Q_1, Q_2

Example 2.1 The true solution of the problem

$$Y'(t) = -Y(t), \qquad Y(0) = 1 \tag{2.6}$$

is $Y(t) = e^{-t}$. Euler's method is given by

$$y_{n+1} = y_n - hy_n, \qquad n \geq 0 \tag{2.7}$$

with $y_0 = 1$ and $t_n = nh$. The solution $y(t)$ for three values of h and selected values of t is given in Table 2.1. To illustrate the procedure, we compute y_1 and y_2 when $h = 0.1$. From (2.7), we obtain

$$\begin{aligned} y_1 &= y_0 - hy_0 = 1 - (0.1)(1) = 0.9, \qquad t_1 = 0.1, \\ y_2 &= y_1 - hy_1 = 0.9 - (0.1)(0.9) = 0.81, \qquad t_2 = 0.2. \end{aligned}$$

For the error in these values, we have

$$\begin{aligned} Y(t_1) - y_1 &= e^{-0.1} - y_1 \doteq 0.004837, \\ Y(t_2) - y_2 &= e^{-0.2} - y_2 \doteq 0.008731. \end{aligned}$$

■

Example 2.2 Solve

$$Y'(t) = \frac{Y(t) + t^2 - 2}{t+1}, \qquad Y(0) = 2 \tag{2.8}$$

whose true solution is

$$Y(t) = t^2 + 2t + 2 - 2(t+1)\log(t+1).$$

Euler's method for this differential equation is

$$y_{n+1} = y_n + \frac{h(y_n + t_n^2 - 2)}{t_n + 1}, \qquad n \geq 0$$

with $y_0 = 2$ and $t_n = nh$. The solution $y(t)$ is given in Table 2.2 for three values of h and selected values of t. A graph of the solution $y_h(t)$ for $h = 0.2$ is given in Figure 2.2. The node values $y_h(t_n)$ have been connected by straight line segments in the graph. Note that the horizontal and vertical scales are different. ■

دقت کنید که در هر دو مثال، رفتار خطا را با کاهش h مشاهده کنید. برای هر ثابت مقدار t، توجه داشته باشـید که با نصـف شـدن h خطاها حدود 2 برابر کاهش می یابد. میزان خطا برای h های 0.2، 0.1 و 0.05 به ترتیب برابر 2.96×10^{-3} ، 1.58×10^{-3} و 8.17×10^{-4} خواهد بود. و این با عوامل متوالی 1.93 و 1.87 کاهش می یابد. برای تمرین باید همین محاسـبه را برای مقادیر دیگر t در هر دو مثال 2.1 و 2.2 انجام دهید. همچنین توجه دا شته با شید که رفتار خطا با افزایش t ممکن ا ست کاملاً متفاوت از رفتار خطای نسبی باشد. در مثال 2.2، خطاهای نسبی ابتدا افزایش یافته و سپس با افزایش t کاهش می یابند.

Q_3

Use the Euler method, $\mathbf{y}_{n+1} = \mathbf{y}_n + hf(x_n, \mathbf{y}_n)$, for a system of equations to solve $y_1' = 2y_1 - 4y_2$, $y_2' = y_1 - 3y_2$, $y_1(0) = 3$, $y_2(0) = 0$. Solve for ten steps with h = 0.1 and plot the solution in the y_1-y_2 plane.

The Euler formula, in vector form, is $\mathbf{y}_{n+1} = \mathbf{y}_n + hf(x_n, \mathbf{y}_n)$. The components of the matrices in this equation are shown below, using the formula for the derivatives.

$$\mathbf{y}_{n+1} = \begin{bmatrix} y_{1,n+1} \\ y_{2,n+1} \end{bmatrix} = \mathbf{y}_n + h\mathbf{f}_n = \begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix} + h\begin{bmatrix} f_{1,n} \\ f_{2,n} \end{bmatrix} = \begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix} + h\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} y_{1,n} \\ y_{2,n} \end{bmatrix}$$

Starting with the initial conditions at x_0 and $\mathbf{y}_0$, we get the following values for the first step.

$$\mathbf{y}_1 = \begin{bmatrix} y_{1,1} \\ y_{2,1} \end{bmatrix} = \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix} + h\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + (0.1)\begin{bmatrix} 2 & -4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + (0.1)\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3.6 \\ 0.3 \end{bmatrix}$$

Continuing in this fashion produces the table below. In this table, we are comparing the numerical results to the exact solution. To obtain the exact solution we can combine the two first order equations into a single equation, $y_1'' + y_1' - 2y_1 = 0$. The solution to this equation is $y_1 = C_1e^x + C_2e^{-2x}$. This gives $y_2 = (C_1/4)e^x + C_2e^{-2x}$. After finding C_1 and C_2 from the initial conditions we have the result that $y_1 = 4e^x - e^{-2x}$ and $y_2 = e^x - e^{-2x}$.

i	x_i	y_{1i}	exact y_{1i}	error y_1	y_{2i}	exact y_{2i}	error y_2	f_1	f_2
0	0.0	3	3	0	0	0	0	6	3
1	0.1	3.6	3.601953	0.001953	0.3	0.28644	-0.01356	6	2.7
2	0.2	4.2	4.215291	0.015291	0.57	0.551083	-0.01892	6.12	2.49
3	0.3	4.812	4.850624	0.038624	0.819	0.801047	-0.01795	6.348	2.355
4	0.4	5.4468	5.517970	0.071170	1.0545	1.042496	-0.01200	6.6756	2.2833
5	0.5	6.11436	6.227006	0.112646	1.28283	1.280842	-0.00199	7.0974	2.26587
6	0.6	6.824100	6.987281	0.163181	1.509417	1.520925	0.011508	7.61053	2.295849
7	0.7	7.585153	7.808414	0.223261	1.739002	1.767156	0.028154	8.21430	2.368148
8	0.8	8.406583	8.700267	0.293684	1.975817	2.023644	0.047828	8.90990	2.479133
9	0.9	9.297573	9.673114	0.375541	2.223730	2.294304	0.070574	9.70023	2.626383
10	1.0	10.26760	10.73779	0.470196	2.486368	2.582947	0.096578	10.58972	2.808491

The exact solution and the solution using Euler's method are plotted in the $y_1 - y_2$ plane on the next page. It is not easy to see on this plot, but the difference between the two lines, which is the error in the solution, is increasing as both values of y_i increase with x.

Q_4, Q_5

Table 5.2 Example of second-order Runge–Kutta method

h	t	$y_h(t)$	Error
0.1	2.0	0.491215673	1.93e − 3
	4.0	−1.407898629	−2.55e − 3
	6.0	0.680696723	5.81e − 5
	8.0	0.841376339	2.48e − 3
	10.0	−1.380966579	−2.13e − 3
0.05	2.0	0.492682499	4.68e − 4
	4.0	−1.409821234	−6.25e − 4
	6.0	0.680734664	2.01e − 5
	8.0	0.843254396	6.04e − 4
	10.0	−1.382569379	−5.23e − 4

Example 5.6 Estimate the error for $h = 0.05$ and $t = 10$ in Table 5.2. Then

$$Y(10) - y_{0.05}(10) \doteq \tfrac{1}{3}[-1.3825669379 - (-1.380966579)] \doteq -5.34 \times 10^{-4}.$$

This compares closely with the actual error of -5.23×10^{-4}. ■

Example 5.7 Consider the problem

$$Y' = \frac{1}{1+x^2} - 2Y^2, \qquad Y(0) = 0 \tag{5.46}$$

with the solution $Y = x/(1+x^2)$. The method (5.28) was used with a fixed stepsize, and the results are shown in Table 5.3. The stepsizes are $h = 0.25$ and $2h = 0.5$. The asymptotic error formula (5.44) becomes

$$Y(x) - y_h(x) = D(x)h^4 + \mathcal{O}(h^5), \tag{5.47}$$

in this case, and this leads to the asymptotic error estimate

$$Y(x) - y_h(x) = \tfrac{1}{15}\left[y_h(x) - y_{2h}(x)\right] + \mathcal{O}(h^5). \tag{5.48}$$

In the table the column labeled "Ratio" gives the ratio of the errors for corresponding node points as h is halved. The last column is an example of formula (5.48). Because $T_n(Y) = \mathcal{O}(h^5)$ for method (5.28), Theorem 5.4 implies that the rate of convergence of $y_h(x)$ to $Y(x)$ is $\mathcal{O}(h^4)$. The theoretical value of "Ratio" is 16, and as h decreases further, this value will be realized more closely. ■

Q_6

Consider the differential equation:

$$y(x) + 3xy + y^2 = 0 \qquad y(1) = 0.5 \quad h = 0.1$$

Solve the differential equation to determine y(1.3) using:

- **a.** Euler Method
- **b.** Second order Taylor series method
- **c.** Second order Runge Kutta method
- **d.** Fourth order Runge-Kutta method
- **e.** Heun's predictor corrector method
- **f.** Midpoint method

Solution:

$$f(x, y) = -3xy - y^2 \qquad y(1) = 0.5$$

a. Euler Method

$$y(1.0) = 0.5$$

$$y(x+h) = y(x) + h\ f(x, y(x))$$

$$y(1.1) = 0.5 + 0.1\left[-3(1)(0.5) - (0.5)^2\right] = 0.3250$$

$$y(1.2) = 0.3250 + 0.1\left[-3(1.1)(0.3250) - (0.3250)^2\right] = 0.2072$$

$$y(1.3) = 0.2072 + 0.1\left[-3(1.2)(0.2072) - (0.2072)^2\right] = 0.1283$$

b. Second Order Taylor Series Method

$$\dot{y} = -3xy - y^2$$

$$\ddot{y} = -3y - 3x(-3xy - y^2) - 2y(-3xy - y^2)$$

$$y(1) = 0.5$$

$$y(x+h) = y(x) + h\ \dot{y} + \frac{h^2}{2!}\ddot{y}$$

$Iteration\ 1: \quad x=1; \quad y=0.5, \quad \dot{y}=-1.75 \ ; \ \ddot{y}=5.5$

$$y(1.1) = 0.5 + 0.1[-1.75] + \frac{0.01}{2}[5.5] = 0.3525$$

$Iteration\ 2: \quad x=1.1; \quad y=0.3525, \quad \dot{y}=-1.2875 \ ; \ \ddot{y}=4.0990$

$$y(1.2) = 0.3525,+0.1[-1.2875] + \frac{0.01}{2}[4.0990] = 0.2442$$

$Iteration\ 3: \quad x=1.2; \quad y=0.2442, \quad \dot{y}=-0.9389 \ ; \ \ddot{y}=3.1061$

$$y(1.3) = 0.2442 + 0.1[-0.9389] + \frac{0.01}{2}[3.1061] = 0.1659$$

c. Second Order Rung-Kutta

$Iteration1: \qquad x=1, y=0.5$

$K_1 = hf(x,y) = 0.1 f(1,0.5) = -0.1750$

$K_2 = hf(x+h, y+K_1) = (0.5)\ f(1+0.1, 0.5+(-0.1750)) = -0.1178$

$$y(x+h) = y(x) + \frac{1}{2}[K1+K1] = 0.5 + 0.5(-0.1750 - 0.1178) = 0.3536$$

$Iteration\ 2: \qquad x=1.1, \quad y=0.3536$

$K_1 = hf(1.1, 0.3536) = -0.1292$

$K_2 = hf(1.1+0.1, 0.3536+(-0.1292)) = -0.0858$

$$y(x+h) = y(x) + \frac{1}{2}[K_1+K_1] = 0.3536 + 0.5(-0.1292 - 0.0858) = 0.2461$$

$Iteration \quad 3: \quad x=1.2\ , \qquad y=0.2461$

$K_1 = hf(x,y) = -0.0947$

$K_2 = hf(x+h, y+K_1) = -0.0613$

$$y(x+h) = y(x) + \frac{1}{2}[K_1+K_2] = 0.2461 + \frac{1}{2}(-0.0947 - 0.0613) = 0.1681$$

d. Fourth Order Rung-Kutta

$Iteration \quad 1: \quad x = 1.0, \quad y = 0.5$

$K_1 = hf(x, y) = -0.175$

$K_2 = hf\left(x + 0.5h, y + 0.5K_1\right) = -0.1470$

$K_3 = hf\left(x + 0.5h, y + 0.5K_2\right) = -0.1525$

$K_4 = hf\left(x + h, y + K_3\right) = -0.1267$

$$y(x+h) = y(x) + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right] = 0.3499$$

$Iteration \quad 2: \quad x = 1.1, \quad y = 0.3499$

$K_1 = hf(x, y) = -0.1277$

$K_2 = hf\left(x + 0.5h, y + 0.5K_1\right) = -0.1069$

$K_3 = hf\left(x + 0.5h, y + 0.5K_2\right) = -0.1111$

$K_4 = hf\left(x + h, y + K_3\right) = -0.0917$

$$y(x+h) = y(x) + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right] = 0.2407$$

$Iteration \quad 3: \quad x = 1.2, \quad y = 0.2407$

$K_1 = hf(x, y) = -0.0924$

$K_2 = hf\left(x + 0.5h, y + 0.5K_1\right) = -0.0767$

$K_3 = hf\left(x + 0.5h, y + 0.5K_2\right) = -0.0800$

$K_4 = hf\left(x + h, y + K_3\right) = -0.0653$

$$y(x+h) = y(x) + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right] = 0.1622$$

e. Heun's Predictor Corrector Method

$$Iteration\ \ 1: \quad x = 1, \quad y(1) = 0.5$$

$$\text{Predictor}: \quad y^0(x+h) = y(x) + h\ f(x, y(x)) = 0.5 + 0.1 * (-1.75) = 0.3250$$

$$Corrector: \quad y(1.1) = y(x) + \frac{h}{2}\Big(f(x, y(x)) + f(x+h, y^0(x+h))\Big)$$

$$= 0.5 + \frac{0.1}{2}\big(-1.75 + -1.1781\big) = 0.3536$$

$$Iteration\ \ 2: \quad x = 1.1, \quad y(1.1) = 0.3536$$

$$\text{Predictor}: \quad y^0(x+h) = y(x) + h\ f(x, y(x)) = 0.3536 + 0.1 * (-1.2919) = 0.2244$$

$$Corrector: \quad y(1.2) = y(x) + \frac{h}{2}\Big(f(x, y(x)) + f(x+h, y^0(x+h))\Big)$$

$$= 0.3536 + \frac{0.1}{2}\big(-1.2919 \quad -0.8582\big) = 0.2461$$

$$Iteration\ \ 3: \quad x = 1.2, \quad y(1.1) = 0.2461$$

$$\text{Predictor}: \quad y^0(x+h) = y(x) + h\ f(x, y(x)) = 0.3536 + 0.1 * (-0.9465) = 0.1514$$

$$Corrector: \quad y(1.3) = y(x) + \frac{h}{2}\Big(f(x, y(x)) + f(x+h, y^0(x+h))\Big)$$

$$= 0.2461 + \frac{0.1}{2}\big(-0.9465 \quad -0.6136\big) = 0.1681$$

f. Midpoint Method

$$Iteration\ 1: \quad x=1, \quad y(1)=0.5$$

$$y_{i+\frac{1}{2}} = y(x)+\frac{h}{2}\ f(x, y(x)) = 0.5+\frac{0.1}{2} * (-1.75) = 0.4125$$

$$y(1.1) = y(x)+hf\left(x+\frac{h}{2}, y_{i+\frac{1}{2}}\right) = 0.3530$$

$$Iteration\ 2: \quad x=1.1, \quad y(1.1)=0.3530$$

$$y_{i+\frac{1}{2}} = y(x)+\frac{h}{2}\ f(x, y(x)) = 0.3530+\frac{0.1}{2} * (-1.2897) = 0.2886$$

$$y(1.2) = y(x)+hf\left(x+\frac{h}{2}, y_{i+\frac{1}{2}}\right) = 0.3530+0.1\,f(1.15, 0.2886) = 0.2452$$

$$Iteration\ 3: \quad x=1.2, \quad y(1.1)=0.2452$$

$$y_{i+\frac{1}{2}} = y(x)+\frac{h}{2}\ f(x, y(x)) = 0.2452+\frac{0.1}{2} * (-0.9427) = 0.1980$$

$$y(1.2) = y(x)+hf\left(x+\frac{h}{2}, y_{i+\frac{1}{2}}\right) = 0.2452+0.1\,f(1.25, 0.198) = 0.1670$$

Q_7

Classical Fourth-Order RK Method

Problem Statement.

(a) Use the classical fourth-order RK method [Eq. (25.40)] to integrate

$$f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$$

using a step size of $h = 0.5$ and an initial condition of $y = 1$ at $x = 0$.

(b) Similarly, integrate

$$f(x, y) = 4e^{0.8x} - 0.5y$$

using $h = 0.5$ with $y(0) = 2$ from $x = 0$ to 0.5.

Solution.

(a) Equations (25.40*a*) through (25.40*d*) are used to compute $k_1 = 8.5$, $k_2 = 4.21875$, $k_3 = 4.21875$ and $k_4 = 1.25$, which are substituted into Eq. (25.40) to yield

$$y(0.5) = 1 + \left\{\frac{1}{6}[8.5 + 2(4.21875) + 2(4.21875) + 1.25]\right\}0.5$$
$$= 3.21875$$

which is exact. Thus, because the true solution is a quartic [Eq. (PT7.16)], the fourth-order method gives an exact result.

(b) For this case, the slope at the beginning of the interval is computed as

$$k_1 = f(0, 2) = 4e^{0.8(0)} - 0.5(2) = 3$$

This value is used to compute a value of y and a slope at the midpoint,

$$y(0.25) = 2 + 3(0.25) = 2.75$$
$$k_2 = f(0.25, 2.75) = 4e^{0.8(0.25)} - 0.5(2.75) = 3.510611$$

This slope in turn is used to compute another value of y and another slope at the midpoint,

$$y(0.25) = 2 + 3.510611(0.25) = 2.877653$$
$$k_3 = f(0.25, 2.877653) = 4e^{0.8(0.25)} - 0.5(2.877653) = 3.446785$$

Next, this slope is used to compute a value of y and a slope at the end of the interval,

$$y(0.5) = 2 + 3.071785(0.5) = 3.723392$$
$$k_4 = f(0.5, 3.723392) = 4e^{0.8(0.5)} - 0.5(3.723392) = 4.105603$$

Finally, the four slope estimates are combined to yield an average slope. This average slope is then used to make the final prediction at the end of the interval.

$$\phi = \frac{1}{6}[3 + 2(3.510611) + 2(3.446785) + 4.105603] = 3.503399$$

$$y(0.5) = 2 + 3.503399(0.5) = 3.751699$$

which compares favorably with the true solution of 3.751521.

Q_8

assuming that at $x = 0$, $y_1 = 4$, and $y_2 = 6$. Integrate to $x = 2$ with a step size of 0.5.

$$\frac{dy_1}{dx} = -0.5y_1 \qquad \frac{dy_2}{dx} = 4 - 0.3y_2 - 0.1y_1$$

Solving Systems of ODEs Using the Fourth-Order RK Method

Problem Statement. Use the fourth-order RK method to solve the ODEs from Example 25.9.

Solution. First, we must solve for all the slopes at the beginning of the interval:

$$k_{1,1} = f_1(0, 4, 6) = -0.5(4) = -2$$
$$k_{1,2} = f_2(0, 4, 6) = 4 - 0.3(6) - 0.1(4) = 1.8$$

where $k_{i,j}$ is the ith value of k for the jth dependent variable. Next, we must calculate the first values of y_1 and y_2 at the midpoint:

$$y_1 + k_{1,1}\frac{h}{2} = 4 + (-2)\frac{0.5}{2} = 3.5$$

$$y_2 + k_{1,2}\frac{h}{2} = 6 + (1.8)\frac{0.5}{2} = 6.45$$

which can be used to compute the first set of midpoint slopes,

$$k_{2,1} = f_1(0.25, 3.5, 6.45) = -1.75$$
$$k_{2,2} = f_2(0.25, 3.5, 6.45) = 1.715$$

These are used to determine the second set of midpoint predictions,

$$y_1 + k_{2,1}\frac{h}{2} = 4 + (-1.75)\frac{0.5}{2} = 3.5625$$

$$y_2 + k_{2,2}\frac{h}{2} = 6 + (1.715)\frac{0.5}{2} = 6.42875$$

which can be used to compute the second set of midpoint slopes,

$$k_{3,1} = f_1(0.25, 3.5625, 6.42875) = -1.78125$$
$$k_{3,2} = f_2(0.25, 3.5625, 6.42875) = 1.715125$$

These are used to determine the predictions at the end of the interval

$$y_1 + k_{3,1}h = 4 + (-1.78125)(0.5) = 3.109375$$
$$y_2 + k_{3,2}h = 6 + (1.715125)(0.5) = 6.857563$$

which can be used to compute the endpoint slopes,

$$k_{4,1} = f_1(0.5, 3.109375, 6.857563) = -1.554688$$
$$k_{4,2} = f_2(0.5, 3.109375, 6.857563) = 1.631794$$

The values of k can then be used to compute [Eq. (25.40)]:

$$y_1(0.5) = 4 + \frac{1}{6}[-2 + 2(-1.75 - 1.78125) - 1.554688]0.5 = 3.115234$$

$$y_2(0.5) = 6 + \frac{1}{6}[1.8 + 2(1.715 + 1.715125) + 1.631794]0.5 = 6.857670$$

منابع:

- California state university 501A Number:15966
- Numerical Methods for Engineers”, Steven C. Chapra and Raymond P. Canale, 7th Edition.
- CISE 301