Cyclotomic numerical semigroups

Cyclotomic exponent sequences of a numerical semigroup

Pedro A. García Sánchez

Joint work with A. Ciolan, A. Herrera-Poyatos and P. Moree

II Encuentro Conjunto España-Brasil de Matemáticas RSME-SEMA-SBM-SBMAC

Partially supported by the project *Estudio y aplicaciones de semigrupos numéricos y afines* MTM2017-84890-P

UNIVERSIDAD DE GRANADA

Departamento de Álgebra

The Hilbert series

Hilbet series of a numerical semigroup

Let S be a numerical semigroup

The Hilbert series is also known as the generating function of S

$$\mathrm{H}_S(x) = \sum_{s \in S} x^s$$

It is well known that

$$\mathrm{H}_S(x) = \frac{\mathrm{K}_S(x)}{\prod_{a \in A}(1 - x^a)},$$

with A the unique minimal generating system of S, and $\mathrm{K}_S(x)$ a polynomial

The degree of $\mathrm{K}_S(x)$ is $\mathrm{F}(S) + \sum_{a \in A} a$

Euler characteristic and K_S

Let n be an element in the numerical semigroup S minimally generated by A

Define

$$\Delta_n = \{F \subset A : n - \sum_{a \in F} a \in S\}$$

and

$$\chi_S(n) = \sum_{F \in \Delta_n} (-1)^{|F|},$$

the Euler characteristic of Δ_n

Then

$$\mathrm{K}_S(x) = \sum_{s \in S} \chi(s) x^s$$

The coefficients of this polynomial can also be computed with the Betti numbers of the semigroup ring of S

Gluings and Hilbert series

Let S_1 and S_2 be two numerical semigroups, and let a_1, a_2 be positive integers such that

- $a_1 \in S_2$ and it is not a minimal generator
- $a_2 \in S_1$ and it is not a minimal generator
- $\gcd(a_1, a_2) = 1$

The $S = a_1 S_1 + a_2 S_2$ is the gluing of S_1 and S_2 (along $a_1 a_2$)

$$\mathrm{H}_{a_1 S_1 +_{a_1 a_2} a_2 S_2}(x) = (1 - x^{a_1 a_2})\, \mathrm{H}_{S_1}(x^{a_1})\, \mathrm{H}_{S_2}(x^{a_2})$$

Hilbert series of complete intersections

Recall that a numerical semigroup S is a complete intersection if

- it is either $\mathbb{N}$ or
- the gluing of two complete intersection numerical semigroups

If $S = a_1S_1 + a_2S_2$ is a gluing, then the underlying graph of $\Delta_{a_1a_2}$ is not connected

The set of elements with associated nonconnected graph are called the Betti elements of S

$$\mathrm{H}_S(x) = \frac{\prod_{b\in\mathrm{Betti}(S)}(1-x^b)^{\mathrm{nc}(\Delta_b)-1}}{\prod_{i=1}^{e}(1-x^{n_i})}$$

Polynomial associated to a numerical semigroup

Numerical semigroup polynomial

Let S be a numerical semigroup, and let $G = \mathbb{N} \setminus S$ be its set of gaps

Observe that

$$\frac{1}{1-x} = \sum_{n \in \mathbb{N}} x^n = \sum_{g \in G} x^g + \mathrm{H}_S(x)$$

Hence

$$\mathrm{P}_S(x) = 1 + (x-1) \sum_{g \in G} x^g = (1-x)\,\mathrm{H}_S(x)$$

is a polynomial, the polynomial associated to S

Complete intersections and polynomials

If S is a complete intersection, then its associated polynomial has this form

$$\mathrm{P}_S(x) = \frac{(1-x)\prod_{b\in B}(1-x^b)^{m_b}}{\prod_{a\in A}(1-x^a)}$$

Since it has all its roots in the unit circle, by Kronecker's lemma, it is a product of cyclotomic polynomials

We say that a numerical semigroup S is *cyclotomic* if its associated polynomial is a product of cyclotomic polynomials

Every complete intersection numerical semigroup is cyclotomic

Is the converse true?

Is every cyclotomic numerical semigroup a complete intersection

Every cyclotomic numerical semigroup is symmetric

We have some computational evidence: all cyclotomic numerical semigroups with Frobenius number less than 71 are complete intersections

Cyclotomic sequences

Cyclotomic sequences

Let f be a polynomial with integer coefficients such that $f(1) = 1$

There exists $e_d \in \mathbb{Z}$, for every $d \in \mathbb{N} \setminus \{0\}$, such that

$$f = \prod_{d \in \mathbb{N}} (1 - x^d)^{e_d}$$

The exponents e_d can be retreived from the roots of f

For the particular case $f = \mathrm{P}_S$, the sequence $(e_1, e_2, \ldots)$ is known as the *exponent sequence* of S

Finite cyclotomic sequences

We say that the cyclotomic sequence $(e_1, e_2, \ldots)$ is finite it if has only finitely many nonzero entries

A numerical semigroup S is cyclotomic if and only if its cyclotomic sequence is finite

$$\Omega = \{d \in \mathbb{Z}^+ : e_d \neq 0\},\ \Omega^* = \Omega \setminus \{1\}$$
$$\Omega_- = \{d \in \Omega : e_d \leq -1\}$$
$$\Omega_+ = \{d \in \Omega : e_d \geq 1\}\ \Omega_+^* = \Omega_+ \setminus \{1\}$$

$$\mathrm{P}_S(x) = \frac{\prod_{d \in \Omega_+} (1 - x^d)^{e_d}}{\prod_{d \in \Omega_-} (1 - x^d)^{-e_d}}$$

Some examples

For $S = \langle 4, 6, 9 \rangle$,

$$\mathrm{P}_S(x) = x^{12} - x^{11} + x^8 - x^7 + x^6 - x^5 + x^4 - x + 1$$
$$= \frac{(1-x)(1-x^{12})(1-x^{18})}{(1-x^4)(1-x^6)(1-x^9)},$$

and thus the exponent sequence of S is

$$(1, 0, 0, -1, 0, -1, 0, 0, -1, 0, 0, 1, 0, 0, 0, 0, 0, 1)$$

For $S = \langle 3, 5, 7 \rangle$, $\mathrm{P}_S(x) = x^5 - x^4 + x^3 - x + 1$, and the first entries of the exponent sequence look like

$$(1, 0, -1, 0, -1, 0, -1, 0, 0, 1, 0, 1, 0, 1, 0, 0, -1, 0, -1, 0, 0, 1, \dots)$$

These sequences can be computed by using `CyclotomicExponentSequence` of the development version of the GAP package `numericalsgps`.

Cyclotomic sequences and generators

Let S be a numerical semigroup

$$\mathrm{P}_S(x) = \frac{\prod_{d \in \Omega_+} (1 - x^d)^{e_d}}{\prod_{d \in \Omega_-} (1 - x^d)^{-e_d}}$$

- Ω_- generates S
- $\Omega_+^* \subseteq S$
- if a is a minimal generator of S, then $e_a = -1$

Cyclotomic sequences: Betti elements

Let S be a numerical semigroup minimally generated by A and with Betti elements B (the set elements $b \in S$ with Δ_b nonconnected)

For $a, b \in \mathbb{Z}$, write

$$a \leq_S b \text{ if } b - a \in S$$

- $\text{Minimals}_{\leq_S}(B) = \text{Minimals}_{\leq_S}(\Omega^* \setminus A) \subseteq \Omega^*_+$
- the elements in $\Omega^* \setminus A$ have at least two factorizations
- for every $b \in \text{Minimals}_{\leq_S}(B)$, the number of factorizations of b is $e_b + 1$, and every two factorizations have disjoint support

If S is a complete intersection, and $b \in B$, then $e_b + 1$ equals the number of connected components of Δ_b

Forest arranged Betti elements

Forests

Let S be a numerical semigroup minimally generated by A and Betti elements B

For a subset X of S, set $\mathcal{H}(X)$ to be the associated Hasse diagram of $(X, \leq_S)$

Assume that the $\mathcal{H}(\Omega^* \setminus A)$ is a directed forest

- $\Omega^*_+ = \Omega^* \setminus A \subseteq B$
- for every $b \in \Omega^* \setminus A$, $e_b + 1$ equals the number of connected components of Δ_b
- $\Omega_- = A$
- In particular, S is cyclotomic

Forests and complete intersections

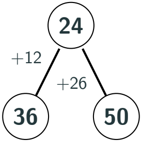
If both $\mathcal{H}(B)$ and $\mathcal{H}(\Omega^* \setminus A)$ are forests, then S is a complete intersection

For $S = \langle 8, 12, 18, 25 \rangle$,

$$\mathrm{P}_S(x) = \frac{(1-x)(1-x^{24})(1-x^{36})(1-x^{50})}{(1-x^{8})(1-x^{12})(1-x^{18})(1-x^{25})}$$

Then $\Omega^* \setminus A = B = \{24, 36, 50\}$

The graph $(B, \leq_S)$ is



In particular if all the elements of $\Omega^* \setminus A$ and B are incomparable via $\leq_S$, then S is a complete intersection

Betti sorted

Let S be a numerical semigroup minimally generated by A and with Betti elements B

Then B is totally ordered with respect to $\leq_S$ (Betti sorted) if and only if $\Omega^* \setminus A$ is totally ordered with respect to $\leq_S$

In this case, S is a complete intersection

A particular case is when B is a singleton, then so is $\Omega^* \setminus A$ (and vice versa)