Tensores y coordenadas:

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Agenda: Tensores y coordenadas

1. Transformaciones, vectores y tensores
2. ¿Cuándo podemos transformar coordenadas?
3. Las componentes contravariantes de un vector transforman....
4. Las componentes covariantes de un vector transforman...
5. Las componentes de un tensor ....
6. Las componentes de contravariantes un tensor transforman ....
7. Las componentes de un tensor transforman..
8. Recapitulando

Puntos y Coordenadas

Consideremos un determinado punto, P, expresado en un sistema de coordenadas particular: $\left(x^1, x^2, \cdots, x^n\right)$ y las coordenadas de ese mismo punto P, expresado en otro sistema de coordenadas $\left(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n\right)$. Ambas representaciones coordenadas de P estarán relacionadas por:

$$\left.\begin{array}{c} \tilde{x}^1 = \tilde{x}^1\left(x^1, x^2, \cdots, x^n\right) \\ \tilde{x}^2 = \tilde{x}^2\left(x^1, x^2, \cdots, x^n\right) \\ \vdots \\ \tilde{x}^n = \tilde{x}^n\left(x^1, x^2, \cdots, x^n\right) \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} x^1 = x^1\left(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n\right) \\ x^2 = x^2\left(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n\right) \\ \vdots \\ x^n = x^n\left(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n\right) \end{array}\right.$$

En una notación más compacta lo que tenemos es:

$$\tilde{x}^i = \tilde{x}^i\left(x^j\right) \Longleftrightarrow x^i = x^i\left(\tilde{x}^j\right), \quad \text{con} \quad i, j = 1, 2, 3, \cdots, n. \qquad (1)$$

¿Cuándo podemos transformar coordenadas?

- Las funciones $x^i = x^i\left(\tilde{x}^m\right)$ y $\tilde{x}^j = \tilde{x}^j\left(x^m\right)$ sean al menos $\mathcal{C}^2$ (función y derivada continua)

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- Que el determinante de la matriz jacobiana sean finito y distinto de cero, esto es

$$\det\left|\frac{\partial x^i\left(\tilde{x}^m\right)}{\partial \tilde{x}^j}\right| \neq 0 \Rightarrow \begin{vmatrix} \frac{\partial x^1}{\partial \tilde{x}^1} & \frac{\partial x^1}{\partial \tilde{x}^2} & \cdots & \frac{\partial x^1}{\partial \tilde{x}^n} \\ \frac{\partial x^2}{\partial \tilde{x}^1} & \frac{\partial x^2}{\partial \tilde{x}^2} & \cdots & \frac{\partial x^2}{\partial \tilde{x}^n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x^n}{\partial \tilde{x}^1} & \frac{\partial x^n}{\partial \tilde{x}^2} & \cdots & \frac{\partial x^n}{\partial \tilde{x}^n} \end{vmatrix} \neq 0 \Rightarrow$$

$$x^i = x^i\left(\tilde{x}^m\right) \Longleftrightarrow \tilde{x}^j = \tilde{x}^j\left(x^m\right) .$$

Ahora bien, una vez más, derivando y utilizando la regla de la cadena:

$$x^i = x^i\left(\tilde{x}^j\left(x^m\right)\right) \Rightarrow \frac{\partial x^i}{\partial x^m} = \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial \tilde{x}^j}{\partial x^m} = \delta^i_m \Rightarrow \mathrm{d}x^i = \frac{\partial x^i}{\partial \tilde{x}^j}\mathrm{d}\tilde{x}^j .$$

Las componentes contravariantes de un vector transforman....

Un conjunto de cantidades $\left\{a^1, a^2, \cdots, a^n\right\}$ se denominarán componentes *contravariantes* de un vector $|a\rangle \in \mathbf{V}$ en un punto P de coordenadas $\left(x^1, x^2, \cdots, x^n\right)$, si:

1. Dada dos bases ortonormales de vectores coordenados: $\{|\mathrm{e}_1\rangle, |\mathrm{e}_2\rangle, \cdots, |\mathrm{e}_n\rangle\}$ y $\{|\tilde{\mathrm{e}}_1\rangle, |\tilde{\mathrm{e}}_2\rangle, \cdots, |\tilde{\mathrm{e}}_n\rangle\}$, se cumple que:

$$|a\rangle = a^j |\mathrm{e}_j\rangle = \tilde{a}^i |\tilde{\mathrm{e}}_i\rangle \;\Rightarrow\; \left\{ \begin{array}{c} \left\langle \mathrm{e}^j \right| a\rangle = a^j \\ \left\langle \tilde{\mathrm{e}}^i \right| a\rangle = \tilde{a}^i \end{array} \right\} \;\Rightarrow\; \tilde{a}^i = a^j \left\langle \tilde{\mathrm{e}}^i \right. |\mathrm{e}_j\rangle .$$

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$$|a\rangle = a^j |\mathrm{e}_j\rangle = \tilde{a}^i |\tilde{\mathrm{e}}_i\rangle \Rightarrow \left\{ \begin{array}{c} \langle \mathrm{e}^j | a\rangle = a^j \\ \langle \tilde{\mathrm{e}}^i | a\rangle = \tilde{a}^i \end{array} \right\} \Rightarrow \tilde{a}^i = a^j \langle \tilde{\mathrm{e}}^i | \mathrm{e}_j\rangle .$$
2. o equivalentemente, bajo una transformación de coordenadas: $x^i = x^i\left(\tilde{x}^j\right)$, con $i, j = 1, 2, 3, \cdots, n$, estas cantidades transforman como:
$$\tilde{a}^i = \frac{\partial \tilde{x}^i}{\partial x^k} a^k \iff a^i = \frac{\partial x^i}{\partial \tilde{x}^k} \tilde{a}^k , \quad \text{con: } \frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^k}{\partial x^l} = \delta^i_l ,$$
y donde las cantidades $\frac{\partial \tilde{x}^i}{\partial x^k}$ y $\frac{\partial x^i}{\partial \tilde{x}^k}$ deberán ser evaluadas en el punto P.

Un conjunto de cantidades $\{b_1, b_2, \cdots, b_n\}$ se denominarán componentes *covariantes* de un vector $\langle b| \in \mathbf{V}^*$ en un punto P de coordenadas $(x^1, x^2, \cdots, x^n)$, si:

1. Dada dos bases de formas: $\{\langle \mathrm{e}^1|, \langle \mathrm{e}^2|, \cdots, \langle \mathrm{e}^n|\}$ y $\{\langle \tilde{\mathrm{e}}^1|, \langle \tilde{\mathrm{e}}^2|, \cdots, \langle \tilde{\mathrm{e}}^n|\}$ se cumple que:

$$\langle b| = b_j \langle \mathrm{e}^j| = \tilde{b}_i \langle \tilde{\mathrm{e}}^i| \Rightarrow \left\{ \begin{array}{c} \langle b| \mathrm{e}_j\rangle = b_j \\ \langle b| \tilde{\mathrm{e}}_i\rangle = \tilde{b}_i \end{array} \right\} \Rightarrow \tilde{b}_i = b_j \langle \mathrm{e}^j |\tilde{\mathrm{e}}_i\rangle .$$

Un conjunto de cantidades $\{b_1, b_2, \cdots, b_n\}$ se denominarán componentes *covariantes* de un vector $\langle b| \in \mathbf{V}^*$ en un punto P de coordenadas $(x^1, x^2, \cdots, x^n)$, si:

1. Dada dos bases de formas: $\left\{\left\langle \mathrm{e}^1\right|, \left\langle \mathrm{e}^2\right|, \cdots, \left\langle \mathrm{e}^n\right|\right\}$ y $\left\{\left\langle \tilde{\mathrm{e}}^1\right|, \left\langle \tilde{\mathrm{e}}^2\right|, \cdots, \left\langle \tilde{\mathrm{e}}^n\right|\right\}$ se cumple que:

$$\langle b| = b_j \left\langle \mathrm{e}^j\right| = \tilde{b}_i \left\langle \tilde{\mathrm{e}}^i\right| \;\Rightarrow\; \left\{ \begin{array}{c} \langle b| \mathrm{e}_j\rangle = b_j \\ \langle b| \tilde{\mathrm{e}}_i\rangle = \tilde{b}_i \end{array} \right\} \;\Rightarrow\; \tilde{b}_i = b_j \left\langle \mathrm{e}^j \,|\tilde{\mathrm{e}}_i\right\rangle .$$

2. o equivalentemente, bajo una transformación de coordenadas $x^i = x^i\left(\tilde{x}^j\right)$ (con $i, j = 1, 2, 3, \cdots, n$) estas cantidades transforman como:

$$\tilde{b}_k = \frac{\partial x^i}{\partial \tilde{x}^k}\, b_i \iff b_k = \frac{\partial \tilde{x}^i}{\partial x^k}\, \tilde{b}_i \,, \text{ con: } \frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^k}{\partial x^l} = \delta^i_l \,, \qquad (2)$$

y donde las cantidades: $\frac{\partial \tilde{x}^i}{\partial x^k}$ y $\frac{\partial x^i}{\partial \tilde{x}^k}$ deberán ser evaluadas en el punto P.

Las componentes de un tensor ....

1. $$T^{ij} = \mathcal{T}\left[\begin{array}{ccc} \left\langle \mathrm{e}^{i}(1)\right| & & \left\langle \mathrm{e}^{j}(2)\right| \\ \downarrow & & \downarrow \\ \bullet & , & \bullet \\ & & \end{array}\right] \Leftrightarrow \mathcal{T} = T^{mn} \left|\mathrm{e}_m(1)\right\rangle \bigotimes \left|\mathrm{e}_n(2)\right\rangle$$

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2. $$T_{ij} = \mathcal{T}\left[\begin{array}{ccc} \left|\mathrm{e}_{i}(1)\right\rangle & & \left|\mathrm{e}_{j}(2)\right\rangle \\ \downarrow & & \downarrow \\ \circ & , & \circ \\ & & \end{array}\right] \Leftrightarrow \mathcal{T} = T_{mn}\left\langle \mathrm{e}^{m}(1)\right| \bigotimes \left\langle \mathrm{e}^{n}(2)\right|$$

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2. $$T_{ij} = \mathcal{T}\left[\begin{array}{cc} |e_i(1)\rangle & |e_j(2)\rangle \\ \downarrow & \downarrow \\ \circ & , \quad \circ \\ & \end{array}\right] \Leftrightarrow \mathcal{T} = T_{mn}\,\langle e^m(1)| \bigotimes \langle e^n(2)|$$

3. $$T^i_{\ j} = \mathcal{T}\left[\begin{array}{cc} |e_j(2)\rangle & \langle e^i(1)| \\ \downarrow & \downarrow \\ \circ & ; \quad \bullet \\ & \end{array}\right] \Leftrightarrow \mathcal{T} = T^m_{\ n}\,\langle e^n(2)| \bigotimes |e_m(1)\rangle$$

Las componentes de un tensor ....

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2. $$T_{ij} = \mathcal{T}\begin{bmatrix} |\mathrm{e}_{i}(1)\rangle & |\mathrm{e}_{j}(2)\rangle \\ \downarrow & \downarrow \\ \circ & , \quad \circ \\ & \end{bmatrix} \Leftrightarrow \mathcal{T} = T_{mn}\,\langle \mathrm{e}^m(1)| \bigotimes \langle \mathrm{e}^n(2)|$$

3. $$T^{i}_{\ j} = \mathcal{T}\begin{bmatrix} |\mathrm{e}_{j}(2)\rangle & \langle \mathrm{e}^{i}(1)| \\ \downarrow & \downarrow \\ \circ & ; \quad \bullet \\ & \end{bmatrix} \Leftrightarrow \mathcal{T} = T^{m}_{\ n}\,\langle \mathrm{e}^n(2)| \bigotimes |\mathrm{e}_m(1)\rangle$$

4. $$T_{i}^{\ j} = \mathcal{T}\begin{bmatrix} |\mathrm{e}_{i}(1)\rangle & \langle \mathrm{e}^{j}(2)| \\ \downarrow & \downarrow \\ \circ & ; \quad \bullet \\ & \end{bmatrix} \Leftrightarrow \mathcal{T} = T_{m}^{\ n}\,\langle \mathrm{e}^m(1)| \bigotimes |\mathrm{e}_n(2)\rangle$$

Generalizamos los conceptos anteriores de la siguiente manera. Dado un conjunto bases para las formas diferenciales $\{\langle x^m(1)|, \langle y^n(2)|\}$, hemos definido las componentes *contravariantes* de un tensor:

$$T^{ij} = \mathcal{T}\left[\begin{array}{ccc} \langle \mathrm{e}^j(1)| & & \langle y^j(2)| \\ \downarrow & & \downarrow \\ \bullet & , & \bullet \\ & & \end{array}\right] \Longleftrightarrow \left\{ T^{ij} \right\} \equiv \left\{ T^{11}, T^{12}, \cdots, T^{1n}; T^{21}, T^{22}, \cdots, T^{2} \right.$$

en esta visión, las componentes contravariantes en un punto P de coordenadas $\left(x^1, x^2, \cdots, x^n\right) \Leftrightarrow x^i = x^i\left(\tilde{x}^j\right)$ (con $i, j = 1, 2, 3, \cdots, n$) transforman como:

$$\tilde{T}^{ij} = \frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^m}\, T^{km} \Longleftrightarrow T^{ij} = \frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial x^j}{\partial \tilde{x}^m}\, \tilde{T}^{km}, \text{ con: } \frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^k}{\partial x^l} = \delta^i_l \,,$$

donde $\frac{\partial \tilde{x}^i}{\partial x^k}$ y $\frac{\partial x^i}{\partial \tilde{x}^k}$ deberán ser evaluadas en el punto P.

Las componentes de un tensor transforman

Si $\{|t_i(1)\rangle, \cdots, |v_k(m)\rangle\}$ y $\{\langle x^e(1)|, \cdots, \langle z^g(n)|\}$ son bases para vectores y formas. Las componentes de un tensor:

$$T_{ijk}^{mn} = \mathcal{T}\left[\begin{array}{ccccccccccccccc} |t_i(1)\rangle & & |u_j(2)\rangle & & & & |v_k(m)\rangle & & \langle x^e(1)| & & \langle y^f(2)| & & & & \langle z^g(n)| \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ \circ & , & \circ, & , & \cdots & , & \circ & ; & \bullet & , & \bullet & , & \cdots & , & \bullet \end{array}\right],$$

serán un conjunto de cantidades:
$\left\{T_{1\cdots 1}^{1\cdots 1}, T_{1\cdots 1}^{2\cdots 1}, \cdots, T_{1\cdots 1}^{\cdots 1}, T_{1\cdots 1}^{\tilde{n}\cdots 1}, T_{2\cdots 1}^{\tilde{n}\cdots 1}, \cdots, T_{\tilde{m}\cdots 1}^{1\cdots 1}, \cdots, T_{\tilde{m}\cdots \tilde{m}}^{\tilde{n}\cdots \tilde{n}}\right\}$ que *contravariantes* y *covariantes* respectivamente, de un tensor mixto en un punto P de coordenadas $\left(x^1, \cdots, x^n\right)$.
Bajo una transformación de coordenadas $x^i = x^i\left(\tilde{x}^j\right)$ con $i, j = 1, \cdots, n$ estas cantidades transforman como:

$$\tilde{T}_{e\cdots g}^{i\cdots k} = \frac{\partial \tilde{x}^i}{\partial x^p} \cdots \frac{\partial \tilde{x}^k}{\partial x^q} \frac{\partial x^a}{\partial \tilde{x}^e} \cdots \frac{\partial x^d}{\partial \tilde{x}^g} \; T_{a\cdots d}^{p\cdots q}$$

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¿ Qué vimos ?

- Coordenadas $\tilde{x}^i = \tilde{x}^i\left(x^j\right) \iff x^i = x^i\left(\tilde{x}^j\right)$

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- Vector: componentes **contravariantes**
  $|a\rangle = a^i \left|\mathrm{e}_i\right\rangle = \tilde{a}^i \left|\tilde{\mathrm{e}}_i\right\rangle \Rightarrow \tilde{a}^i = a^j \left\langle\tilde{\mathrm{e}}^i \left|\mathrm{e}_j\right.\right\rangle \Leftrightarrow \tilde{a}^i = \frac{\partial \tilde{x}^i}{\partial x^k}\ a^k$

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- Coordenadas $\tilde{x}^i = \tilde{x}^i\left(x^j\right) \iff x^i = x^i\left(\tilde{x}^j\right)$
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- Formas: componentes **covariantes** $\langle b| = b_i\left\langle \mathrm{e}^i\right| = \tilde{b}_i\left\langle \tilde{\mathrm{e}}^i\right| \Rightarrow \tilde{b}_i = b_j\left\langle \mathrm{e}^j \,|\tilde{\mathrm{e}}_i\right\rangle \Leftrightarrow \tilde{b}_k = \frac{\partial x^i}{\partial \tilde{x}^k}\,b_i$

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  $|a\rangle = a^i\,|\mathrm{e}_i\rangle = \tilde{a}^i\,|\tilde{\mathrm{e}}_i\rangle \Rightarrow \tilde{a}^i = a^j\left\langle\tilde{\mathrm{e}}^i\,|\mathrm{e}_j\right\rangle \Leftrightarrow \tilde{a}^i = \frac{\partial\tilde{x}^i}{\partial x^k}\;a^k$
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  $\langle b| = b_i\left\langle\mathrm{e}^i\right| = \tilde{b}_i\left\langle\tilde{\mathrm{e}}^i\right| \Rightarrow \tilde{b}_i = b_j\left\langle\mathrm{e}^j\,|\tilde{\mathrm{e}}_i\right\rangle \Leftrightarrow \tilde{b}_k = \frac{\partial x^i}{\partial\tilde{x}^k}\;b_i$
- Tensores $\mathcal{T} = T^m{}_n\,\langle\mathrm{e}^n(2)|\bigotimes|\mathrm{e}_m(1)\rangle = \tilde{T}^k{}_l\left\langle\tilde{\mathrm{e}}^k(2)\right|\bigotimes|\tilde{\mathrm{e}}_l(1)\rangle$
  entonces $\tilde{T}^i{}_j = \frac{\partial\tilde{x}^i}{\partial x^k}\frac{\partial x^m}{\partial\tilde{x}^j}\;T^k{}_m$