Chapter 3

Interpolation, Extrapolation and Curve Fitting

Fatemeh Baharifard

- Interpolation
  - Non-Equally Spaced
    - Lagrange Method
    - Newton Divided Differences
  - Equally Spaced
    - Newton Forward Differences Method
    - Newton Backward Differences Method
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      - Forward Central
      - Backward Central
- Extrapolation
- Curve Fitting

Interpolation Example

x	$\sin x$
30°	0.5
31°	0.515038
32°	0.529919

$$\sin 30.5^\circ = ?$$

Interpolation Example

x	$\sin x$
30°	0.5
31°	0.515038
32°	0.529919

$$\sin 30.5^{\circ} = \frac{0.5 + 0.515038}{2} = 0.507519$$

Why Interpolation ?

Hundreds of years ago, when there was not any computer, people manually calculated the value of functions such as sinus at certain points like 30° and 31° and then used interpolation methods to calculate its values at intermediate points.

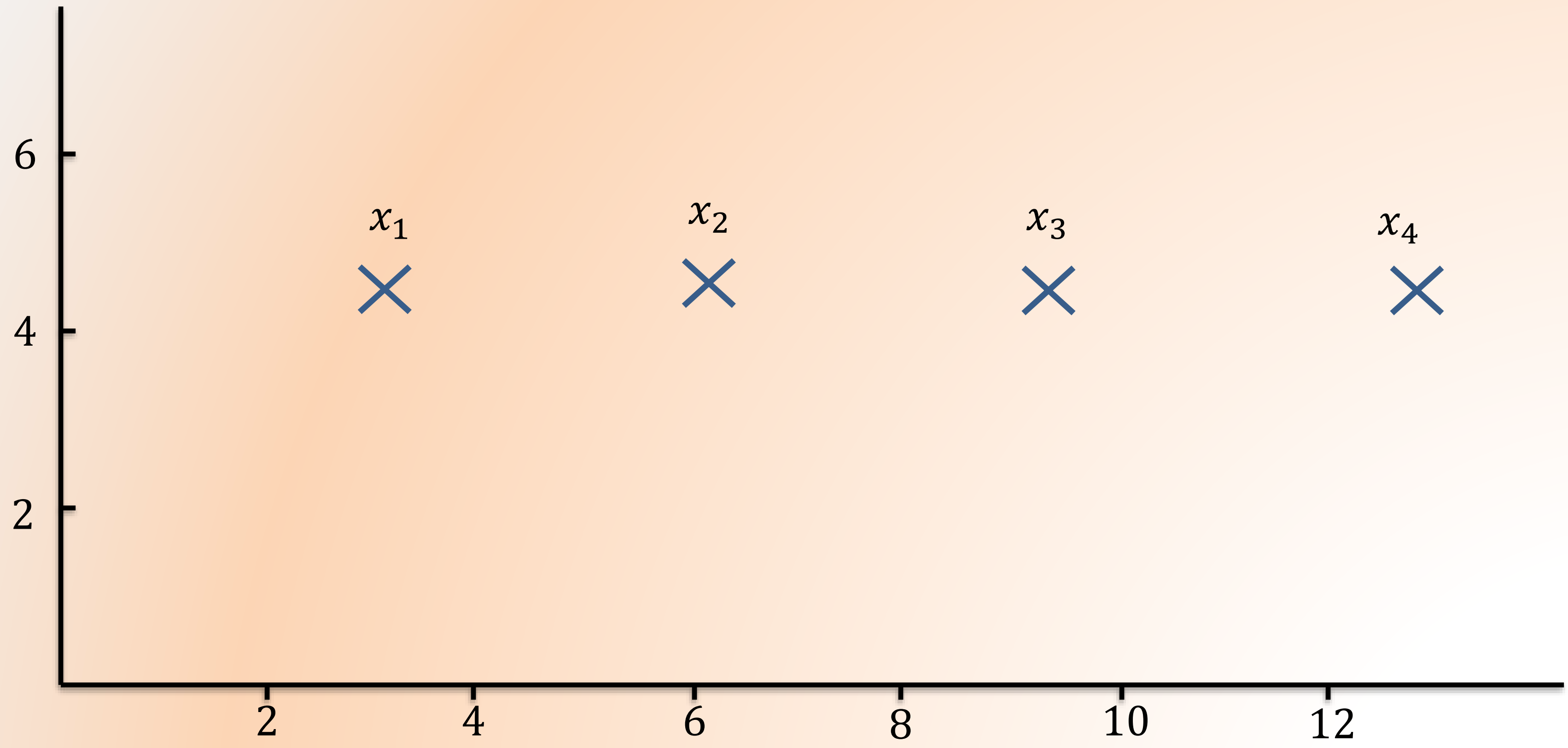
Why Interpolation ?

Hundreds of years ago, when there was not any computer, people manually calculated the value of functions such as sinus at certain points like 30° and 31° and then used interpolation methods to calculate its values at intermediate points.

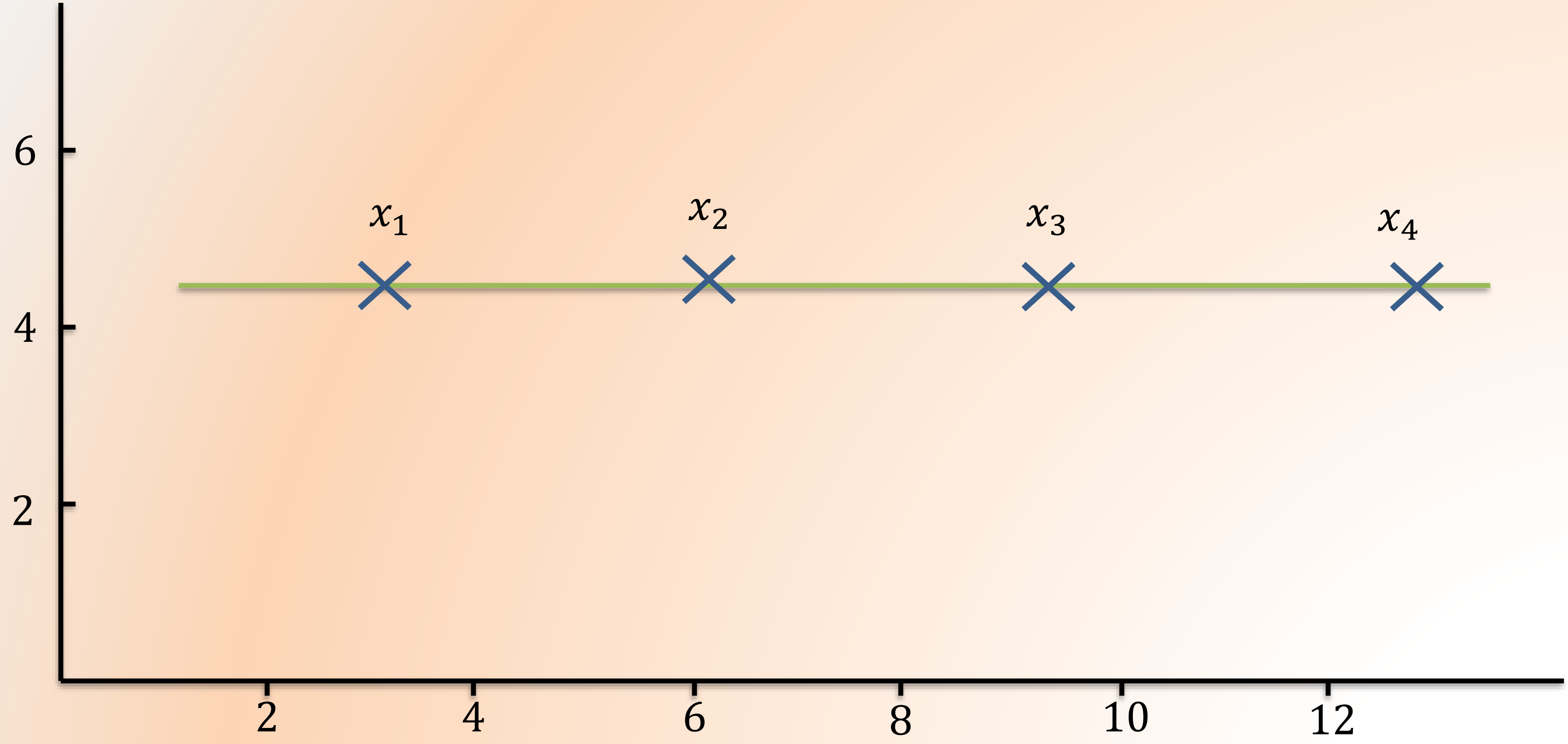
- We have several points and we need a function that passes through all of them.

- We have a function with a strange form and we need a fairly accurate polynomial approximation for it.

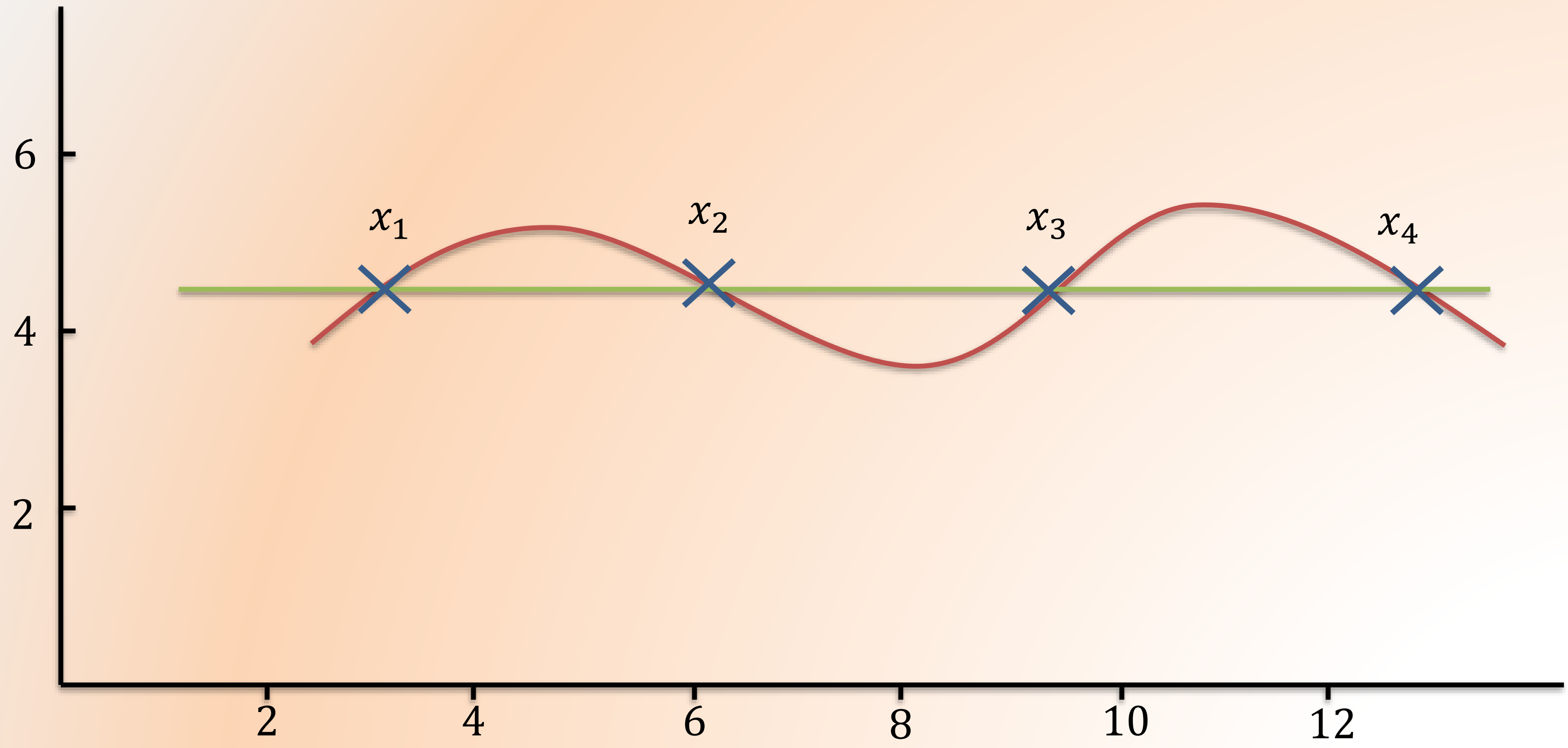
Interpolation



Interpolation



Interpolation



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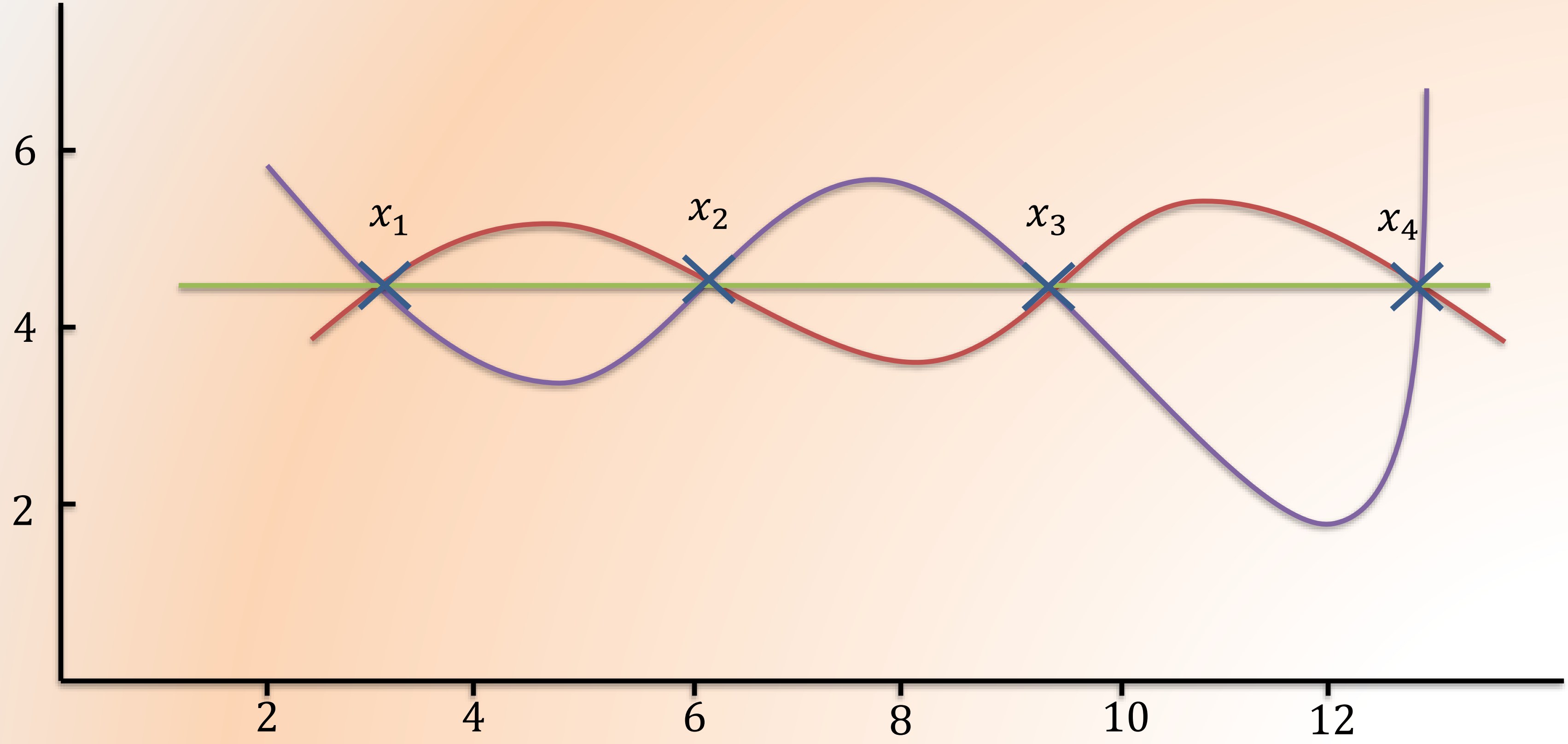
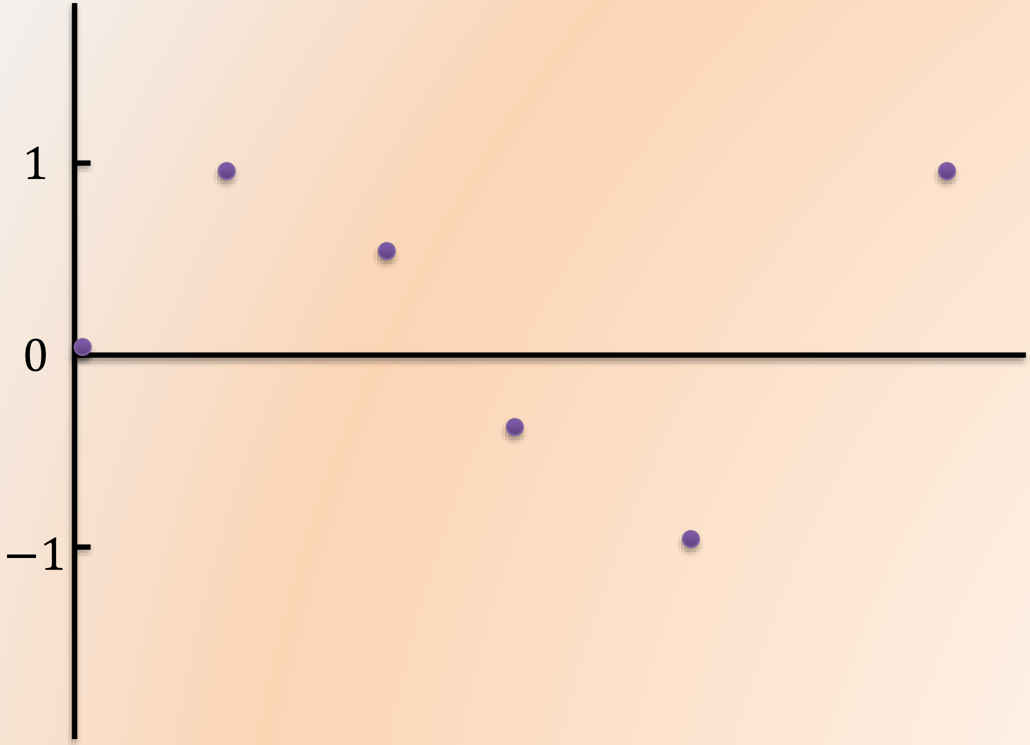


Interpolation



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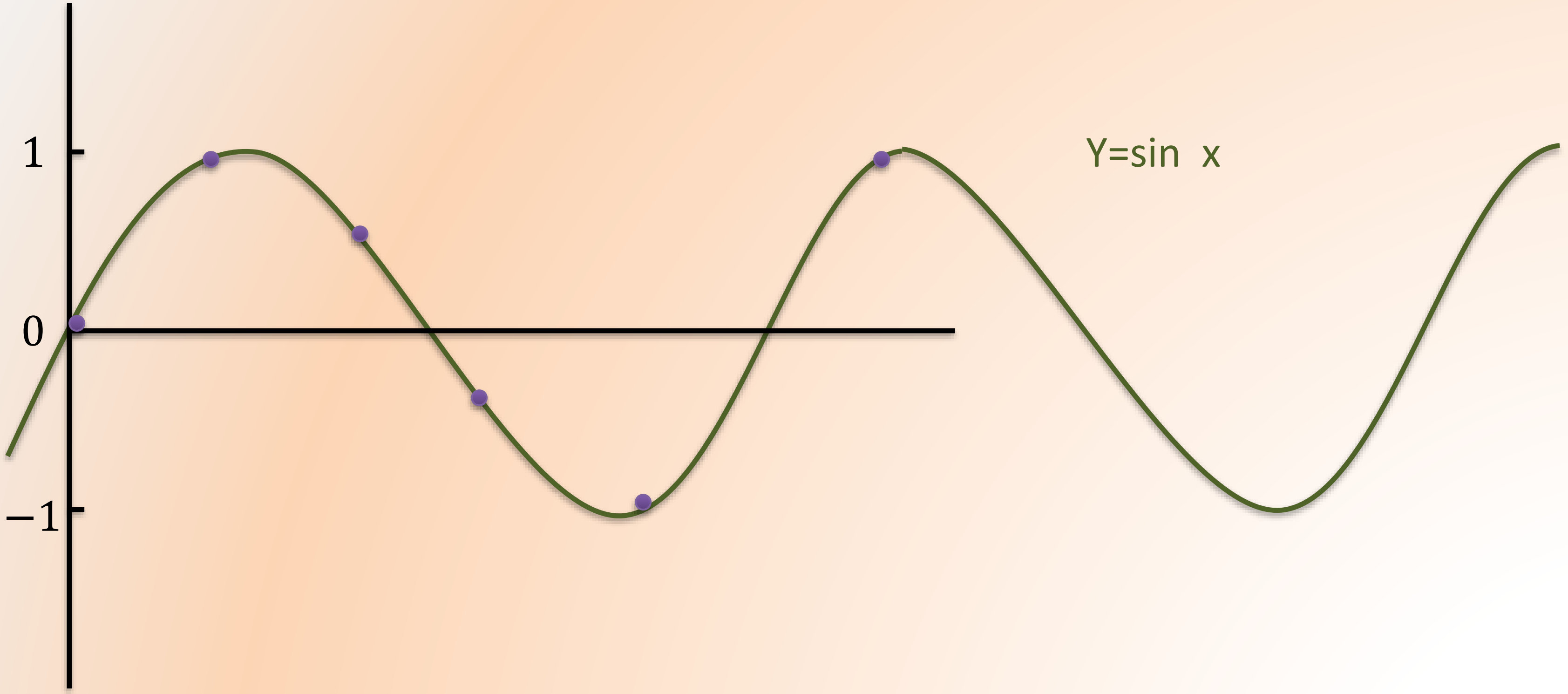

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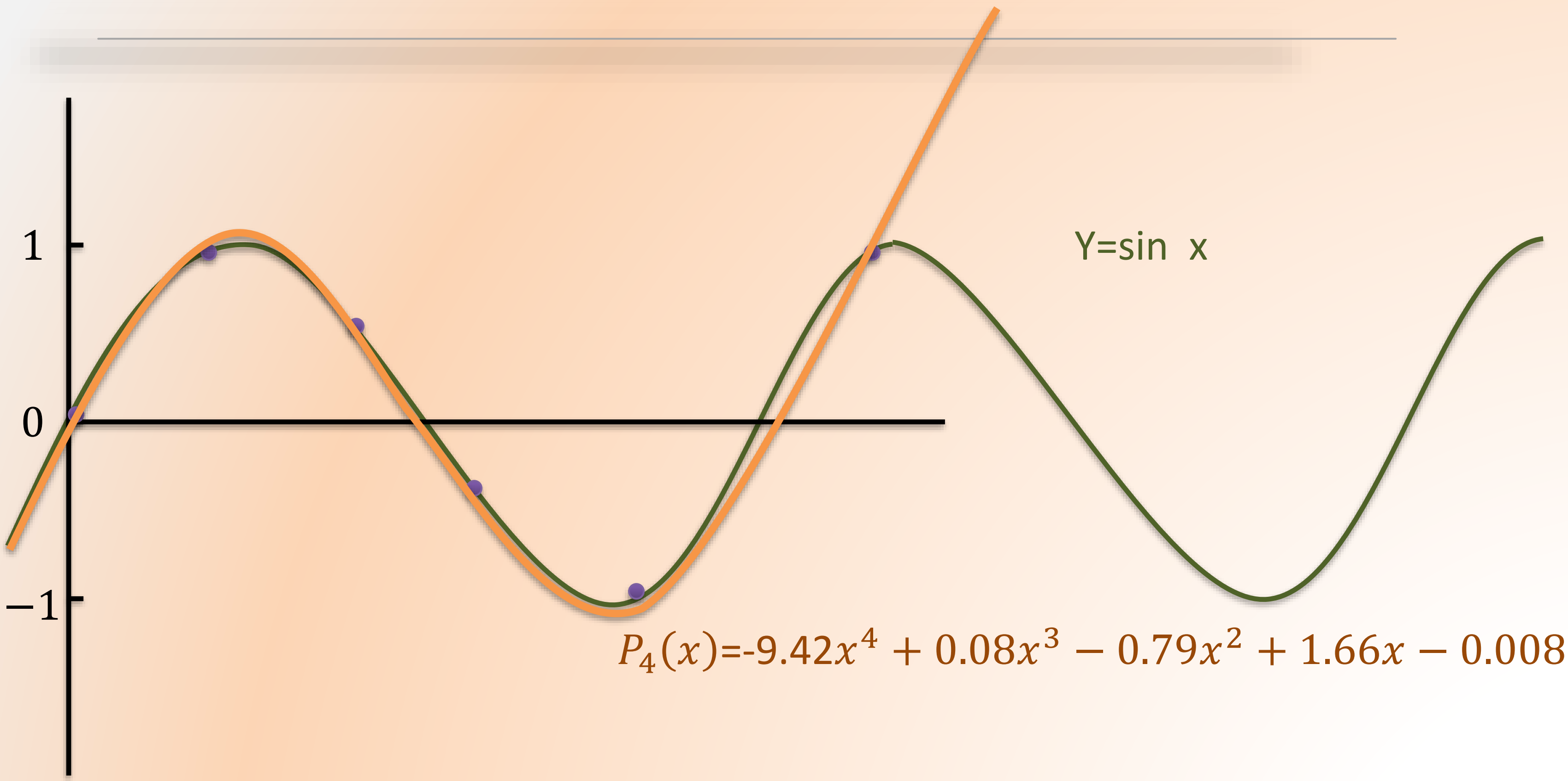
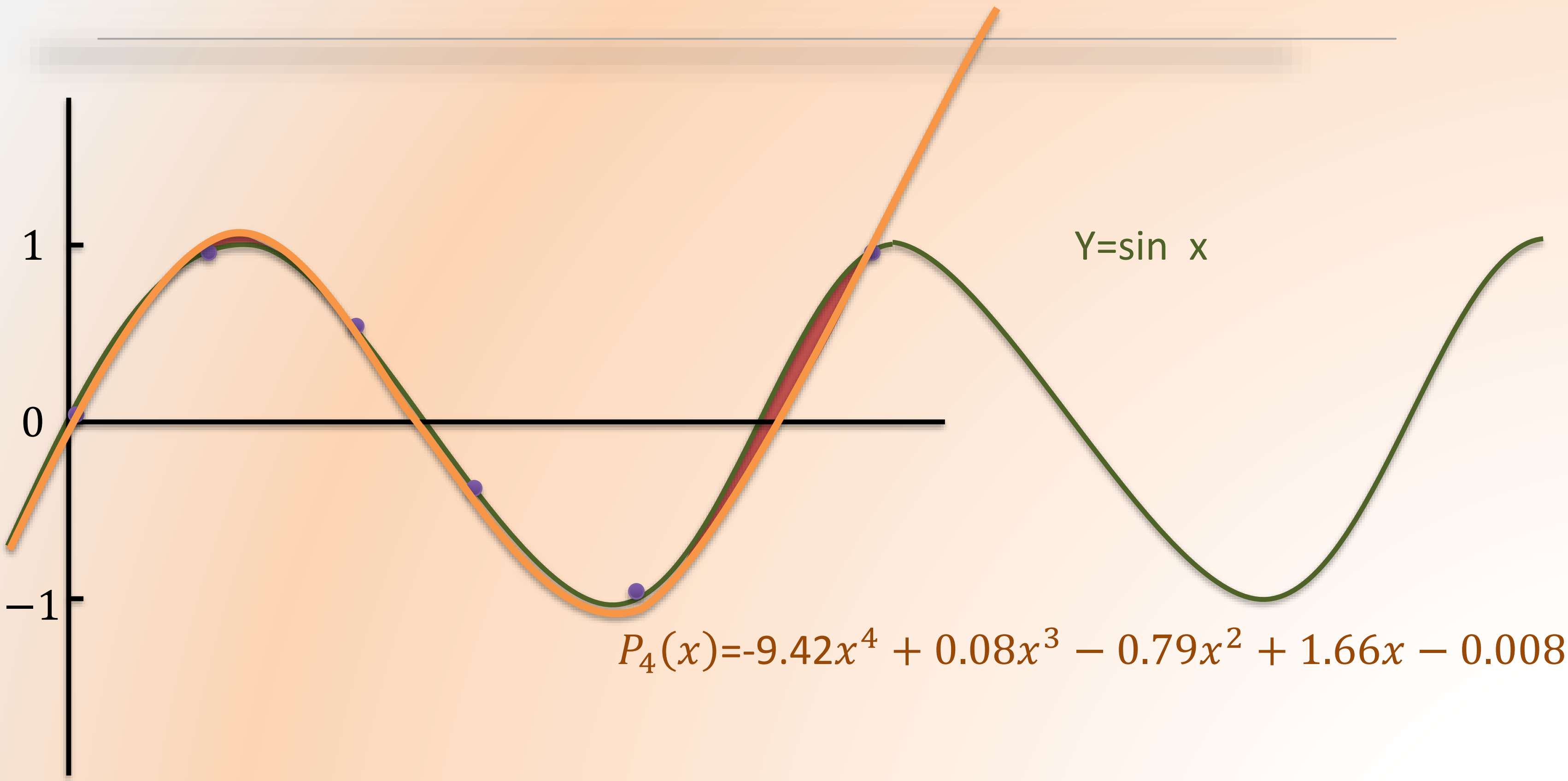


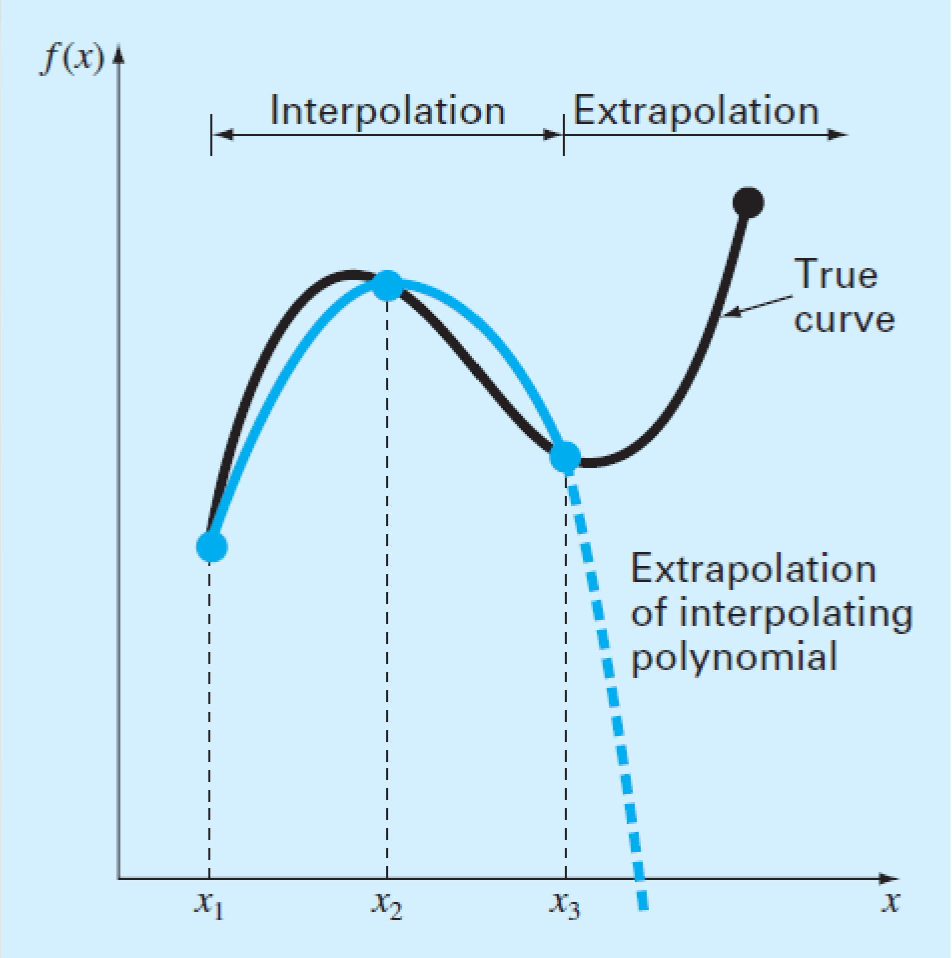
$P_4(x)$=-9.42$x^4 + 0.08x^3 - 0.79x^2 + 1.66x - 0.008$

Interpolation

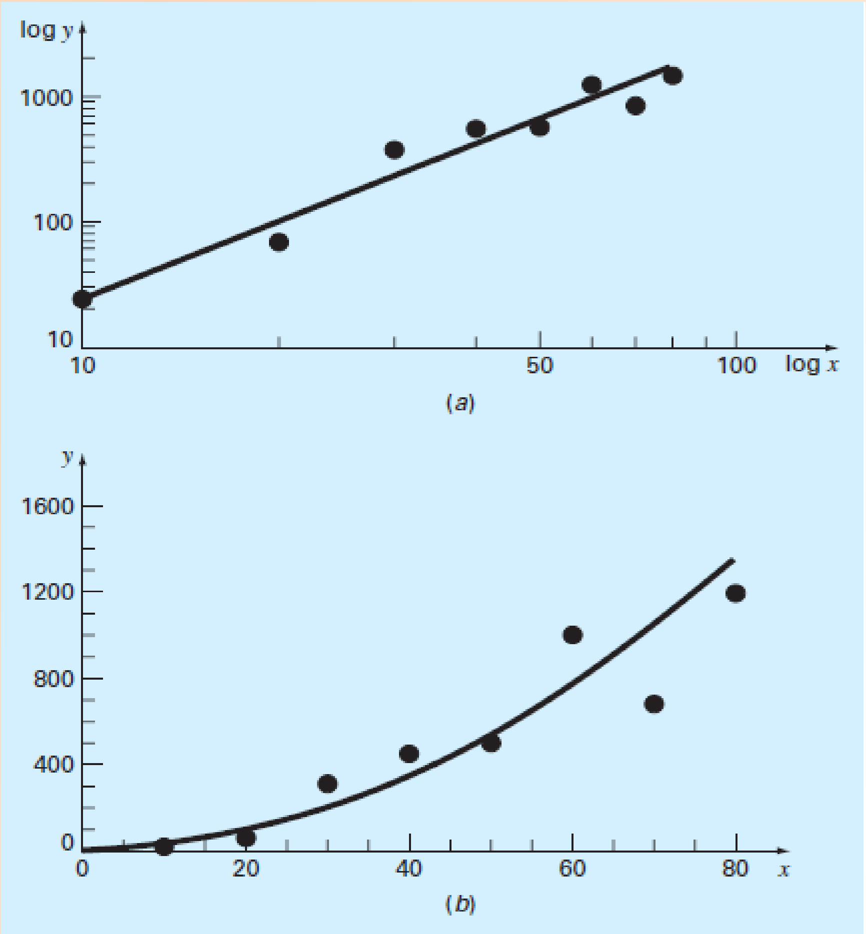


$P_4(x)$=-9.42$x^4 + 0.08x^3 - 0.79x^2 + 1.66x - 0.008$

Interpolation & Extrapolation



Curve Fitting



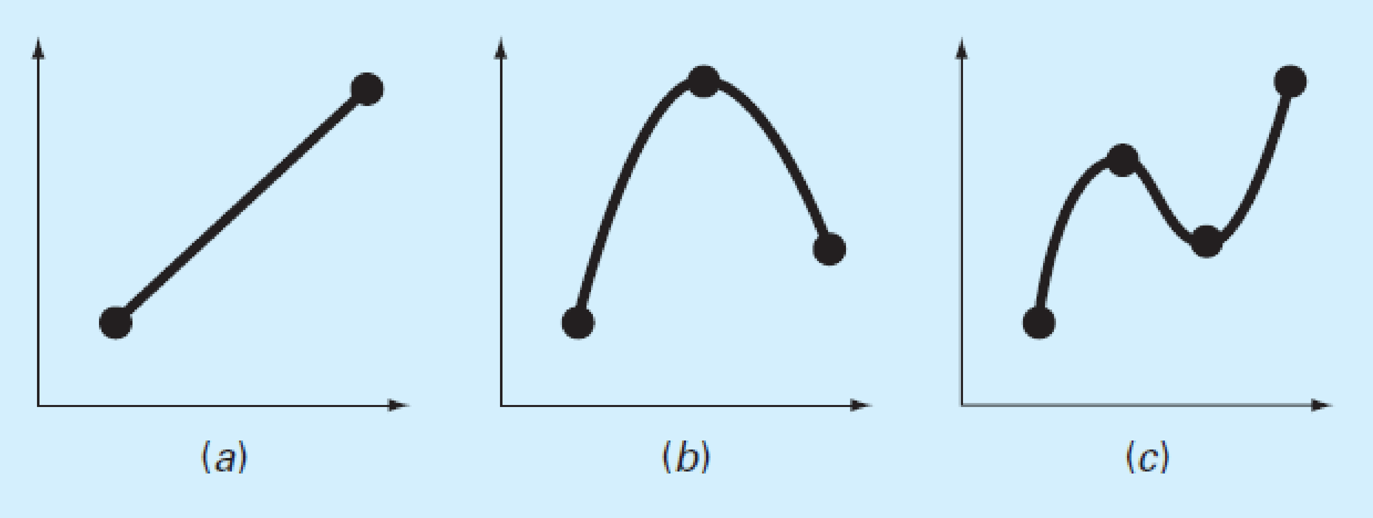
Interpolation

Definition: If we have $n + 1$ distinct points (x_0 to x_n) from $f(x)$, interpolation polynomial is a polynomial at most from n degree that passes through all these points.

$$\boldsymbol{P_n(x_i) = f(x_i)}$$

- Increasing the number of points does not necessarily reduce the error!

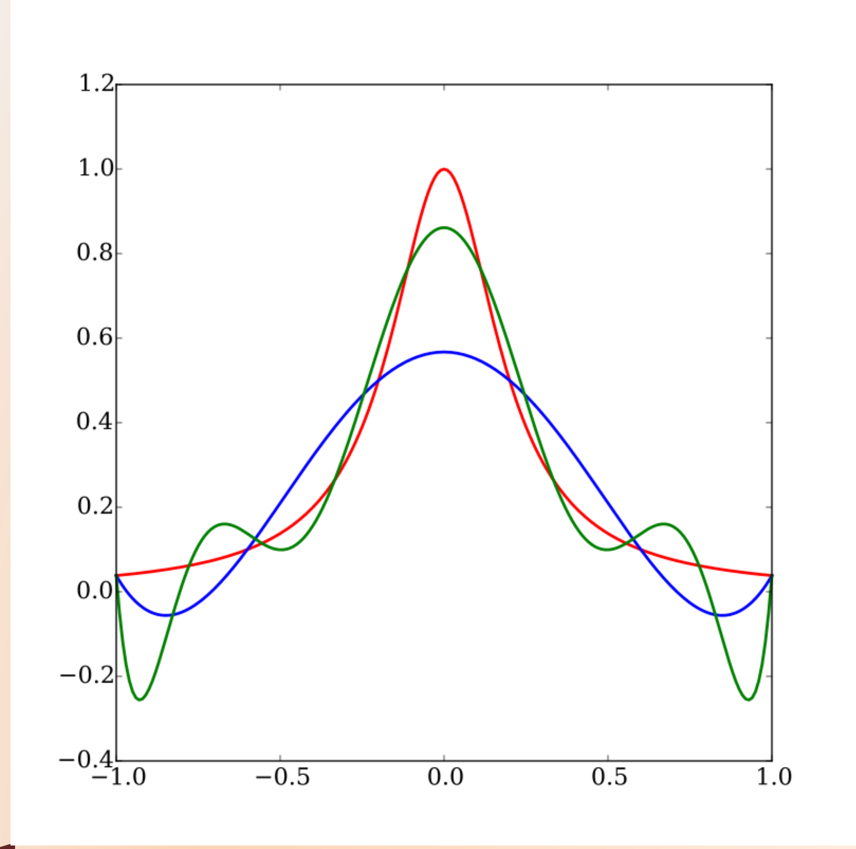
Interpolation



Examples of interpolating polynomials:
(*a*) first-order (linear) connecting two points,
(*b*) second-order (quadratic or parabolic) connecting three points,
(*c*) third-order (cubic) connecting four points.

Runge's phenomenon

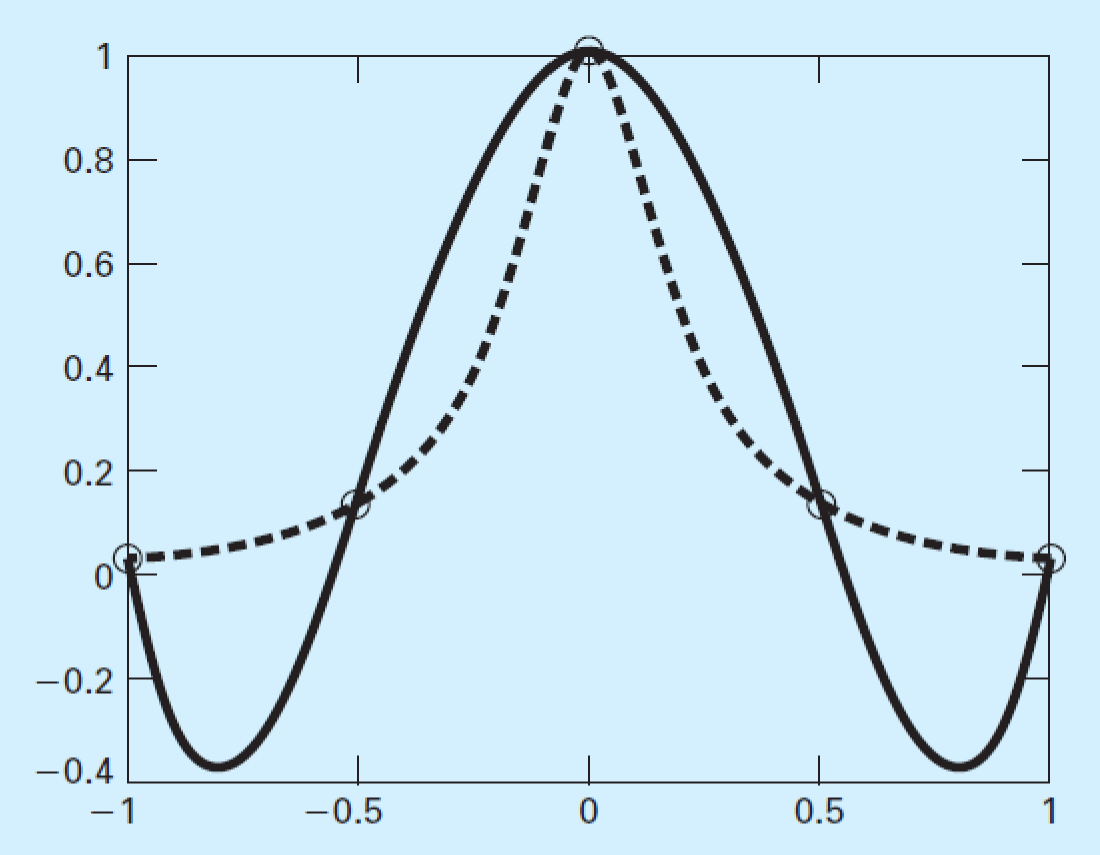
higher degrees does not always improve accuracy!



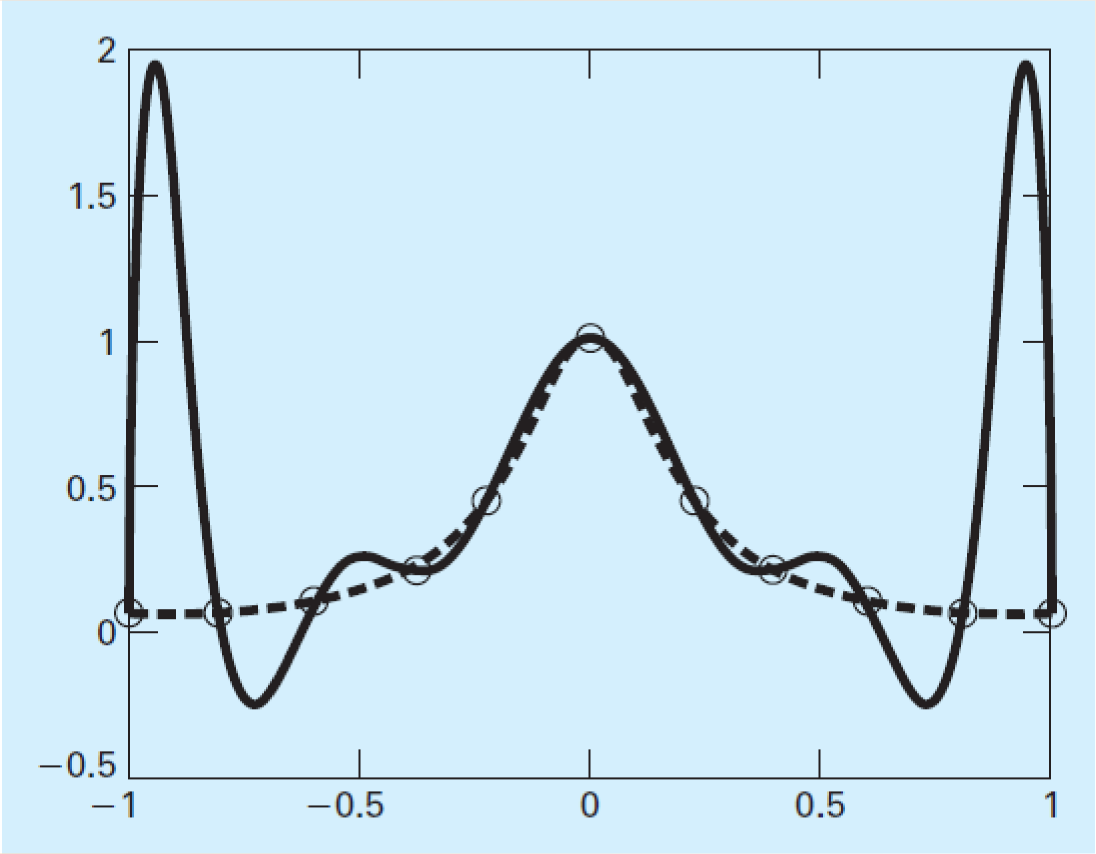
$$f(x) = \frac{1}{1+25x^2}$$

The red curve is the Runge function.
The blue curve is a 5th-order interpolating polynomial (using six equally spaced interpolating points).
The green curve is a 9th-order interpolating polynomial (using ten equally spaced interpolating points).

Runge's phenomenon



Runge's phenomenon



Interpolation

Actually we want to find $P_n(x)$ that is an approximation for $f(x)$ and $n+1$ points are used to calculate it.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

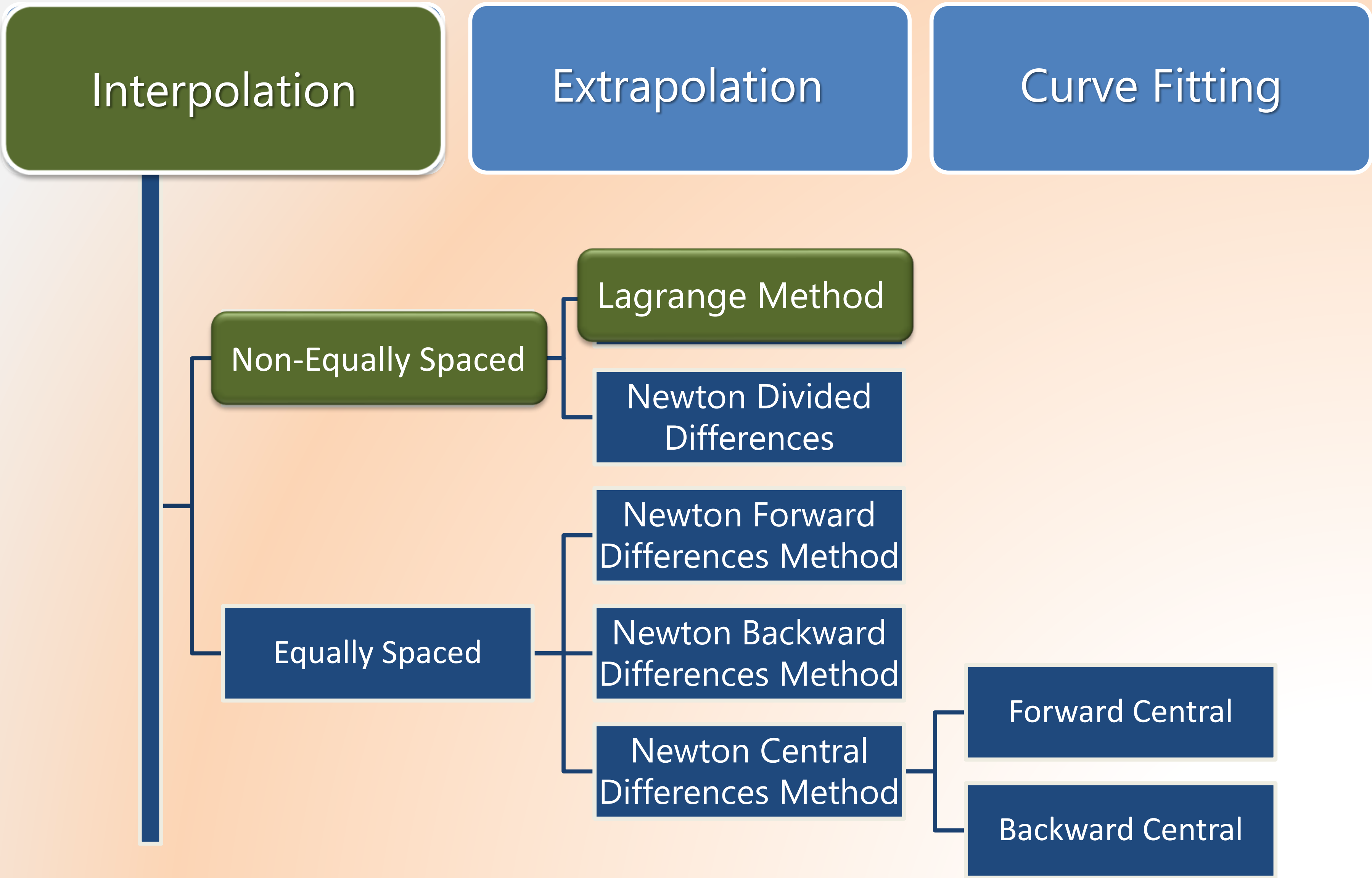
$P_n(x)$ must be passed through all of this $n+1$ points, so we have $n+1$ equations for $n+1$ unknowns.

$$\begin{cases} f(x_0) = a_0 + a_1 x_0 + a_2 {x_0}^2 + \cdots + a_n {x_0}^n \\ f(x_1) = a_0 + a_1 x_1 + a_2 {x_1}^2 + \cdots + a_n {x_1}^n \\ \qquad \cdots \\ f(x_n) = a_0 + a_1 x_n + a_2 {x_n}^2 + \cdots + a_n {x_n}^n \end{cases}$$

These equations have a unique answer for a_0 to a_n.

Interpolation

When we have $\forall i \; x_i - x_{i-1} = x_{i+1} - x_i$, then we use Newton forward, backward or central differences method and in other cases, we use other methods, such as Lagrange method or Newton divided differences method.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Lagrange Method

$$P_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$$

Lagrange Method

$$P_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$$

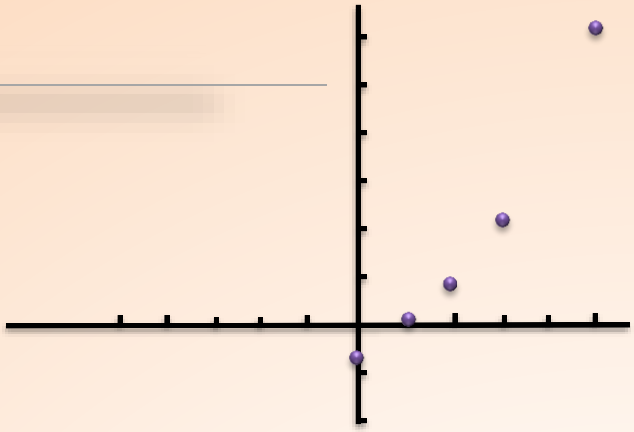
$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) =$$

$$= \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}$$

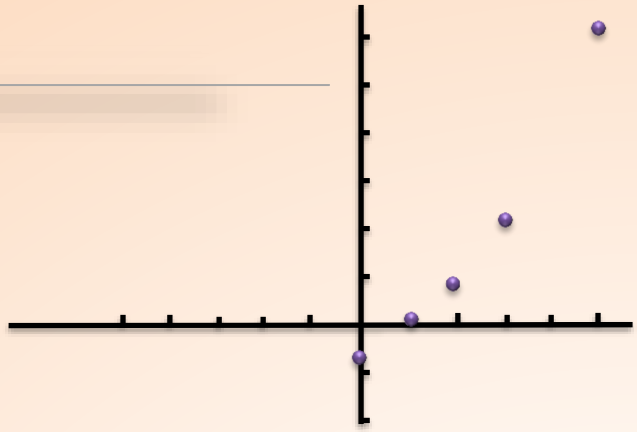
Lagrange Method

Example: find $f(2.5)$.

x_i	0	1	2	3	5
$f(x_i)$	-3	0	5	12	32



Lagrange Method



Example: find $f(2.5)$.

x_i	0	1	2	3	5
$f(x_i)$	-3	0	5	12	32

Solution:

$$L_0(x) = \frac{(x-1)(x-2)(x-3)(x-5)}{(0-1)(0-2)(0-3)(0-5)}$$

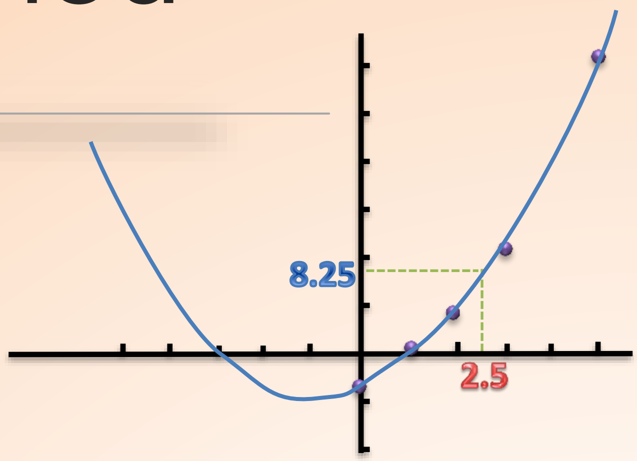
$$L_1(x) = \frac{(x-0)(x-2)(x-3)(x-5)}{(1-0)(1-2)(1-3)(1-5)}$$

$$L_2(x) = \frac{(x-0)(x-1)(x-3)(x-5)}{(2-0)(2-1)(2-3)(2-5)}$$

$$L_3(x) = \frac{(x-0)(x-1)(x-2)(x-5)}{(3-0)(3-1)(3-2)(3-5)}$$

$$L_4(x) = \frac{(x-0)(x-1)(x-2)(x-3)}{(5-0)(5-1)(5-2)(5-3)}$$

Lagrange Method



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$$L_4(x) = \frac{(x-0)(x-1)(x-2)(x-3)}{(5-0)(5-1)(5-2)(5-3)}$$

$$P_4(x) = -3L_0(x) + 0L_1(x) + 5L_2(x) + 12L_3(x) + 32L_4(x)$$

$$P_4(x) = x^2 + 2x - 3 \rightarrow f(2.5) = P(2.5) = 8.25$$

Lagrange Method

- Selecting different x_is results in different interpolated polynomials.

Cons:

1. It's fairly hard and time-consuming to calculate $l_i(x)$s.
2. We can not understand the degree of interpolation polynomial from degree of $l_i(x)$s.
3. It is not a stable method and it is completely dependent on x_is. Plus, if we want to add an x_i, we must redo all calculations from the beginning.

Lagrange Method

Error of method:

$$\varepsilon_n = f(x) - P_n(x) \leq (x - x_0)(x - x_1) \dots (x - x_n) \frac{f^{n+1}(t)}{(n+1)!}$$

$$x_0 \leq t \leq x_n \quad | \quad t = maximize \quad |f^{n+1}(x_i)|$$

An Example

- Compute the value of $\sqrt{115}$ using Lagrange method.
- What's the maximum error in calculating $\sqrt{115}$?

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- What's the maximum error in calculating $\sqrt{115}$?

Solution:

$$f(x) = \sqrt{x}$$

x_i	100	121	144
$\sqrt{x_i}$	10	11	12

$$L_0(x) = \frac{(x-121)(x-144)}{(100-121)(100-144)} \qquad L_1(x) = \frac{(x-100)(x-144)}{(121-100)(121-144)}$$

$$L_2(x) = \frac{(x-100)(x-121)}{(144-100)(144-121)}$$

$$P_2(x) = 10L_0(x) + 11L_1(x) + 12L_2(x)$$

$$\Rightarrow f(115) = P_2(115) = 10.7228$$

An Example

- Compute the value of $\sqrt{115}$ using Lagrange method.
- What's the maximum error in calculating $\sqrt{115}$?

Solution:

$$f(x) = \sqrt{x}$$

x_i	100	121	144
$\sqrt{x_i}$	10	11	12

$$\varepsilon_n \leq (x - x_0)(x - x_1)(x - x_2)\left(\frac{F^{(n+1)}(t)}{(n+1)!}\right)$$

$$f(x) = \sqrt{x} \to f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \to f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \to f^{(3)}(x) = \frac{3}{8}x^{-\frac{5}{2}}$$

$$\varepsilon_2 \leq (115 - 100)(115 - 121)(115 - 144)\left(\frac{\frac{3}{8}(100)^{-\frac{5}{2}}}{(2+1)!}\right) \to \varepsilon_2 \leq 0.00163$$

An Example

- Compute the value of $\sqrt{115}$ using Lagrange method.
- What's the maximum error in calculating $\sqrt{115}$?

Solution:

$$f(x) = \sqrt{x}$$

x_i	100	121	144
$\sqrt{x_i}$	10	11	12

$$\varepsilon_2 \leq (115-100)(115-121)(115-144)\left(\frac{\frac{3}{8}(100)^{-\frac{5}{2}}}{(2+1)!}\right) \rightarrow \varepsilon_2 \leq 0.00163$$ **(upper bound)**

$\sqrt{115}$=10.7238,
$f(115) = P_2(115) = 10.7228$,
10.7238 - 10.7228 = 0.001 **(Real error)**

Interpolation | Extrapolation | Curve Fitting

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Newton Divided Differences Method

Newton's form of the interpolation polynomial is:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

Newton Divided Differences Method

Divided differences are :

$$f[x_k] = f(x_k)$$

$$f[x_k, x_{k+1}] = \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k}$$

$$f[x_k, x_{k+1}, x_{k+2}] = \frac{f[x_{k+1}, x_{k+2}] - f[x_k , x_{k+1}]}{x_{k+2} - x_k}$$

$$f\left[x_k, x_{k+1}, \dots , x_{k+j}\right] = \frac{f\left[x_{k+1}, \dots , x_{k+j}\right] - f\left[x_k , \dots , x_{k+j-1}\right]}{x_{k+j} - x_k}$$

Newton Divided Differences Method

x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	2^{nd} order	3^{rd} order	4^{th} order
x_0	$f[x_0]$				
x_1	$f[x_1]$				
x_2	$f[x_2]$				
x_3	$f[x_3]$				
x_4	$f[x_4]$				
.					
. .					

Newton Divided Differences Method

x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	2^{nd} order	3^{rd} order	4^{th} order
x_0	$f[x_0]$				
x_1	$f[x_1]$	$f[x_0, x_1]$			
x_2	$f[x_2]$	$f[x_1, x_2]$			
x_3	$f[x_3]$	$f[x_2, x_3]$			
x_4	$f[x_4]$	$f[x_3, x_4]$			
.					
. .					

Newton Divided Differences Method

x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	2^{nd} order	3^{rd} order	4^{th} order
x_0	$f[x_0]$				
x_1	$f[x_1]$	$f[x_0, x_1]$			
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$		
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$		
x_4	$f[x_4]$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$		
.					
. .					

Newton Divided Differences Method

x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	2^{nd} order	3^{rd} order	4^{th} order
x_0	$f[x_0]$				
x_1	$f[x_1]$	$f[x_0, x_1]$			
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$		
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$	
x_4	$f[x_4]$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$f[x_0, x_1, x_2, x_3, x_4]$
.					
. .					

Newton Divided Differences Method

x_k	$f[x_k]$	$f[x_k, x_{k+1}]$	2^{nd} order	3^{rd} order	4^{th} order
x_0	$f[x_0]$	$\rightarrow a_0$			
x_1	$f[x_1]$	$f[x_0, x_1]$	$\rightarrow a_1$		
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	$\rightarrow a_2$	
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$	$\rightarrow a_3$
x_4	$f[x_4]$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$f[x_0, x_1, x_2, x_3, x_4]$
.					a_4
. .					

Newton Divided Differences Method

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

This method has solved problems of Lagrange method:

Error of method:

$$\varepsilon_n = f(x) - P_n(x) \le (x - x_0)(x - x_1) \ldots (x - x_n)\frac{f^{n+1}(t)}{(n+1)!}$$

$$x_0 \le t \le x_n \quad | \quad t = maximize \quad |f^{n+1}(x_i)|$$

Newton Divided Differences Method

- The degree of interpolation polynomial is clarified when values in a column of the table become equal (so values in next column will be 0).
- With adding a x_i, only one row will be added to the table.
- Calculations are much simpler.

Newton Divided Differences Method

Example:

x_i	1	2	3	4	5
$f(x_i)$	-3	0	15	48	105

Newton Divided Differences Method

Example:

x_i	1	2	3	4	5
$f(x_i)$	-3	0	15	48	105

Solution:

x_i	$f(x_i)$	1st order	2nd order	3rd order	4th order
1	-3				
2	0	$\frac{-3-0}{1-2}=3$			
3	15	$\frac{0-15}{2-3}=15$	6		
4	48	$\frac{15-48}{3-4}=33$	9	1	
5	105	$\frac{48-105}{4-5}=57$	12	1	0

Newton Divided Differences Method

$P_3(x) = -3 + 3\,(x-1) + 6\,(x-1)(x-2) + 1\,(x-1)(x-2)(x-3)$

$= x^3 - 4x$

Solution:

x_i	$f(x_i)$	1st order	2nd order	3rd order	4th order
1	-3 → a_0				
2	0	$\frac{-3-0}{1-2} = 3$ → a_1			
3	15	$\frac{0-15}{2-3} = 15$	6 → a_2		
4	48	$\frac{15-48}{3-4} = 33$	9	1 → a_3	
5	105	$\frac{48-105}{4-5} = 57$	12	1	0

Newton Divided Differences Method

Example:

x_i	1	2	3	4	5	6
$f(x_i)$	-3	0	15	48	105	192

Solution:

x_i	$f(x_i)$	1st order	2nd order	3rd order	4th order
1	-3 $\rightarrow a_0$				
2	0	$\frac{-3-0}{1-2} = 3 \rightarrow a_1$			
3	15	$\frac{0-15}{2-3} = 15$	6 $\rightarrow a_2$		
4	48	$\frac{15-48}{3-4} = 33$	9	1 $\rightarrow a_3$	
5	105	$\frac{48-105}{4-5} = 57$	12	1	0
6	192	$\frac{105-192}{5-6} = 87$	15	1	0

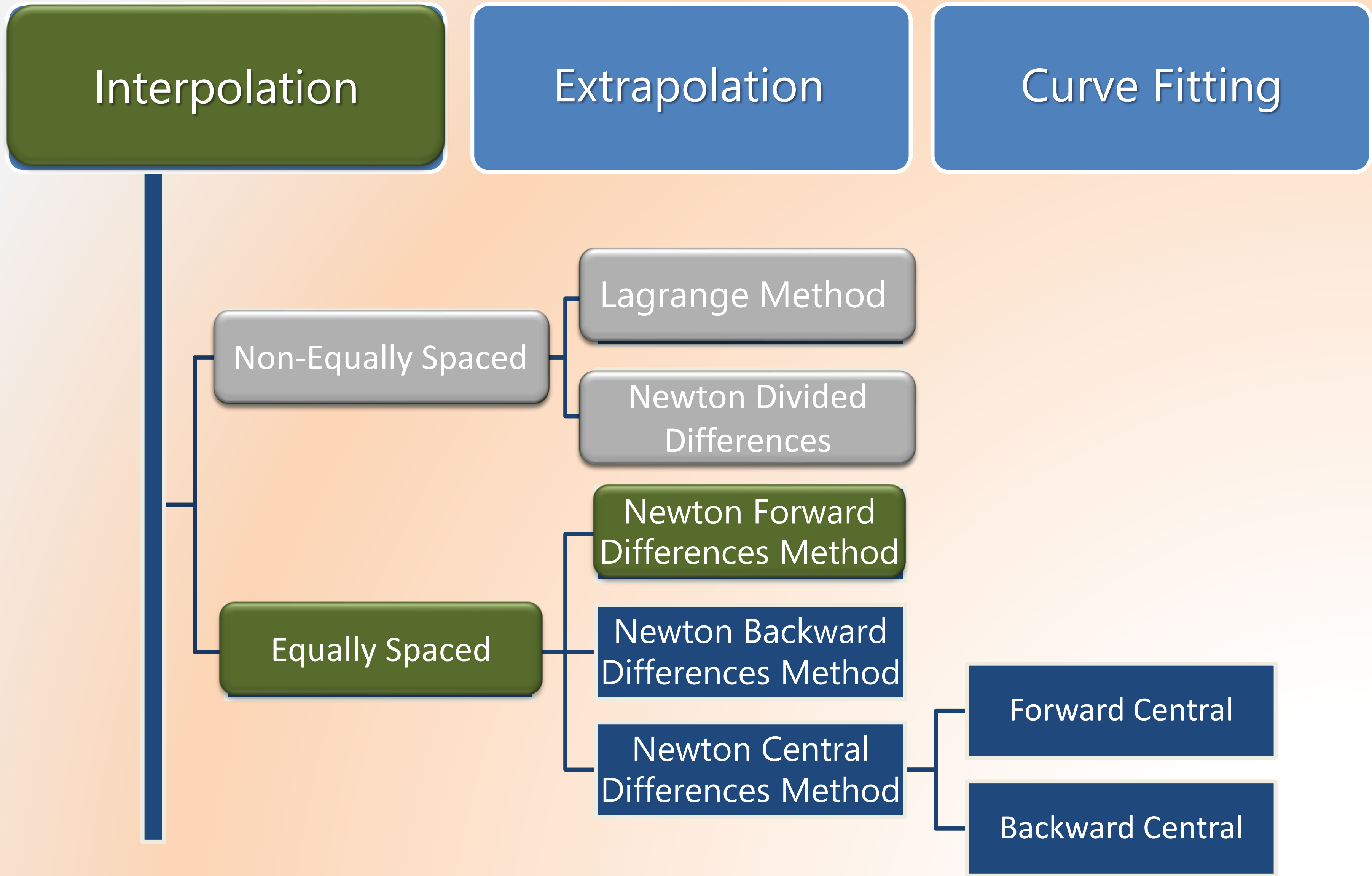
Newton Divided Differences Method

$$P_3(x) = -3 + 3\,(x-1) + 6\,(x-1)(x-2) + 1\,(x-1)(x-2)(x-3)$$

$$= x^3 - 4x$$

Solution:

x_i	$f(x_i)$	1st order	2nd order	3rd order	4th order
1	-3 → a_0				
2	0	$\frac{-3-0}{1-2} = 3$ → a_1			
3	15	$\frac{0-15}{2-3} = 15$	6 → a_2		
4	48	$\frac{15-48}{3-4} = 33$	9	1 → a_3	
5	105	$\frac{48-105}{4-5} = 57$	12	1	0
6	192	$\frac{105-192}{5-6} = 87$	15	1	0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Newton Forward Differences Method

In this method, we have:

$$P_n(x) = \sum_{s=0}^{n} \binom{r}{s} \Delta^s f_0$$

$$= f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \cdots + \frac{r(r-1)\ldots(r-(n-1))}{n!}\Delta^n f_0$$

$$x = x_0 + rh \rightarrow r = \frac{x - x_0}{h}$$

And :

$$\Delta f_i = f_{i+1} - f_i$$

$$\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$$

$$\vdots$$

$$\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i$$

Newton Forward Differences Method

$\boldsymbol{x_i}$	$\boldsymbol{f_i}$	$\Delta \boldsymbol{f_i}$	$\Delta^2 \boldsymbol{f_i}$	$\Delta^3 \boldsymbol{f_i}$
x_0	f_0			
x_1	f_1	$\Delta f_0 = f_1 - f_0$		
x_2	f_2	$\Delta f_1 = f_2 - f_1$	$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$	
x_3	f_3	$\Delta f_2 = f_3 - f_2$	$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$	$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$
…				

Newton Forward Differences Method

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
x_0	$\boxed{f_0}$			
x_1	f_1	$\boxed{\Delta f_0 = f_1 - f_0}$		
x_2	f_2	$\Delta f_1 = f_2 - f_1$	$\boxed{\Delta^2 f_0 = \Delta f_1 - \Delta f_0}$	
x_3	f_3	$\Delta f_2 = f_3 - f_2$	$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$	$\boxed{\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0}$
…				

Newton Forward Differences Method

$$f[x_k, \dots, x_{k+j}] = \frac{\Delta^j f_k}{j!\, h^j}$$

$x = x_0 + sh \implies (x - x_0) = rh$

$(x - x_1) = (x - x_0 - h) = rh - h = (r-1)h$

$(x - x_2) = (x - x_1 - h) = (r-1)h - h = (r-2)h$

$(x - x_{n-1}) = (r - n + 1)h$

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1})$$

$$P_n(x) = f_0 + \frac{\Delta f_0}{h}(rh) + \frac{\Delta^2 f_0}{2!\, h^2}(rh)(r-1)h + \cdots + \frac{\Delta^n f_0}{n!\, h^n}(rh)(r-1)h \dots (r-n+1)h$$

Newton Forward Differences Method

Error of method:

$$\varepsilon_n = f(x) - P_n(x) \leq (x - x_0)(x - x_1) \dots (x - x_n)\frac{f^{n+1}(t)}{(n+1)!}$$

$$x_0 \leq t \leq x_n \quad | \quad t = maximize \quad |f^{n+1}(x_i)|$$

Error of method:

$$\varepsilon_n = f(x) - P_n(x) \leq \frac{h^{n+1}}{(n+1)!} r(r-1) \dots (r-n) f^{n+1}(t)$$

$$x_0 \leq t \leq x_n \quad | \quad t = \ maximize \quad |f^{n+1}(x_i)|$$

An Example

Calculate the forward differences of $f(0.25)\ and\ f(0.35)$ according to following data.

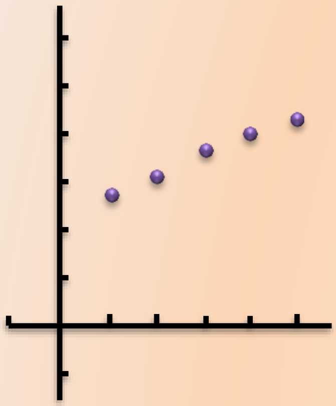
x_i	0.1	0.2	0.3	0.4	0.5
$f(x_i)$	1.4	1.56	1.76	2	2.28

An Example

Calculate the forward differences of $f(0.25)$ and $f(0.35)$ according to following data.

x_i	0.1	0.2	0.3	0.4	0.5
$f(x_i)$	1.4	1.56	1.76	2	2.28

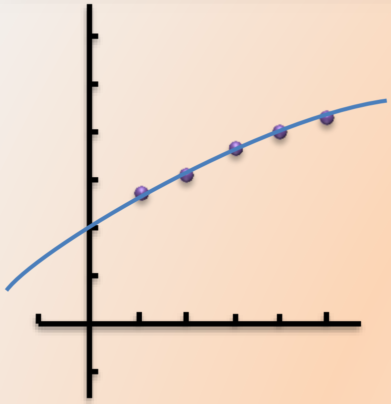
Solution:



x_i	$f(x_i)$	1st order	2nd order	3rd order
0.1	1.4			
0.2	1.56	0.16		
0.3	1.76	0.2	0.04	
0.4	2	0.24	0.04	0
0.5	2.28	0.28	0.04	0

$$P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \cdots + \frac{r(r-1)\ldots(r-(n-1))}{n!}\Delta^n f_0$$

An Example



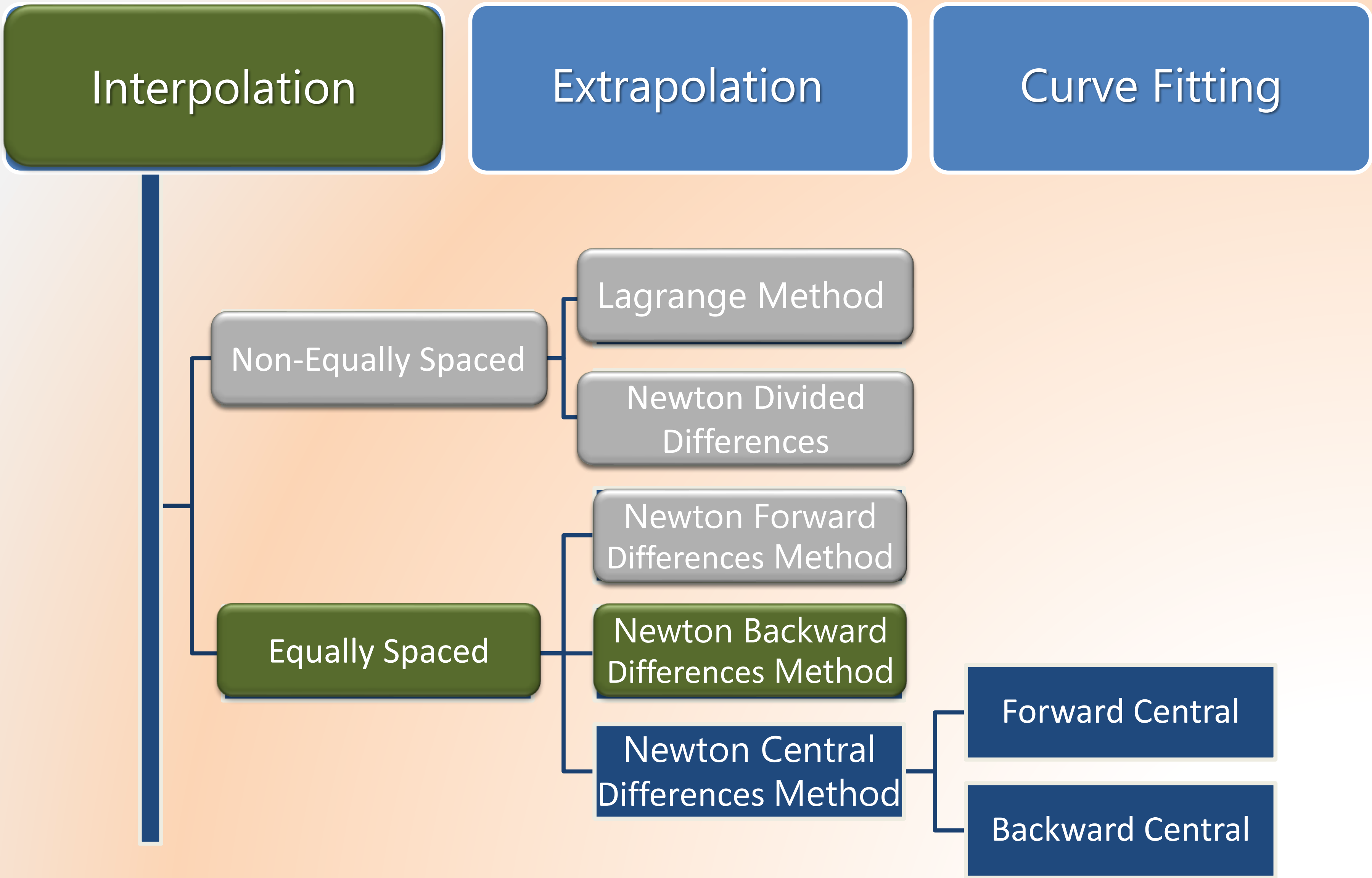
x_i	$f(x_i)$	1st order	2nd order	3rd order
0.1	1.4			
0.2	1.56	0.16		
0.3	1.76	0.2	0.04	
0.4	2	0.24	0.04	0
0.5	2.28	0.28	0.04	0

Solution: forward:

$$h = 0.1 \rightarrow x_0 = 0.1 \rightarrow r = \frac{x-0.1}{0.1}$$

$$P_2(x) = 1.4 + \left(\frac{x-0.1}{0.1}\right)(0.16) + \left(\frac{x-0.1}{0.1}\right)\left(\frac{x-0.1}{0.1} - 1\right)\left(\frac{0.04}{2!}\right)$$

$$= 2x^2 + x + 1.28 \rightarrow \begin{cases} f(0.25) = 1.655 \\ f(0.35) = 1.875 \end{cases}$$


Interpolation
Extrapolation
Curve Fitting
Lagrange Method
Non-Equally Spaced
Newton Divided Differences
Newton Forward Differences Method
Equally Spaced
Newton Backward Differences Method
Forward Central
Newton Central Differences Method
Backward Central

Newton Backward Differences Method

In this method, we have:

$$P_n(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \cdots + \frac{r(r+1)\dots(r+n-1)}{n!}\nabla^n f_n$$

$$x = x_n + rh \rightarrow r = \frac{x - x_n}{h}$$

And:

$$\nabla f_i = f_i - f_{i-1}$$

$$\nabla^2 f_i = \nabla f_i - \nabla f_{i-1}$$

$$\vdots$$

$$\nabla^k f_i = \nabla^{k-1} f_i - \nabla^{k-1} f_{i-1}$$

Newton Backward Differences Method

x_i	f_i	∇f_i	$\nabla^2 f_i$	$\nabla^3 f_i$
$\mathbf{x_0}$	f_0	$\nabla f_1 = f_1 - f_0$	$\nabla^2 f_2 = \nabla f_2 - \nabla f_1$	$\nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2$
$\mathbf{x_1}$	f_1	$\nabla f_2 = f_2 - f_1$	$\nabla^2 f_3 = \nabla f_3 - \nabla f_2$	
$\mathbf{x_2}$	f_2	$\nabla f_3 = f_3 - f_2$		
$\mathbf{x_3}$	f_3			
…				

Newton Backward Differences Method

x_i	f_i	∇f_i	$\nabla^2 f_i$	$\nabla^3 f_i$
x_0	f_0	$\nabla f_1 = f_1 - f_0$	$\nabla^2 f_2 = \nabla f_2 - \nabla f_1$	$\boxed{\nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2}$
x_1	f_1	$\nabla f_2 = f_2 - f_1$	$\boxed{\nabla^2 f_3 = \nabla f_3 - \nabla f_2}$	
x_2	f_2	$\boxed{\nabla f_3 = f_3 - f_2}$		
x_3	$\boxed{f_3}$			
…				

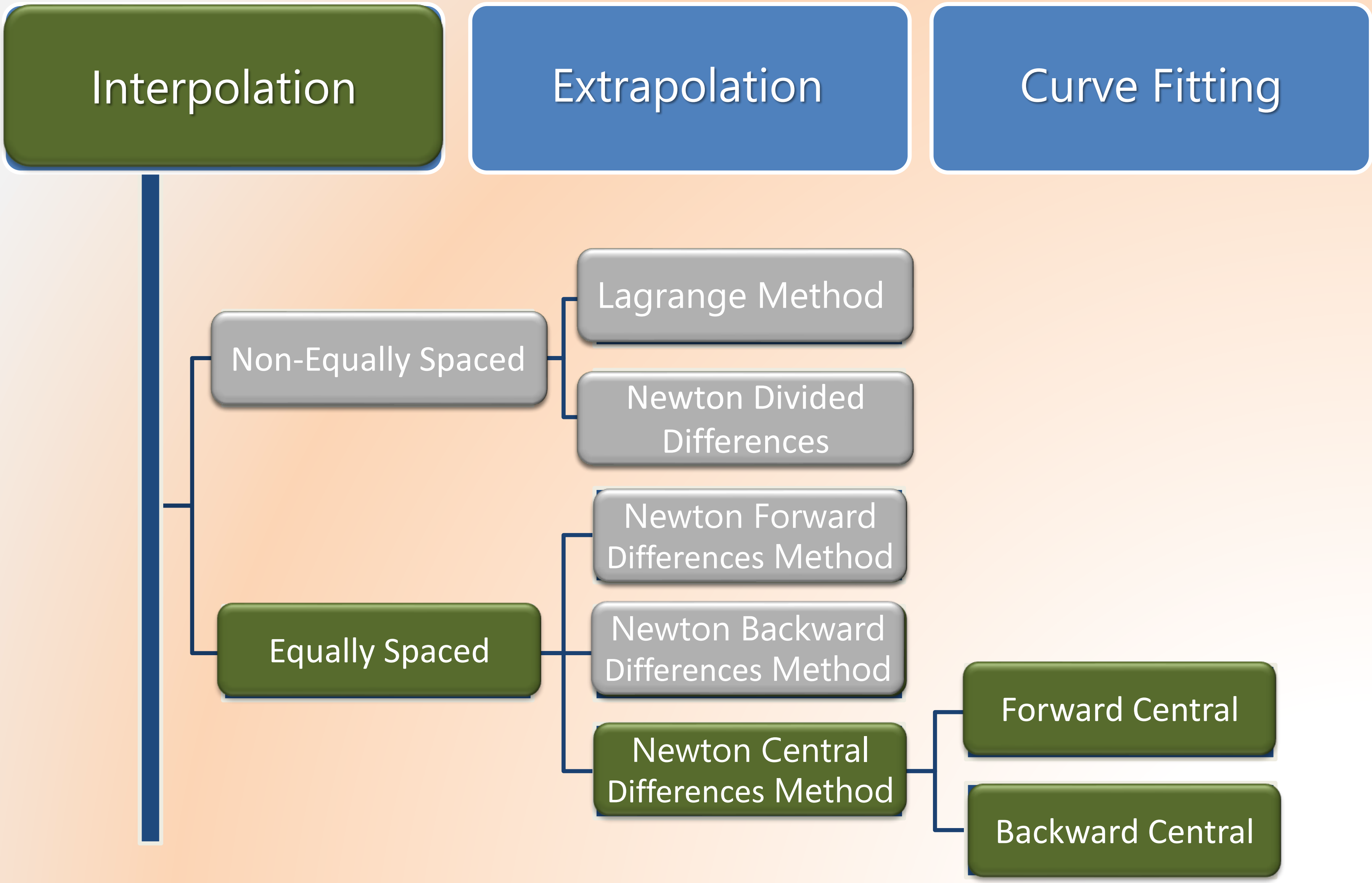
Newton Backward Differences Method

x_i	f_i	∇f_i	$\nabla^2 f_i$	$\nabla^3 f_i$
x_0	f_0	$\nabla f_1 = f_1 - f_0$	$\nabla^2 f_2 = \nabla f_2 - \nabla f_1$	$\boxed{\nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2}$
x_1	f_1	$\nabla f_2 = f_2 - f_1$	$\boxed{\nabla^2 f_3 = \nabla f_3 - \nabla f_2}$	
x_2	f_2	$\boxed{\nabla f_3 = f_3 - f_2}$		
x_3	$\boxed{f_3}$			
...				

Error of method:

$$\varepsilon_n = f(x) - P_n(x) \le \frac{h^{n+1}}{(n+1)!} r(r+1)(r+2)\ldots(r+n) f^{n+1}(t)$$

$$x_0 \le t \le x_n \; and \; \forall i \; |f^{n+1}(x_i)| \le f^{n+1}(t)$$


Interpolation
Extrapolation
Curve Fitting
Non-Equally Spaced
Lagrange Method
Newton Divided Differences
Equally Spaced
Newton Forward Differences Method
Newton Backward Differences Method
Newton Central Differences Method
Forward Central
Backward Central

Newton Central Differences Method

Newton Forward Central   Newton Backward Central

$$P_n(x) = f_{\frac{n}{2}} + r\delta f_{\frac{n+1}{2}} + \frac{r(r-1)}{2!}\delta^2 f_{\frac{n}{2}} + \frac{r(r-1)(r+1)}{3!}\delta^3 f_{\frac{n+1}{2}}$$

$$+ \cdots$$

$$+ \frac{r(r-1)(r+1)(r-2)(r+2)\ldots\left(r-\left(\frac{n}{2}-1\right)\right)\left(r+\left(\frac{n}{2}-1\right)\right)}{n!}\delta^n f_{\frac{n}{2}}$$

$$x = x_{\frac{n}{2}} + rh \rightarrow r = \frac{x - x_{\frac{n}{2}}}{h}$$

Newton Central Differences Method

Newton Forward Central Newton Backward Central

$$P_n(x) = f_{\frac{n}{2}} + r\delta f_{\frac{n-1}{2}} + \frac{r(r+1)}{2!}\delta^2 f_{\frac{n}{2}} + \frac{r(r+1)(r-1)}{3!}\delta^3 f_{\frac{n-1}{2}} + \cdots + \frac{r(r+1)(r-1)(r+2)(r-2)\ldots\left(r+\left(\frac{n}{2}-1\right)\right)\left(r-\left(\frac{n}{2}-1\right)\right)}{n!}\delta^n f_{\frac{n}{2}}$$

$$x = x_{\frac{n}{2}} + rh \rightarrow r = \frac{x - x_{\frac{n}{2}}}{h}$$

Newton Central Differences Method

In this method, we have:

$$\delta f_i = f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}$$

$$\delta^2 f_i = \delta f_{i+\frac{1}{2}} - \delta f_{i-\frac{1}{2}}$$

$$\vdots$$

$$\delta^k f_i = \delta^{k-1} f_{i+\frac{1}{2}} - \delta^{k-1} f_{i-\frac{1}{2}}$$

Newton Central Differences Method

$\boldsymbol{x_i}$	$\boldsymbol{f_i}$	$\boldsymbol{\delta f_{i+\frac{1}{2}}}$	$\boldsymbol{\delta^2 f_{i+1}}$	$\boldsymbol{\delta^3 f_{i+\frac{3}{2}}}$	$\boldsymbol{\delta^4 f_{i+2}}$
x_0	f_0				
		$\delta f_{\frac{1}{2}} = f_1 - f_0$			
x_1	f_1		$\delta^2 f_1 = \delta f_{\frac{3}{2}} - \delta f_{\frac{1}{2}}$		
		$\delta f_{\frac{3}{2}} = f_2 - f_1$		$\delta^3 f_{\frac{3}{2}} = \delta^2 f_2 - \delta^2 f_1$	
x_2	f_2		$\delta^2 f_2 = \delta f_{\frac{5}{2}} - \delta f_{\frac{3}{2}}$		$\delta^4 f_2 = \delta^3 f_{\frac{5}{2}} - \delta^3 f_{\frac{3}{2}}$
		$\delta f_{\frac{5}{2}} = f_3 - f_2$		$\delta^3 f_{\frac{5}{2}} = \delta^2 f_3 - \delta^2 f_2$	
x_3	f_3		$\delta^2 f_3 = \delta f_{\frac{7}{2}} - \delta f_{\frac{5}{2}}$		
		$\delta f_{\frac{7}{2}} = f_4 - f_3$			
x_4	f_4				

Newton Central Differences Method

x_i	f_i	$\delta f_{i+\frac{1}{2}}$	$\delta^2 f_{i+1}$	$\delta^3 f_{i+\frac{3}{2}}$	$\delta^4 f_{i+2}$
x_0	f_0				
		$\delta f_{\frac{1}{2}} = f_1 - f_0$			
x_1	f_1		$\delta^2 f_1 = \delta f_{\frac{3}{2}} - \delta f_{\frac{1}{2}}$		
		$\delta f_{\frac{3}{2}} = f_2 - f_1$		$\delta^3 f_{\frac{3}{2}} = \delta^2 f_2 - \delta^2 f_1$	
x_2	$\boxed{f_2}$		$\boxed{\delta^2 f_2 = \delta f_{\frac{5}{2}} - \delta f_{\frac{3}{2}}}$		$\boxed{\delta^4 f_2 = \delta^3 f_{\frac{5}{2}} - \delta^3 f_{\frac{3}{2}}}$
		$\boxed{\delta f_{\frac{5}{2}} = f_3 - f_2}$		$\boxed{\delta^3 f_{\frac{5}{2}} = \delta^2 f_3 - \delta^2 f_2}$	
x_3	f_3		$\delta^2 f_3 = \delta f_{\frac{7}{2}} - \delta f_{\frac{5}{2}}$		
		$\delta f_{\frac{7}{2}} = f_4 - f_3$			
x_4	f_4				

Forward Central

Newton Central Differences Method

Backward Central

x_i	f_i	$\delta f_{i+\frac{1}{2}}$	$\delta^2 f_{i+1}$	$\delta^3 f_{i+\frac{3}{2}}$	$\delta^4 f_{i+2}$
x_0	f_0				
		$\delta f_{\frac{1}{2}} = f_1 - f_0$			
x_1	f_1		$\delta^2 f_1 = \delta f_{\frac{3}{2}} - \delta f_{\frac{1}{2}}$		
		$\boxed{\delta f_{\frac{3}{2}} = f_2 - f_1}$		$\boxed{\delta^3 f_{\frac{3}{2}} = \delta^2 f_2 - \delta^2 f_1}$	
x_2	$\boxed{f_2}$		$\boxed{\delta^2 f_2 = \delta f_{\frac{5}{2}} - \delta f_{\frac{3}{2}}}$		$\boxed{\delta^4 f_2 = \delta^3 f_{\frac{5}{2}} - \delta^3 f_{\frac{3}{2}}}$
		$\delta f_{\frac{5}{2}} = f_3 - f_2$		$\delta^3 f_{\frac{5}{2}} = \delta^2 f_3 - \delta^2 f_2$	
x_3	f_3		$\delta^2 f_3 = \delta f_{\frac{7}{2}} - \delta f_{\frac{5}{2}}$		
		$\delta f_{\frac{7}{2}} = f_4 - f_3$			
x_4	f_4				

Newton Central Differences Method

x_i	f_i	$\delta f_{i+\frac{1}{2}}$	$\delta^2 f_{i+1}$	$\delta^3 f_{i+\frac{3}{2}}$	$\delta^4 f_{i+2}$
x_0	f_0				
		$\delta f_{\frac{1}{2}} = f_1 - f_0$			
x_1	f_1		$\delta^2 f_1 = \delta f_{\frac{3}{2}} - \delta f_{\frac{1}{2}}$		
		$\delta f_{\frac{3}{2}} = f_2 - f_1$		$\delta^3 f_{\frac{3}{2}} = \delta^2 f_2 - \delta^2 f_1$	
x_2	f_2		$\delta^2 f_2 = \delta f_{\frac{5}{2}} - \delta f_{\frac{3}{2}}$		$\delta^4 f_2 = \delta^3 f_{\frac{5}{2}} - \delta^3 f_{\frac{3}{2}}$
		$\delta f_{\frac{5}{2}} = f_3 - f_2$		$\delta^3 f_{\frac{5}{2}} = \delta^2 f_3 - \delta^2 f_2$	
x_3	f_3		$\delta^2 f_3 = \delta f_{\frac{7}{2}} - \delta f_{\frac{5}{2}}$		
		$\delta f_{\frac{7}{2}} = f_4 - f_3$			
x_4	f_4				

Backward Central

Forward Central

Newton Forward, Central and Backward Differences Methods

- If interpolated point is close to x_0, we use forward differences.
- If it is close to x_n, we use backward differences.
- If it is in the middle of points, we use central differences method.

An Example

Calculate the forward, backward and central differences of $f(0.25)$ and $f(0.35)$ according to following data.

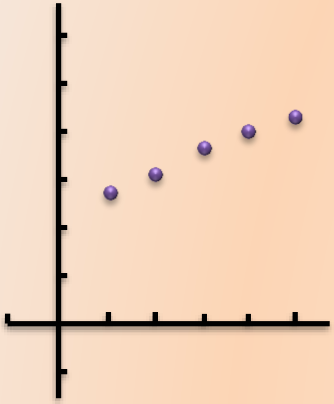
x_i	0.1	0.2	0.3	0.4	0.5
$f(x_i)$	1.4	1.56	1.76	2	2.28

An Example

Calculate the forward, backward and central differences of $f(0.25)$ *and* $f(0.35)$ according to following data.

x_i	0.1	0.2	0.3	0.4	0.5
$f(x_i)$	1.4	1.56	1.76	2	2.28

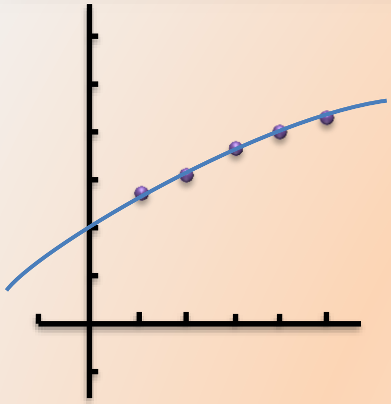
Solution:



x_i	$f(x_i)$	1st order	2nd order	3rd order
0.1	1.4			
0.2	1.56	0.16		
0.3	1.76	0.2	0.04	
0.4	2	0.24	0.04	0
0.5	2.28	0.28	0.04	0

$$P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \cdots + \frac{r(r-1)\dots(r-(n-1))}{n!}\Delta^n f_0$$

An Example



x_i	$f(x_i)$	1st order	2nd order	3rd order
0.1	1.4			
0.2	1.56	0.16		
0.3	1.76	0.2	0.04	
0.4	2	0.24	0.04	0
0.5	2.28	0.28	0.04	0

Solution: forward:

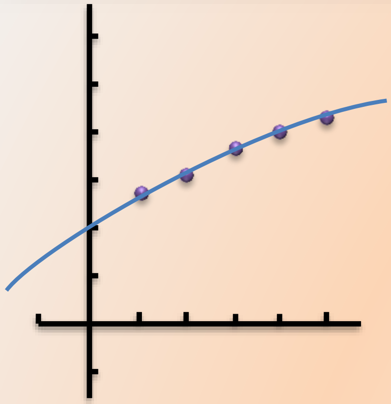
$$h = 0.1 \rightarrow x_0 = 0.1 \rightarrow r = \frac{x-0.1}{0.1}$$

$$P_2(x) = 1.4 + \left(\frac{x-0.1}{0.1}\right)(0.16) + \left(\frac{x-0.1}{0.1}\right)\left(\frac{x-0.1}{0.1} - 1\right)\left(\frac{0.04}{2!}\right)$$

$$= 2x^2 + x + 1.28 \rightarrow \begin{cases} f(0.25) = 1.655 \\ f(0.35) = 1.875 \end{cases}$$

$$P_n(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \cdots + \frac{r(r+1)\dots(r+n-1)}{n!}\nabla^n f_n$$

An Example



x_i	$f(x_i)$	1st order	2nd order	3rd order
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0.2	1.56	0.16		
0.3	1.76	0.2	0.04	
0.4	2	0.24	0.04	0
0.5	2.28	0.28	0.04	0

Solution: backward:

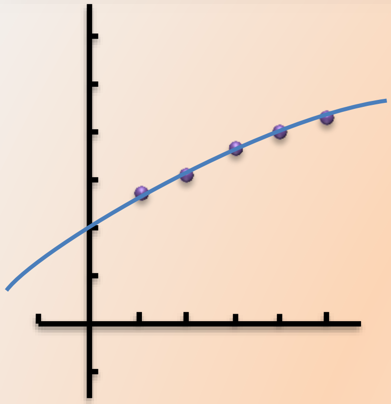
$$h = 0.1 \rightarrow x_n = 0.5 \rightarrow r = \frac{x - 0.5}{0.1}$$

$$P_2(x) = 2.28 + \left(\frac{x-0.5}{0.1}\right)(0.28) + \left(\frac{x-0.5}{0.1}\right)\left(\frac{x-0.5}{0.1} + 1\right)\left(\frac{0.04}{2!}\right)$$

$$= 2x^2 + x + 1.28 \rightarrow \begin{cases} f(0.25) = 1.655 \\ f(0.35) = 1.875 \end{cases}$$

$$P_n(x) = f_{\frac{n}{2}} + r\delta f_{\frac{n+1}{2}} + \frac{r(r-1)}{2!}\delta^2 f_{\frac{n}{2}} + \frac{r(r-1)(r+1)}{3!}\delta^3 f_{\frac{n+1}{2}} + \cdots + \frac{r(r-1)(r+1)(r-2)(r+2)\ldots\left(r-\left(\frac{n}{2}-1\right)\right)\left(r+\left(\frac{n}{2}-1\right)\right)}{n!}\delta^n f_{\frac{n}{2}}$$

An Example



x_i	$f(x_i)$	1st order	2nd order	3rd order
0.1	1.4			
0.2	1.56	0.16		
0.3	1.76	0.2	0.04	
0.4	2	0.24	0.04	0
0.5	2.28	0.28	0.04	0

Solution: forward central:

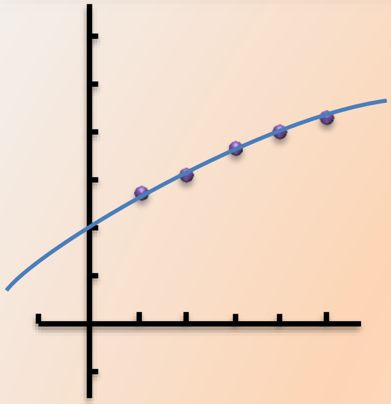
$$h = 0.1 \rightarrow x_{\frac{n}{2}} = 0.3 \rightarrow r = \frac{x-0.3}{0.1}$$

$$P_2(x) = 1.76 + \left(\frac{x-0.3}{0.1}\right)(0.24) + \left(\frac{x-0.3}{0.1}\right)\left(\frac{x-0.3}{0.1}-1\right)\left(\frac{0.04}{2!}\right)$$

$$= 2x^2 + x + 1.28 \rightarrow \begin{cases} f(0.25) = 1.655 \\ f(0.35) = 1.875 \end{cases}$$

$$P_n(x) = f_{\frac{n}{2}} + r\delta f_{\frac{n-1}{2}} + \frac{r(r+1)}{2!}\delta^2 f_{\frac{n}{2}} + \frac{r(r+1)(r-1)}{3!}\delta^3 f_{\frac{n-1}{2}} + \cdots + \frac{r(r+1)(r-1)(r+2)(r-2)\ldots\left(r+\left(\frac{n}{2}-1\right)\right)\left(r-\left(\frac{n}{2}-1\right)\right)}{n!}\delta^n f_{\frac{n}{2}}$$

An Example



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0.2	1.56	0.16		
0.3	1.76	0.2	0.04	
0.4	2	0.24	0.04	0
0.5	2.28	0.28	0.04	0

Solution: backward central:

$$h = 0.1 \rightarrow x_{\frac{n}{2}} = 0.3 \rightarrow r = \frac{x-0.3}{0.1}$$

$$P_2(x) = 1.76 + \left(\frac{x-0.3}{0.1}\right)(0.2) + \left(\frac{x-0.3}{0.1}\right)\left(\frac{x-0.3}{0.1}+1\right)\left(\frac{0.04}{2!}\right)$$

$$= 2x^2 + x + 1.28 \rightarrow \begin{cases} f(0.25) = 1.655 \\ f(0.35) = 1.875 \end{cases}$$

Important Note

In case of having even number of nodes, you have to swap the formulas of newton central forward method with those of newton central backward method.

Newton Central Differences Method

x_i	f_i	$\delta f_{i+\frac{1}{2}}$	$\delta^2 f_{i+1}$	$\delta^3 f_{i+\frac{3}{2}}$	$\delta^4 f_{i+2}$
x_0	f_0				
		$\delta f_{\frac{1}{2}} = f_1 - f_0$			
x_1	f_1		$\delta^2 f_1 = \delta f_{\frac{3}{2}} - \delta f_{\frac{1}{2}}$		
		$\delta f_{\frac{3}{2}} = f_2 - f_1$		$\delta^3 f_{\frac{3}{2}} = \delta^2 f_2 - \delta^2 f_1$	
x_2	f_2		$\delta^2 f_2 = \delta f_{\frac{5}{2}} - \delta f_{\frac{3}{2}}$		$\delta^4 f_2 = \delta^3 f_{\frac{5}{2}} - \delta^3 f_{\frac{3}{2}}$
		$\delta f_{\frac{5}{2}} = f_3 - f_2$		$\delta^3 f_{\frac{5}{2}} = \delta^2 f_3 - \delta^2 f_2$	
x_3	f_3		$\delta^2 f_3 = \delta f_{\frac{7}{2}} - \delta f_{\frac{5}{2}}$		
		$\delta f_{\frac{7}{2}} = f_4 - f_3$			
x_4	f_4				

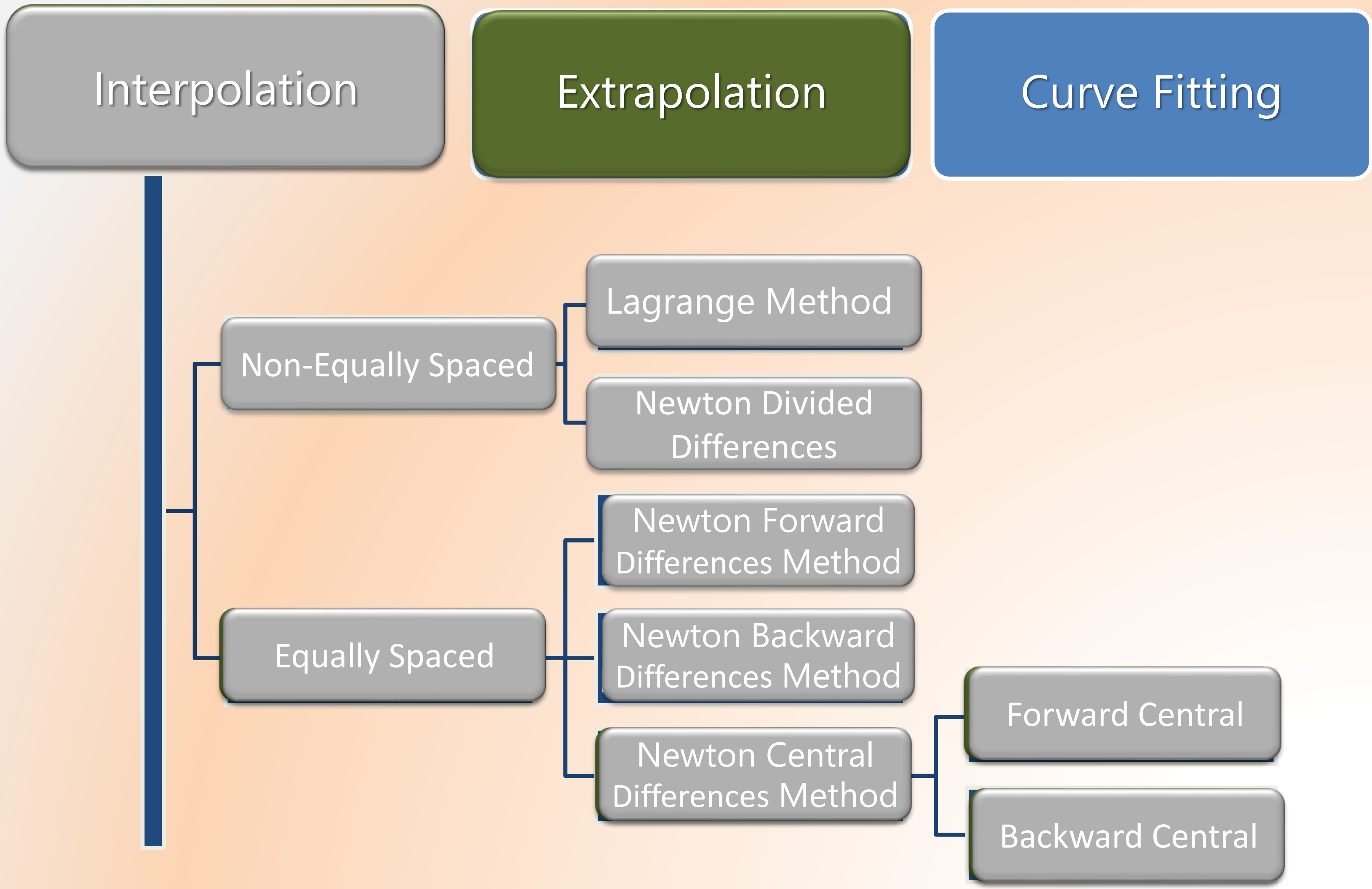
Backward Central

Forward Central

Newton Central Differences Method

x_i	f_i	$\delta f_{i+\frac{1}{2}}$	$\delta^2 f_{i+1}$	$\delta^3 f_{i+\frac{3}{2}}$	$\delta^4 f_{i+2}$
x_0	f_0				
		$\delta f_{\frac{1}{2}} = f_1 - f_0$			
x_1	f_1		$\delta^2 f_1 = \delta f_{\frac{3}{2}} - \delta f_{\frac{1}{2}}$		
		$\delta f_{\frac{3}{2}} = f_2 - f_1$		$\delta^3 f_{\frac{3}{2}} = \delta^2 f_2 - \delta^2 f_1$	
x_2	f_2		$\delta^2 f_2 = \delta f_{\frac{5}{2}} - \delta f_{\frac{3}{2}}$		
		$\delta f_{\frac{5}{2}} = f_3 - f_2$			
x_3	f_3				

Backward Central


Interpolation
Extrapolation
Curve Fitting
Non-Equally Spaced
Lagrange Method
Newton Divided Differences
Equally Spaced
Newton Forward Differences Method
Newton Backward Differences Method
Newton Central Differences Method
Forward Central
Backward Central

Extrapolation

Extrapolation is similar to interpolation, but in extrapolation, value of the point that we want to calculate is located outside the set of known data points (Error may be high here).

We can use all interpolation methods for extrapolation.

Interpolation

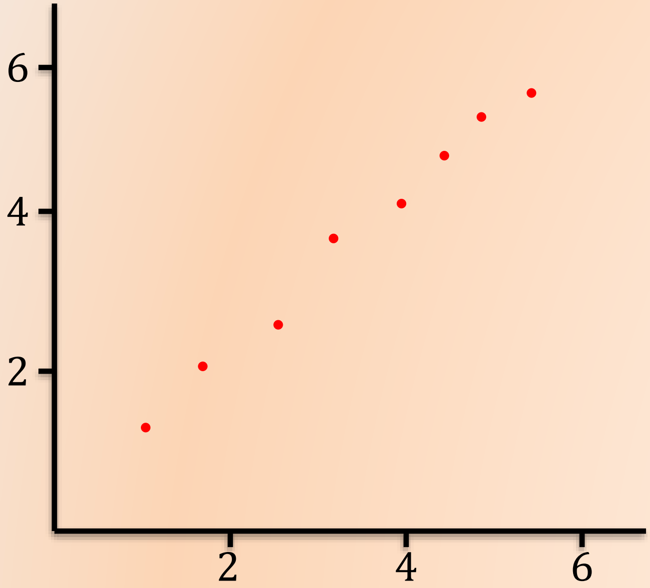
Extrapolation

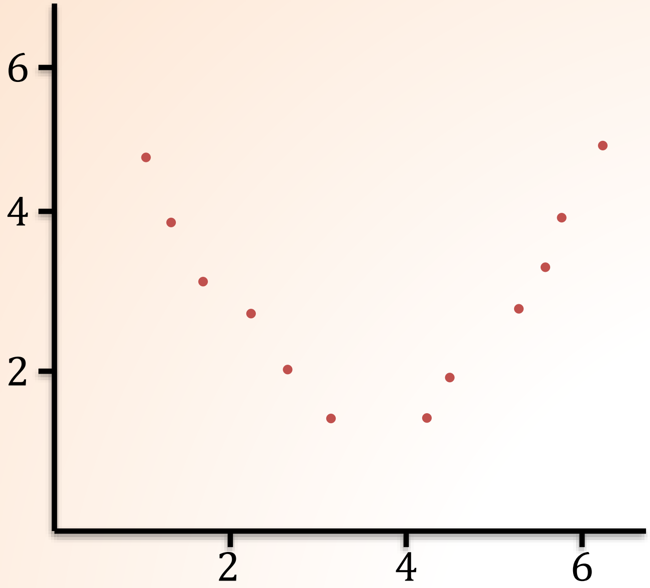
Curve Fitting

- **Fit to Polynomials**
- Some Curves
- Fit to Other Functions with and without Linearization

Curve Fitting

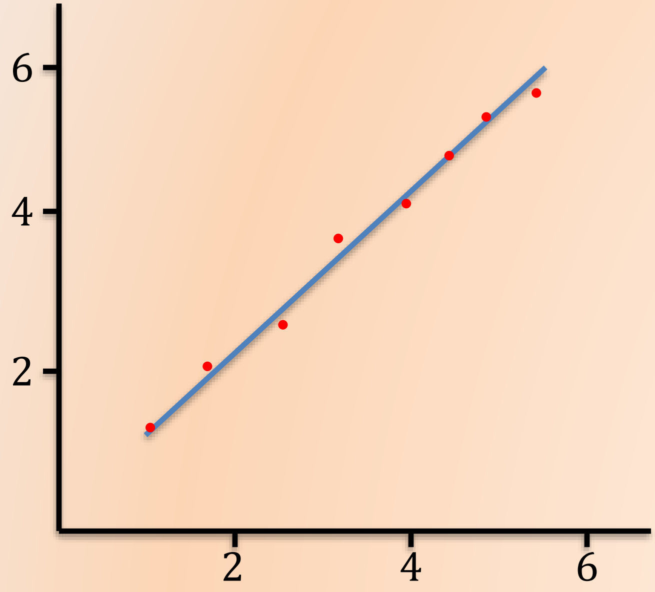
Curve fitting is the process of constructing a curve or a mathematical function that has the best fit to a series of data points.

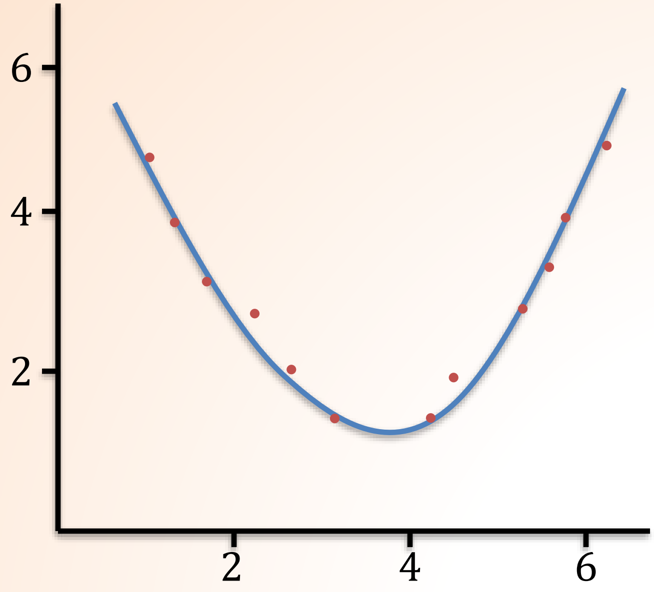


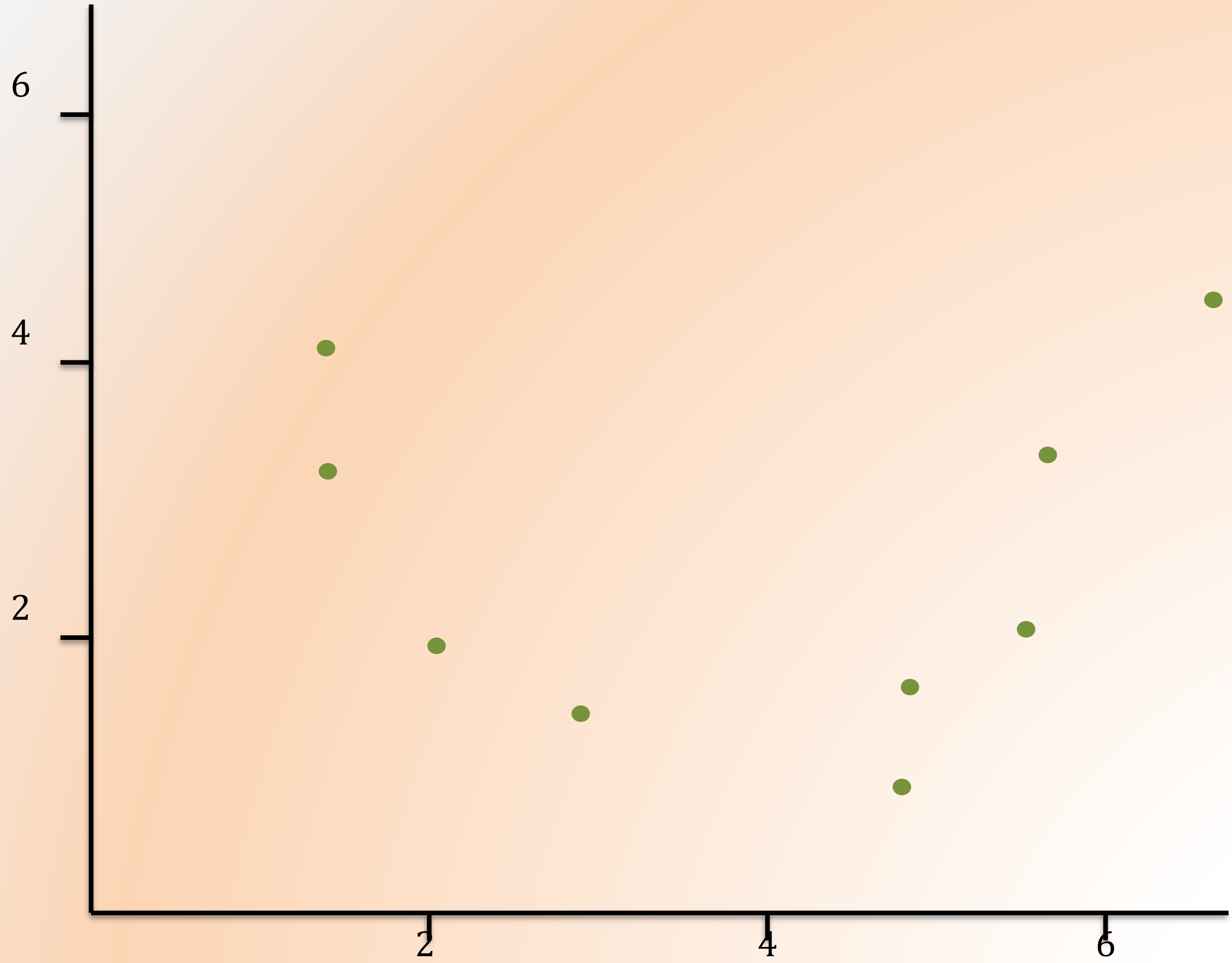


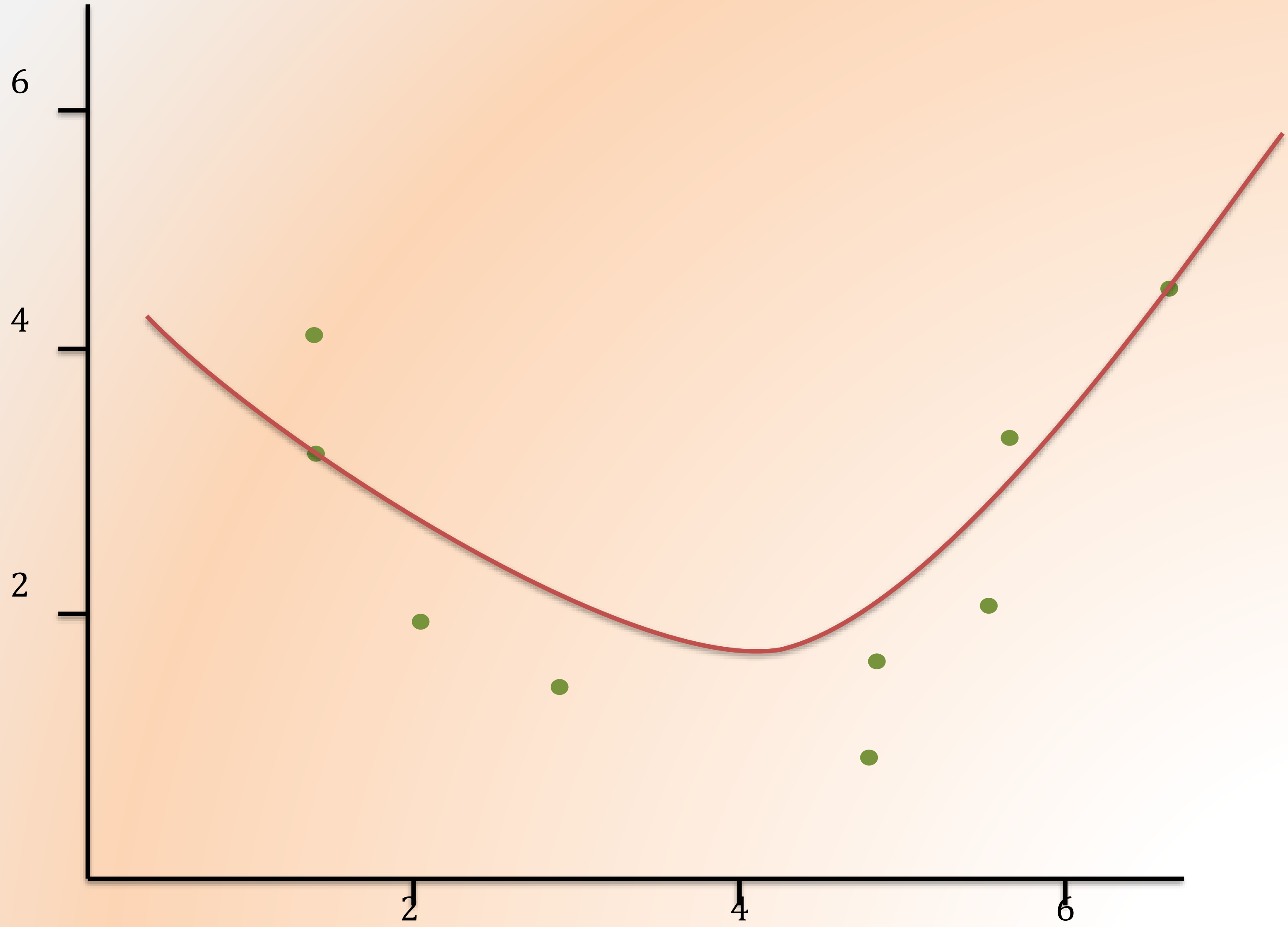
Curve Fitting

Curve fitting is the process of constructing a curve or a mathematical function that has the best fit to a series of data points.






6
4
2
2
4
6



6
4
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2
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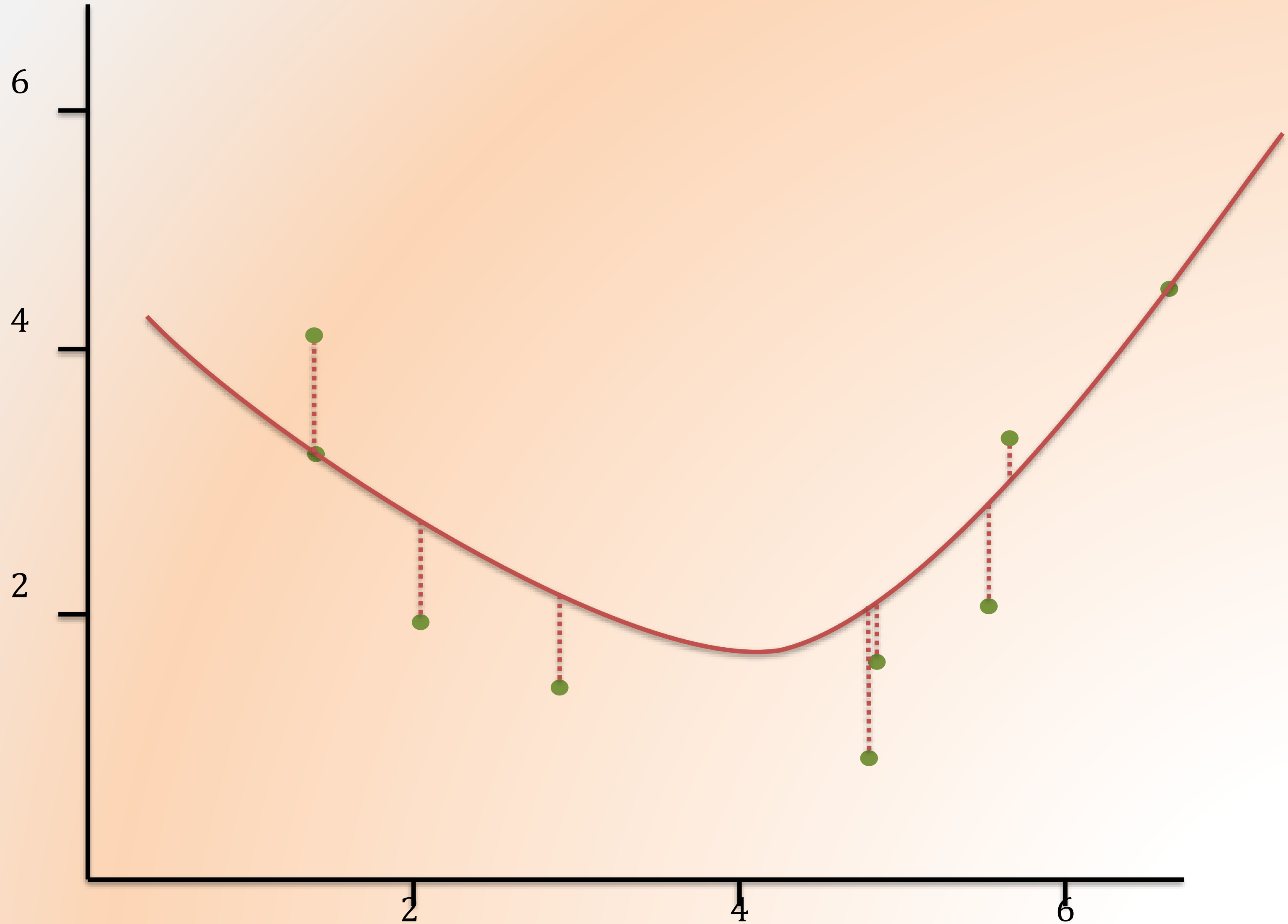
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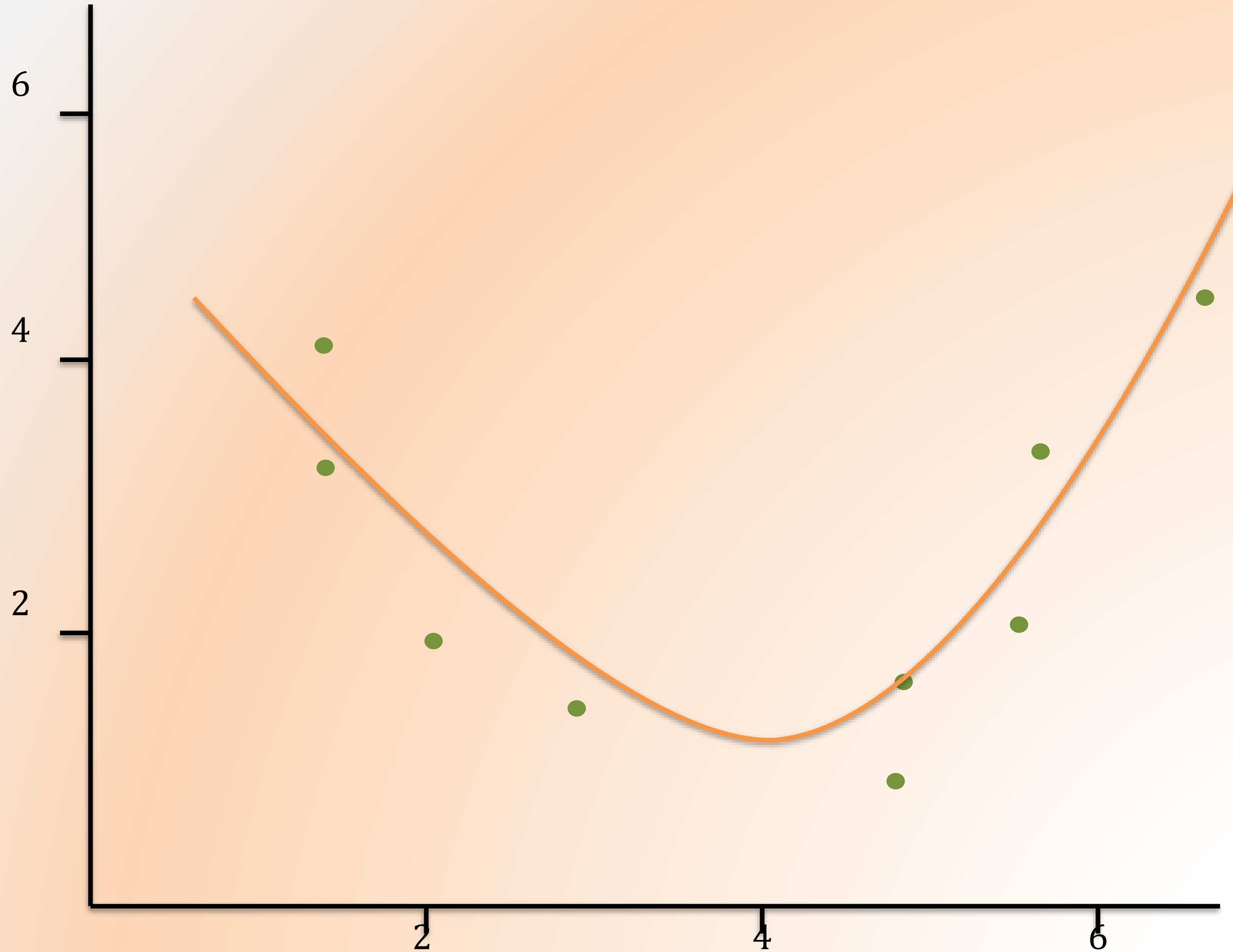
6
4
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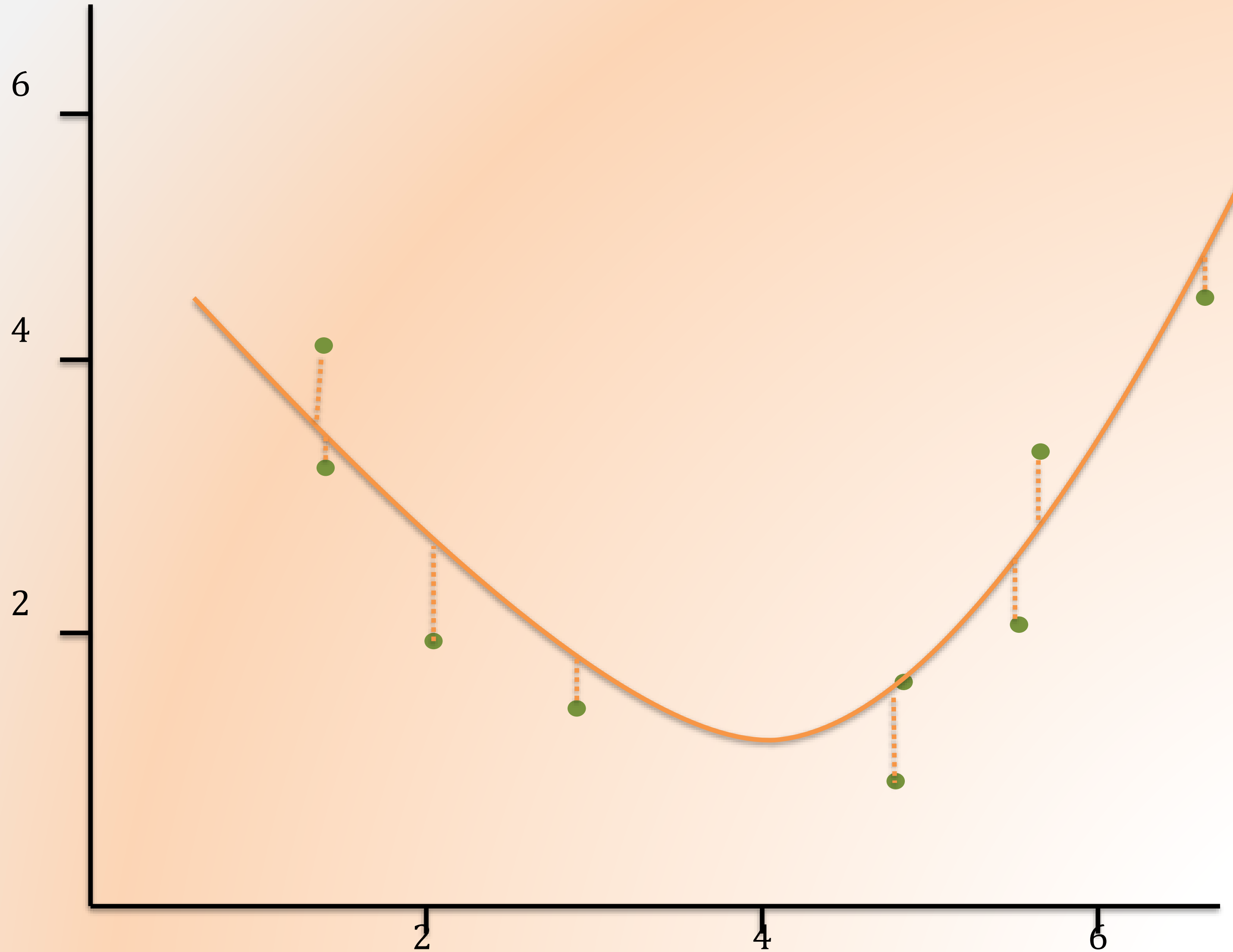
6
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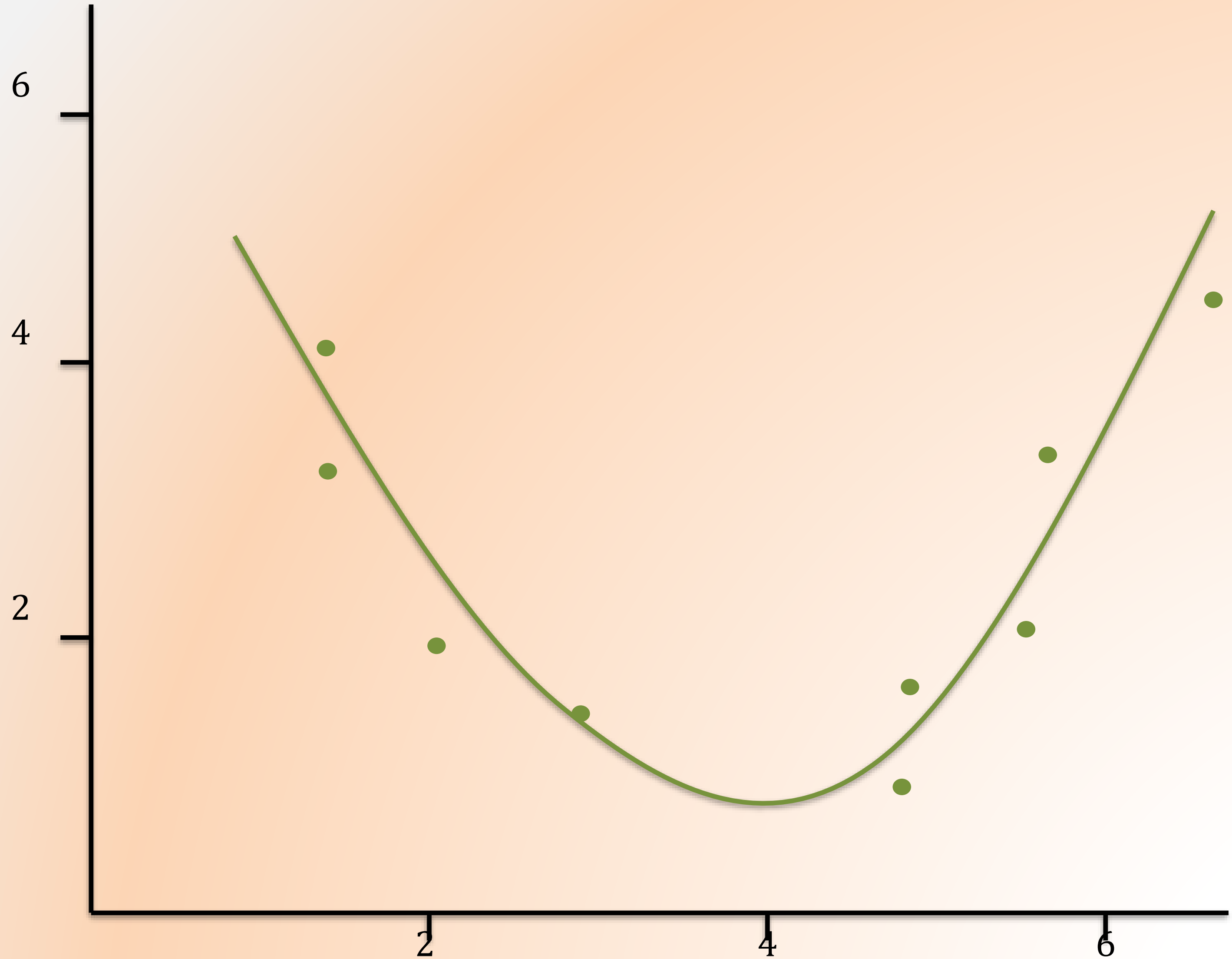
4
6


6
4
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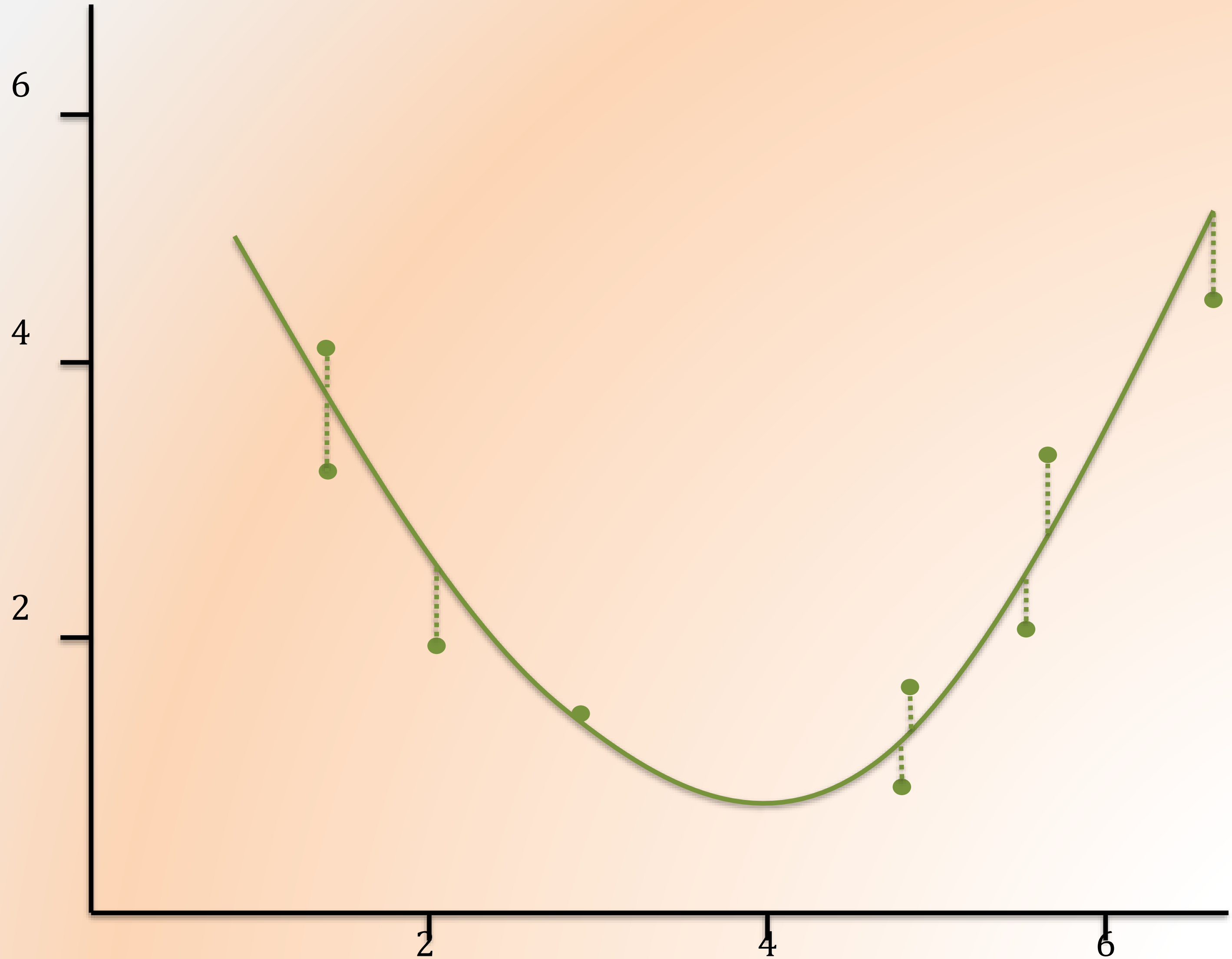
4
6


6
4
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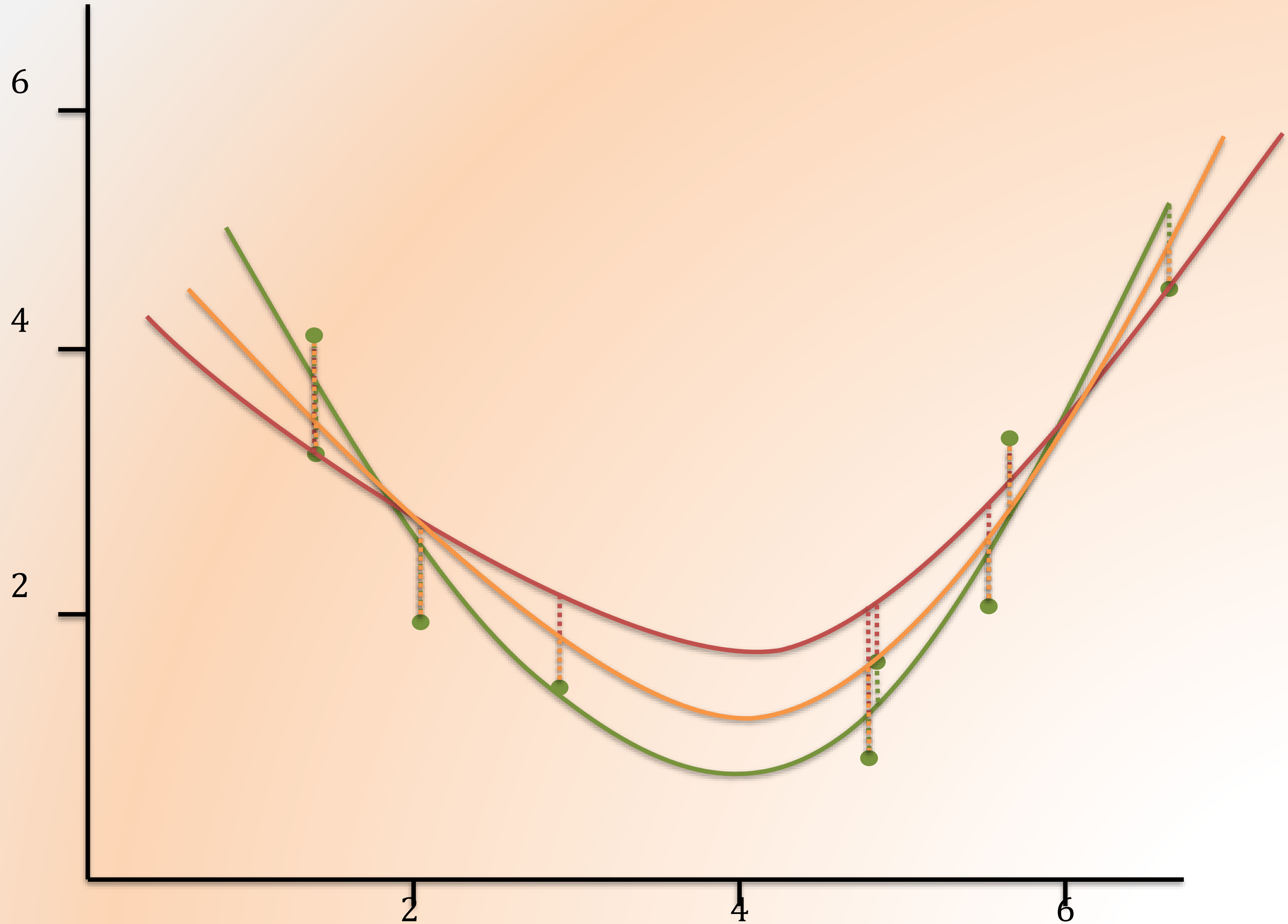
2
4
6


6
4
2
2
4
6

Least Squares Fit

Number of x_is is n+1. As a result, in case of interpolating, interpolated polynomial would be of degree n; On the other hand, m is the degree of the fitted curve. Our curve must minimize the value of RMS in order to minimize:

$$\mathrm{E} = \sum_{i=0}^{n} [y_i - P_m(x_i)]^2$$

Least Squares Fit

Number of x_is is n+1. As a result, in case of interpolating, interpolated polynomial would be of degree n; On the other hand, m is the degree of the fitted curve. Our curve must minimize the value of RMS in order to minimize:

$$\mathrm{E} = \sum_{i=0}^{n} [y_i - P_m(x_i)]^2$$

We need some measure in order to choose the best fit; Here, our measure is **RMS** (Root Mean Square):

$$\sqrt{\frac{1}{n}\sum_{i=0}^{n} (y_i - P_m(x_i))^2}$$

Least Squares Fit

$$P_m(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

$$\mathrm{E} = \sum_{i=0}^{n} [y_i - P_m(x_i)]^2$$

$$\frac{\partial E}{\partial a_i} = 0 \quad i = 0,1,\ldots,m$$

Least Squares Fit

$$E = \sum_{i=0}^{n} [y_i - P_m(x_i)]^2$$

$$\frac{\partial E}{\partial a_i} = 0 \quad i = 0,1, \dots, m$$

$$\begin{cases} a_0 S_{x^0} + a_1 S_{x^1} + a_2 S_{x^2} + \cdots + a_m S_{x^m} = S_y \\ a_0 S_{x^1} + a_1 S_{x^2} + a_2 S_{x^3} + \cdots + a_n S_{x^{m+1}} = S_{xy} \\ \dots \\ a_0 S_{x^m} + a_1 S_{x^{m+1}} + a_2 S_{x^{m+2}} + \cdots + a_n S_{x^{2m}} = S_{x^m y} \end{cases}$$

Least Squares Fit

$$\mathrm{E} = \sum_{i=0}^{n} [y_i - P_m(x_i)]^2$$

$$\frac{\partial E}{\partial a_i} = 0 \quad i = 0,1, \dots, m$$

$$(n+1)$$

$$\begin{cases} a_0 S_{x^0} + a_1 S_{x^1} + a_2 S_{x^2} + \cdots + a_m S_{x^m} = S_y \\ a_0 S_{x^1} + a_1 S_{x^2} + a_2 S_{x^3} + \cdots + a_n S_{x^{m+1}} = S_{xy} \\ \dots \\ a_0 S_{x^m} + a_1 S_{x^{m+1}} + a_2 S_{x^{m+2}} + \cdots + a_n S_{x^{2m}} = S_{x^m y} \end{cases}$$

$$S_{x^0} = \sum_{i=0}^{n} x_i^0 = n + 1$$

Least Squares Fit

$$E = \sum_{i=0}^{n} [y_i - P_m(x_i)]^2$$

$$\frac{\partial E}{\partial a_i} = 0 \quad i = 0,1,\dots,m$$

$(n+1)$

$$\begin{cases} a_0 S_{x^0} + a_1 S_{x^1} + a_2 S_{x^2} + \cdots + a_m S_{x^m} = S_y \\ a_0 S_{x^1} + a_1 S_{x^2} + a_2 S_{x^3} + \cdots + a_n S_{x^{m+1}} = S_{xy} \\ \cdots \\ a_0 S_{x^m} + a_1 S_{x^{m+1}} + a_2 S_{x^{m+2}} + \cdots + a_n S_{x^{2m}} = S_{x^m y} \end{cases}$$

$$S_{x^0} = \sum_{i=0}^{n} x_i^0 = n+1 \,, S_{x^1} = \sum_{i=0}^{n} x_i^1 \,, S_{x^2} = \sum_{i=0}^{n} {x_i}^2 \,, S_{x^k y} = \sum_{i=0}^{n} {x_i}^k y_i$$

Least Squares Fit

Suppose that $\boldsymbol{m} = \mathbf{1}$, so we will have linear fit:

$$\begin{cases} a_0(n+1) + a_1\left(\sum_{i=0}^{n} x_i\right) = \sum_{i=0}^{n} y_i \\ a_0\left(\sum_{i=0}^{n} x_i\right) + a_1\left(\sum_{i=0}^{n} {x_i}^2\right) = \sum_{i=0}^{n} x_i y_i \end{cases}$$

$$\boxed{P_1(x) = a_0 + a_1 x}$$

Least Squares Fit

Suppose that $\boldsymbol{m} = \mathbf{2}$, so we will have:

$$\begin{cases} a_0(n+1) + a_1\left(\sum_{i=0}^{n} x_i\right) + a_2\left(\sum_{i=0}^{n} {x_i}^2\right) = \sum_{i=0}^{n} y_i \\ a_0\left(\sum_{i=0}^{n} x_i\right) + a_1\left(\sum_{i=0}^{n} {x_i}^2\right) + a_2\left(\sum_{i=0}^{n} {x_i}^3\right) = \sum_{i=0}^{n} x_i y_i \\ a_0\left(\sum_{i=0}^{n} {x_i}^2\right) + a_1\left(\sum_{i=0}^{n} {x_i}^3\right) + a_2\left(\sum_{i=0}^{n} {x_i}^4\right) = \sum_{i=0}^{n} {x_i}^2 y_i \end{cases}$$

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

An Example

m =2

x_i	-4	-3	-2	-1	0	1	2	3	4	5
y_i	38	20	11	3	-1	2	6	14	26	44

An Example

x_i	-4	-3	-2	-1	0	1	2	3	4	5	5
y_i	38	20	11	3	-1	2	6	14	26	44	163
x_i^2											
x_i^3											
x_i^4											
$x_i\ y_i$											
$x_i^2\ y_i$											

An Example

											Σ
x_i	-4	-3	-2	-1	0	1	2	3	4	5	5
y_i	38	20	11	3	-1	2	6	14	26	44	163
x_i^2	16	9	4	1	0	1	4	9	16	25	85
x_i^3											
x_i^4											
$x_i \ y_i$											
$x_i^2 \ y_i$											

An Example

											Σ
x_i	-4	-3	-2	-1	0	1	2	3	4	5	5
y_i	38	20	11	3	-1	2	6	14	26	44	163
x_i^2	16	9	4	1	0	1	4	9	16	25	85
x_i^3	-64	-27	-8	-1	0	1	8	27	64	125	125
x_i^4											
$x_i\ y_i$											
$x_i^2\ y_i$											

An Example

											Σ
x_i	-4	-3	-2	-1	0	1	2	3	4	5	5
y_i	38	20	11	3	-1	2	6	14	26	44	163
x_i^2	16	9	4	1	0	1	4	9	16	25	85
x_i^3	-64	-27	-8	-1	0	1	8	27	64	125	125
x_i^4	256	81	16	1	0	1	16	81	256	625	1333
$x_i \ y_i$											
$x_i^2 \ y_i$											

An Example

											Σ
x_i	-4	-3	-2	-1	0	1	2	3	4	5	5
y_i	38	20	11	3	-1	2	6	14	26	44	163
x_i^2	16	9	4	1	0	1	4	9	16	25	85
x_i^3	-64	-27	-8	-1	0	1	8	27	64	125	125
x_i^4	256	81	16	1	0	1	16	81	256	625	1333
$x_i \ y_i$	-152	-60	-22	-3	0	2	12	42	104	220	143
$x_i^2 \ y_i$											

An Example

											Σ
x_i	-4	-3	-2	-1	0	1	2	3	4	5	5
y_i	38	20	11	3	-1	2	6	14	26	44	163
x_i^2	16	9	4	1	0	1	4	9	16	25	85
x_i^3	-64	-27	-8	-1	0	1	8	27	64	125	125
x_i^4	256	81	16	1	0	1	16	81	256	625	1333
$x_i\ y_i$	-152	-60	-22	-3	0	2	12	42	104	220	143
$x_i^2\ y_i$	608	180	44	3	0	2	24	126	416	1100	2503

An Example

	$\sum$
x_i	5
y_i	163
x_i^2	85
x_i^3	125
x_i^4	1333
$x_i\ y_i$	143
$x_i^2\ y_i$	2503

$$\begin{cases} a_0(n+1) + a_1\left(\sum_{i=0}^{n} x_i\right) + a_2\left(\sum_{i=0}^{n} {x_i}^2\right) = \sum_{i=0}^{n} y_i \\ a_0\left(\sum_{i=0}^{n} x_i\right) + a_1\left(\sum_{i=0}^{n} {x_i}^2\right) + a_2\left(\sum_{i=0}^{n} {x_i}^3\right) = \sum_{i=0}^{n} x_i y_i \\ a_0\left(\sum_{i=0}^{n} {x_i}^2\right) + a_1\left(\sum_{i=0}^{n} {x_i}^3\right) + a_2\left(\sum_{i=0}^{n} {x_i}^4\right) = \sum_{i=0}^{n} {x_i}^2 y_i \end{cases}$$

$$\Rightarrow \begin{cases} 10a_0 + 5a_1 + 85a_2 = 163 \\ 5a_0 + 85a_1 + 125a_2 = 143 \\ 85a_0 + 125a_1 + 1333a_2 = 2503 \end{cases}$$

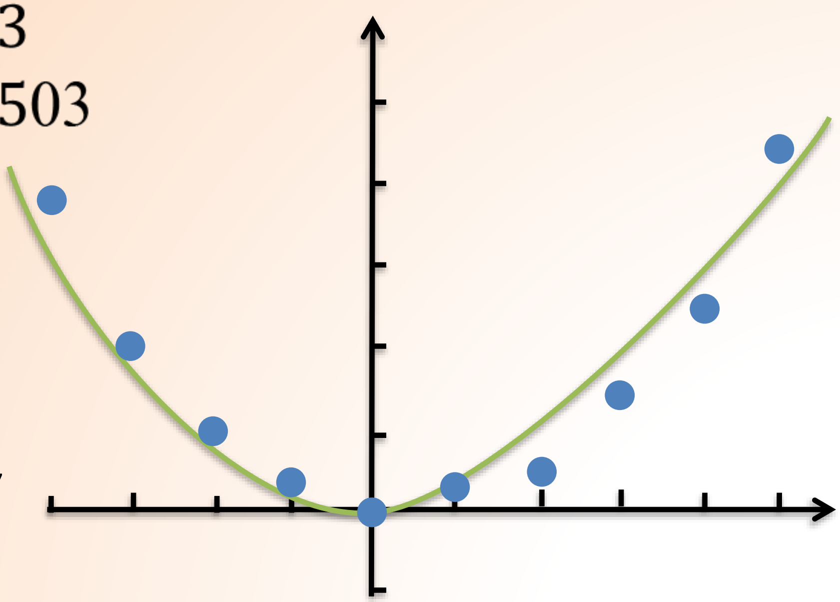
An Example

x_i	-4	-3	-2	-1	0	1	2	3	4	5
y_i	38	20	11	3	-1	2	6	14	26	44

$$\Rightarrow \begin{cases} 10a_0 + 5a_1 + 85a_2 = 163 \\ 5a_0 + 85a_1 + 125a_2 = 143 \\ 85a_0 + 125a_1 + 1333a_2 = 2503 \end{cases}$$

$$\Rightarrow \begin{cases} a_0 = -0.07 \\ a_1 = -1.25 \\ a_2 = 2 \end{cases}$$

$$\Rightarrow P_2(x) = 2x^2 - 1.25x - 0.07$$



Interpolation

Extrapolation

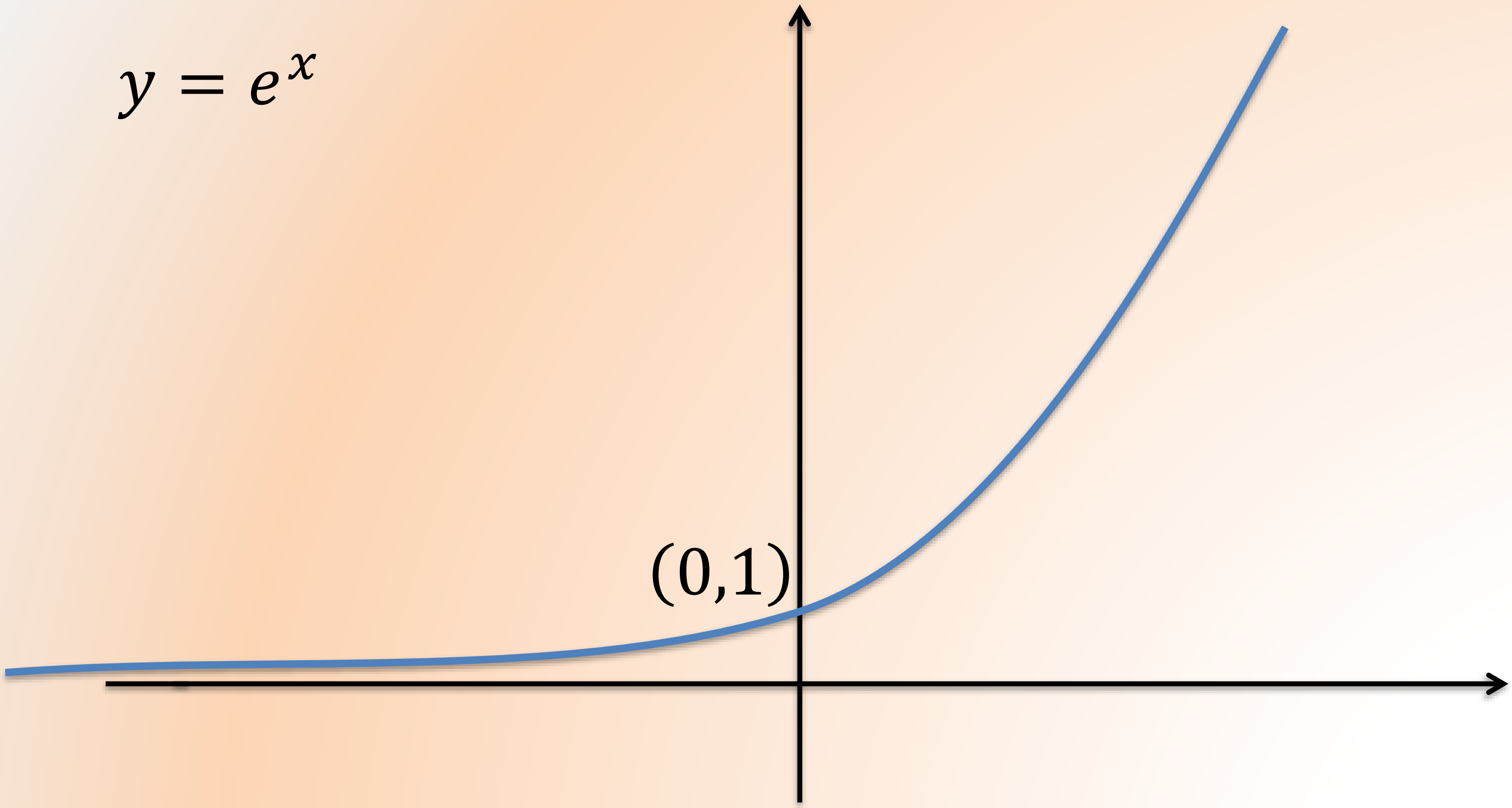
Curve Fitting

Fit to Polynomials

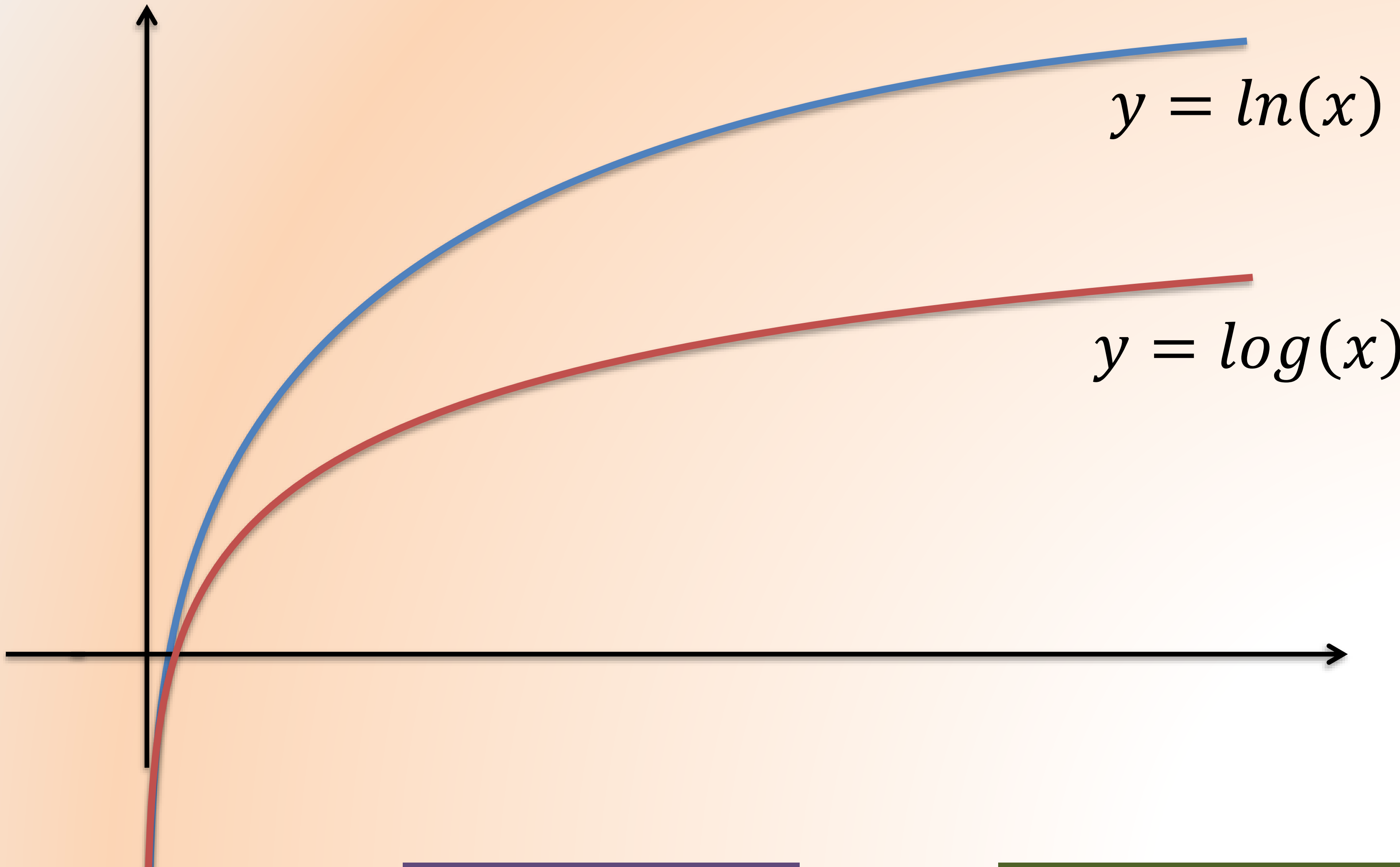
Some curves

Fit to Other Functions with and without Linearization

Exponential

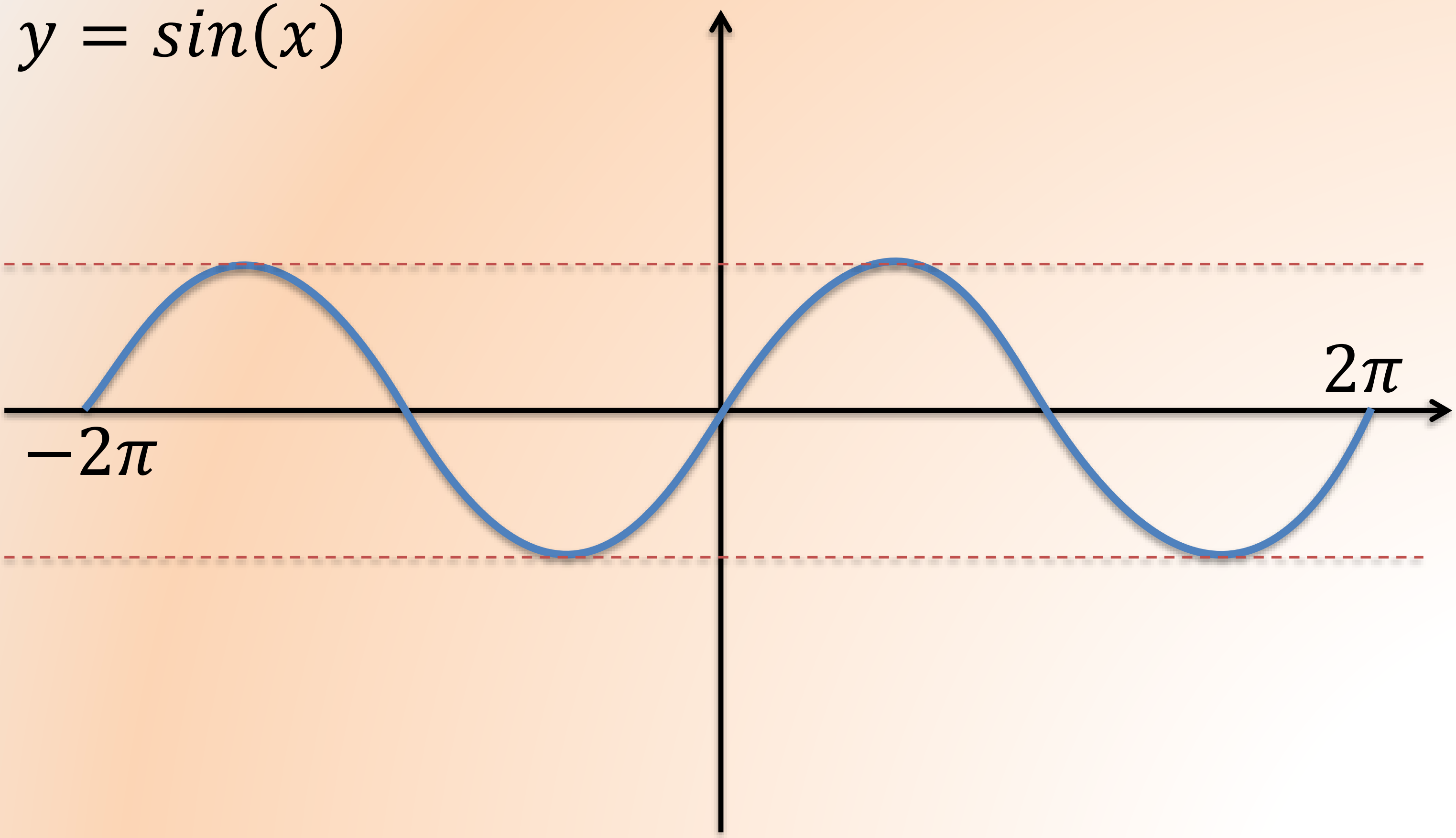


Logarithmic

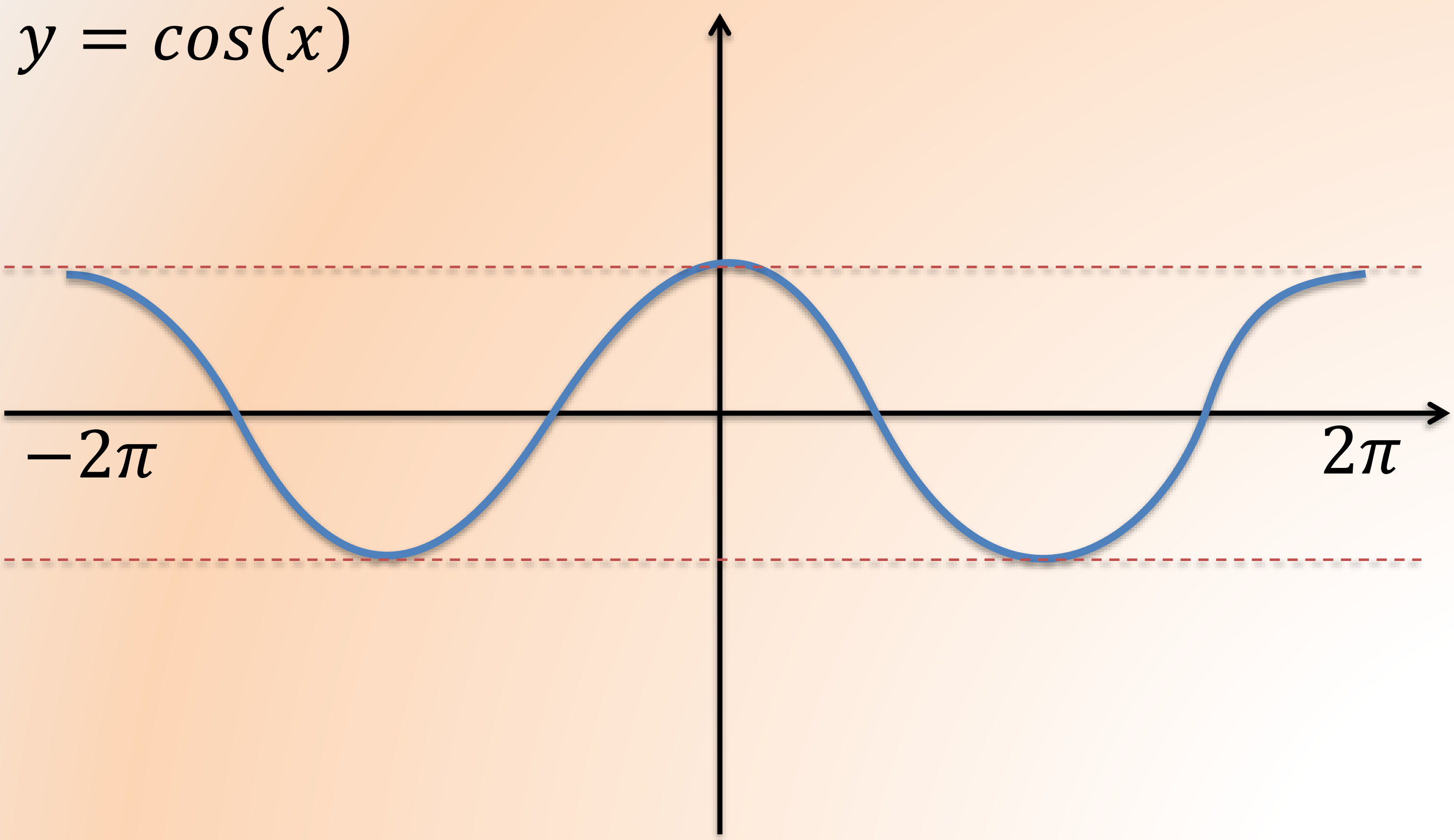


Trigonometric Functions

$y = sin(x)$

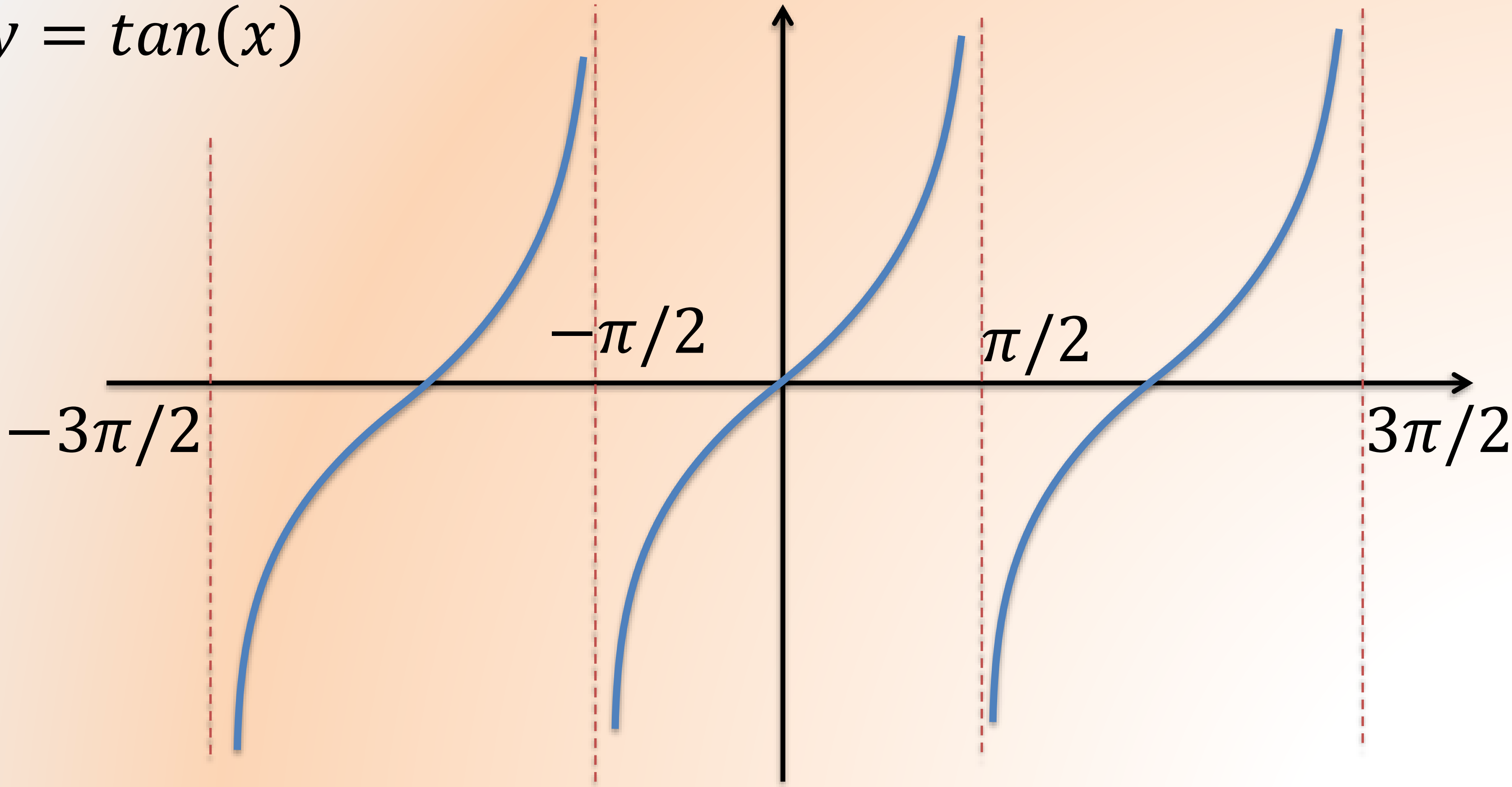


Trigonometric Functions



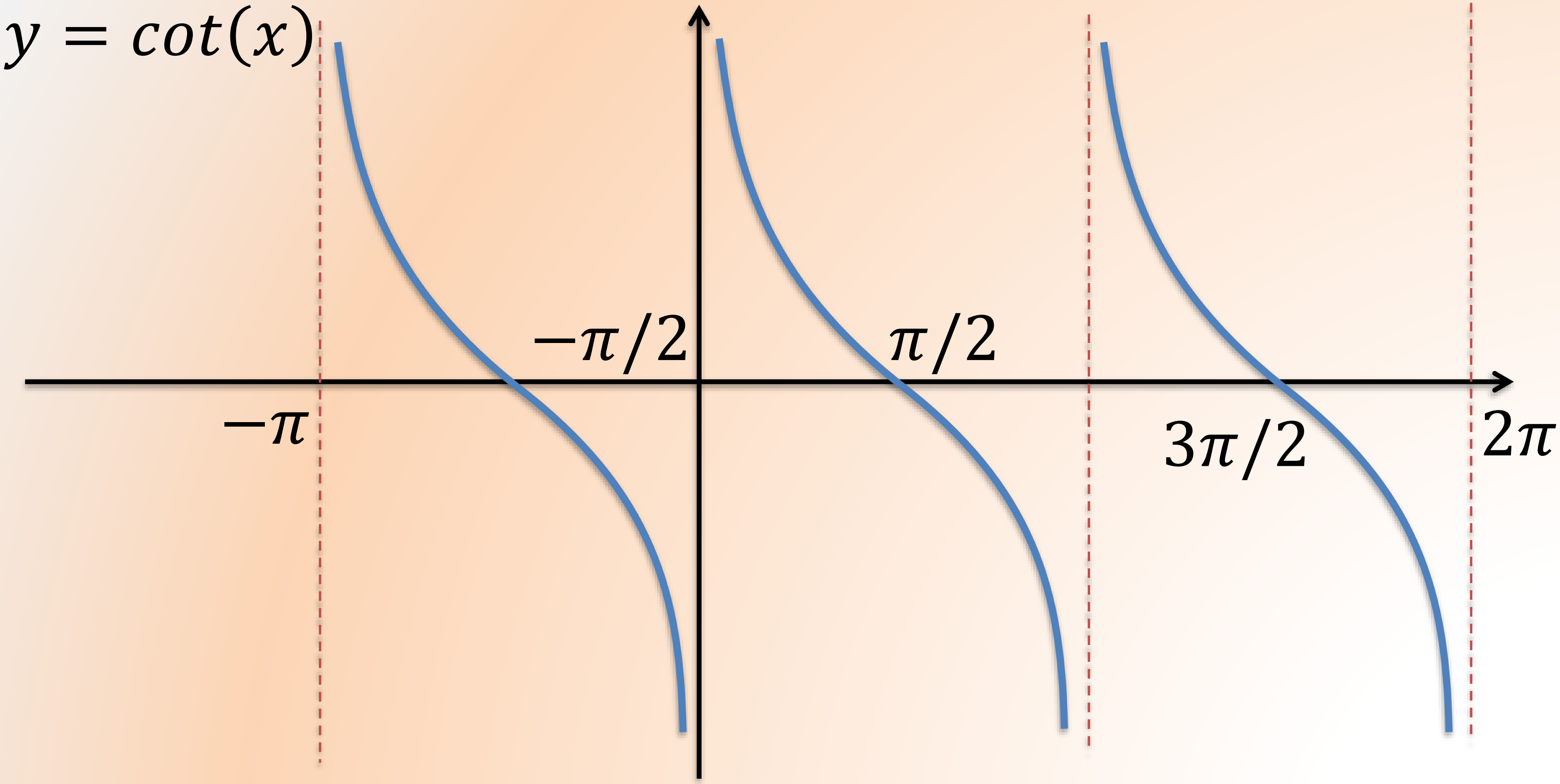
Trigonometric Functions

$y = tan(x)$



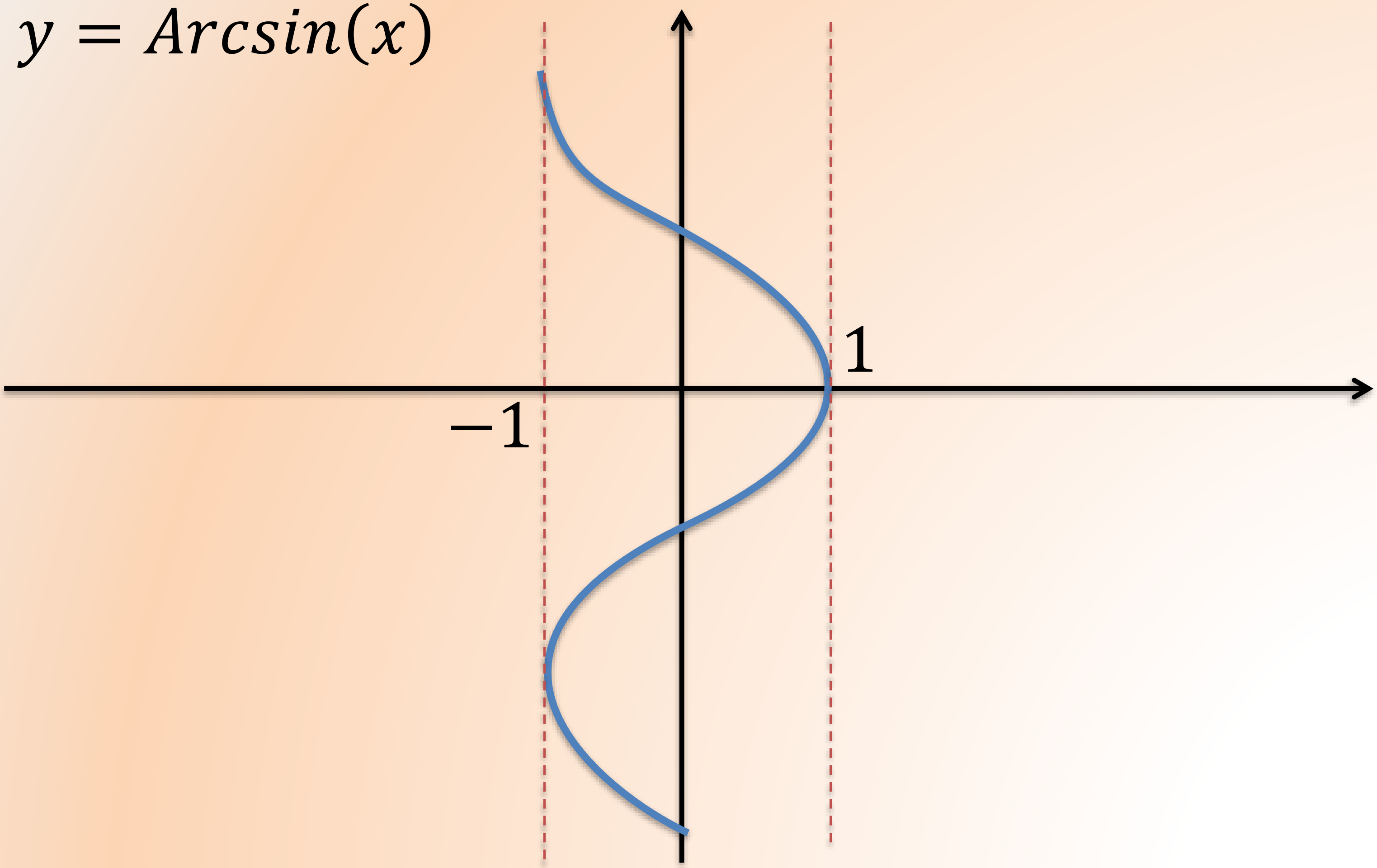
Trigonometric Functions

$y = cot(x)$



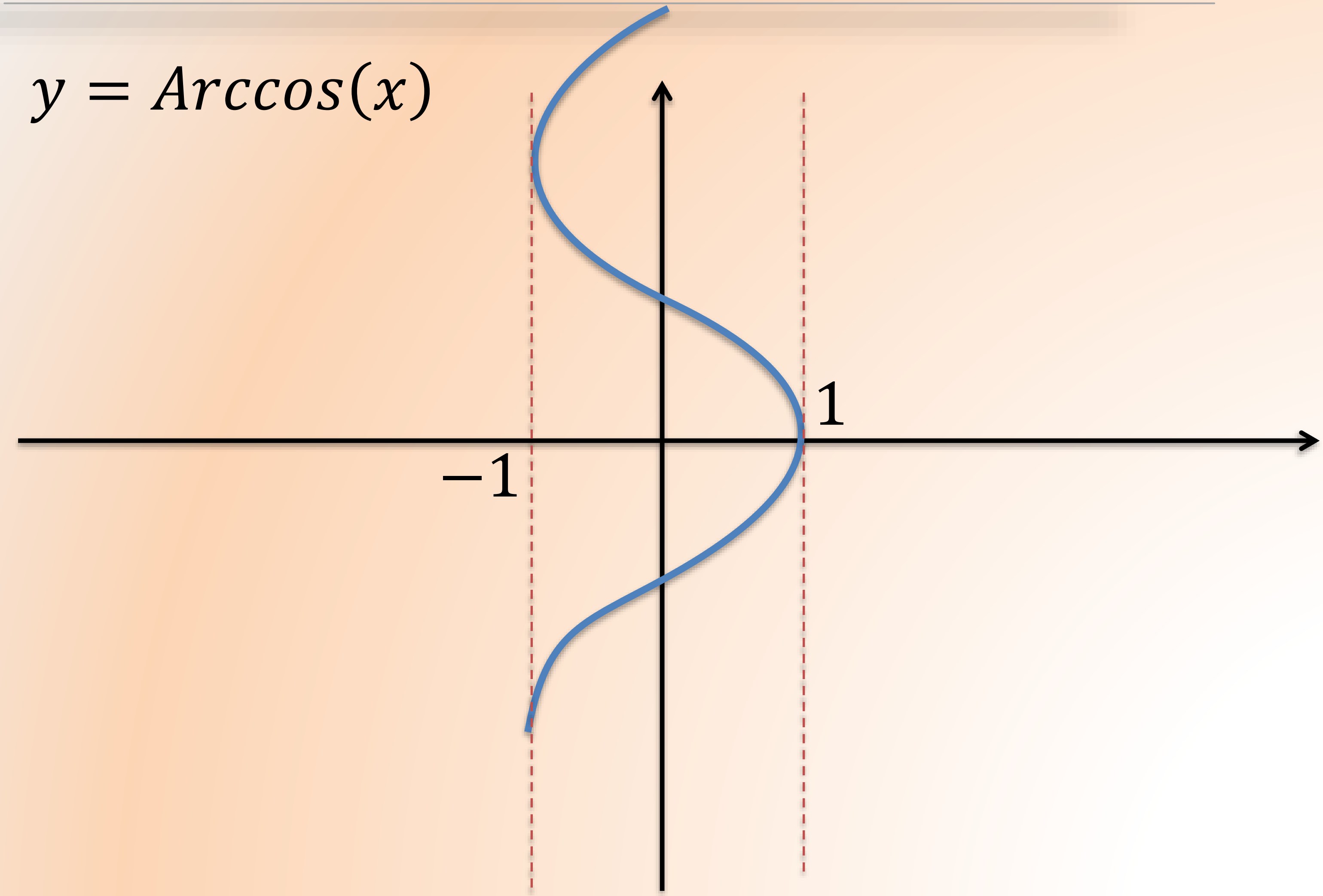
Inverse Trigonometric Functions

$y = Arcsin(x)$



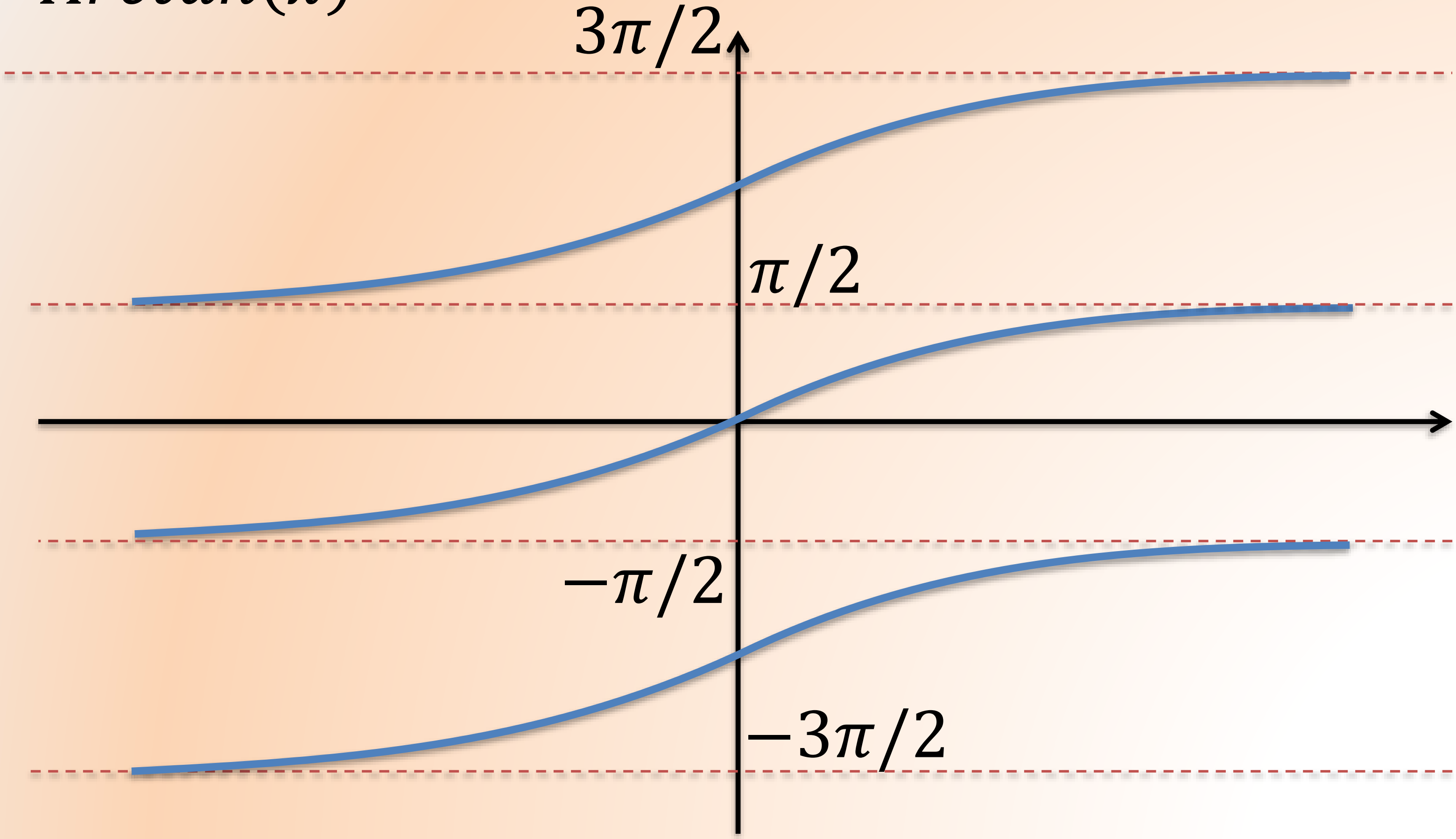
Inverse Trigonometric Functions

$y = Arccos(x)$



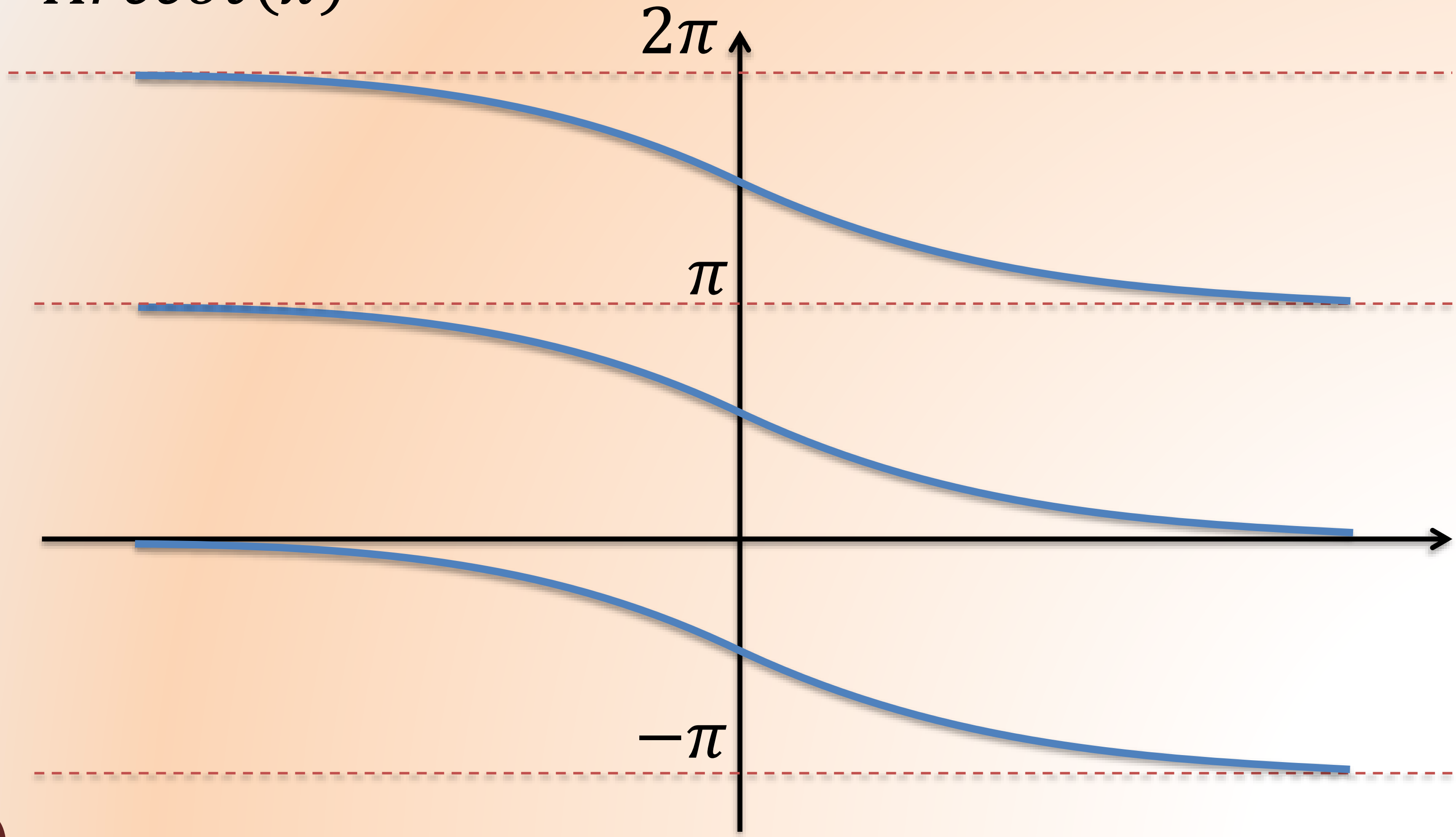
Inverse Trigonometric Functions

$y = Arctan(x)$

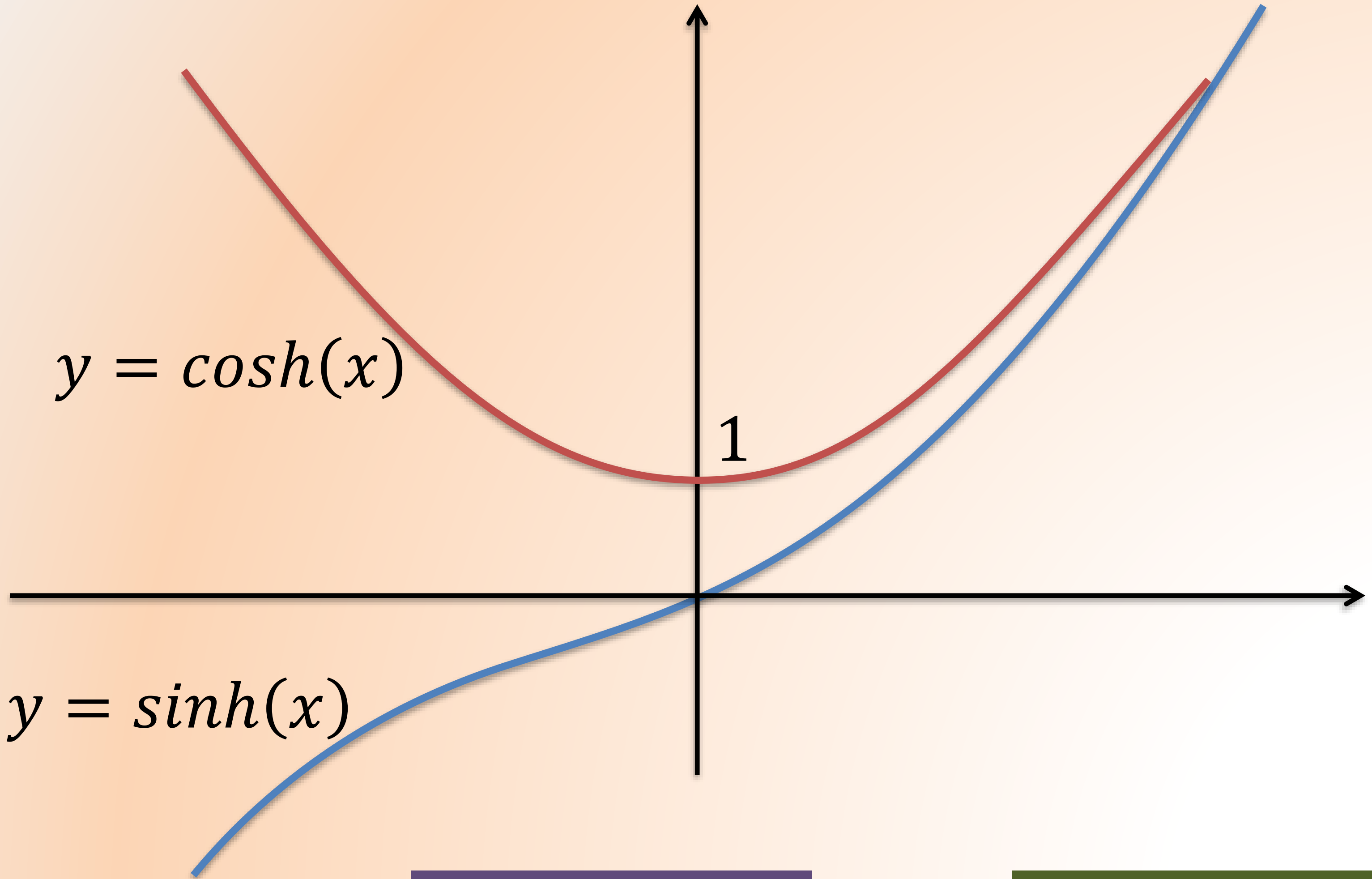


Inverse Trigonometric Functions

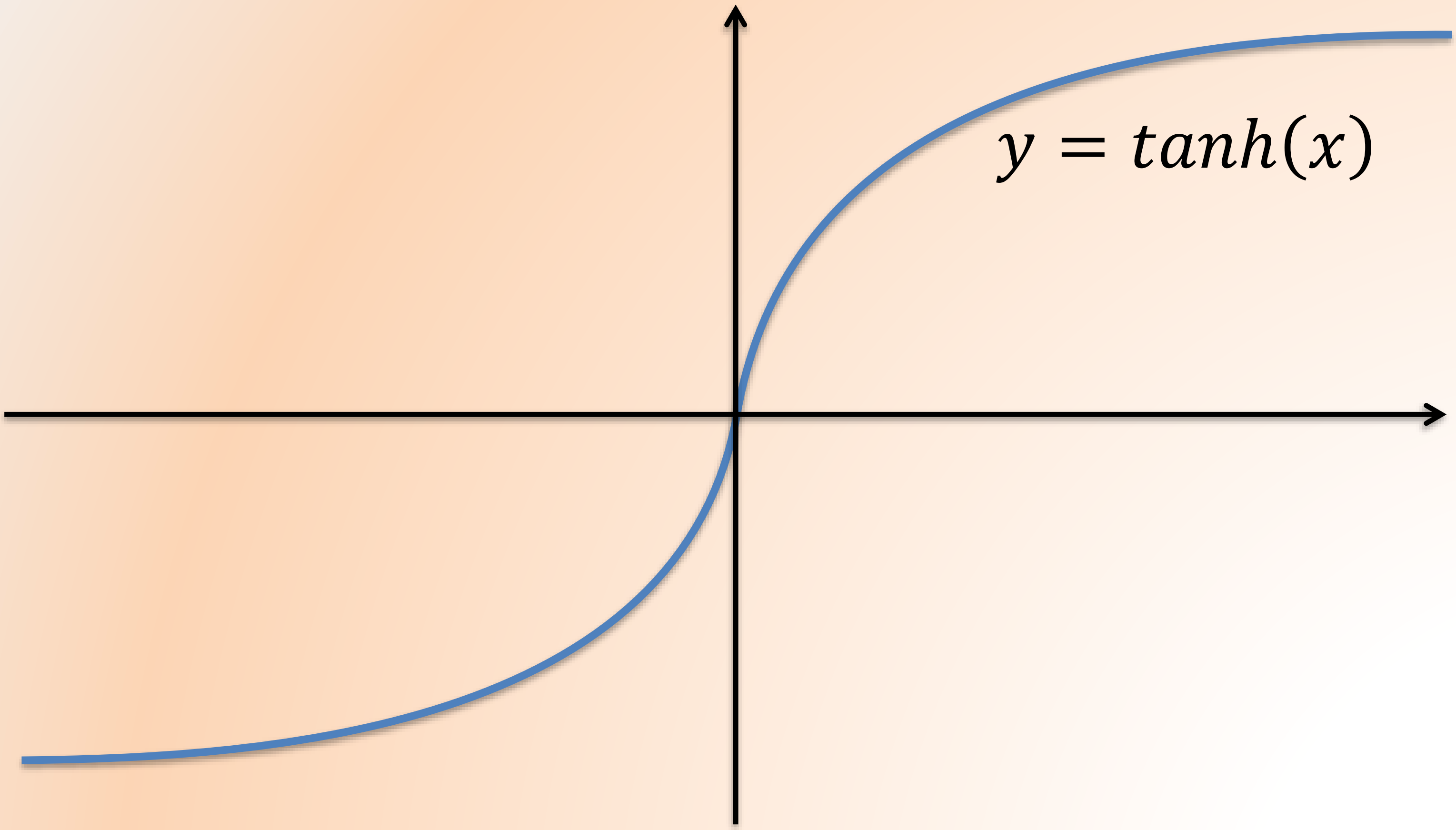
$y = Arccot(x)$



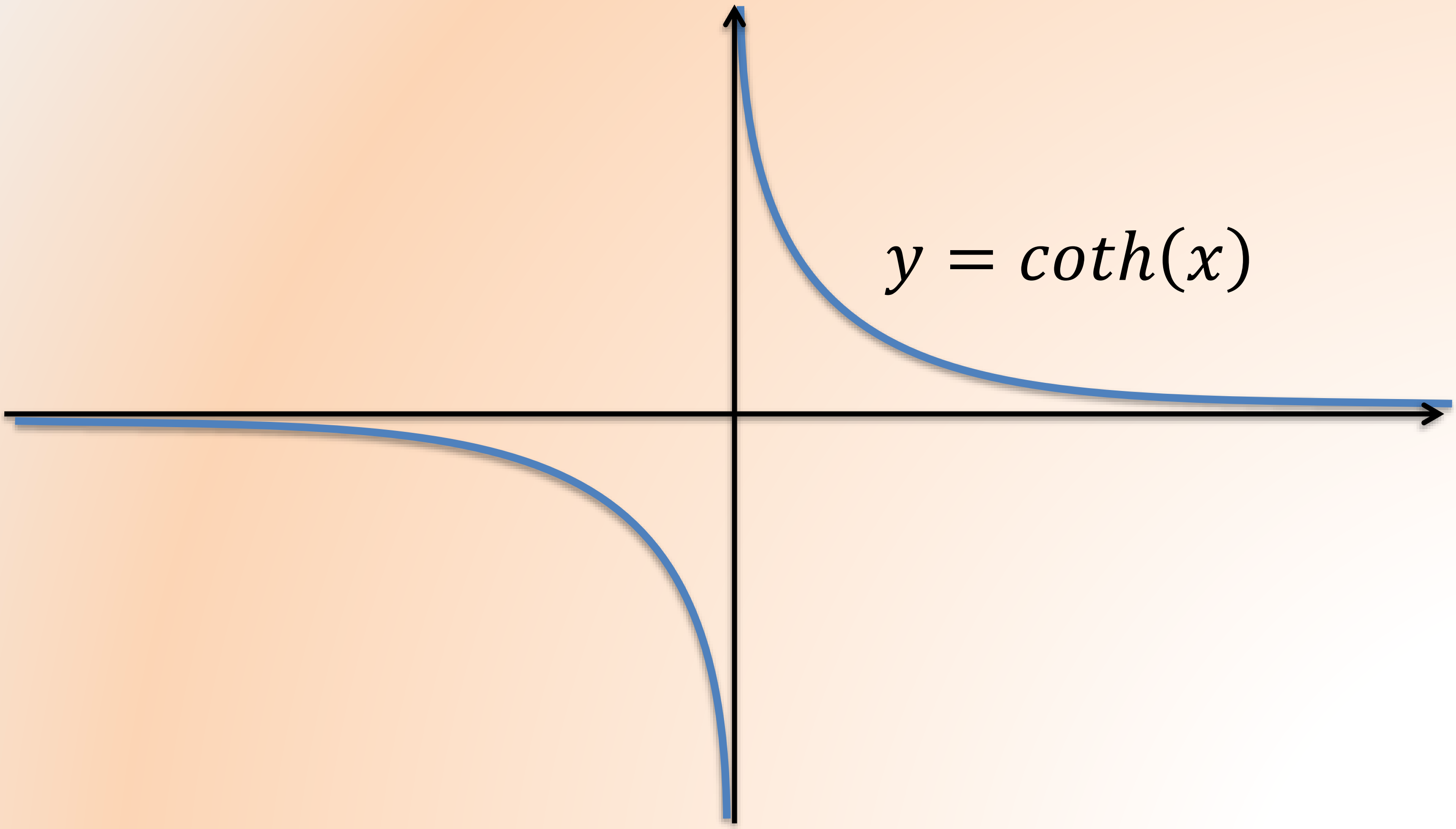
Hyperbolic



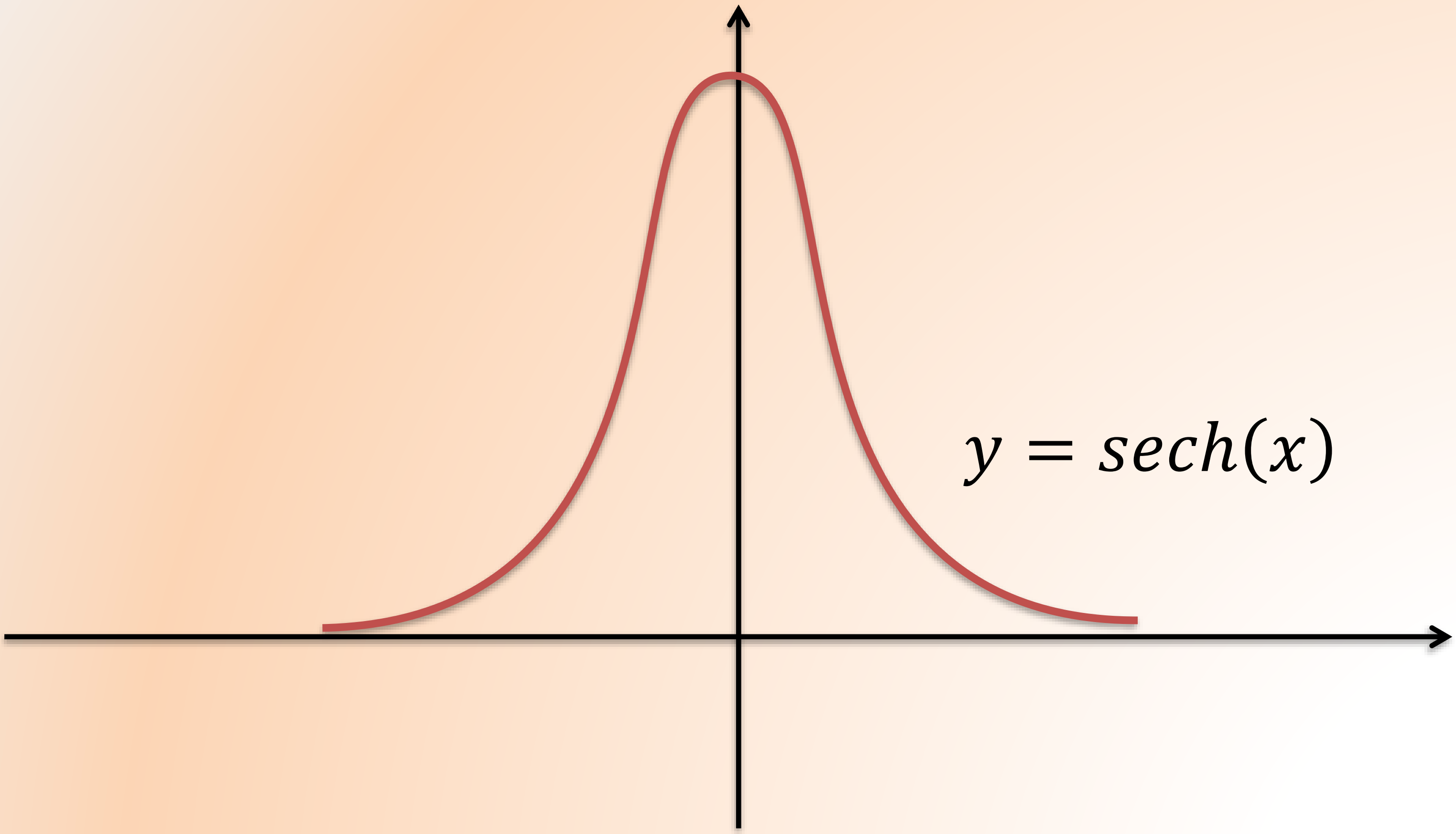
Hyperbolic



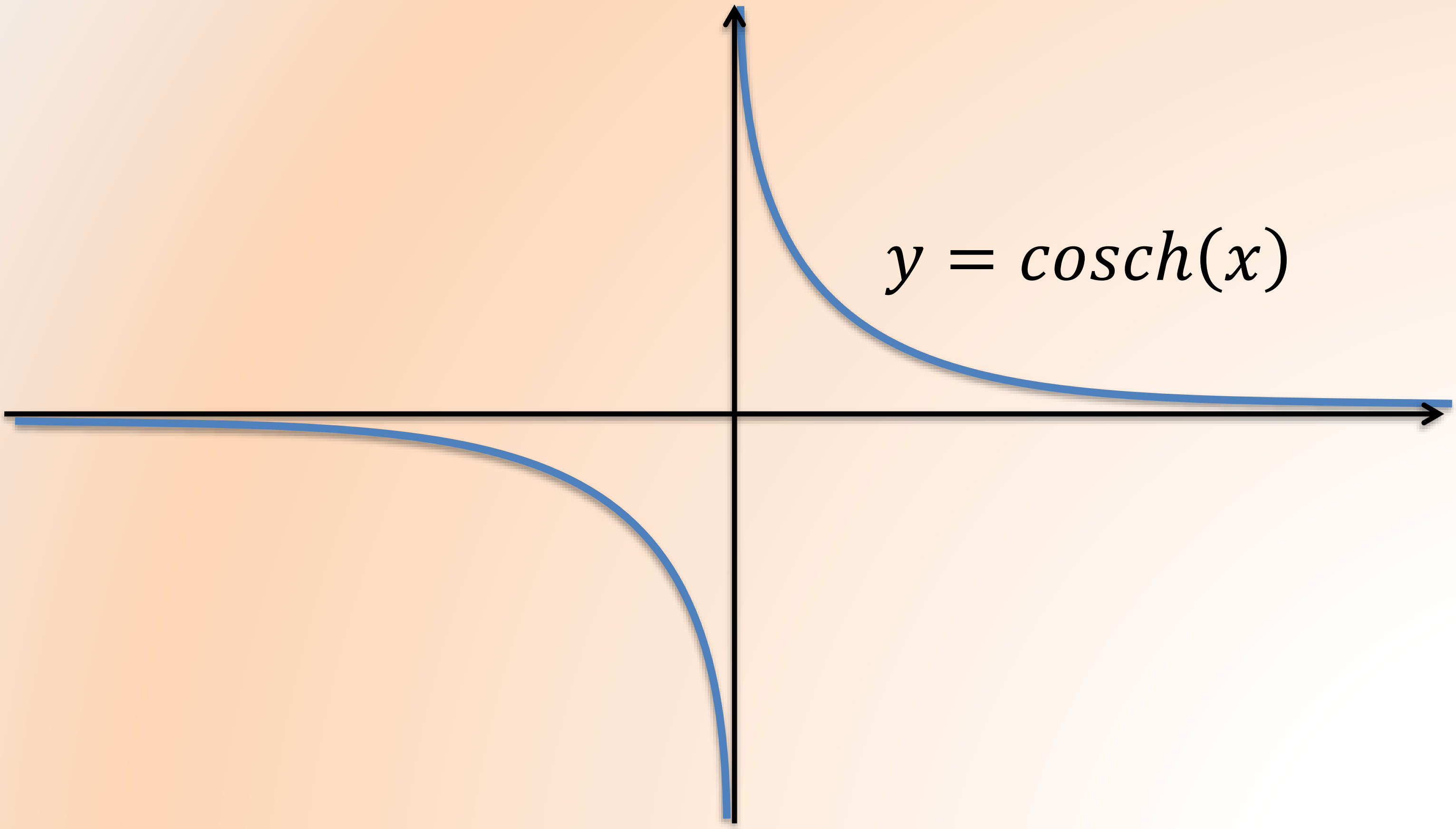
Hyperbolic



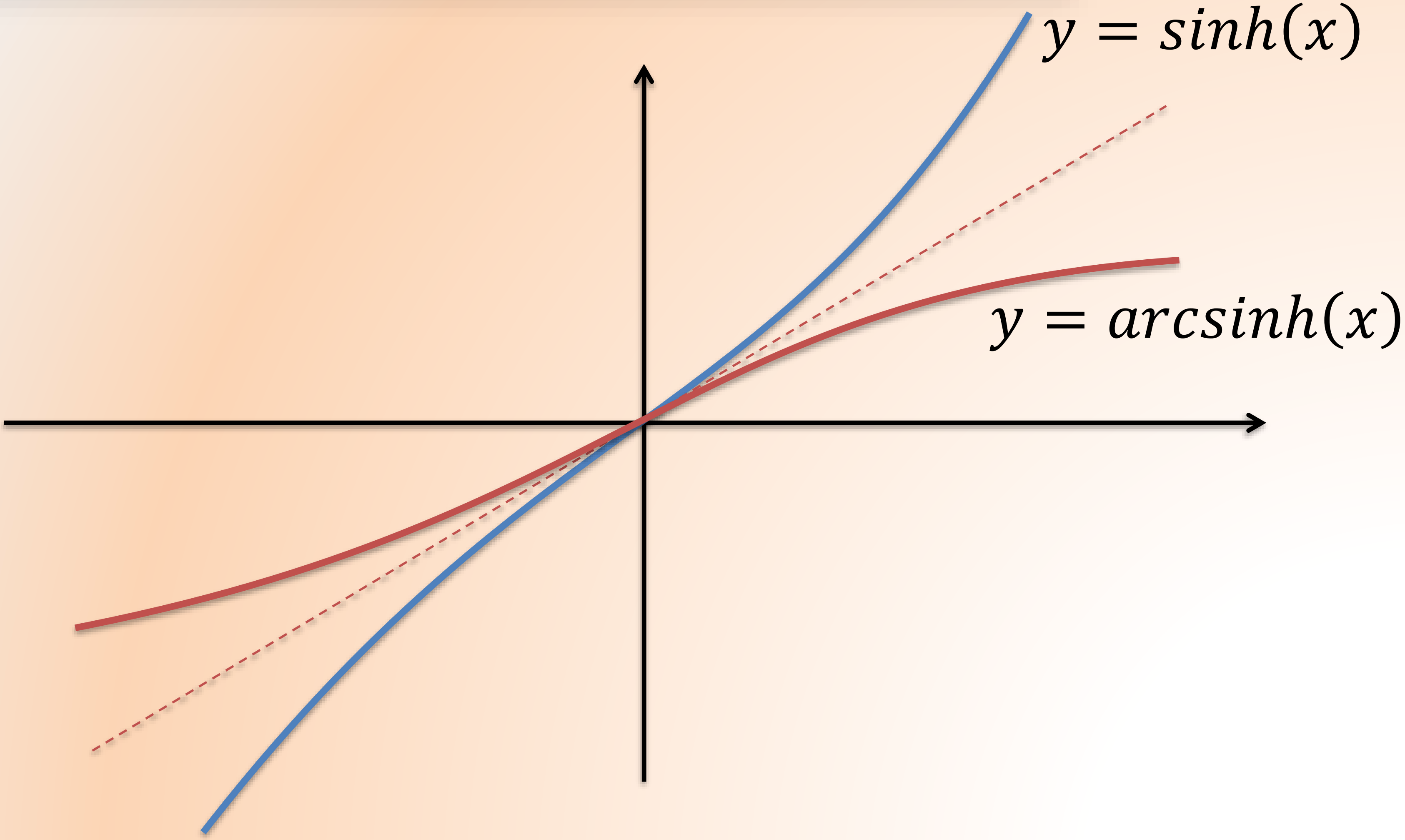
Hyperbolic



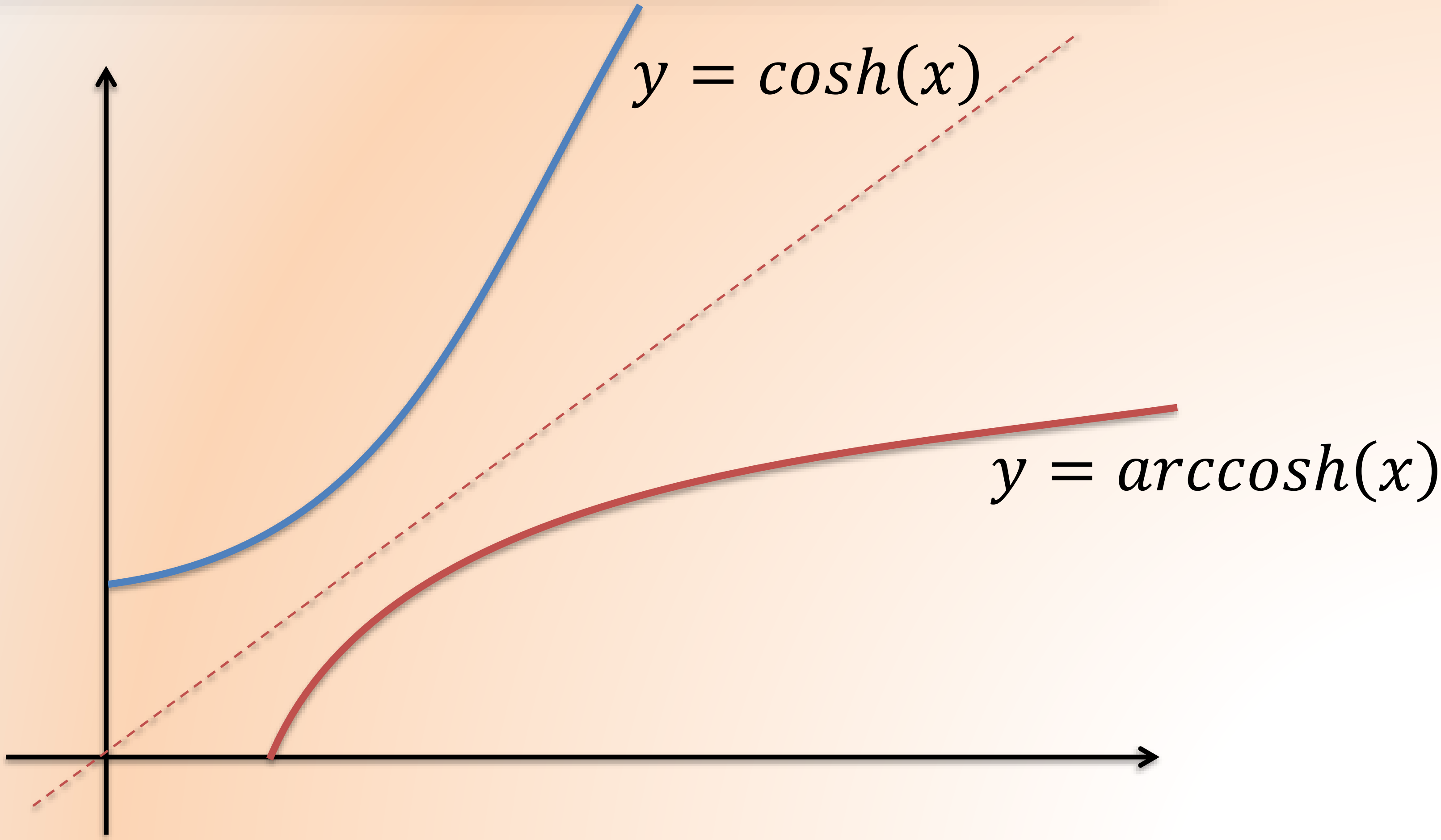
Hyperbolic



Inverse Hyperbolic

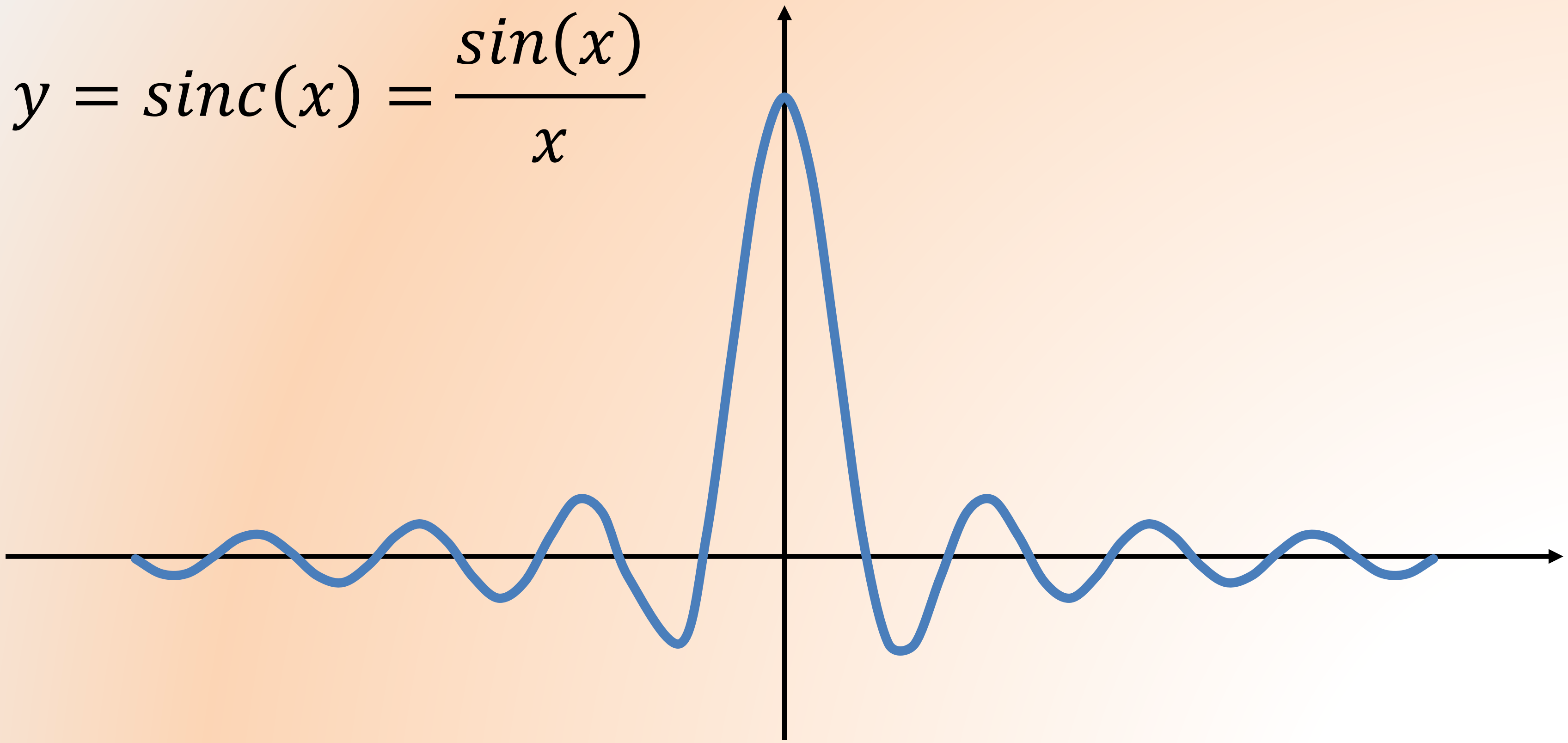


Inverse Hyperbolic



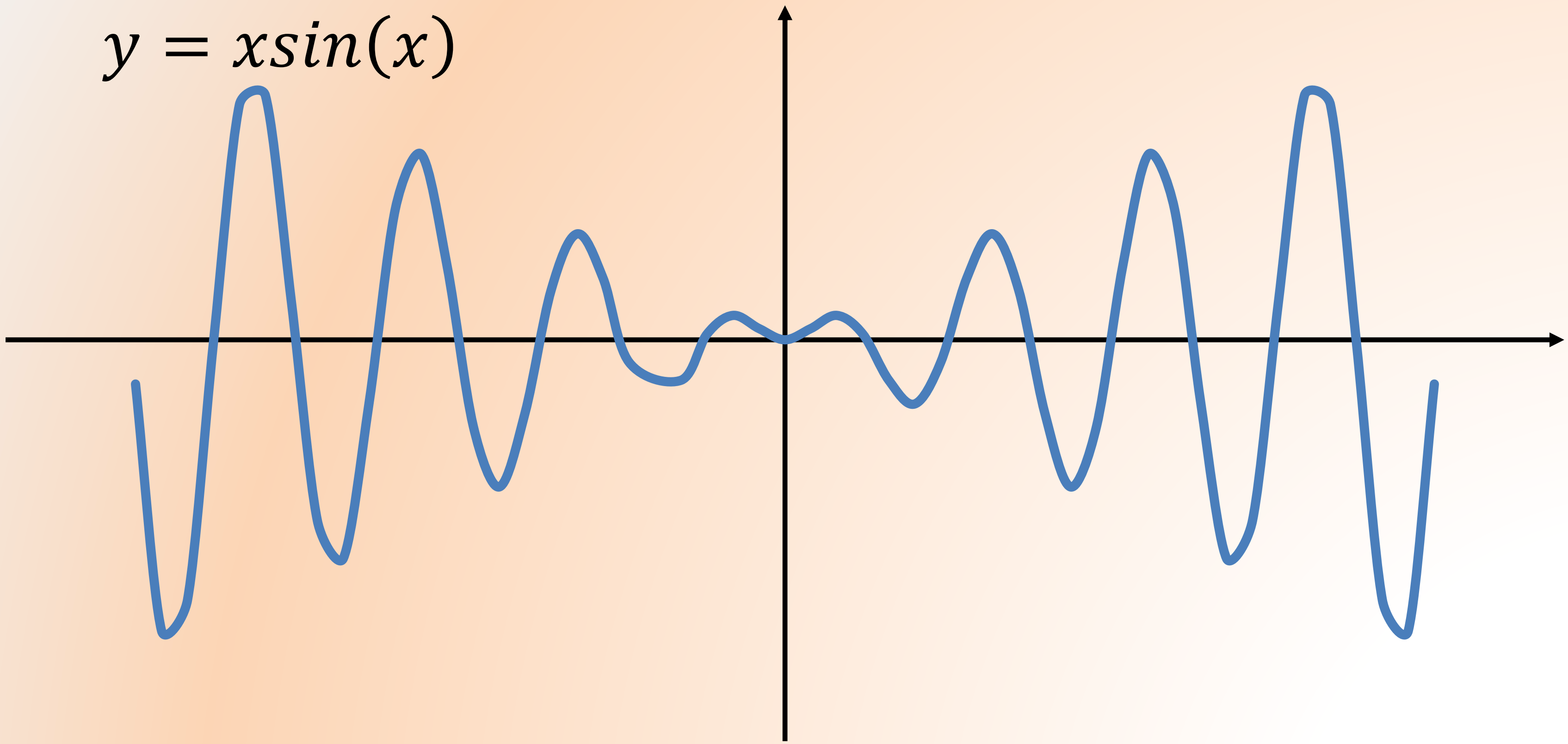
Sinc Function

$$y = sinc(x) = \frac{sin(x)}{x}$$



xSin(x)

$$y = xsin(x)$$



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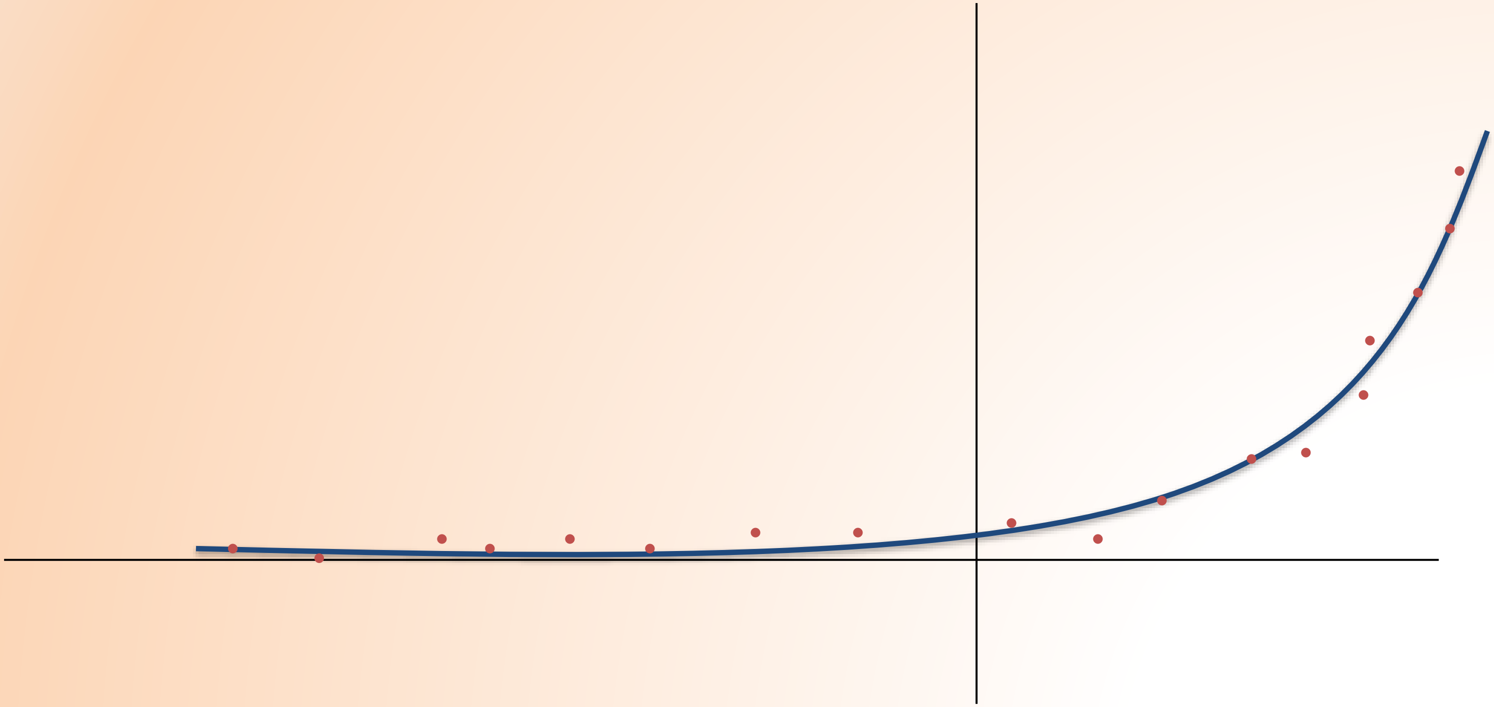
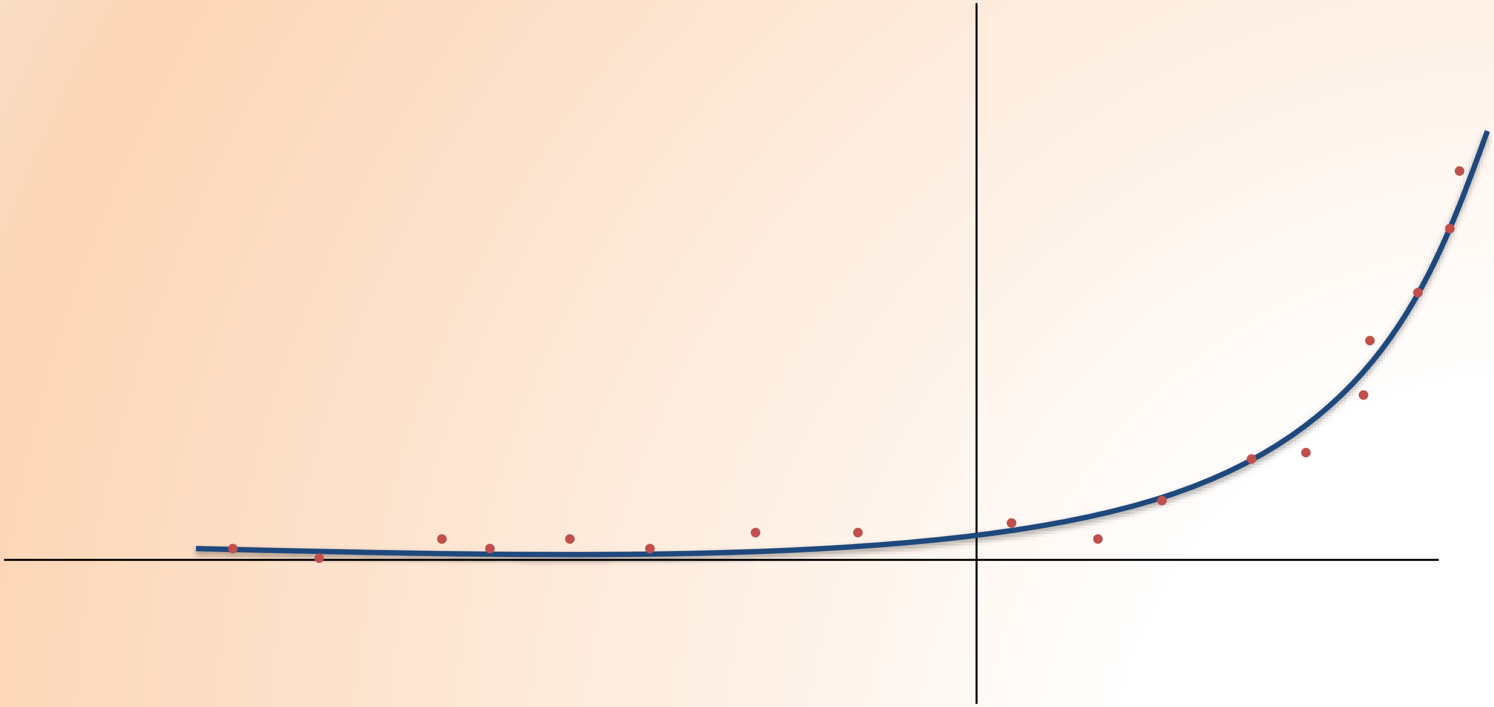
Exponential Fit

$$y = ae^{bx}$$



Exponential Fit

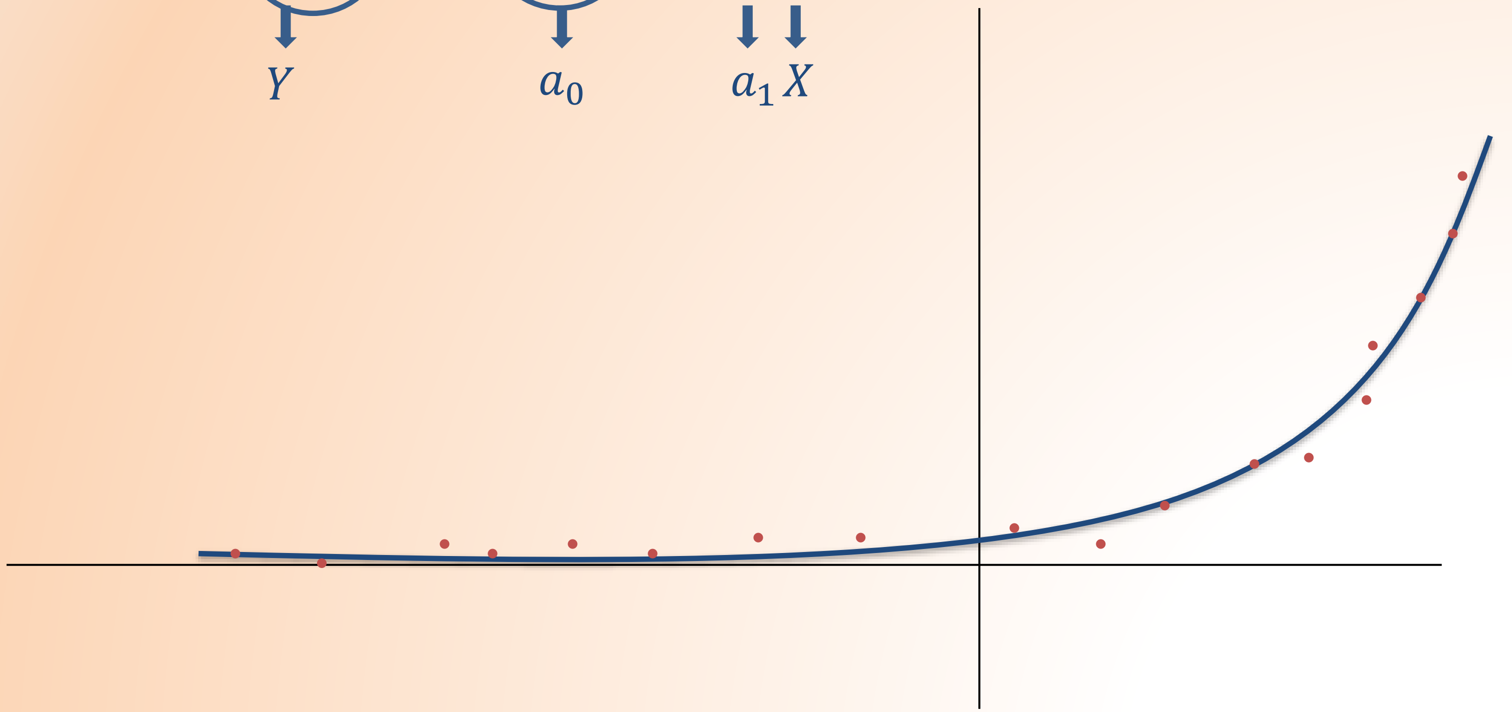
$$\boldsymbol{y} = ae^{bx} \;\; \rightarrow \ln y = \ln a + bx$$



Exponential Fit

$$y = ae^{bx} \rightarrow \ln y = \ln a + bx$$

$$\underbrace{\ln y}_{Y} = \underbrace{\ln a}_{a_0} + \underbrace{b}_{a_1}\underbrace{x}_{X}$$



An Example

Fit the best exponential curve (ae^{bx}) to the tabular data below:

x_i	**0**	**1**	**2**
y_i	1	2	6

Solution:

$$y = ae^{bx} \quad \Rightarrow lny = lna + bx$$

$$Y = a_0 + a_1 X \Longrightarrow \begin{cases} Y = lny \\ a_0 = lna \\ a_1 = b \\ X = x \end{cases}$$

$$\begin{cases} a_0(n+1) + a_1\left(\sum_{i=0}^{n} x_i\right) = \sum_{i=0}^{n} y_i \\ a_0\left(\sum_{i=0}^{n} x_i\right) + a_1\left(\sum_{i=0}^{n} {x_i}^2\right) = \sum_{i=0}^{n} x_i y_i \end{cases}$$

An Example

x_i	y_i	$X_i = x_i$	$Y_i = lny_i$	X_i^2	X_iY_i
0	1	0	0	0	0
1	2	1	0.69	1	0.69
2	6	2	1.79	4	3.58
		$S_X = \mathbf{3}$	$S_Y = \mathbf{2.48}$	$S_{X^2} = \mathbf{5}$	$S_{XY} = \mathbf{4.27}$

$$\begin{cases} a_0(n+1) + a_1\left(\sum_{i=0}^{n} x_i\right) = \sum_{i=0}^{n} y_i \\ a_0\left(\sum_{i=0}^{n} x_i\right) + a_1\left(\sum_{i=0}^{n} {x_i}^2\right) = \sum_{i=0}^{n} x_i y_i \end{cases}$$

An Example

x_i	y_i	$X_i = x_i$	$Y_i = lny_i$	X_i^2	X_iY_i
0	1	0	0	0	0
1	2	1	0.69	1	0.69
2	6	2	1.79	4	3.58
		$S_X = 3$	$S_Y = 2.48$	$S_{X^2} = 5$	$S_{XY} = 4.27$

$$\Rightarrow \begin{cases} 3a_0 + 3a_1 = 2.48 \\ 3a_0 + 5a_1 = 4.27 \end{cases} \Rightarrow \begin{cases} a_0 = -0.07 \\ a_1 = 0.9 \end{cases}$$

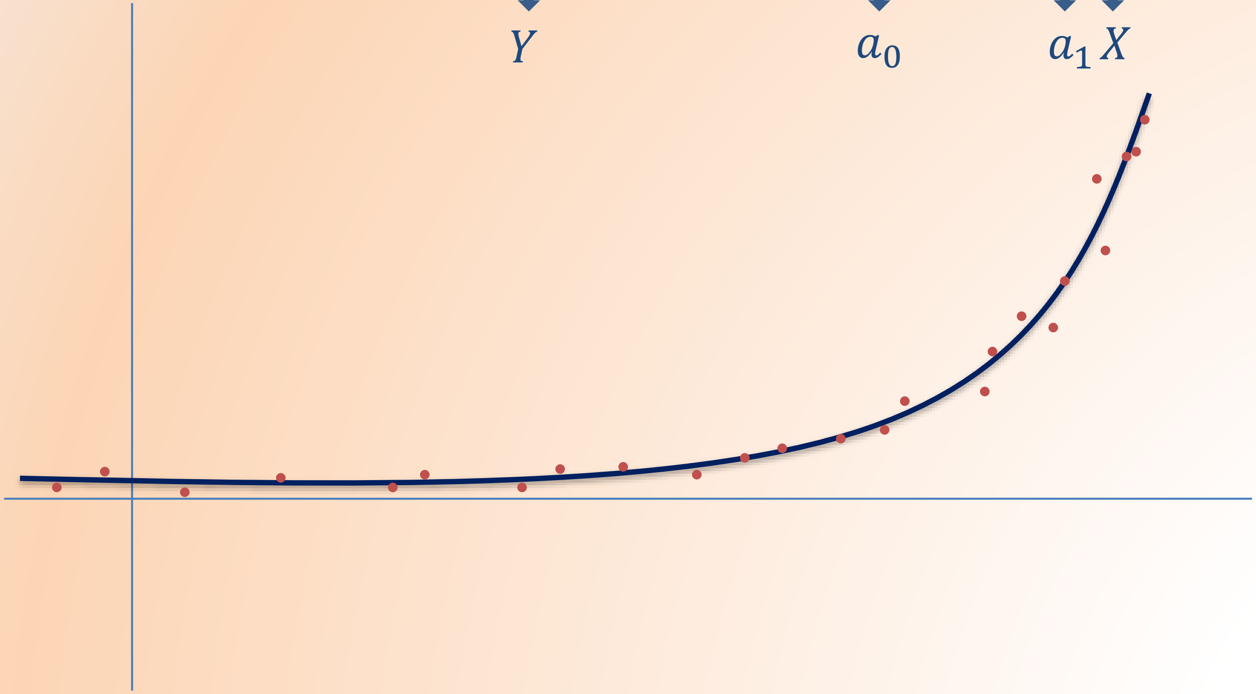
$$Y = a_0 + a_1 X \quad \Rightarrow \begin{cases} Y = lny \\ a_0 = lna \\ a_1 = b \\ X = x \end{cases}$$

$$\Rightarrow \begin{cases} b = a_1 = 0.9 \\ a = e^{a_0} = e^{-0.07} = 0.93 \end{cases} \Rightarrow y = 0.93e^{0.9x}$$

Linearization

$$y = \alpha a^{\beta x} \rightarrow \log_a y = log_a^{\alpha} + \beta x$$

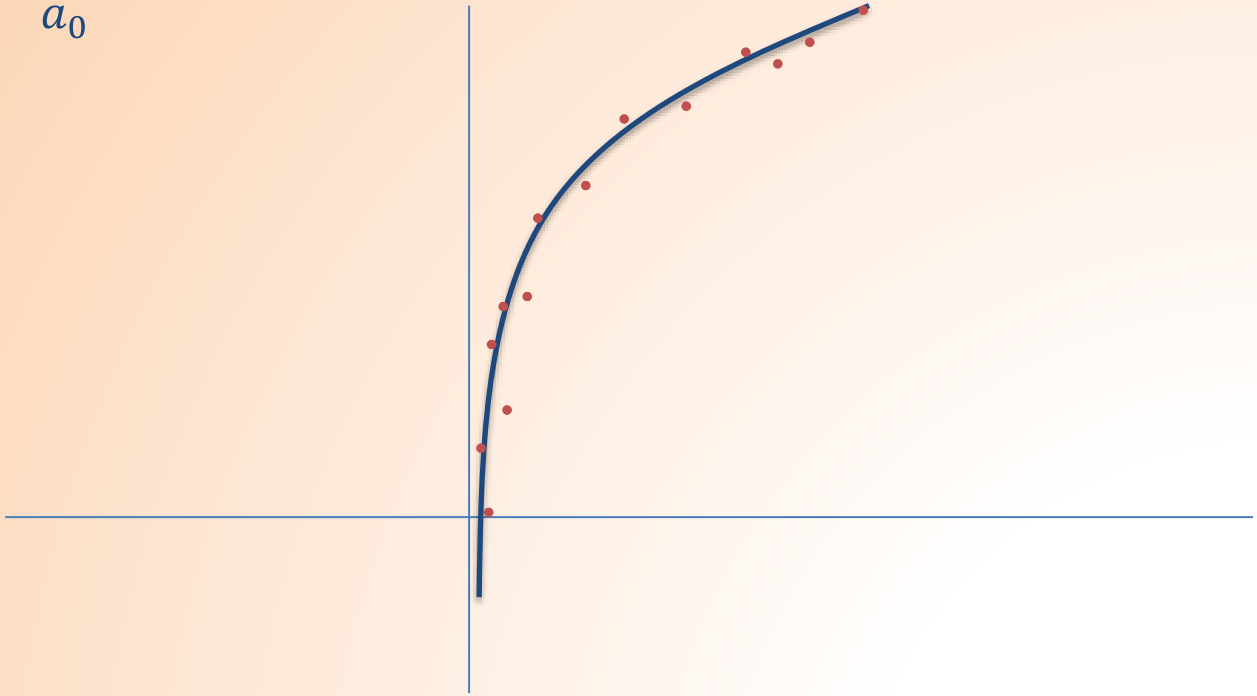
Y a_0 a_1 X



Linearization

$$y = a \ln x + b$$

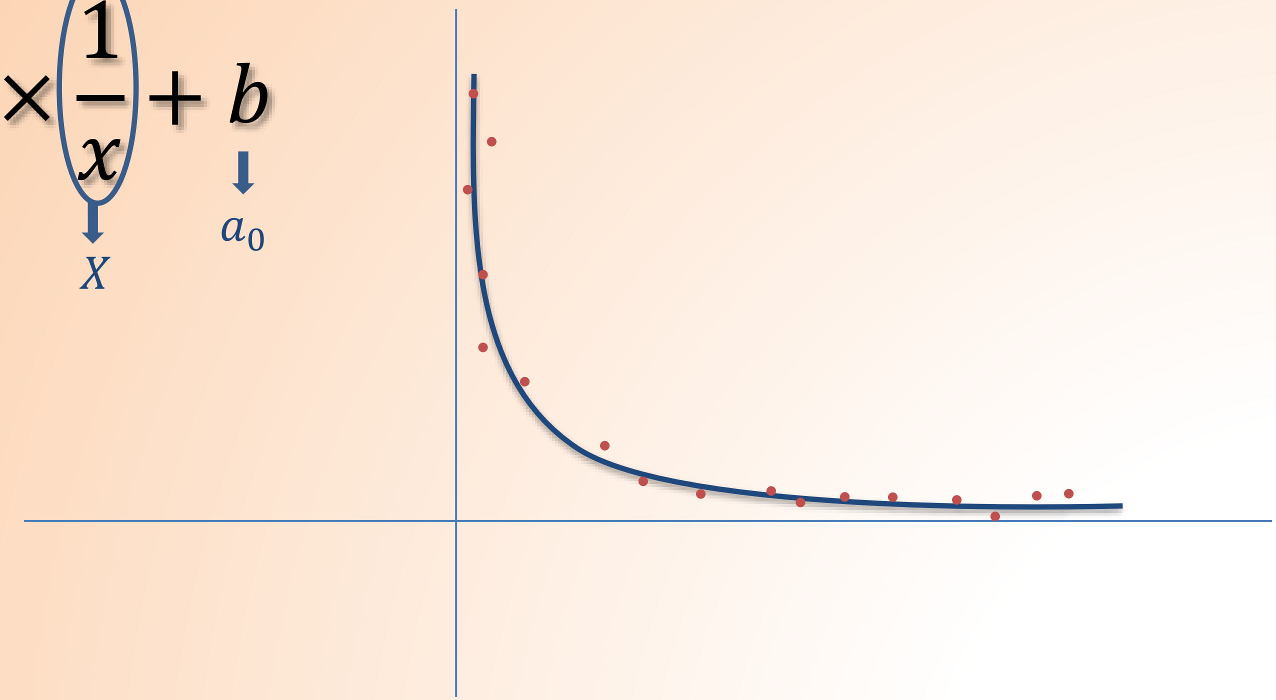
$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ Y & a_1 & X & a_0 \end{array}$$



Linearization

$$y = \frac{a}{x} + b$$

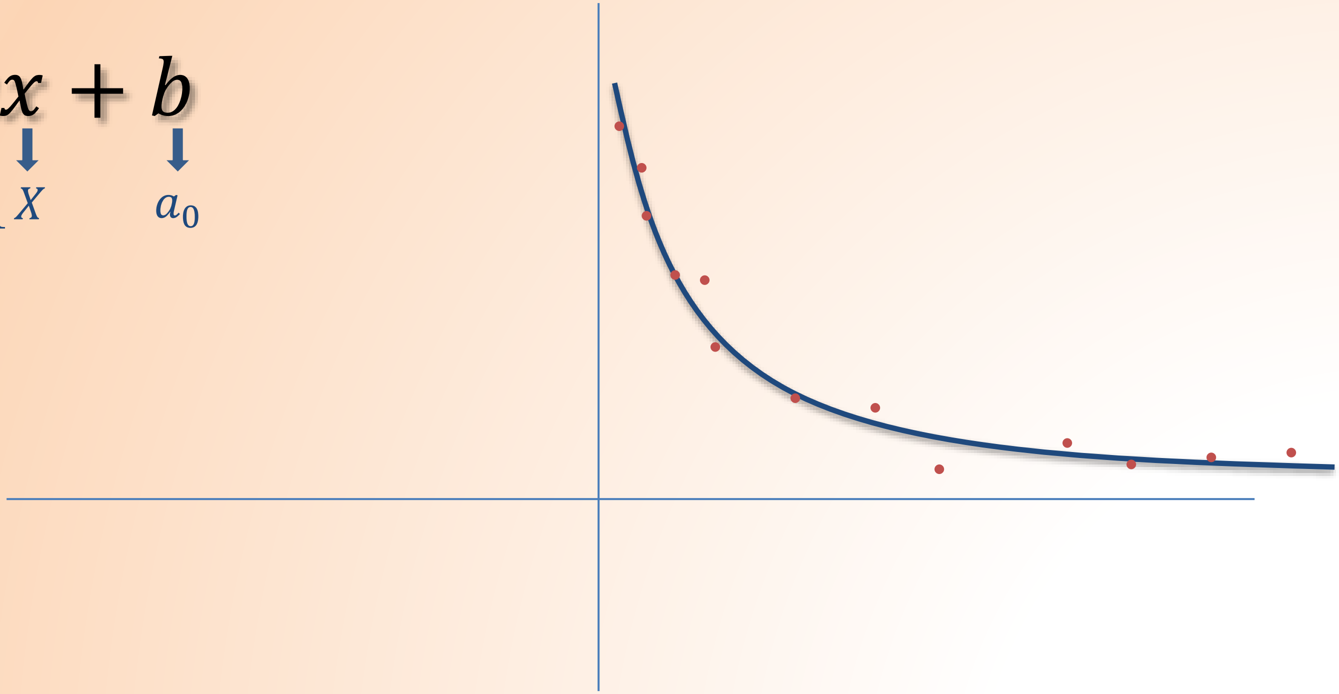
$$\rightarrow \underbrace{y}_{Y} = \underbrace{a}_{a_1} \times \underbrace{\frac{1}{x}}_{X} + \underbrace{b}_{a_0}$$



Linearization

$$y = \frac{1}{ax+b}$$

$$\rightarrow \underbrace{\frac{1}{y}}_{Y} = \underbrace{a}_{a_1}\underbrace{x}_{X} + \underbrace{b}_{a_0}$$

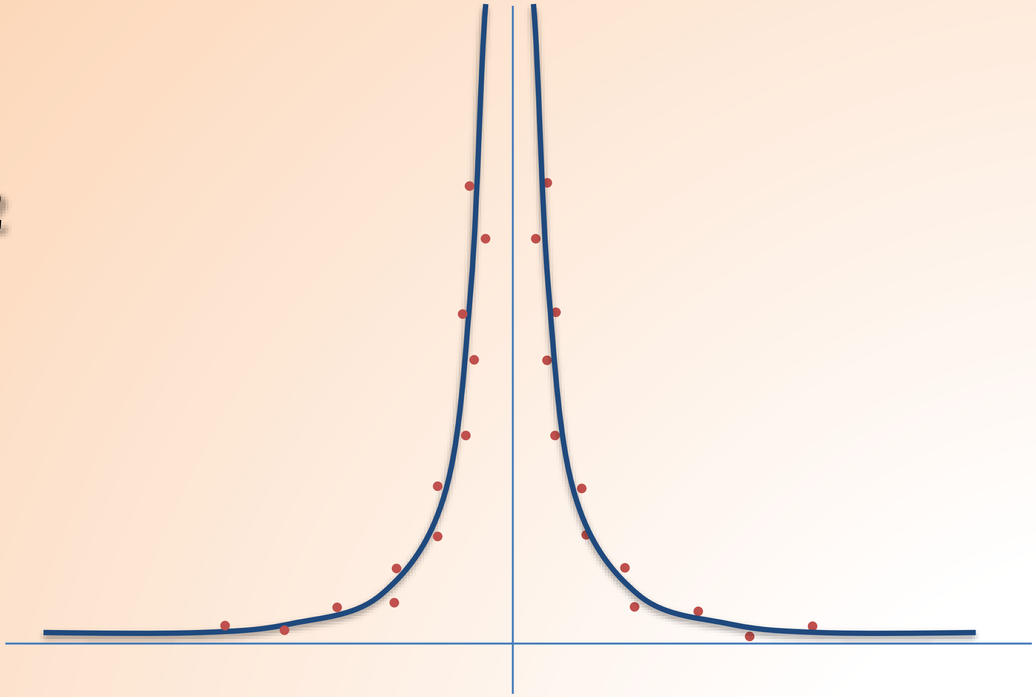


Linearization

$$y = \frac{1}{(ax+b)^2}$$

$$\rightarrow \frac{1}{y} = (ax+b)^2$$

$$\rightarrow \underbrace{\sqrt{\frac{1}{y}}}_{Y} = \underbrace{a}_{a_1}\underbrace{x}_{X} + \underbrace{b}_{a_0}$$



Least Squares Fit

$$F(x) = a_0 f_0 + a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$$

Consider number of (x_i, y_i) is m:

$$\mathrm{E} = \sum_{i=1}^{m} [y_i - F(x_i)]^2$$

$$\frac{\partial E}{\partial a_i} = 0 \quad i = 0,1, \dots, n$$

Least Squares Fit

$$
\begin{bmatrix}
\sum_{i=1}^{m} f_0 f_0 & \sum_{i=1}^{m} f_0 f_1 & \cdots & \sum_{i=1}^{m} f_0 f_n \\
\sum_{i=1}^{m} f_1 f_0 & \sum_{i=1}^{m} f_1 f_1 & \cdots & \sum_{i=1}^{m} f_1 f_n \\
\vdots & \vdots & & \vdots \\
\sum_{i=1}^{m} f_n f_0 & \sum_{i=1}^{m} f_n f_1 & \cdots & \sum_{i=1}^{m} f_n f_n
\end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}
=
\begin{bmatrix}
\sum_{i=1}^{m} f_0 y_i \\
\sum_{i=1}^{m} f_1 y_i \\
\vdots \\
\sum_{i=1}^{m} f_n y_i
\end{bmatrix}
$$

با اســـتفاده از داده‌های جدولی زیر یک تابع به صـــورت $g(x)=a\operatorname{Sin}x+b\operatorname{Cos}x$ به روش کمترین مربعات برازش کنید.

x_i	۰/۰۵	۰/۱	۰/۱۵	۰/۲۰
f_i	۰/۵۲۹۴	۰/۹۴۱۵	۱/۱۴۷۵	۱/۱۰۹۳

حل:

$$y=a\operatorname{Sin}x+b\operatorname{Cos}x \Rightarrow D=\sum(y-a\operatorname{Sin}x-b\operatorname{Cos}x)^2$$

$$\begin{cases}\dfrac{\partial D}{\partial a}=0\Rightarrow -2\sum \operatorname{Sin}x(y-a\operatorname{Sin}x-b\operatorname{Cos}x)=0\Rightarrow \sum y\operatorname{Sin}x-\sum a\operatorname{Sin}^2x-\sum b\operatorname{Sin}x\operatorname{Cos}x=0\\ \dfrac{\partial D}{\partial b}=0\Rightarrow -2\sum \operatorname{Cos}x(y-a\operatorname{Sin}x-b\operatorname{Cos}x)=0\Rightarrow \sum y\operatorname{Cos}x-\sum a\operatorname{Sin}x\operatorname{Cos}x-\sum b\operatorname{Cos}^2x=0\end{cases}$$

$$\Rightarrow\begin{cases}a\sum \operatorname{Sin}^2x+b\sum \operatorname{Sin}x\operatorname{Cos}x=\sum y\operatorname{Sin}x\\ a\sum \operatorname{Sin}x\operatorname{Cos}x+b\sum \operatorname{Cos}^2x=\sum y\operatorname{Cos}x\end{cases}$$

اکنون جدول زیر را تشکیل می‌دهیم:

	x_i	y_i	Sin^2x_i	$\operatorname{Sin}x_i\operatorname{Cos}x_i$	Cos^2x_i	$y_i\operatorname{Sin}x_i$	$y_i\operatorname{Cos}x_i$
	۰/۰۵	۰/۵۲۹۴	۰/۰۰۲۵	۰/۰۴۹۹	۰/۹۹۷۵	۰/۰۲۶۵	۰/۵۲۸۷
	۰/۱	۰/۹۴۱۵	۰/۰۱۰۰	۰/۰۹۹۳	۰/۹۹۰۰	۰/۰۹۴۰	۰/۹۳۶۸
	۰/۱۵	۱/۱۴۷۵	۰/۰۲۲۳	۰/۱۴۷۸	۰/۹۷۷۷	۰/۱۷۱۵	۱/۱۳۴۶
	۰/۲۰	۱/۱۰۹۳	۰/۰۳۹۵	۰/۱۹۴۷	۰/۹۶۰۵	۰/۲۲۰۴	۱/۰۸۷۲
$\sum$	–	–	۰/۰۷۴۳	۰/۴۹۱۷	۳/۹۲۵۷	۰/۵۱۲۴	۳/۶۸۷۳

حال اعداد به دست آمده را در دستگاه قرار می‌دهیم:

$$\begin{cases}a(0/0743)+b(0/4917)=0/5124\\ a(0/4917)+b(3/9257)=3/6873\end{cases}\Rightarrow a=3/9768,\ b=0/4412$$

با قرار دادن در فرم $y=a\operatorname{Sin}x+b\operatorname{Cos}x$، خواهیم داشت:

$$y=3/9768\operatorname{Sin}x+0/4412\operatorname{Cos}x$$


Any questions?