

CHAPTER VII

Numerical Solution of Ordinary Differential Equations (ODE)

I. Definition

An equation that consists of derivatives is called a differential equation. Differential equations have applications in all areas of science and engineering. Mathematical formulation of most of the physical and engineering problems lead to differential equations. So, it is important for engineers and scientists to know how to set up differential equations and solve them.

Differential equations are of two types

- 1) ordinary differential equation (ODE)
- 2) partial differential equations (PDE).

An ordinary differential equation is that in which all the derivatives are with respect to a single independent variable. Examples of ordinary differential equation include

- 1) $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 0, \frac{dy}{dx}(0) = 2, y(0) = 4,$
- 2) $\frac{d^3 y}{dx^3} + 3\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + y = \sin x, \frac{d^2 y}{dx^2}(0) = 12, \frac{dy}{dx}(0) = 2, y(0) = 4$

Note: In this first part, we will see how to solve ODE of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

In another section, we will discuss how to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

But first, How to write a first order differential equation in the above form?

Example 1

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x), y(0) = 5$$

is rewritten as

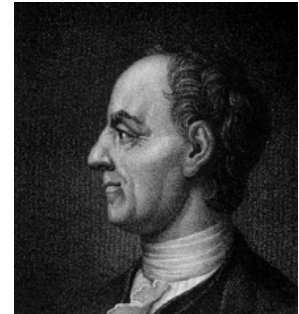
$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2 y^2}{e^y}, y(0) = 5$$

In this case

$$f(x, y) = \frac{2\sin(3x) - x^2 y^2}{e^y}$$

II. Euler's Method

Swiss mathematician who was tutored by Johann Bernoulli. He worked at the Petersburg Academy and Berlin Academy of Science. He had a phenomenal memory, and once did a calculation in his head to settle an argument between students whose computations differed in the fiftieth decimal place. Euler lost sight in his right eye in 1735, and in his left eye in 1766. Nevertheless, aided by his phenomenal memory (and having practiced writing on a large slate when his sight was failing him), he continued to publish his results by dictating them. Euler was the most prolific mathematical writer of all times finding time (even with his 13 children) to publish over 800 papers in his lifetime. He won the Paris Academy Prize 12 times. When asked for an explanation why his memoirs flowed so easily in such huge quantities, Euler is reported to have replied that his pencil seemed to surpass him in intelligence. François Arago said of him "He calculated just as men breathe, as eagles sustain themselves in the air" (Beckmann 1971, p. 143; Boyer 1968, p. 482).



We will use Euler's method to solve an ODE under the form:

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

At $x = 0$, we are given the value of $y = y_0$. Let us call $x = 0$ as x_0 . Now since we know the slope of y with respect to x , that is, $f(x, y)$, then at $x = x_0$, the slope is $f(x_0, y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0) = y_0$.

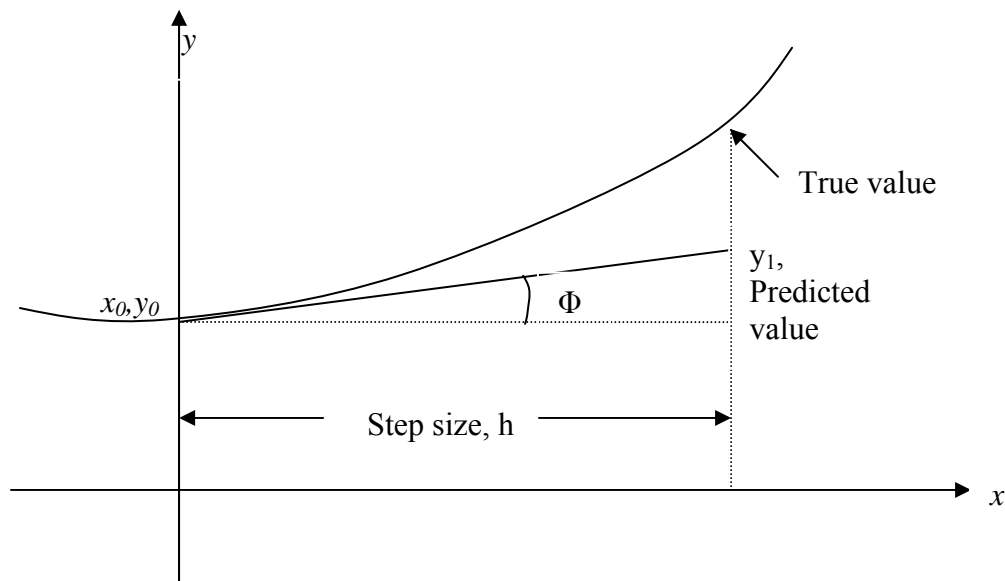


Figure 1. Graphical interpretation of the first step of Euler's method.

So the slope at $x=x_0$ as shown in the figure above

$$\begin{aligned}\text{Slope} &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= f(x_0, y_0)\end{aligned}$$

Thus

$$y_1 = y_0 + f(x_0, y_0) (x_1 - x_0)$$

If we consider $x_1 - x_0$ as a step size h , we get

$$y_1 = y_0 + f(x_0, y_0) h.$$

We are able now to use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2 , which is the predicted value at x_2 ,

$$y_2 = y_1 + f(x_1, y_1) h$$

$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y = y_i$ at x_i , then

$$y_{i+1} = y_i + f(x_i, y_i) h$$

This formula is known as the Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

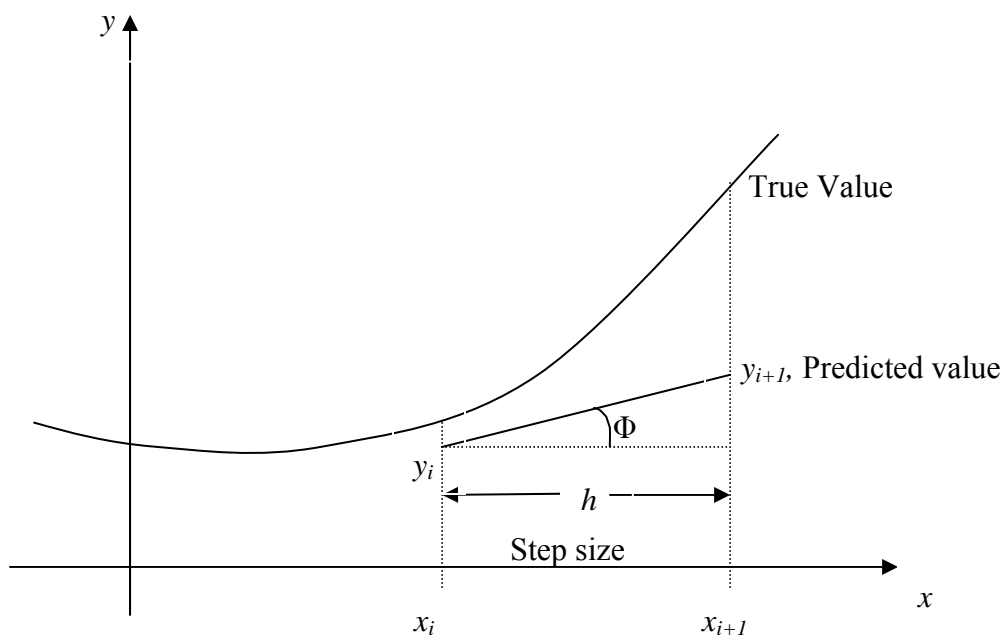


Figure 2. General graphical interpretation of Euler's method.

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of $h = 240$ seconds.

- Additional resources to analyze the above example

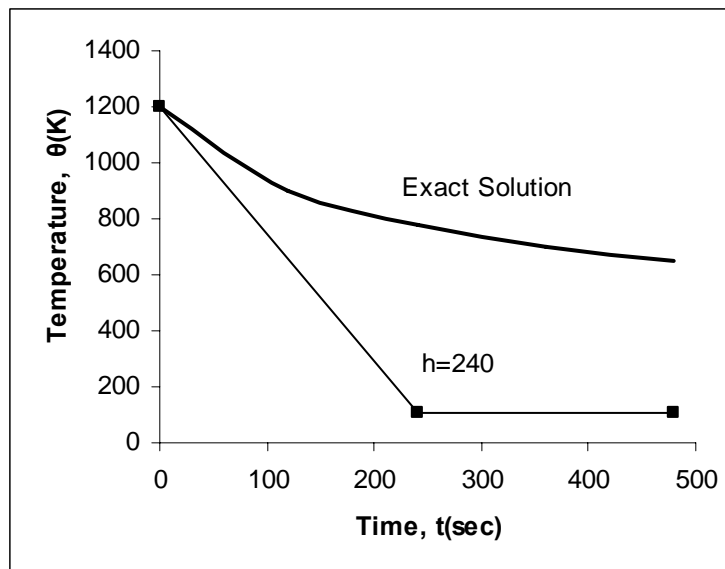


Figure 3. Comparing exact and Euler's method.

The problem was solved again using a smaller step size. The results are given below in Table 1.

For smaller steps

Step size, h	$\theta(480)$	E_t	$ \epsilon_t \%$
480	-987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.77	14.806	2.2864

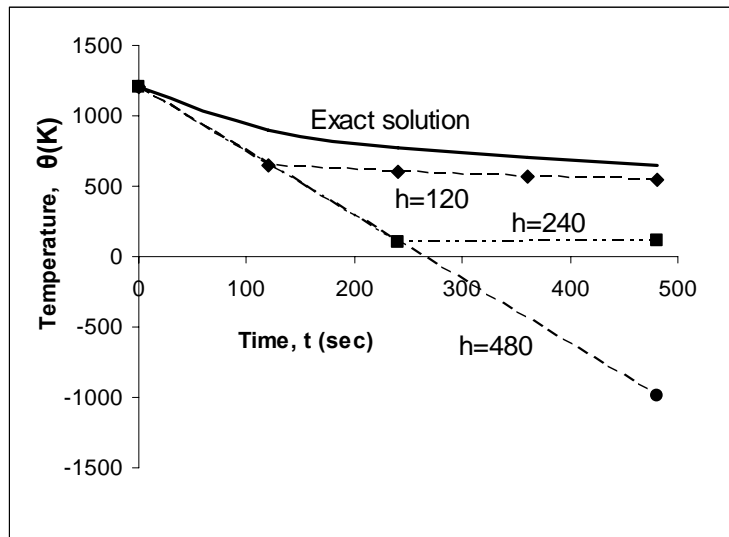


Figure 4. Comparison of Euler's method with exact solution for different step sizes.

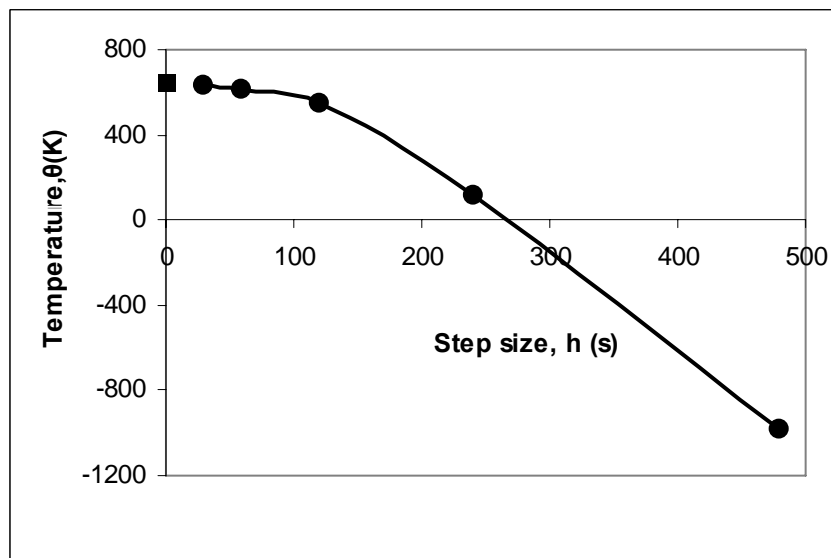


Figure 5. Effect of step size in Euler's method.

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as:

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.333 \times 10^{-2} \theta) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation is

$$\theta = 647.57 K$$

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \frac{dy}{dx} \bigg|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!} \frac{d^2 y}{dx^2} \bigg|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \frac{d^3 y}{dx^3} \bigg|_{x_i, y_i} (x_{i+1} - x_i)^3 + \dots$$

$$y_{i+1} = y_i + f(x_i, y_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \dots$$

As you can see the first two terms of the Taylor series

$$y_{i+1} = y_i + f(x_i, y_i)h \quad \text{are the Euler's method.}$$

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \dots$$

The true error hence is approximately proportional to the square of the step size, that is, as the step size is halved, the true error gets approximately quartered. However from Table 1, we see that as the step size gets halved, the true error only gets approximately halved. This is because the true error being proportioned to the square of the step size is the local truncation error, that is, error from one point to the next. The global truncation error is however proportional only to the step size as the error keeps propagating from one point to another.

II. Runge-Kutta 2nd order

Euler's method was derived from Taylor series as:

$$y_{i+1} = y_i + f(x_i, y_i)h$$

This can be considered to be Runge-Kutta 1st order method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \dots$$

Now let us consider a 2nd order method formula. This new formula would include one more term of the Taylor series as follows:

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2$$

Let us now apply this to a simple example:

$$\frac{dy}{dx} = e^{-2x} - 3y, y(0) = 5$$

$$f(x, y) = e^{-2x} - 3y$$

Now since y is a function of x ,

$$\begin{aligned} f'(x, y) &= \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y} \frac{dy}{dx} \\ &= \frac{\partial}{\partial x}(e^{-2x} - 3y) + \frac{\partial}{\partial y}[(e^{-2x} - 3y)](e^{-2x} - 3y) \\ &= -2e^{-2x} + (-3)(e^{-2x} - 3y) \\ &= -5e^{-2x} + 9y \end{aligned}$$

The 2nd order formula would be

$$\begin{aligned} y_{i+1} &= y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2 \\ y_{i+1} &= y_i + (e^{-2x_i} - 3y_i)h + \frac{1}{2!}(-5e^{-2x_i} + 9y_i)h^2 \end{aligned}$$

You could easily notice the difficulty of having to find $f'(x, y)$ in the above method. What Runge and Kutta did was write the 2nd order method as

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

where

$$\begin{aligned} k_1 &= f(x_i, y_i) \\ k_2 &= f(x_i + p_1 h, y_i + q_{11} k_1 h) \end{aligned}$$

This form allows us to take advantage of the 2nd order method without having to calculate $f'(x, y)$.

But, how do we find the unknowns a_1 , a_2 , p_1 and q_{11} ? Equating the above equations:

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2 \text{ and } y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

gives three equations.

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

Since we have 3 equations and 4 unknowns, we can assume the value of one of the unknowns. The other three will then be determined from the three equations. Generally the value of a_2 is chosen to evaluate the other three constants. The three values generally used for a_2 are $\frac{1}{2}$, 1 and $\frac{2}{3}$, and are known as Heun's Method, Midpoint method and Ralston's method, respectively.

II.1. Heun's method

Here we choose $a_2 = \frac{1}{2}$, giving

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2 \right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1h)$$

This method is graphically explained in Figure 6.

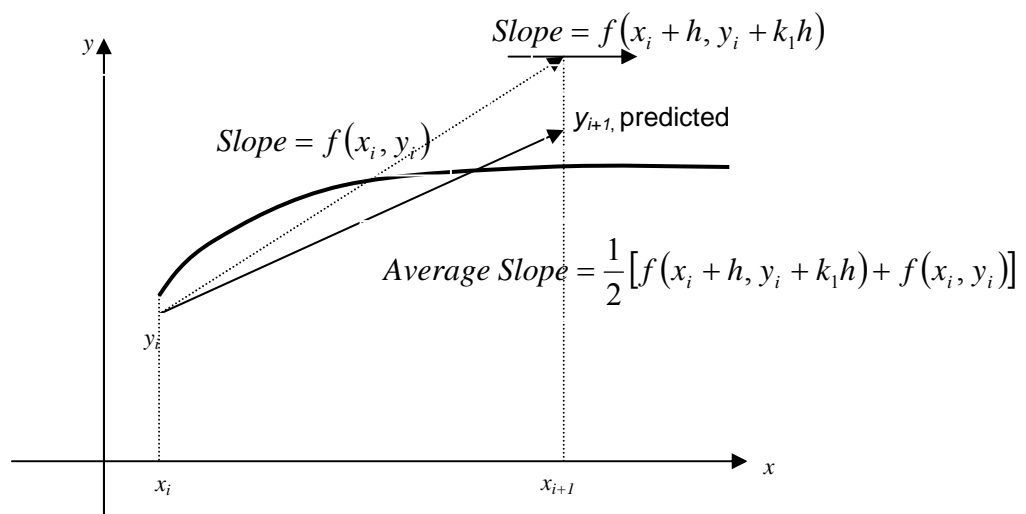


Figure 6. Runge-Kutta 2nd order method (Heun's method).

II.2. Midpoint method

Here we choose $a_2 = 1$, giving

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

II.3. Ralston's method

Here we choose $a_2 = \frac{2}{3}$, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} (\theta^4 - 81 \times 10^8), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Runge-Kutta 2nd order method (with Heun's method). Assume a step size of $h = 240$ seconds.

- Additional resources to analyze the above example

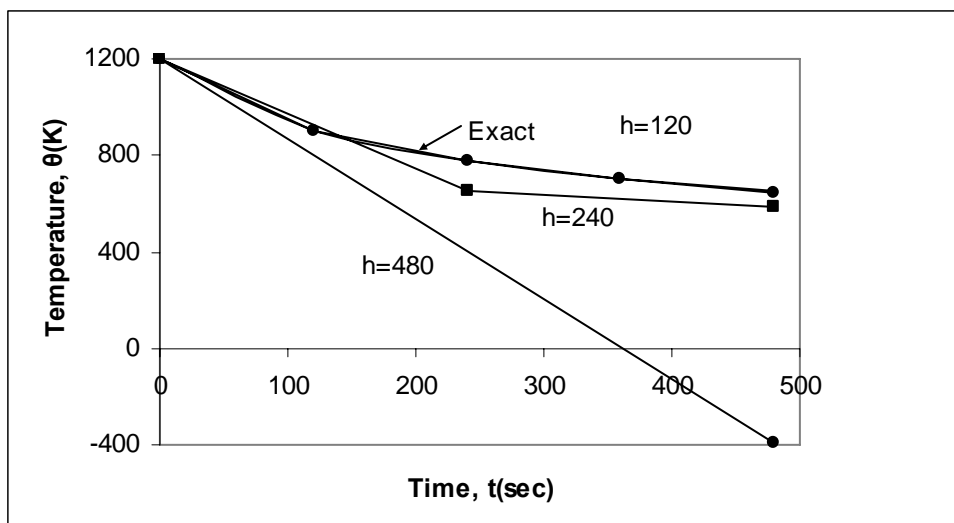


Figure 7. Heun's method results for different step sizes.

Using smaller step size would increase the accuracy of the result as given in Table 1 and Figure 3 below.

Effect of step size for Heun's method			
Step size, h	$\theta(480)$	E_t	$ \epsilon_t \%$
480	-393.87	1041.4	160.82
240	584.27	63.304	9.7756
120	651.35	-3.7762	0.58313
60	649.91	-2.3406	0.36145
30	648.21	-0.63219	0.097625

Now, let us compare Euler's method and Runge-Kutta 2nd order method results:

Comparison of Euler and the Runge-Kutta methods				
Step size, h	$\theta(480)$			
	Euler	Heun	Midpoint	Ralston
480	-987.84	-393.87	1208.4	449.78
240	110.32	584.27	976.87	690.01
120	546.77	651.35	690.20	667.71
60	614.97	649.91	654.85	652.25
30	632.77	648.21	649.02	648.61

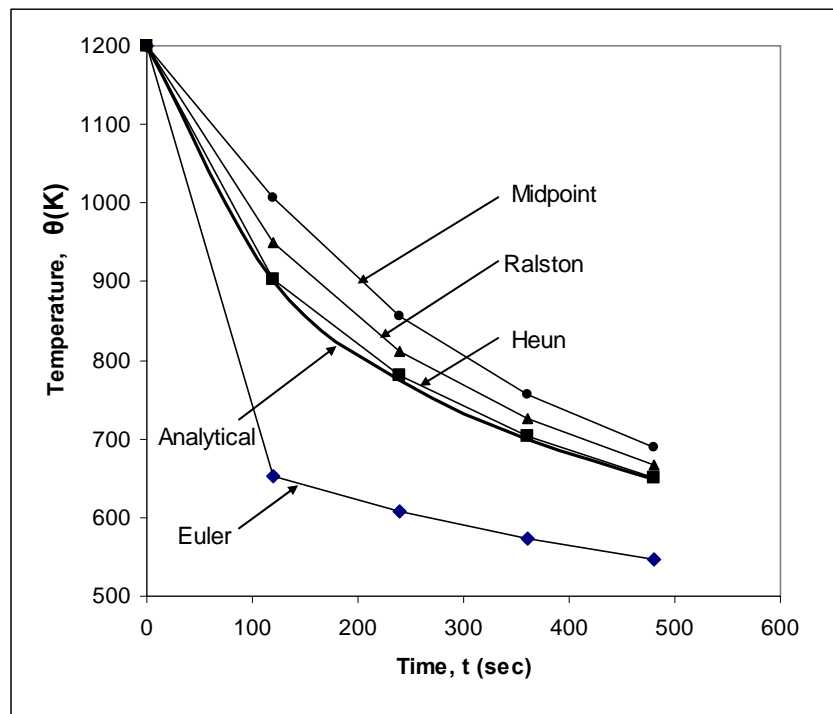


Figure 8. Comparison of Euler and Runge Kutta methods with exact results over time.

NOTE: How do these three methods compare with results obtained if we found $f'(x, y)$ directly?

We know that since we are including first three terms in the series, if the solution is a polynomial of order two or less (that is, quadratic, linear or constant), any of the three methods are exact. But for any other case the results will be different.

Consider the following example

$$\frac{dy}{dx} = e^{-2x} - 3y, y(0) = 5$$

If we directly find the $f'(x, y)$, the first three terms of Taylor series gives

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2$$

where

$$f(x, y) = e^{-2x} - 3y$$

$$f'(x, y) = -5e^{-2x} + 9y$$

For a step size of $h = 0.2$, using Heun's method, we find

$$y(0.6) = 1.0930$$

The exact solution

$$y(x) = e^{-2x} + 4e^{-3x}$$

gives

$$\begin{aligned} y(0.6) &= e^{-2(0.6)} + 4e^{-3(0.6)} \\ &= 0.96239 \end{aligned}$$

Then the absolute relative true error is

$$\begin{aligned} |\epsilon_t| &= \left| \frac{0.96239 - 1.0930}{0.96239} \right| \times 100 \\ &= 13.571\% \end{aligned}$$

For the same problem, the results from the Euler and the three Runge-Kutta method are given below

Comparison of Euler's and Runge-Kutta 2 nd order methods						
	$y(0.6)$					
	Exact	Euler	Direct 2nd	Heun	Midpoint	Ralston
Value	0.96239	0.4955	1.0930	1.1012	1.0974	1.0994
$ \epsilon_t $ %		48.514	13.571	14.423	14.029	14.236

III. Runge-Kutta 4th order

Runge-Kutta 4th order method is based on the following

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)h$$

where knowing the value of $y = y_i$ at x_i , we can find the value of $y = y_{i+1}$ at x_{i+1} , and

$$h = x_{i+1} - x_i$$

The above equation is equated to the first five terms of Taylor series

$$\begin{aligned} y_{i+1} &= y_i + \frac{dy}{dx} \Big|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!} \frac{d^2y}{dx^2} \Big|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \frac{d^3y}{dx^3} \Big|_{x_i, y_i} (x_{i+1} - x_i)^3 \\ &+ \frac{1}{4!} \frac{d^4y}{dx^4} \Big|_{x_i, y_i} (x_{i+1} - x_i)^4 \end{aligned}$$

Knowing that $\frac{dy}{dx} = f(x, y)$ and $x_{i+1} - x_i = h$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f'''(x_i, y_i)h^4$$

Based on equating the above equations, one of the popular solutions used is

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h\right)$$

$$k_4 = f(x_i + h, y_i + k_3h)$$

Solving the same example as above, but considering Runge-Kutta 4th order method, gives the following results:

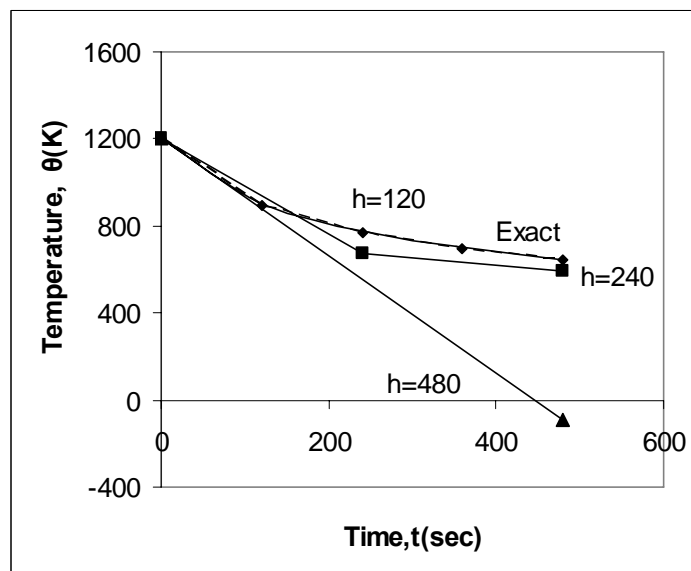


Figure 9. Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.

The effect of step size on the value of the calculated temperature at $t=480$ seconds, can be seen on the following table and figure.

Value of temperature at time, $t=480$ s for different step sizes

Step size, h	$\theta(480)$	E_t	$ \varepsilon_t \%$
480	-90.278	737.85	113.94
240	594.91	52.660	8.1319
120	646.16	1.4122	0.21807
60	647.54	0.033626	0.0051926
30	647.57	0.00086900	0.00013419

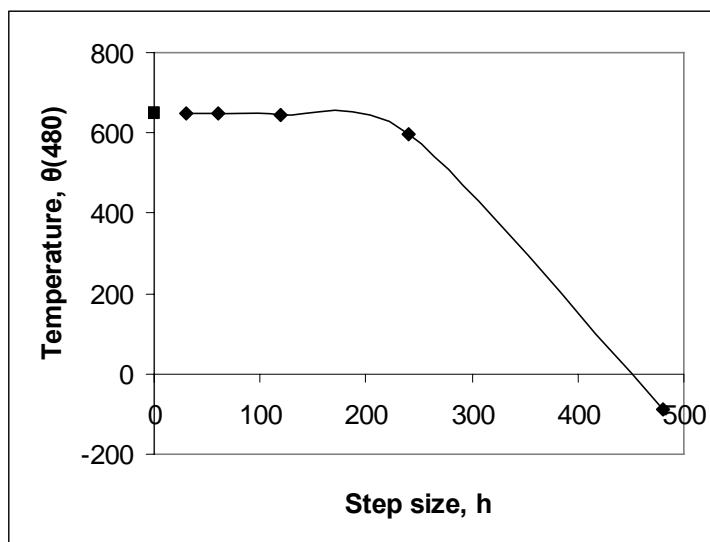


Figure 10. Effect of step size in Runge-Kutta 4th order method.

We can also compare the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method) and Runge-Kutta 4th order method.

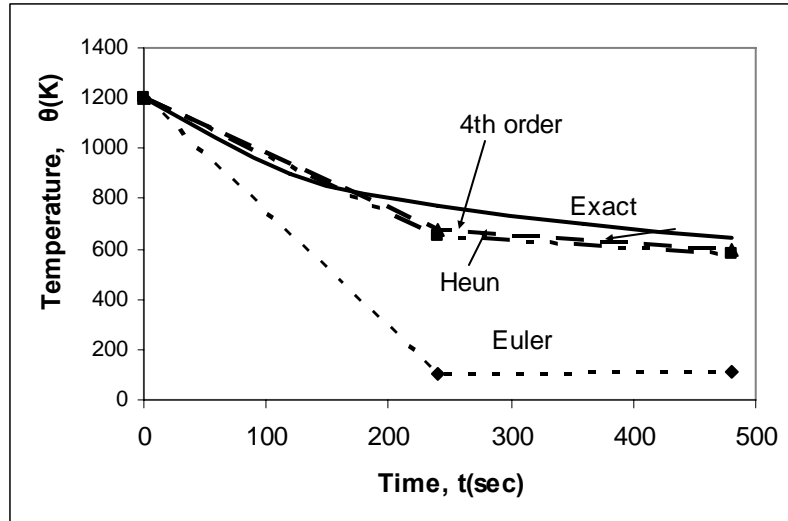


Figure 11. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

There are other versions of the fourth order method just like there are several versions of the second order methods. Formula developed by Kutta is

$$y_{i+1} = y_i + \frac{1}{8}(k_1 + 3k_2 + 3k_3 + k_4)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{3}h, y_i + \frac{1}{3}hk_1\right)$$

$$k_3 = f\left(x_i + \frac{2}{3}h, y_i - \frac{1}{3}hk_1 + hk_2\right)$$

$$k_4 = f(x_i + h, y_i + hk_1 - hk_2 + hk_3)$$

IV. Solving higher order ODE

Until now, we have solved using Euler's and Runge-Kutta methods first order ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

But now, the question is: what do we do to solve simultaneous or coupled differential equations or differential equations that are higher than first order? For example an n^{th} order differential equations:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

By assuming

$$\begin{aligned}
 y &= z_1 \\
 \frac{dy}{dx} &= \frac{dz_1}{dx} = z_2 \\
 \frac{d^2y}{dx^2} &= \frac{dz_2}{dx} = z_3 \\
 &\vdots \\
 \frac{d^{n-1}y}{dx^{n-1}} &= \frac{dz_{n-1}}{dx} = z_n \\
 \frac{d^ny}{dx^n} &= \frac{dz_n}{dx} = \frac{1}{a_n} \left(-a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} \dots - a_1 \frac{dy}{dx} - a_0 y + f(x) \right) \\
 &= \frac{1}{a_n} (-a_{n-1} z_n \dots - a_1 z_2 - a_0 z_1 + f(x))
 \end{aligned}$$

The above equations represent n first order differential equations as follows

$$\begin{aligned}
 \frac{dz_1}{dx} &= z_2 = f_1(z_1, z_2, \dots, x) \\
 \frac{dz_2}{dx} &= z_3 = f_2(z_1, z_2, \dots, x) \\
 &\vdots \\
 \frac{dz_n}{dx} &= \frac{1}{a_n} (-a_{n-1} z_n \dots - a_1 z_2 - a_0 z_1 + f(x))
 \end{aligned}$$

Example

Rewrite the following differential equation as a set of first order differential equations:

$$3 \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 5y = e^{-x}, y(0) = 5, y'(0) = 7$$

The ordinary differential equation would be rewritten as follows. Assume

$$\frac{dy}{dx} = z,$$

Then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

Substituting this in the given second order ordinary differential equation gives

$$\begin{aligned}
 3 \frac{dz}{dx} + 2z + 5y &= e^{-x} \\
 \frac{dz}{dx} &= \frac{1}{3} (e^{-x} - 2z - 5y).
 \end{aligned}$$

The set of two simultaneous first order ordinary differential equations complete with the initial conditions then is

$$\frac{dy}{dx} = z, y(0) = 5$$

$$\frac{dz}{dx} = \frac{1}{3}(e^{-x} - 2z - 5y), z(0) = 7.$$

Now one can apply any of numerical methods used for first order ordinary differential equations to solve the above first order differential equations.