Chapter 5

Ordinary Differential Equations

Fatemeh Baharifard

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First-Order ODE

Second & Higher Order ODE

- One-Step Method
  - Taylor's Method
  - Euler's Method
  - Modified Euler's Method
  - Runge–Kutta's Method
    - 2nd Order
      - Modified Euler's Method
      - Midpoint's Method
      - Heun's Method
    - 3rd Order
    - 4th Order
- Multi-Steps Method
  - Adams–'Bashforth's Method
  - Adams–Moulton's Method
  - Predictor-Corrector Method

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Differential Equations

An ordinary differential equation is a relation that contains x, y and one or more of their derivatives.

Solving an ordinary differential equation means eliminating these derivatives.

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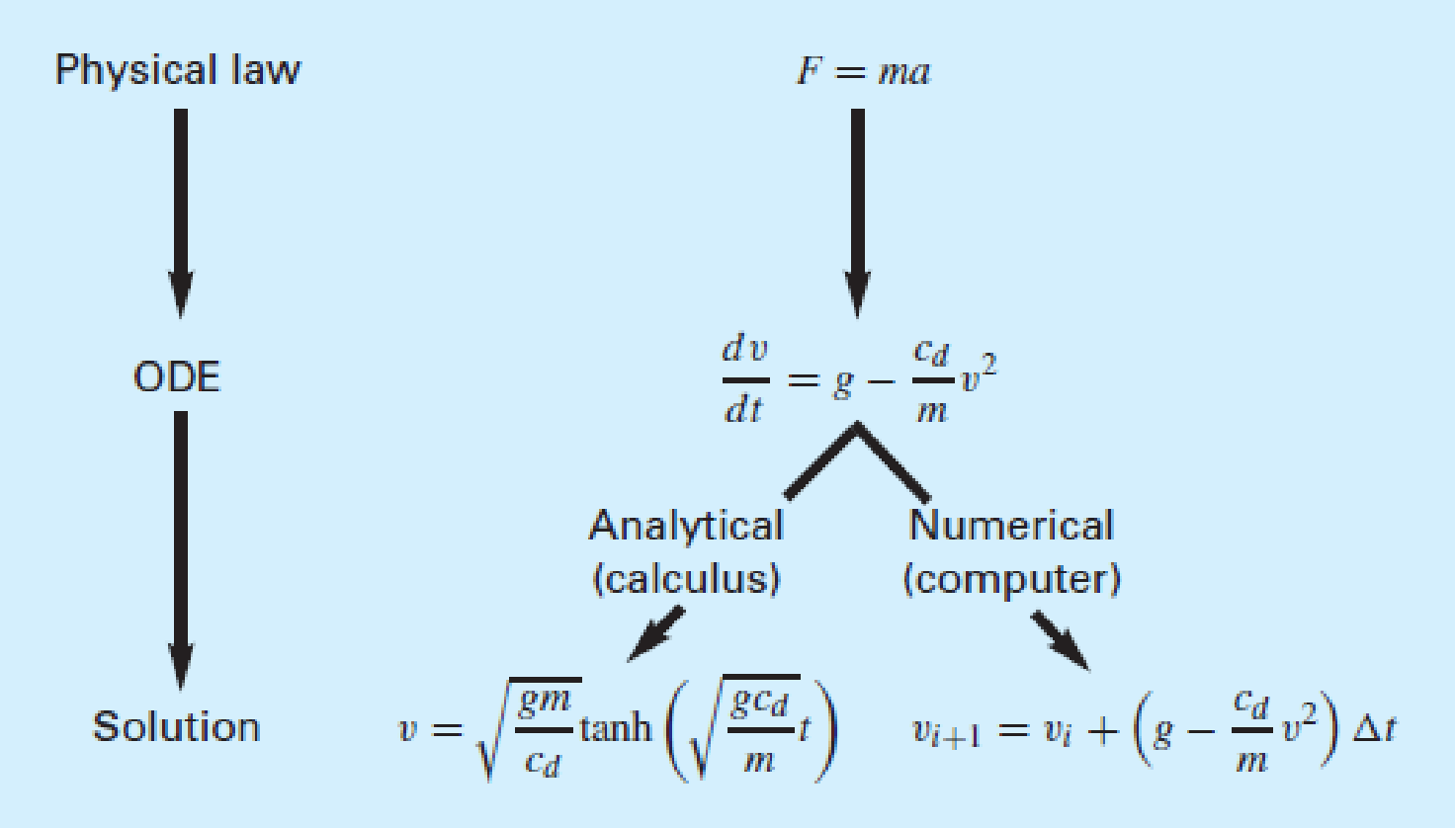
Differential Equations



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Differential Equations

ODE: Ordinary Differential Equation

involves one independent variable

PDE: Partial Differential Equation

involves two or more independent variables

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Differential Equations

ODE	PDE
$\frac{dy}{dx} = x\sin\left(x^2\right)\cos y$	$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = e^{x-y}$
$\frac{dy}{dx} = y\csc x + e^x$	$3\frac{\partial u}{\partial x^2} + 7\frac{\partial u}{\partial x \partial y} + 6\frac{\partial u}{\partial y^2} = 0$
$\frac{d^2y}{dx^2} + 4xy\frac{dy}{dx} + 5y = 0$	$\frac{\partial v}{\partial t} - k\frac{\partial v}{\partial x^2} = v$

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Ordinary Differential Equations

These equations can be classified according to their orders:

- First order example : $\frac{dy}{dx} = \frac{x}{y}$
- Second order example : $\frac{d^2y}{dx^2} = -\frac{dy}{dx} - y$
- Higher order example : $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = \frac{dx}{dy} + x + y$

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Ordinary Differential Equations

a simple fourth-order polynomial

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$$

if we differentiate it, we obtain an ODE:

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

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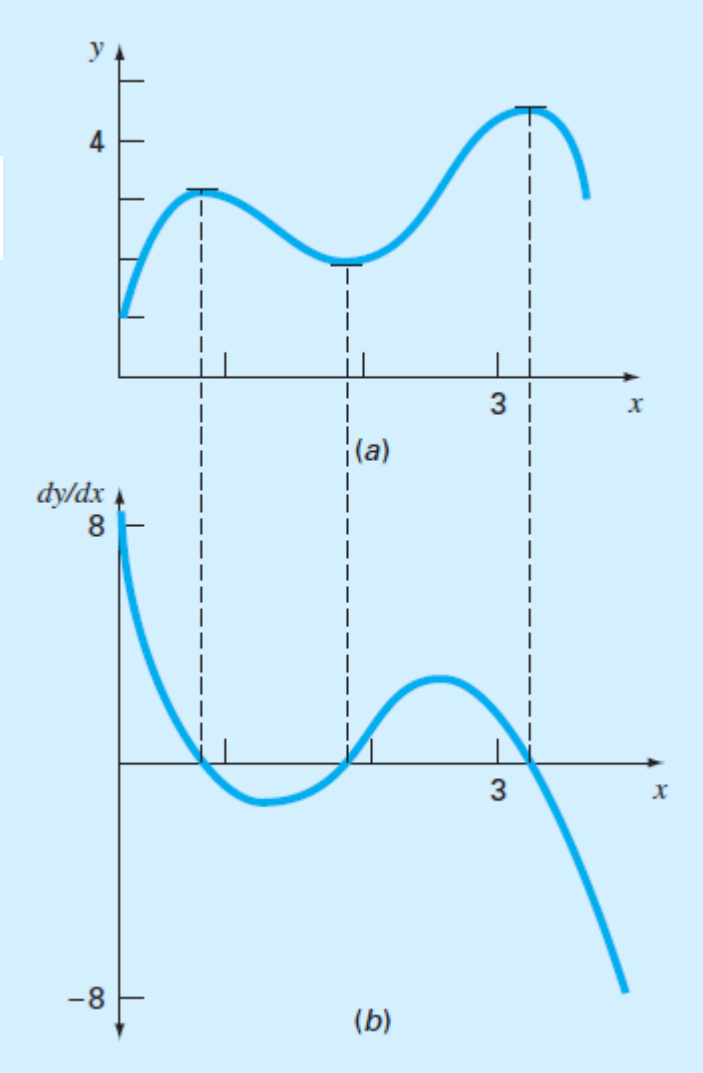
Ordinary Differential Equations

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$$

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$



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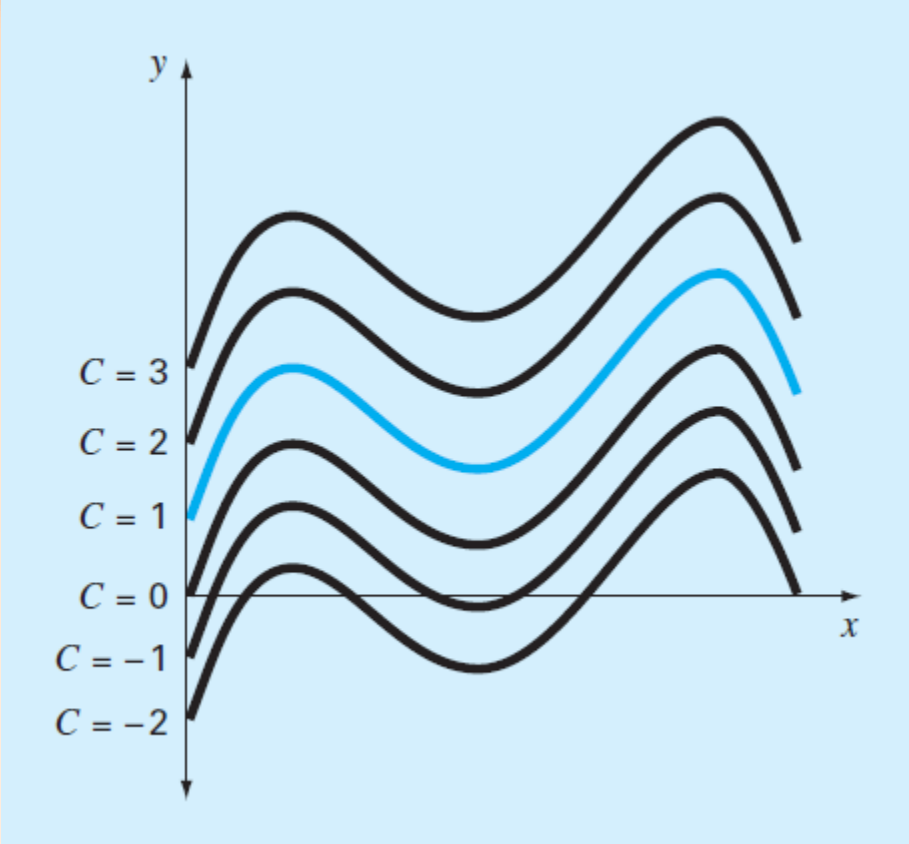
Ordinary Differential Equations

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

$$y = \int (-2x^3 + 12x^2 - 20x + 8.5)\, dx$$

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + C$$



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Ordinary Differential Equations

For an *n*th-order differential equation, *n* conditions are required to obtain a unique solution.

initial-value problem (IVP): all conditions are specified at the same value of the independent variable

boundary-value problem (BVP): conditions occurs at different values of the independent variable.

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Differential Equations

We consider Ordinary Differential Equations with initial value condition

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Differential Equations

The general form of a first-order differential equation is:

$$\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

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Solving Differential Equations

An example:

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{x}{y} \\ y(0) = 1 \end{cases}$$

Solution:

$$\frac{dy}{dx} = \frac{x}{y} \rightarrow x\,dx = y\,dy \rightarrow \int x\,dx = \int y\,dy \rightarrow \frac{x^2}{2} + C = \frac{y^2}{2}$$

$$\rightarrow \mathrm{C} = \frac{1}{2} \rightarrow y^2 = x^2 + 1$$

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Numerical Methods

Usually it is hard to use previous method; So, we use numerical methods.

In order to use numerical methods, we need the value of y in $n+1$ points, $x_0, x_1, \dots, x_n$, to be able to calculate the value of y at the point x_{n+1} such that:

$$x_i - x_{i-1} = h$$

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Taylor's Method

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

$$x_1 = x_0 + h$$

$$y(x_1) = y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{1}{2!}h^2 y''(x_0) + \cdots$$

$$y' = f(x, y) = f$$

$$y'' = f_x + f_y y' = f_x + f_y f$$

$$y''' = f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_y f_x + f_y^2 f$$

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Taylor's Method

$$y(x_1) = y(x_0 + h) \simeq y(x_0) + hy'(x_0) + \cdots + \frac{h^k}{k!}y^{(k)}(x_0) = y_1$$

$$y(x_2) = y(x_1 + h) \simeq y(x_1) + hy'(x_1) + \cdots + \frac{h^k}{k!}y^{(k)}(x_1)$$

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Taylor's Method

$$y(x_{i+1}) = y_{i+1} = y_i + hy_i{}' + \frac{h^2}{2!}y_i{}'' + \cdots + \frac{h^n}{n!}y_i{}^{(n)}$$

Accuracy of this method is high but sometimes it is hard to differentiate.

The error is $O(h^n)$.

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An Example

$$\begin{cases} y' = x + y \\ y(0) = 1 \end{cases}$$

$y(0.5) = ?$

Solution:

$f(x,y) = x + y \,,\; y_0 = 1 \,,\; x_0 = 0$

$$y' = f(x,y) = x + y$$

$$y'' = f'(x,y) = 1 + y' = 1 + x + y$$

$$y''' = f''(x,y) = y'' = 1 + x + y$$

$$y^{(4)} = f'''(x,y) = y''' = 1 + x + y$$

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An Example

$h = ۰/۱ \,,\, k = ۴$

$$y_{n+۱} = y_n + ۰/۱(x_n + y_n) + \frac{(۰/۱)^۲}{۲!}(۱ + x_n + y_n) + \frac{(۰/۱)^۳}{۶}(۱ + x_n + y_n)$$
$$+ \frac{(۰/۱)^۴}{۲۴}(۱ + x_n + y_n)$$
$$\simeq ۰/۰۰۵۱۷ + ۰/۱۰۵۱۷x_n + ۱/۱۰۵۱۷y_n$$

$$y_۱ = ۰/۰۰۵۱۷ + ۰/۱۰۵۱۷(۰) + ۱/۱۰۵۱۷(۱) = ۱/۱۱۰۳۴$$
$$y_۲ = ۰/۰۰۵۱۷ + ۰/۱۰۵۱۷(۰/۱) + ۱/۱۰۵۱۷(۱/۱۱۰۳۴) = ۱/۲۴۲۸۰$$
$$y_۳ = ۰/۰۰۵۱۷ + ۰/۱۰۵۱۷(۰/۲) + ۱/۱۰۵۱۷(۱/۲۴۲۸۰) = ۱/۳۹۹۷۱$$
$$y_۴ = ۰/۰۰۵۱۷ + ۰/۱۰۵۱۷(۰/۳) + ۱/۱۰۵۱۷(۱/۳۹۹۷۱) = ۱/۵۸۳۶۴$$
$$y_۵ = ۰/۰۰۵۱۷ + ۰/۱۰۵۱۷(۰/۴) + ۱/۱۰۵۱۷(۱/۵۸۳۶۴) = ۱/۷۹۷۴۳$$

$$y(۰/۵) \simeq y_۵ = ۱/۷۹۷۴۳$$

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Euler's Method

This method is just like Taylor Method with $n = 1$:

$$y(x_{i+1}) = y_{i+1} = y_i + hy_i{}'$$

The error is $O(h)$.

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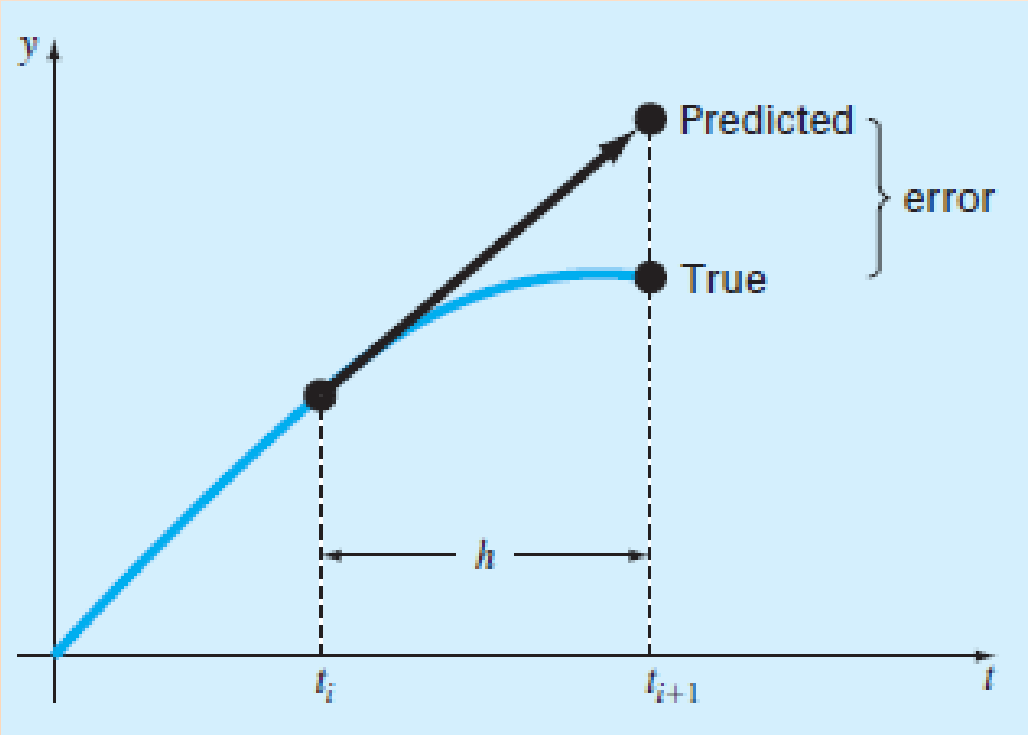
Euler's Method



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Modified Euler's Method

predictor

$$y_{i+1}{}'^{*} = f(x_{i+1}, y_{i+1}{}^{*})$$

corrector

$$y_{i+1} = y_i + \frac{h}{2}(y_i{}' + y_{i+1}{}'^{*})$$

($y_{i+1}{}^{*}$ is computed using Euler's method.)

The error is $O(h^2)$.

It is a *Predictor-Corrector* method, because at first it *predicts* answer by another method and then *corrects* it!

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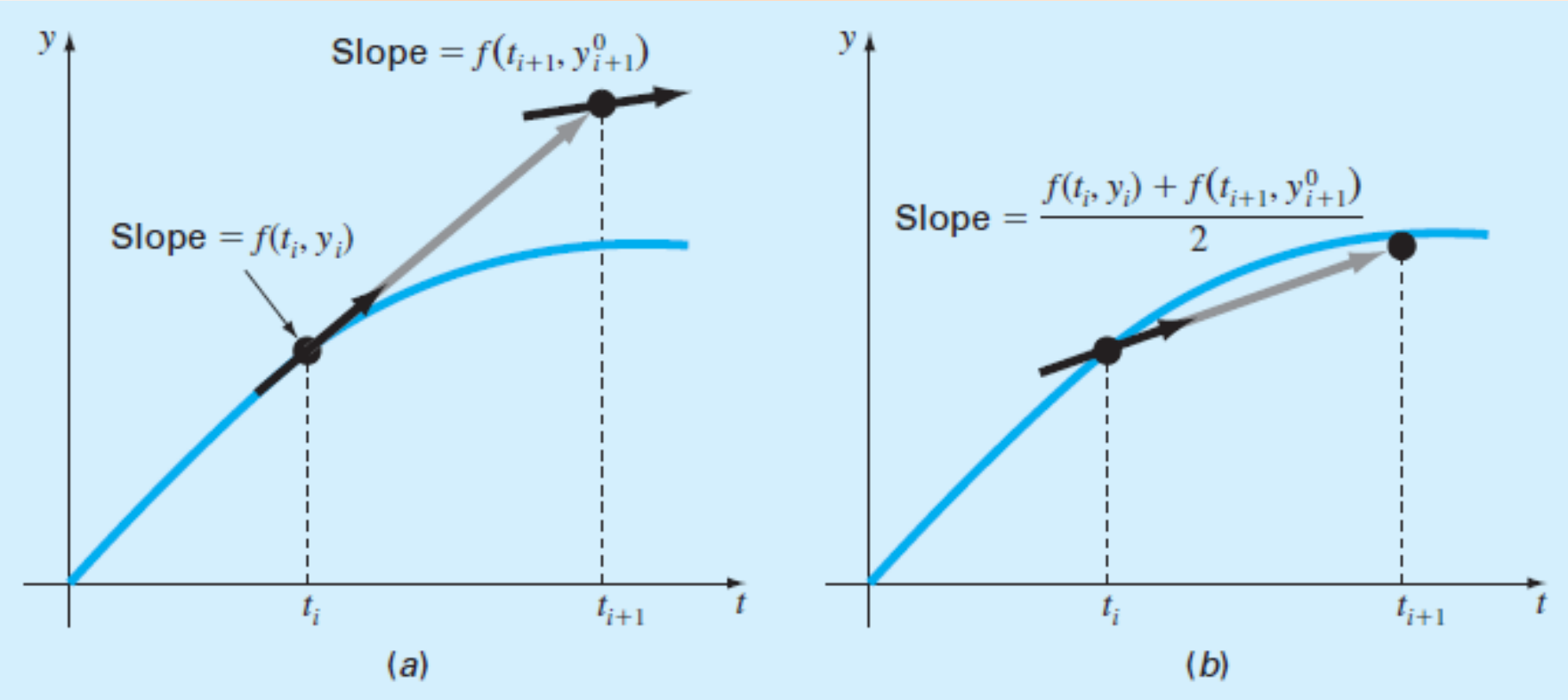
Modified Euler's Method



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An Example

$$\begin{cases} y' = x^2 + y \\ y(0) = 1 \\ h = 0.1 \end{cases} \Rightarrow y(0.1) = ?$$

$$y^*(0.1) = y_0 + 0.1\, f(0,1) \Rightarrow$$
$$= 1 + 0.1(0^2 + 1) = 1.1$$

predictor

$$y(0.1) = 1 + \frac{0.1}{2}(f(0,1) + f(0.1\,, 1.1)) \Rightarrow$$
$$= 1 + 0.05(1 + 1.11) = 1.1055$$

corrector

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Runge–Kutta's Methods

In second order Runge-Kutta's Method we have:

$$y_{i+1} = y_i + ak_1 + bk_2$$
$$k_1 = hy_i{}' = hf(x_i, y_i), k_2 = hf(x_i + \alpha h, y_i + \beta k_1)$$

For finding a, b, α and β we have:

$$\begin{cases} a + b = 1 \\ b\alpha = \dfrac{1}{2} \\ b\beta = \dfrac{1}{2} \end{cases}$$

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Runge–Kutta's Methods

Previous equations represent three equations with four unknown parameters, we can assign any value to one of the parameters.

Following methods are some of the popular choices and the names associated with the resulting formulas.

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Runge–Kutta's Methods

First method is just like Modified Euler's method:

$$a = b = \frac{1}{2}, \qquad \alpha = \beta = 1$$

$$k_2 = hf(x_{i+1}, y_i + k_1)$$

$$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)$$

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Runge–Kutta's Methods

Second method is Mid-Point method:

$$a = 0, b = 1, \qquad \alpha = \beta = \frac{1}{2}$$

$$k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$$

$$y_{i+1} = y_i + k_2$$

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Runge–Kutta's Methods

Third method is Heun's method:

$$a = \frac{1}{4}, \qquad b = \frac{3}{4}, \qquad \alpha = \beta = \frac{2}{3}$$

$$k_2 = hf\left(x_i + \frac{2}{3}h, y_i + \frac{2}{3}k_1\right)$$

$$y_{i+1} = y_i + \frac{1}{4}(k_1 + 3k_2)$$

Error is $O(h^2)$ in all 2nd order Runge-Kutta methods.

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Runge–Kutta's Methods

We have third-order Runge-Kutta's method with $O(h^3)$ error:

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$$

$$k_3 = hf(x_i + h, y_i + 2k_2 - k_1)$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

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Runge–Kutta's Methods

And we have fourth-order Runge-Kutta's method with:

$k_1 = hf(x_i, y_i)$
$k_2 = hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$
$k_3 = hf(x_i + \frac{h}{2}, y_i + \frac{k_2}{2})$
$k_4 = hf(x_i + h, y_i + k_3)$
$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

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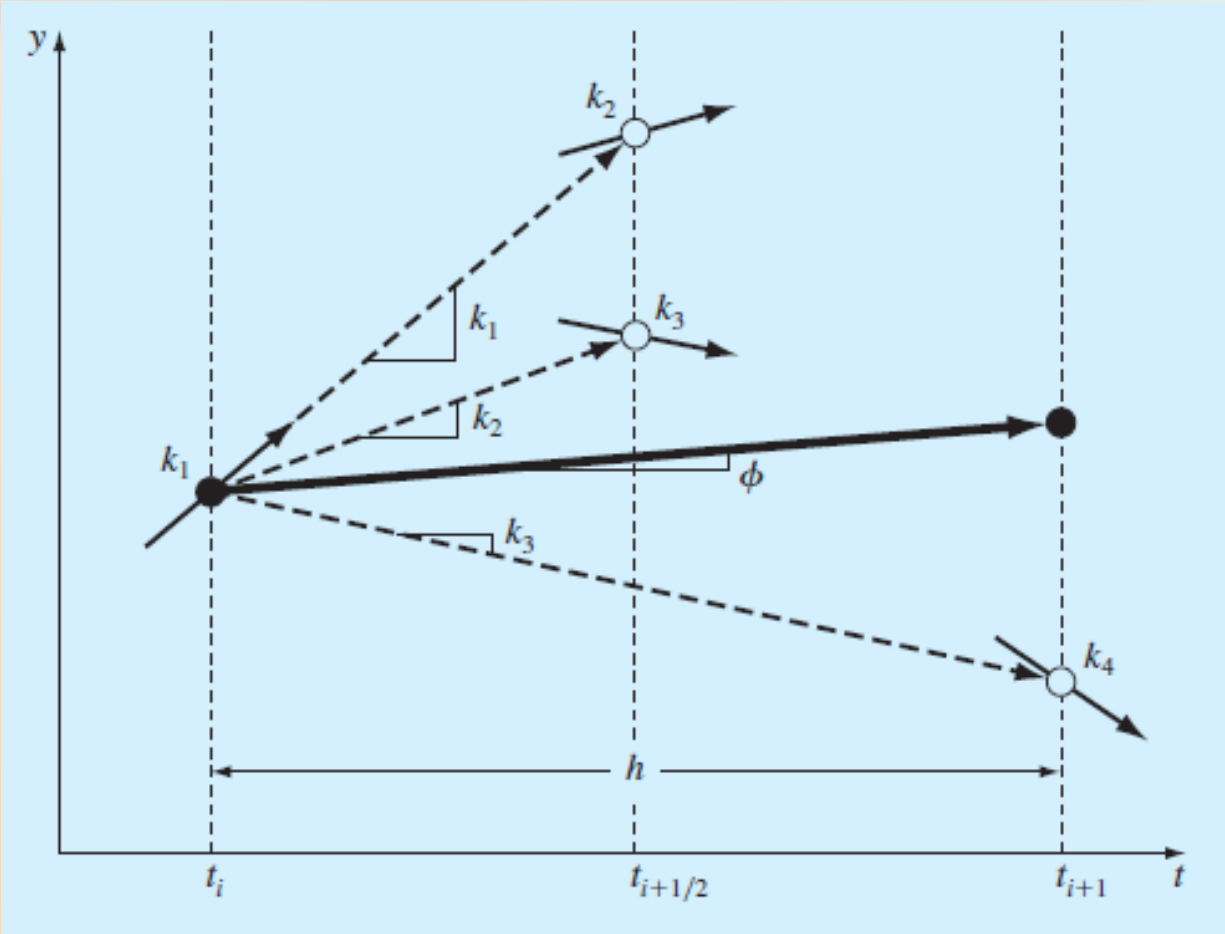
Runge–Kutta's Methods



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Runge–Kutta’s Methods

$h = 0.2$ ، را با روش رونگ کوتا مرتبه ۴

$$\begin{cases} y' = y + x^2 \\ y(0) = 1 \end{cases} \qquad 0 \le x \le 1$$

$$f(x,y) = y + x^2$$

$$k_1 = 0.2\, f(x_0, y_0) = 0.2\, f(0, 1) = 0.2$$

$$k_2 = 0.2\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\, f(0.1, 1 + 0.1) = 0.222$$

$$k_3 = 0.2\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\, f(0.1, 1 + 0.111) = 0.22421$$

$$k_4 = 0.2\, f(x_0 + h, y_0 + k_3) = 0.2\, f(0.2, 1 + 0.22421) = 0.25684$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.22421$$

k	x_k	y_k (EU)	y_k (RKF)	y
۱	۰٫۲	۱٫۲۰۰۰۰	۱٫۲۲۴۲۱	۱٫۲۲۴۲۱
۲	۰٫۴	۱٫۴۴۸۰۰	۱٫۵۱۵۴۷	۱٫۵۱۵۴۷
۳	۰٫۶	۱٫۷۹۹۹۰	۱٫۹۰۲۴۴	۱٫۹۰۲۴۴
۴	۰٫۸	۲٫۱۹۵۵۲	۲٫۴۳۲۲۰	۲٫۴۳۲۲۲
۵	۱	۲٫۷۹۲۹۲	۳٫۱۵۴۸۱	۳٫۱۵۴۸۵

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Multi-Steps Methods

All previous methods were *single-step* or direct methods that refer to only one previous point and its derivative to determine the current value.

But we have some *multi-steps* methods that refer to several previous points and derivative values.

One of these multi-steps methods families is Adams methods family.

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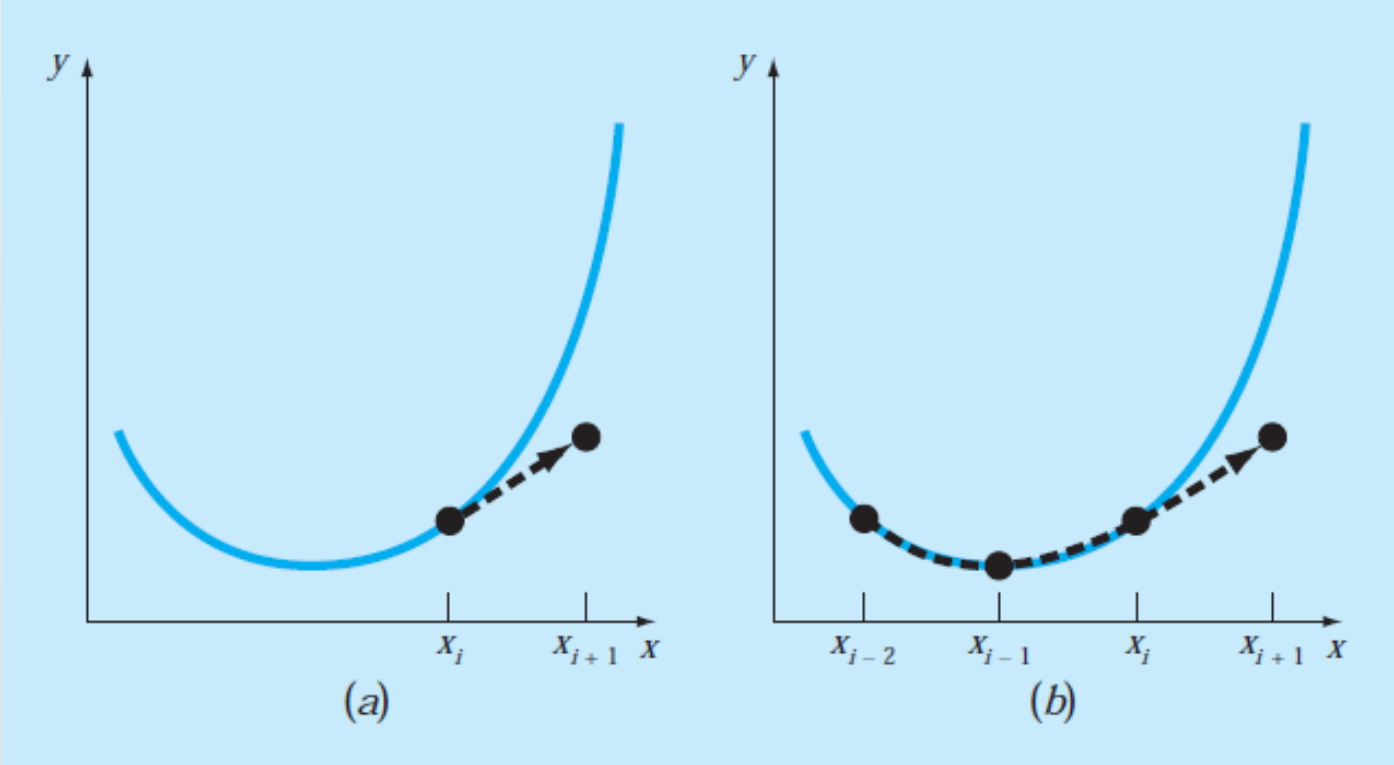
Multi-Steps Methods



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Explicit Method

Second-order Adams-Bashforth method with $O(h^2)$ error:

$$y_{i+1} = y_i + \frac{h}{2}[3f_i - f_{i-1}]$$

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Explicit Method

Third-order Adams-Bashforth method with $O(h^3)$ error:

$$y_{i+1} = y_i + \frac{h}{12}[23f_i - 16f_{i-1} + 5f_{i-2}]$$

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Explicit Method

Fourth-order Adams-Bashforth method with $O(h^4)$ error:

$$y_{i+1} = y_i + \frac{h}{24}[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}]$$

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Implicit Method

Second-order Adams-Moulton method with $O(h^2)$ error:

$$y_{i+1} = y_i + \frac{h}{2}[f_{i+1} + f_i]$$

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Implicit Method

Third-order Adams-Moulton method with $O(h^3)$ error:

$$y_{i+1} = y_i + \frac{h}{12}[5f_{i+1} + 8f_i - f_{i-1}]$$

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Implicit Method

Fourth-order Adams-Moulton method with $O(h^4)$ error:

$$y_{i+1} = y_i + \frac{h}{24}[9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}]$$

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Predictor-Corrector Method

Second-order Adams method

$$y_{i+1}{}^{*} = y_i + \frac{h}{2}(3f_i - f_{i-1})$$

predictor

$$y_{i+1} = y_i + \frac{h}{2}[f_{i+1}{}^{*} + f_i]$$

corrector

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Predictor-Corrector Method

Third-order Adams method

$$y_{i+1}{}^{*} = y_i + \frac{h}{12}(23f_i - 16f_{i-1} + 5f_{i-2})$$

predictor

$$y_{i+1} = y_i + \frac{h}{12}[5f_{i+1}{}^{*} + 8f_i - f_{i-1}]$$

corrector

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Predictor-Corrector Method

Fourth-order Adams method

$$y_{i+1}{}^{*} = y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3})$$

predictor

$$y_{i+1} = y_i + \frac{h}{24}[9f_{i+1}{}^{*} + 19f_i - 5f_{i-1} + f_{i-2}]$$

corrector

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Predictor-Corrector Method

f_is are calculated using one of the previous methods, such as Runge-Kutta. Note that the order of error of this method and the Adams method should be identical.

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An Example

Solve

$$\frac{dy}{dx} = 2x + y^2 x \qquad y(0) = 2$$

$h = 0.1$, *Use* $2-step$ Predictor corrector Method

compute y(0.2)

We need two initial conditions to use the

$2-step$ Predictor corrector Method

We will first use RK2 to estimate y(0.1)

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An Example

We need two initial conditions

Use RK2 to compute $y(0.1)$ *then we can use*

the Predictor corrector Method

$$\frac{dy}{dx} = 2x + y^2 x \qquad y(0) = 2, \quad h = 0.1,$$

$$K1 = 0.1(0) = 0$$

$$K2 = 0.1(0.2 + 0.4) = 0.06$$

$$y(0.1) = 2 + 0.5(0.06) = 2.03$$

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An Example

$$\frac{dy}{dx} = 2x + y^2 x \qquad y_{i-1} = y(0) = 2, y_i = y(0.1) = 2.03, \qquad h = 0.1$$

$$\text{Predictor: } y_{i+1} = y_i + h\left(\frac{3}{2} f(y_i) - \frac{1}{2} f(y_{i-1})\right)$$

$$= 2.03 + 0.1\left(\frac{3}{2}\left(2(0.1) + 2.03^2(0.1)\right) - \frac{1}{2}(0+0)\right) = 2.1218$$

$$\text{Corrector: } y_{i+1} = y_i + h\left(\frac{1}{2} f(y_{i+1}) + \frac{1}{2} f(y_i)\right)$$

$$= 2.03 + 0.1\left(\frac{1}{2} f(y_{i+1}) + \frac{1}{2}\left(2(0.1) + 2.03^2(0.1)\right)\right) = 2.1256$$

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First-Order ODE

Second & Higher Order ODE

- Euler's Method
- 4th Order Runge-Kutta's Method

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2nd and Higher Order ODEs

Any n^{th}-order ODE can be written into a 1st order ODE or a system of n first-order ODEs with n initial values.

An example: $\begin{cases} 5y''' + 3x^2y + cos(xy') - 2xy'' = 0 \\ y(0) = 1 \\ y'(0) = -2 \\ y''(0) = 3.5 \end{cases}$

$$\begin{cases} \dfrac{du_1}{dx} = u_2 \\ \dfrac{du_2}{dx} = u_3 \\ \dfrac{du_3}{dx} = -\dfrac{1}{5}[3x^2u_1 + cos(xu_2) - 2xu_3] \end{cases} \qquad \begin{cases} u_1(0) = 1 \\ u_2(0) = -2 \\ u_3(0) = 3.5 \end{cases}$$

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2nd and Higher Order ODEs

Any n^{th}-order ODE has n initial values, otherwise the equation can't be solved.

For example, this equation describes a mechanical device which is supposed to move an item on a specific path:

$$mx''(t) + cx'(t) + kx(t) = f(t)$$

In these set of problems, usually $x(t_0)$ and $x'(t_0)$ are known in $t_0 = 0$.

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An Example

$$\begin{cases} xy'' - y' + 4x^3 y = 0 \\ y(1) = 1 \\ y'(1) = 2 \end{cases}$$

$$u_1 = y(x) \Rightarrow u_1(1) = y(1) = 1$$

$$u'_1 = y'(x) = u_2 \Rightarrow u_2(1) = y'(1) = 2$$

$$u'_2 = y''(x) = \frac{1}{x}[y' - 4x^3 y]$$

$$\Rightarrow \begin{cases} \dfrac{du_1}{dx} = u_2 \\ \dfrac{du_2}{dx} = \dfrac{1}{x}[u_2 - 4x^3 u_1] \end{cases}, \begin{cases} u_1(1) = 1 \\ u_2(1) = 2 \end{cases}$$

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Systems of ODEs

A system of n first-order ODE is as below and it has n initial values.

$$\begin{cases} u'_1(x) = \dfrac{du_1}{dx} = f\big(x, u_1(x), u_2(x), \dots,, u_n(x)\big), u_1(x_0) = a_1 \\ u'_2(x) = \dfrac{du_2}{dx} = g\big(x, u_1(x), u_2(x), \dots,, u_n(x)\big), u_2(x_0) = a_2 \\ \vdots \\ u'_n(x) = \dfrac{du_n}{dx} = m\big(x, u_1(x), u_2(x), \dots,, u_n(x)\big), u_n(x_0) = a_n \end{cases}$$

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Systems of ODEs

Solving such systems means removing derivatives such that x and y can be written in terms of t.
For example:

$$\begin{cases} \dfrac{dx}{dt} = x + 2y, & x(0) = 6 \\ \dfrac{dy}{dt} = 3x + 2y, & y(0) = 4 \end{cases}$$

$$\begin{cases} x(t) = 4e^{4t} + 2e^{-t} \\ y(t) = 6e^{4t} - 2e^{-t} \end{cases}$$

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Converting to System of n first-order ODEs

$$u_1 = y \rightarrow u_1(x_0) = a_1$$
$$u_1{}' = y' = u_2 \rightarrow u_2(x_0) = a_2$$
$$\dots$$
$$u_{n-1}{}' = y^{(n-1)} = u_n \rightarrow u_n(x_0) = a_n$$
$$u_n{}' = F(x, u_1, \dots, u_n)$$

$$\begin{cases} u_1{}' = u_2 \\ u_2{}' = u_3 \\ \dots \\ u_{n-1}{}' = u_n \\ u_n{}' = F(x, u_1, \dots, u_n) \end{cases}$$

$$\begin{cases} u_1(x_0) = a_1 \\ u_2(x_0) = a_2 \\ \dots \\ u_n(x_0) = a_n \end{cases}$$

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An Example

$$\begin{cases} xy'' - y' + 4x^3 y = 0 \\ y(1) = 1 \\ y'(1) = 2 \end{cases}$$

$$u_1 = y \Rightarrow u_1(1) = y(1) = 1 \Rightarrow u_1{}' = y' = u_2$$

$$\Rightarrow u_2(1) = y'(1) = 2 \Rightarrow u_2{}' = y'' = \frac{1}{x}[y' - 4x^3 y]$$

$$\Rightarrow \begin{cases} \dfrac{du_1}{dx} = u_2 \\ \dfrac{du_2}{dx} = \dfrac{1}{x}[u_2 - 4x^3 u_1] \end{cases}, \begin{cases} u_1(1) = 1 \\ u_2(1) = 2 \end{cases}$$

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First-Order ODE

Second & Higher Order ODE

- Euler's Method
- 4th Order Runge-Kutta's Method

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Numerical Methods

Consider this 2-equation system with 2 initial values (The following methods are applicable to any n-equation system of DE):

$$\begin{cases} \dfrac{dy}{dx} = f(x,y,z), & y(x_0) = y_0 \\ \dfrac{dz}{dx} = g(x,y,z), & z(x_0) = z_0 \end{cases} \Rightarrow \begin{cases} dy = f(x,y,z)dx \\ dz = g(x,y,z)dx \end{cases}$$

All first-order methods can be used to solve n-equation systems.

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Euler's Method

In Euler's method, we just need to apply these subtractions:

$$\begin{cases} dx = x_{k+1} - x_k = h \\ dy = y_{k+1} - y_k \\ dz = z_{k+1} - z_k \end{cases}$$

$$\Rightarrow \begin{cases} y_{k+1} - y_k \approx f(x_k, y_k, z_k)(x_{k+1} - x_k) \\ z_{k+1} - z_k \approx g(x_k, y_k, z_k)(x_{k+1} - x_k) \end{cases}$$

$$\Rightarrow \begin{cases} y_{k+1} = y_k + hf(x_k, y_k, z_k) \\ z_{k+1} = z_k + hg(x_k, y_k, z_k) \end{cases}$$

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First-Order ODE

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4th Order Runge–Kutta's Method

In this method we have:

$$\Rightarrow \begin{cases} y_{k+1} = y_k + \dfrac{h}{6}(k_{f_1} + 2k_{f_2} + 2k_{f_3} + k_{f_4}) \\ z_{k+1} = z_k + \dfrac{h}{6}(k_{g_1} + 2k_{g_2} + 2k_{g_3} + k_{g_4}) \end{cases}$$

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4th Order Runge–Kutta's Method

Values of $k_i s$ are computed this way:

$$\begin{cases} k_{f_1} = f(x_k, y_k, z_k) \\ k_{f_2} = f(x_k + \frac{h}{2}, y_k + \frac{h}{2}k_{f_1}, z_k + \frac{h}{2}k_{g_1}) \\ k_{f_3} = f(x_k + \frac{h}{2}, y_k + \frac{h}{2}k_{f_2}, z_k + \frac{h}{2}k_{g_2}) \\ k_{f_4} = f(x_k + h, y_k + hk_{f_3}, z_k + hk_{g_3}) \end{cases} \qquad \begin{cases} k_{g_1} = g(x_k, y_k, z_k) \\ k_{g_2} = g(x_k + \frac{h}{2}, y_k + \frac{h}{2}k_{f_1}, z_k + \frac{h}{2}k_{g_1}) \\ k_{g_3} = g(x_k + \frac{h}{2}, y_k + \frac{h}{2}k_{f_2}, z_k + \frac{h}{2}k_{g_2}) \\ k_{g_4} = g(x_k + h, y_k + hk_{f_3}, z_k + hk_{g_3}) \end{cases}$$

Note that the entries are identical in all rows.

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An Example

$$\begin{cases} \dfrac{dx}{dt} = x + 2y, x(0) = 6 \\ \dfrac{dy}{dt} = 3x + 2y, y(0) = 4 \end{cases}, h = 0.02 \Longrightarrow \begin{cases} x(0.02) = ? \\ y(0.02) = ? \end{cases}$$

$$\begin{cases} x(0.02) = 6 + \dfrac{0.02}{6}(14 + 2 \times 14.66 + 2 \times 14.6854 + 15.372852) = 6.29354551 \\ y(0.02) = 4 + \dfrac{0.02}{6}(26 + 2 \times 26.94 + 2 \times 26.9786 + 27.96268) = 4.53932490 \end{cases}$$

Compare with exact answers:

$$\begin{cases} x(0.02) = 6.293545617 \\ y(0.02) = 4.539325059 \end{cases}$$

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An Example

$$\begin{cases} y''(x) + 4y'(x) + 5y(x) = 0 \\ y(0) = 3 \\ y'(0) = -5 \end{cases} \Longrightarrow y(0.1) = ?$$

$$y(x) = u_1(x), u_1(0) = 3, y'(x) = u_2(x), u_2(0) = -5$$

$$\begin{cases} u_1'(x) = u_2(x) = f(x, u_1, u_2) \\ u_2'(x) = -4u_2(x) - 5u_1(x) = g(x, u_1, u_2) \end{cases}$$

$$\begin{cases} k_{f_1} = f(0,3,-5) = -5 \\ k_{f_2} = f(0.05,2.75,-4.75) = -4.75 \\ k_{f_3} = f(0.05,2.7625,-4.7375) = -4.7375 \\ k_{f_4} = f(0.1,2.52625,-4.48625) = -4.48625 \end{cases}$$

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An Example

$$\begin{cases} y''(x) + 4y'(x) + 5y(x) = 0 \\ y(0) = 3 \\ y'(0) = -5 \end{cases} \Longrightarrow y(0.1) = ?$$

$$\begin{cases} k_{g_1} = g(0,3,-5) = 5 \\ k_{g_2} = g(0.05,2.75,-4.75) = 5.25 \\ k_{g_3} = g(0.05,2.7625,-4.7375) = 5.1375 \\ k_{g_4} = g(0.1,2.52625,-4.48625) = 5.31375 \end{cases}$$

So, we have:

$$y(0.1) = u_1(0.1) = u_1(0) + \frac{0.1}{6}\left[k_{f_1} + 2k_{f_2} + 2k_{f_3} + k_{f_4}\right] = 2.52564$$

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Any questions?