Paréntesis de Poisson

Luis A. Núñez

Escuela de Física, Facultad de Ciencias,
Universidad Industrial de Santander, Santander, Colombia

1 de abril de 2025

Agenda

- Definición
- Propiedades
- Dos ejemplos
- Paréntesis de Poisson y Transformaciones Canónicas

Paréntesis de Poisson: Definición

- Sea una función $f\left(q_i, p_i, t\right)$ en el espacio de fase $\left(q_i, p_i\right)$, $i = 1, \ldots, s$, de un sistema mecánico con un Hamiltoniano $\mathcal{H}\left(q_i, p_i, t\right)$.

Paréntesis de Poisson: Definición

- Sea una función $f\left(q_i, p_i, t\right)$ en el espacio de fase $\left(q_i, p_i\right)$, $i = 1, \ldots, s$, de un sistema mecánico con un Hamiltoniano $\mathcal{H}\left(q_i, p_i, t\right)$.
- La derivada total de f es $\frac{df}{dt} = \sum_{i=1}^{s}\left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right) + \frac{\partial f}{\partial t}$.

Paréntesis de Poisson: Definición

- Sea una función $f\left(q_i, p_i, t\right)$ en el espacio de fase $\left(q_i, p_i\right)$, $i = 1, \ldots, s$, de un sistema mecánico con un Hamiltoniano $\mathcal{H}\left(q_i, p_i, t\right)$.
- La derivada total de f es $\frac{df}{dt} = \sum_{i=1}^{s}\left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right) + \frac{\partial f}{\partial t}$.
- Sus ecuaciones de Hamilton son $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$

Paréntesis de Poisson: Definición

- Sea una función $f(q_i, p_i, t)$ en el espacio de fase (q_i, p_i), $i = 1, \ldots, s$, de un sistema mecánico con un Hamiltoniano $\mathcal{H}(q_i, p_i, t)$.
- La derivada total de f es $\frac{df}{dt} = \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$.
- Sus ecuaciones de Hamilton son $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$
- Sustituyendo, tenemos $\frac{df}{dt} = \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) + \frac{\partial f}{\partial t}$

Paréntesis de Poisson: Definición

- Sea una función $f(q_i, p_i, t)$ en el espacio de fase (q_i, p_i), $i = 1, \ldots, s$, de un sistema mecánico con un Hamiltoniano $\mathcal{H}(q_i, p_i, t)$.
- La derivada total de f es $\frac{df}{dt} = \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$.
- Sus ecuaciones de Hamilton son $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$
- Sustituyendo, tenemos $\frac{df}{dt} = \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) + \frac{\partial f}{\partial t}$
- Entonces el paréntesis de Poisson de la función f para un sistema con hamiltoniano $\mathcal{H}$ es $\{f, \mathcal{H}\} \equiv \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right)$.

- Sea una función $f(q_i, p_i, t)$ en el espacio de fase (q_i, p_i), $i = 1, \ldots, s$, de un sistema mecánico con un Hamiltoniano $\mathcal{H}(q_i, p_i, t)$.
- La derivada total de f es $\frac{df}{dt} = \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$.
- Sus ecuaciones de Hamilton son $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$
- Sustituyendo, tenemos $\frac{df}{dt} = \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) + \frac{\partial f}{\partial t}$
- Entonces el paréntesis de Poisson de la función f para un sistema con hamiltoniano $\mathcal{H}$ es $\{f, \mathcal{H}\} \equiv \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right)$.
- Con lo cual $\frac{df}{dt} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$

Paréntesis de Poisson: Definición

- Sea una función $f(q_i, p_i, t)$ en el espacio de fase (q_i, p_i), $i = 1, \ldots, s$, de un sistema mecánico con un Hamiltoniano $\mathcal{H}(q_i, p_i, t)$.
- La derivada total de f es $\frac{df}{dt} = \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$.
- Sus ecuaciones de Hamilton son $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$
- Sustituyendo, tenemos $\frac{df}{dt} = \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) + \frac{\partial f}{\partial t}$
- Entonces el paréntesis de Poisson de la función f para un sistema con hamiltoniano $\mathcal{H}$ es $\{f, \mathcal{H}\} \equiv \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right)$.
- Con lo cual $\frac{df}{dt} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$
- Si f no depende explícitamente del tiempo (cantidad conservada), tenemos $\{f, \mathcal{H}\} = 0$

Paréntesis de Poisson: Definición

- Sea una función $f(q_i, p_i, t)$ en el espacio de fase (q_i, p_i), $i = 1, \ldots, s$, de un sistema mecánico con un Hamiltoniano $\mathcal{H}(q_i, p_i, t)$.
- La derivada total de f es $\frac{df}{dt} = \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$.
- Sus ecuaciones de Hamilton son $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$
- Sustituyendo, tenemos $\frac{df}{dt} = \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) + \frac{\partial f}{\partial t}$
- Entonces el paréntesis de Poisson de la función f para un sistema con hamiltoniano $\mathcal{H}$ es $\{f, \mathcal{H}\} \equiv \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right)$.
- Con lo cual $\frac{df}{dt} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$
- Si f no depende explícitamente del tiempo (cantidad conservada), tenemos $\{f, \mathcal{H}\} = 0$
- Para $f(q_i, p_i, t)$ y $g(q_i, p_i, t)$ podemos definir el paréntesis de Poisson de f y g como $\{f, g\} \equiv \sum_{i=1}^{s} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$.

Paréntesis de Poisson: Propiedades 1/2

- El paréntesis de Poisson puede ser considerado como una operación entre dos funciones definidas en un espacio algebraico que asigna otra función en ese espacio.

Paréntesis de Poisson: Propiedades 1/2

- El paréntesis de Poisson puede ser considerado como una operación entre dos funciones definidas en un espacio algebraico que asigna otra función en ese espacio.
- El paréntesis de Poisson, como operación algebraica, posee las siguientes propiedades (características de un álgebra de Lie):
  - $\{f, g\} = -\{g, f\}, \quad \{f, f\} = 0$ (antisimetría).
  - $\{f, c\} = 0$, si $c =$ cte.
  - $\{af_1 + bf_2, g\} = a\{f_1, g\} + b\{f_2, g\}$, $a, b =$ ctes, un operador lineal.
  - $\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}$, (no asociativo).
  - $\{f, \{g, \mathcal{H}\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ la identidad de Jacobi.

- Como p_i y q_i representan coordenadas independientes tenemos

$$\{q_i, f\} = \sum_{k=1}^{s} \left(\frac{\partial q_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \cancelto{0}{\frac{\partial q_k}{\partial p_k}} \frac{\partial f}{\partial q_k} \right) = \sum_{k=1}^{s} \delta_{ik} \frac{\partial f}{\partial p_k} = \frac{\partial f}{\partial p_i}$$

$$\{p_i, f\} = \sum_{k=1}^{s} \left(\cancelto{0}{\frac{\partial p_i}{\partial q_k}} \frac{\partial f}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = -\sum_{k=1}^{s} \delta_{ik} \frac{\partial f}{\partial q_k} = -\frac{\partial f}{\partial q_i}.$$

Paréntesis de Poisson: Propiedades 2/2

- Como p_i y q_i representan coordenadas independientes tenemos

$$\{q_i, f\} = \sum_{k=1}^{s} \left(\frac{\partial q_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \cancelto{0}{\frac{\partial q_k}{\partial p_k}} \frac{\partial f}{\partial q_k} \right) = \sum_{k=1}^{s} \delta_{ik} \frac{\partial f}{\partial p_k} = \frac{\partial f}{\partial p_i}$$

$$\{p_i, f\} = \sum_{k=1}^{s} \left(\cancelto{0}{\frac{\partial p_i}{\partial q_k}} \frac{\partial f}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = -\sum_{k=1}^{s} \delta_{ik} \frac{\partial f}{\partial q_k} = -\frac{\partial f}{\partial q_i}.$$

- Si $f = p_j$, ó $f = q_j$, $\Rightarrow \{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}$

Paréntesis de Poisson: Propiedades 2/2

- Como p_i y q_i representan coordenadas independientes tenemos

$$\{q_i, f\} = \sum_{k=1}^{s} \left(\frac{\partial q_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \cancelto{0}{\frac{\partial q_k}{\partial p_k}} \frac{\partial f}{\partial q_k} \right) = \sum_{k=1}^{s} \delta_{ik} \frac{\partial f}{\partial p_k} = \frac{\partial f}{\partial p_i}$$

$$\{p_i, f\} = \sum_{k=1}^{s} \left(\cancelto{0}{\frac{\partial p_i}{\partial q_k}} \frac{\partial f}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = -\sum_{k=1}^{s} \delta_{ik} \frac{\partial f}{\partial q_k} = -\frac{\partial f}{\partial q_i}.$$

- Si $f = p_j$, ó $f = q_j$, $\Rightarrow \{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}$
- Las ecuaciones de Hamilton pueden escribirse como
$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \{q_i, \mathcal{H}\} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = \{p_i, \mathcal{H}\}$

Paréntesis de Poisson: Propiedades 2/2

- Como p_i y q_i representan coordenadas independientes tenemos

$$\{q_i, f\} = \sum_{k=1}^{s} \left(\frac{\partial q_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \cancelto{0}{\frac{\partial q_k}{\partial p_k}} \frac{\partial f}{\partial q_k} \right) = \sum_{k=1}^{s} \delta_{ik} \frac{\partial f}{\partial p_k} = \frac{\partial f}{\partial p_i}$$

$$\{p_i, f\} = \sum_{k=1}^{s} \left(\cancelto{0}{\frac{\partial p_i}{\partial q_k}} \frac{\partial f}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = -\sum_{k=1}^{s} \delta_{ik} \frac{\partial f}{\partial q_k} = -\frac{\partial f}{\partial q_i}.$$

- Si $f = p_j$, ó $f = q_j$, $\Rightarrow \{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}$
- Las ecuaciones de Hamilton pueden escribirse como
  $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \{q_i, \mathcal{H}\} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = \{p_i, \mathcal{H}\}$
- En Mecánica Cuántica, la operación $[A, B] = AB - BA$ es el conmutador de los operadores u observables A y B.

Paréntesis de Poisson: Propiedades 2/2

- Como p_i y q_i representan coordenadas independientes tenemos
$$\{q_i, f\} = \sum_{k=1}^{s} \left(\frac{\partial q_i}{\partial q_k}\frac{\partial f}{\partial p_k} - \cancelto{0}{\frac{\partial q_k}{\partial p_k}}\frac{\partial f}{\partial q_k} \right) = \sum_{k=1}^{s} \delta_{ik}\frac{\partial f}{\partial p_k} = \frac{\partial f}{\partial p_i}$$
$$\{p_i, f\} = \sum_{k=1}^{s} \left(\cancelto{0}{\frac{\partial p_i}{\partial q_k}}\frac{\partial f}{\partial p_k} - \frac{\partial p_i}{\partial p_k}\frac{\partial f}{\partial q_k} \right) = -\sum_{k=1}^{s} \delta_{ik}\frac{\partial f}{\partial q_k} = -\frac{\partial f}{\partial q_i}.$$
- Si $f = p_j$, ó $f = q_j$, $\Rightarrow \{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}$
- Las ecuaciones de Hamilton pueden escribirse como
$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \{q_i, \mathcal{H}\} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = \{p_i, \mathcal{H}\}$
- En Mecánica Cuántica, la operación $[A, B] = AB - BA$ es el conmutador de los operadores u observables A y B.
- La estructura algebraica de la Mecánica Clásica, expresada en las propiedades de los paréntesis de Poisson, se preserva en la Mecánica Cuántica. En particular, $\{q_i, p_j\} = i\hbar\delta_{ij}$, donde $\hbar$ es la constante de Planck (dividida por 2π).

Dos Ejemplos

- Calcular $\{r, \mathbf{p}\}$, donde $r = \left(x^2 + y^2 + z^2\right)^{1/2}$
  - $\{r, \mathbf{p}\} = \{r, p_x\}\,\hat{\mathbf{x}} + \{r, p_y\}\,\hat{\mathbf{y}} + \{r, p_z\}\,\hat{\mathbf{z}}$
  - $\{r, p_x\} = \sum_{i=1}^{3}\left(\frac{\partial r}{\partial q_i}\frac{\partial p_x}{\partial p_i} - \frac{\partial r}{\partial p_i}\frac{\partial p_x}{\partial q_i}\right) = \frac{\partial r}{\partial x}\frac{\partial p_x}{\partial p_x} = \frac{x}{r}$
  - $\{r, p_y\} = \frac{y}{r}, \quad \{r, p_z\} = \frac{z}{r}$
  - luego $\{r, \mathbf{p}\} = \frac{x}{r}\hat{\mathbf{x}} + \frac{y}{r}\hat{\mathbf{y}} + \frac{z}{r}\hat{\mathbf{z}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$

Dos Ejemplos

- Calcular $\{r, \mathbf{p}\}$, donde $r = \left(x^2 + y^2 + z^2\right)^{1/2}$
  - $\{r, \mathbf{p}\} = \{r, p_x\}\,\hat{\mathbf{x}} + \{r, p_y\}\,\hat{\mathbf{y}} + \{r, p_z\}\,\hat{\mathbf{z}}$
  - $\{r, p_x\} = \sum_{i=1}^{3} \left(\frac{\partial r}{\partial q_i}\frac{\partial p_x}{\partial p_i} - \frac{\partial r}{\partial p_i}\frac{\partial p_x}{\partial q_i}\right) = \frac{\partial r}{\partial x}\frac{\partial p_x}{\partial p_x} = \frac{x}{r}$
  - $\{r, p_y\} = \frac{y}{r}, \quad \{r, p_z\} = \frac{z}{r}$
  - luego $\{r, \mathbf{p}\} = \frac{x}{r}\hat{\mathbf{x}} + \frac{y}{r}\hat{\mathbf{y}} + \frac{z}{r}\hat{\mathbf{z}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$
- Dadas las componentes del momento angular $\mathbf{L} = \mathbf{r} \times \mathbf{p}$:

  $L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x$

  Calcular los paréntesis de Poisson para las componentes de $\mathbf{p}$ y $\mathbf{L}$:

  $\{p_y, L_x\} = -\frac{\partial L_x}{\partial y} = -p_z; \quad \{p_x, L_x\} = -\frac{\partial L_x}{\partial x} = 0;$

  $\{p_z, L_y\} = -\frac{\partial L_y}{\partial z} = -p_x$

  $\{L_x, L_y\} = \sum_{i=1}^{3} \left(\frac{\partial L_x}{\partial q_i}\frac{\partial L_y}{\partial p_i} - \frac{\partial L_x}{\partial p_i}\frac{\partial L_y}{\partial q_i}\right)$

  $\{L_x, L_y\} =$

  $\left(\frac{\partial L_x}{\partial x}\frac{\partial L_y}{\partial p_x} - \frac{\partial L_x}{\partial p_x}\frac{\partial L_y}{\partial x}\right) + \left(\frac{\partial L_x}{\partial y}\frac{\partial L_y}{\partial p_y} - \frac{\partial L_x}{\partial p_y}\frac{\partial L_y}{\partial y}\right) + \left(\frac{\partial L_x}{\partial z}\frac{\partial L_y}{\partial p_z} - \frac{\partial L_x}{\partial p_z}\frac{\partial L_y}{\partial z}\right)$

  $\{L_x, L_y\} = xp_y - yp_x = L_z; \quad \{L_y, L_z\} = L_x \quad \{L_z, L_x\} = L_y$

Dos Ejemplos

- Calcular $\{r, \mathbf{p}\}$, donde $r = \left(x^2 + y^2 + z^2\right)^{1/2}$
  - $\{r, \mathbf{p}\} = \{r, p_x\}\, \hat{\mathbf{x}} + \{r, p_y\}\, \hat{\mathbf{y}} + \{r, p_z\}\, \hat{\mathbf{z}}$
  - $\{r, p_x\} = \sum_{i=1}^{3} \left(\frac{\partial r}{\partial q_i} \frac{\partial p_x}{\partial p_i} - \frac{\partial r}{\partial p_i} \frac{\partial p_x}{\partial q_i} \right) = \frac{\partial r}{\partial x} \frac{\partial p_x}{\partial p_x} = \frac{x}{r}$
  - $\{r, p_y\} = \frac{y}{r}, \quad \{r, p_z\} = \frac{z}{r}$
  - luego $\{r, \mathbf{p}\} = \frac{x}{r}\hat{\mathbf{x}} + \frac{y}{r}\hat{\mathbf{y}} + \frac{z}{r}\hat{\mathbf{z}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$
- Dadas las componentes del momento angular $\mathbf{L} = \mathbf{r} \times \mathbf{p}$:
  $L_x = y p_z - z p_y, \quad L_y = z p_x - x p_z, \quad L_z = x p_y - y p_x$
  Calcular los paréntesis de Poisson para las componentes de $\mathbf{p}$ y $\mathbf{L}$:
  $\{p_y, L_x\} = -\frac{\partial L_x}{\partial y} = -p_z; \quad \{p_x, L_x\} = -\frac{\partial L_x}{\partial x} = 0;$
  $\{p_z, L_y\} = -\frac{\partial L_y}{\partial z} = -p_x$
  $\{L_x, L_y\} = \sum_{i=1}^{3} \left(\frac{\partial L_x}{\partial q_i} \frac{\partial L_y}{\partial p_i} - \frac{\partial L_x}{\partial p_i} \frac{\partial L_y}{\partial q_i} \right)$
  $\{L_x, L_y\} =$
  $\left(\frac{\partial L_x}{\partial x} \frac{\partial L_y}{\partial p_x} - \frac{\partial L_x}{\partial p_x} \frac{\partial L_y}{\partial x} \right) + \left(\frac{\partial L_x}{\partial y} \frac{\partial L_y}{\partial p_y} - \frac{\partial L_x}{\partial p_y} \frac{\partial L_y}{\partial y} \right) + \left(\frac{\partial L_x}{\partial z} \frac{\partial L_y}{\partial p_z} - \frac{\partial L_x}{\partial p_z} \frac{\partial L_y}{\partial z} \right)$
  $\{L_x, L_y\} = x p_y - y p_x = L_z; \quad \{L_y, L_z\} = L_x \quad \{L_z, L_x\} = L_y$
- Entonces, $\{L_i, L_j\} = \epsilon_{ijk} L_k$. En Mecánica Cuántica, $[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$.

Paréntesis de Poisson y Transformaciones 1/2

Para una transformación canónica $Q_j = Q_j(q_k, p_k, t)$ y $P_i = P_i(q_k, p_k, t)$ se cumple que $\{Q_i, Q_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}$

- Sea $Q = q + e^p \quad \& \quad P = p$

Paréntesis de Poisson y Transformaciones 1/2

Para una transformación canónica $Q_j = Q_j(q_k, p_k, t)$ y $P_i = P_i(q_k, p_k, t)$ se cumple que $\{Q_i, Q_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}$

- Sea $Q = q + e^p \quad \& \quad P = p$
- $\{Q, Q\} = \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q} = (1)(e^p) - (e^p)(1) = 0;$

Paréntesis de Poisson y Transformaciones 1/2

Para una transformación canónica $Q_j = Q_j(q_k, p_k, t)$ y $P_i = P_i(q_k, p_k, t)$ se cumple que $\{Q_i, Q_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}$

- Sea $Q = q + e^p \quad \& \quad P = p$
- $\{Q, Q\} = \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q} = (1)(e^p) - (e^p)(1) = 0;$
- $\{P, P\} = \frac{\partial P}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial P}{\partial q} = (0)(1) - (1)(0) = 0;$ y finalmente

Para una transformación canónica $Q_j = Q_j(q_k, p_k, t)$ y $P_i = P_i(q_k, p_k, t)$ se cumple que $\{Q_i, Q_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}$

- Sea $Q = q + e^p \quad \& \quad P = p$
- $\{Q, Q\} = \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q} = (1)(e^p) - (e^p)(1) = 0;$
- $\{P, P\} = \frac{\partial P}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial P}{\partial q} = (0)(1) - (1)(0) = 0;$ y finalmente
- $\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$. Calculando cada una $\frac{\partial Q}{\partial q} = 1$, $\frac{\partial Q}{\partial p} = e^p$, $\frac{\partial P}{\partial q} = 0$, y $\frac{\partial P}{\partial p} = 1$, obtenemos $\{Q, P\} = (1)(1) - (e^p)(0) = 1$.

Paréntesis de Poisson y Transformaciones 1/2

Universidad Industrial de Santander
CONSTRUIMOS FUTURO

Para una transformación canónica $Q_j = Q_j(q_k, p_k, t)$ y $P_i = P_i(q_k, p_k, t)$ se cumple que $\{Q_i, Q_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}$

- Sea $Q = q + e^p \quad \& \quad P = p$
- $\{Q, Q\} = \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q} = (1)(e^p) - (e^p)(1) = 0;$
- $\{P, P\} = \frac{\partial P}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial P}{\partial q} = (0)(1) - (1)(0) = 0$; y finalmente
- $\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$. Calculando cada una $\frac{\partial Q}{\partial q} = 1$, $\frac{\partial Q}{\partial p} = e^p$, $\frac{\partial P}{\partial q} = 0$, y $\frac{\partial P}{\partial p} = 1$, obtenemos $\{Q, P\} = (1)(1) - (e^p)(0) = 1$.
- Por lo tanto, como la tranformación cumple con $\{Q, Q\} = 0, \quad \{P, P\} = 0, \quad$ y $\quad \{Q, P\} = 1$. **Es canónica**

Consideremos la siguiente transformación

$P_1 = p_1 - 2p_2, \quad Q_1 = q_1$ y $P_2 = -2q_1 - q_2, \quad Q_2 = p_2$

- $\{Q_1, P_1\} = \sum_{i=1}^{s} \left(\frac{\partial Q_1}{\partial q_i} \frac{\partial P_1}{\partial p_i} - \frac{\partial Q_1}{\partial p_i} \frac{\partial P_1}{\partial q_i} \right)$

  $\{Q_1, P_1\} = \left(\frac{\partial Q_1}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left(\frac{\partial Q_1}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 1$

Consideremos la siguiente transformación

$P_1 = p_1 - 2p_2, \quad Q_1 = q_1$ y $P_2 = -2q_1 - q_2, \quad Q_2 = p_2$

- $\{Q_1, P_1\} = \sum_{i=1}^{s} \left(\frac{\partial Q_1}{\partial q_i} \frac{\partial P_1}{\partial p_i} - \frac{\partial Q_1}{\partial p_i} \frac{\partial P_1}{\partial q_i} \right)$

  $\{Q_1, P_1\} = \left(\frac{\partial Q_1}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left(\frac{\partial Q_1}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 1$

- $\{Q_2, P_2\} = \left(\frac{\partial Q_2}{\partial q_1} \frac{\partial P_2}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_2}{\partial q_1} \right) + \left(\frac{\partial Q_2}{\partial q_2} \frac{\partial P_2}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_2}{\partial q_2} \right) = 1$

Consideremos la siguiente transformación

$P_1 = p_1 - 2p_2, \quad Q_1 = q_1$ y $P_2 = -2q_1 - q_2, \quad Q_2 = p_2$

- $\{Q_1, P_1\} = \sum_{i=1}^{s} \left(\frac{\partial Q_1}{\partial q_i} \frac{\partial P_1}{\partial p_i} - \frac{\partial Q_1}{\partial p_i} \frac{\partial P_1}{\partial q_i} \right)$

  $\{Q_1, P_1\} = \left(\frac{\partial Q_1}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left(\frac{\partial Q_1}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 1$
- $\{Q_2, P_2\} = \left(\frac{\partial Q_2}{\partial q_1} \frac{\partial P_2}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_2}{\partial q_1} \right) + \left(\frac{\partial Q_2}{\partial q_2} \frac{\partial P_2}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_2}{\partial q_2} \right) = 1$
- $\{Q_2, P_1\} = \left(\frac{\partial Q_2}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left(\frac{\partial Q_2}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 0$

Consideremos la siguiente transformación

$P_1 = p_1 - 2p_2, \quad Q_1 = q_1$ y $P_2 = -2q_1 - q_2, \quad Q_2 = p_2$

- $\{Q_1, P_1\} = \sum_{i=1}^{s} \left(\frac{\partial Q_1}{\partial q_i}\frac{\partial P_1}{\partial p_i} - \frac{\partial Q_1}{\partial p_i}\frac{\partial P_1}{\partial q_i} \right)$

  $\{Q_1, P_1\} = \left(\frac{\partial Q_1}{\partial q_1}\frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1}\frac{\partial P_1}{\partial q_1} \right) + \left(\frac{\partial Q_1}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2}\frac{\partial P_1}{\partial q_2} \right) = 1$
- $\{Q_2, P_2\} = \left(\frac{\partial Q_2}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial Q_2}{\partial p_1}\frac{\partial P_2}{\partial q_1} \right) + \left(\frac{\partial Q_2}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial Q_2}{\partial p_2}\frac{\partial P_2}{\partial q_2} \right) = 1$
- $\{Q_2, P_1\} = \left(\frac{\partial Q_2}{\partial q_1}\frac{\partial P_1}{\partial p_1} - \frac{\partial Q_2}{\partial p_1}\frac{\partial P_1}{\partial q_1} \right) + \left(\frac{\partial Q_2}{\partial q_2}\frac{\partial P_1}{\partial p_2} - \frac{\partial Q_2}{\partial p_2}\frac{\partial P_1}{\partial q_2} \right) = 0$
- $\{Q_1, P_2\} = \left(\frac{\partial Q_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial Q_1}{\partial p_1}\frac{\partial P_2}{\partial q_1} \right) + \left(\frac{\partial Q_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial Q_1}{\partial p_2}\frac{\partial P_2}{\partial q_2} \right) = 0$

Paréntesis de Poisson y Transformaciones 2/2

Consideremos la siguiente transformación

$P_1 = p_1 - 2p_2, \quad Q_1 = q_1$ y $P_2 = -2q_1 - q_2, \quad Q_2 = p_2$

- $\{Q_1, P_1\} = \sum_{i=1}^{s} \left(\frac{\partial Q_1}{\partial q_i} \frac{\partial P_1}{\partial p_i} - \frac{\partial Q_1}{\partial p_i} \frac{\partial P_1}{\partial q_i} \right)$

  $\{Q_1, P_1\} = \left(\frac{\partial Q_1}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left(\frac{\partial Q_1}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 1$
- $\{Q_2, P_2\} = \left(\frac{\partial Q_2}{\partial q_1} \frac{\partial P_2}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_2}{\partial q_1} \right) + \left(\frac{\partial Q_2}{\partial q_2} \frac{\partial P_2}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_2}{\partial q_2} \right) = 1$
- $\{Q_2, P_1\} = \left(\frac{\partial Q_2}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left(\frac{\partial Q_2}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 0$
- $\{Q_1, P_2\} = \left(\frac{\partial Q_1}{\partial q_1} \frac{\partial P_2}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_2}{\partial q_1} \right) + \left(\frac{\partial Q_1}{\partial q_2} \frac{\partial P_2}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_2}{\partial q_2} \right) = 0$
- Por lo tanto, la tranformación **Es canónica**