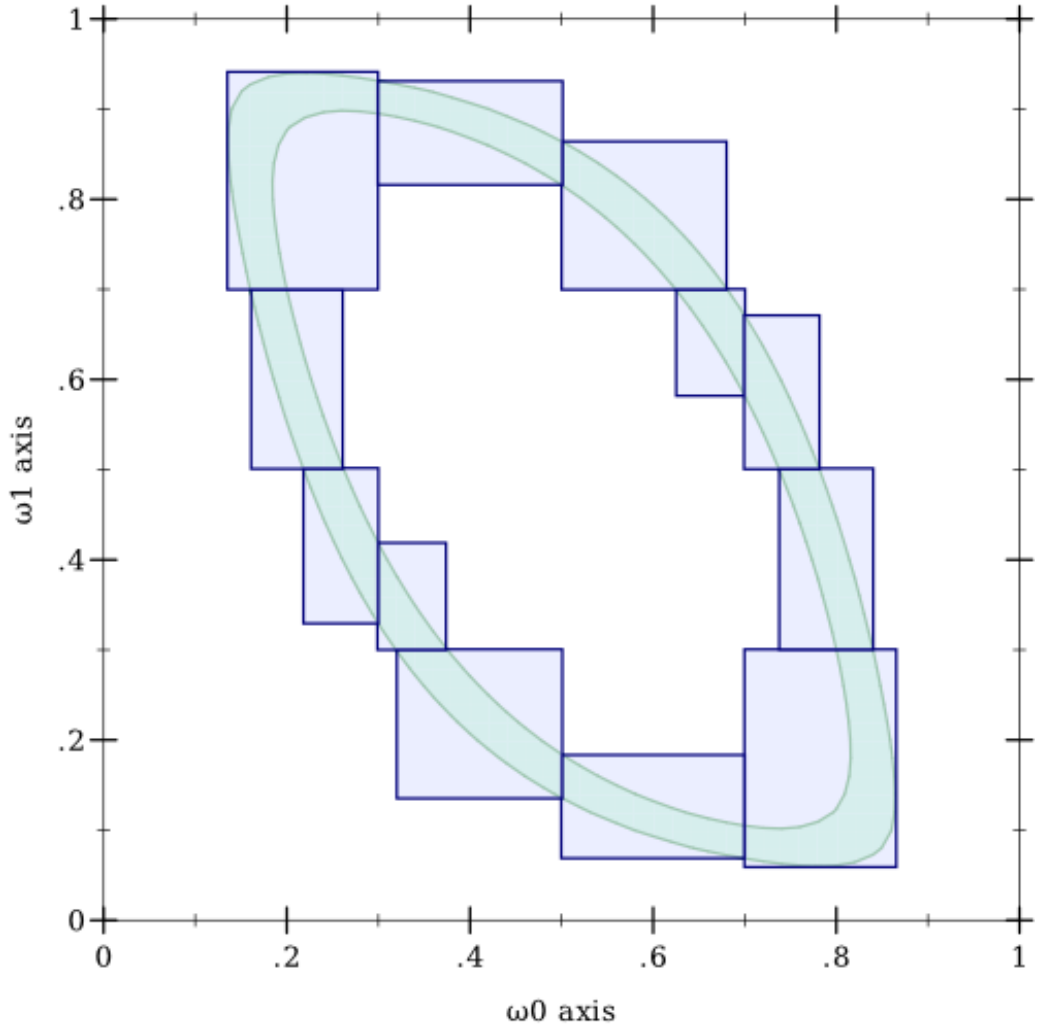

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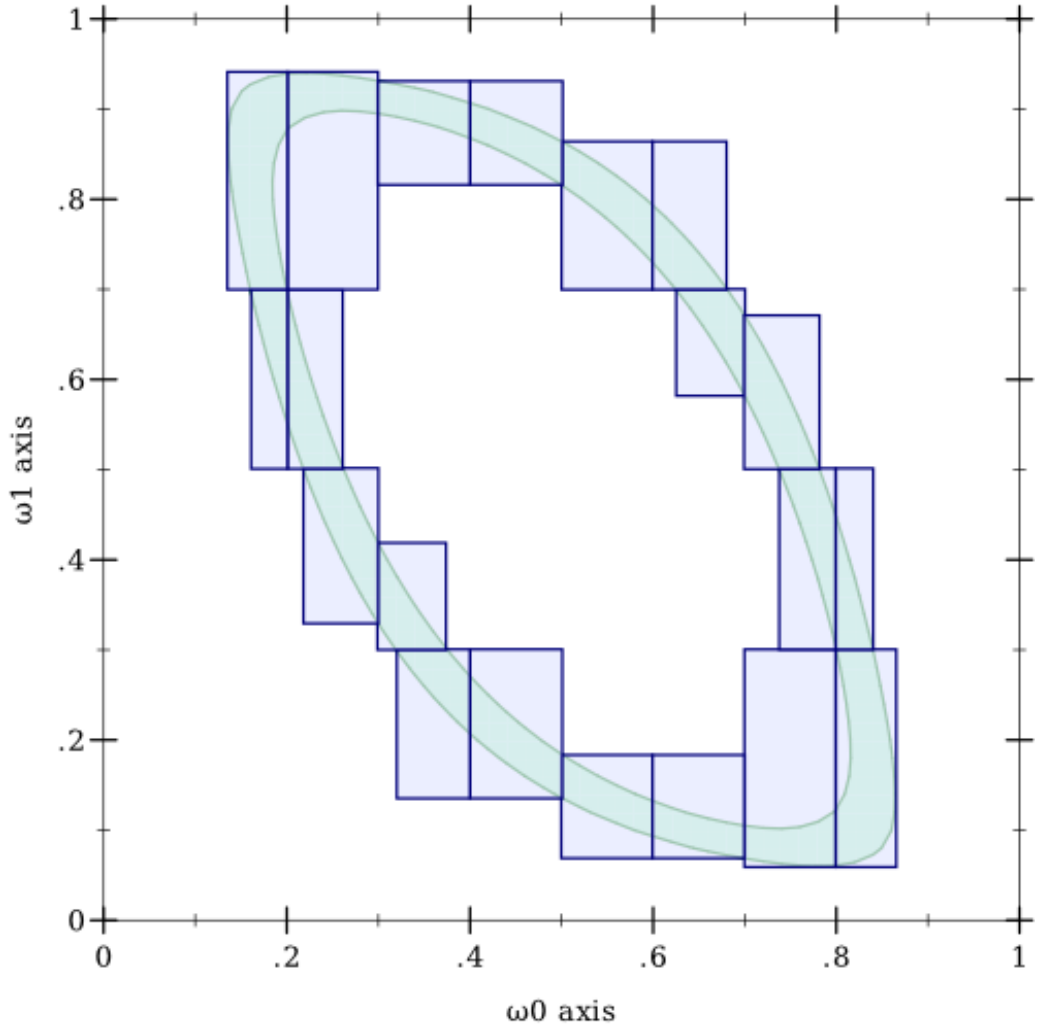

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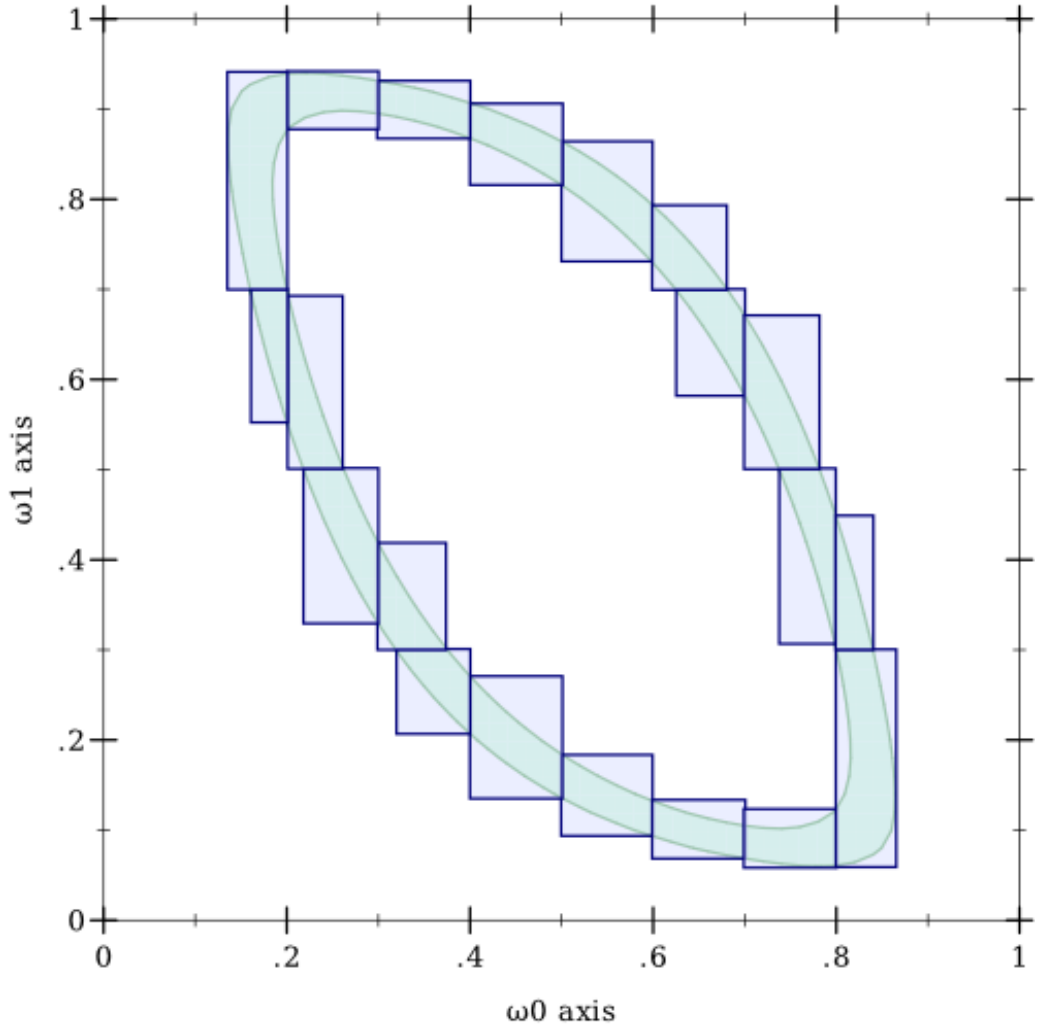

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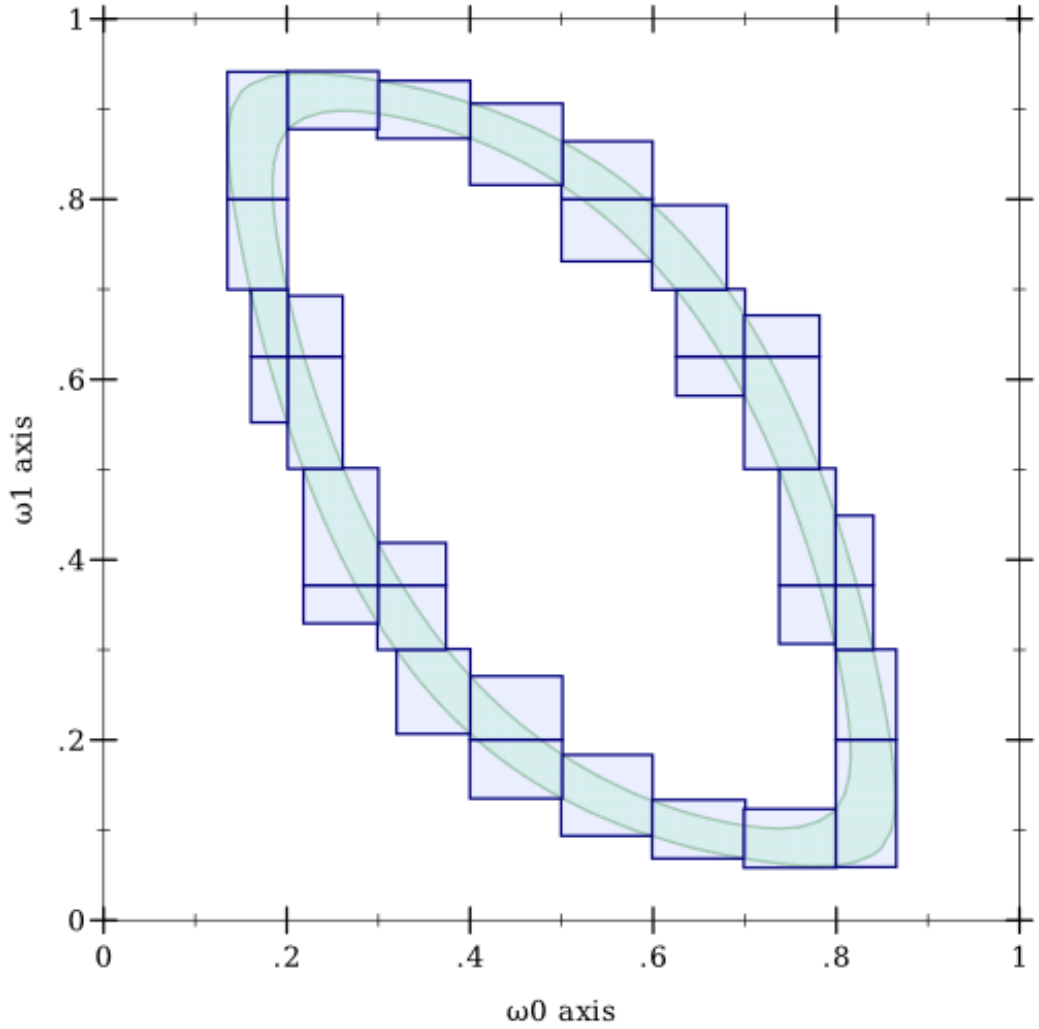

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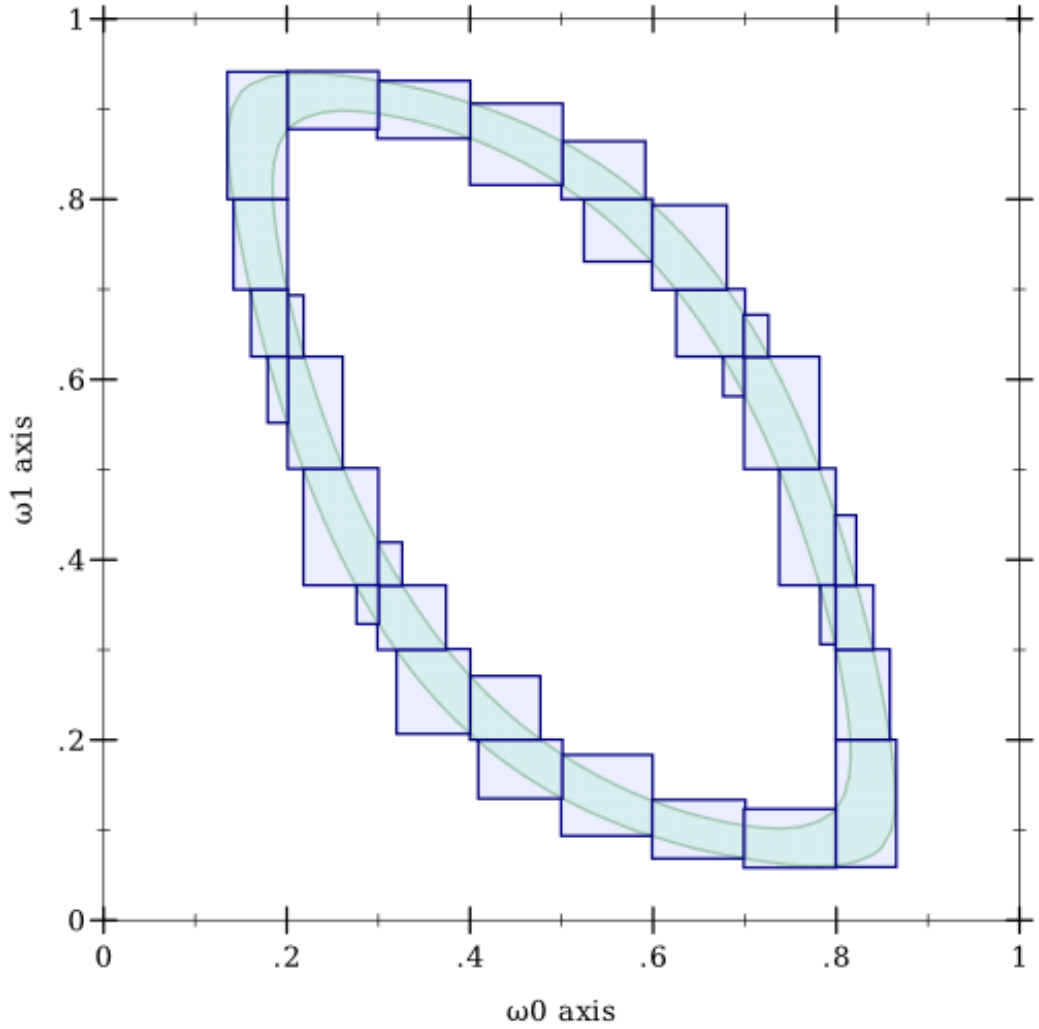

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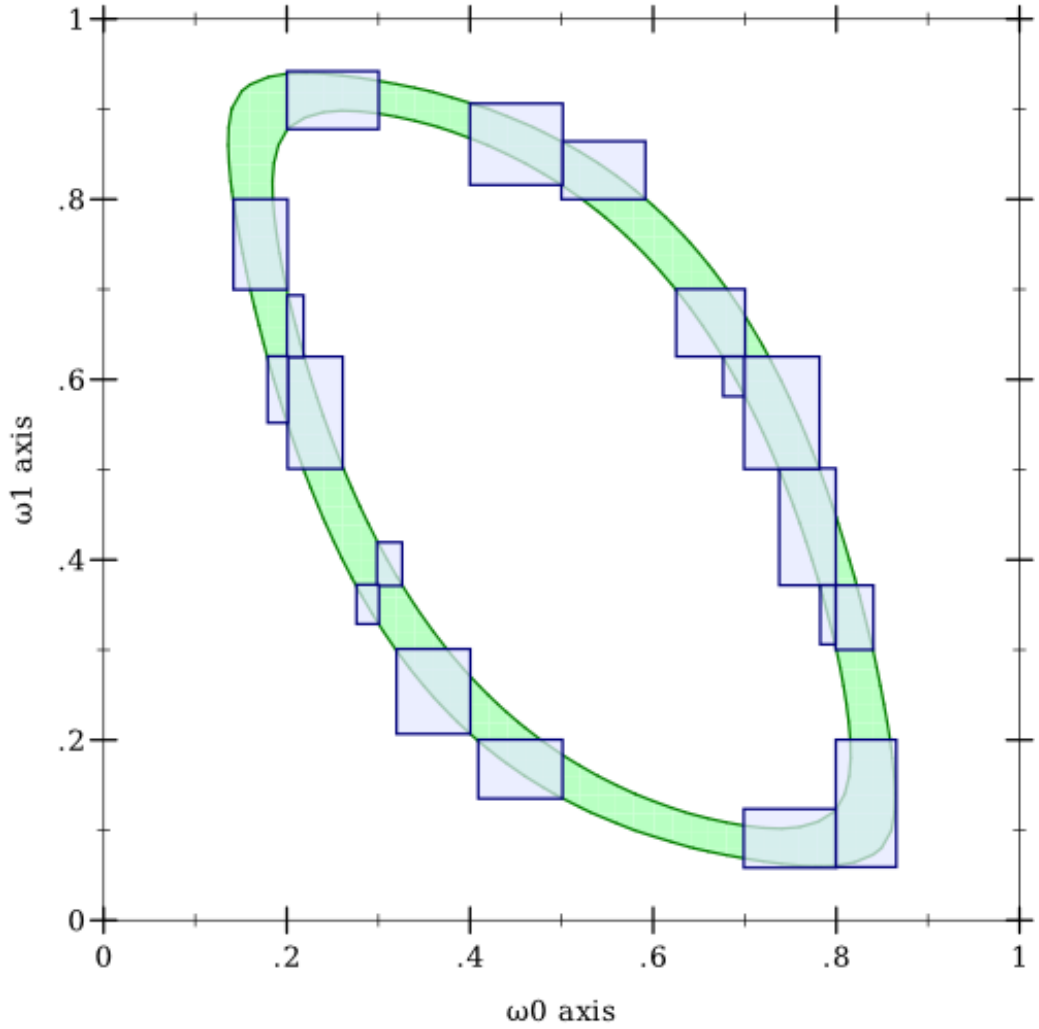

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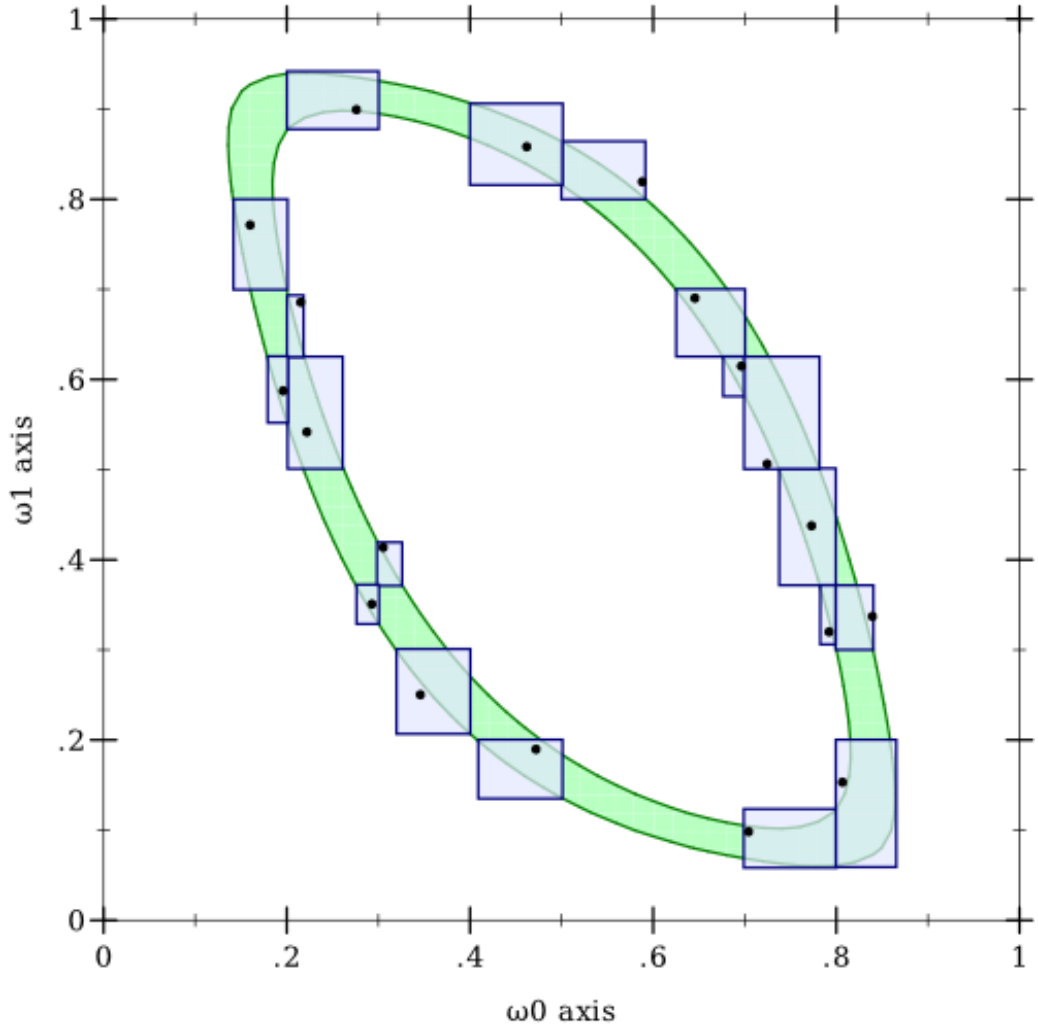

Crazy Idea is Actually Feasible If...

- Standard interpretation of programs as pure functions from a random source
- Efficient way to compute preimage sets
- Efficient representation of arbitrary sets
- Efficient way to compute volumes of preimage sets

- Proof of correctness w.r.t. standard interpretation

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Standard Interpretation

- Grammar:

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e & ::= & x\ e \,|\, \text{if}\ e\ e\ e \,|\, \text{let}\ e\ e \,|\, \text{env}\ n \,|\, \langle e, e \rangle \,|\, \delta\ e \,|\, v \\
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- Easy implementation in any language with lambdas

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- Advantage: proofs about all programs by structural induction
- Can preimages be computed compositionally?

Pair Preimages

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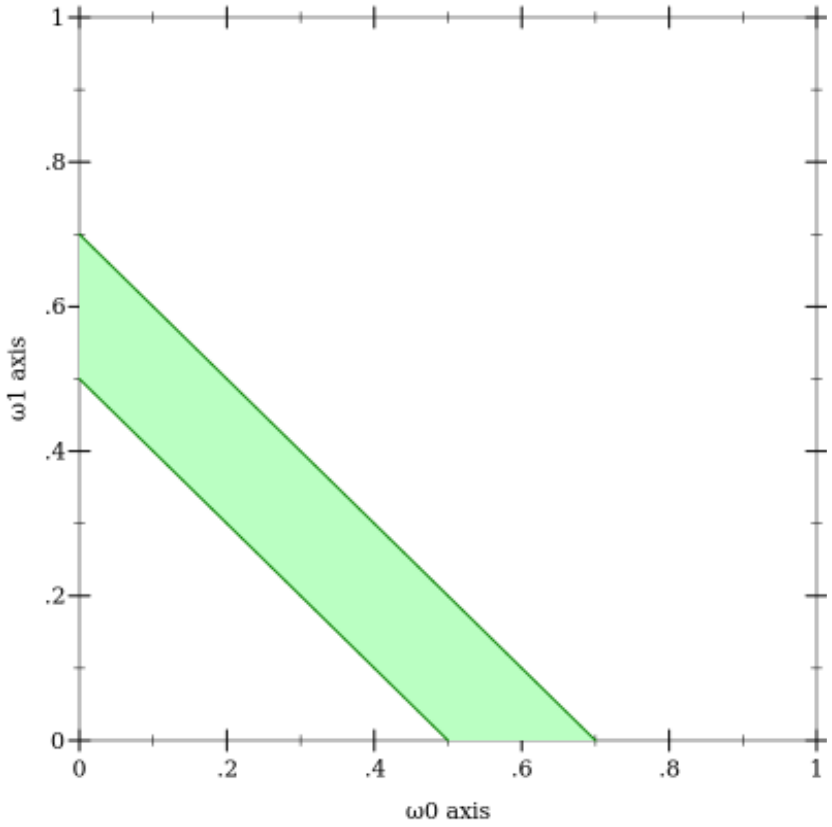

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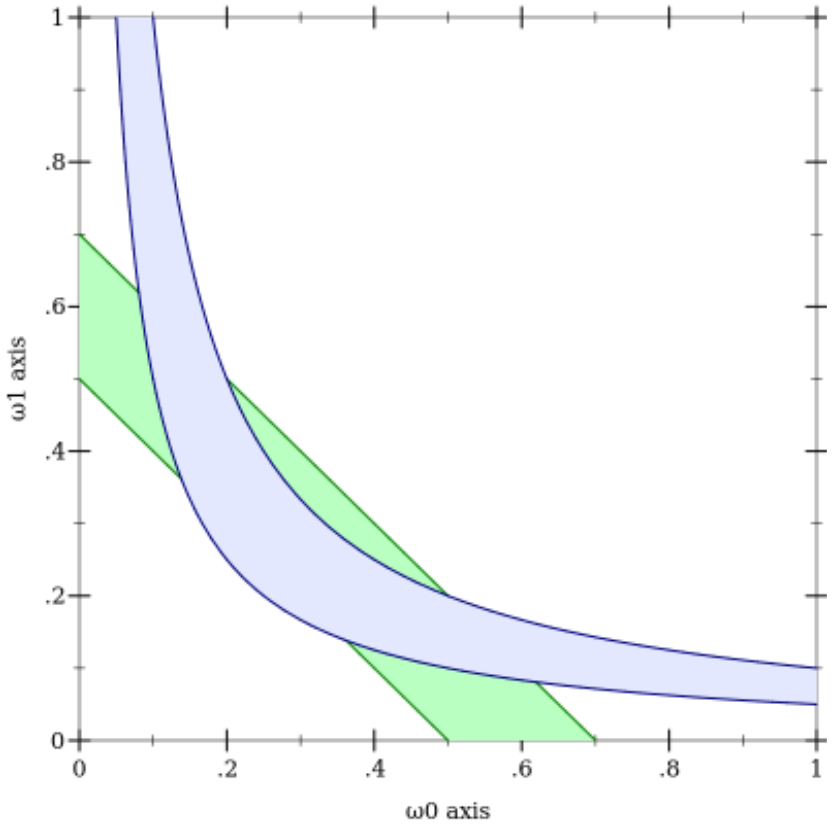
$f_1^{-1}([0.5, 0.7])$ and $f_2^{-1}([0.05, 0.1])$:



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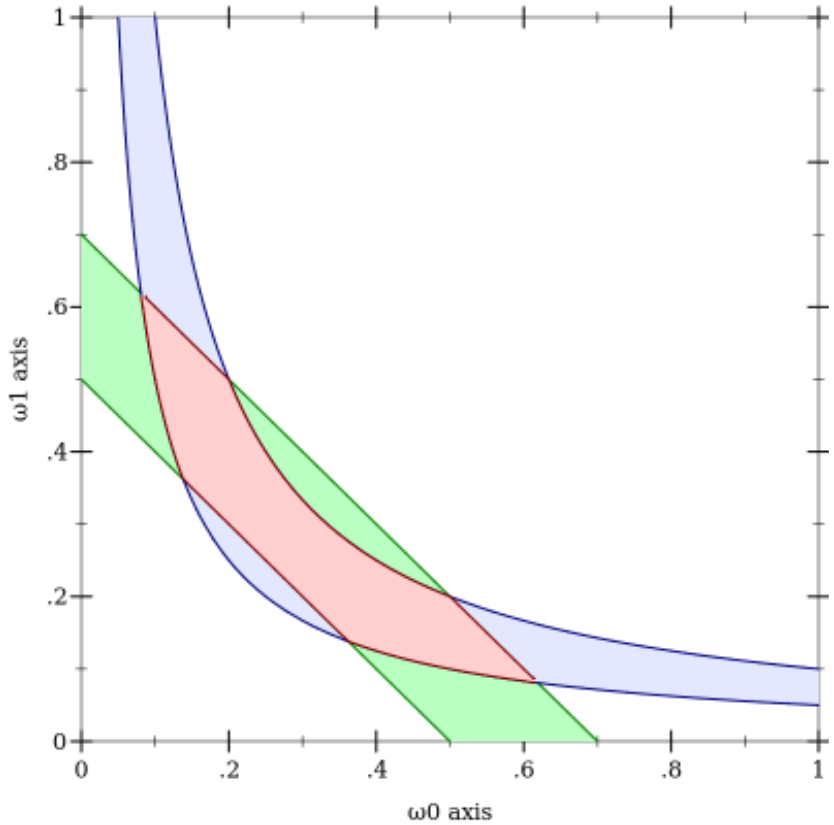
$f_1^{-1}([0.5, 0.7])$ and $f_2^{-1}([0.05, 0.1])$ and $f^{-1}([0.5, 0.7] \times [0.05, 0.1])$:



Nonstandard Interpretation: Computing Preimages

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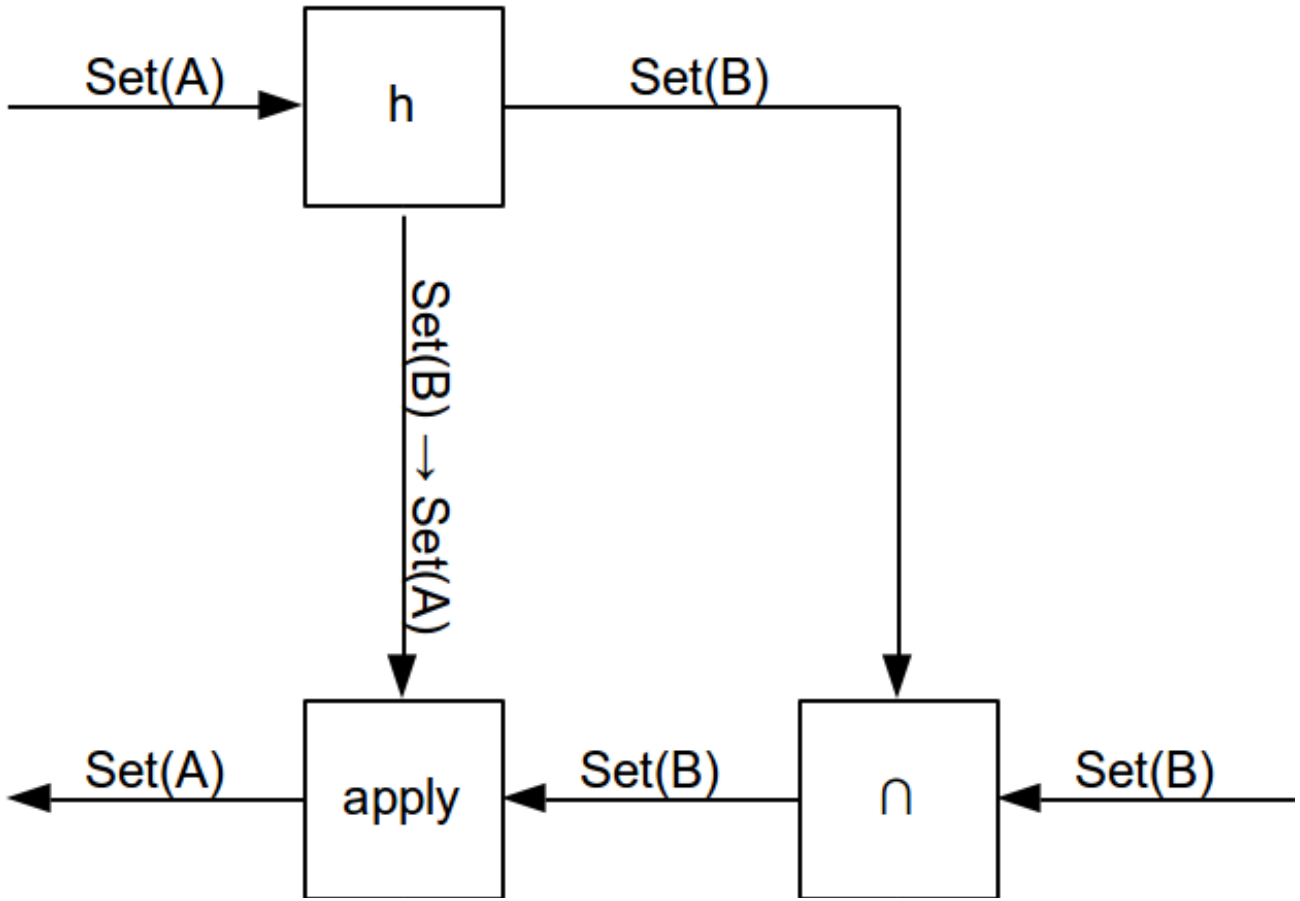

Nonstandard Interpretation: Preimages Under Pairing

- Pairing types:

$$\mathrm{pair} : (A \to B_1) \to (A \to B_2) \to (A \to \langle B_1, B_2 \rangle)$$

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- Similar theorems for every kind of expression

Nonstandard Interpretation: Correctness

Theorem. For all programs p, $[\![p]\!]_{\mathrm{pre}}$ computes preimages under $[\![p]\!]$.

Proof. By structural induction on program terms.

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- Q. Where do I get a computer that runs them?
  - A. Nowhere, but we'll approximate them soon.
- Q. Where did you get a λ-calculus that could operate on arbitrary, possibly infinite sets, anyway?
  - A. Well...

Lambda-ZFC

N. Toronto and J. McCarthy. Computing in Cantor's Paradise With λ_{ZFC}. FLOPS 2012

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λ calculus

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Infinite sets and operations

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λ calculus + Infinite sets and operations = λ_{ZFC}

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- Can express uncountably infinite operations, can't solve its own halting problem
- Can use contemporary mathematical theorems directly

Rectangular Approximation

- A *rectangle* is
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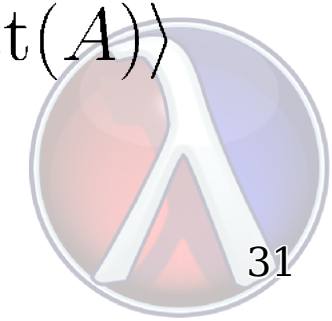

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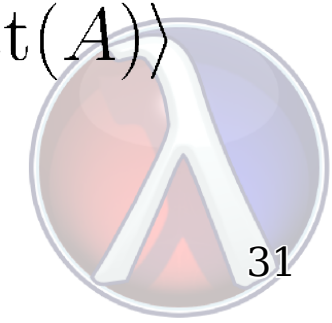

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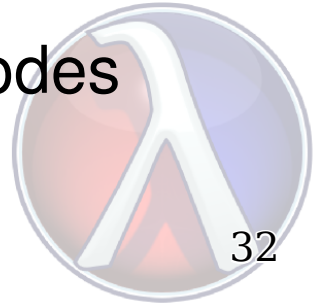
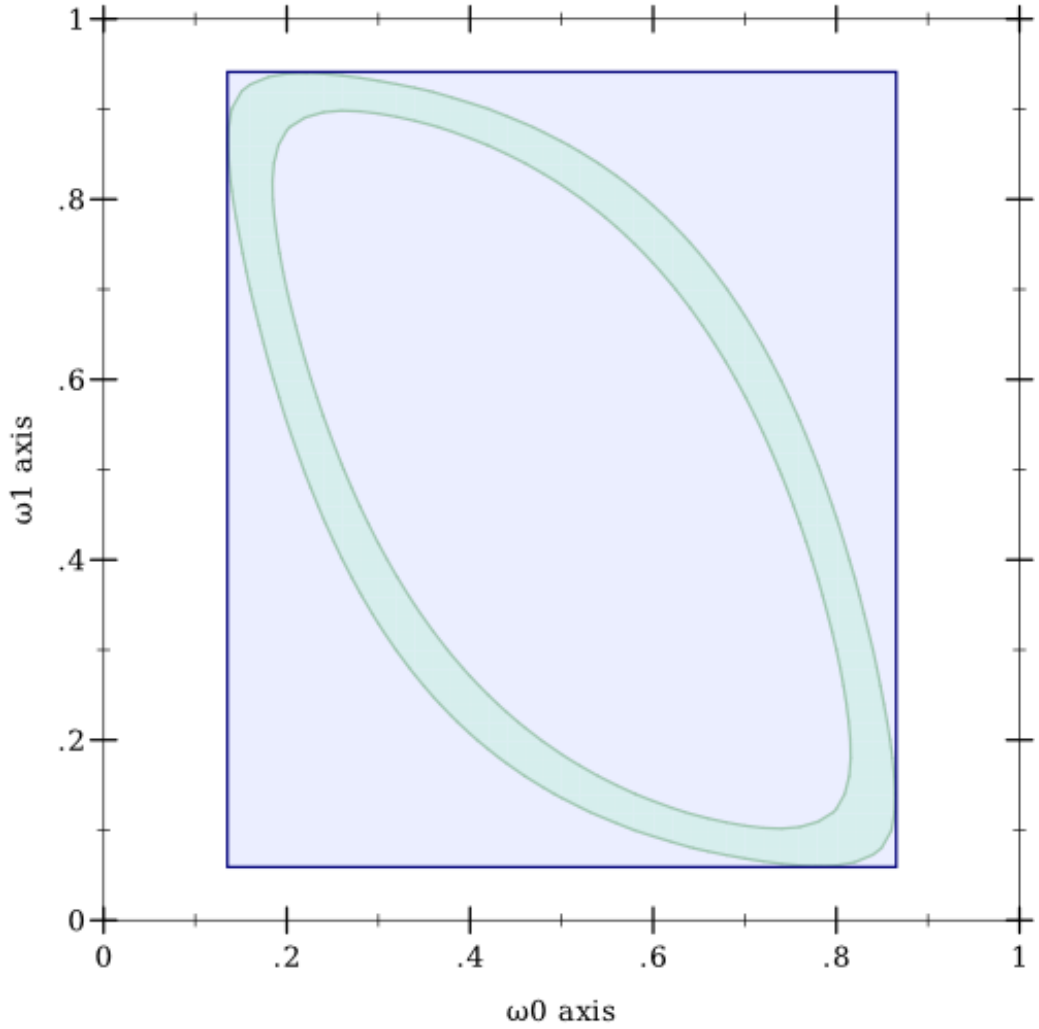
Theorem (decreasing). $[\![\cdot]\!]'_{\mathrm{pre}}$ never returns preimages larger than the given subdomain.

- Consequence: Refining preimage partitions never explodes



In Practice...

Theorems prove this always works:



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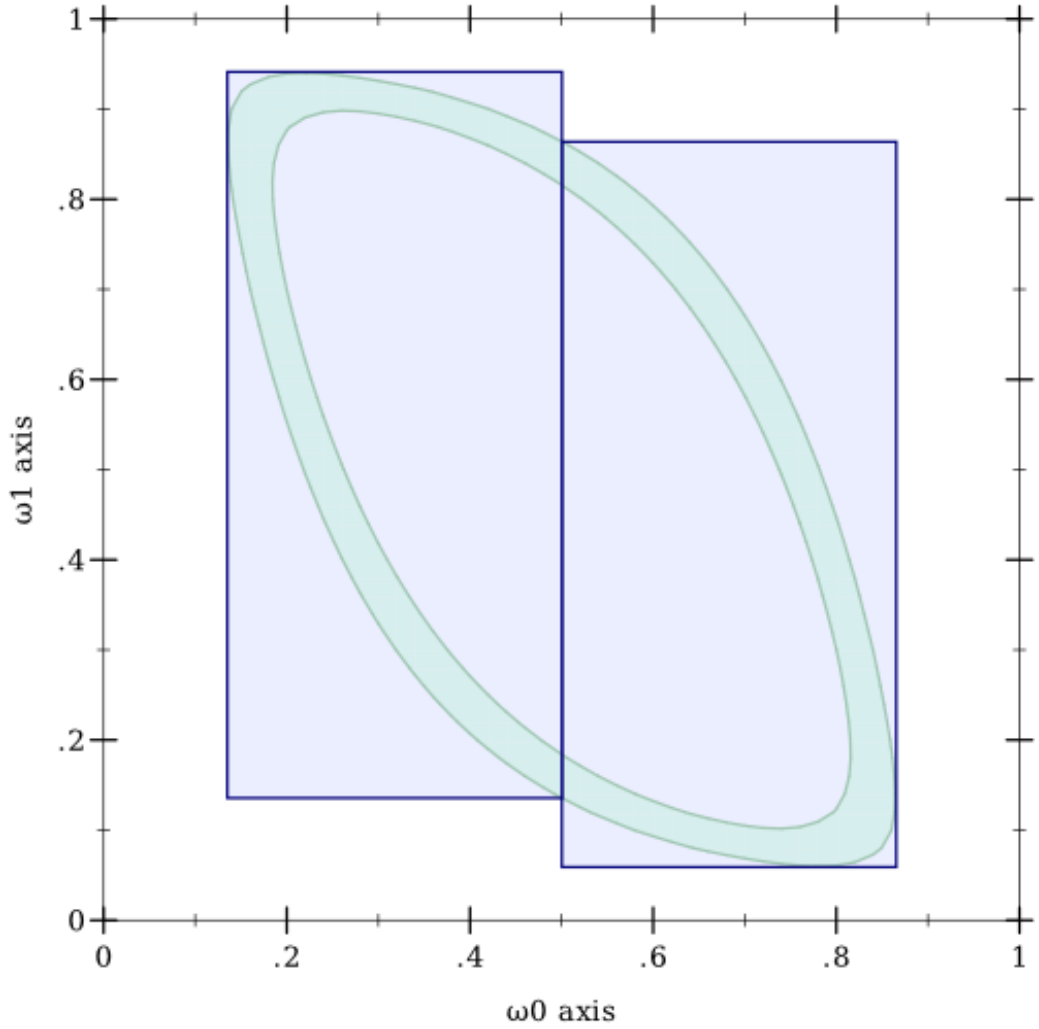

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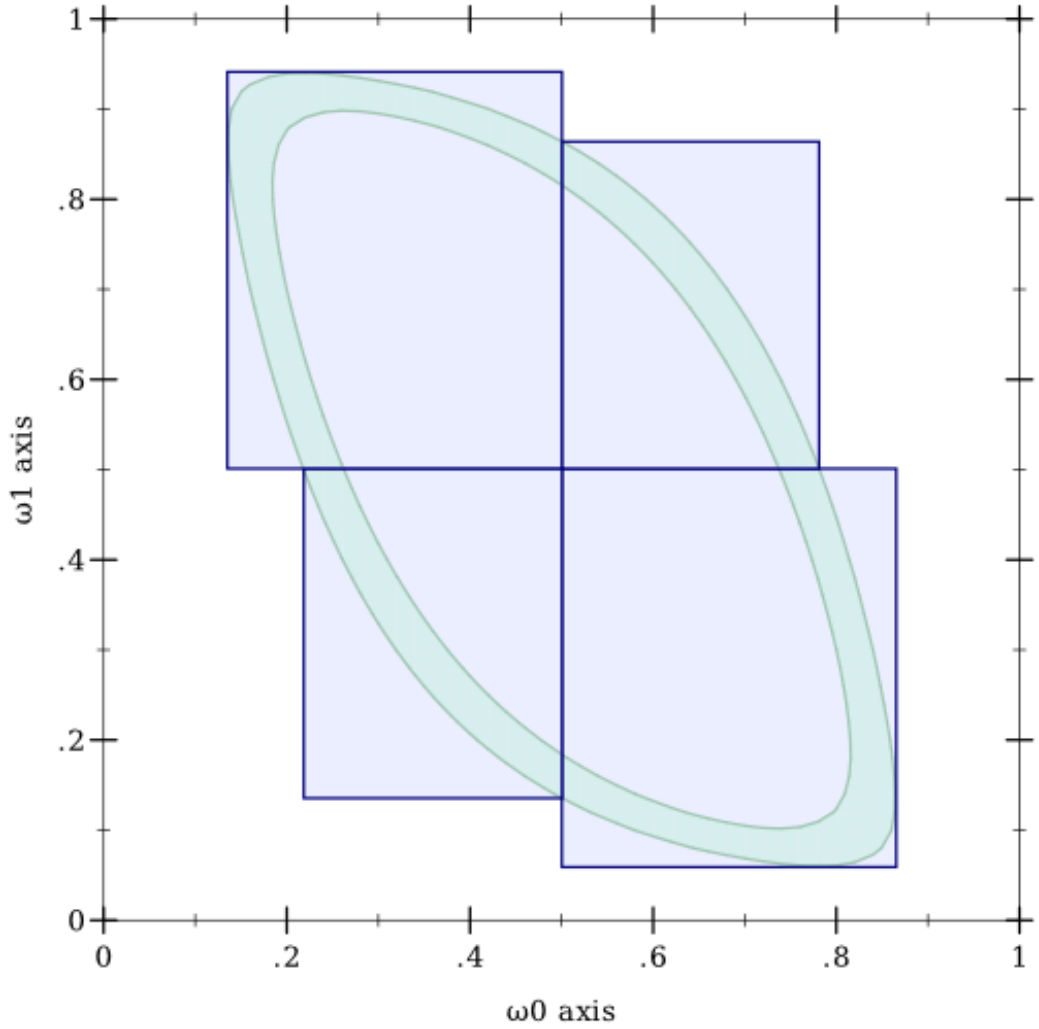

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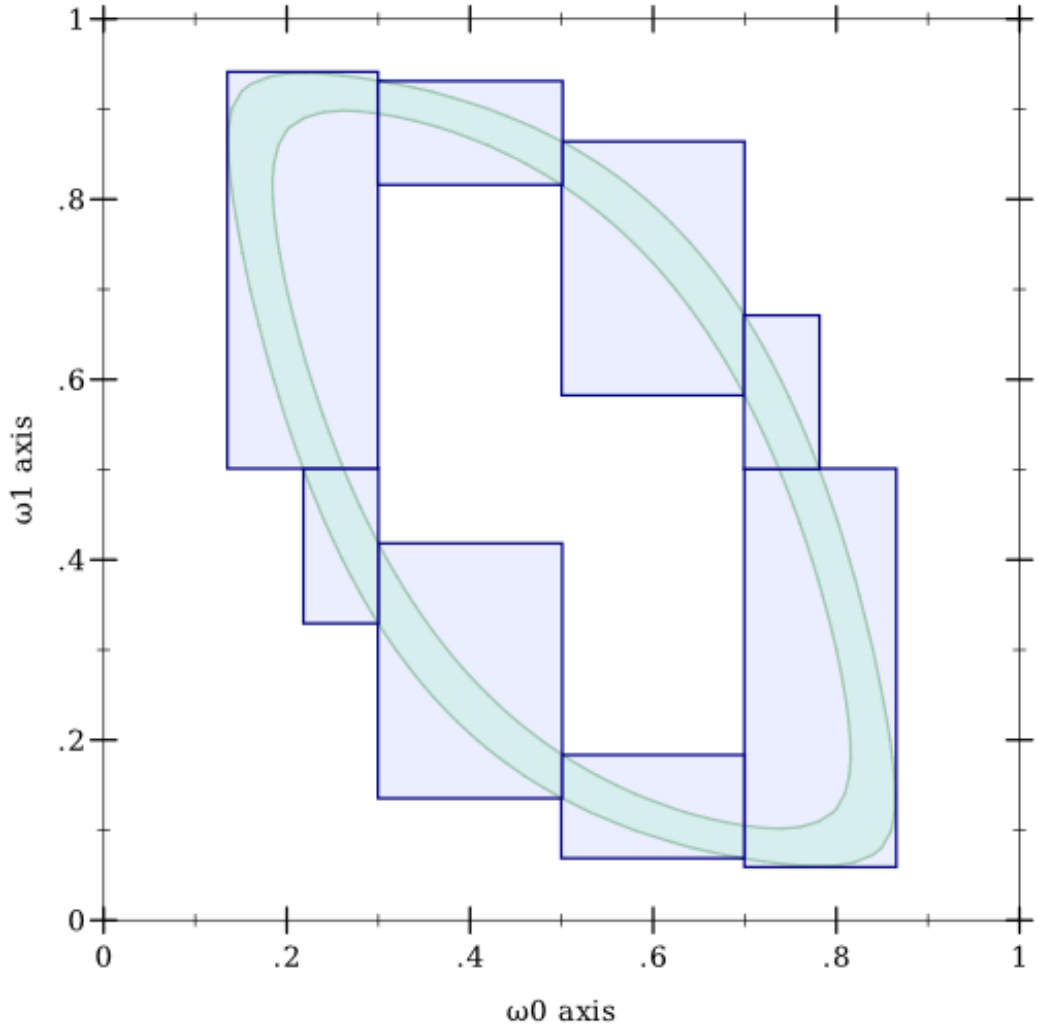

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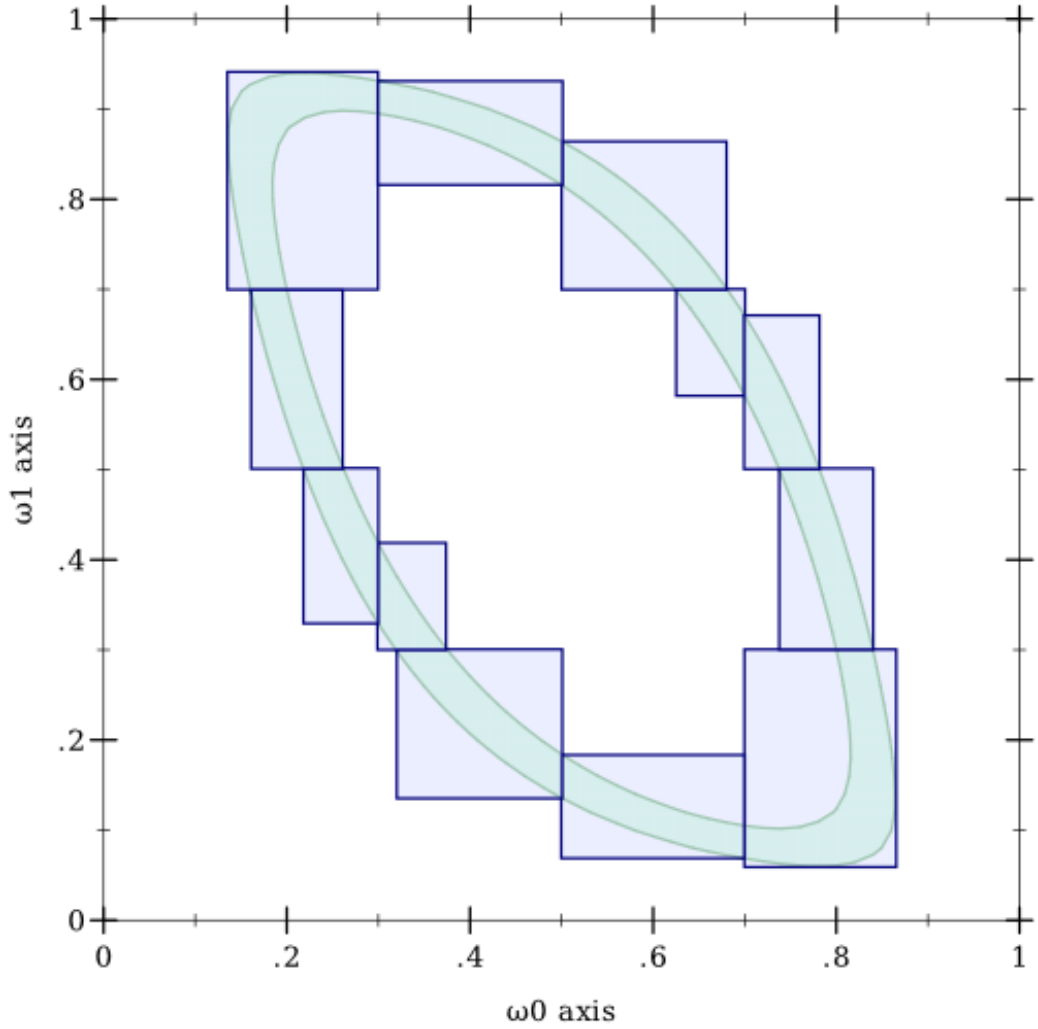

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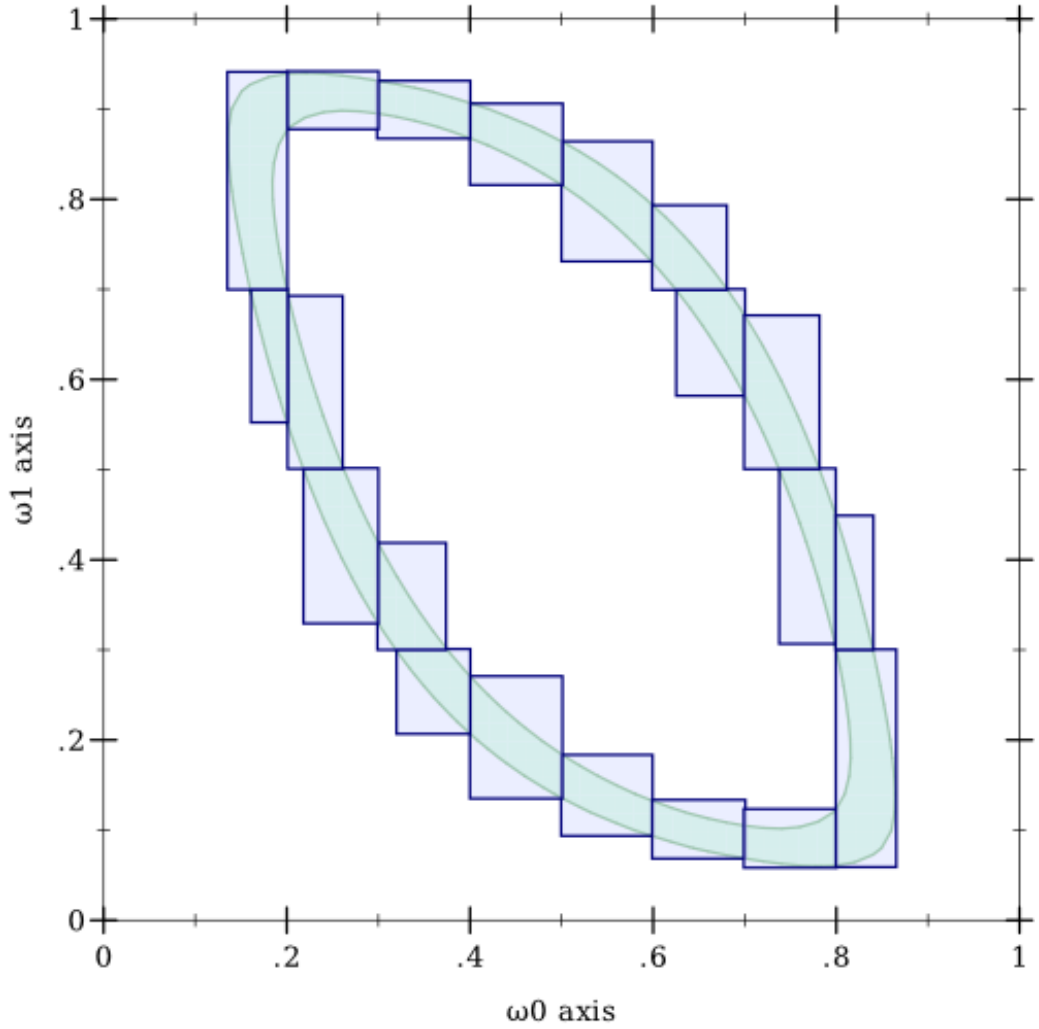

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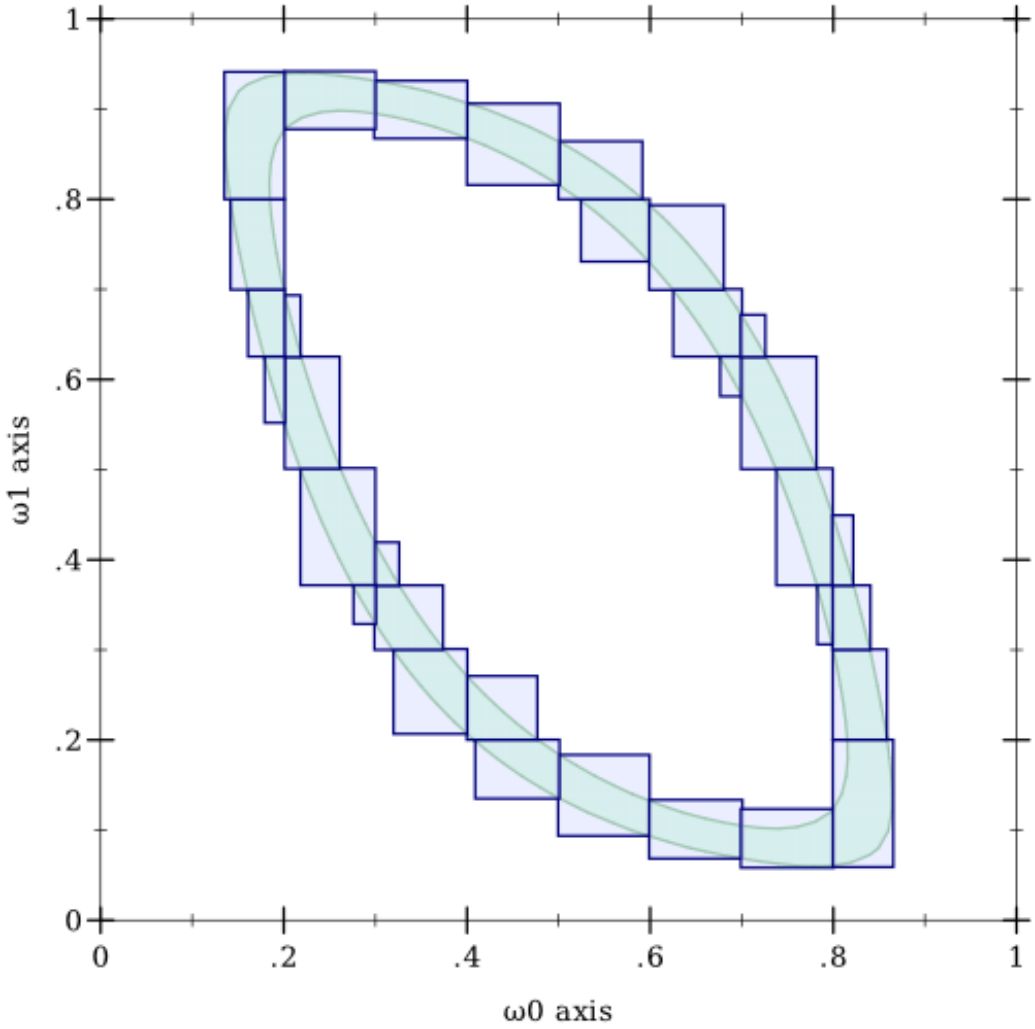

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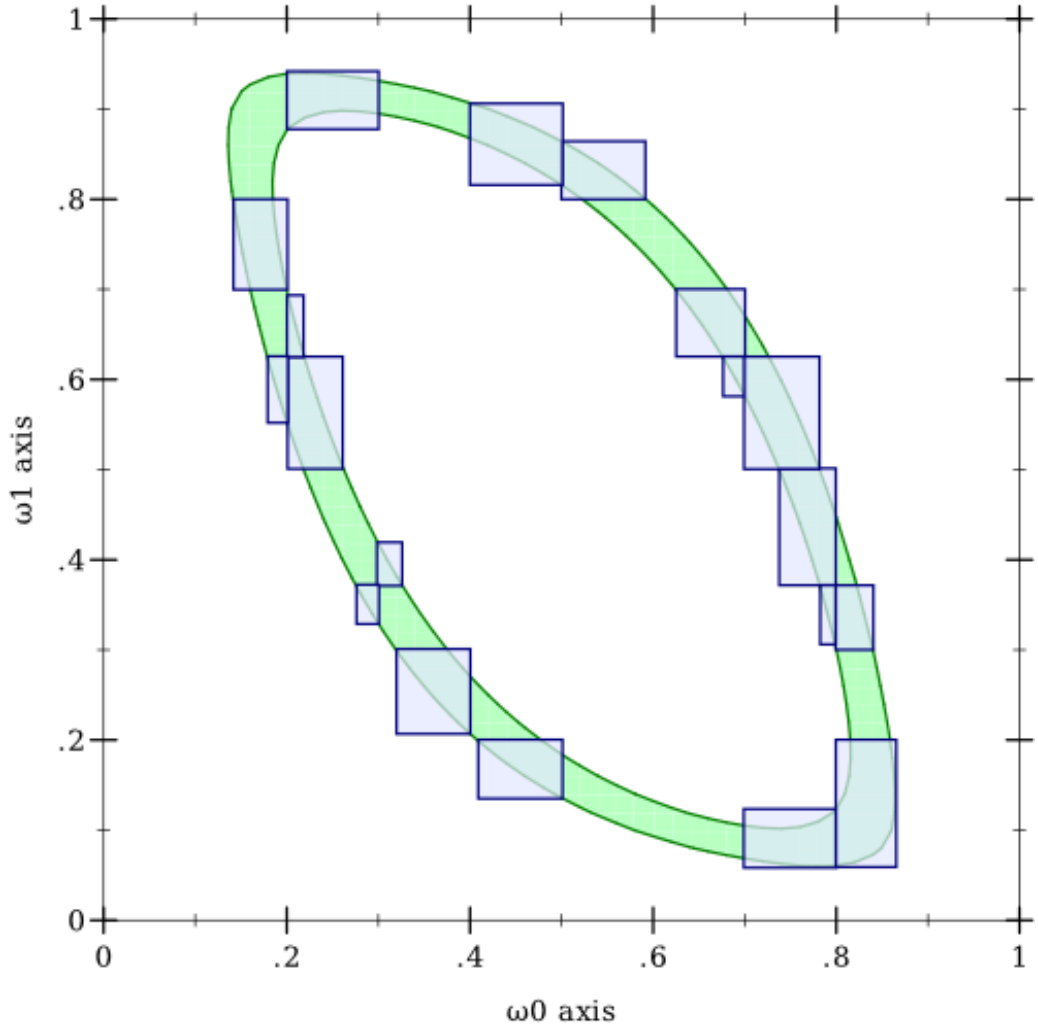

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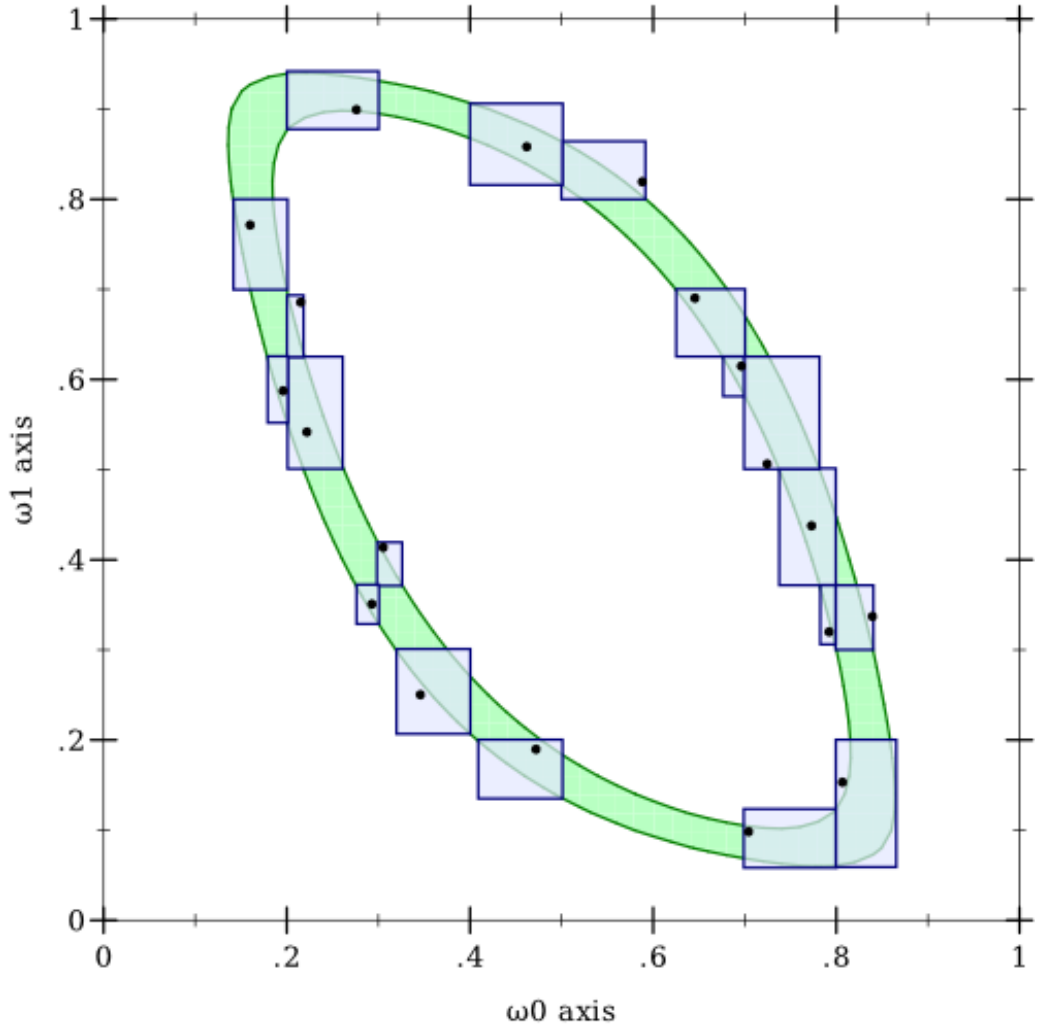

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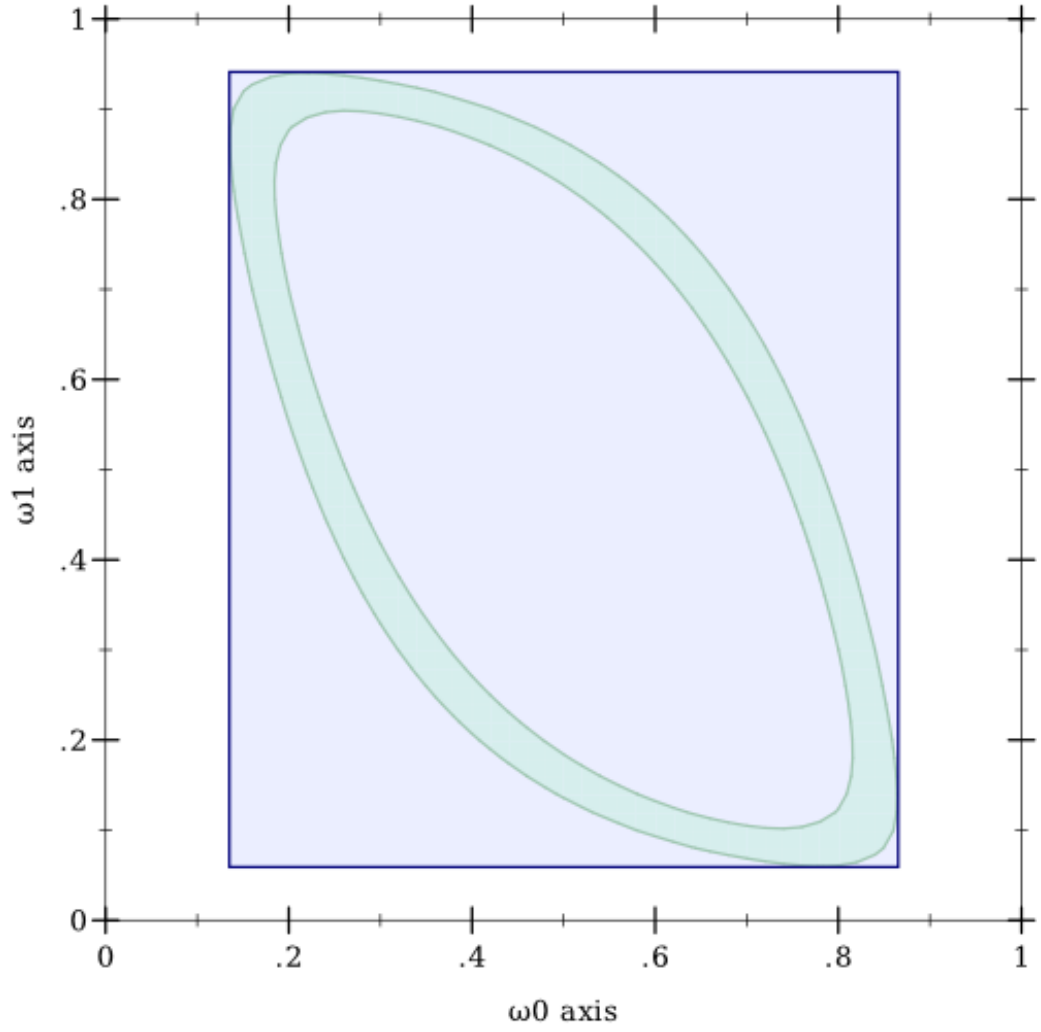
First, refine using preimage computation:



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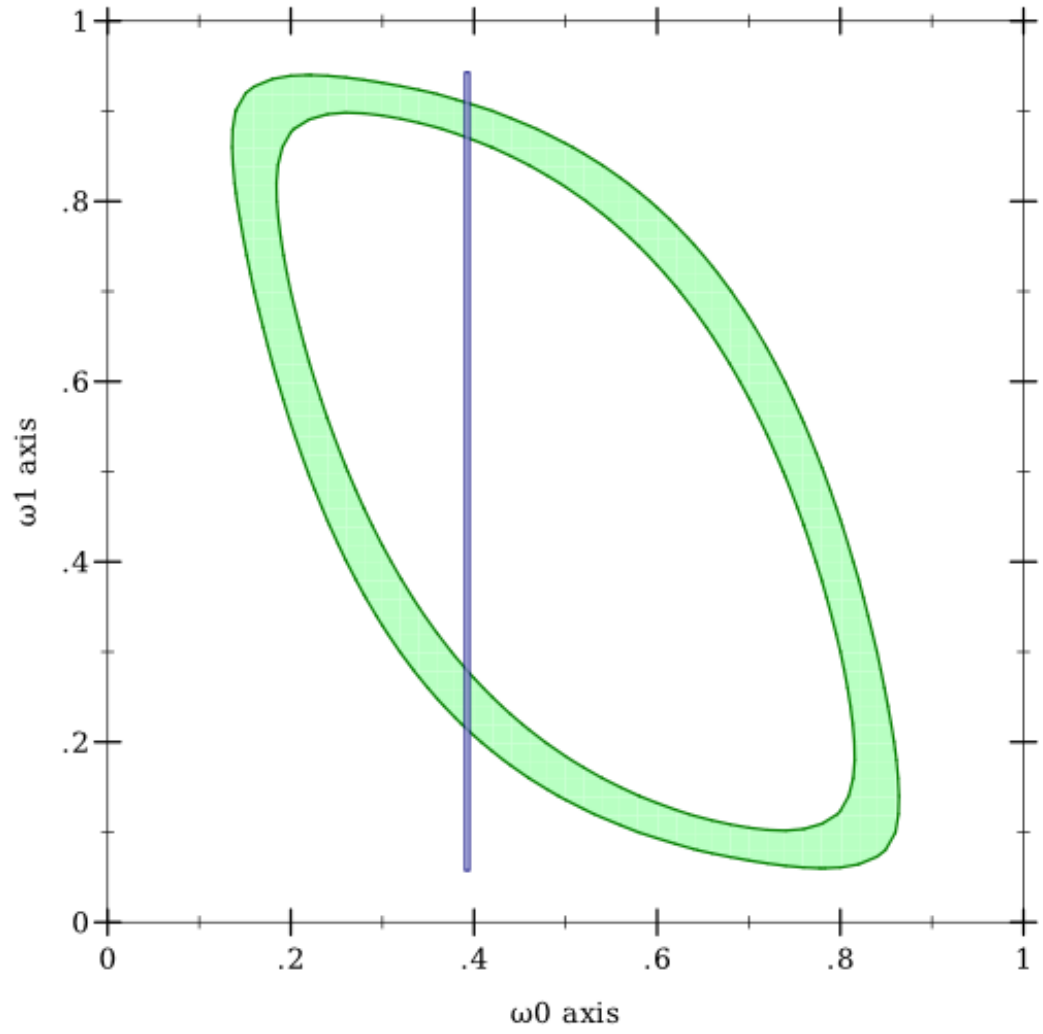
Second, randomly choose from arbitrarily fine partition:



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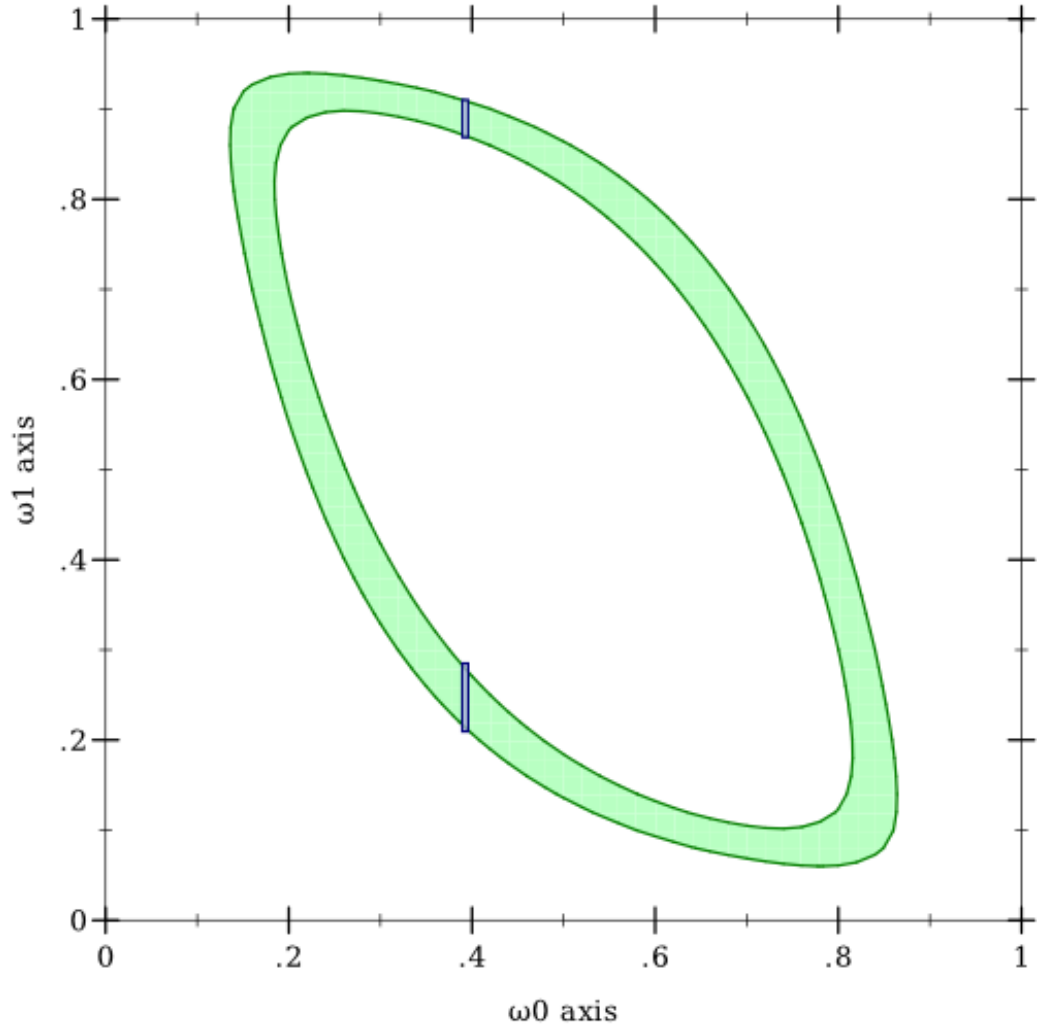
Third, refine again:



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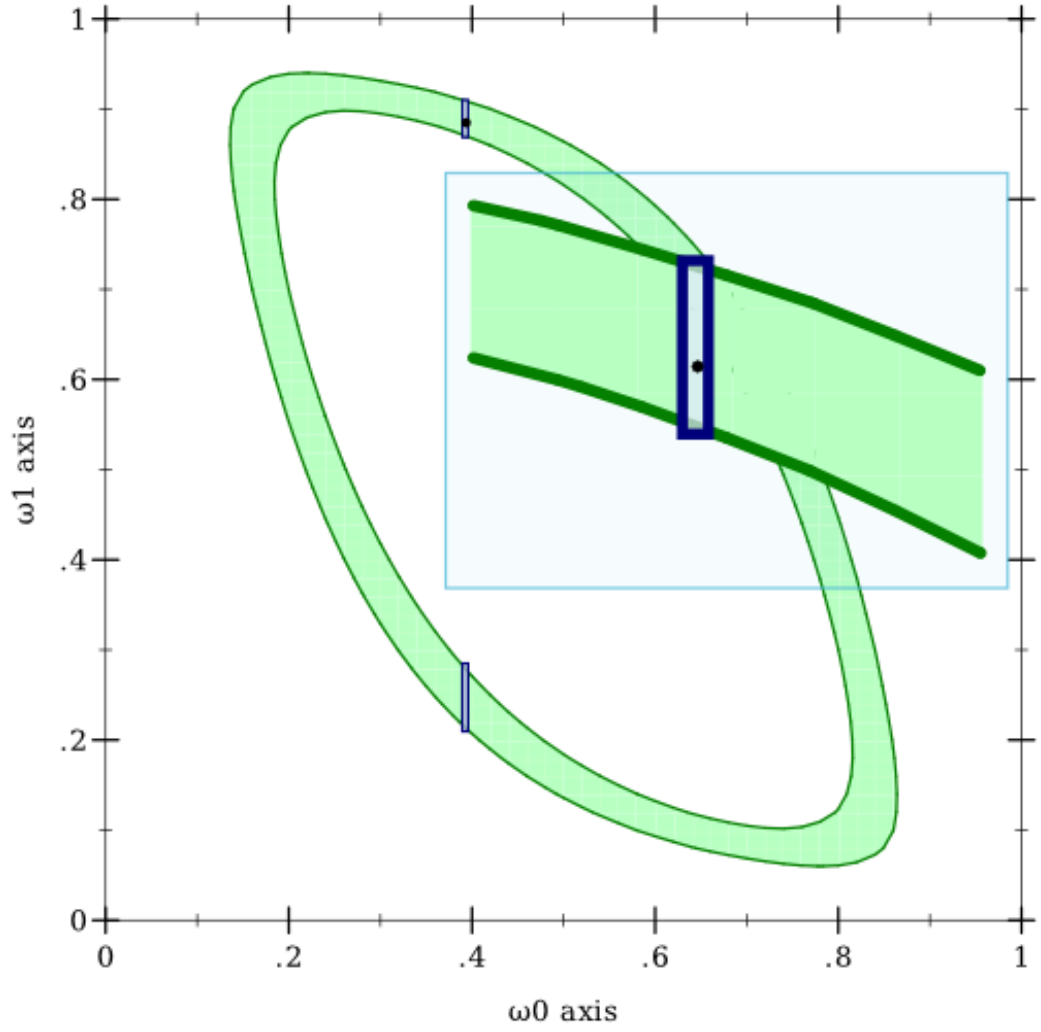
Fourth, sample uniformly:



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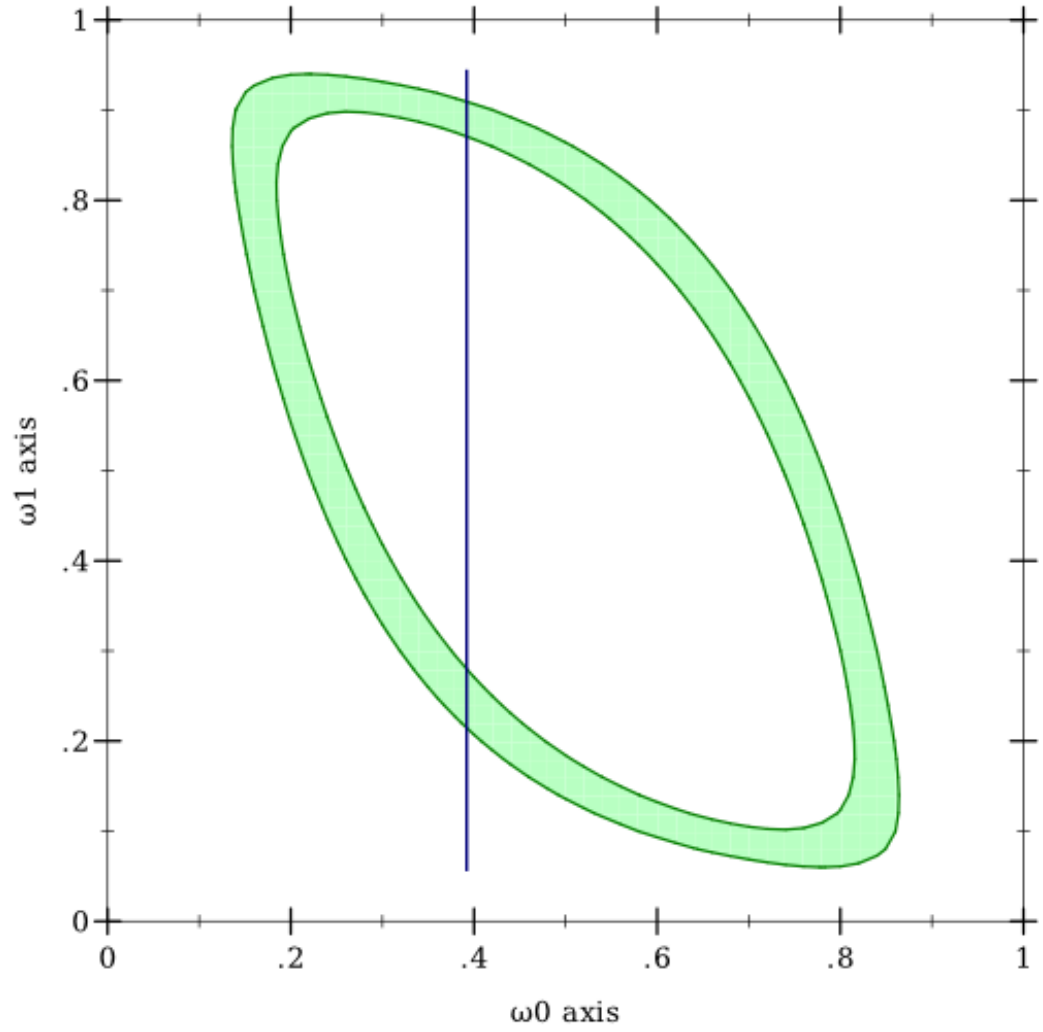
Do process “in the limit”; i.e. choose $[\omega_0, \omega_0] \times \Omega_1$:



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- Or a bound on its implementations' error to overapproximate upward- and downward-rounded implementations:

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> (ivl (norm-inv-cdf- 0.8) (norm-inv-cdf+ 0.8))
(ivl 0.8416212335729142 0.8416212335729149)
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What About Infinite-Dimensional Models?

- General recursion, programs that halt with probability 1; e.g.

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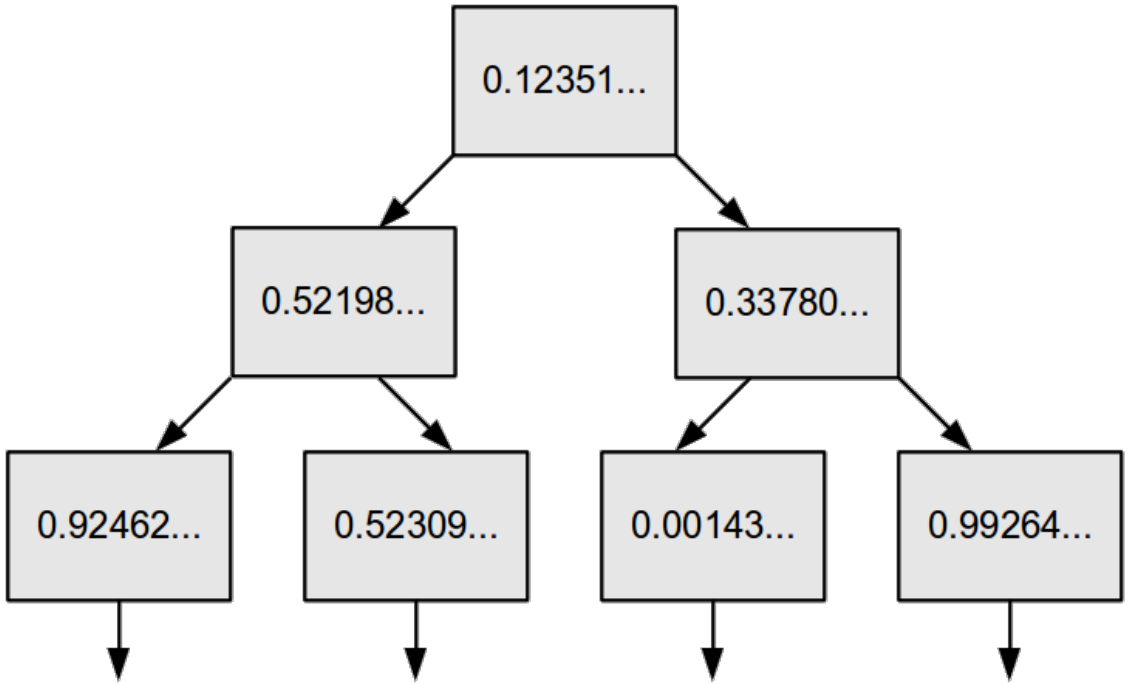
- Random domain: an infinite tree of uniform random numbers

Domain Values

- Values $\omega \in \Omega$ are infinite binary trees:



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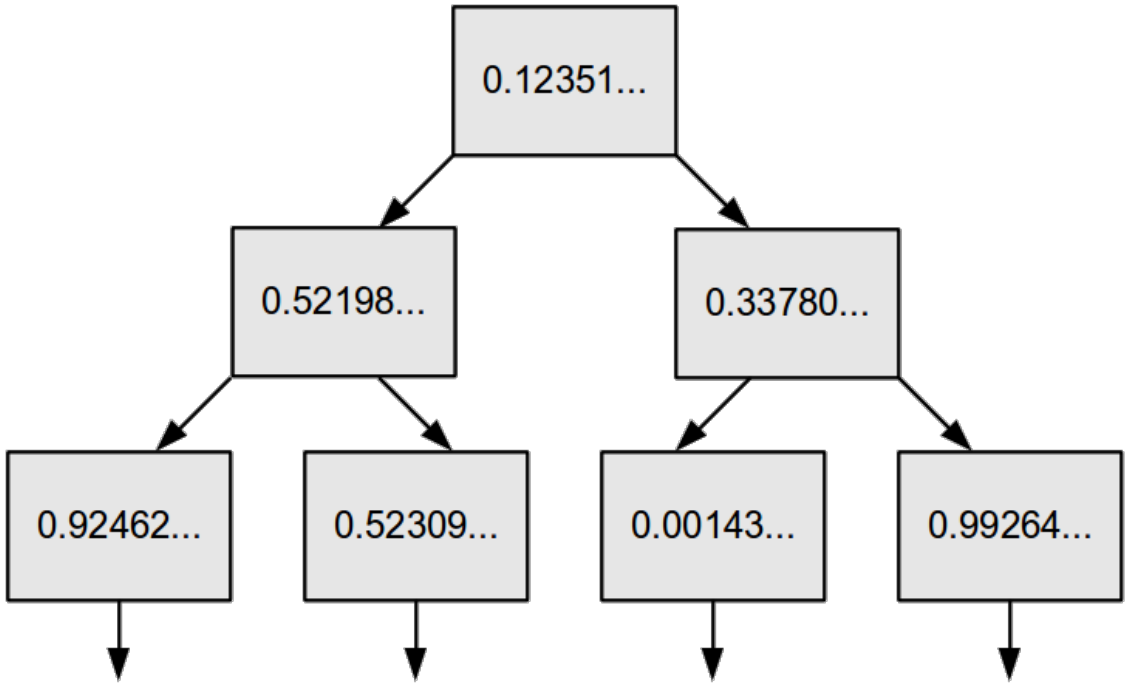

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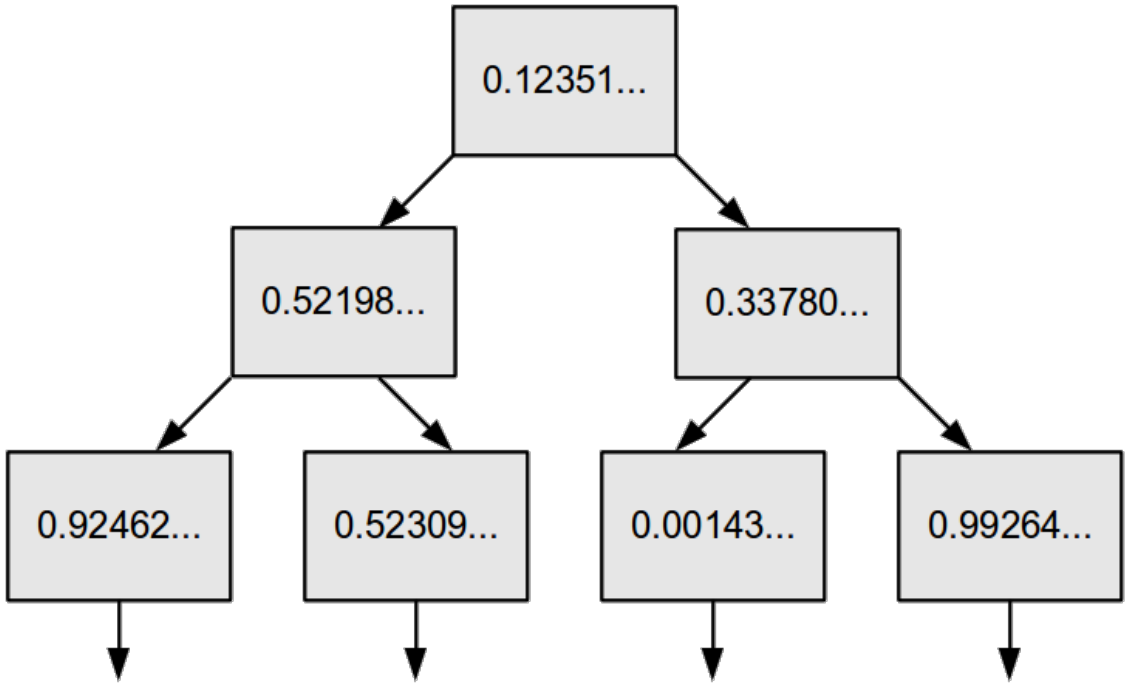

- Implemented using lazy trees of random values
- No probability *density* for this, but there is a *measure*

Domain Rectangles

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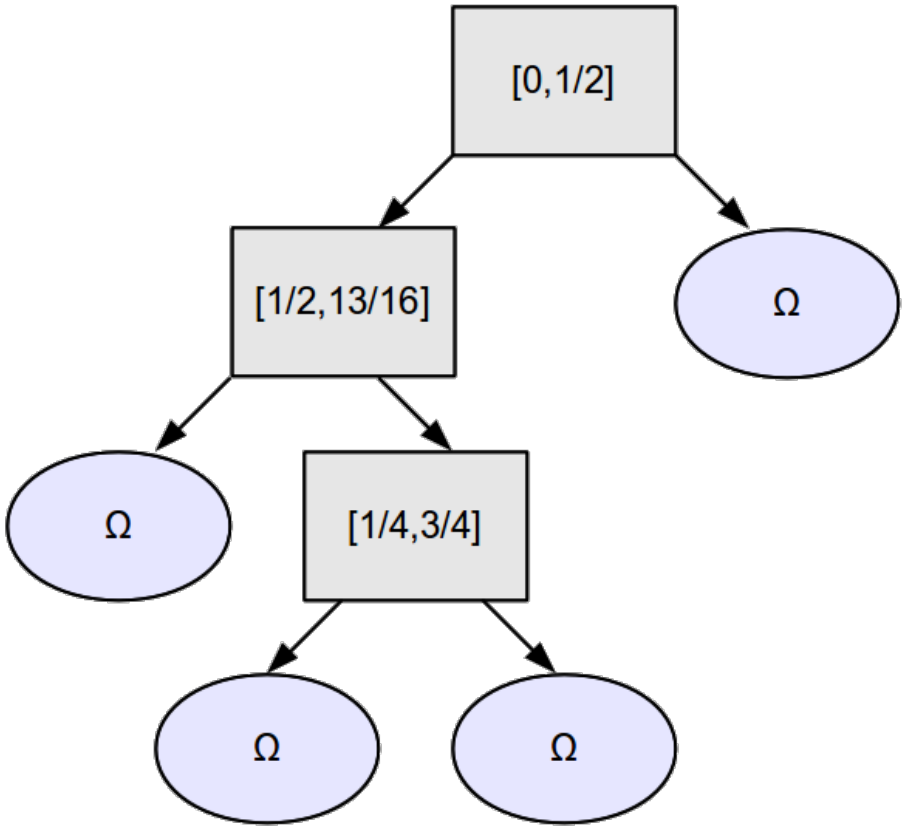

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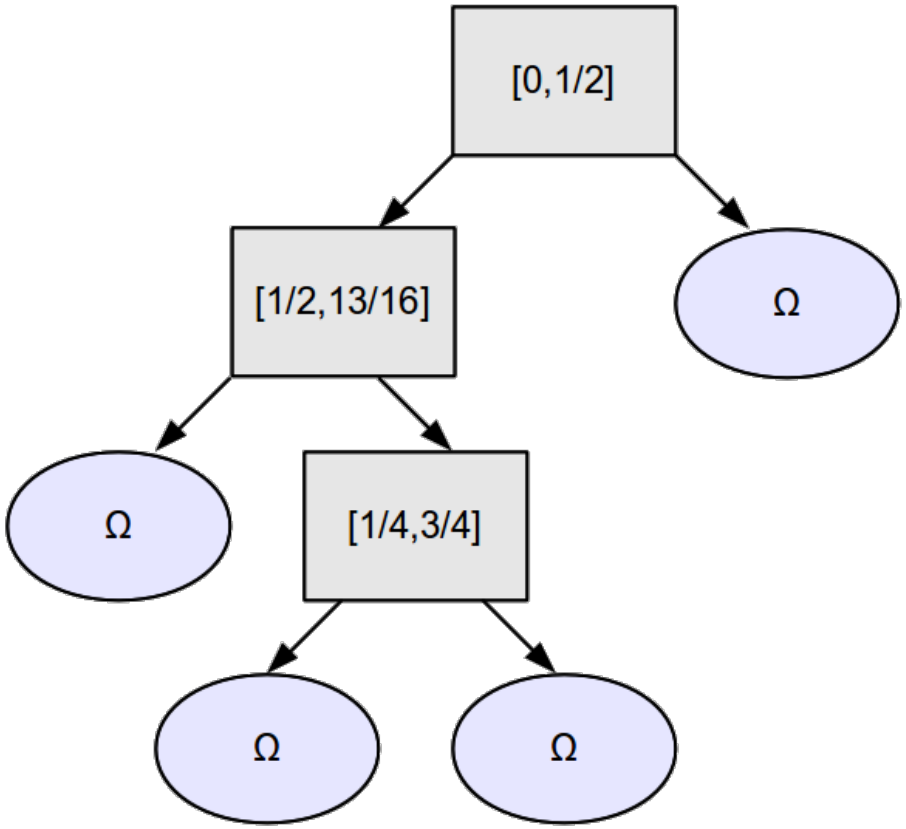

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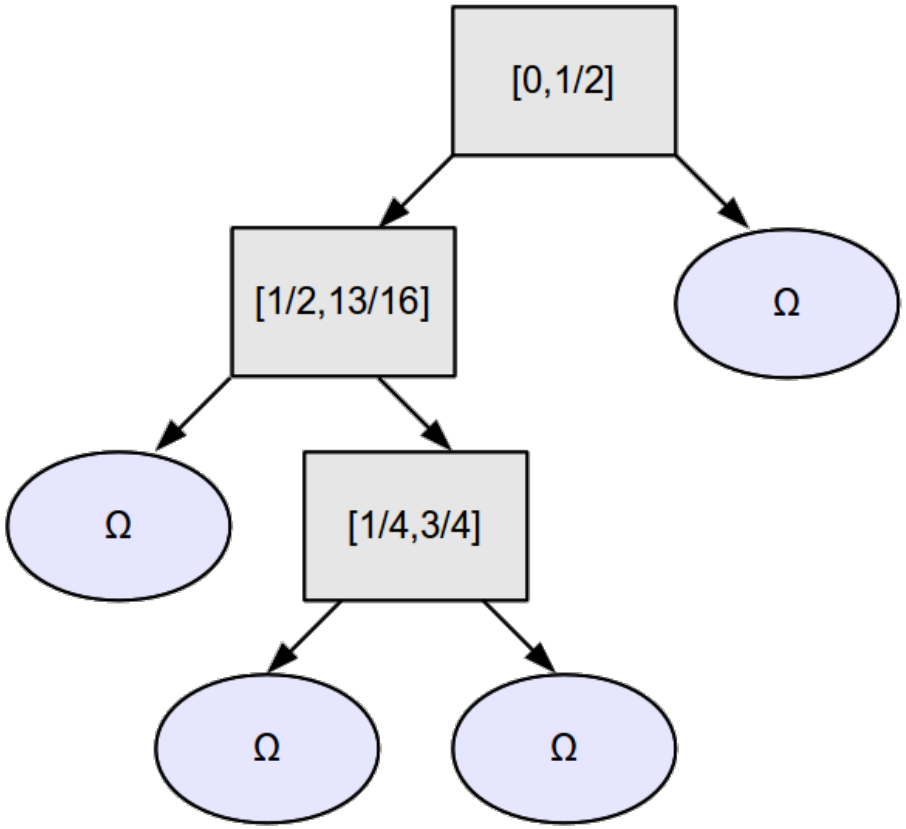

- Only finitely many axes are not $[0, 1]$
- Implemented using finite trees of real sets

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- Solution: Language with measure-theoretic semantics

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  - Directly implementable, given a rectangular set library

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  - Correct for any program, any positive-probability condition (up to floating-point error)

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- Goal: Sample in the preimage of

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(set-list reals reals (interval (- 1.0 ε) (+ 1.0 ε)))
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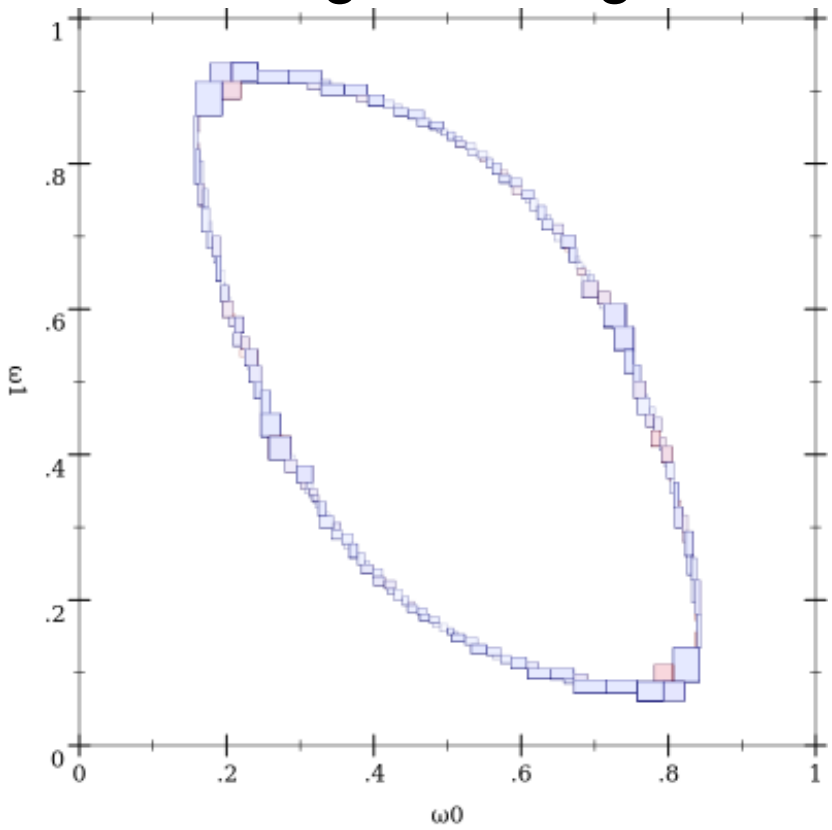
Demo: Normal-Normal With Circular Condition

For ε = 0.01:

Preimage rectangles



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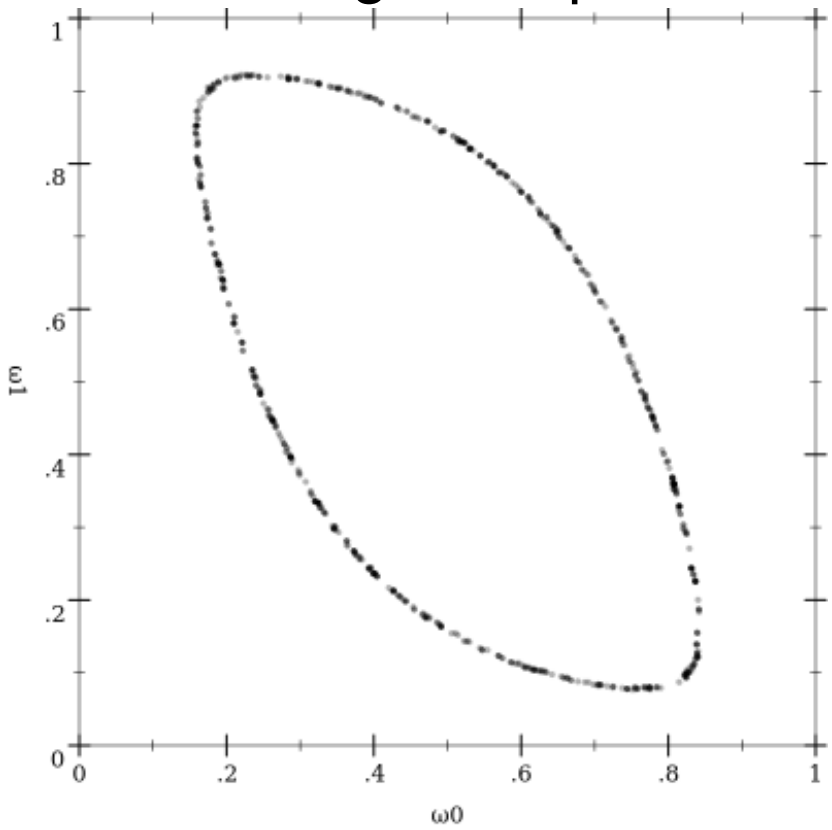

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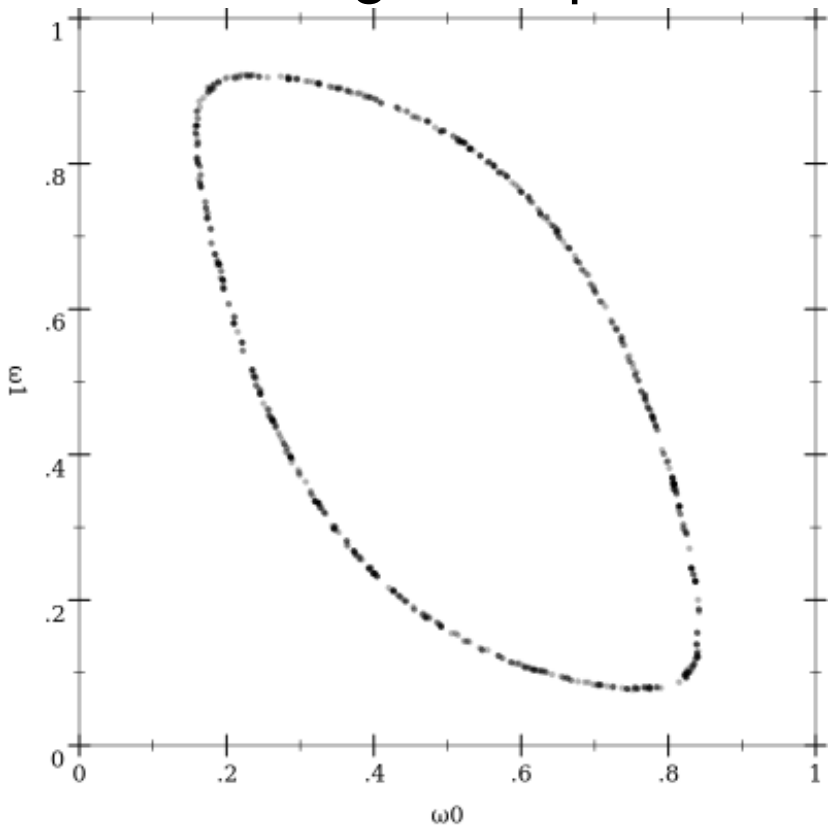
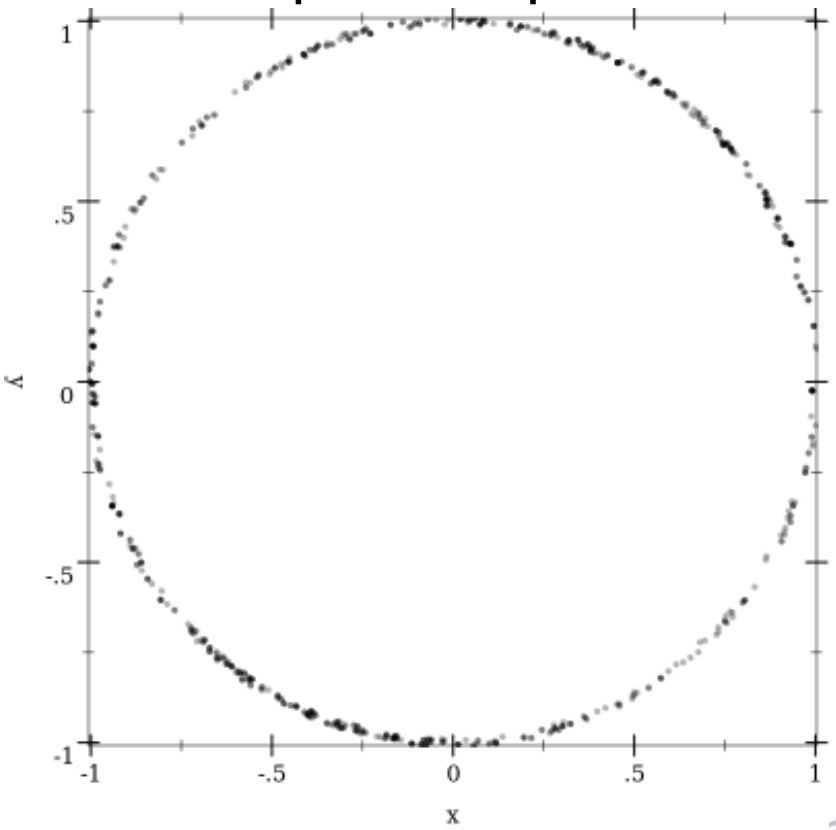


Output samples



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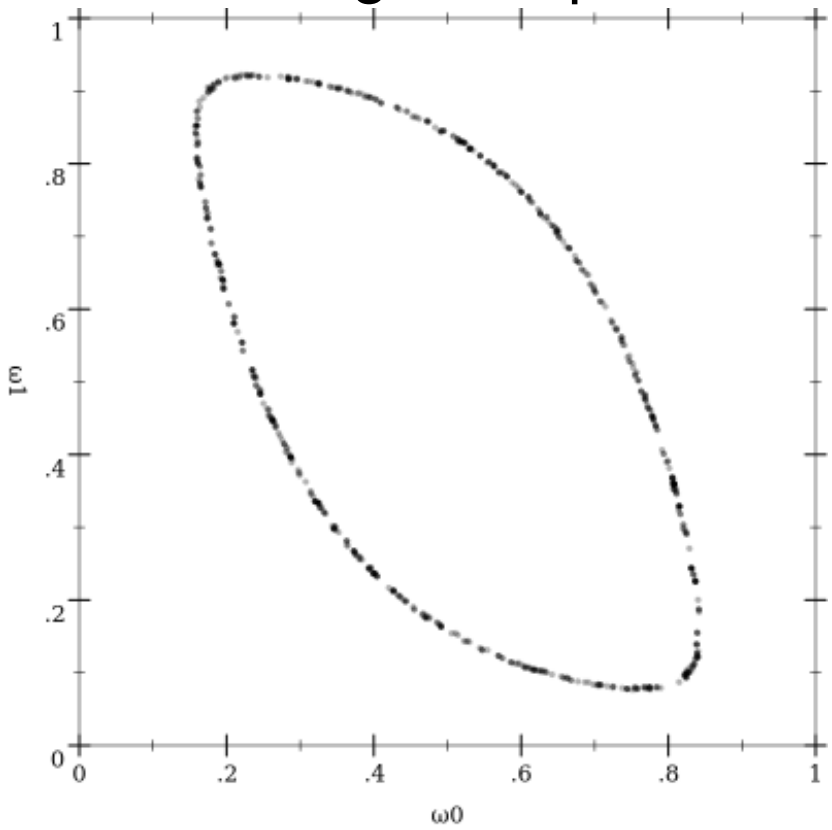

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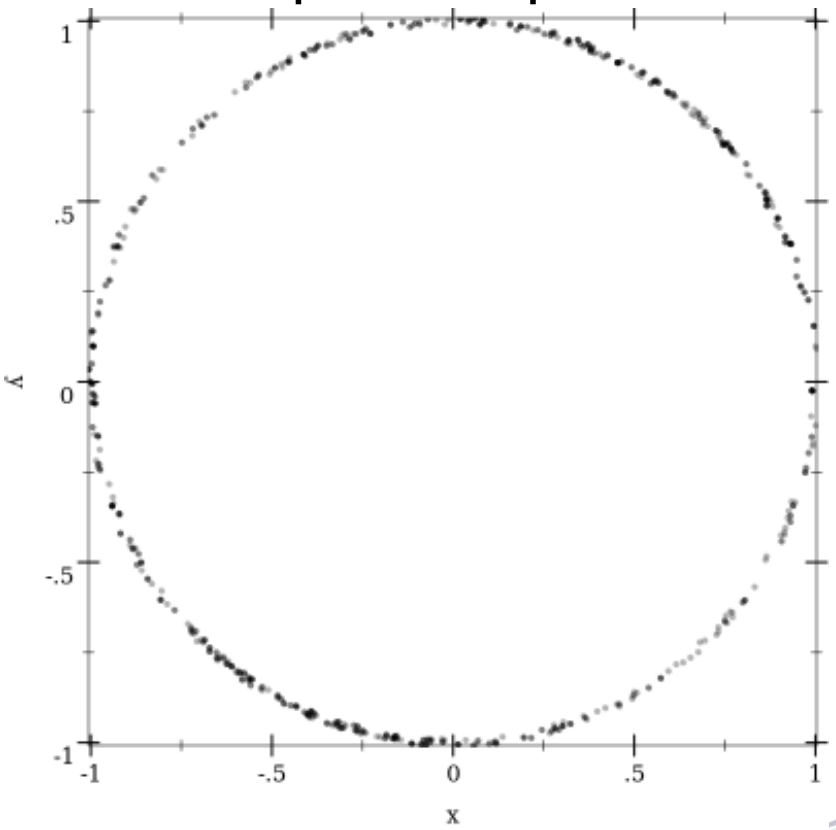

- Works fine with much smaller ε

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(define/drbayes e
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```
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Demo: Thermometer

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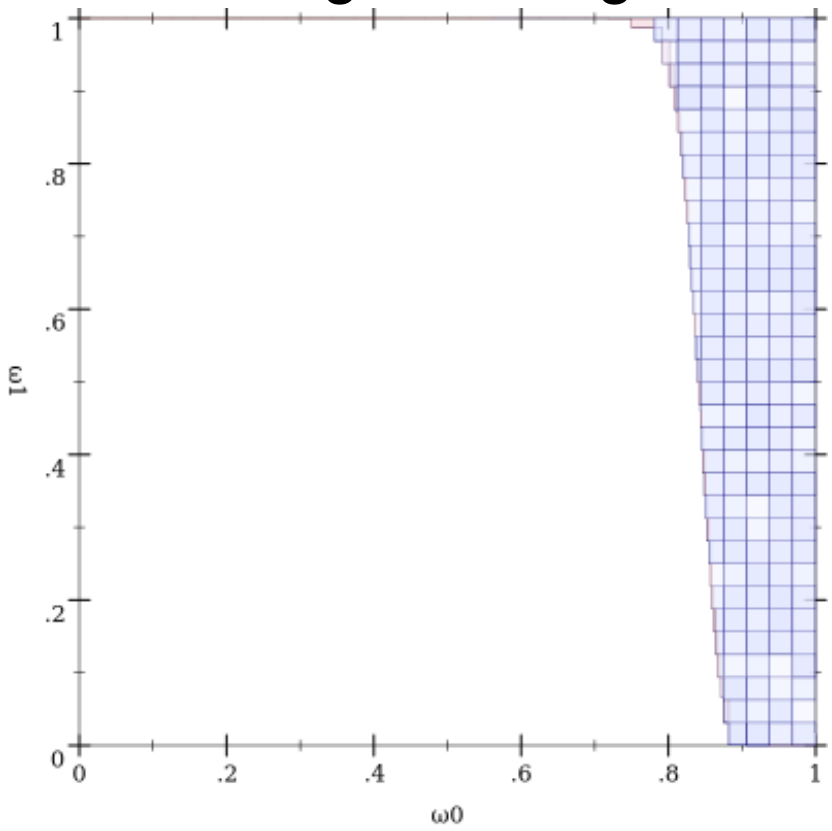

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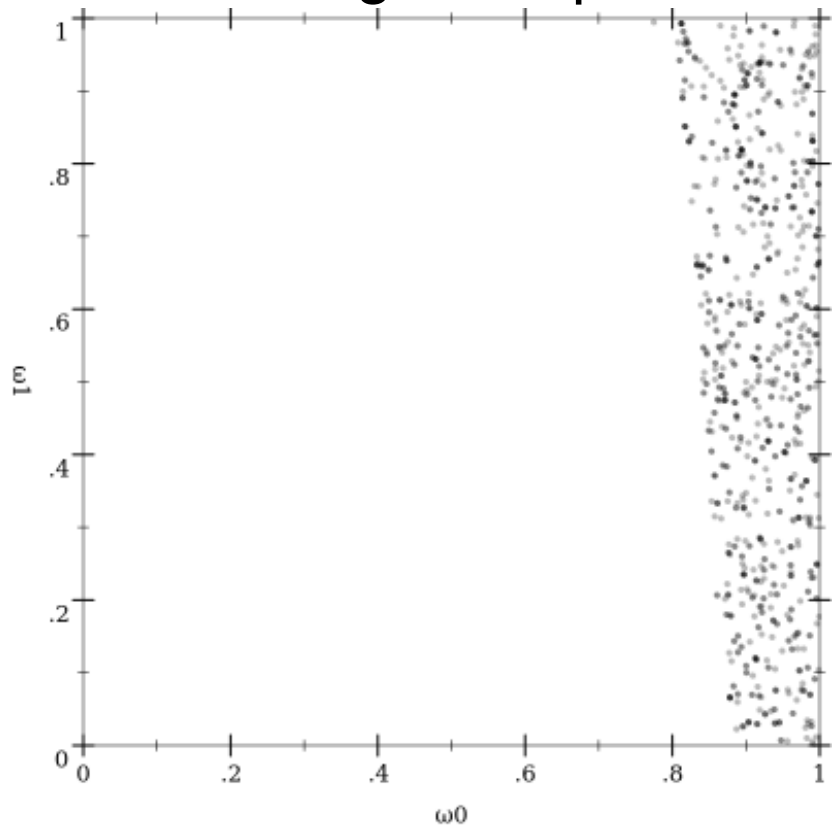

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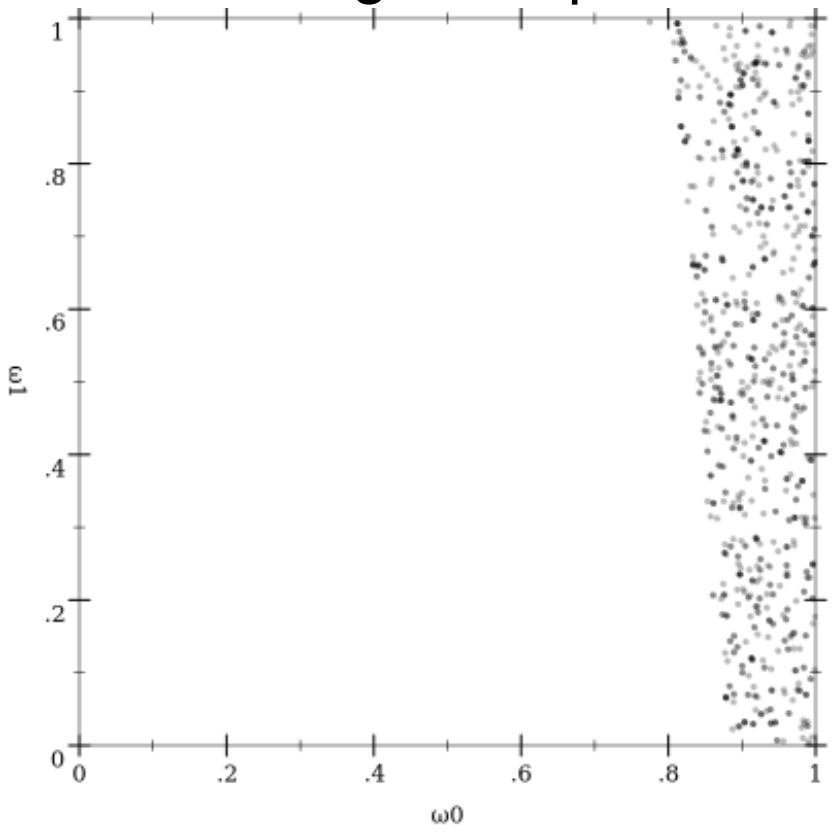
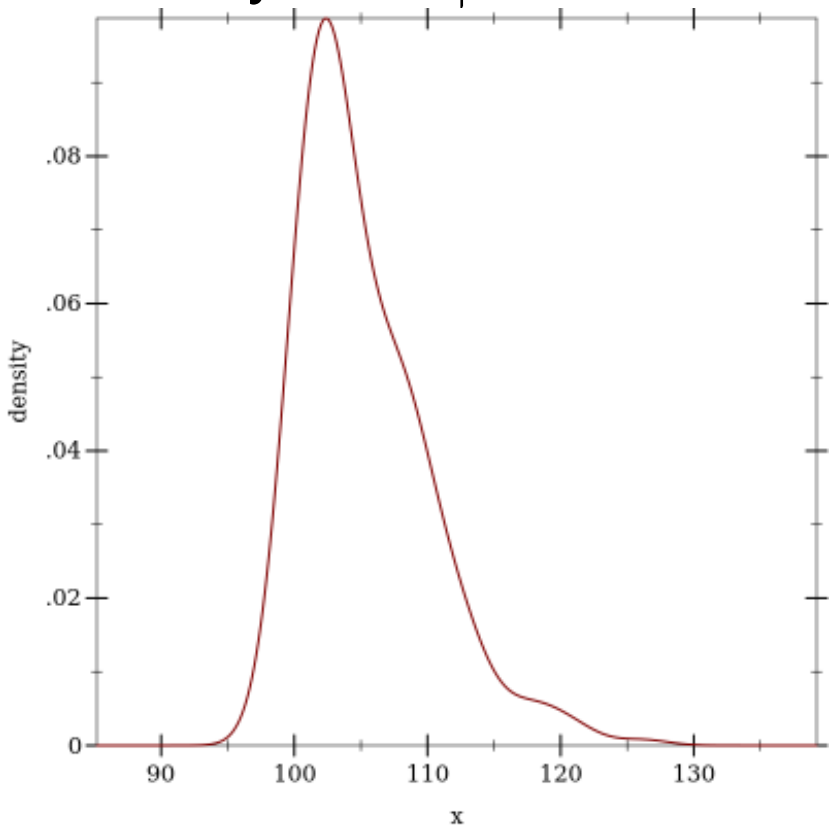


Density of $X \mid Y' = 100$



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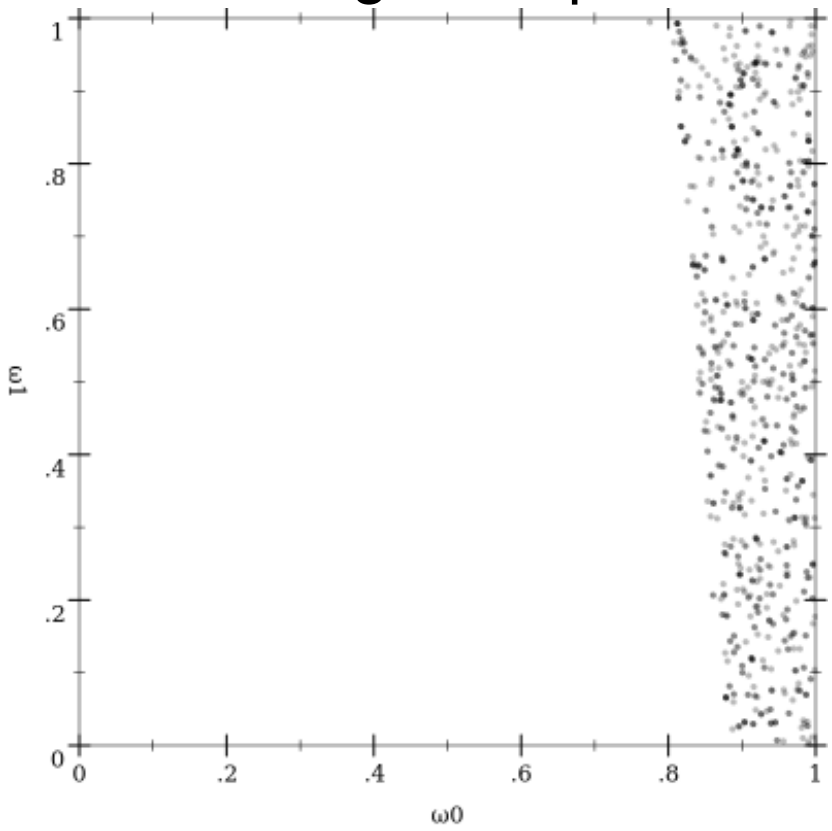

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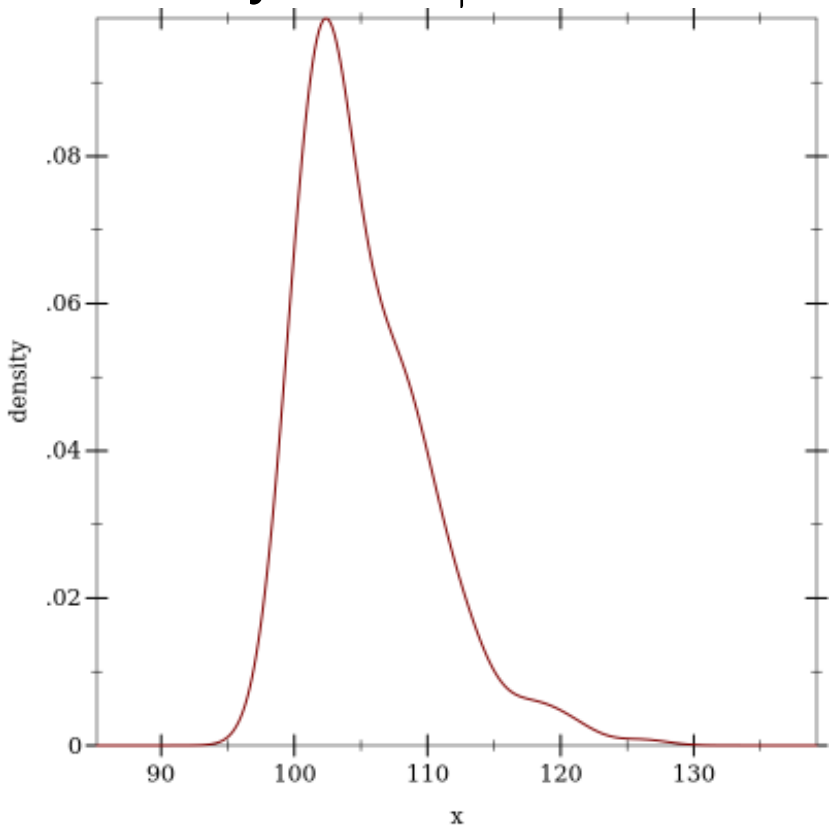
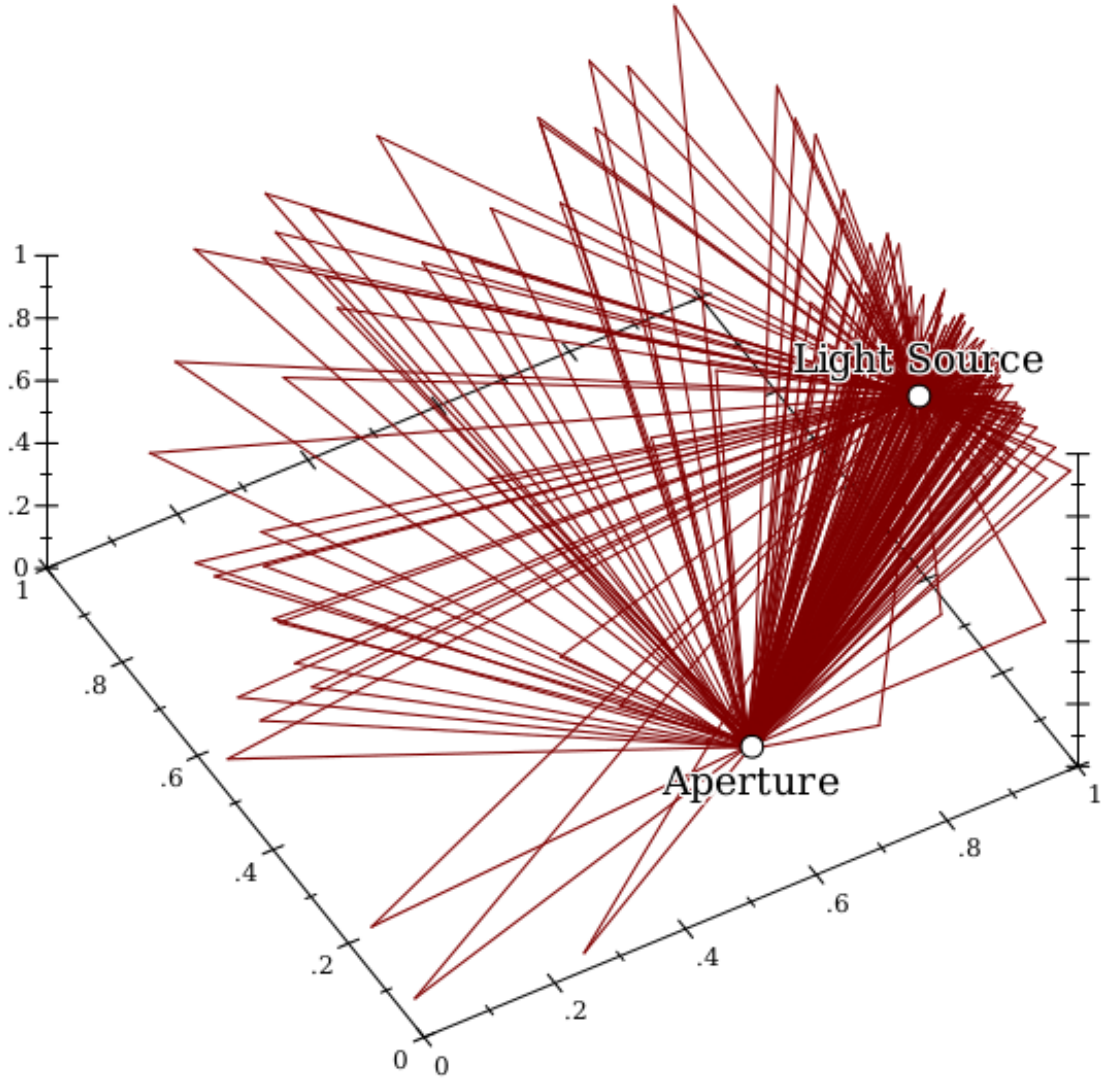


Calculated from samples: mean 105.1, stddev 4.6

Demo: Stochastic Ray Tracing

- Idea: Model light transmission and reflection, condition on paths that pass through aperture



Demo: Stochastic Ray Tracing

- Part of the implementation (totals ~50 lines):

```
(define/drbayes (ray-plane-intersect p0 v n d)
  (let ([denom  (- (vec-dot v n))])
    (if (positive? denom)
        (let ([t  (/ (+ d (vec-dot p0 n)) denom)])
          (if (positive? t)
              (collision t (vec+ p0 (vec-scale v t)) n)
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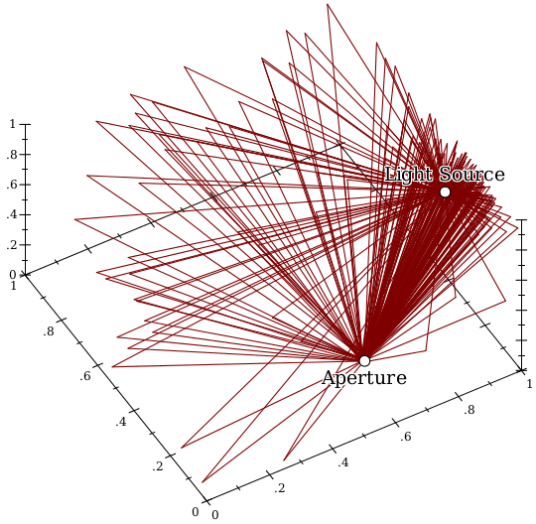
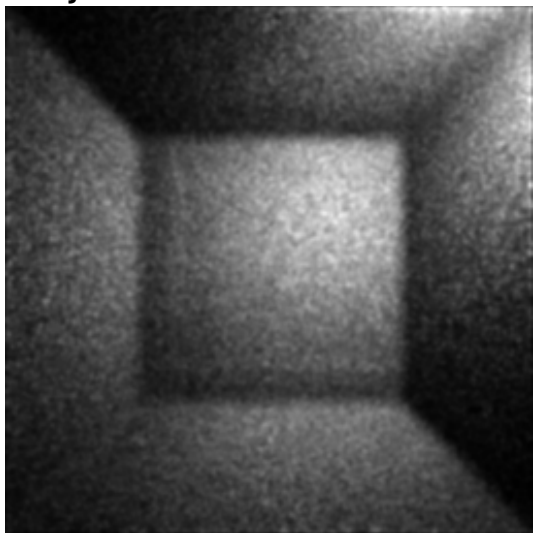
- Constrained light path outputs:

Paths Through Aperture



Projected and Accumulated



Future Work

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Future Work

Untitled 3 - DrRacket*

Untitled 3 (define ...) Run Stop

```
#lang drbayes

(define (dist x y θ d)
  (- (+ (* x (cos θ))
        (* y (sin θ)))
     d))

(define (prof d σ v+ v-)
  (+ (* 1/2 (- v+ v-))
     (erf (/ d (* (sqrt 2) σ)))
     (* 1/2 (+ v+ v-))))

(define (edge x y θ d v+ v- σ)
  (prof (dist x y θ d) σ v+ v-))

(define (scene-edge i j)
  (let ([θ  (uniform (- pi) pi)]
        [d  (uniform -3 3)]
        [σ  (beta 1.6 1)]
        [v+  (uniform 0 1)]
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    (λ (x y)
      (edge (- x (+ i 1/2)) (- y (+ j 1/2)) θ d v+ v- σ))))

(define (image-point i j)
  (normal (mean (map scene-edge (n9-x i j) (n9-y i j))) w))

(define (scene-reg i j)
  (exp (* -1/2 (/ (variance (map scene-edge (n9-x i j) (n9-y i j))) γ^2))))

(define (reconstruct img)
  ....)
```

```
Welcome to DrRacket, version 6.0.0.1--2013-12-12(c321f6dd/d) [3m].
Language: racket; memory limit: 1024 MB.
> (reconstruct
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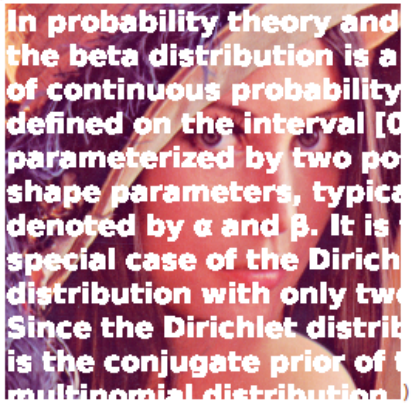

```
>
```

Determine language from source 7:2 639.69 MB

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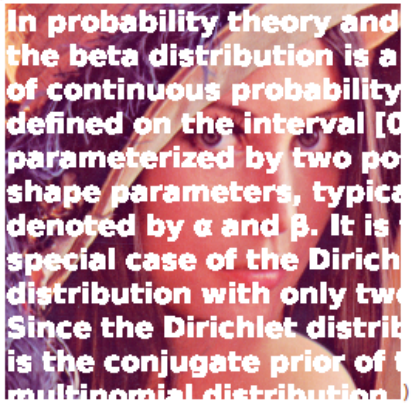

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Determine language from source 7:2 639.69 MB

... and much, much more...