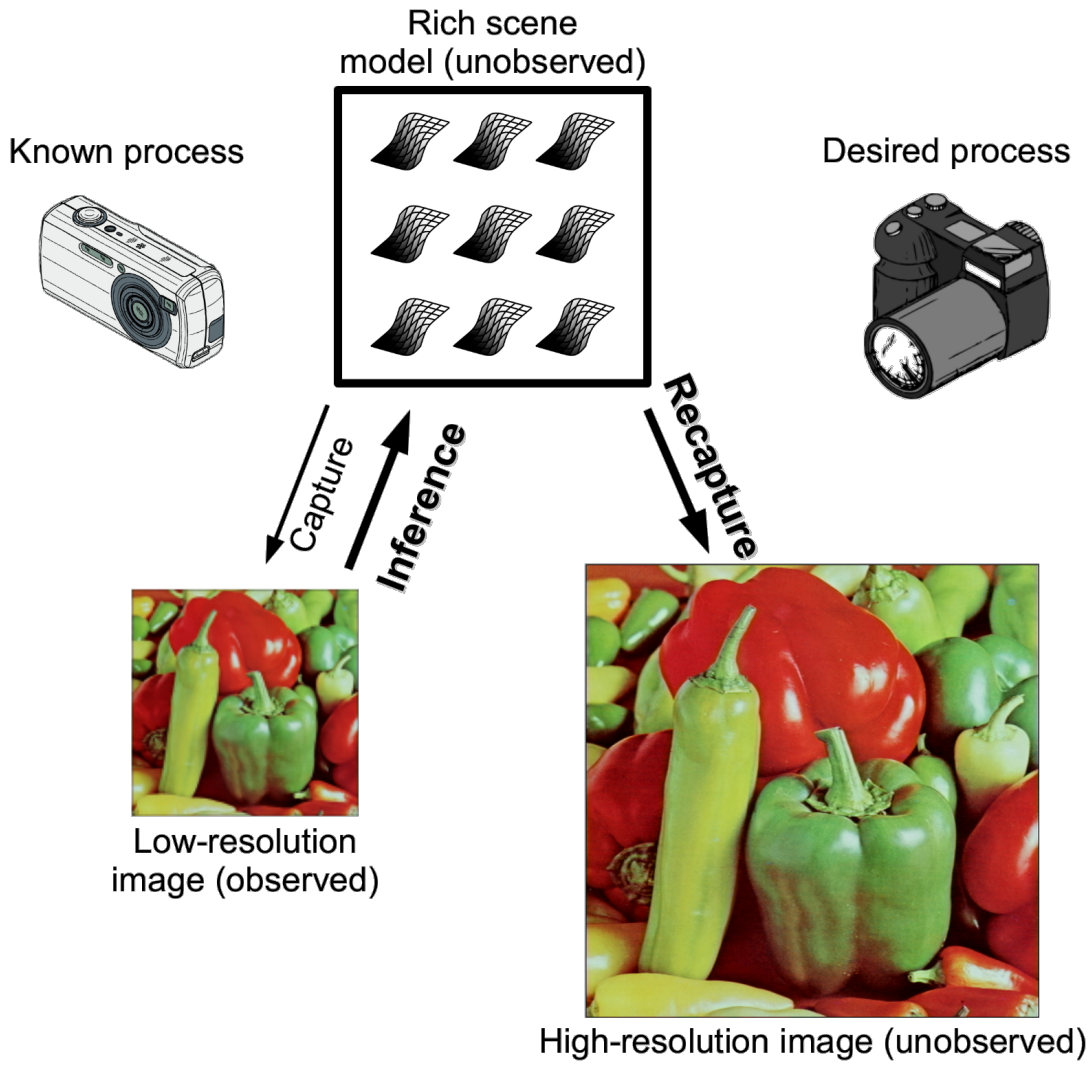
Running Probabilistic Programs Backward

Neil Toronto

PLT @ Brigham Young University

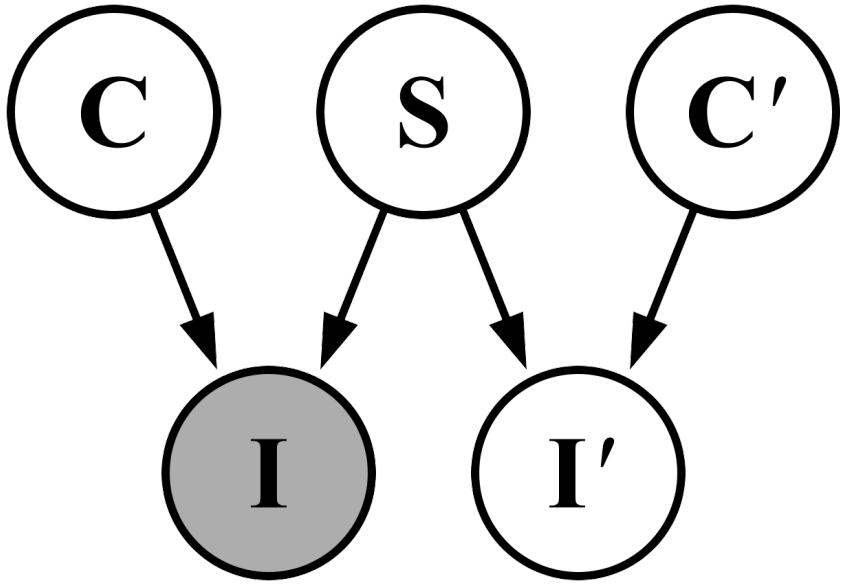
Master's Research: Super-Resolution



Toronto et al. Super-Resolution via Recapture and Bayesian Effect Modeling. CVPR 2009

Master's Research: Super-Resolution

- Bayesian Edge Inference (BEI) graphical model:



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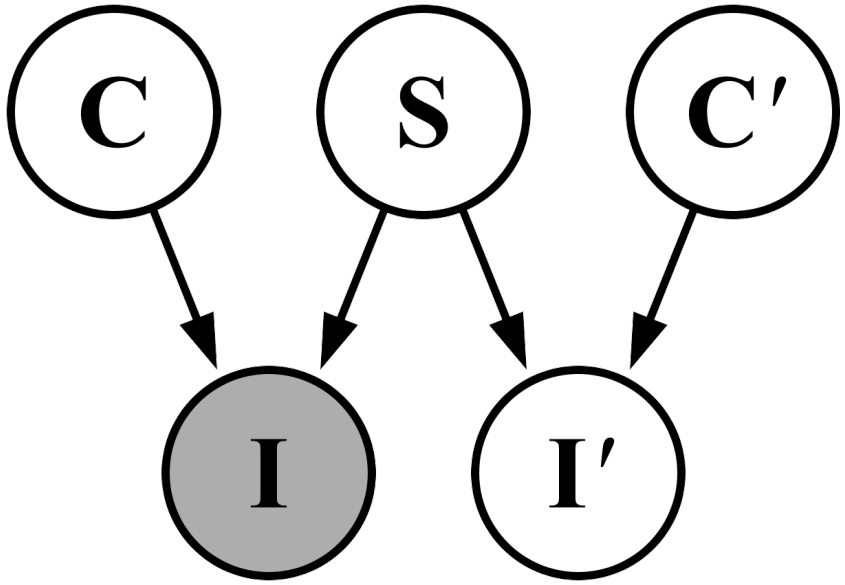

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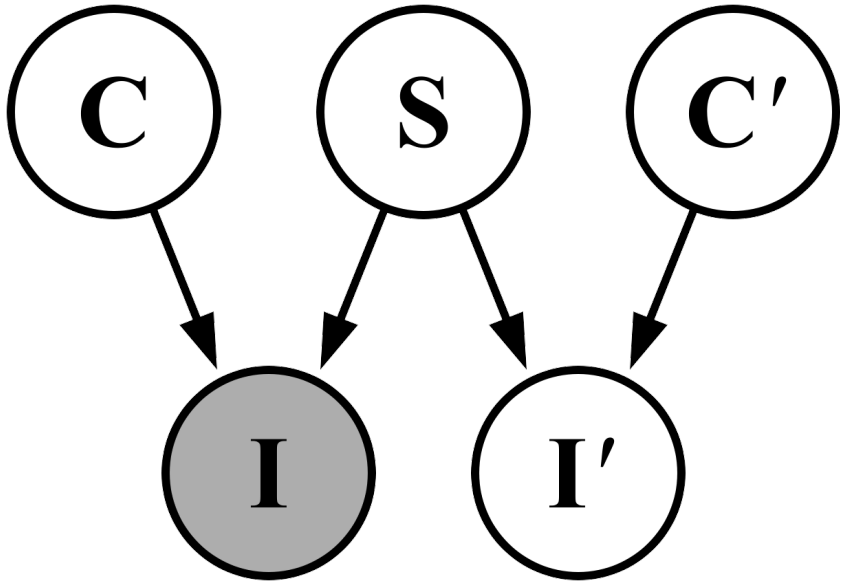

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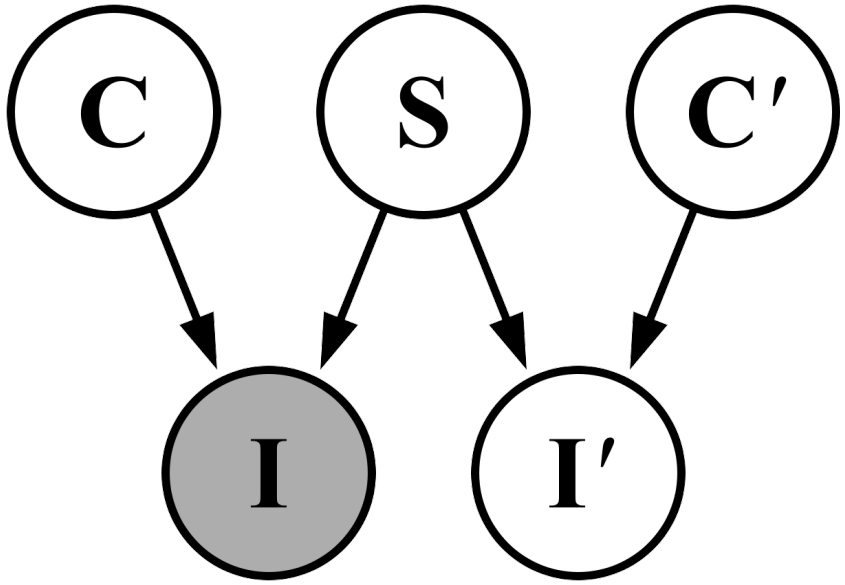

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- Query implementation: 600 lines of evil Python

Main Results: Super-Resolution

Resolution Synthesis

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Local Correlation

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Bayesian Edge Inference

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Bayesian Edge Inference

- Also beat state-of-the-art on “objective” measures

Pleasantly Surprising Results: Inpainting

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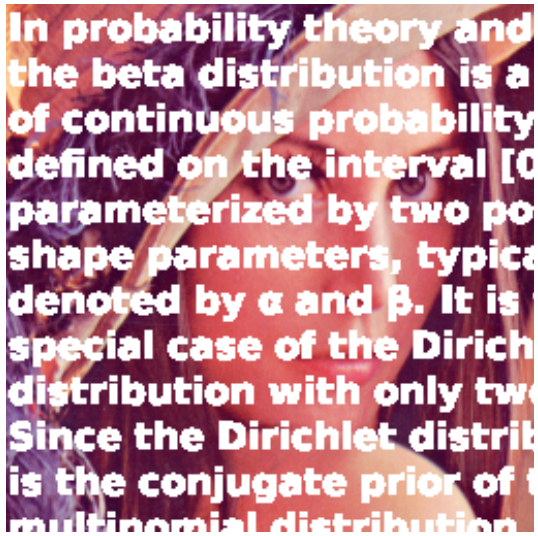

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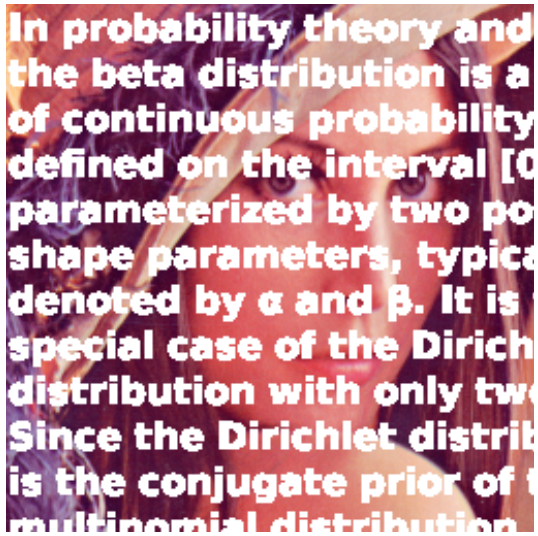

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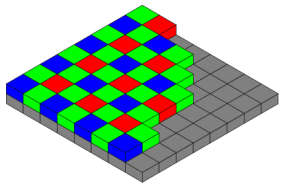
Bayer filter: 

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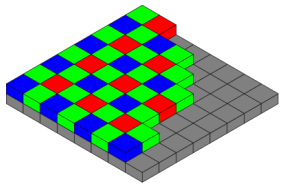
Bayer filter: 

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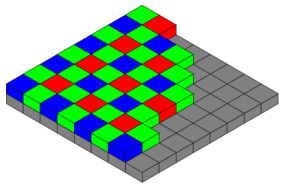
Bayer filter: 

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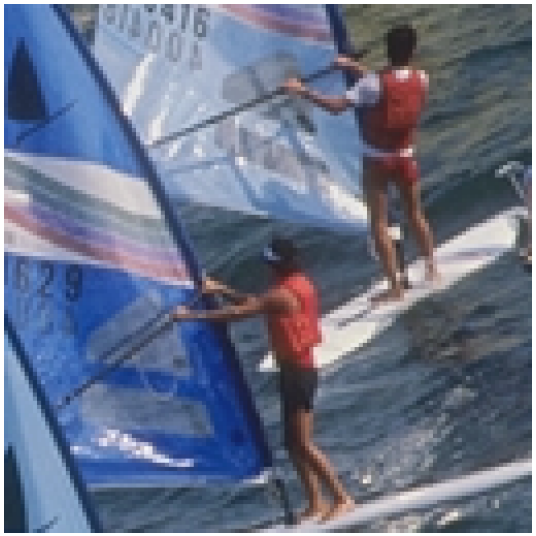

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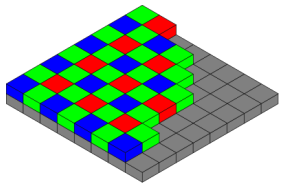
Bayer filter: 

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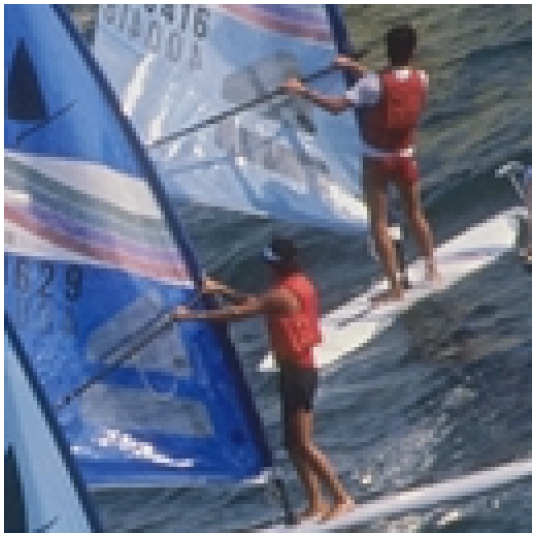

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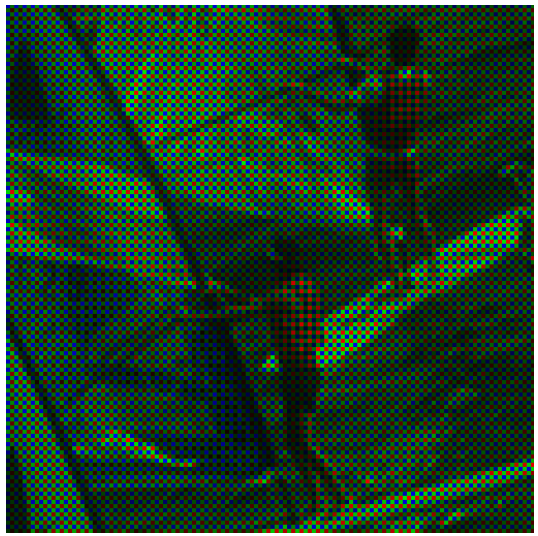

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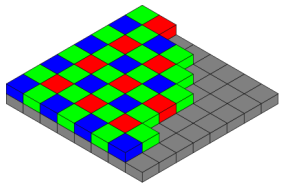
Bayer filter: 

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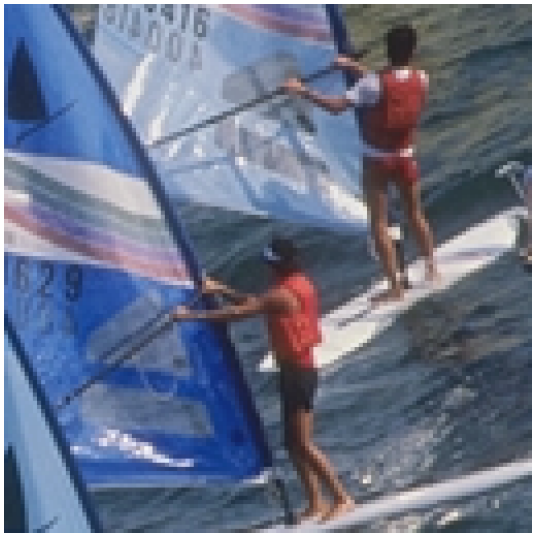

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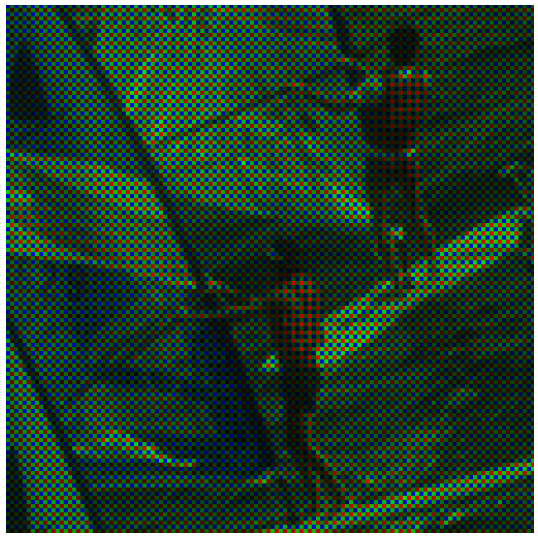

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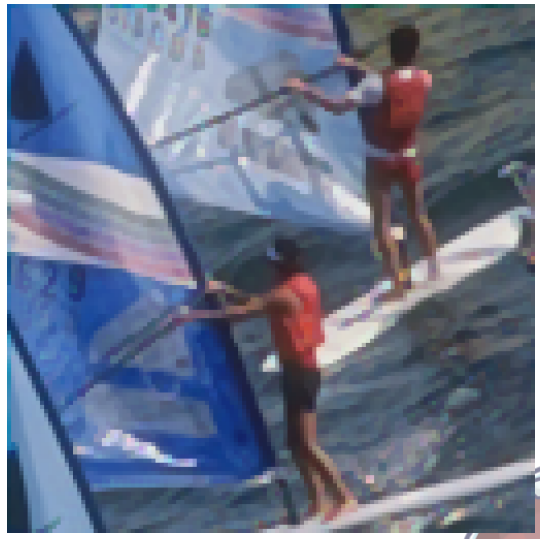

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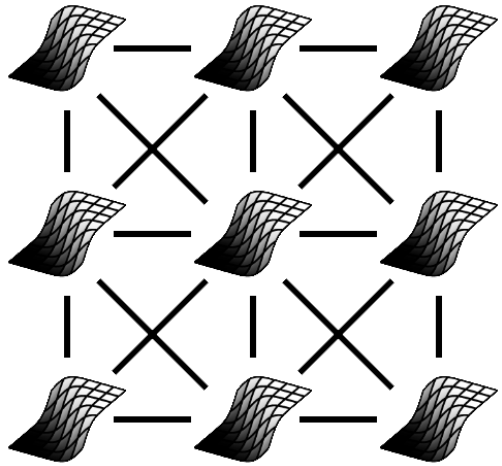

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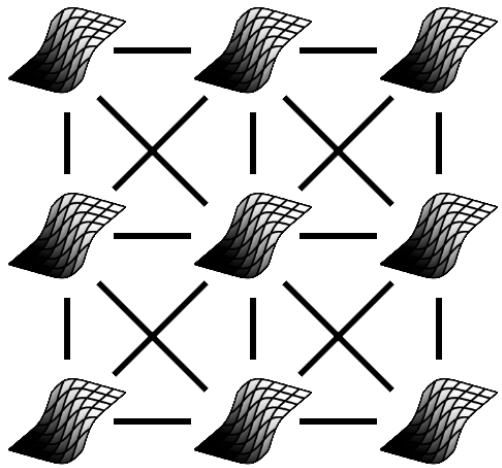

Reason 2: “To *approximate* blurring with a spatially varying point-spread function (PSF), we assign each facet a Gaussian PSF and convolve each analytically *before combining outputs*.”

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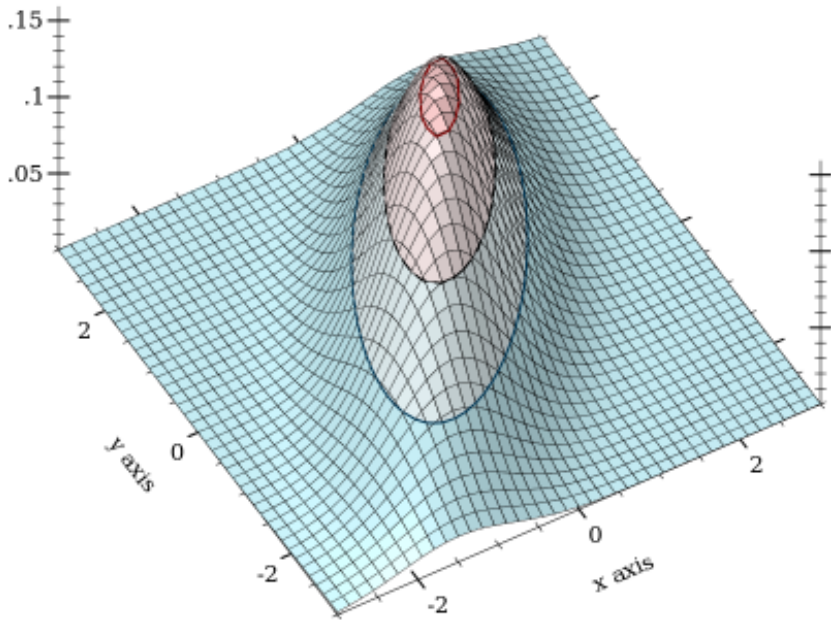
- Density model $f : \mathbb{R} \times \mathbb{R} \to [0, \infty)$:



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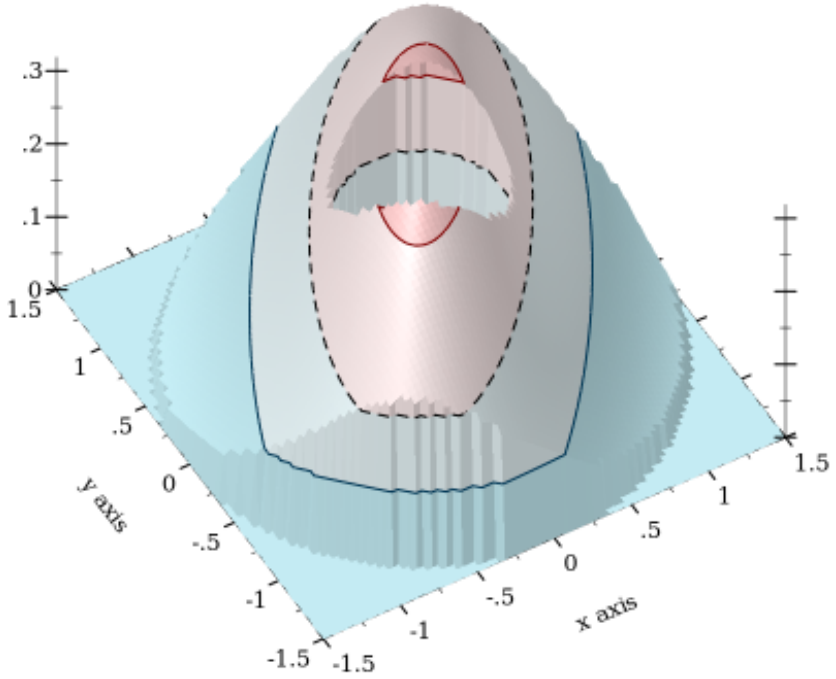
Conditional density with $\epsilon = 0.5$:



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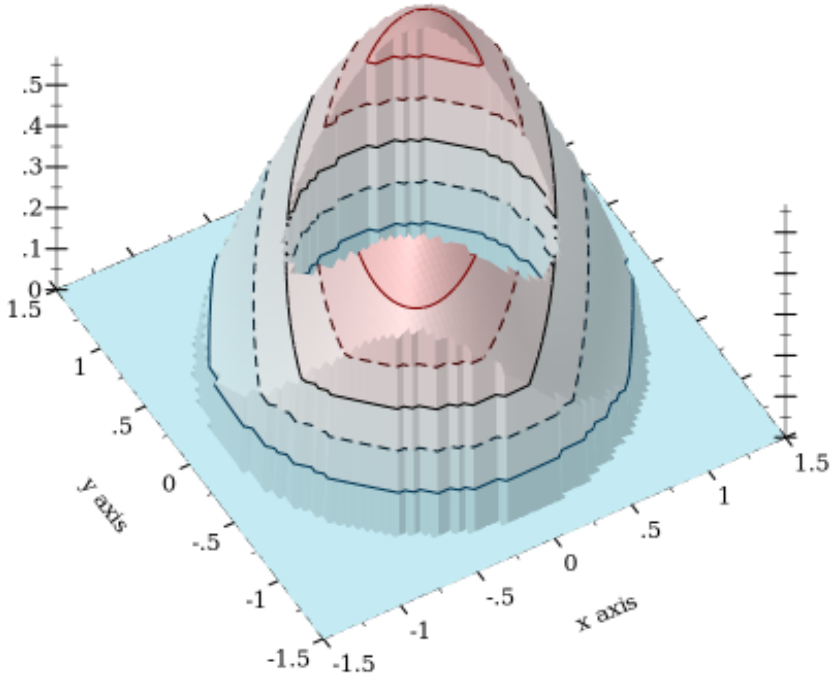

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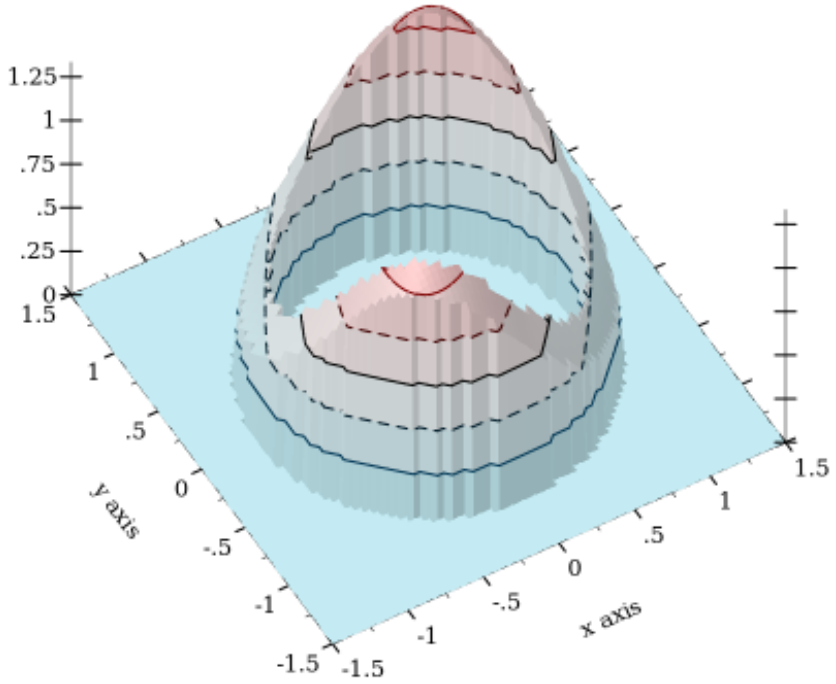
Conditional density with $\epsilon = 0.1$:



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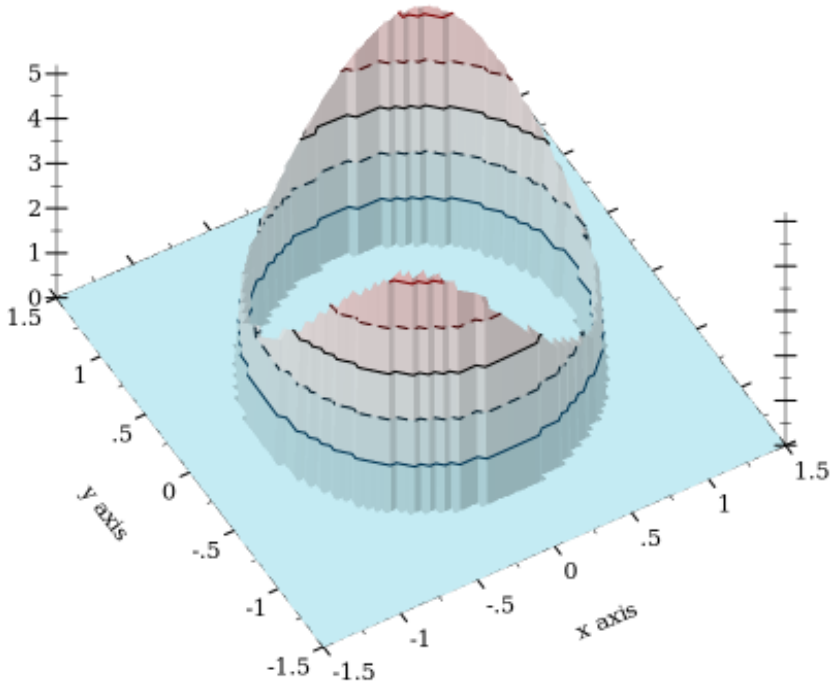

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There are tricks to get around limitations, but none are generally applicable...

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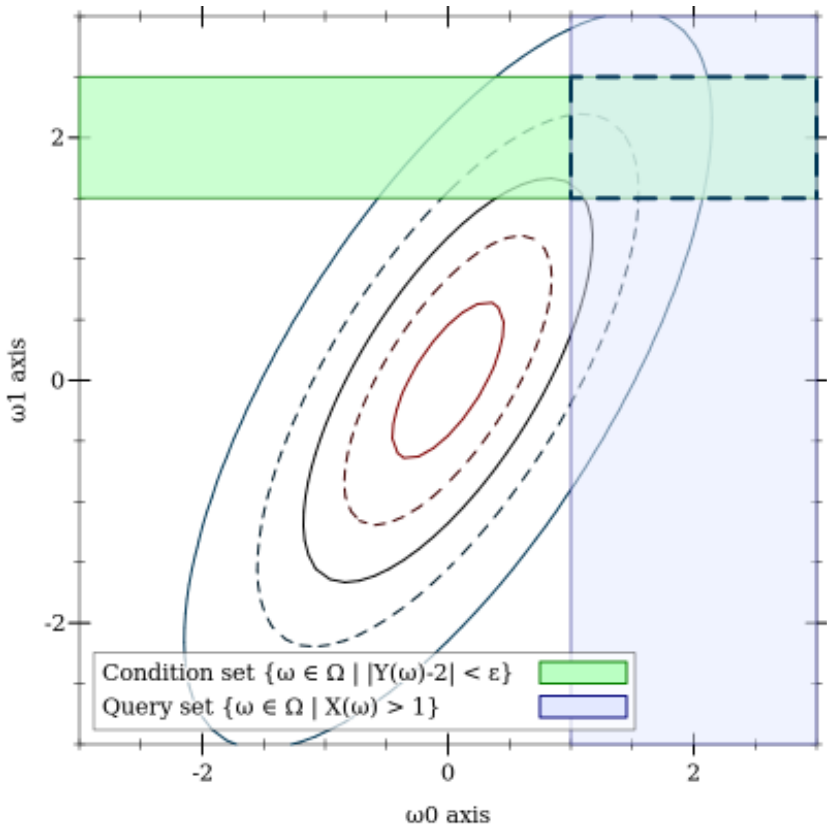

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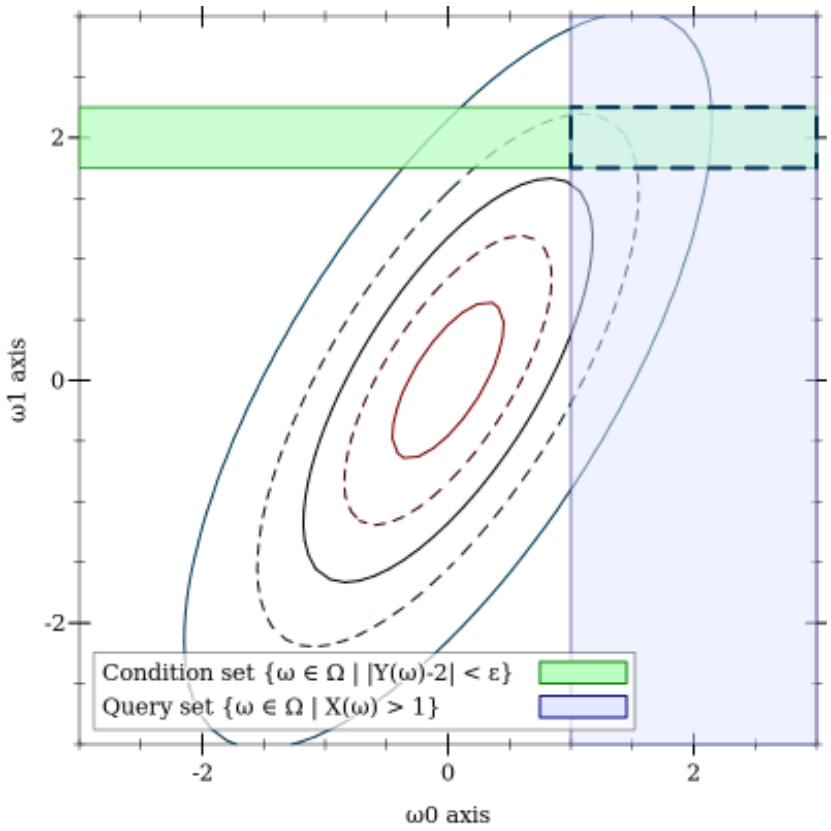

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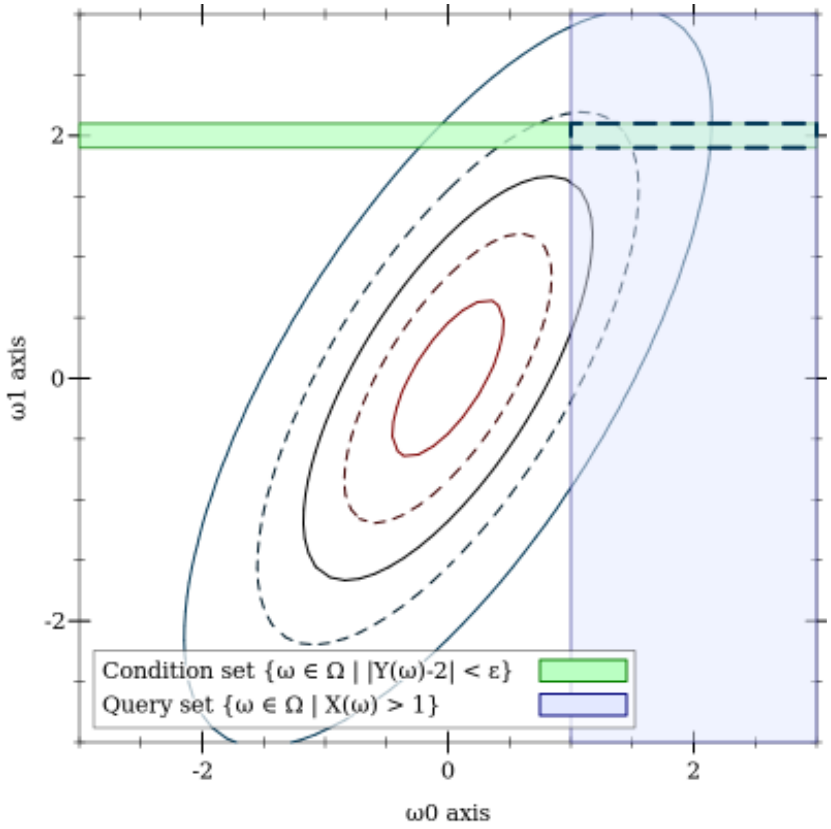

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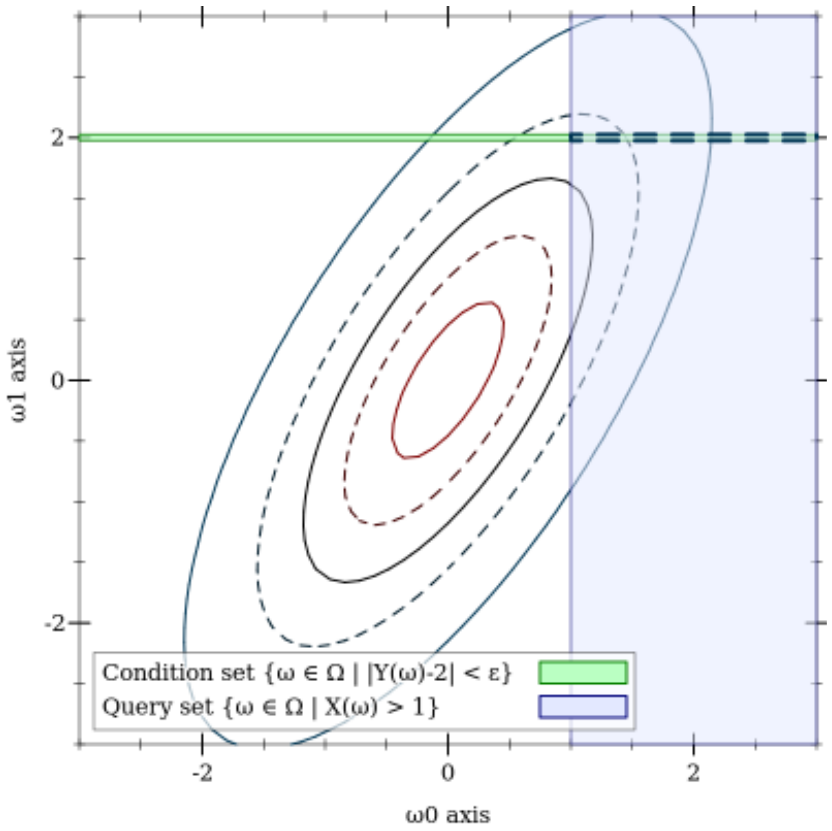

Zero-Probability Conditions (Circular)

- Can differentiate w.r.t. any random variable:

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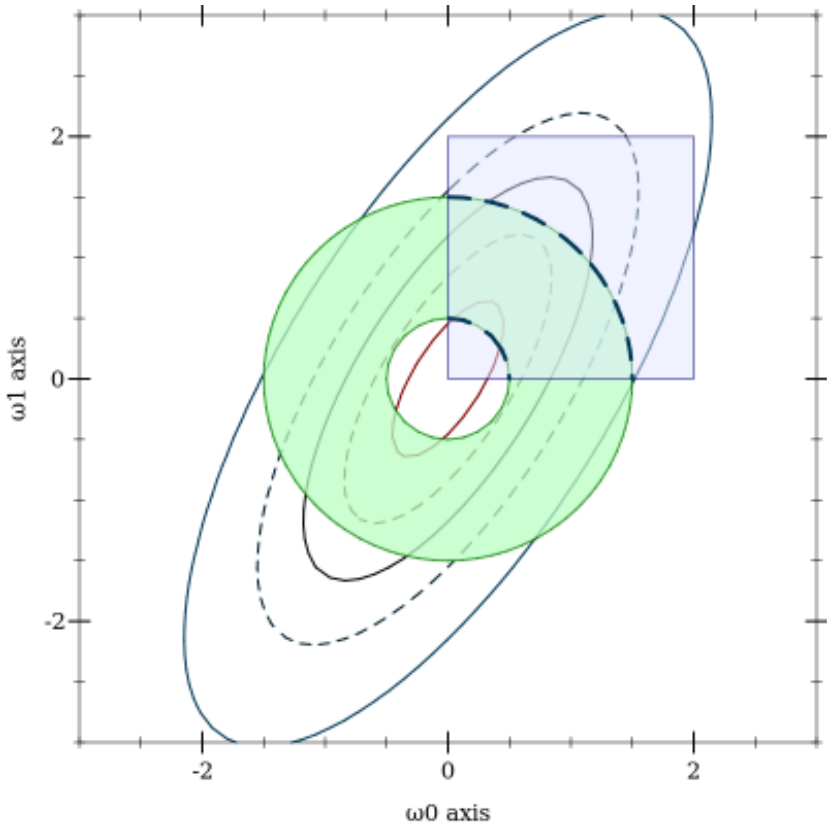

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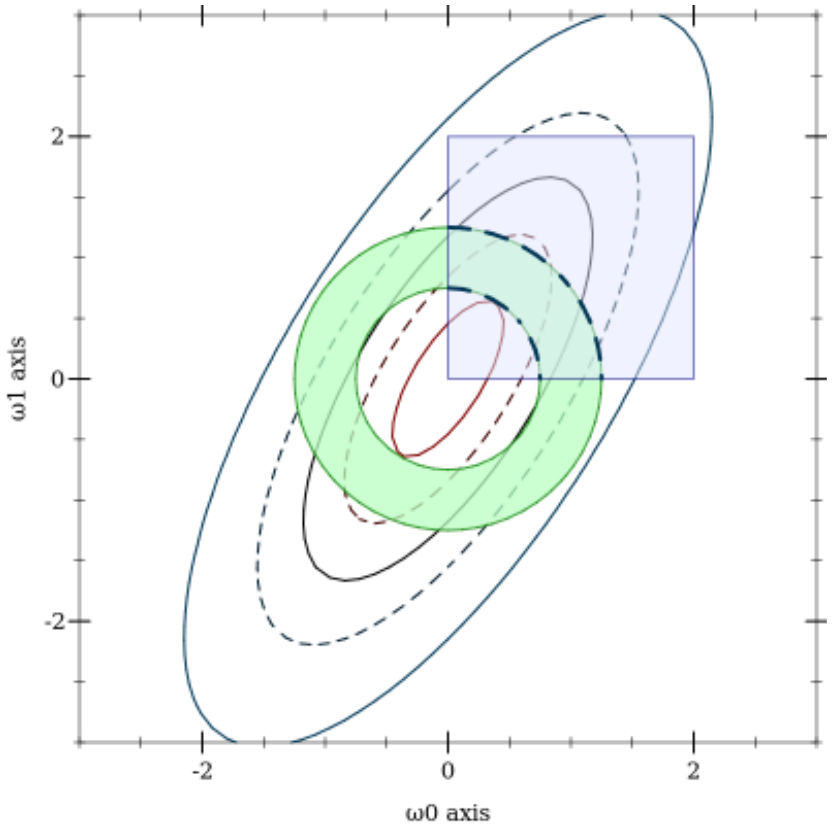

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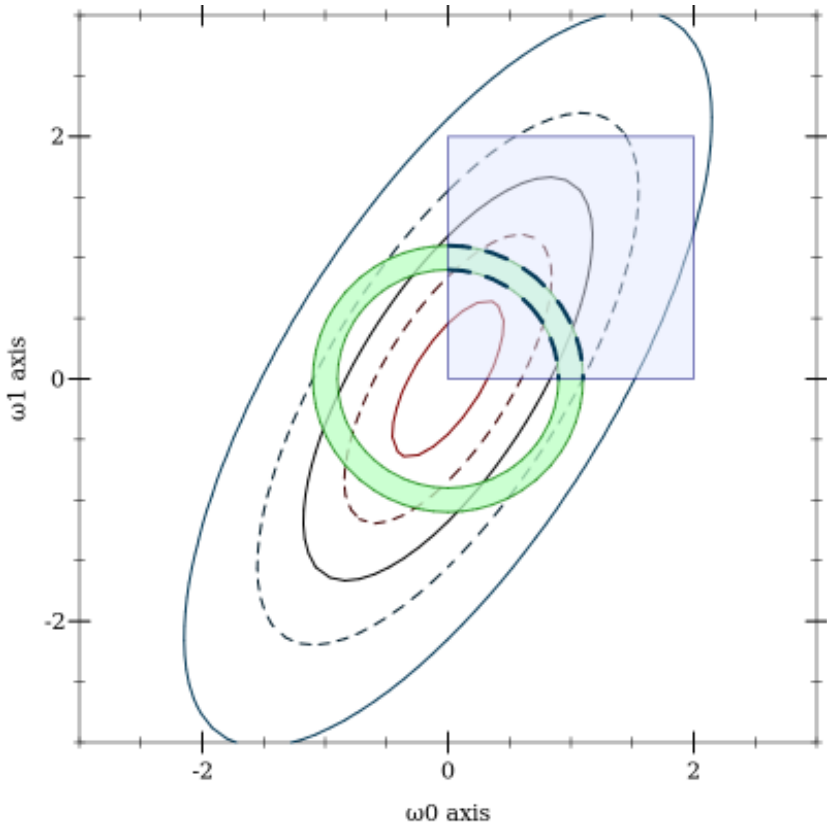

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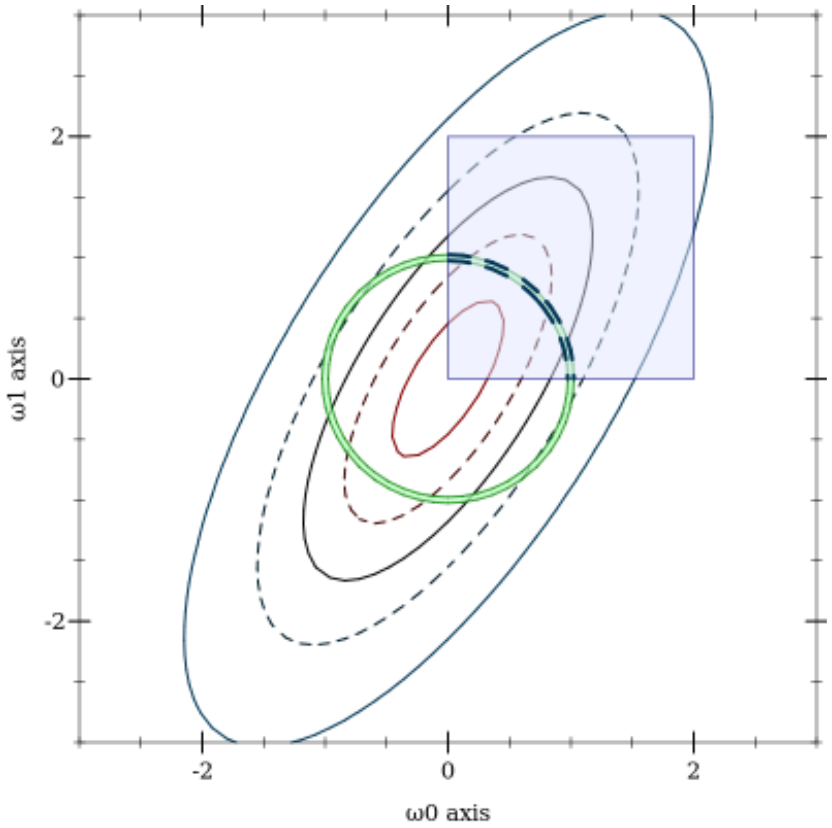

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- What do preimages under Z look like? How hard would they be to represent and sample from uniformly?

Preimages Under the Unit Circle Condition

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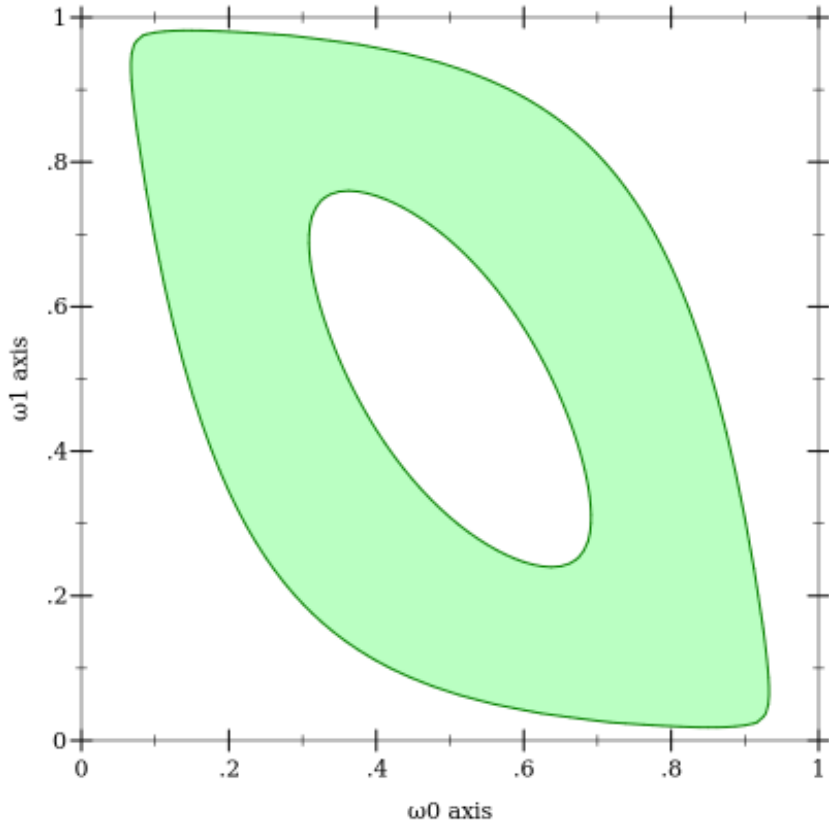
$Z^{-1}([0, \epsilon)), \epsilon = 0.5$:



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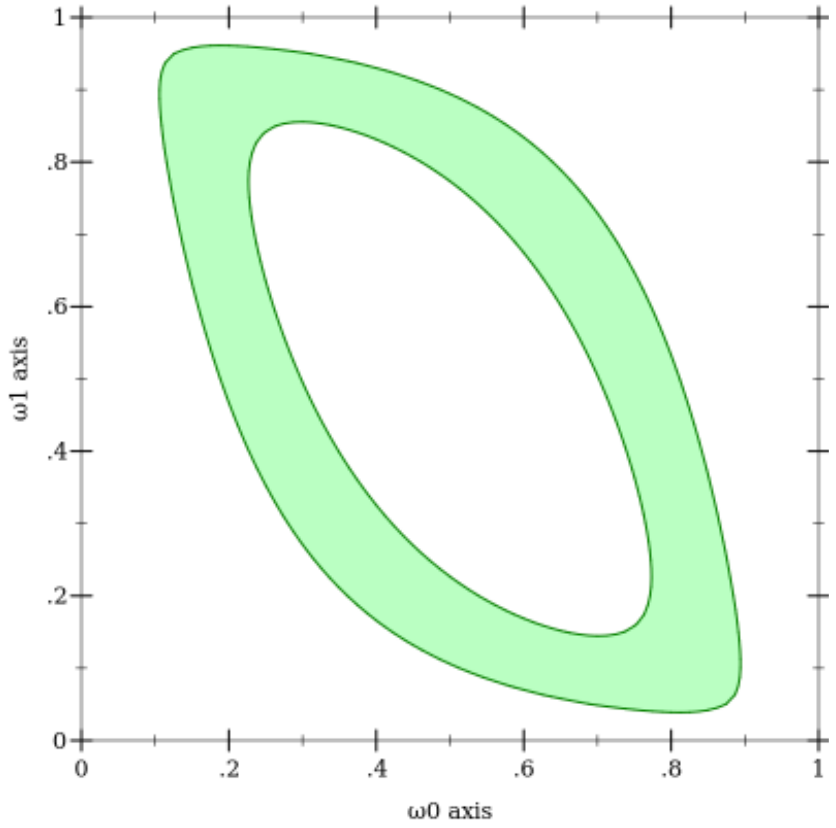

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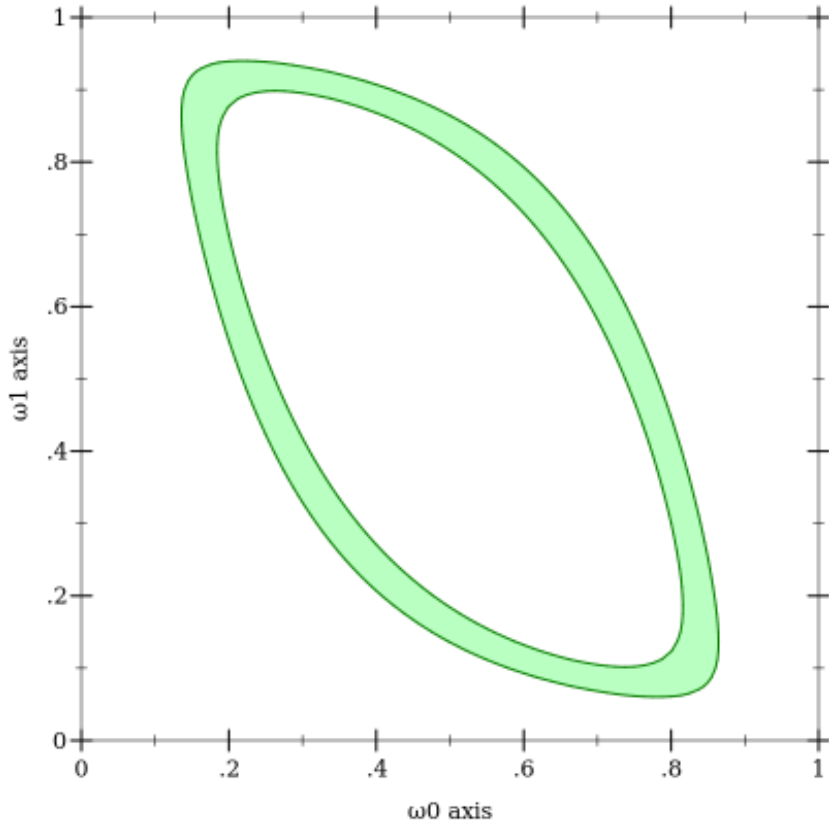

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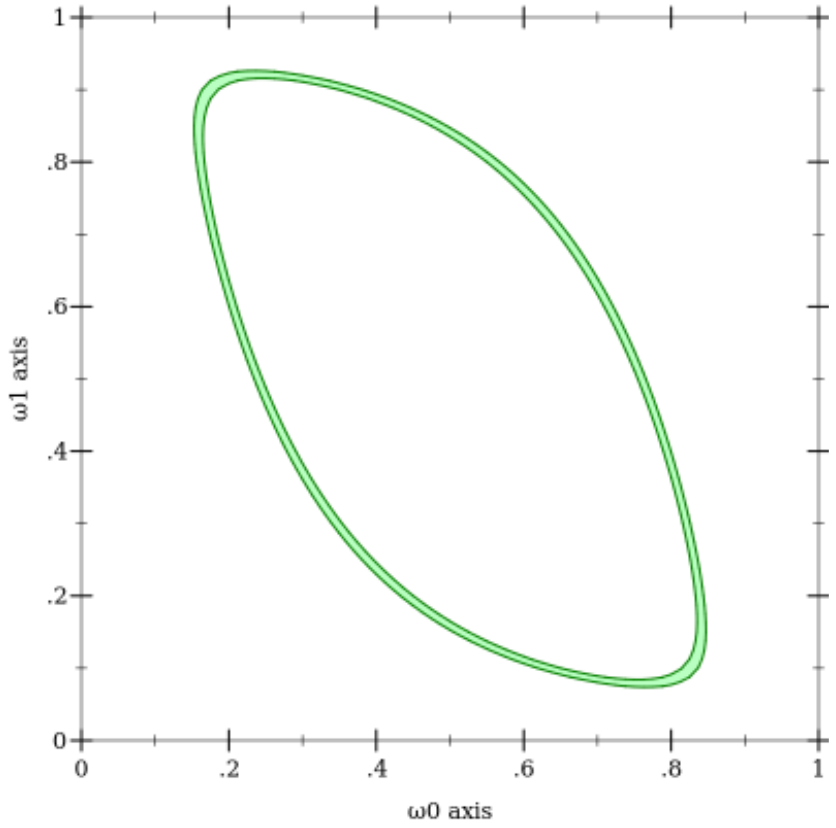

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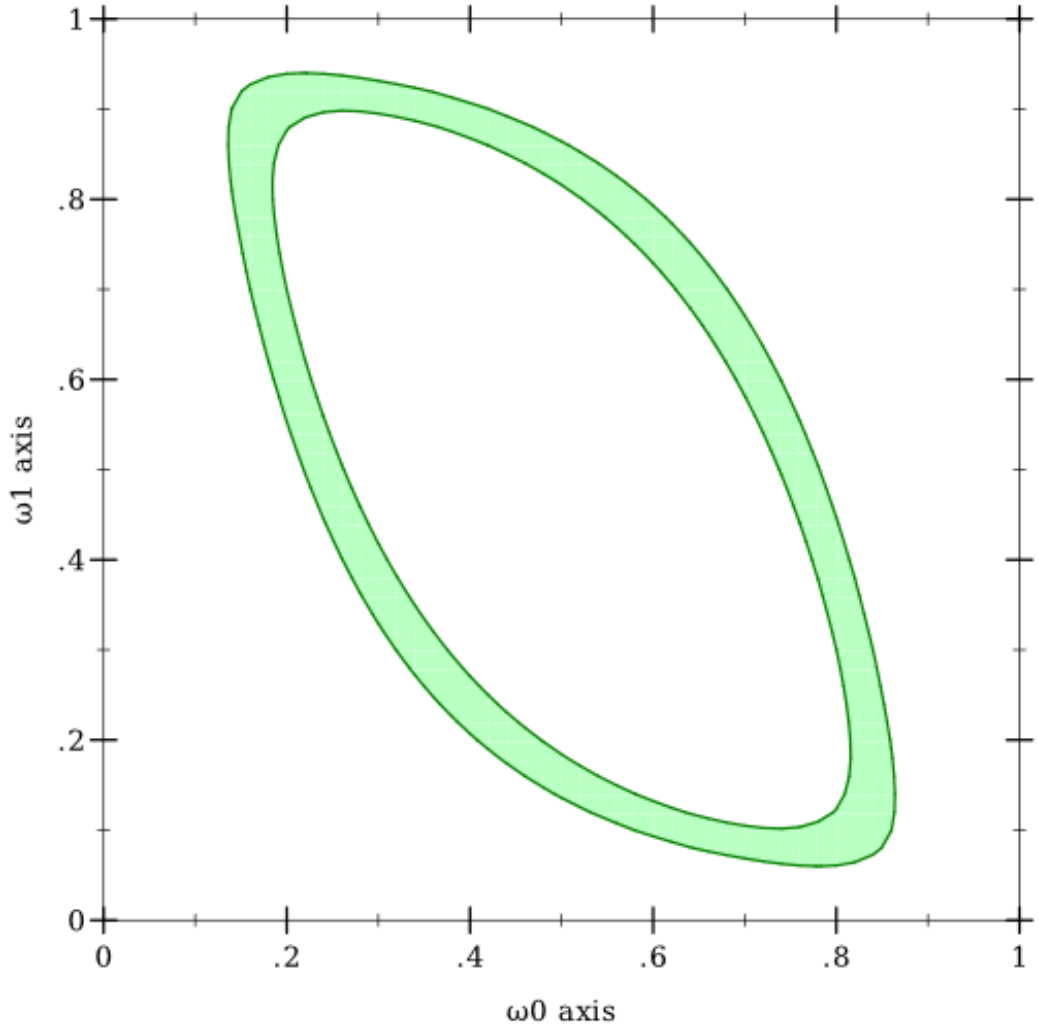
- Exact implementation is out of reach, but...

What About Approximating?

Conservative approximation with rectangles:



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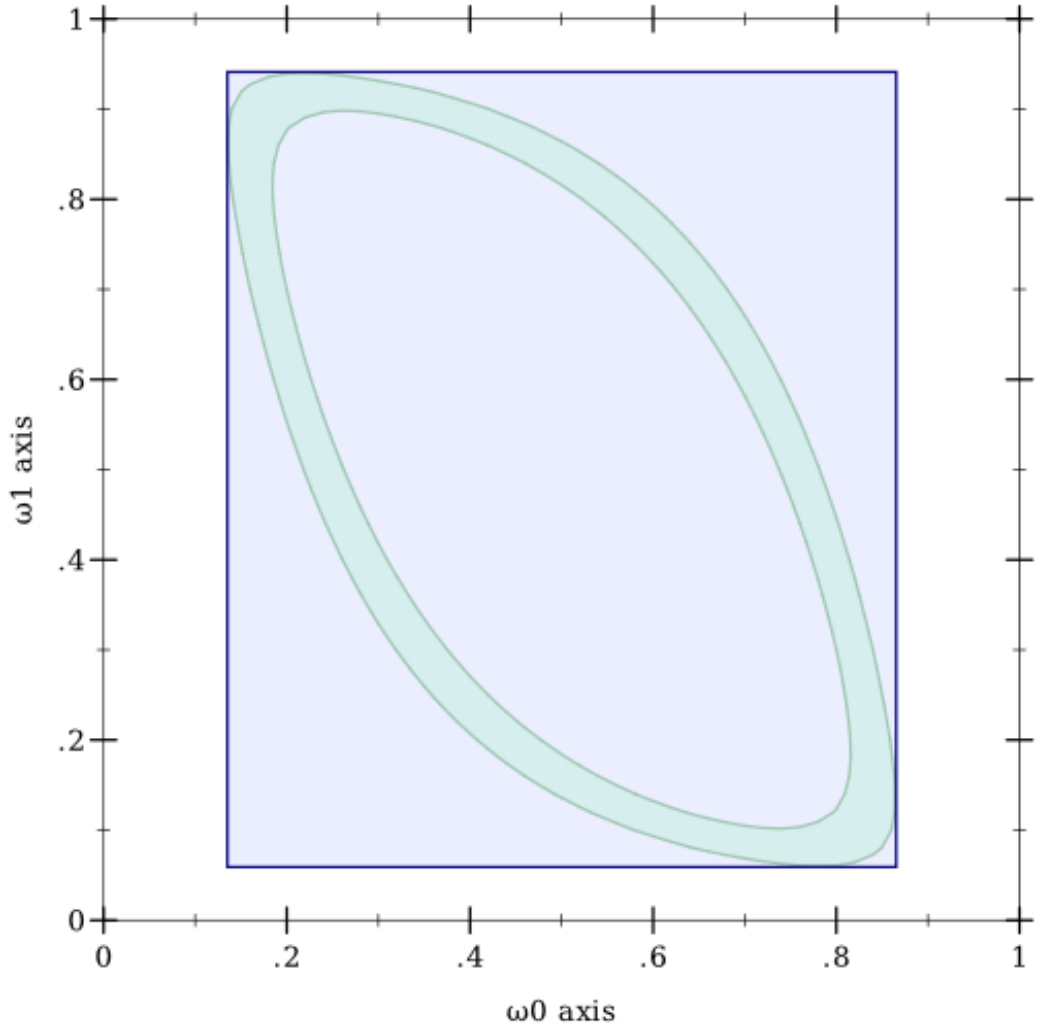

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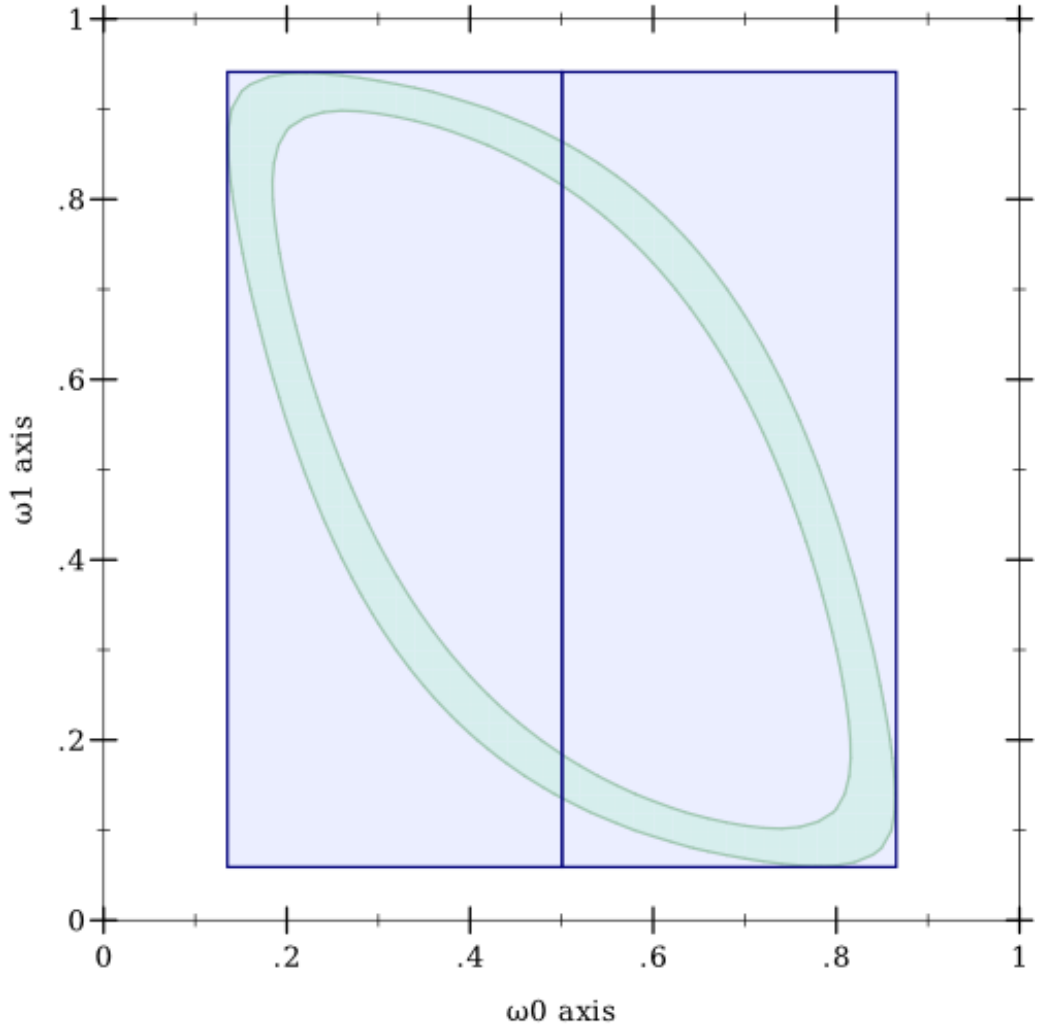

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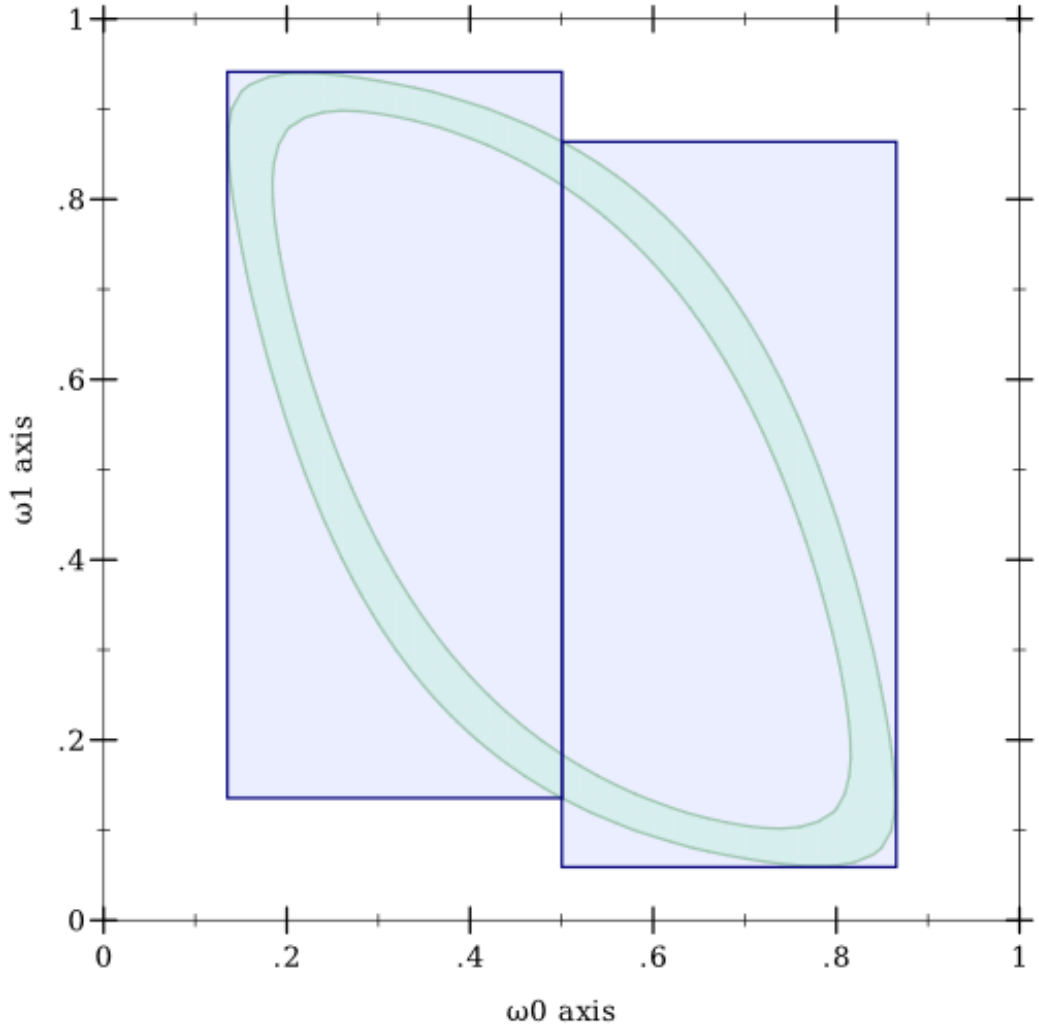

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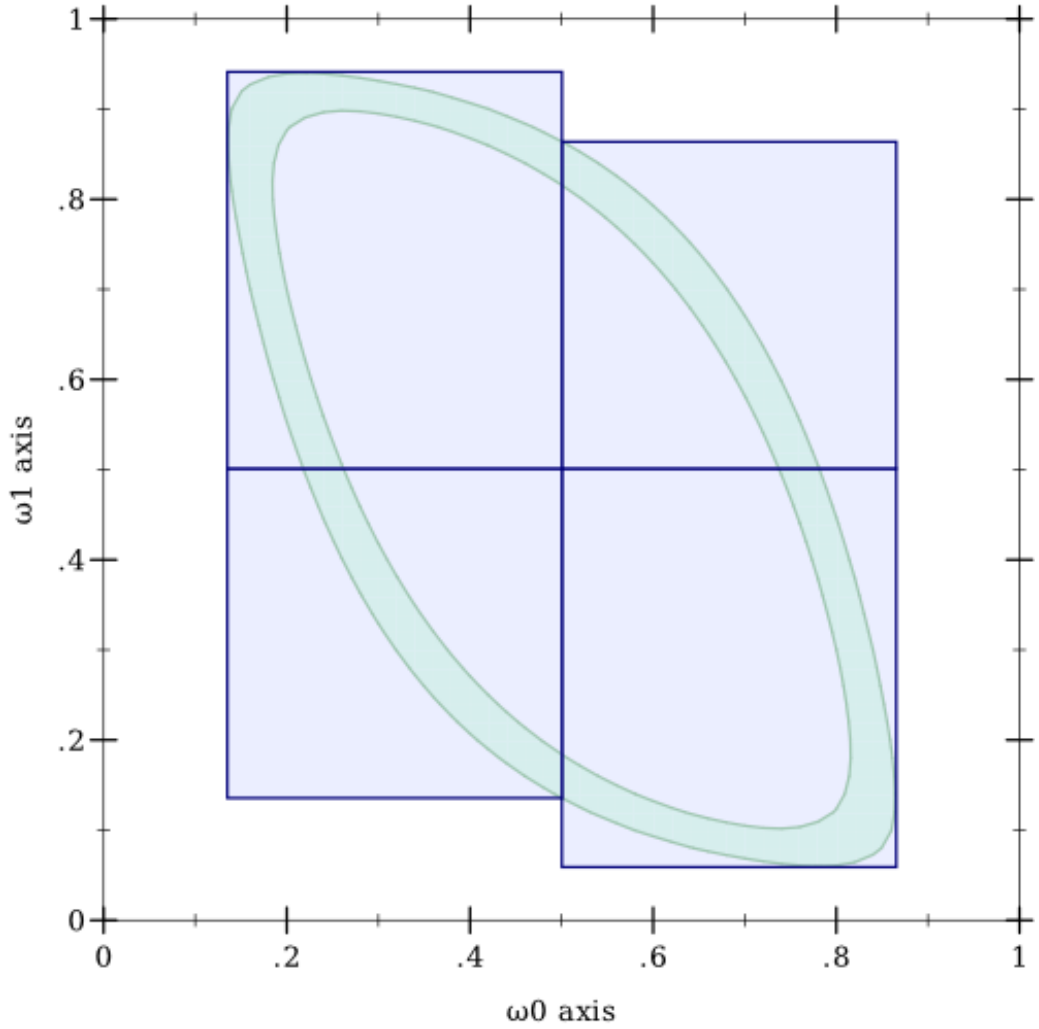

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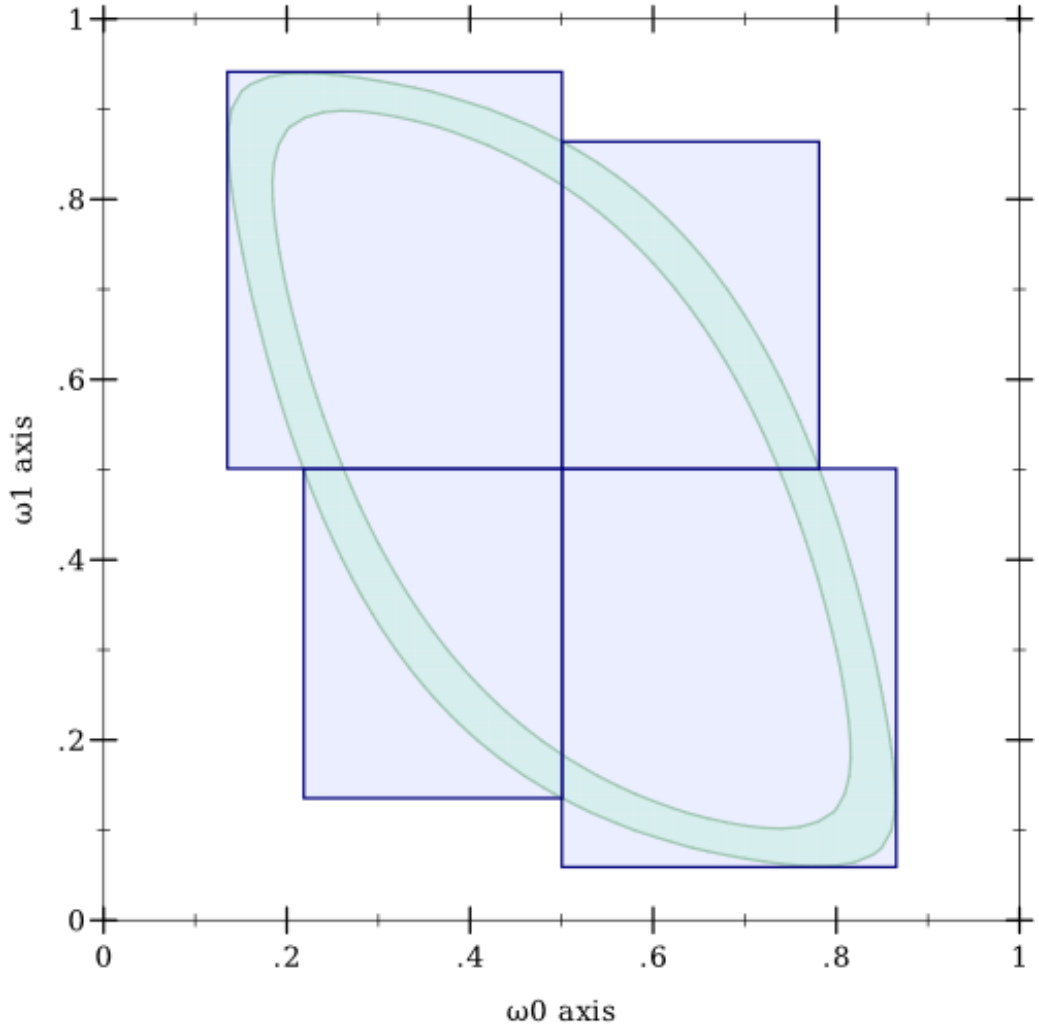

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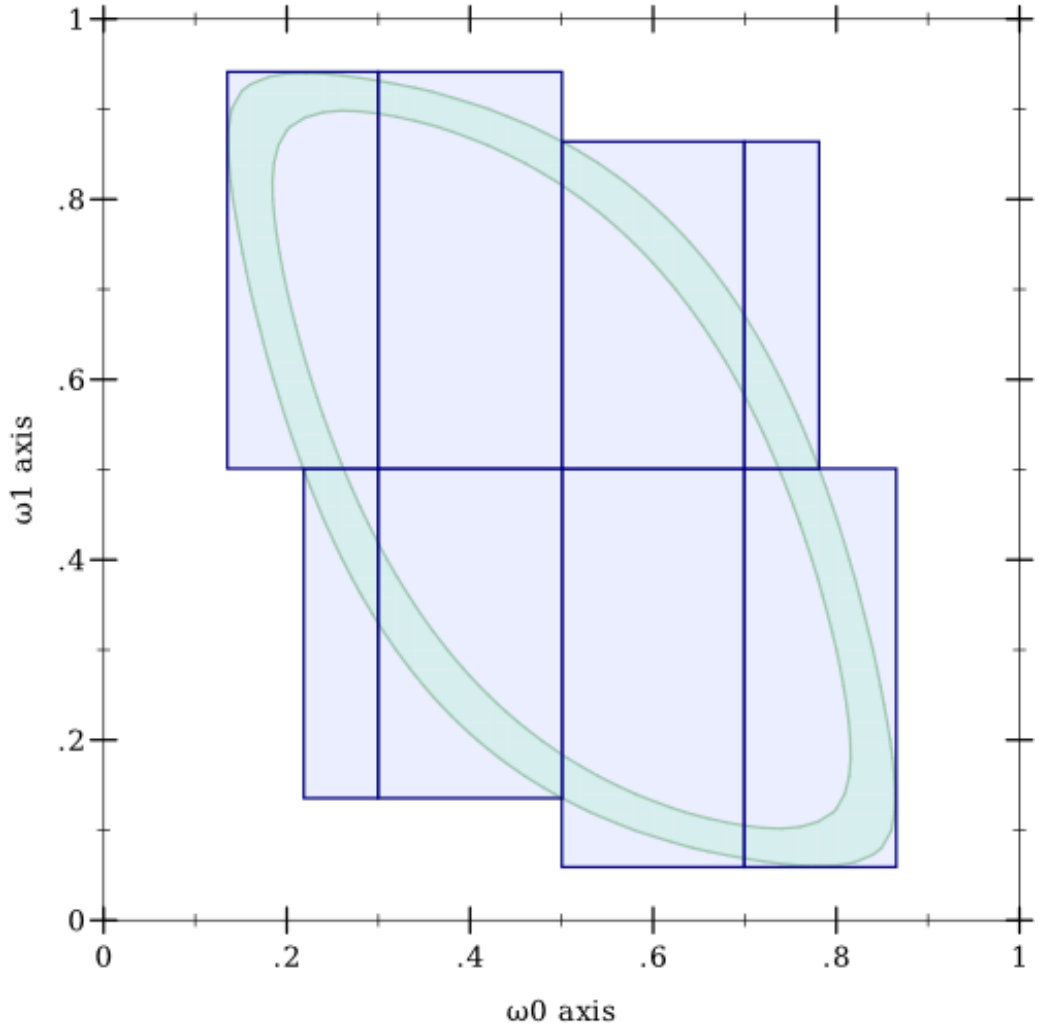

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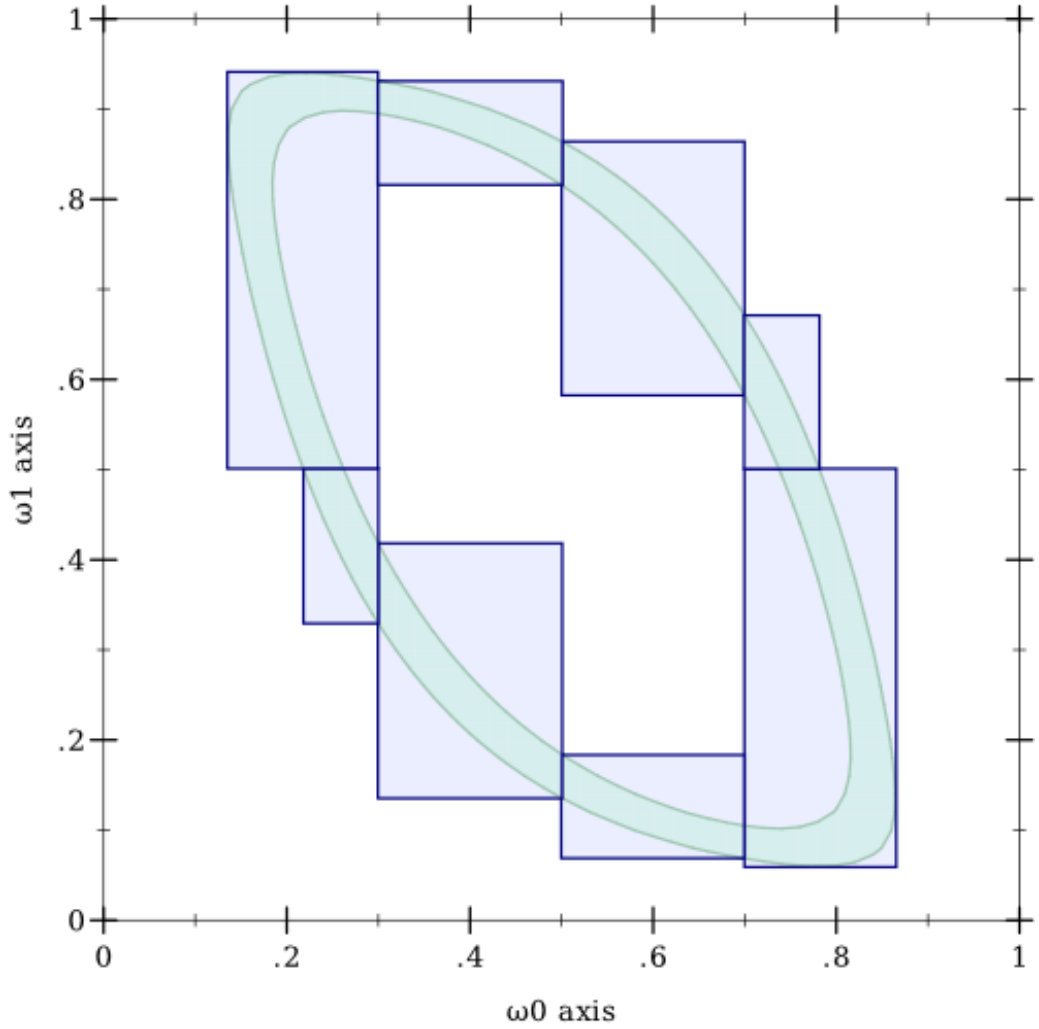

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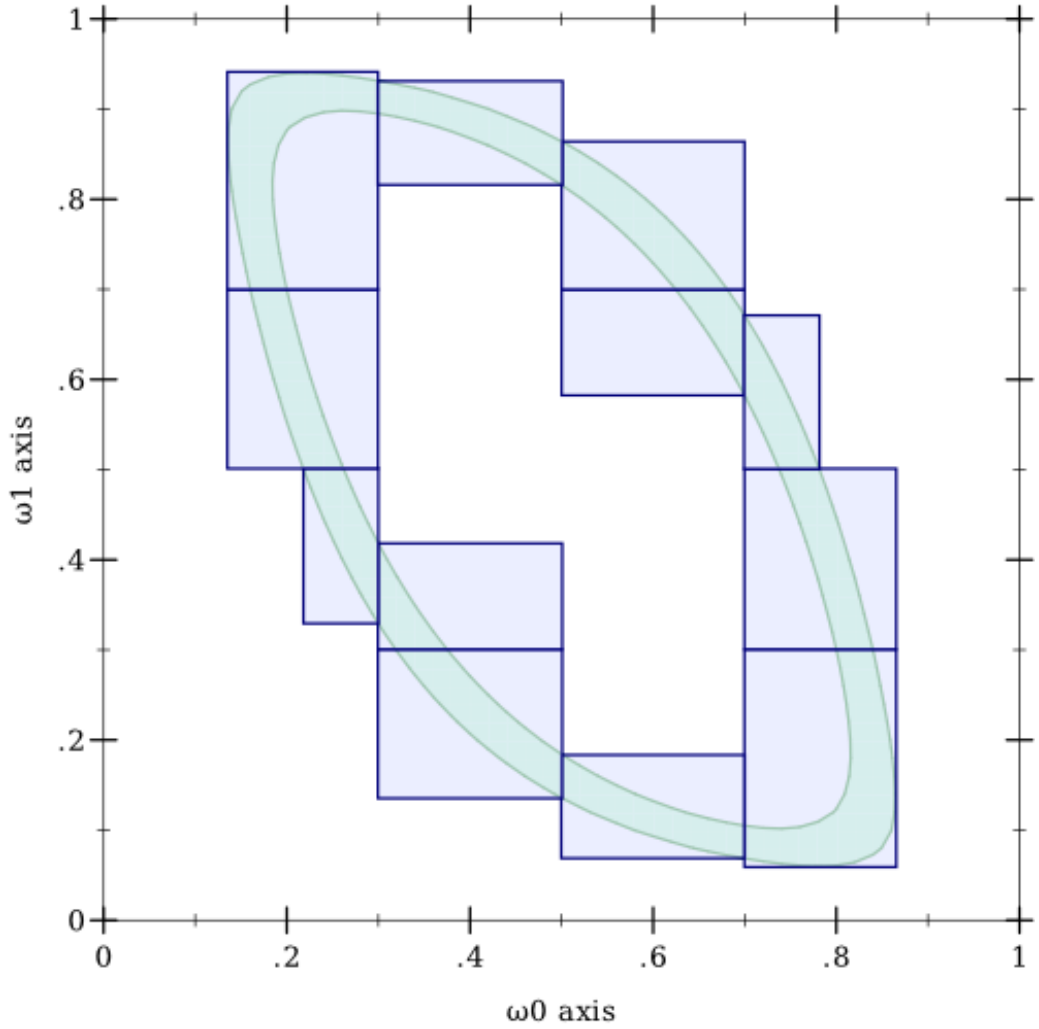

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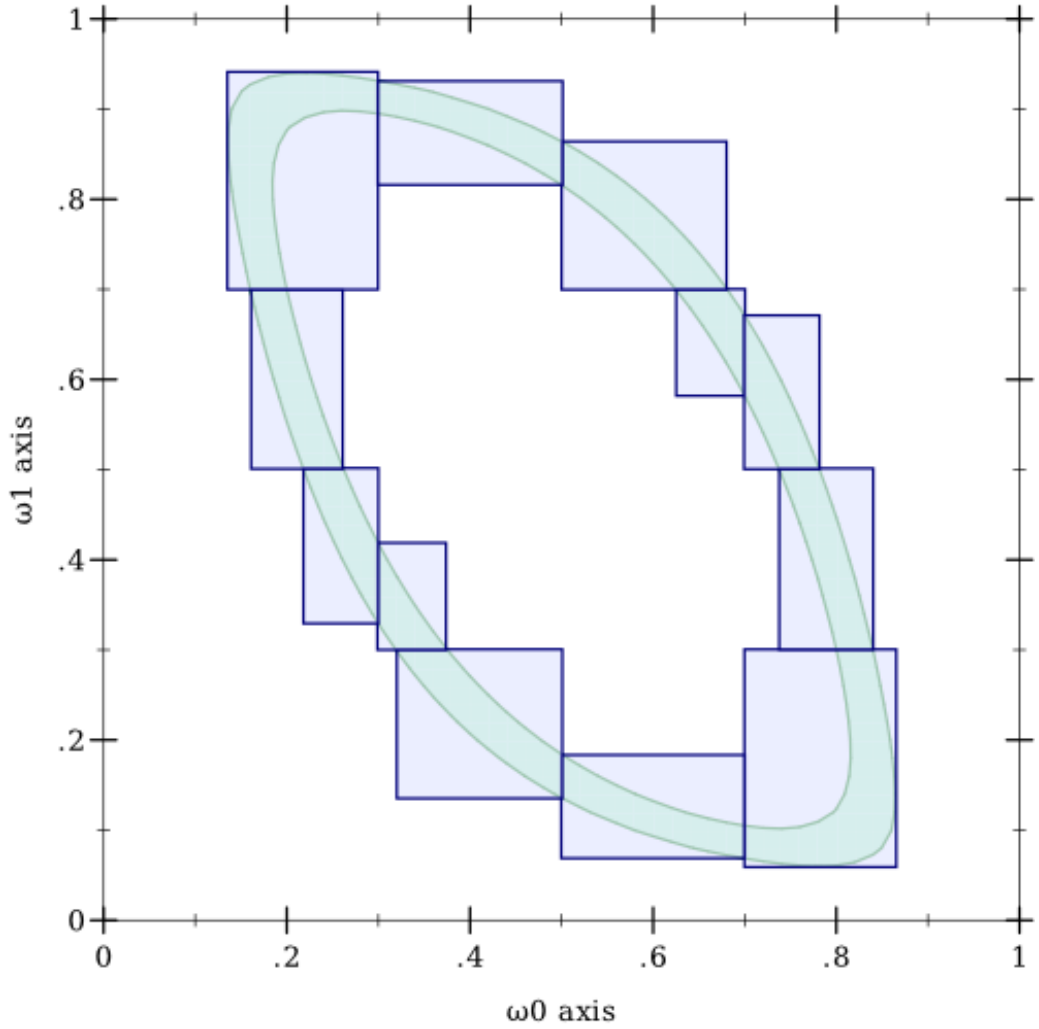

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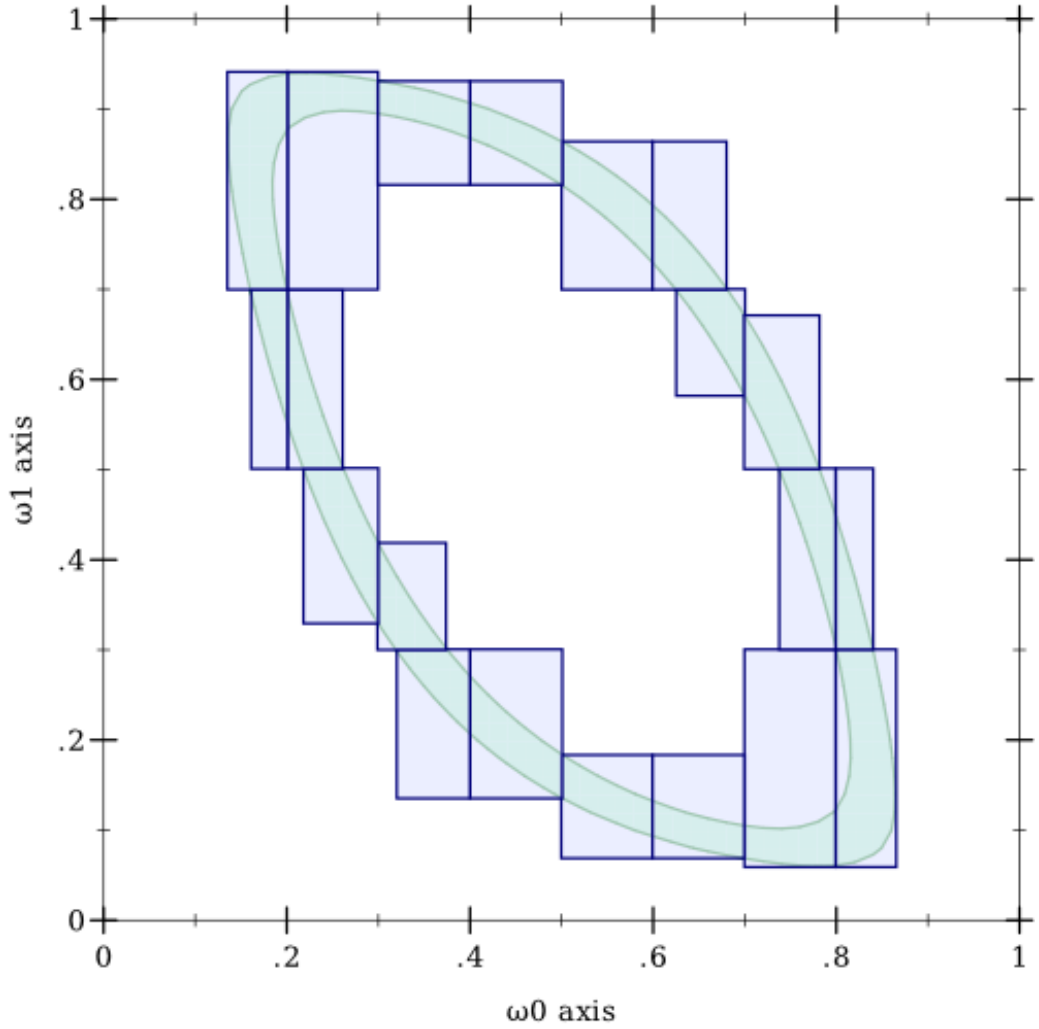

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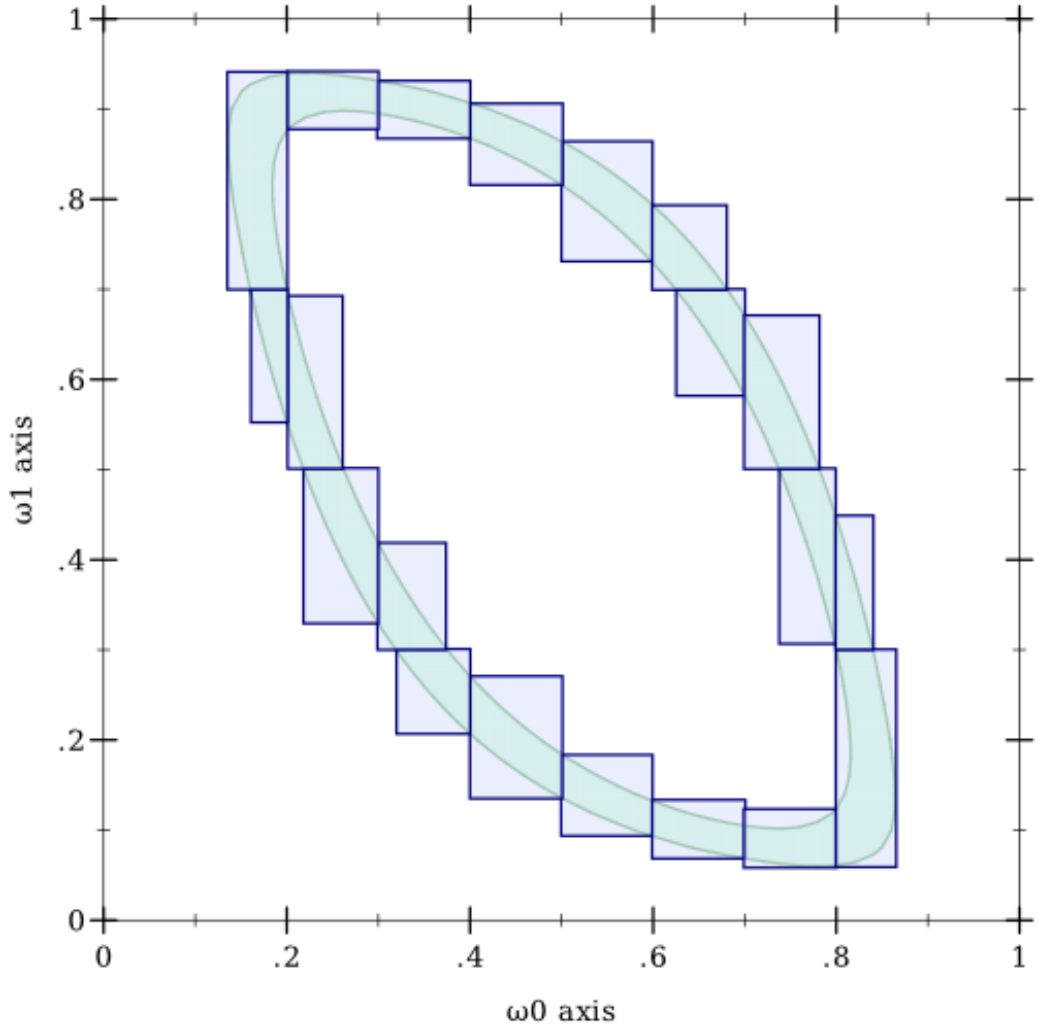

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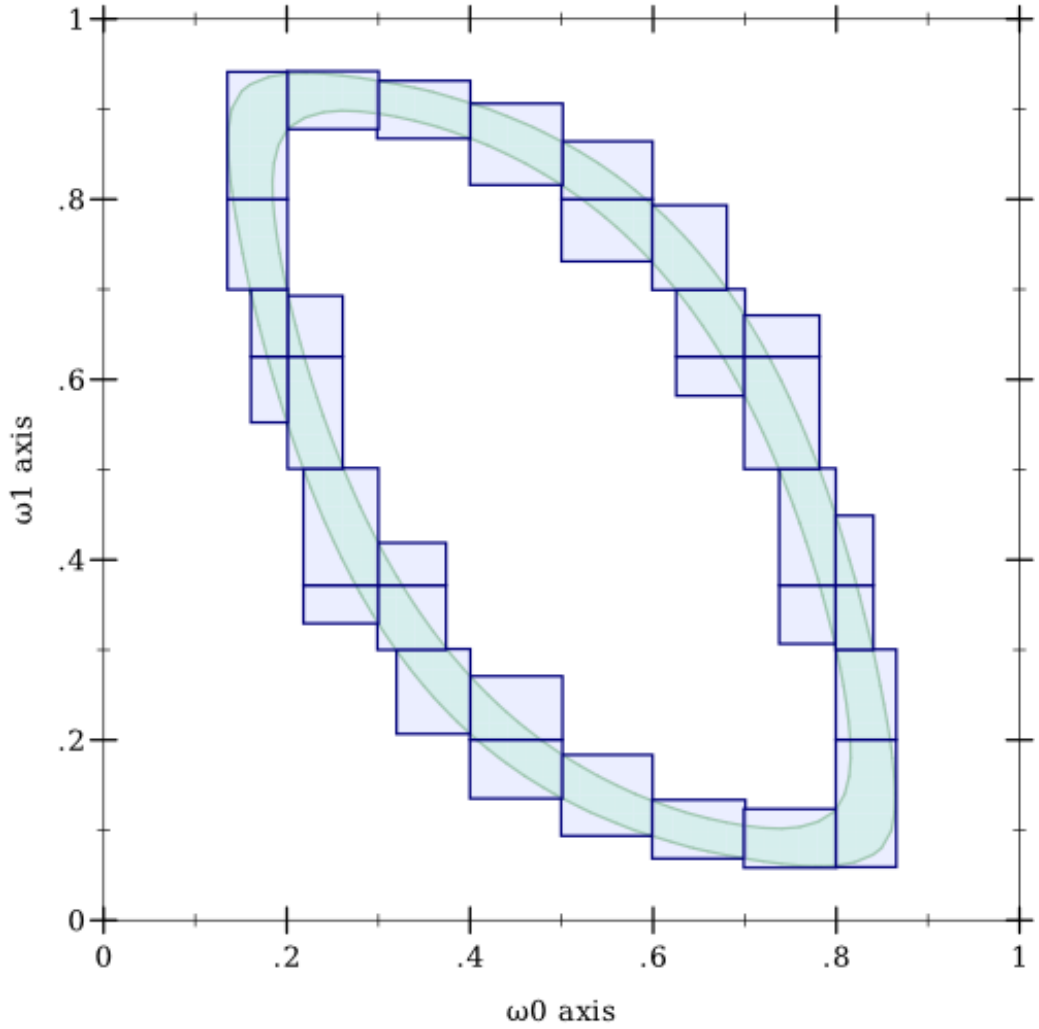

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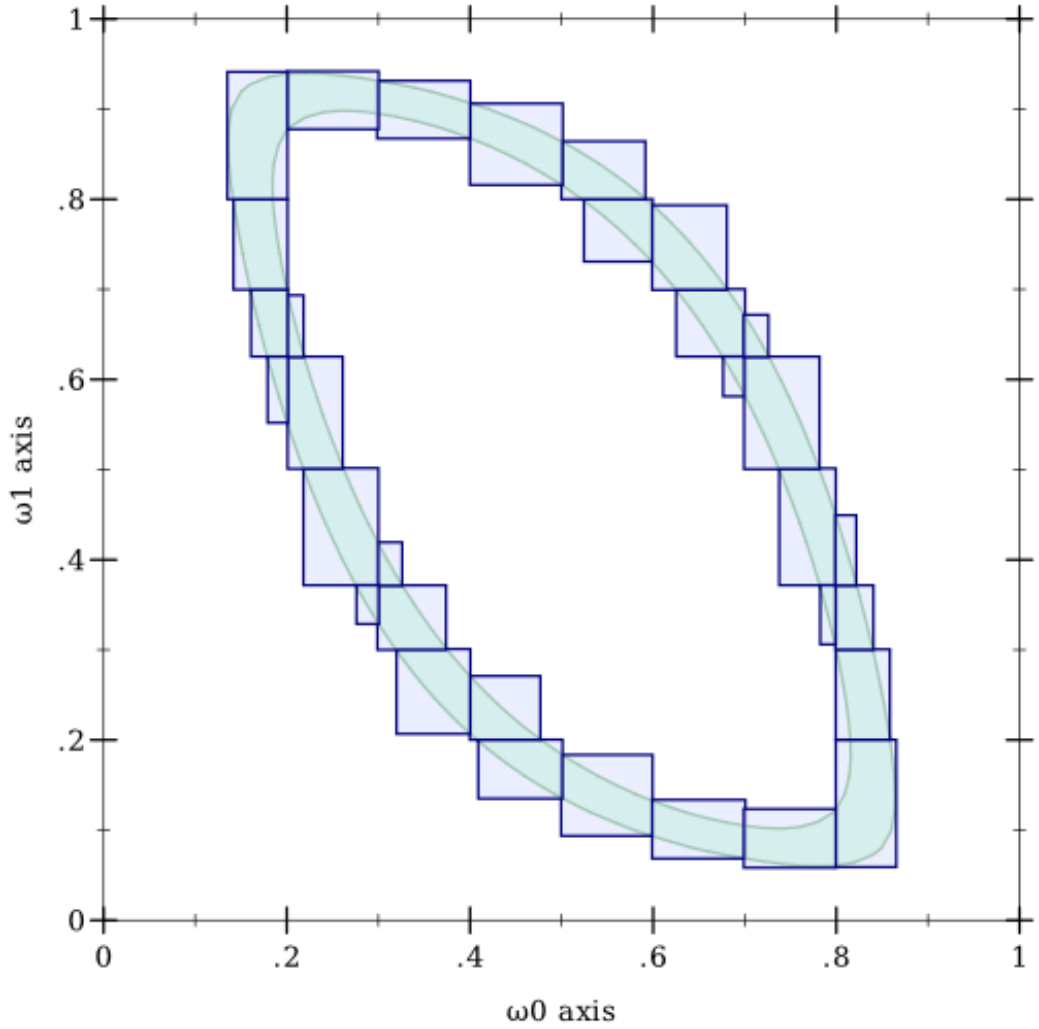

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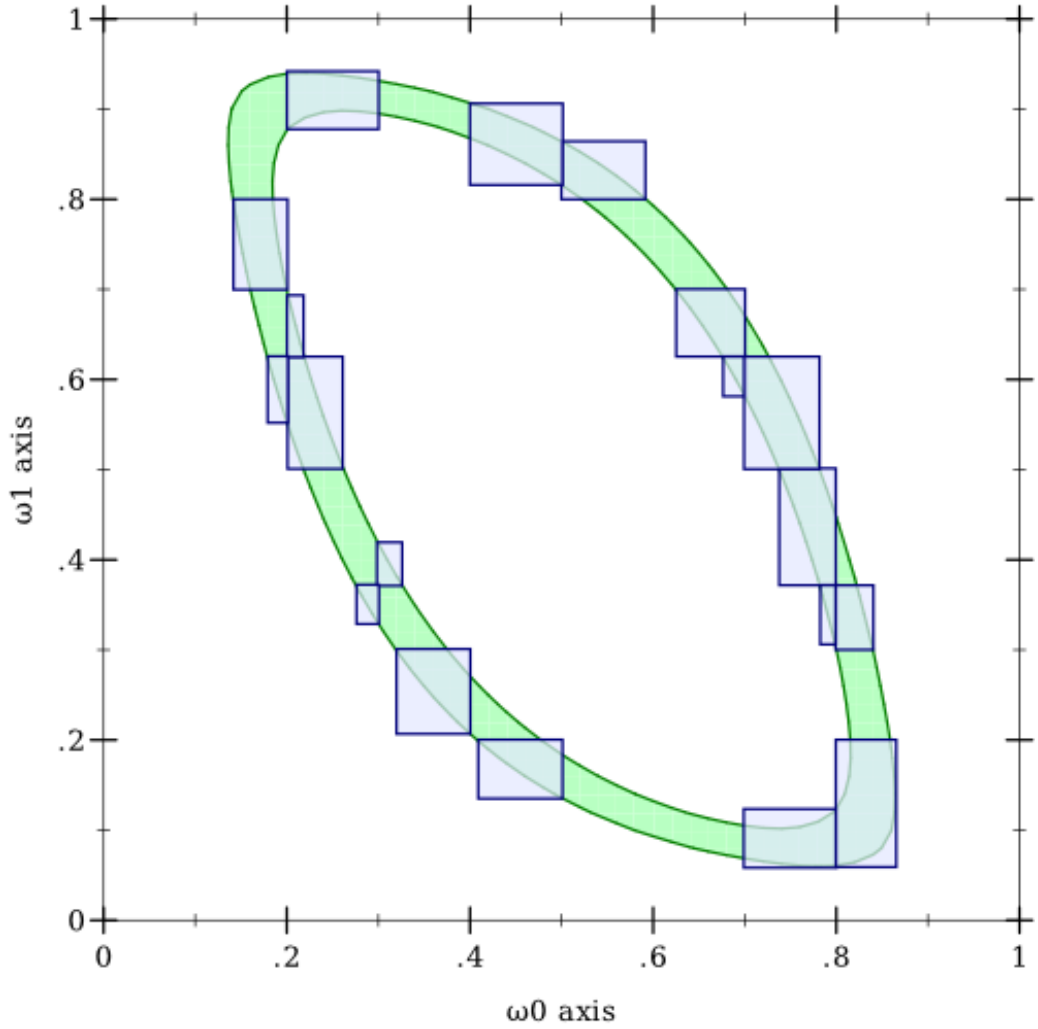
Sampling to overcome exponential explosion:



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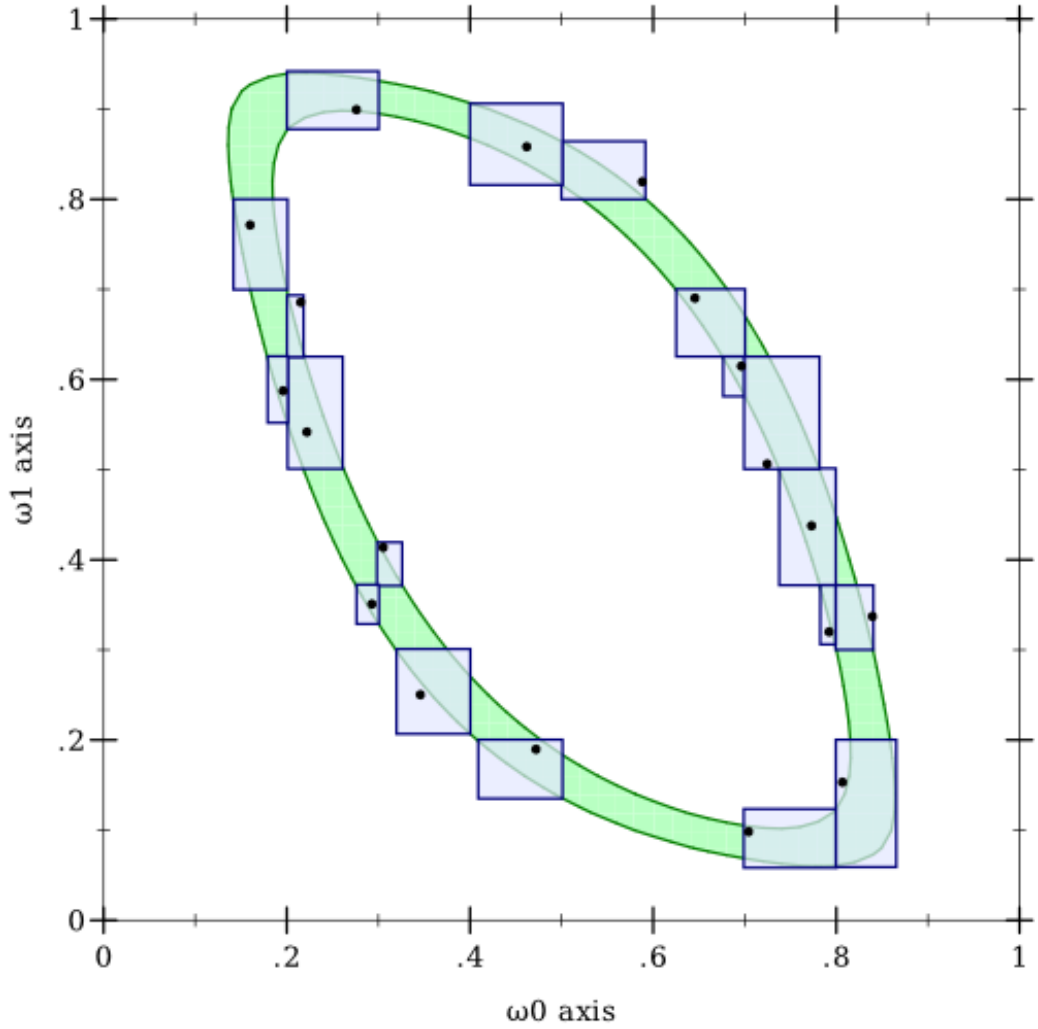

For Conditioned Sampling, We Need...

- Standard interpretation of programs as pure functions from a random source
- Efficient way to compute preimage sets
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  - Efficient way to determine whether a domain sample is actually in the preimage (just use standard interpretation)
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Moar Precision: Standard Interpretation

- Grammar:

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- Easy implementation: $[\![\cdot]\!]$ becomes a Racket macro

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- Short version: If you stick to standard measurable sets, the results of applying a random function to a random value can't have a sensible distribution
- It's okay, we can defunctionalize lambdas away (closures are just products)

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then $\mathrm{pair}_{\mathbf{pre}}\ h_1\ h_2$ computes preimages under $a \mapsto \langle f_1(a), f_2(a) \rangle$.

Pair Preimages

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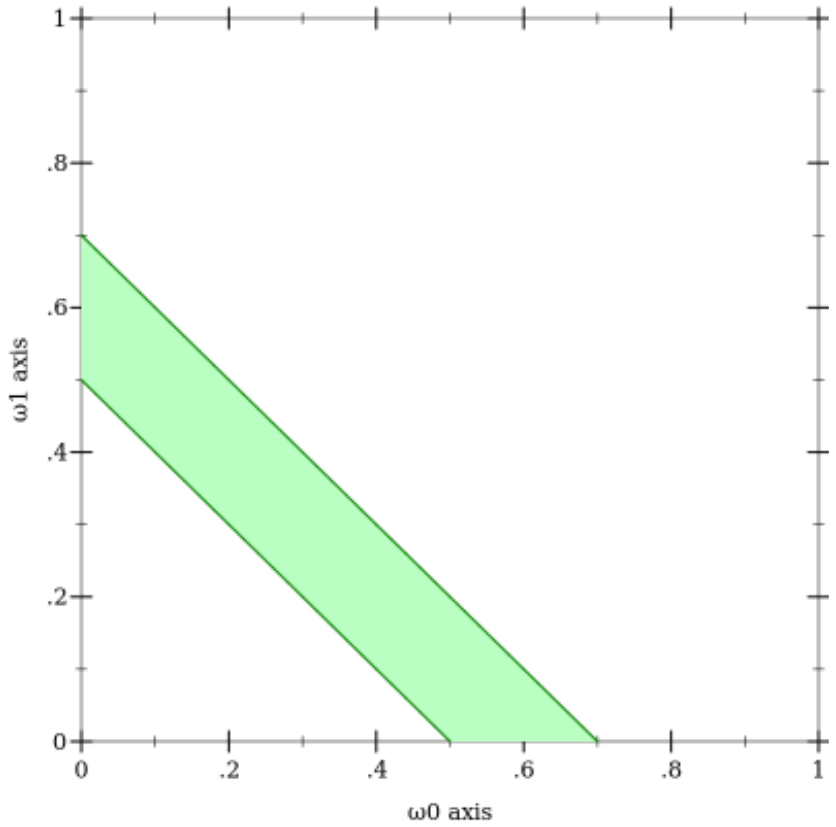

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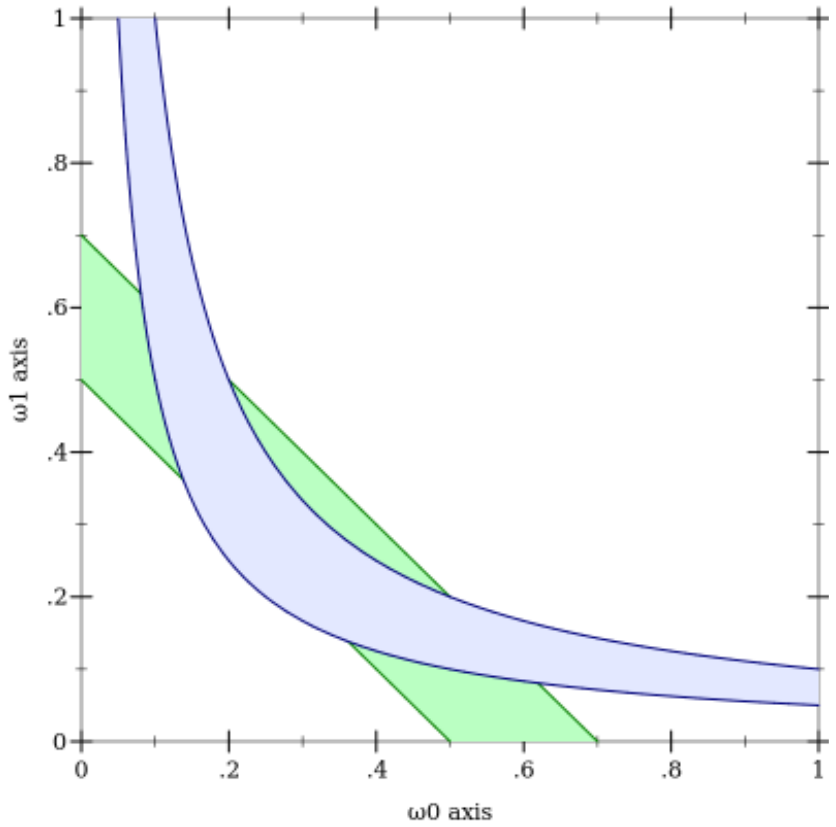

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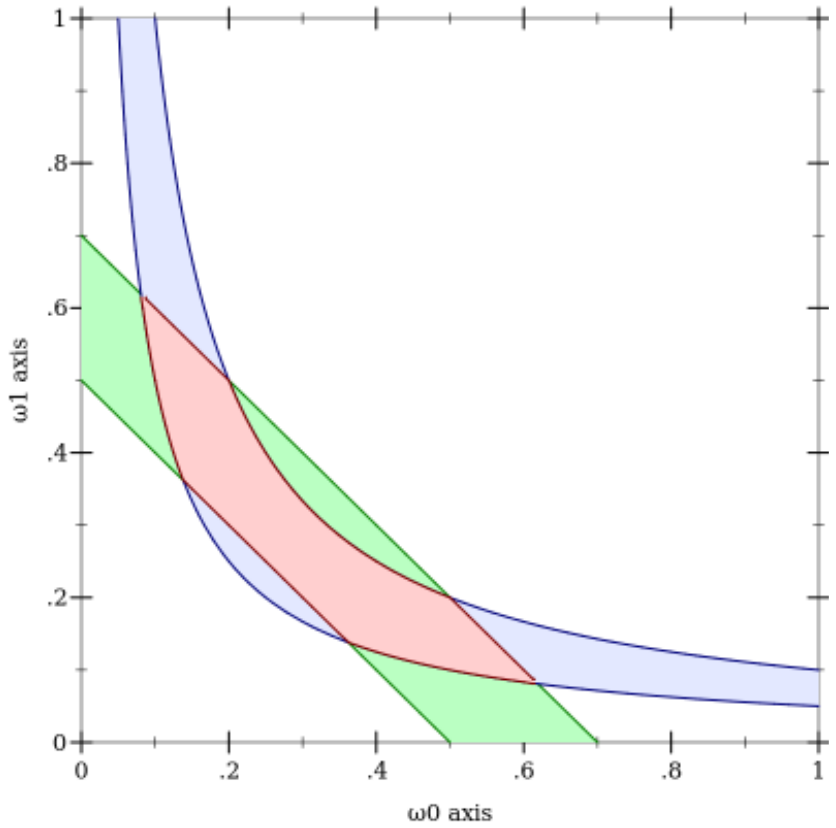

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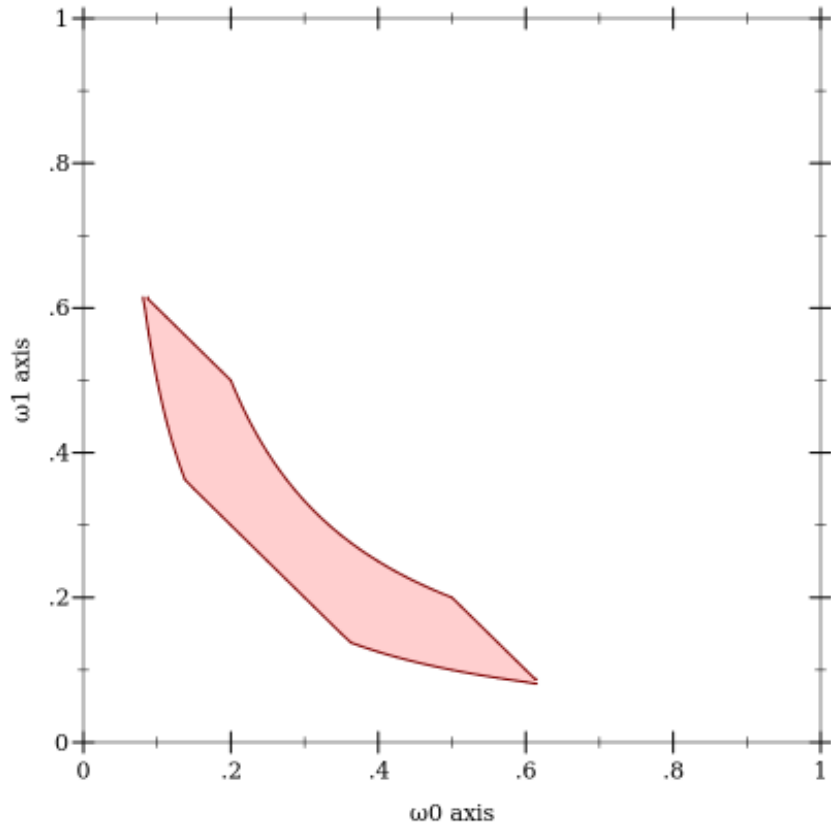

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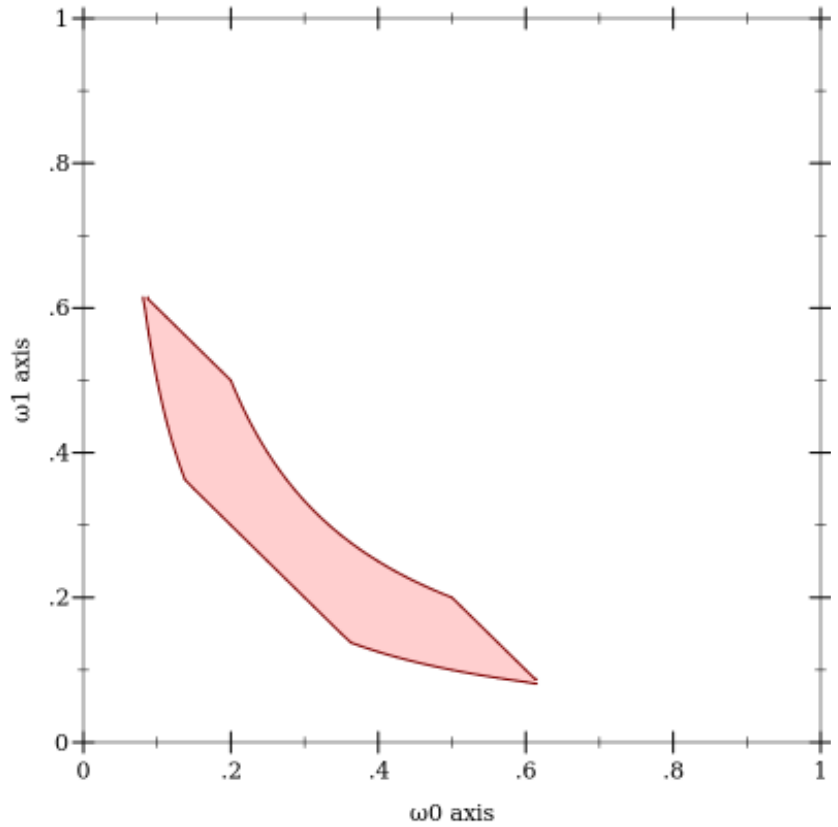

- pair_{pre} does this, but for arbitrary output sets

Compositional Semantics

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Corollary. If

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Nonstandard Interpretation Correctness

Theorem. For all programs p, $[\![p]\!]_{\mathbf{pre}}$ computes preimages under $[\![p]\!]$.

Proof. By structural induction on program terms.

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  - A. Nowhere, but we can approximate them.
- Q. Where did you get a λ-calculus that could operate on arbitrary, possibly infinite sets, anyway?
  - A. Well...

Lambda-ZFC

N. Toronto and J. McCarthy. Computing in Cantor's Paradise With λ_{ZFC}. FLOPS 2012

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λ calculus

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λ calculus

+

Set theory

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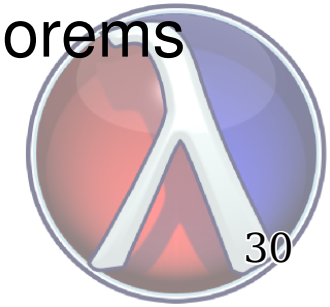

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- Applies to all probabilistic language work to date and in the forseeable future

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- Derive $[\![\cdot]\!]'_{\mathbf{pre}} : p \to (\Omega \underset{\mathbf{pre}}{\rightsquigarrow}' B)$

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Theorem (sound). $[\![\cdot]\!]'_{\mathrm{pre}}$ computes overapproximations of the preimages computed by $[\![\cdot]\!]_{\mathrm{pre}}$.

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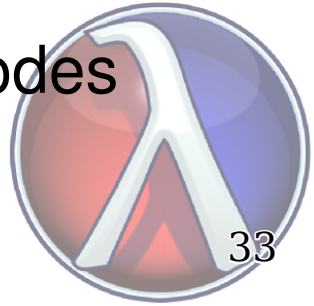
Theorem (decreasing). $[\![\cdot]\!]'_{\mathrm{pre}}$ never returns preimages larger than the given subdomain.

- Consequence: Refining preimage partitions never explodes



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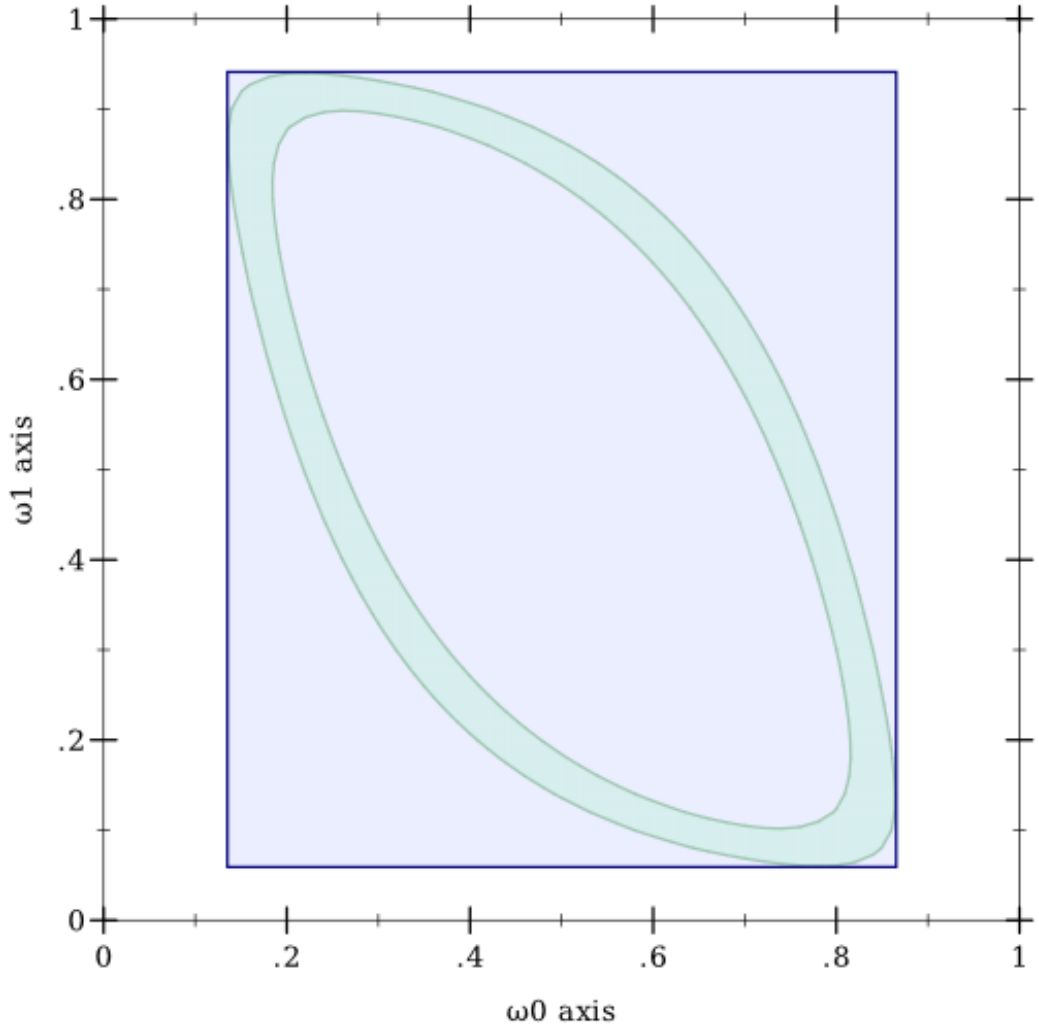

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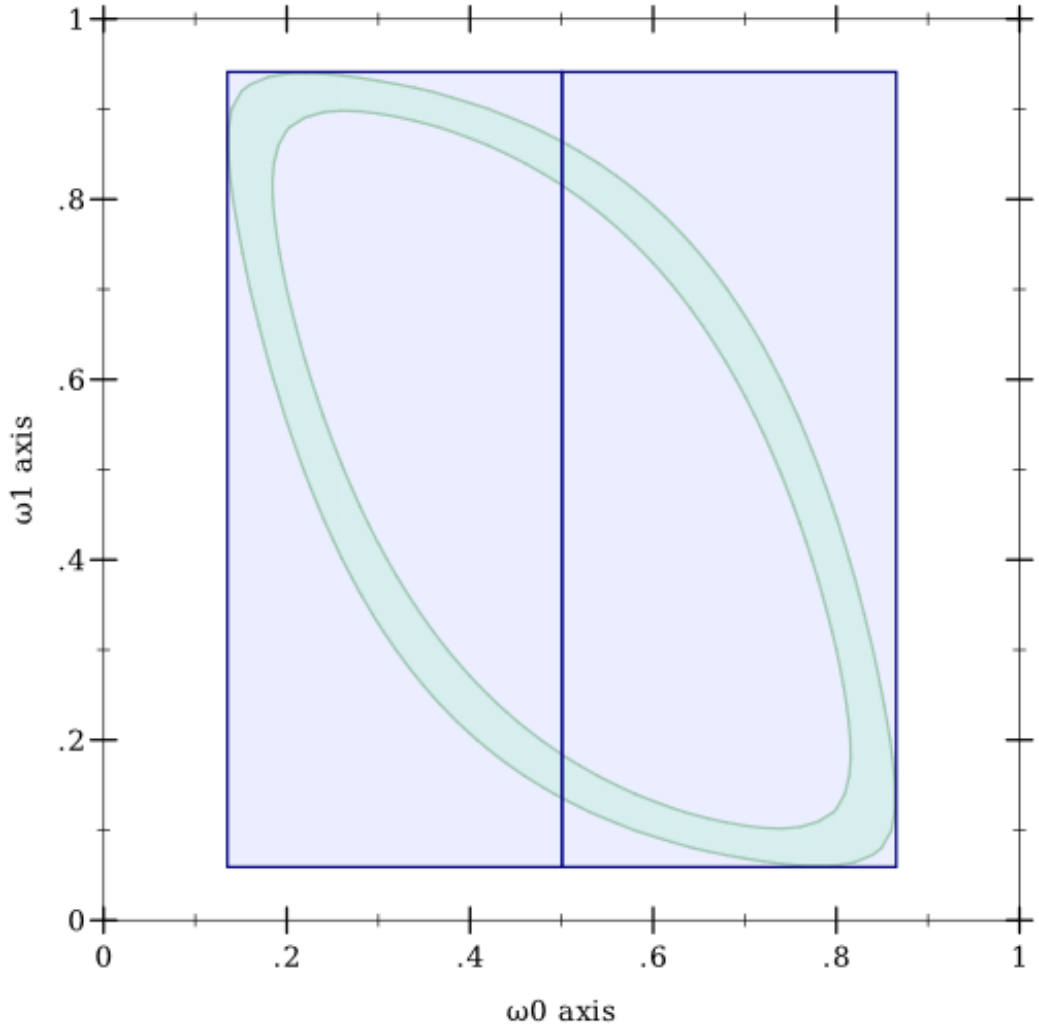

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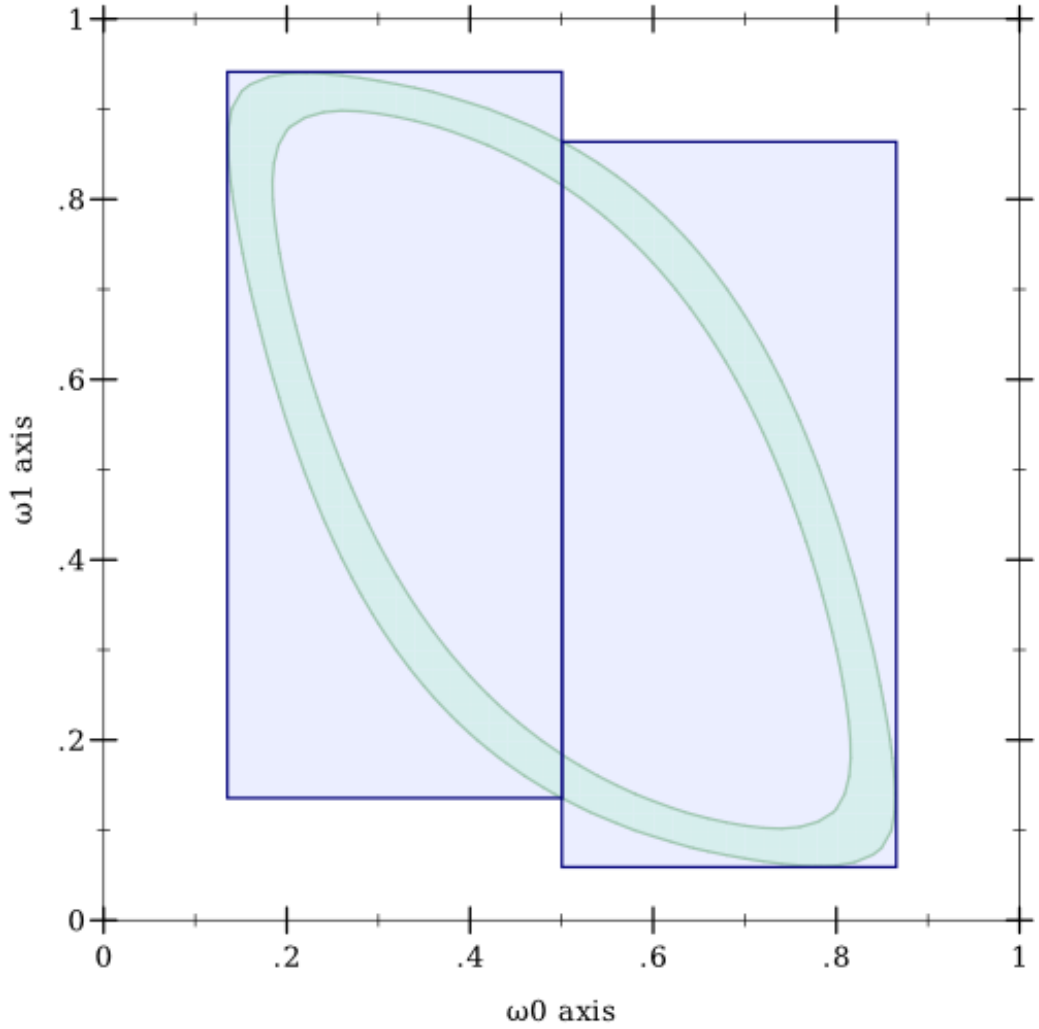

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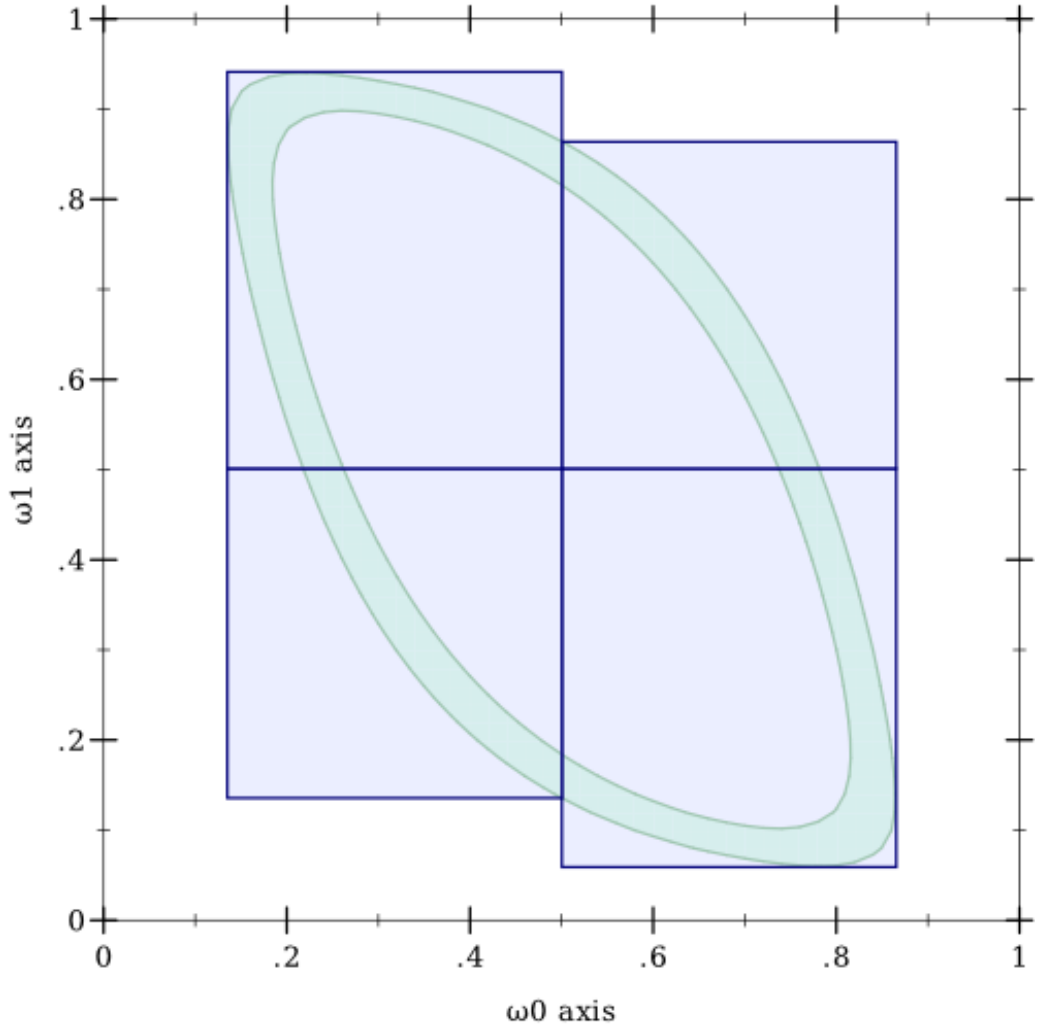

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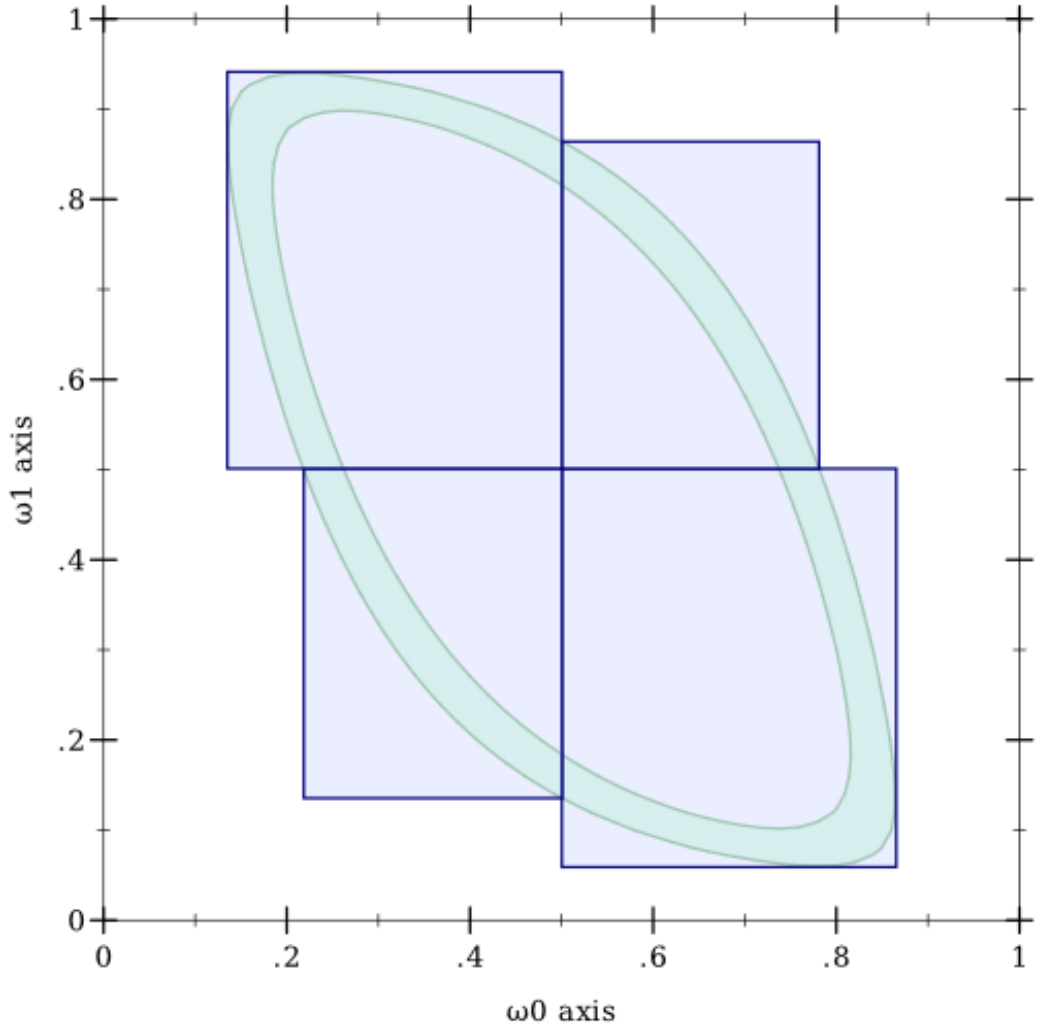

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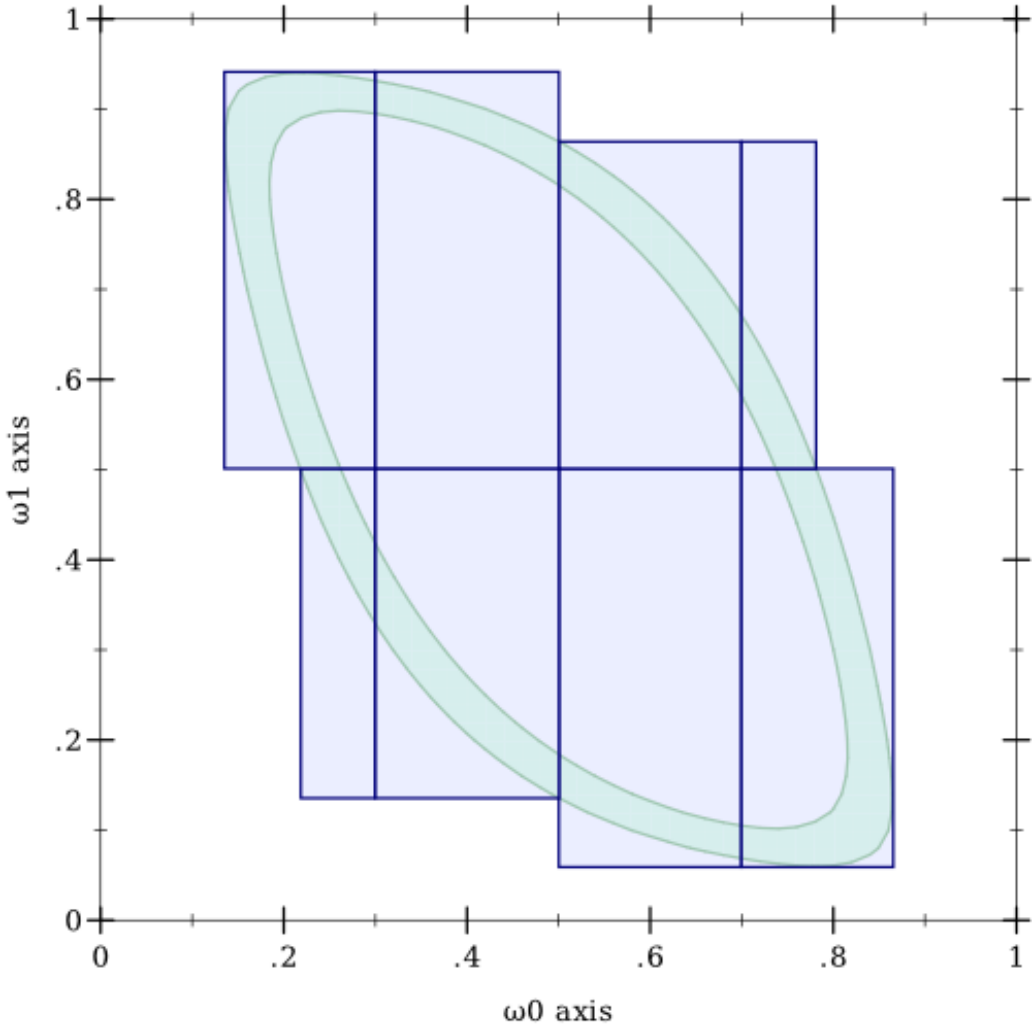

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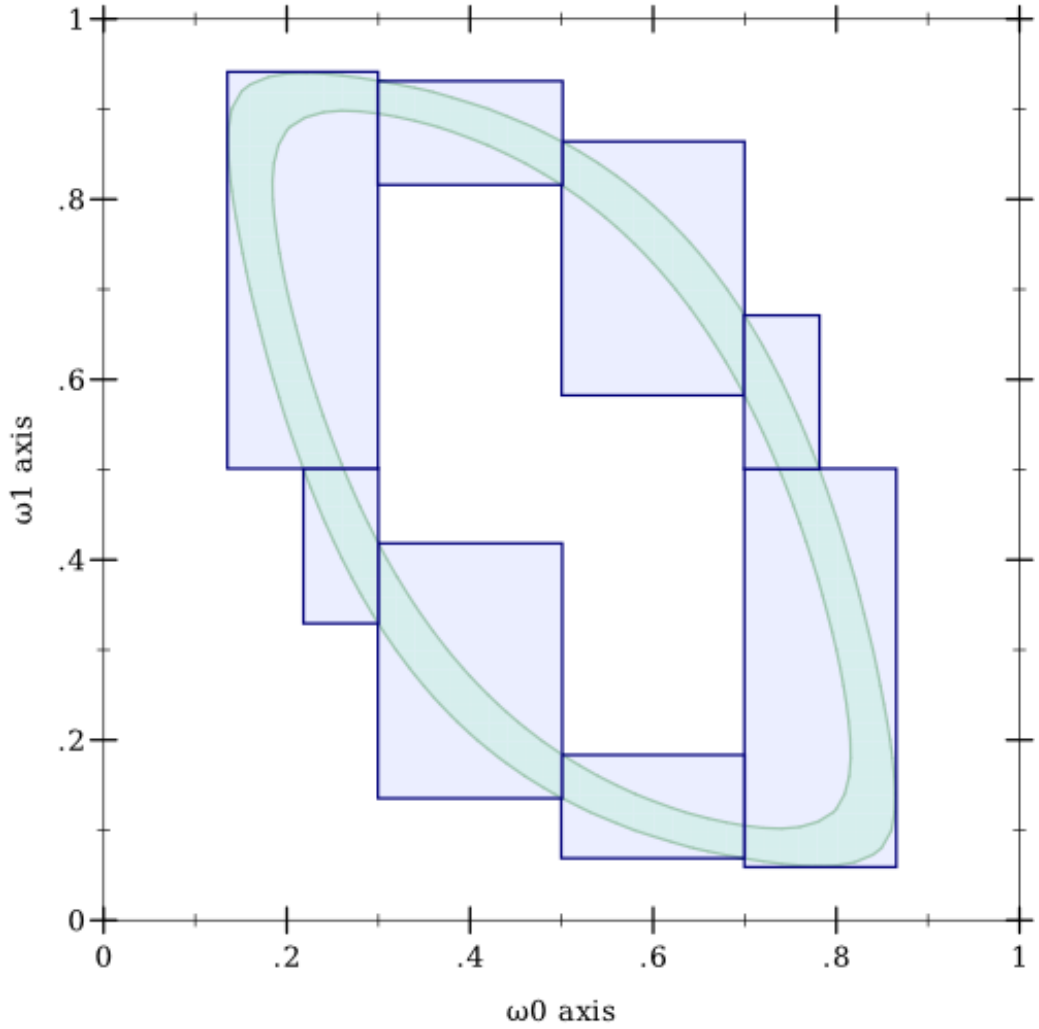

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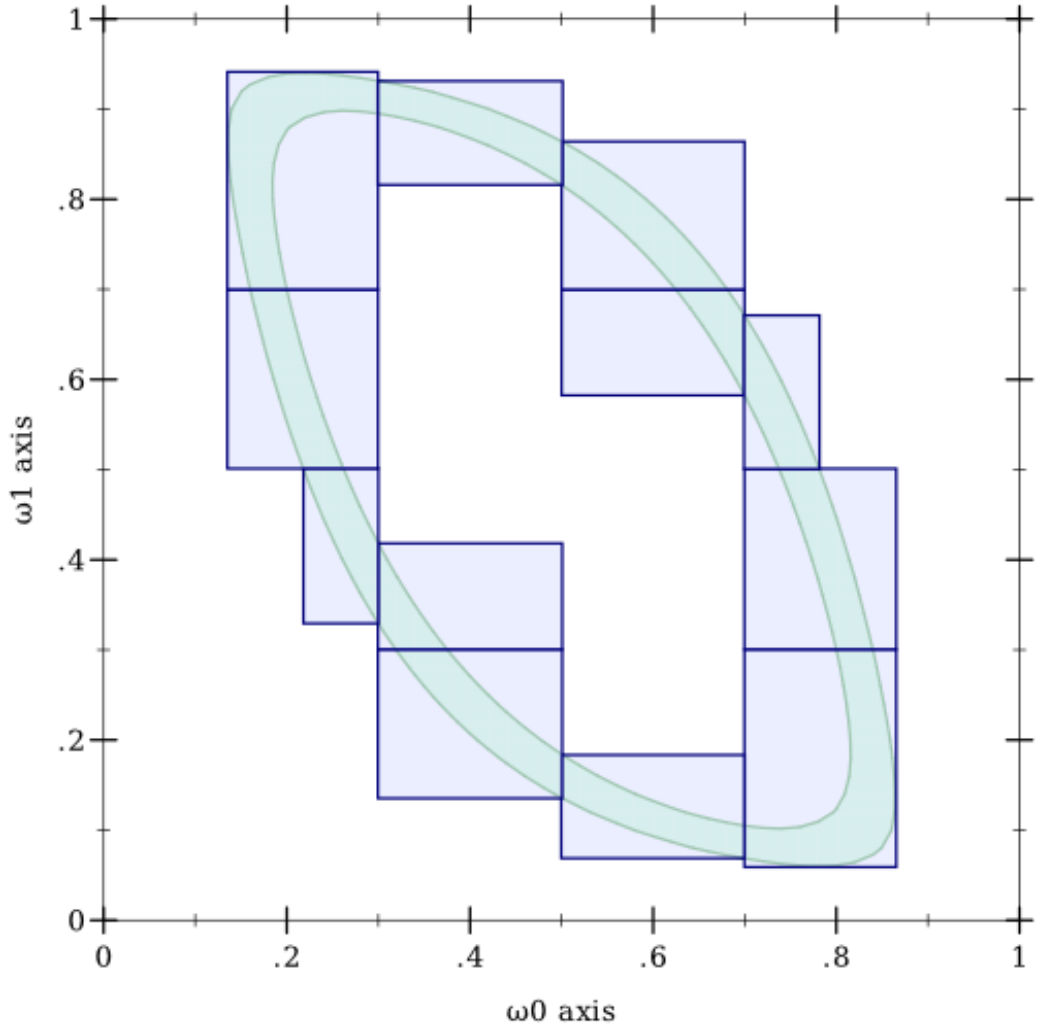

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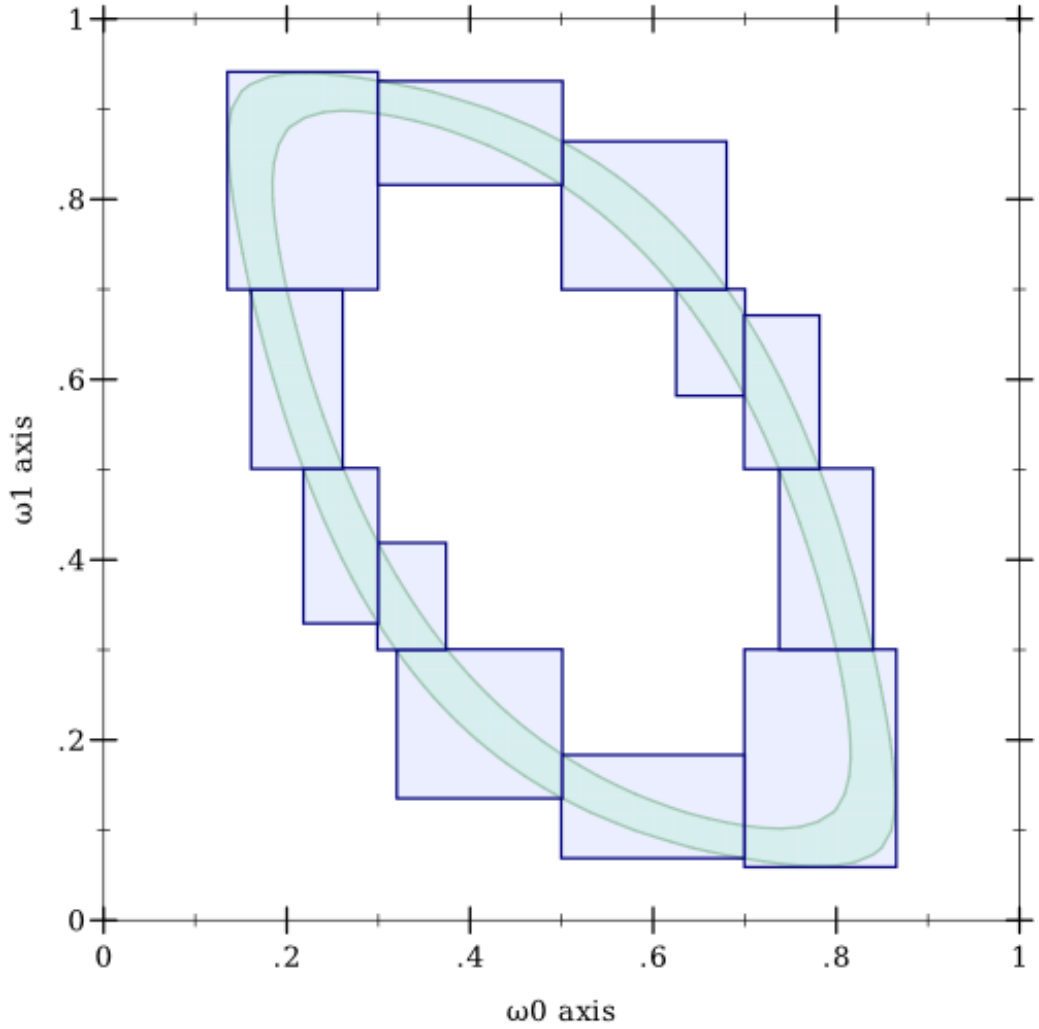

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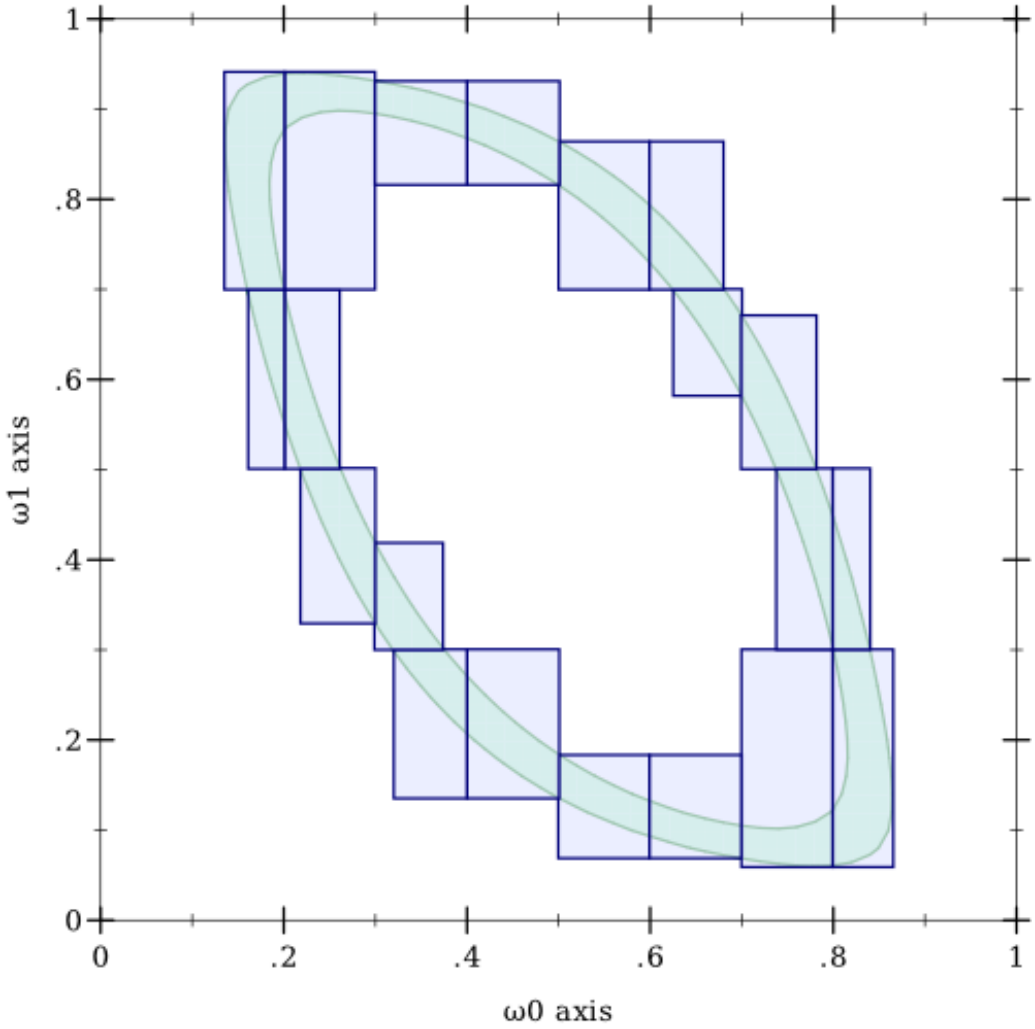

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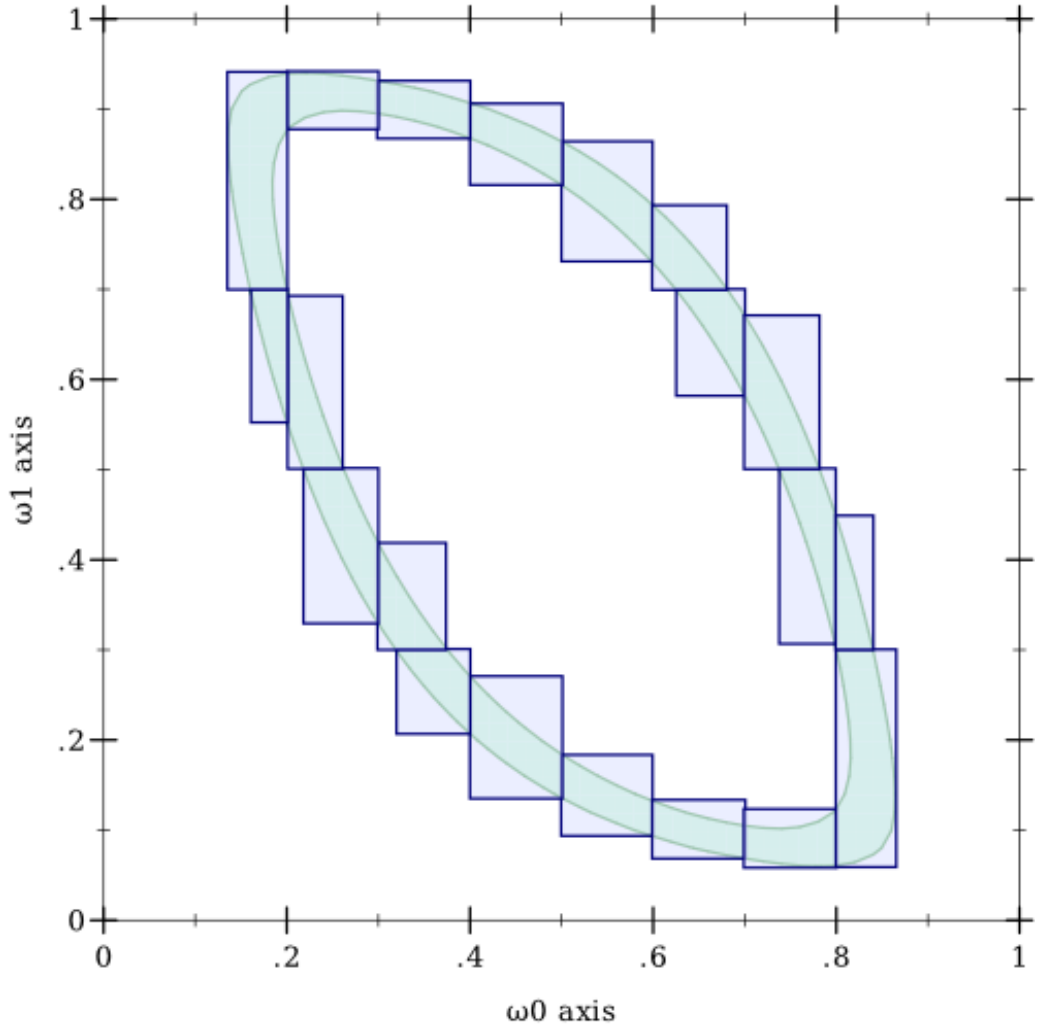

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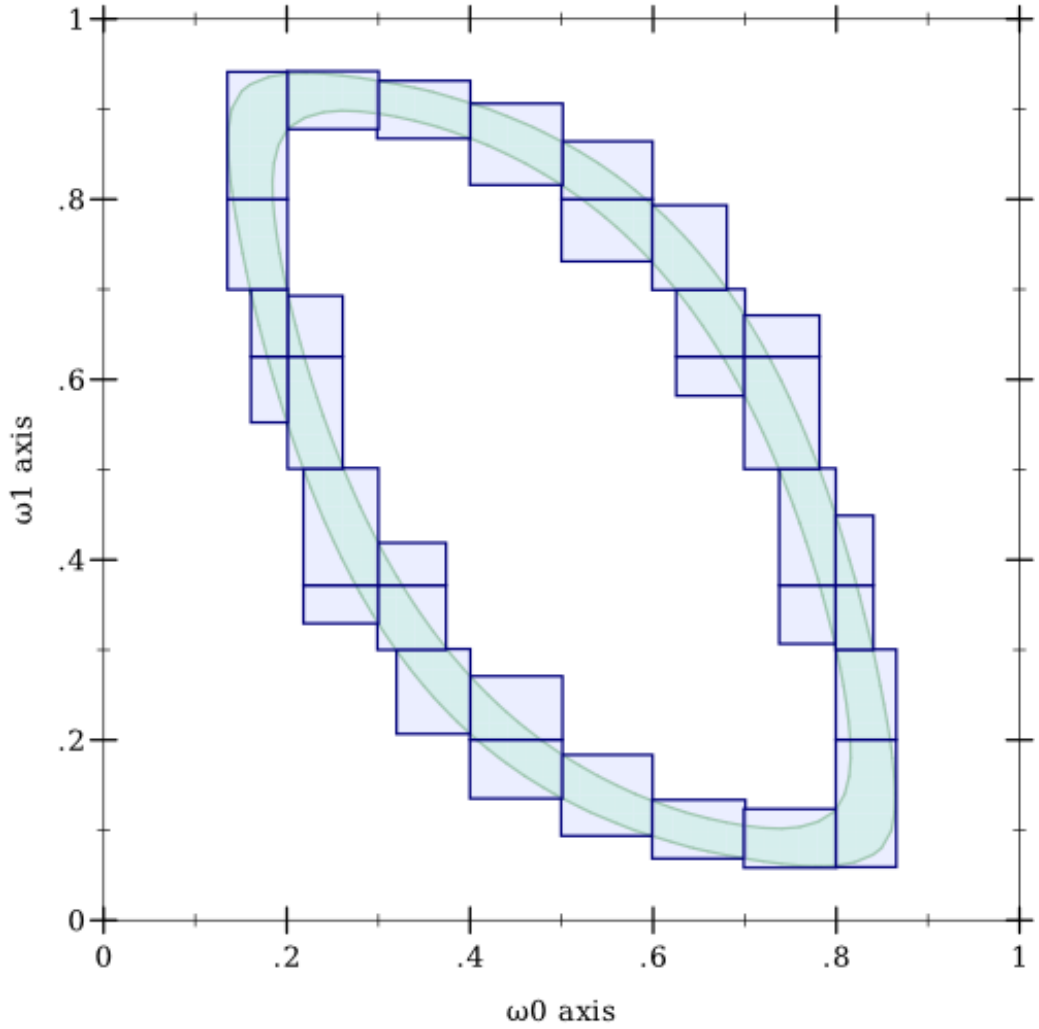

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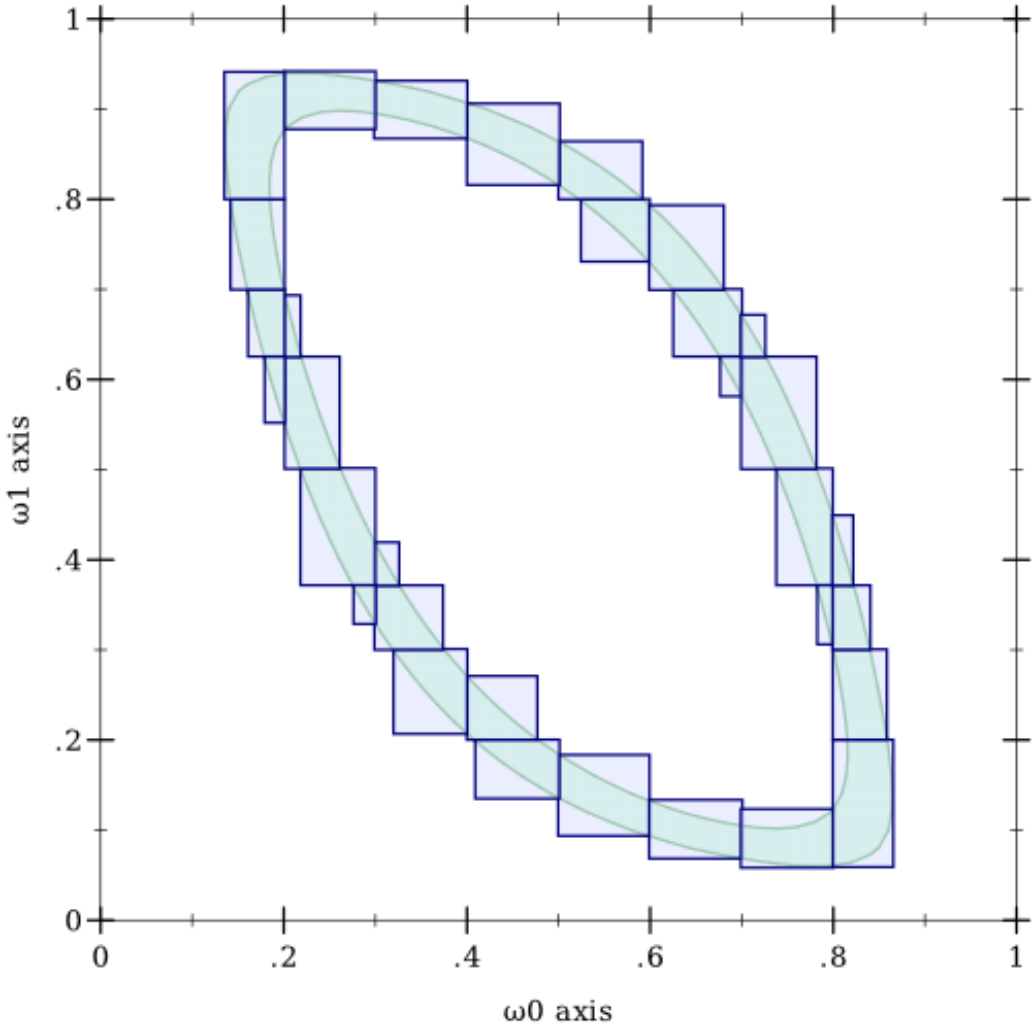

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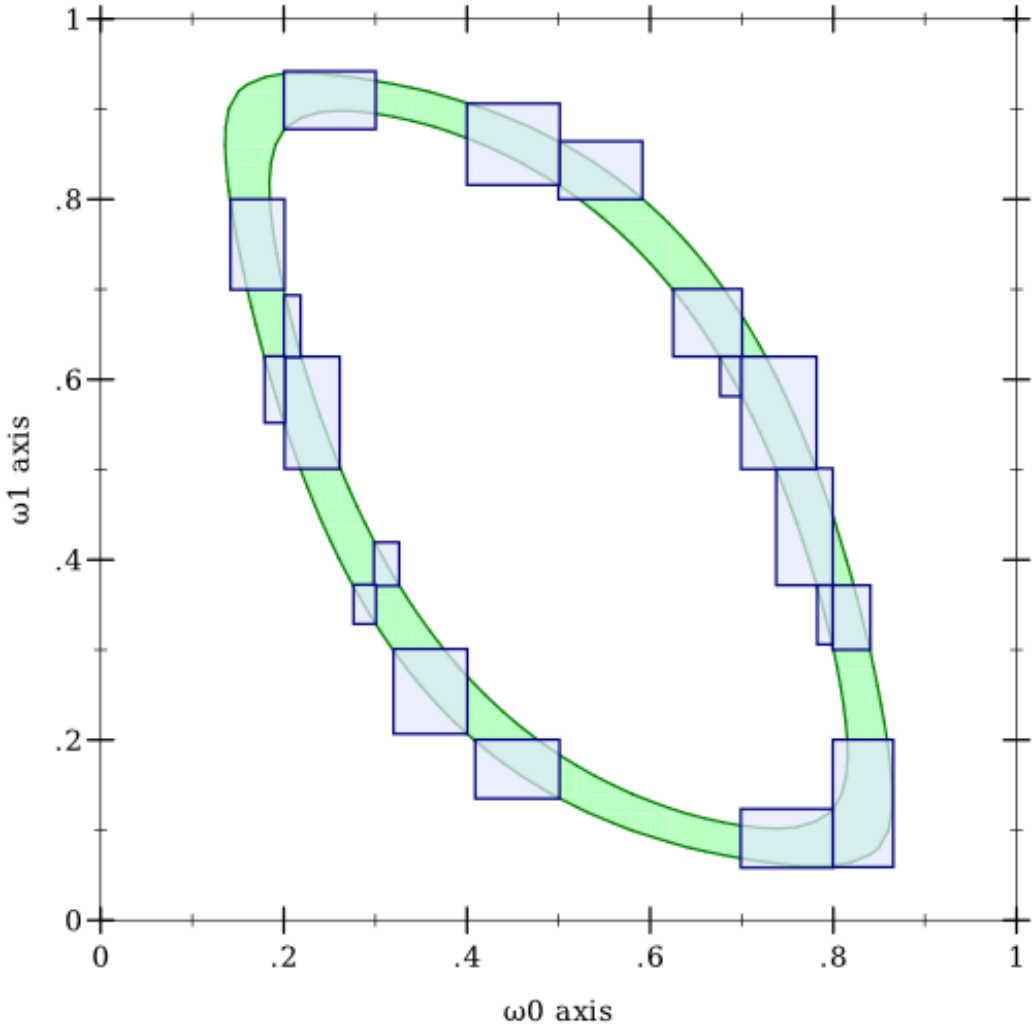

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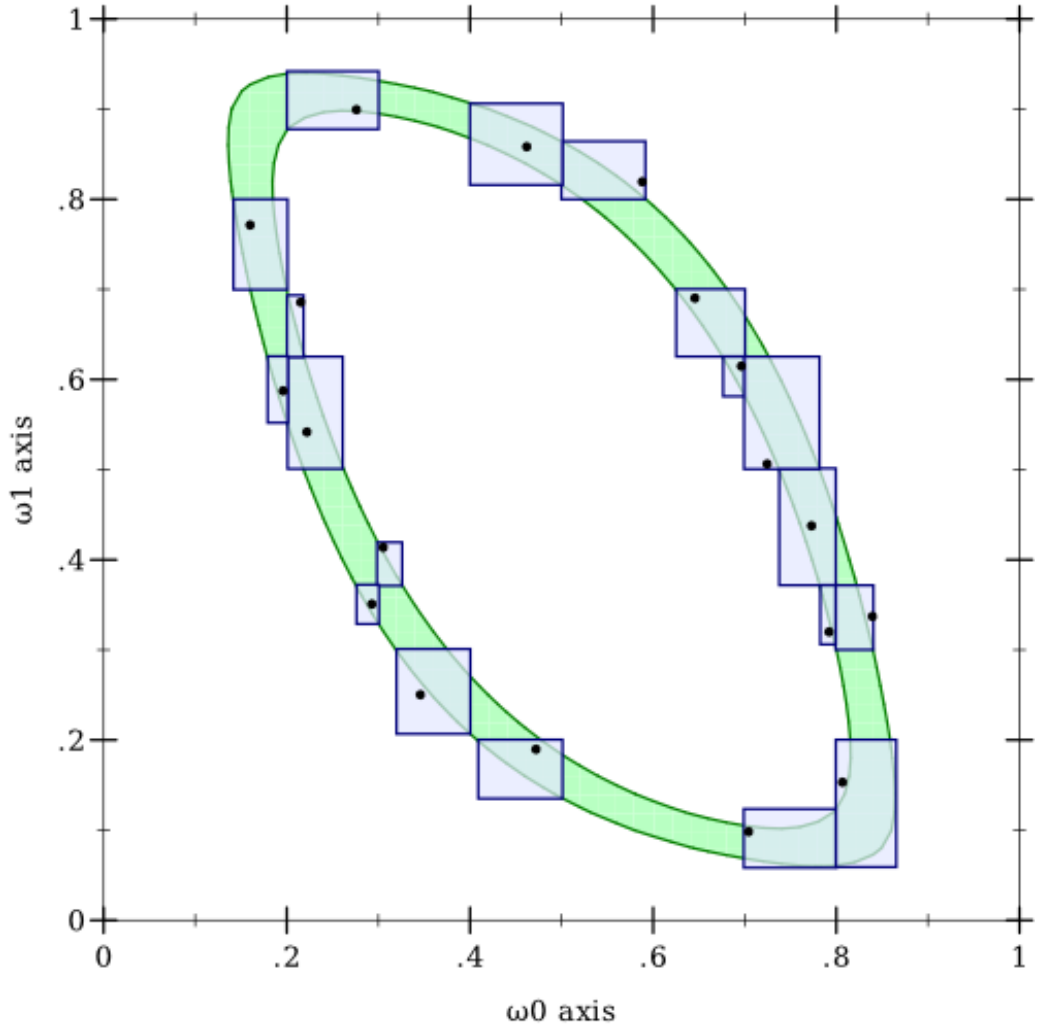

Algorithm 2

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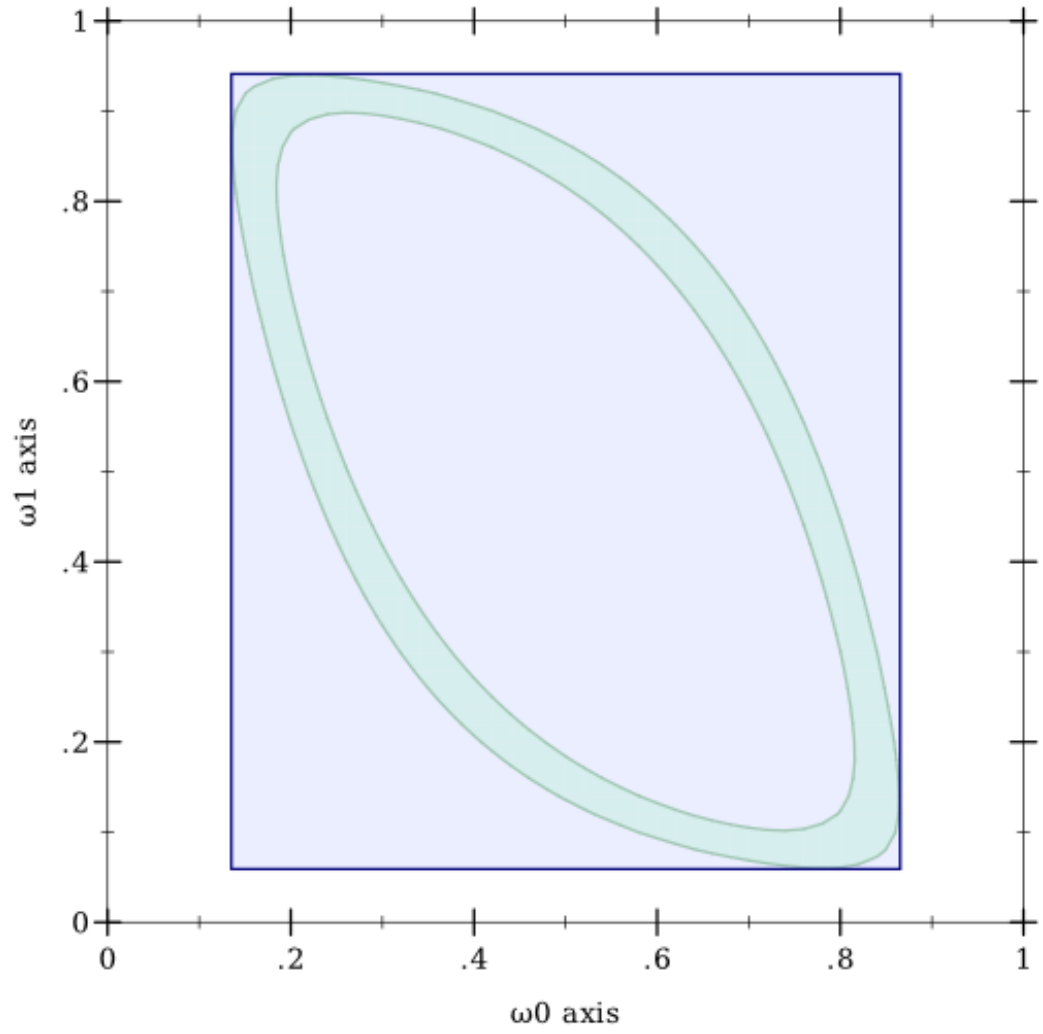
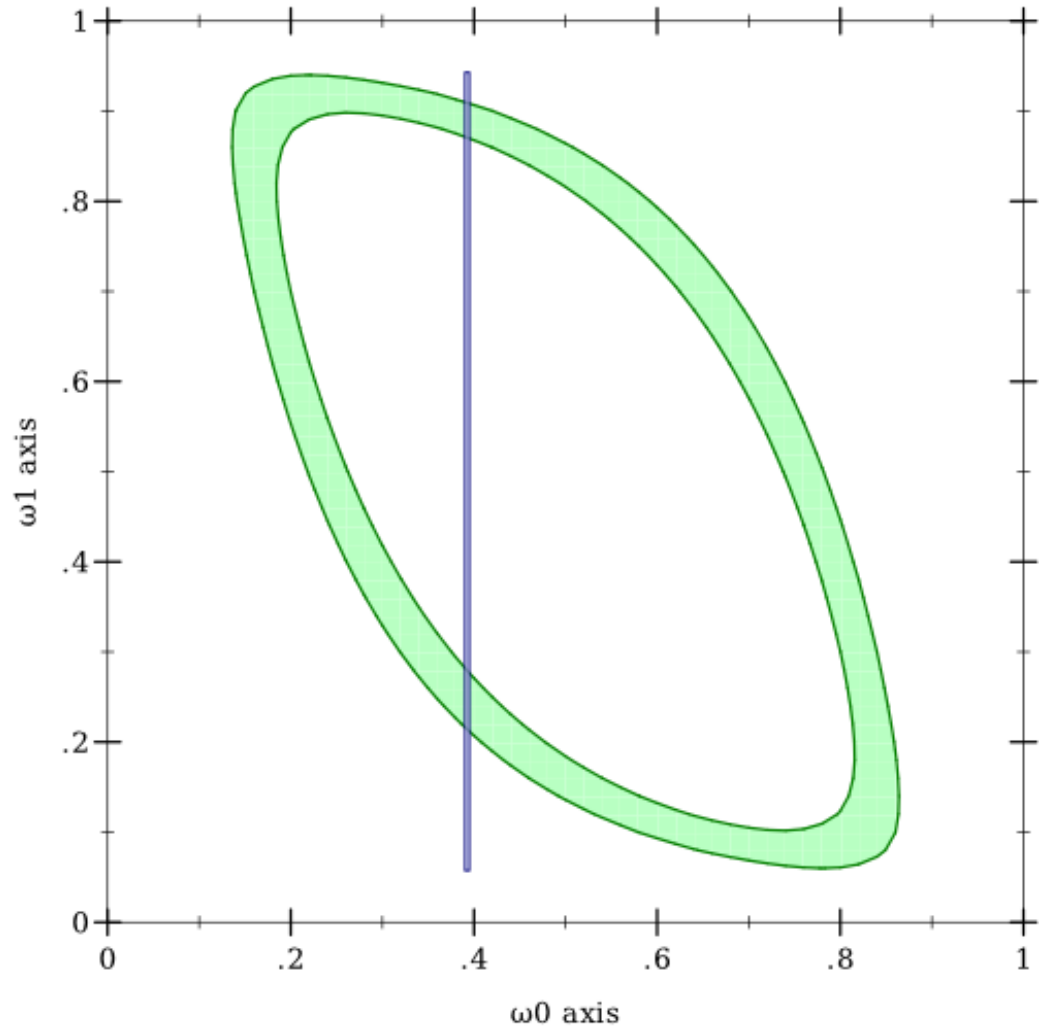
First, refine using preimage computation:



Algorithm 2

- Alternative to arbitrarily low-rate rejection sampling:

Second, randomly choose from arbitrarily fine partition:



Algorithm 2

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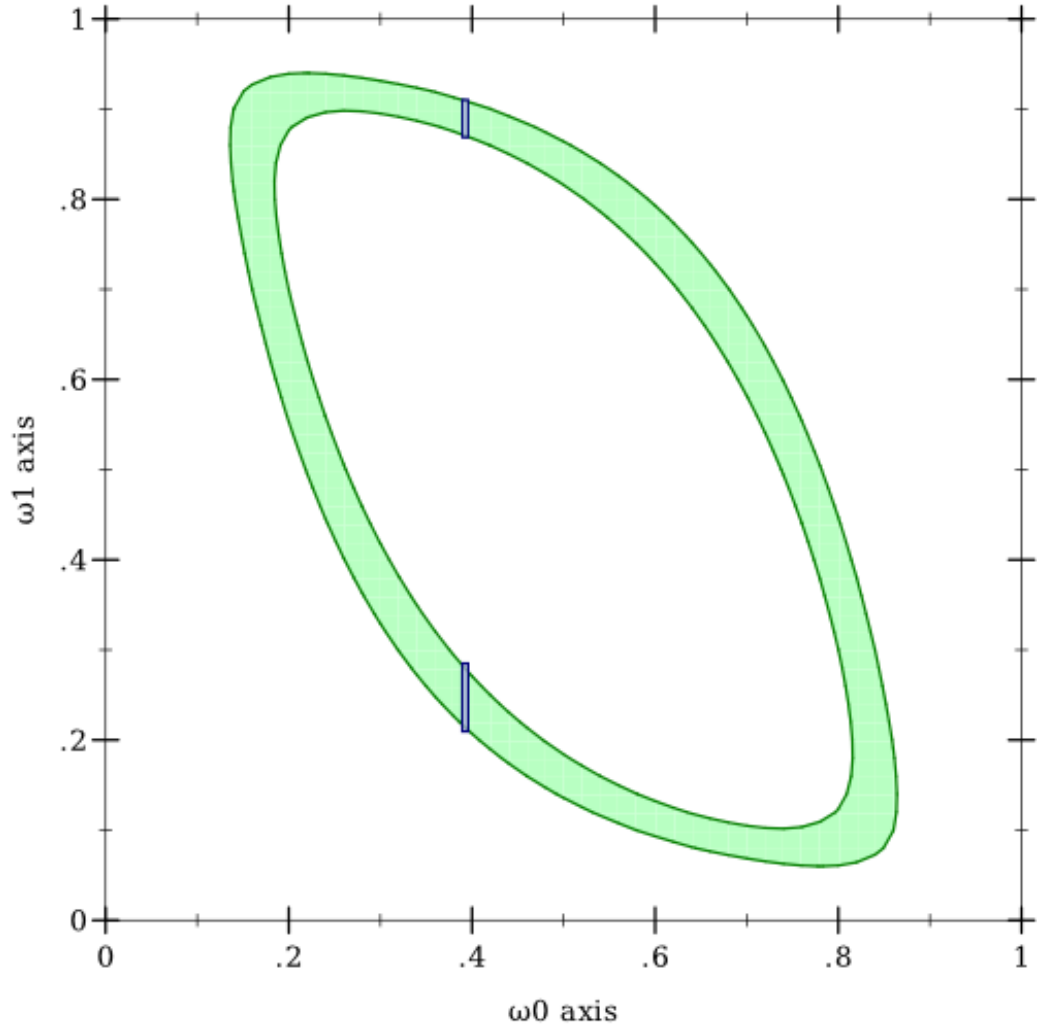
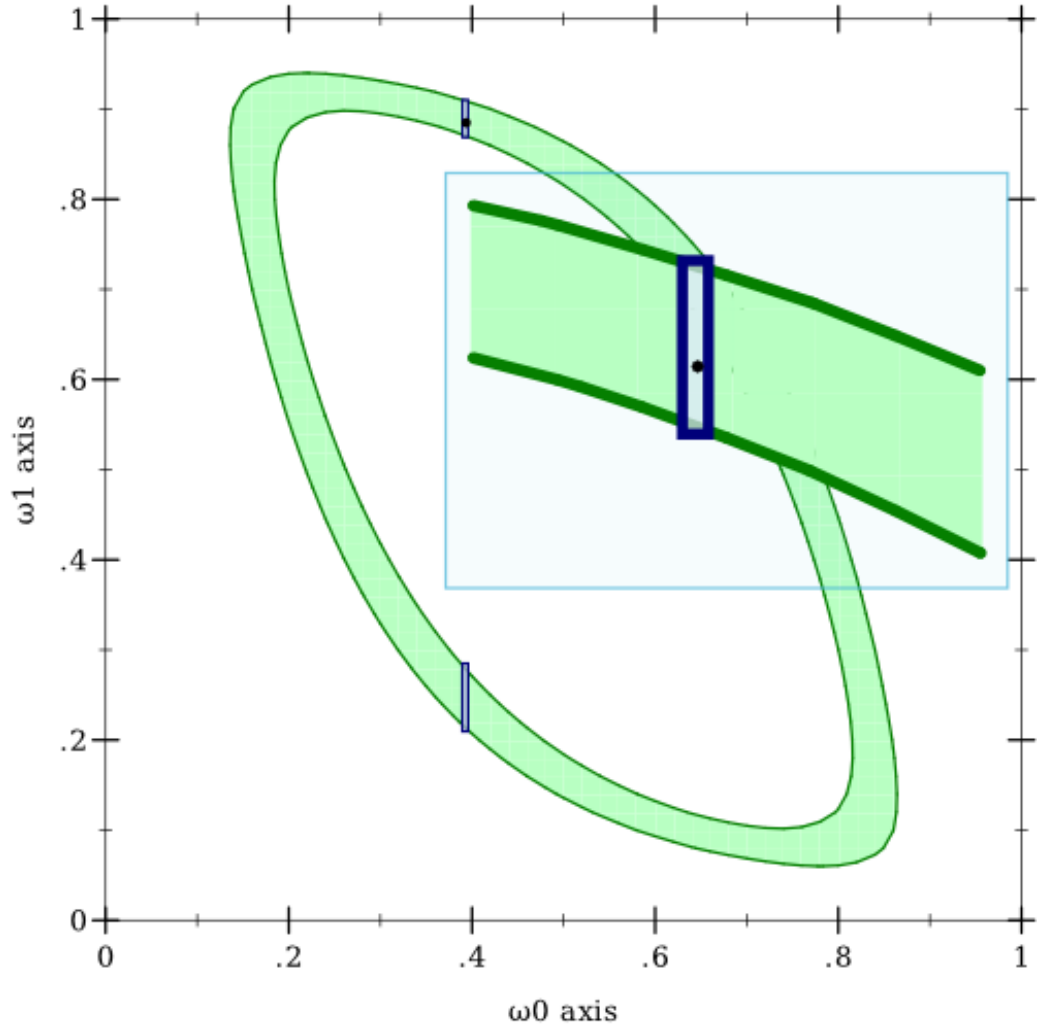
Third, refine again:



Algorithm 2

- Alternative to arbitrarily low-rate rejection sampling:

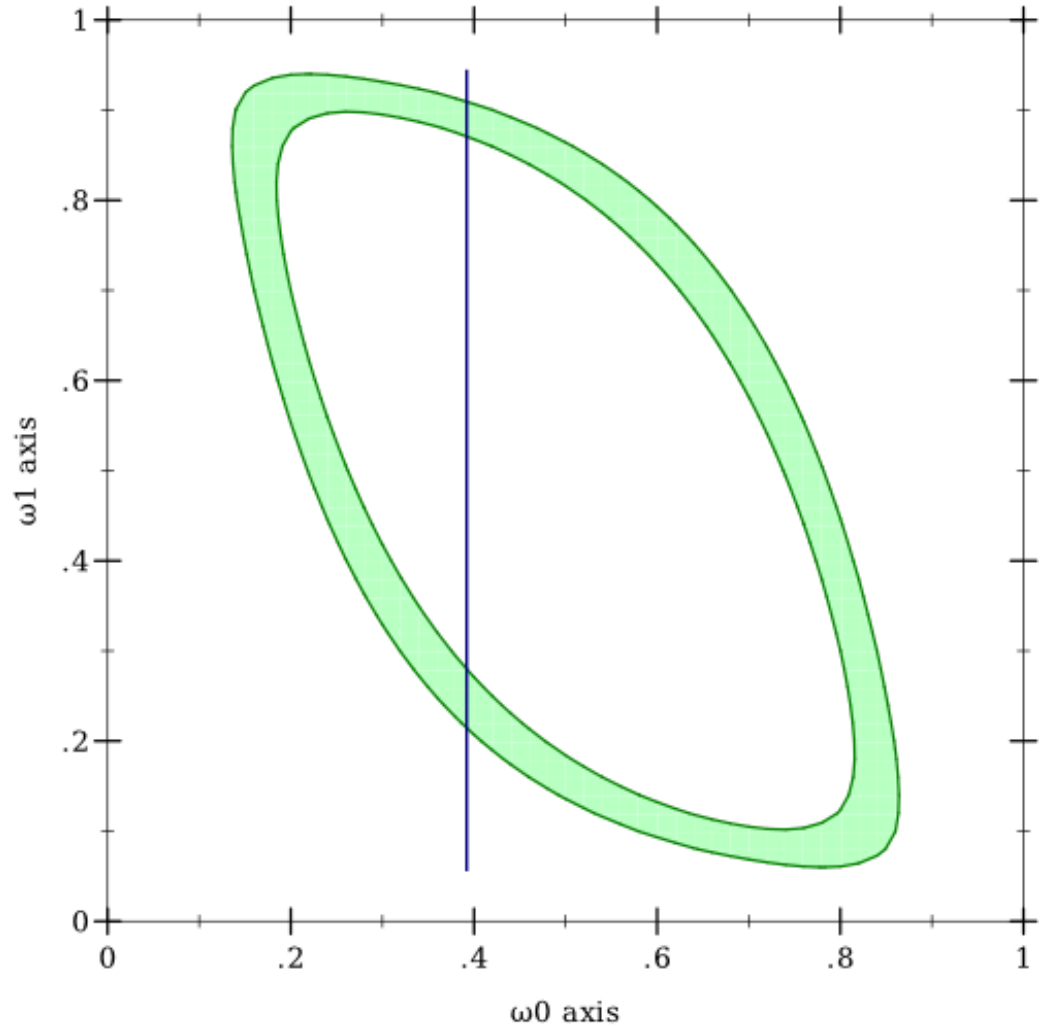
Fourth, sample uniformly:



Algorithm 2

- Alternative to arbitrarily low-rate rejection sampling:

Do process “in the limit”; i.e. choose $[\omega_0, \omega_0] \times \Omega_1$:



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- Or a bound on its implementations' error to overapproximate upward- and downward-rounded implementations:

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> (ivl (norm-inv-cdf- 0.8) (norm-inv-cdf+ 0.8))
(ivl 0.8416212335729142 0.8416212335729149)
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8
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- Computing preimages often decides branching; it's randomly forced otherwise

Domain Values

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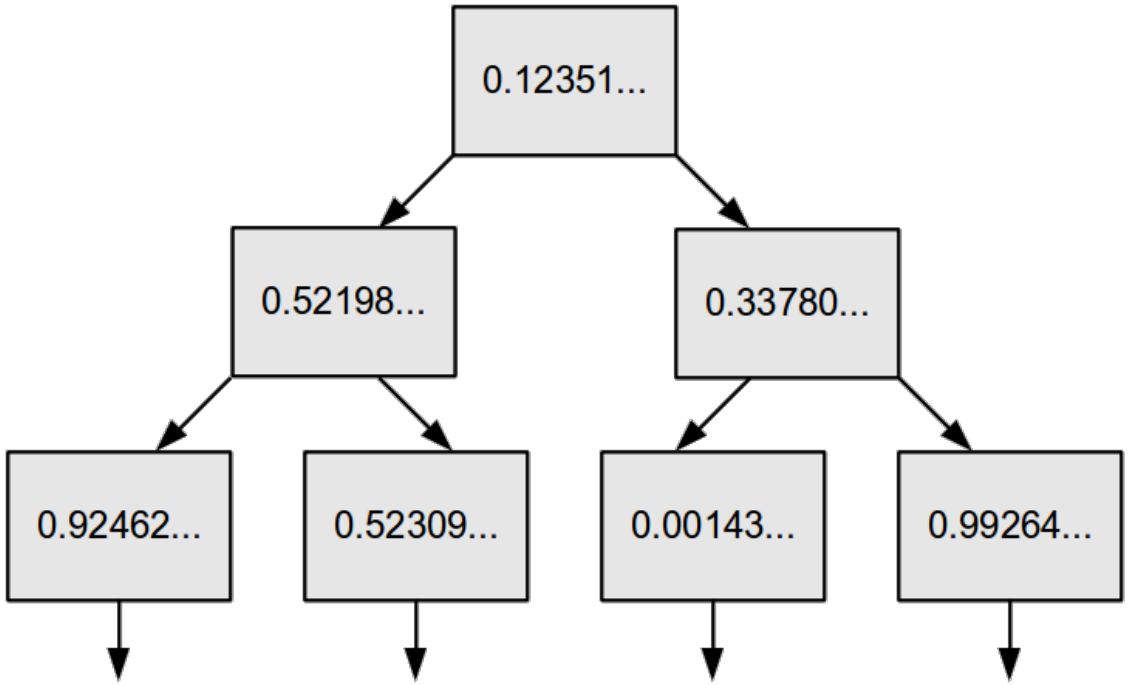

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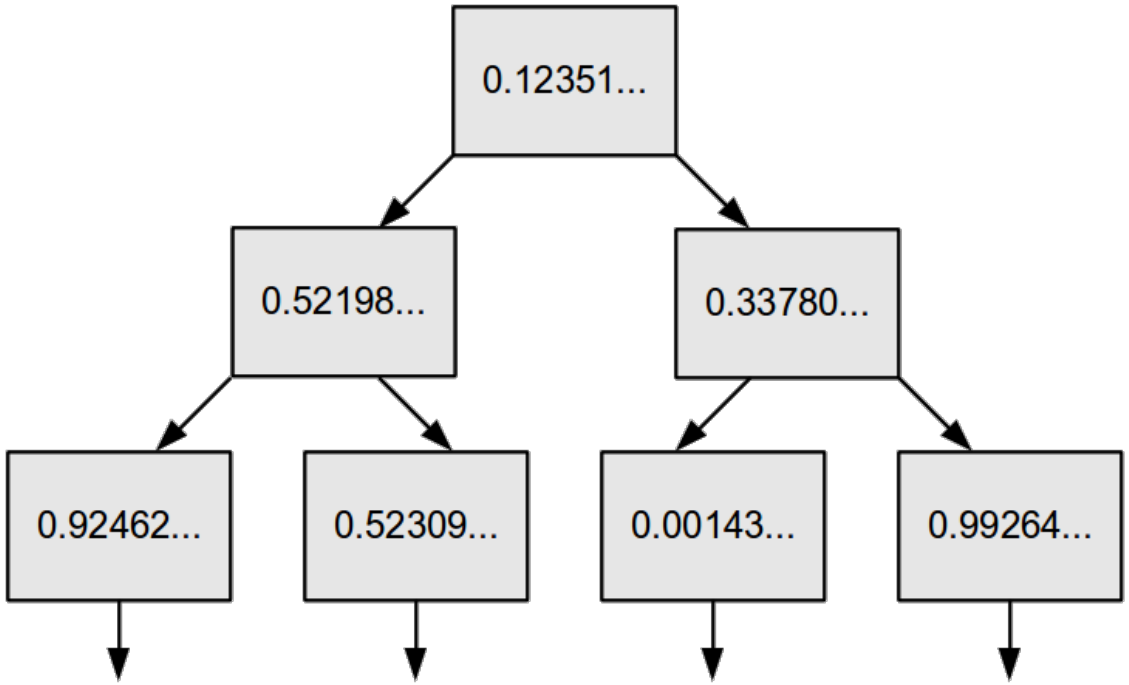

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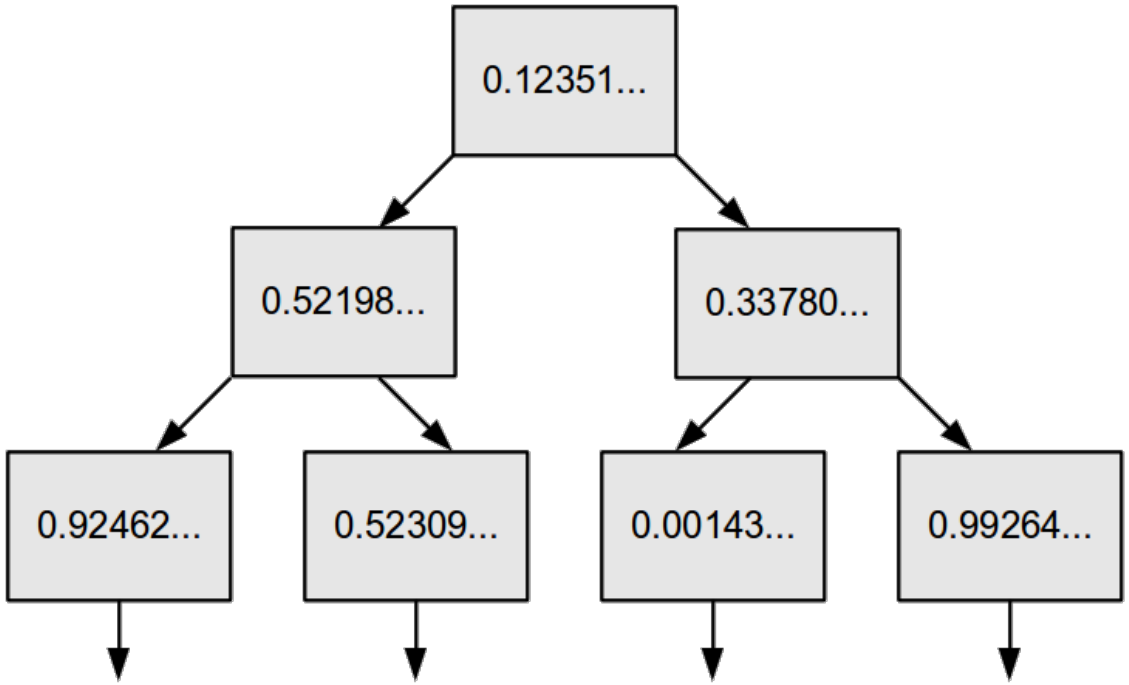

- Implemented using lazy trees of random values
- Each $t \in T$ is an infinite tree of branch choices

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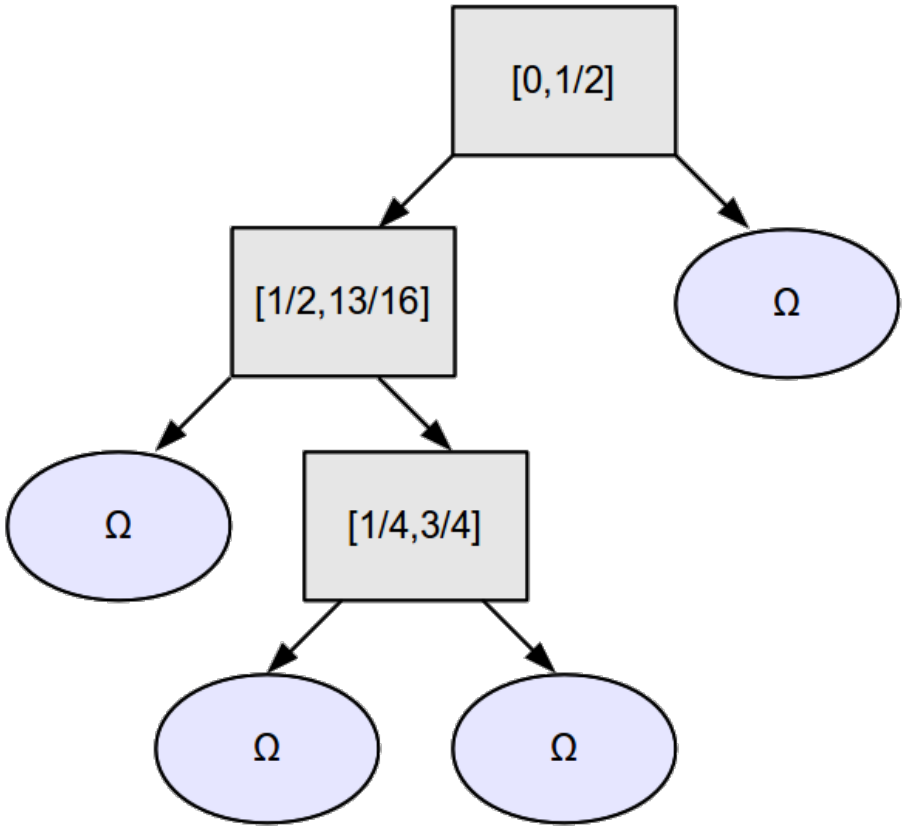

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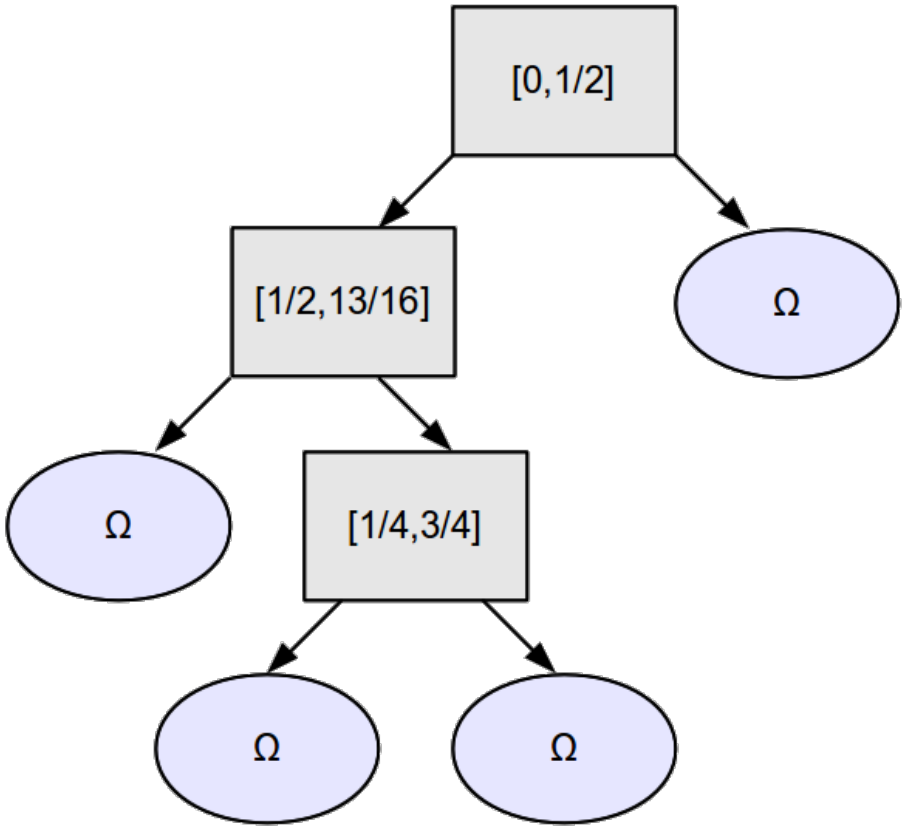

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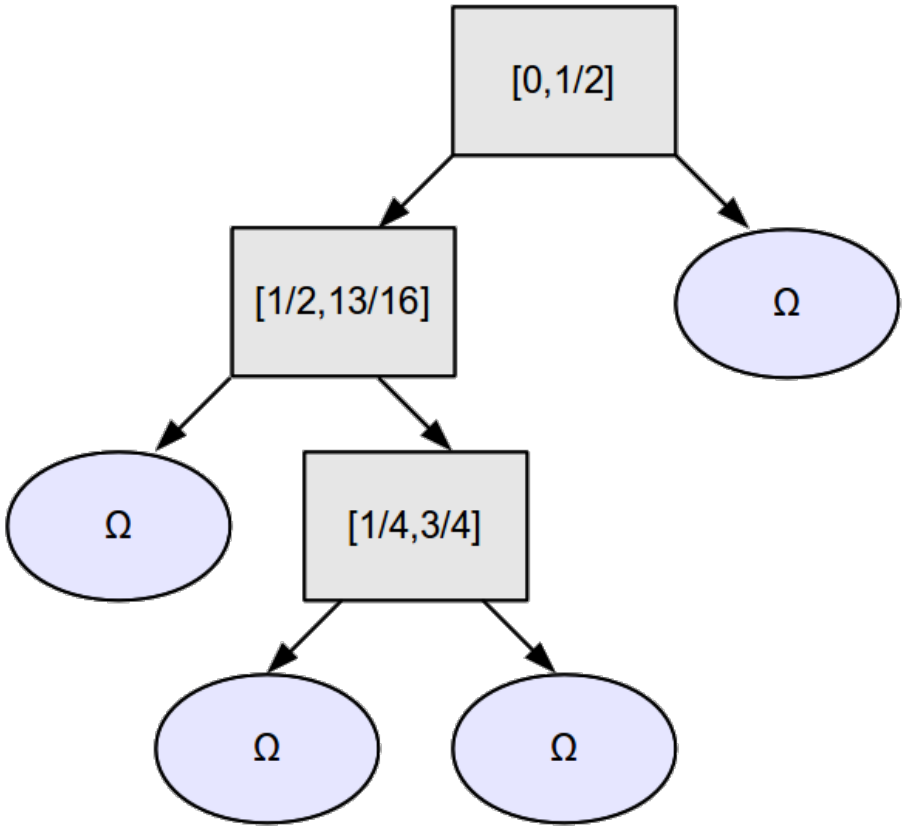

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Demos

- Normal-Normal with observation
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- Thermometer
- Normal-Normals with varying ε
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- Discover, characterize and overcome current limitations