شماره سوال: ۱

$$P_2(x) = P_2(x_0+sh) = f_0 + s(f_1-f_0) + \frac{s(s-1)}{2}(f_2-2f_1+f_0)$$

$$f(x) = P_2(x) + E_2(x)$$

$$E_2(x) = E_2(x_0+sh) = \frac{f^{(3)}(\eta(x))}{3!}(sh)((s-1)h)((s-2)h) \qquad x = x_0+sh$$

$$\frac{dP_2(x)}{dx} = \frac{dP_2}{ds}(s) \times \frac{ds}{dx} = \frac{1}{h}\left[f_1-f_0 + \frac{2s-1}{2}(f_2-2f_1+f_0)\right]$$

$$f'(x_0) \simeq \frac{dP_2(x_0)}{dx} = \frac{1}{h}\left[f_1-f_0 - 1/2\,(f_2-2f_1+f_0)\right]$$

$$= \frac{1}{2h}(-3f_0+4f_1-f_2)$$ الف)

ب)
$$E_2(x) = \frac{f^{(3)}(\eta(x_0+sh))}{3!}\, s(s-1)(s-2)h^3$$

$$\frac{dE_2(x)}{dx} = \frac{dE_2}{ds} \times \frac{ds}{dx} = \frac{1}{3!h}\Big(\frac{d}{ds} f^{(3)}(\eta(x_0+sh))\; s(s-1)(s-2)h^3$$

$$+ f^{(3)}(\eta(x_0+sh))\, h^3\big((s-1)(s-2)+s(s-2)+s(s-1)\big)\Big)$$

$$\Rightarrow \frac{dE_2(x_0)}{dx} = \frac{1}{3!h}\left(h^3 f^{(3)}(\eta(x_0))(2)\right) = \frac{h^2}{3} f^{(3)}(\eta(x_0))$$

خطای تعیین از مرتبه $O(h^2)$ است.

الف)

$$\overbrace{a \qquad\qquad b}^{h} \qquad h = b - a$$

$$\alpha_1 I = \alpha_1 F_1(h) + \alpha_1 C_1 h^P + O(h^{P+1})$$

$$\alpha_2 I = \alpha_2 F_2(h) + \alpha_2 C_1 h^P + O(h^{P+1})$$

$$(\alpha_1 + \alpha_2) I = \alpha_1 F_1(h) + \alpha_2 F_2(h) + (\alpha_1 C_1 + \alpha_2 C_2) h^P + O(h^{P+1})$$

$$I = \frac{\alpha_1}{\alpha_1 + \alpha_2} F_1(h) + \frac{\alpha_2}{\alpha_1 + \alpha_2} F_2(h) + \frac{\alpha_1 C_1 + \alpha_2 C_2}{\alpha_1 + \alpha_2} h^P + O(h^{P+1})$$

If $\frac{\alpha_1 C_1 + \alpha_2 C_2}{\alpha_1 + \alpha_2} = 0$, then $I = \frac{\alpha_1 F_1(h) + \alpha_2 F_2(h)}{\alpha_1 + \alpha_2} + O(h^{P+1})$

$$\alpha_1 C_1 + \alpha_2 C_2 = 0 \qquad \alpha_1 = C_2 \ , \ \alpha_2 = -C_1$$

کافیه دو تا رو کم کنیم

$$C_2 I = C_2 F_1(h) + C_2 C_1 h^P + O(h^{P+1})$$

$$C_1 I = C_1 F_2(h) + C_1 C_2 h^P + O(h^{P+1})$$

$$(C_2 - C_1) I = C_2 F_1(h) - C_1 F_2(h) + O(h^{P+1})$$

$$I = \frac{C_2}{C_2 - C_1} F_1(h) - \frac{C_1}{C_2 - C_1} F_2(h) + O(h^{P+1})$$

ب)

ذوزنقه‌ای: $I_A = \frac{b-a}{2}[f(a) + f(b)] \qquad E = -\frac{(b-a)^3}{12} f''(\theta)$

مستطیلی: $I_A = (b-a) f\left(\frac{a+b}{2}\right) \qquad E = \frac{(b-a)^3}{24} f''(\theta)$

$$f(x)=f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)+\frac{1}{2}f''\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)^2$$

$$+\frac{1}{3!}f^{(3)}\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)^3+\frac{1}{4!}f^{(4)}(\theta_x)\left(x-\frac{a+b}{2}\right)^4$$

$$I=\int_a^b f(x)dx=\int_a^b f\left(\frac{a+b}{2}\right)dx+\underline{f'\left(\frac{a+b}{2}\right)\int_a^b\left(x-\frac{a+b}{2}\right)dx}+$$

$$\frac{1}{2!}f''\left(\frac{a+b}{2}\right)\int_a^b\left(x-\left(\frac{a+b}{2}\right)\right)^2dx+\underline{\frac{1}{3!}f^{(3)}\left(\frac{a+b}{2}\right)\int_a^b\left(x-\frac{a+b}{2}\right)^3dx}$$

$$+\frac{1}{4!}\int_a^b f^{(4)}(\theta_x)\left(x-\frac{a+b}{2}\right)^4dx$$

$$=(b-a)f\left(\frac{a+b}{2}\right)+\frac{1}{2!}\frac{1}{3}f''\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)^3\Bigg|_a^b$$

$$+\frac{1}{4!}\frac{1}{5}f^{(4)}(\theta_x)\left(x+\frac{a+b}{2}\right)^5\Bigg|_a^b$$

$$\Rightarrow I=(b-a)f\left(\frac{a+b}{2}\right)+\frac{(b-a)^3}{24}f''\left(\frac{a+b}{2}\right)+O(h^5)$$

$$C_1=\frac{1}{24}f''\left(\frac{a+b}{2}\right)$$ که رسید

چندجمله‌ای درون‌یاب را در نقاط $a, b, \frac{a+b}{2}$:

$$f_2(x)=f(a)+f[a,b](x-a)+f[a,b,\tfrac{a+b}{2}](x-a)(x-b)$$

$$+f[a,b,\tfrac{a+b}{2},\tfrac{2(a+b)}{3}](x-a)(x-b)\left(x-\frac{a+b}{2}\right)$$

$$+f[a,b,\tfrac{a+b}{2},\tfrac{2(a+b)}{3}](x-a)(x-b)\left(x-\frac{a+b}{2}\right)\left(x-\tfrac{2}{3}(a+b)\right)$$

$$\int_a^b p(x)\,dx = (b-a)f(a) + \frac{f[a,b]}{2}(b-a)^2 + \tfrac{1}{6} f[a,b,\tfrac{a+b}{2}](a-b)^3$$

$$+ f[a,b,\tfrac{a+b}{2},\tfrac{2}{3}(a+b)] \cancel{\int_a^b (x-a)(x-b)(x-\tfrac{a+b}{2})\,dx}^{\,0}$$

$$+ \int f[a,b,\tfrac{a+b}{2},\tfrac{2}{3}(a+b)](x-a)(x-b)(x-\tfrac{a+b}{2})(x-\tfrac{2}{3}(a+b))\,dx$$

$$I = (b-a)\frac{f(a)+f(b)}{2} + -\frac{f[a,b,\frac{a+b}{2}]}{6}(\underbrace{b-a}_{h})^3 + O(h^5)$$

$$C_2 = -\frac{f[a,b,\frac{a+b}{2}]}{6}$$ ترم خطا

$$I_A = \frac{-24}{4f[a,b,\frac{a+b}{2}] - f''(\frac{a+b}{2})}\left[-\frac{f[a,b,\frac{a+b}{2}]}{6}(b-a)f(\tfrac{a+b}{2}) + \frac{1}{24}f''(\tfrac{a+b}{2})(b-a)\frac{f(a)+f(b)}{2}\right]$$

۲) روش سیمپسون

$$I_A = \frac{b-a}{6}\left[f(a) + 4f(\tfrac{a+b}{2}) + f(b)\right]$$

$$E = -\frac{(b-a)^5}{2880} f^{(4)}(\theta)$$

[illegible]

$$I_A = \overbrace{\frac{24 f[a,b,\frac{a+b}{2}]}{4f[a,b,\frac{a+b}{2}] - f''(\frac{a+b}{2})}}^{4/6}(b-a)f(\tfrac{a+b}{2}) + \underbrace{\frac{f''(\frac{a+b}{2})}{4f[a,b,\frac{a+b}{2}] - f''(\frac{a+b}{2})}}_{1/6}(b-a)(f(a)+f(b))$$

$$4f[a,b,\frac{a+b}{2}] = 4f[a,b,\frac{a+b}{2}] - f''(\frac{a+b}{2})$$

$$6f''(\frac{a+b}{2}) = 4f[a,b,\frac{a+b}{2}] - f''(\frac{a+b}{2})$$

$$\Rightarrow \boxed{12f[a,b,\frac{a+b}{2}] = 3f''(\frac{a+b}{2})}$$

x^* نقطه ثابت تابع g است $\Rightarrow$ $g(x^*) = x^*$

$$g(x) = g(x^*) + g'(x^*)(x - x^*) + g''(x^*)(x - x^*)^2 + \cdots + \frac{g^{(p-1)}(x^*)}{(p-1)!}(x - x^*)^{p-1} + \frac{g^{(p)}(\eta(x))}{p!}(x - x^*)^p$$

$$g'(x^*) = \cdots = g^{(p-1)}(x^*) = 0 \Rightarrow g(x) = g(x^*) + \frac{g^{(p)}(\eta(x^*))}{p!}(x - x^*)^p$$

با توجه به دنباله $\{x^i\}_{i=0}^{\infty}$ که داریم: $x_{n+1} = g(x_n)$ ؛

$$x^{n+1} = g(x^n) = x^* + \frac{g^{(p)}(\eta(x^*))}{p!}(x^n - x^*)^p$$

$$|x^{n+1} - x^*| = \frac{1}{p!}\left|g^{(p)}(\eta(x^*))\right| |x^n - x^*|^p$$

$$\Rightarrow \frac{|x^{n+1} - x^*|}{|x^n - x^*|^p} = \frac{1}{p!}\left|g^{(p)}(\eta(x^n))\right|$$

چون $g^{(p)}(x)$ در $(x^* - \epsilon, x^* + \epsilon)$ پیوسته است و $\{x^i\}_{i=0}^{\infty} \longrightarrow x^*$ بنابراین

$$\{g^{(p)}(\eta(x^n))\} \longrightarrow \{g^{(p)}(x^*)\}$$

بنابراین

$$\lim_{x^n \to x^*} \frac{|x^{n+1} - x^*|}{|x^n - x^*|^p} = \frac{1}{p!}\left|g^{(p)}(x^*)\right|$$

بنابراین مرتبه همگرایی برابر با p است با ثابت $c = \frac{1}{p!}|g^{(p)}(x^*)|$ ، مرتبه همگرایی p است.

الف)

$$A(\bar{x}+d)=b+r \Rightarrow A\bar{x}+Ad=b+r \Rightarrow Ad=r \Rightarrow d=A^{-1}r$$

$$\Rightarrow \|d\|=\|A^{-1}r\| \le \|A^{-1}\|\|r\|$$

$$A\bar{x}=b \Rightarrow \|b\|=\|A\bar{x}\| \le \|A\|\|x\| \Rightarrow \frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|}$$

$$\Rightarrow \frac{\|d\|}{\|x\|} \le \|A^{-1}\|\|A\|\frac{\|r\|}{\|b\|}$$

ب)

$$e=x-\bar{x} \Rightarrow \underbrace{Ax}_{b}-A\bar{x}=Ae \Rightarrow \underbrace{b-A\bar{x}}_{r}=Ae$$

$$\Rightarrow \begin{cases} Ae=r \\ Ae+A\bar{x}=b \Rightarrow A(\bar{x}+e)=b \end{cases}$$

با استفاده از بسازی قسمت الف):

$$Ax=b$$

$$Ae=b-A\bar{x}=r \qquad \Rightarrow \frac{\|x\|}{\|e\|} \le K(A)\frac{\|b\|}{\|r\|}$$

$$Ae+Ax=A(e+x)=r+b$$

$$\Rightarrow \frac{\|e\|}{\|x\|} \ge \frac{1}{K(A)}\frac{\|r\|}{\|b\|}$$

$$Ae=r \Rightarrow A^{-1}r=e \Rightarrow \|e\|=\|A^{-1}r\| \le \|A^{-1}\|\|r\|$$

$$Ax=b \Rightarrow \|b\| \le \|A\|\|x\|$$

$$\Rightarrow \|e\|\|b\| \le \|A\|\|A^{-1}\|\|r\|\|x\|$$

$$\Rightarrow \frac{\|e\|\|b\|}{\|x\|} \le K(A)\|r\|$$

$$\Rightarrow \frac{\|e\|}{\|x\|} \le K(A)\frac{\|r\|}{\|b\|}$$

$p(x) = y'(x)$

$$\begin{cases} \dfrac{dy}{dx} = p(x) = f_1(x, y, p) \\ \dfrac{dp}{dx} = -xp(x) - x^2 y(x) = f_2(x, y, p) \end{cases} \qquad \begin{aligned} y(0) &= 0 \\ p(0) &= 1 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} y' \\ p' \end{bmatrix} = \begin{bmatrix} p \\ -xp + x^2 y \end{bmatrix}$$

$$\begin{cases} y(0 + 0.1) = y(0) + \dfrac{h}{2}\Big[f_1(0, y(0), p(0)) + f_1\big(0.1, y(0) + h f_1(0, y(0), p(0)), p(0) + h f_2(0, y(0), p(0))\big)\Big] \\ p(0 + 0.1) = p(0) + \dfrac{h}{2}\Big[f_2(0, y(0), p(0)) + f_2\big(0.1, y(0) + h f_1(0, y(0), p(0)), p(0) + h f_2(0, y(0), p(0))\big)\Big] \end{cases}$$

$$y(0.1) = y(0) + \frac{h}{2}\Big[p(0) + p(0) + h(-0 p(0) - 0^2 y(0))\Big] = h/2\,[2] = h = 0.1$$

$$p(0.1) = p(0) + \frac{h}{2}\Big[0 \underbrace{- 0.1(1 + 0)}_{-0.1} \underbrace{- (0.1)^2 (0 + p(0))}_{-(0.1)^2}\Big] = 1 + h/2\,(-0.11)$$

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