EXTENDED FLRW MODELS, NON-ABELIAN GAUGE FIELDS AND THE WEAK COSMOLOGICAL PRINCIPLE

Master thesis by

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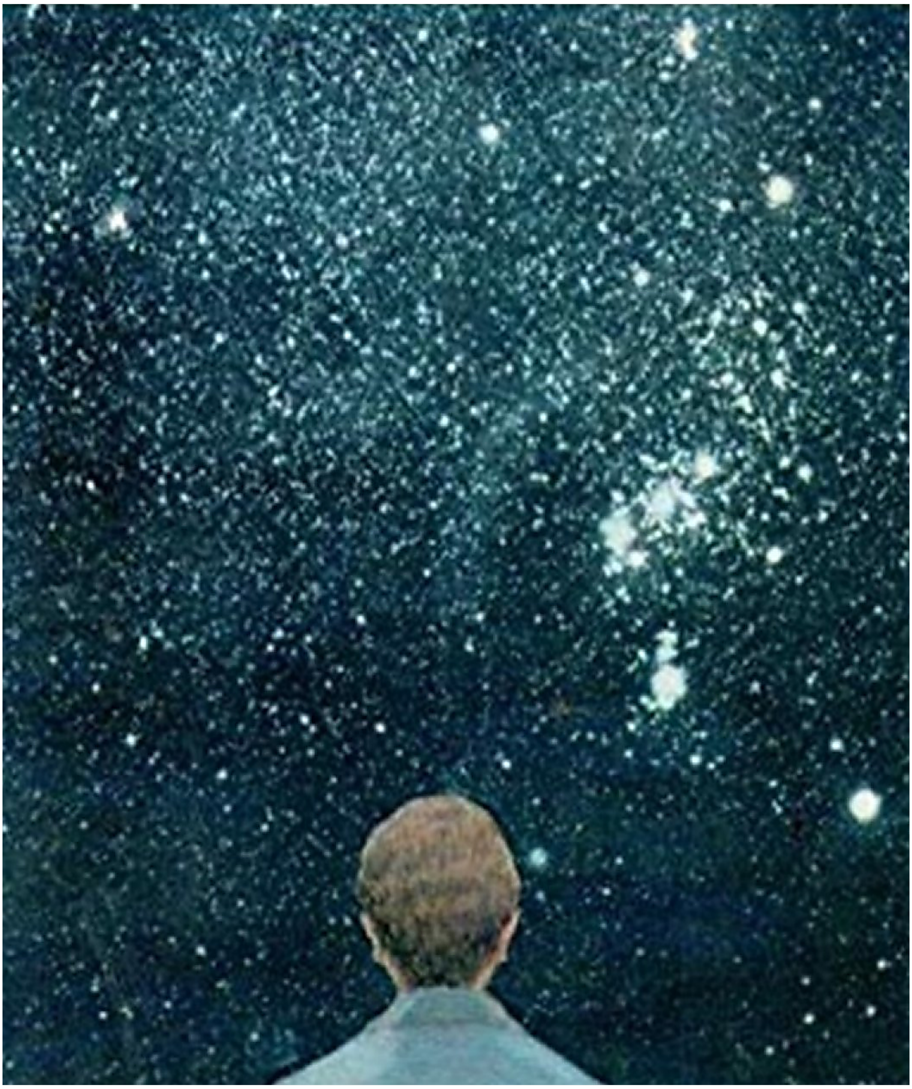
To my mother

the Aleph of my life



“Unfortunately, these young people do not understand that sacrificing one’s life is, in many cases, the easiest thing in the world, whereas, for example, dedicating five or six years of their beautiful youth to study and science –even if only to multiply their energies in the service of truth and to reach the dreamed–of goal is a sacrifice beyond their strength.

Fiódor Dostoyevski. The Brothers Karamazov (1880)

Abstract

In the context of the Bianchi universes, i.e., homogeneous but anisotropic cosmologies, we study the conditions to obtain vanishing shear that satisfy the cosmological principle. The anisotropic curvature has to be balanced by the anisotropic stresses of the non-perfect fluid that dominates the universe's energy density. Such a fluid has been considered to describe an n-form with or without potential. The scenario is successfully realised only for the free 0-form and 2-form cases, which are equivalent via Hodge duality.

N. H. B.

As a way of gratitude

The death of others is the announcement of my own end.

—*In some remote place in Puerto Caicedo, Putumayo, Colombia* (2025).

First of all, I acknowledge...

A modo de gratitud

La muerte de los demás, es el anuncio de mi propio fin.

—En algún lugar recóndito de puerto Caicedo, Putumayo, Colombia (2025).

Primero me gustaría agradecer...

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PART II: Matter model

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INTRODUCTION

“When I heard the learn’d astronomer,
When the proofs, the figures, were ranged in columns before me,
When I was shown the charts and diagrams, to add, divide, and measure them,
When I sitting heard the astronomer where he lectured with much applause in the lecture-room,
How soon unaccountable I became tired and sick,
Till rising and gliding out I wander’d off by myself,
In the mystical moist night-air, and from time to time,
Look’d up in perfect silence at the stars.

Walt Whitman. When I Heard the Learn’d Astronomer (1867)

Chapter 1

Prolegomenon

On the tree of science: Verisimilitude, but not truth; appearance of freedom, but not freedom. Thanks to these two fruits, the tree of science is in no danger of being mistaken for the tree of life.

—*Friedrich Nietzsche. The traveller and its shadow* (1880).

Cosmology is the study of the physical universe at large, projected onto the scientific method through mathematics [1]. It is understood as the theory that deals with the cosmos involving mathematical and physical aspects. Strictly speaking, such a study proposes and tests mathematical theories for the physical universe on a large scale, and for structure formation. It is a purely scientific exercise related to the Big Bang theory, probably preceded by an earlier phase of accelerated expansion.

In this context, the Earth is a tiny speckle in a vast cosmic arena [2]. Thus, we are not in a privileged place to do cosmological observations, contrary to pre-Copernican thinking[1]. It is just the Copernican principle which expresses the anti-anthropocentric point of view:

> Principle 1 (The Copernican principle [3]). The Earth is not in a central, specially-favoured position in the physical universe.

This means that our cosmological observations do not have any special properties with respect to other observations made anywhere in the universe, which implies that the universe must look essentially the same

[1] Strictly, they used to think we were at the outskirts, or lowest point of the cosmos.

everywhere. This consequence of the Copernican principle is known as the weak cosmological principle:

> Principle 2 (The weak cosmological principle [3]). The universe presents the same aspect from every point.

Moreover, a different assumption often made is that spacetime is *isotropic* about us, this means that it looks the same in every spatial direction, which implies that it is spherically symmetric about us [4]. Again, the Copernican principle ensures this fact in a stronger version of the previous one:

> Principle 3 (The strong cosmological principle [3]). The universe is isotropic around every point.

This implies that spacetime must be spatially homogeneous[2] [5]. On the other hand, both versions of the cosmological principle imply that the universe must be homogeneous; however, this sentence is valid only on large scales. Furthermore, these principles should be understood as expressing the idea that *there exists an averaging scale* at which the universe is homogeneous [1].

The fact that spatial homogeneity and isotropy about us imply isotropy everywhere [6], suggests models with a highly symmetric geometry known as Friedmann-Lemaître-Robertson-Walker (FLRW) models [7–11]. Nowadays, there is no known feature of the cosmos which *definitely* contradicts the hypothesis that our universe has approximately an FLRW geometry. So, why then do we not simply devote our investigations to the problem of selecting the FLRW universe whose parameters, like the Hubble constant, density, etc., best fit the observed universe? This is, after all, the aim towards which modern observational and theoretical work is directed. Although FLRW is a good model, there seem to be some good reasons

[2]Concepts like isotropy, homogeneity, and spatial homogeneity, among others, will be discussed in more detail in the subsection 2.1, and mathematically formalised in subsection 2.4.

to consider models that are not exactly like this. There are experimental facts that ratify this idea:

- ⋆ Observations of fairly uniform distribution of He^4 suggest that the universe evolved through the helium formation phase with an abundance that agrees with the FLRW model prediction [12, 13]. However, if the observations of very low helium content in some stars are correct, the conventional picture of the early stages of the universe must be modified [14].
- ⋆ Any observed anisotropy of the background radiation or discrete source distributions would prove that the universe was anisotropic. In particular, anisotropy could have observable consequences on the polarisation and spectrum of the cosmic microwave background radiation [15, 16]. While these effects would be difficult to account for by FLRW models, other plausible models might account for them.

Moreover, the physical aspects of our universe at the largest possible cosmological scales were measured with incredible accuracy by cosmic microwave background radiation (CMB) probes like COBE [17], WMAP [18–20], and Planck [21–23]. They suggest that the universe is homogeneous and isotropic on large scales, and the ΛCDM model seems to describe such a scenario with high accuracy [24]. Specifically, the homogeneous and isotropic character of the universe has been confirmed by Planck with a level of deviation from isotropy quite compatible with zero [25]. Despite this, the same observations that describe some "anomalies"[3] [27] suggest a possible deviation from isotropy and homogeneity at some point in the evolutionary history of the universe [21, 22].

As we saw briefly, the cosmological principle leads to the conclusion that the universe is spatially homogeneous and isotropic, and it is implemented at

[3]However, the origin of these features is still uncertain and today is the theme of many scientific discussions. It is argued that the anomalies could be a consequence of new physics or, perhaps, are merely statistical fluctuations [26].

the background level. However, the facts discussed above suggest, as an explanation, from a cosmological point of view, a possible violation of isotropy during the evolution of the universe. Here is where the *Bianchi models* provide a natural framework since they describe spatially homogeneous cosmological models that break three-dimensional rotation invariance [6]. Such a breaking can be realized in many ways: the fundamental observers' congruence may possess shear [28, 29] and vorticity [29, 30], the matter sector may possess anisotropic stress [31, 32] and tilt [33, 34], or the spatial sections of homogeneity may possess anisotropic curvature [35]. Thus, according to Einstein's ideas, the various types of anisotropies do not evolve independently: they must be intimately connected through Einstein's field equations. Furthermore, models that can generate and sustain an anisotropic phase of expansion have gained attention [36, 37]. Hence, a deeper analysis of anisotropic spacetimes has become a necessity. As an example, it could be considered that the evolution of the shear tensor, which measures the difference between rates of expansion in different directions, is "sourced" by the tensors that describe the anisotropic stress of matter and anisotropic curvature [38].

In addition to the above evidence, nowadays there exists an observational fact about the accelerated expansion the universe is experiencing [39, 40]. Commonly, such expansion is thought to be sourced by the cosmological constant Λ [41]. However, in spite of its success, this explanation has many problems when confronted with observational evidence [42]. Thus, there exists a huge discrepancy between theory and observations about the value of Λ, usually referred to as the *cosmological constant problem* [41]. So, in conjunction with other discrepancies such as the Hubble tension [43], this invites us to suggest that new dynamical degrees of freedom must be considered, this being one of the most popular alternatives to the so-called quintessence models based on scalar fields [44, 45]. Despite the success of the theories mentioned above, some other theories built with other type of fields -for instance, vector fields [46–49], p-forms [36], and so on- have a richer phenomenology and provide many cosmological consequences which have not been fully explored. Among these proposals, we are interested in non-Abelian gauge fields: they are the link between cosmology and the phenomenology of particle physics [50–54].

Considering the preceding arguments, Bianchi cosmological models—recognised as the most general spatially homogeneous frameworks encompassing open, flat, and closed Friedmann-Lemaître-Robertson-Walker (FLRW) models as particular instances—present a rigorous context for evaluating both standard cosmological assumptions and compelling alternative scenarios. In this thesis, special attention is devoted to a specific subset of Bianchi cosmologies, namely orthogonal spacetimes characterised by anisotropic spatial curvature yet exhibiting isotropic expansion. Such models, known as extended FLRW cosmologies, display shear-free fluid flows analogous to conventional FLRW models and remain consistent with the weak cosmological principle.

The existence of shear-free cosmological solutions with anisotropic spatial curvature was initially investigated by Mimoso and Crawford in [55] and subsequently revisited in [56, 57]. It has been demonstrated that, within spatially homogeneous models involving non-tilted perfect fluids, the shear-free condition necessarily leads to FLRW metrics [58]. Thus, shear-free orthogonal Bianchi models possessing anisotropic spatial curvature inherently require the presence of imperfect matter components, specifically fluids exhibiting anisotropic stresses.

Naturally, from everything previously stated, we perceive that the assumption of isotropy in light of the experimental facts about the expansion of the universe is too strong, which invites us to think about the postulation of a less restrictive cosmological principle that, however, respects the Copernican principle and the weak cosmological principle. As a consequence, we are interested in understanding how to implement these ideas and what the consequences would be. An specific contribution toward addressing this challenge, this thesis aims to identify particular configurations of non-Abelian gauge fields which, in conjunction with the universe's standard matter content, are capable of sustaining the anisotropies characteristic of extended FLRW shear-free cosmologies. This investigation thus extends and complements previous analyses such as the one presented in [38], contributing further insights into alternative cosmological models that align with observational evidence yet relax the stringent isotropy constraints characteristic of conventional FLRW models.

Summary (Philosophical justification): This project intends to contribute to the longstanding investigation of how likely the observed universe is [59]. Specifically, it aims to contribute to answering the question: Can the spatially anisotropic solutions to general relativity be dynamically distinguished from FLRW cosmologies? We shall assume General Relativity (GR) to be the correct theory of gravity.

However, before discussing in detail the problem that this work shall intend to resolve, it is of capital importance to get acquainted with some background material like the geometrical concepts discussed here.

Finally, throughout the text we assume a torsion-free, foliable and Lorentzian manifold with signature $+2$. Additionally, Greek indexes $\{\alpha, \beta, \gamma, ...\}$ generally run over space-time components whereas Latin indices $\{a, b, c, ...\}$ run over spatial components only. We employ e_μ referring to a general basis vector and $\tilde{\omega}^\mu$ as a general one-form. Ultimately, we shall assume a space-time foliation such that $e_0 = \partial_0 = \partial_t$ is orthonormal to spatial hypersurfaces Σ, and the 4-velocity of our congruence of fundamental observers u will be aligned along e_0.

Chapter 2
Bianchi models

> *But my Lord has shewn me the intestines of all my countrymen in the Land of Two Dimensions by taking me with him into the Land of Three. What therefore more easy than now to take his servant on a second journey into the blessed region of the Fourth Dimension, where I shall look down with him once more upon this land of Three Dimensions, and see the inside of every three- dimensioned house, the secrets of the solid earth, the treasures of the mines in Spaceland, and the intestines of every solid living creature, even of the noble and adorable Spheres.*
>
> —*Edwin A. Abbott. Flatland: a romance of many dimensions* (1884).

This chapter builds extensively on chapter 1 in *Cosmological models from a geometric point of view* [6], chapter 3 in *3+1 Formalism in General Relativity* [60], chapter 4 in *Relativity on Curved Manifolds* [61], chapter 4 in *Relativistic Cosmology* [4], chapter 3 in *Lecture notes in Lie groups and Lie algebras* [62], chapter 2 in *Tales from Wonderland* [1] and chapter 15 in *Einstein's General Theory of Relativity* [63].

2.1 An intuitive geometrical perspective of the universe

Having declared general relativity as the theoretical framework governing the cosmological models analysed herein, we shall now proceed to discuss the main characteristics encountered throughout this work.

Like in other geometrical theories of gravity, every physical quantity is defined via a geometric quantity and vice versa[4]. Therefore, describing a physical symmetry is equivalent to describing a geometrical symmetry. In this section, we give an informal but intuitive definition of the most essential symmetries that the universe might possess. Thus, we shall assume what is in some sense a physical symmetry by the assumption that the laws of physics are the same at every point of spacetime[5].

As mentioned in the prolegomenon, we impose the fulfilment of the weak cosmological principle as a consequence of the Copernican one. I mean, the Universe is the same at every point; therefore, an observer would have no way of telling where he/she was in spacetime, and all physical quantities would be the same at every point. Thus, we would guarantee no overall evolution in such a universe. It would be in a "steady state". Such four-dimensional spacetime is called *homogeneous* [6].

The type of homogeneity that we will use is the *spatial*, where every point lies in a homogeneous three-dimensional section of the spacetime, specifically, a space-like hypersurface. That is, the tangent vector of any curve lying in a homogeneous section will be everywhere space-like. When the manifold that describes a spatially-homogeneous universe is foldable[6], this spacetime evolves. Hence, in this situation, as we go from one space hypersurface to the next, the physical quantities may alter.

Isotropy, as an equivalence in all directions of the spacetime, is the other symmetry that we want to introduce. If the spacetime as a whole does not distinguish between two or more directions at a point, then it exhibits isotropy. Of course, if the symmetry is a continuous symmetry, then the isotropy is also continuous, like the rotational symmetry of the sphere

[4] "Matter tells spacetime how to curve, and curved spacetime tells matter how to move" [64].
[5] For more information, see chapter 2 of [3].
[6] In other words, that satisfy the Frobenius theorem for vector fields [65].

about the origin. For instance, we can imagine a time-like direction at a point where all spatial directions perpendicular to it are equivalent, so an observer at that point, which has a 4-velocity along the time-like direction, will see that all spatial directions are equivalent. In this situation, the spacetime is said to be *spherically symmetric* about that point. Moreover, if for a given time direction at a point, there exists a space direction such that every direction in the two-dimensional surface perpendicular to both the space and time directions is equivalent, the space is said to be *rotationally symmetric* [6]. Moreover, a space that has rotational symmetry at every point is called *locally rotationally symmetric* [4]. Besides, a space that is spherically symmetric about every point usually is called *isotropic.*

What is the observational evidence about the geometry of the universe? There are three main deductions about geometry that may be made from observation [6]:

★ *The universe is undergoing an overall evolution. Consequently, it is not a homogeneous spacetime* [66–68].

★ *The universe is spatially homogeneous* [69–71].

★ *The universe is isotropic about us. The evidence for this concerns both the isotropy of discrete sources and the isotropy of the cosmic microwave background radiation* [72–77].

To formalise mathematically all of these concepts and the kinematic properties of the cosmological models, it is of capital importance to develop a few things about the $3 + 1$ formalism in General Relativity.

2.2 Geometry of hypersurfaces

2.2.1 Hypersurface embedded in spacetime and induced metric

Let $\mathcal{M}$ be a 4-dimensional manifold[7]. A submanifold Σ is the image of a 3-dimensional manifold $\hat{\Sigma}$ by an embedding[8] $\Phi : \hat{\Sigma} \mapsto \mathcal{M}$, defined by[9]

$$\Sigma = \Phi(\hat{\Sigma}). \tag{2.1}$$

Let γ be a curve in $\hat{\Sigma}$, such that the map Φ takes curves $\gamma \in \hat{\Sigma}$ and maps them to curves $\Phi(\gamma) \in \Sigma$. Hence, Φ induces a map between tangent spaces, i.e. takes vectors in $\hat{\Sigma}$ and gives as a result vectors in Σ. This map is called the *push-forward map* Φ_* [61]. To study its action on vectors, let $x^\alpha = (t, x, y, z)$ be the local coordinates of a chart in $\mathcal{M}$, so

$$\begin{aligned} \Phi_* : \mathcal{T}_p(\hat{\Sigma}) \quad &\longrightarrow \quad \mathcal{T}_p(\mathcal{M}) \\ v^i = (v^x, v^y, v^z) &\longmapsto \quad \Phi_*(v^i) = (0, v^x, v^y, v^z), \end{aligned} \tag{2.2}$$

with p being a point in $\mathcal{M}$ and $v^i = (v^x, v^y, v^z)$ being the components of the vector v with respect to the coordinate basis $\partial/\partial x^i \in \mathcal{T}_{\Phi(p)}(\hat{\Sigma})$ associated with the coordinates $x^i = (x, y, z)$ on the 3-manifold $\hat{\Sigma}$ [60].

Additionally, the map Φ induces another map, Φ^*, called the *pull-back map* which, in contrast to Φ_*, maps one-forms in $\mathcal{T}^*_p(\mathcal{M})$ to one-forms in $\mathcal{T}^*_p(\hat{\Sigma})$ as follows

$$\begin{aligned} \Phi^* : \mathcal{T}^*_p(\mathcal{M}) \quad &\longrightarrow \quad \mathcal{T}^*_p(\hat{\Sigma}) \\ \tilde{\omega} \quad &\longmapsto \quad \Phi^*(\tilde{\omega}) : \; \mathcal{T}_p(\hat{\Sigma}) \; \to \; \mathbb{R} \\ &\qquad\qquad\qquad\qquad v \;\; \mapsto \; \langle \tilde{\omega}, \Phi_*(v) \rangle \, . \end{aligned} \tag{2.3}$$

[7]For further details about what a Manifold is, see the Appendix A.1.
[8]This means that $\Phi : \hat{\Sigma} \mapsto \Sigma$ is a homeomorphism [78].
[9]Given the homeomorphic character of (2.1), we can guarantee that each hypersurface does not intersect itself [60].

Thanks to (2.3), the pull-back map can be extended to the multi-linear forms on $\mathcal{T}_{\Phi(p)}(\mathcal{M})$: let $\mathbb{A}$ a n-linear form on $\mathcal{T}_{\Phi(p)}(\mathcal{M})$, then $\Phi^*\mathbb{A}$ will be the n-linear form on $\mathcal{T}_{\Phi(p)}(\mathcal{M})$ defined by [61]

$$\Phi^*\mathbb{A}(v_1, ..., v_n) = \mathbb{A}(\Phi_*(v_1), ..., \Phi_*(v_n)). \tag{2.4}$$

Of fundamental importance in the next subsection will be the map (2.4) on the metric tensor of the spacetime $\mathcal{M}$, denoted by $\mathbf{g}$. The map $\mathbf{h}$ is called the *induced metric on* Σ[10] and is defined as

$$\mathbf{h} = \Phi^*\mathbf{g}, \tag{2.5}$$

such that

$$\forall u, v \in \mathcal{T}_p(\hat{\Sigma}) \otimes \mathcal{T}_p(\hat{\Sigma}) \implies \langle u, v\rangle = \mathbf{g}(\Phi_*(u), \Phi_*(v)) = \mathbf{h}(u, v). \tag{2.6}$$

So, the next step consists in studying this and other operators.

2.2.2 Transverse and parallel projector operators

One of the key ingredients in cosmology is to define a unique family of fundamental observers, whose motion represents the average motion of matter in the universe [4]; observers that we shall represent by a narrow congruence of time-like curves γ representing the points at fixed positions in each hypersurface Σ -or laboratory-. Therefore, the vector v is a member of the tangent vector field $\mathcal{V}$ to this congruence such that, under a parametrization τ, satisfies $\langle v, v\rangle = -1$ on the congruence. Thus, we can see the hypersurface Σ defined above as the set of points on the congruence having the same value of the parameter τ, so it will represent the observer's three-dimensional space-laboratory at some given time [61].

We take a point $p \in \gamma$, and define $\mathcal{V}|_p = v$. Additionally, let $\boldsymbol{\pi}$ and $\mathbf{h}$ be endomorphisms onto $\mathcal{T}_p(\mathcal{M})$, defined as

[10] Or 3-metric for simplicity.

$$\boldsymbol{\pi}(w) = -\langle u, w\rangle\, u, \tag{2.7}$$

$$\mathbf{h}(w) = w - \boldsymbol{\pi}(w), \tag{2.8}$$

for all $w \in \mathcal{V}$. In terms of components, we have

$$\boldsymbol{\pi}(w)^\alpha = \pi^\alpha{}_\beta\, w^\beta : \qquad \pi^\alpha{}_\beta = -\, u^\alpha\, u_\beta, \tag{2.9}$$

$$\mathbf{h}(w)^\alpha = h^\alpha{}_\beta\, w^\beta : \qquad h^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha\, u_\beta. \tag{2.10}$$

Now, we define the subspaces $\mathcal{T}_{\perp p}(\mathcal{M})$ and $\mathcal{T}_{\parallel p}(\mathcal{M})$ by [61]:

$$\mathcal{T}_{\perp p}(\mathcal{M}) = \Big\{ n \in \mathcal{T}_p(\mathcal{M}) \quad | \quad \langle n, v\rangle = 0 \Big\}, \tag{2.11}$$

$$\mathcal{T}_{\parallel p}(\mathcal{M}) = \Big\{ u \in \mathcal{T}_p(\mathcal{M}) \quad | \quad \exists \lambda \in \mathbb{R}, u = \lambda v \Big\}, \tag{2.12}$$

called the orthogonal and parallel projections of the tangent space of $\mathcal{M}$, respectively. By means of the definitions (2.11) and (2.12), we see that

$$\begin{aligned} \boldsymbol{\pi} : \mathcal{T}_p(\mathcal{M}) &\longmapsto \mathcal{T}_{\parallel p}(\mathcal{M}), \\ \mathbf{h} : \mathcal{T}_p(\mathcal{M}) &\longmapsto \mathcal{T}_{\perp p}(\mathcal{M}), \end{aligned}$$

so these operators are identity maps when restricted to $\mathcal{T}_{\parallel p}(\mathcal{M})$ and $\mathcal{T}_{\perp p}(\mathcal{M})$, respectively. Therefore, thanks to the definition of the operators **h** and $\boldsymbol{\pi}$, we have that, for all vectors $w \in \mathcal{T}_p(\mathcal{M})$, we can always decompose them as[11]

$$w = \boldsymbol{h}(w) + \boldsymbol{\pi}(w). \tag{2.13}$$

As a consequence, we can decompose the tangent space $\mathcal{T}_p(\mathcal{M})$ into their orthogonal and parallel parts as

$$\mathcal{T}_p(\mathcal{M}) = \mathcal{T}_{\perp p}(\mathcal{M}) \oplus \mathcal{T}_{\parallel p}(\mathcal{M}). \tag{2.14}$$

[11] Moreover, we can show that this decomposition is unique and that the correspondence between $\mathcal{T}_p(\mathcal{M})$ and $\mathcal{T}_{\parallel p}(\mathcal{M}) \oplus \mathcal{T}_{\perp p}(\mathcal{M})$ is an isomorphism. For more details, see [79].

The maps $\boldsymbol{h}$ and $\boldsymbol{\pi}$ are called the *parallel* and *transverse* operators of v at p [61]. In the case of the spacetime modelled as a 4-manifold, the orthogonal tangent space $\mathcal{T}_{\perp p}(\mathcal{M})$ is the tangent space of the 3-dimensional hypersurface Σ defined in the sub-subsection 2.2.1, and the parallel tangent space $\mathcal{T}_{\parallel p}(\mathcal{M})$ is associated to the time-like wordline of the fundamental observer, then the vector n that spans $\mathcal{T}_{\parallel p}(\mathcal{M})$ is called the *normal time-like vector*.

Furthermore, thanks to (2.14), at each point $p \in \Sigma$, the space of all space-time vectors can be orthogonally decomposed as

$$\mathcal{T}_p(\mathcal{M}) = \mathcal{T}_p(\Sigma) \oplus \mathrm{span}(n), \tag{2.15}$$

where span(n) means the 1-dimensional subspace of $\mathcal{T}_p(\mathcal{M})$ generated by the normal vector n [60].

In the sub-subsection 2.2.1 we noticed that the embedding Φ of $\hat{\Sigma}$ in $\mathcal{M}$ induces the push-forward (2.2) and the pull-back (2.3), but does not induce the inverse mappings[12]. However, the orthogonal projector operator $\boldsymbol{h}$ provides naturally this "reverse" mapping: from its definition, $\boldsymbol{h}$ is a mapping $\mathcal{T}_{\Phi(p)}(\mathcal{M}) \to \mathcal{T}_p(\hat{\Sigma})$, so we can construct with it a mapping $\boldsymbol{h}^* : \mathcal{T}_p^*(\hat{\Sigma}) \to \mathcal{T}_{\Phi(p)}^*(\mathcal{M})$ such that, for any 1-form $\tilde{\omega} \in \mathcal{T}_{\Phi(p)}^*(\hat{\Sigma})$, we shall have

$$\begin{aligned} \boldsymbol{h}^*(\tilde{\omega}) :\ \mathcal{T}_{\Phi(p)}(\mathcal{M}) &\longrightarrow \quad \mathbb{R} \\ v \quad &\longmapsto \langle \tilde{\omega}, \boldsymbol{h}(v) \rangle \,, \end{aligned} \tag{2.16}$$

which defines a linear form in $\mathcal{T}_{\Phi(p)}^*(\mathcal{M})$. Of course, the definition (2.16) can be extended to any multilinear form $\mathbb{A}$ -as we did in (2.4)- acting on $\mathcal{T}_p(\hat{\Sigma})$ as follows

$$\begin{aligned} \boldsymbol{h}^*(\mathbb{A}) :\ \mathcal{T}_{\Phi(p)}(\mathcal{M}) \otimes ... \otimes \mathcal{T}_{\Phi(p)}(\mathcal{M}) &\longrightarrow \qquad \mathbb{R} \\ (v_1, ..., v_n) \qquad &\longmapsto \mathbb{A}(\boldsymbol{h}(v_1), ..., \boldsymbol{h}(v_n)). \end{aligned} \tag{2.17}$$

[12] Namely, the push-forward map provides a bridge from $\mathcal{T}_p(\hat{\Sigma})$ to $\mathcal{T}_{\Phi(p)}(\mathcal{M})$ and the pull-back one from $\mathcal{T}_{\Phi(p)}^*(\mathcal{M})$ to $\mathcal{T}_p^*(\hat{\Sigma})$, but this does not mean that it provides a bridge in a reverse way in both cases.

As we did in (2.5), let us apply this definition to a metric tensor: the induced metric $\boldsymbol{h}$. Hence, $\boldsymbol{h}^*(\boldsymbol{h})$ constitutes a bilinear form on $\mathcal{M}$, which coincides with (2.5) when its arguments are tangent vectors of Σ and it also gives zero if any of its arguments is a vector in $\mathcal{T}_{\perp p}(\mathcal{M})$, i.e. a vector parallel to the normal vector n [60]. Since it constitutes an "extension" of (2.5) to all vectors in $\mathcal{T}_{\Phi(p)}(\mathcal{M})$, we shall denote it by the same symbol:

$$\boldsymbol{h} \coloneqq \boldsymbol{h}^*(\boldsymbol{h}). \tag{2.18}$$

Fortunately, this can be expressed in terms of the spacetime metric $\boldsymbol{g}$ and the dual of the normal vector, denoted as $\tilde{n}$, according to

$$\boldsymbol{h} = \boldsymbol{g} + \tilde{n} \otimes \tilde{n}; \tag{2.19}$$

or in components as

$$h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta, \tag{2.20}$$

that coincides with (2.10) since n is parallel to u.

Finally, thanks to (2.13) and (2.15), the space-time distance between points p and p' infinitesimally close in $\mathcal{M}$ can be decomposed as [61]

$$\delta s^2 = g_{\alpha\beta}\, \delta x^\alpha\, \delta x^\beta = -(\delta t)^2 + (\delta l)^2, \tag{2.21}$$

with

$$(\delta t)^2 = -\pi_{\alpha\beta}\, \delta x^\alpha\, \delta x^\beta, \tag{2.22}$$

$$(\delta l)^2 = \; h_{\alpha\beta}\, \delta x^\alpha\, \delta x^\beta. \tag{2.23}$$

This corresponds to the infinitesimal time interval and space distance between p and p', respectively, as measured by $\mathcal{V}$ as in special relativity[13].

[13] To check that this is the correct interpretation, see the chapter 9 in De Felice's book [61].

2.3 Kinematic properties of cosmological models

In cosmology, a physically motivated choice of preferred motion exists for the matter components of the cosmological model under study. Generally, such an alternative corresponds to a preferred 4-velocity field u^α that generates a preferred congruence of fundamental observers [4]. For this reason, it is fundamental to study the 3+1 decomposition of the spacetime $\mathcal{M}$ about u^α as we did in the former section.

Once this is done, we shall study the kinematics of the cosmological models to describe how the geometrical evolution is with respect to the proper time of the fundamental observers' congruence. Therefore, the fundamental observers will be comoving with the matter-defined 4-velocity u^α; however, if we change our choice of fundamental 4-velocity, the kinematics of the model will transform in a well-defined way [3]. Thus, to describe the spacetime geometry, it is necessary to use comoving-type coordinates adapted to the fundamental observers, which are defined locally as follows [4]:

1) We choose a hypersurface Σ that intersects each fundamental observer worldline and label each integral curve at the point where it intersects Σ with y^i coordinates.
2) We extend this labelling off Σ by maintaining the same labelling for the wordlines at later and earlier times. Thus, the y^i are comoving coordinates: the value of the coordinates is maintained along each worldline.
3) We define a time coordinate t along the fluid flow lines.

In this way, the couple (y^i, t) are said to be comoving coordinates adapted to the flow lines[14]. Even more, this choice of "preferred coordinates" preserves (i) the time transformations $\{t' = t'(t, y^i), y^{i'} = y^i\}$, corresponding

[14]In general, the surfaces $t = cte$ will not be orthogonal to the fundamental worldline; indeed, in general, it is not possible to choose a time coordinate for which these hypersurfaces are orthogonal [4].

to a new choice of time hypersurface and (ii) coordinate transformations $\{t' = t, y^{i'} = y^{i'}(y^i)\}$. Also is convenient to use a *normalized time* s, such that the coordinates $x^\alpha = (s, y^i)$ are called *normalized comoving coordinates*, where s measures the proper time from Σ along the wordlines.

In cosmology, the matter components allow us to make a physically motivated choice of preferred motion[15], which implies a preferred 4-velocity u^μ at each point [4], as a unit time-like vector

$$u^\mu = \frac{dx^\mu}{d\tau} \;\Rightarrow\; \langle u, u \rangle = -1, \tag{2.24}$$

or in normalised comoving coordinates as

$$u^\mu = \delta^\mu_0 \;\Leftrightarrow\; \frac{ds}{d\tau} = 1, \frac{dy^i}{d\tau} = 0. \tag{2.25}$$

In general, this vector will be given by

$$u^\mu = \left(\frac{\partial x^\mu}{\partial s}\right)_{y^i = cte}. \tag{2.26}$$

Notice that this vector is closely linked to the normal vector n and the subspace $\mathcal{T}_{\perp p}(\mathcal{M})$ defined in (2.11).

Let $\gamma(\lambda)$ be a curve on Σ with coordinates (s, y^i), such that it links a set of fundamental observer worldlines in that hypersurface. At all times, the same curve links the same congruence. Strictly speaking, the curve is dragged along by the world lines from Σ to any other hypersurface where $s = cte$. Naturally, we define the vector $\beta^\mu = (dx^\mu/d\lambda)\,\delta\lambda$ tangent to this curve, given in the comoving coordinates by $\beta^\mu = (0, \delta y^i)$ with $\delta y^i = (dy^i/d\lambda)\,\delta\lambda$, such that it links the same pair of infinitesimally closed worldlines. In general coordinates x^μ, this vector will given by

$$\beta^\mu = \left(\frac{\partial x^\mu}{\partial y^i}\right)_{s=cte} \delta y^i. \tag{2.27}$$

[15] We could, for example, choose the cosmic microwave background frame in which the dipole radiation vanishes or the frame in which the total momentum density of all components vanishes.

However, this does not imply that (2.27) will be in general orthogonal to the fundamental observers' worldlines. So, it will represent spatial and temporal displacements between neighbours' fundamental observers [4].

Through (2.27), we define the relative position vector as the orthogonal projection of β^μ with respect to u^μ by means of the $\boldsymbol{h}$ operator defined in (2.19), and denoted as $\beta^{\langle a\rangle}$. This projected vector shall represent a spacetime displacement between neighbours' wordlines if and only if the relative velocity between them is not too large.

Once defined (2.27), the relative velocity will correspond to its time derivative projected orthogonally concerning u^μ to produce a vector in its rest frame $\dot\beta^{\langle a\rangle} := v^a$. So, we define the relative velocity as $v^a = \dot\beta^{\langle a\rangle}$. Even more, as a consequence of (2.26) and (2.27), v^a takes the form

$$v^a = \mathcal{H}^a{}_b\,\beta^{\langle b\rangle} : \qquad \mathcal{H}^a{}_b := h_a^c h_b^d \nabla_d\, u_c = \bar\nabla_b\, u_a, \tag{2.28}$$

where the operator $\bar\nabla_b$ is the 4-dimensional covariant derivative projected onto the three-dimensional hypersurface Σ [4]. As an important fact, one can think about the $\mathcal{H}^a{}_b$ tensor as measuring the failure of β^a to be parallel-transported along the fundamental observers' congruence.

To describe the cosmological kinematic quantities, we have to split $\mathcal{H}^a{}_b$ in its irreducible parts:

$$\mathcal{H}_{ab} = \mathcal{H}_{(ab)} + \mathcal{H}_{[ab]} = \Theta_{ab} + \omega_{ab}, \tag{2.29}$$

where $\Theta_{ab} = \Theta_{(ab)} = \bar\nabla_{(a}u_{b)}$ and $\omega_{ab} = \omega_{[ab]} = \bar\nabla_{[b}u_{a]}$ are known as the *expansion* and *vorticity* tensors, respectively[16]. Moreover,

$$\Theta_{ab} = \Theta_{\langle ab\rangle} = \frac{1}{3}\Theta^c{}_c\, h_{ab} := \sigma_{ab} + \frac{1}{3}\,\Theta\, h_{ab}, \tag{2.30}$$

where σ_{ab} -known as the *shear tensor*- is the projected symmetric tracefree (PSTF) part of Θ_{ab}, i.e. $\sigma_{ab} = \bar\nabla_{<a}u_{a>}$. On the other hand, Θ corresponds

[16] Further details about its definitions can be found in Appendix C.

to the *volume expansion tensor*, as the trace part $\Theta = \bar{\nabla}_a u^a$[17]. Let us interpret the physical meaning of each of these tensor quantities through different types of cosmological models [4]:

★ *A pure expansion universe* ($\omega_{ab} = \sigma_{ab} = 0$): in this situation, if we consider a sphere of galaxies of radius δl (see eq.(2.23)) around us at time t, the distances to all galaxies at time $t + \delta t$ have increased by $dl = \Theta \delta l \delta t/3$ and their directions have all remained unchanged, so the galaxies then form a larger/shorter sphere with each galaxy lying in the same direction as before. Therefore, we have a distortion-free expansion without any rotation.

★ *A pure shear universe* ($\omega_{ab} = \Theta = 0$): in this case, the galaxies sphere at time t will suffer a change in one of its j-principal axis directions, so it will have changed an amount $dl = \sigma_j \delta l \delta t$ and the galaxies' directions remain unchanged. Thus, the galaxies form an ellipsoid, expanded in one direction but contracted in the others, such that the volume is preserved. Hence, we have a pure distortion without any rotation or change of volume.

★ *A pure vorticity universe* ($\sigma_{ab} = \Theta = 0$): in this case, the galaxies experience a rotation that preserves the distances, such that the unique possibles changes are due to pure rotation. So, this represents a pure rotation without distortion or expansion.

2.4 Homogeneous and anisotropic cosmologies

In this subsection, we shall formalise mathematically the concepts given intuitively in subsection 2.1 by means of all the mathematical and physical machinery given in the subsequent subsections, plus another mathematical tool, which is the theory of Lie groups and Lie algebras, that we shall

[17] All of this splitting is invariant because these are tensor equations. Thus, the splitting will be the same irrespective of what coordinates, or frames are used [4].

discuss.

Stationarity means that there exists a Killing vector field which is everywhere time-like, and then the space exhibits time independence. *Staticity*, for its part, additionally requires that the time-like Killing vector field be also hypersurface-orthogonal. If there is a group of motions that leave a point p fixed on the manifold, this is called an *isotropy group*. *Homogeneity* occurs when a group acts transitively on the whole of spacetime; specifically, *spatial homogeneity* occurs when a group acts transitively on spatial sections [6]. All of these things will be formalized in this section to construct the Bianchi models, the heart of this investigation project.

As mentioned in the prolegomenon 1, we will trust or assume the veracity of the Copernican principle, i.e. we shall use cosmologies with three-dimensional spatially homogeneous hypersurface Σ sections. So, we will focus on 4-manifolds with 3-dimensional spatially homogeneous hypersurfaces of simultaneity. However, what does it mean to study a *cosmology*?

> Definition 1 (A cosmology [1]). A cosmology consists of the quartet $(\boldsymbol{\mathcal{M}}, \boldsymbol{g}, \boldsymbol{u}, \boldsymbol{\Gamma})$, with $\boldsymbol{\mathcal{M}}$ being the four-dimensional spacetime manifold, $\boldsymbol{g}$ being the spacetime metric tensor, $\boldsymbol{u}$ being the time-like velocity field of the fundamental observers' congruence and $\boldsymbol{\Gamma}$ being the matter content.

2.4.1 Homogeneity

We think of a manifold $\mathcal{M}$ as possessing a symmetry if the geometry is invariant under a certain transformation that is an endomorphism on $\mathcal{M}$; namely, if the metric is the same, in some sense, from one point to another. Symmetries of the metric are called *isometries* [80].

Definition 2 (Isometry [81]). A diffeomorphism $\varphi : \mathcal{M} \mapsto \mathcal{M}$ will be said to be an *isometry* if it carries the metric into itself, that is, if the mapped metric $\varphi^* \boldsymbol{g}$ by the pull-back map φ^*, is equal to $\boldsymbol{g}$ at every point. Then, the map φ^* preserves the scalar products, as

$$\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y})|_p = \varphi^* \boldsymbol{g}(\varphi_* \boldsymbol{X}, \varphi_* \boldsymbol{Y})|_{\varphi(p)}, \tag{2.31}$$

with $\boldsymbol{X}$ and $\boldsymbol{Y}$ being vector fields on $\mathcal{T}_p(\mathcal{M})$.

Moreover, the set of all one-parameter diffeomorphisms that are isometries forms a group under the binary operation of composition[18], called the *isometry group*.

Definition 3 (Isometry group [63]). The isometry group is denoted by $Isom(\mathcal{M})$ and defined by

$$Isom(\mathcal{M}) := \Big\{ \varphi : \mathcal{M} \longmapsto \mathcal{M} \mid \varphi \text{ is an isometry} \Big\}. \tag{2.32}$$

With this definition, we can formalise mathematically the homogeneous spaces presented in 2.1 as follows.

Definition 4 (Homogeneous spaces [63]). If for each pair of points $p, q \in \mathcal{M}$ there exists a $\varphi \in Isom(\mathcal{M})$ so that $\varphi(p) = q$, then we say that $\mathcal{M}$ is a homogeneous space. In other words, a homogeneous space is a space where you can get from one point to any other point using an isometry.

Thus, homogeneity is a measure of how similar a manifold looks as we move from point to point. To the one-parameter group of diffeomorphisms, a notion of an underlying vector field is attached, such that this vector field is at every point p tangent to the orbit[19] of p [1].

Indeed, if the local one-parameter group of diffeomorphisms φ_t generated by a vector field $\boldsymbol{X}$ is a group of isometries, we call the vector field $\boldsymbol{X}$ a

[18] To check the proof, see chapter 2 in De Felice's book [61].
[19] To check what an orbit is, see the Appendix A.2.

Killing vector field. Thus, the Lie derivative of the metric tensor $\boldsymbol{g}$ with respect to $\boldsymbol{X}$ is

$$\mathcal{L}_{\boldsymbol{X}}\, \boldsymbol{g} = \lim_{t \to 0} \frac{1}{t} \left(\boldsymbol{g} - \varphi_t^* \boldsymbol{g} \right) = 0, \tag{2.33}$$

since $\boldsymbol{g} = \varphi_t^* \boldsymbol{g}$ [81]. In conclusion, on a homogeneous manifold, there always exist Killing vectors -denoted as $\{\boldsymbol{\xi}_\mu\}$- generating isometries by connecting any two points on the manifold; these vectors are fundamental for understanding the notion of *isotropy*.

2.4.2 Isotropy

At any point p on $\mathcal{M}$, the tangent space $\mathcal{T}_p(\mathcal{M})$ is a vector space with the same dimension n of the manifold. Thus, any collection of n-linearly independent vectors in $\mathcal{T}_p(\mathcal{M})$ is a basis for this space [65].

A n-dimensional manifold will be homogeneous if the number of Killing vectors $\{\boldsymbol{\xi}_\mu\}$ is equal to or larger than the dimension of the manifold [63]. Denoting the number of Killing vectors as m, we shall require that $m \geq n$ as a homogeneity requirement. So, the case where $m > n$ means that not all Killing vectors can be linearly independent: the points where this happens are called *singular points* [6]. Furthermore, the difference

$$d = m - n, \tag{2.34}$$

tells us the measure of how many transformations are left that will leave the metric invariant upon having subtracted the n transformations that satisfy the homogeneity requirement on an n-dimensional manifold. In other words, d corresponds to the infinitesimal transformations which leave p fixed. For this reason, d is said to be a measure of what we call *isotropy* [1][20].With all these ideas in mind, we define the *isotropy subgroup* of a point $p \in \mathcal{M}$ as follows:

[20]Many examples of this idea are given in section C of [6].

Definition 5 (Isotropy subgroup [1]). Take a point $p \in \mathcal{M}$. Then, the isotropy subgroup of p is defined as

$$Iso_p(\mathcal{M}) = \Big\{\varphi \in Isom(\mathcal{M}) \,|\, \varphi(p) = p\Big\}. \tag{2.35}$$

Hence, the isotropy group is a subgroup of the isometry group (2.32) that leaves the point p fixed [63]. Moreover, the isometry group and isotropy subgroup are *Lie groups*[21] associated with a *Lie Algebra*[22].

The importance of the concept of Lie algebra is that there is a certain finite-dimensional Lie algebra intimately associated with each Lie group and that properties of the Lie group are reflected in properties of its Lie algebra. The connection between these two topics is given by the following proposition[23]:

Proposition 2.1 (The Lie algebra of a Lie group [63]). Let $\mathcal{G}$ be a Lie group. Then the tangent space of $\mathcal{G}$ at the identity element $\mathcal{T}_e(\mathcal{G})$ is a Lie algebra, i.e.

$$\mathfrak{g} = \mathcal{T}_e(\mathcal{G}). \tag{2.36}$$

As we saw, the Killing vectors $\{\boldsymbol{\xi}_\mu\}$ are the generators of the isometries on the manifold at a particular point p. Moreover, $\{\boldsymbol{\xi}_\mu\}$ corresponds to an element of the Lie algebra of the isometry Lie group $Isom(\mathcal{M})$. Furthermore, the Killing vector field of the whole manifold forms a finite-dimensional vector space, such that its algebra is isomorphic to the Lie algebra of

[21] A Lie group is a group $\mathcal{G}$ which is also a manifold with a $\mathcal{C}^\infty$ structure, such that for all $x, y \in \mathcal{G}$, we have that $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are $\mathcal{C}^\infty$ functions [82]. This topic will be deepened throughout the development of the thesis.

[22] A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ together with a bilinear operator $[\quad,\quad] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called *bracket* such that for all $x, y, z \in \mathfrak{g}$ the bilinear operator is anti-commutative and satisfy the Jacobi identity [83]. Again, this topic will be deepened throughout the development of the thesis.

[23] To check the proof, see section 3.3 in [62].

$Iso(\mathcal{M})$ [63].

Until now, we have two vector fields: the Killing vectors field $\{\boldsymbol{\xi}_\mu\}$ and the vector field associated with the Lie algebra at the identity element e as a consequence of the theorem 2.1, denoted as $\{\boldsymbol{e}_\mu\}$. Since both arise from the same Lie group, we may wonder if they represent the same algebra. We assume that both vector fields coincide at a point $p \in \mathcal{M}$; of course, this choice is always possible since they are linearly independent by assumption and both span the tangent space at every point. Thus, there will exist, by the inverse function theorem for manifolds[24], an invertible Jacobian matrix as a map 1-1 between these vector fields [85]. However, this matrix is position-dependent. Quite the contrary happens with the structure constants, so we can evaluate them without trouble at the point p.

Let $\mathcal{D}^\lambda_{\mu\nu}$ be the structure coefficients of the Killing vector, and $\mathcal{C}^\lambda_{\mu\nu}$ the structure coefficients of the Lie algebra at the identity element of the Lie groups, such that

$$[\boldsymbol{\xi}_\mu, \boldsymbol{\xi}_\alpha] = \mathcal{D}^\lambda_{\mu\nu}\, \boldsymbol{\xi}_\lambda \qquad \text{and} \qquad [\boldsymbol{e}_\mu, \boldsymbol{e}_\alpha] = \mathcal{C}^\lambda_{\mu\nu}\, \boldsymbol{e}_\lambda. \tag{2.37}$$

The structure constants are just different representations of the same Lie algebra, so if we choose that the vector fields $\boldsymbol{\xi}_\mu$ and $\boldsymbol{e}_\alpha$ coincide at one point, then the structure constants will differ only by a sign[25]. Once the Lie algebra of the Killing vector fields has been studied, we can understand the algebraic properties of the isometry group $Isom(\mathcal{M})$ and the isotropy subgroup $Iso_p(\mathcal{M})$. With all of this in mind, we come back for a while to the definition of the eq.(2.34): when not all the Killing vectors at p are linearly independent, they have to span a tangent space of dimension

[24]Suppose M and N are smooth manifolds, and $F : M \to N$ is a smooth map. If $p \in M$ is a point such that the differential of F at p, dF_p, is invertible, then there are connected neighbourhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism [84].

[25]Strictly speaking, we say that $\boldsymbol{e}_\alpha$ defines a left-invariant frame, while $\boldsymbol{\xi}_\mu$ defines the right-invariant one [62]. To see the proof of this statement, see Chapter 15 in [63].

$s < r$, with r being the dimension of $Isom(\mathcal{M})$. So, analogously as we did in (2.34), we can have a measure of isotropy as the difference.

$$d = r - s, \tag{2.38}$$

which represents the dimension of the isotropy subgroup $Iso_p(\mathcal{M})$. Thus, the Killing vector fields that vanish at p form a subgroup of dimension d that leaves p fixed, according to (2.34). With all of these tools, we can classify both the isotropic and homogeneous properties of the space as follows [1] :

- ⋆ The dimension d of $Iso_p(\mathcal{M})$ of the manifold $(\mathcal{M}, \boldsymbol{g})$ determines the *isotropic properties* of the manifold.
- ⋆ The dimension s of the orbit of $Isom(\mathcal{M})$, i.e. the $\dim((\boldsymbol{\xi}_\mu))$ at a point p determines the *homogeneity properties* of the manifold.

According to definition 1, we are interested in four dimensional manifolds. Furthermore, the case $s = 4$ has to correspond to a static universe because there is no change with respect to time. Moreover, we shall study expanding cosmological models so that we shall impose that the dimension of the orbit at $p \in \mathcal{M}$ under $Isom(\mathcal{M})$ is equal to the dimension of the spatial hypersurface Σ, i.e. we impose $s = 3$.

Once specified s, we must also specify the dimension d of the isotropy group [1]:

- ★ $\boldsymbol{d = 3}$: *Isotropic*. These models correspond to $r = 3$ and are maximally symmetric, known as the Friedmann-Lemaître-Robertson-Walker models.
- ★ $\boldsymbol{d = 1}$: *Locally rotationally symmetric (LRS)*. In this case, we must have $r = 4$, such that $Iso_p(\mathcal{M}) = SO(2)$.
- ★ $\boldsymbol{d = 0}$: *Anisotropic*. They are called **the Bianchi models**.

In this research work, we will focus only on the Bianchi models. So, we shall restrict our study to consider a manifold that has

$$\boxed{s = 3 \qquad \text{and} \qquad d = 0.} \tag{2.39}$$

Along the document, we shall see that the algebraic properties of the Lie group can be understood in terms of the corresponding Lie algebra, thanks to the theorem 2.1. To understand the total anisotropy case, we shall classify the three-dimensional Lie-algebras by means of the orthonormal frame.

2.5 Orthonormal frame formalism

A cosmological model $(\mathcal{M}, \mathbf{g}, \mathbf{u}, \mathbf{\Gamma})$ describes the universe via a Lorentzian metric $\mathbf{g}$, an expanding observer congruence with 4-velocity $\mathbf{u}$, and matter content $\mathbf{\Gamma}$. Its dynamics results from interactions between matter and geometry, governed by the Einstein field equations.

Based on the ideas of Ellis and MacCallum [86], it is useful to express the cosmological models in terms of a vector basis $\{e_\mu\}$ and its dual 1 -form basis $\{\omega^\mu\}$, ensuring mutual orthogonality and unit length, with e_0 being timelike. The metric tensor components then satisfy the relation

$$\mathbf{g}(e_\mu, e_\nu) = \eta_{\mu\nu} : \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1). \tag{2.40}$$

Relative to this orthonormal frame, the line element takes the form

$$ds^2 = \eta_{\mu\nu}\, \omega^\mu \otimes \omega^\nu. \tag{2.41}$$

For any given basis of vector fields e_μ, the commutators $[e_\mu, e_\nu]$ also form vector fields and can therefore be expressed as a linear combination of the basis vectors:

$$[e_\mu, e_\nu] = \gamma^\lambda{}_{\mu\nu}\, e_\lambda : \qquad \gamma^\lambda{}_{\mu\nu} = \gamma^\lambda{}_{\mu\nu}\,(x^\beta), \tag{2.42}$$

where the coefficients $\gamma^{\lambda}{}_{\mu\nu}$ are called *the commutation functions of the basis*[26].

By means the Levi-Civita connection to define the covariant derivatives ∇, the connection components relative to a basis $\{e_\mu\}$ are the set of functions $\Gamma^{\beta}{}_{\mu\nu}$ defined by writing the vector fields $\nabla_{e_\nu} e_\mu$ as linear combinations of the basis vectors:

$$\nabla_{e_\nu} e_\mu = \Gamma^{\beta}{}_{\mu\nu}\, e_\beta.$$

Since the connection is assumed to be torsion-free and compatible with the metric ($\nabla g = 0$), it follows that the connection components are related to the derivatives of the metric components and the commutation functions by means of [87]

$$\Gamma_{\beta\mu\nu} = \frac{1}{2}\Big[e_\mu(g_{\beta\nu}) + e_\nu(g_{\mu\beta}) - e_\beta(g_{\nu\mu}) + \gamma^{\alpha}{}_{\nu\mu}\, g_{\beta\alpha} + \gamma^{\alpha}{}_{\beta\nu}\, g_{\mu\alpha} - \gamma^{\alpha}{}_{\mu\beta}\, g_{\nu\alpha}\Big], \tag{2.43}$$

where $\Gamma_{\beta\mu\nu} = g_{\beta\omega}\, \Gamma^{\omega}_{\mu\nu}$.

If we use an orthonormal frame (2.41), the 24 commutation functions $\gamma^{\lambda}{}_{\mu\nu}$ are the basic variables and the gauge freedom present in the Einstein equations is reduced to an arbitrary Lorentz transformation, representing the freedom in the choice of the orthonormal frame [88]. Additionally, when we apply the usual Jacoby identities for vector fields to the basis vectors $\{e_\mu\}$, they yield a set of 16 identities summarised as

$$e_{[\mu}\gamma^{\beta}_{\nu\omega]} - \gamma^{\beta}_{\alpha[\mu}\gamma^{\alpha}_{\nu\omega]} = 0. \tag{2.44}$$

These identities, combined with the Einstein field equations, yield first-order evolution equations for certain commutation functions and impose constraints involving only spatial derivatives. This approach, known as

[26] The connection between these functions and the structure constants in (2.37) will be relevant for the classification of Bianchi models in the following chapter.

the *orthonormal frame formalism*[27], naturally facilitates the application of dynamical systems methods due to its direct formulation in terms of first-order evolution equations.

Essentially, the orthonormal frame formalism provides a 1+3 decomposition of the Einstein field equations into evolution and constraint equations, relative to the timelike vector field $\mathbf{e}_0$ in an orthonormal frame $\{\mathbf{e}_\mu\}$. In cosmology, $\mathbf{e}_0$ is typically chosen as the fundamental 4-velocity $\mathbf{u}$ or, in models with an isometry group, as the normal to the spacelike group orbits. The expansion rate scalar Θ plays a crucial role in the evolution equations, but in cosmological applications, it is usually replaced by the Hubble scalar $H = \frac{1}{3}\Theta$ [88].

2.5.1 Conmutation functions as variables

In the orthonormal frame in which the metric has components $\eta_{\mu\nu}$ and the basic variables are the commutation functions $\gamma^{\beta}{}_{\mu\nu}$, the components of the Levi-Civita connection given in (2.43) reduce to

$$\Gamma_{\beta\mu\nu} = \frac{1}{2}\left(\gamma^{\alpha}{}_{\nu\mu}\,\eta_{\beta\alpha} + \gamma^{\alpha}{}_{\beta\nu}\,\eta_{\mu\alpha} - \gamma^{\alpha}{}_{\mu\beta}\,\eta_{\nu\alpha}\right). \tag{2.45}$$

We shall represent the commutation functions in terms of fundamental algebraic quantities, ensuring that some possess clear physical or geometrical meaning. Specifically, the commutation function with a time index can be expressed through the kinematic properties of the timelike congruence $(u = e_0)$, including the scalar expansion Θ, the four-acceleration $\dot{u}_\mu$, the shear $\sigma_{\mu\nu}$, and the vorticity vector ω_μ, as introduced in Section 2.3. Additionally, it incorporates the quantity

$$\Omega^a = \frac{1}{2}\,\epsilon^{abc} e^{\mu}_{b} e_{c\mu;\nu} u^{\nu}, \tag{2.46}$$

[27] The pioneers behind this formalism in the cosmological context were Ellis [89], Ellis & MacCallum [86] and McCallum [6].

which represents the local angular velocity of the spatial frame $\{e_a\}$ relative to a Fermi-propagated spatial frame[28]. Furthermore, it can be shown that the commutation functions can be rewritten as [88]

$$\begin{aligned}
\gamma^a_{\ 0b} &= -\sigma_b^{\ a} - H\delta_b^{\ a} - \epsilon^a_{\ bc}(\omega^c + \Omega^c),\\
\gamma^0_{\ 0a} &= \dot{u}_a,\\
\gamma^0_{\ ab} &= -2\epsilon_{ab}^{\ \ c}\omega_c.
\end{aligned} \tag{2.47}$$

Here, σ_{ab}, ω_a, and $\dot{u}_a$ represent the nonzero components of $\sigma_{\mu\nu}$, ω_μ, and $\dot{u}_\mu$ in the orthonormal frame, respectively. On the other hand, the spatial components of the commutation functions can be decomposed into a symmetric two-index quantity n_{ab} and a one-index quantity a_b, given by [88]:

$$\gamma^c_{ab} = \varepsilon_{abm}n^{mc} + a_a\delta^c_b - a_b\delta^c_a. \tag{2.48}$$

The decomposition introduced, in conjunction with the orthonormal frame formalism, will play a fundamental role in the systematic classification of Bianchi cosmologies as formulated by Schücking, Behr, and Bianchi [90]. A comprehensive analysis of this classification will be undertaken in the subsequent chapter.

[28]For further details, see Chapter 6 in [64].

Chapter 3

Bianchi classification

This chapter builds extensively on chapter 4 in *Cosmological models from a geometric point of view* [6], chapter 15 in *Einstein's General Theory of Relativity* [63], chapter 1 and 6 in *Dynamical Systems in Cosmology* [88], *A Class of Homogeneous Cosmological Models* [86], *On the Three-Dimensional Spaces which Admit a Continuous Group of Motions* [91] and *The Bianchi models: then and now* [92].

In section 2.2, we assume that the four-dimensional manifold $\mathcal{M}$ can be foliated in three-dimensional spatial sections:

$$\mathcal{M} = \mathbb{R} \times \Sigma_t.$$

The time variable is represented by $\mathbb{R}$, with each Σ_t corresponding to a three-dimensional homogeneous hypersurface labelled by a specific time parameter. However, *how many different possibilities do we have for Σ_t under these conditions?* The answer depends on how many different Lie algebras we have in three dimensions. What Bianchi really did was to classify the three-dimensional Lie algebras. Still, the Lie algebra can be taken as that of the isometry group, so that it can be shown that both turn out to be equivalent [92].

Definition 6 (Bianchi cosmology [88]). A Bianchi cosmology $(\mathcal{M}, \mathbf{g}, \mathbf{u}, \mathbf{\Gamma})$ is a model whose metric admits a three-dimensional group of isometries acting simply[a] transitively on spacelike hypersurfaces, which are surfaces of homogeneity Σ_t in spacetime.

[a]What act simply transitively means is a matter of discussion in Appendix A.2.

Thus, any Bianchi model admits a Lie algebra of Killing vector fields with basis ξ_i and structure constants $\mathcal{C}^i_{jk}$ such that satisfy the condition (2.37).

The group orbits, referred to as surfaces of homogeneity, have associated tangent vectors $\boldsymbol{\xi}_i$. The fundamental 4-velocity $\mathbf{u}$ may align perpendicularly with these orbits, characterizing orthogonal (non-tilted) models, or it may have a component along them, defining tilted models. We shall focus on non-tilted models.

Following the Bianchi ideas, we shall classify the different Lie algebras of the Killing vector fields, and hence the associated isometry group [91]. This is equivalent to classifying the structure constants $\mathcal{C}^i_{jk}$, which transform as a tensor of rank $(1,2)$ under a change of the basis of the Lie algebra and satisfy the following two restrictions:

$$\mathcal{C}^i_{jk} = -\mathcal{C}^i_{kj} \qquad \text{and} \qquad \mathcal{C}^i_{j[k}\mathcal{C}^n_{lm]} = 0. \tag{3.1}$$

The first condition reflects the antisymmetry of the structure constants, while the second is equivalent to the fulfilment of the Jacobi identity. The Bianchi classification is based on the distinct types of three-dimensional Lie algebras, each designated by a Roman numeral from I to IX. Utilizing any of these algebras, one can formulate a spatially homogeneous cosmological model. However, the defining feature of each Bianchi model lies in its specific structure constants, which are determined through a decomposition originally attributed to Bianchi [91], Schücking [90] and Behr [93][29].

3.1 Bianchi-Schücking-Behr classification

Based on the decomposition made for the commutation functions in (2.48), we shall decompose the structure constants in terms of the trace-free part and trace part by means of

$$\mathcal{C}^k_{ij} = \varepsilon_{ijm}\hat{n}^{mk} + \hat{a}_i\delta^k_j - \hat{a}_j\delta^k_i, \tag{3.2}$$

[29] This classification has an interesting history; for details, see [94].

where $\hat{n}^{mk} = \hat{n}^{km}$ and $\hat{a}_i$ are constants. The hats distinguish these quantities from the corresponding ones in (2.48), which are not constant in general [88]. The antisymmetric nature of $\mathcal{C}^k_{ij}$ becomes manifest when expressed in this form, while the Jacobi identity (3.1) is equivalently rewritten as

$$\hat{n}^{ij}\hat{a}_j = 0. \tag{3.3}$$

By allowing an arbitrary linear transformation of the tangent space at a given point in spacetime, the tensors $\mathcal{C}^k_{ij}$ -and consequently, the associated isometry groups- are categorized into ten distinct equivalence classes. These classes are systematically characterized using the quantities $\hat{n}^{ij}$ and $\hat{a}_j$ [59]. The classification begins by distinguishing between class A, where $\hat{a}_j = 0$, and class B, where $\hat{a}_j \neq 0$.

From equation (3.3), there remains the freedom to perform an arbitrary time-dependent rotation of the basis $\{e_a\}$ within each homogeneous hypersurface Σ_t. This freedom allows for $\hat{n}^{ij}$ to be rendered diagonal at each Σ_t, while also permitting a suitable relabelling or reversal of the coordinate axes to modify the signs of its diagonal elements. Consequently, an appropriate basis can always be selected such that

$$\hat{n}_{ij} = \mathrm{diag}\,(\hat{n}_1, \hat{n}_2, \hat{n}_3) \quad \text{and} \quad \hat{a}^i = (\hat{a}, 0, 0), \tag{3.4}$$

and (3.3) then reads

$$\hat{n}_1\hat{a} = 0. \tag{3.5}$$

This enables the classification of each Bianchi model class by examining the eigenvalue signatures of the symmetric tensor $\hat{n}^{ij}$, as outlined in table 3.1. The designation of these equivalence classes adheres to the Bianchi-Schücking-Behr classification framework [94].

In table 3.1, the last column shows which equivalence classes admit one of the three Robertson-Walker models. Each equivalence class of tensors

$\mathcal{C}_{ij}^{k}$ forms a linear submanifold of the space of all 3-index tensors. The dimensions of these submanifolds are given in the fourth column.

Group Class	Group Type	Eigenvalues of $\hat{n}^{ij}$	Dimension	Contains Robertson-Walker?
A	I	$0, 0, 0$	0	k=0
	II	$+, 0, 0$	3	
	VI_0	$0, +, -$	5	$k = 0$
	VII_0	$0, +, +$	5	
	VIII	$-, +, +$	6	$k = +1$
	IX	$+, +, +$	6	
B	V	$0, 0, 0$	3	$k = -1$
	IV	$0, 0, +$	5	
	VI_h	$0, +, -$	6	$k = -1$
	VII_h	$0, +, +$	6	

Table 3.1: Classification of the Bianchi cosmologies into ten equivalence classes using the eigenvalues of $\hat{n}^{ij}$. *Taken from* [59].

The VI_h and VII_h classes are unique in that they admit a further refinement into five-dimensional equivalence classes, each distinguished by a parameter $h \neq 0$. This parameter is determined through the relation [59]

$$\hat{a}_b\, \hat{a}_c = \frac{h}{2}\, \epsilon_{bik}\, \epsilon_{cjl}\, \hat{n}^{ij}\, \hat{n}^{kl}. \tag{3.6}$$

Or taking into account (3.5), it can be rewritten as

$$\hat{a}^2 = h\, \hat{n}_2\, \hat{n}_3, \tag{3.7}$$

so that h is well defined if and only if $\hat{n}_2\, \hat{n}_3 \neq 0$ in class B models. Note that $h < 0$ in type VI_h and $h > 0$ in type VII_h. Additionally, the type VI_h with $h = -1$ is sometimes referred to as Bianchi type III [63].

3.2 How to construct a Bianchi cosmology?

Let $\hat{n}$ denote the unit vector field orthogonal to the orbits of the isometry group $\text{Isom}(M)$ (see Definition 3)[30]. Consequently, $\hat{n}$ is tangent to a geodesic congruence and satisfies the condition $\hat{n}_{[\mu,\nu]} = 0$. This naturally leads to the definition of the time coordinate t, given by [88]:

[30] In practice, it will be the 4-velocity of the fundamental observes.

$$n_\mu = -t_{,\mu},$$

such that the group orbits are contained within some hypersurface Σ_t. Furthermore, the unit vector $\hat{n}$ remains invariant under the action of the isometry group defined in (2.37); that is[31],

$$[\xi_a, \hat{n}] = 0. \tag{3.8}$$

We can select a triad of spacelike vector fields $\{e_a\}$ that are tangent to the group orbits, satisfying the orthogonality condition $\mathbf{g}(\hat{n}, e_a) = 0$. Additionally, these vectors are either Lie-dragged along the Killing vector fields or, conversely, the Killing vector fields are Lie-dragged along them:

$$\pounds_{\mathbf{e}_j}\xi_i = -\pounds_{\xi_i}\mathbf{e}_j = 0. \tag{3.9}$$

The frame $\{\hat{n}, e_a\}$ is commonly referred to as the *group-invariant* frame [88]. Consequently, the components of the metric in this frame, as well as the commutation functions, must satisfy the following conditions:

$$g_{00} = -1, \qquad g_{0i} = 0, \qquad g_{ij} = g_{ij}(t) \quad \text{and} \quad \gamma^{\lambda}{}_{\mu\nu} = \gamma^{\lambda}{}_{\mu\nu}(t). \tag{3.10}$$

Since the congruence $\hat{n}$ is hypersurface-orthogonal, the vector fields $\{e_a\}$ form a Lie algebra with structure constants $\gamma^{a}{}_{bc}$, which remain constant within each hypersurface Σ_t. Moreover, this algebra is isomorphic to the Lie algebra of the Killing vector fields, as established in (2.37). Consequently, the classification of Bianchi cosmologies can be performed using either the structure constants $\mathcal{C}^{k}{}_{ij}$ (i.e., $\hat{n}^{ij}$ and $\hat{a}_i$) or the spatial commutation functions $\gamma^{k}{}_{ij}$ (i.e., n^{ij} and a_i). The remaining arbitrariness in the frame choice corresponds to a time-dependent linear transformation, explicitly given by

$$\tilde{e}_i = \Lambda_i^{\,j}(t)\, e_j. \tag{3.11}$$

[31] In fact, the opposite is also true: the isometry group is invariant under the unit vector $\hat{n}$ [63].

By exploiting this remaining freedom, we can define a set of *time-independent* spatial vector fields $\{E_i\}$ that satisfy the condition given in (3.8). Consequently, the commutation functions remain constant, allowing us to apply a constant linear transformation that equates them to the structure constants of the corresponding Lie algebra,

$$[E_i, E_j] = \mathcal{C}^k_{\ ij}\, E_k. \tag{3.12}$$

In this approach, known as the *metric approach* [88], the basic variables are the frame components of the metric $g_{ij}(t)$[32]. The one-forms ω^i, which are dual to the frame vectors E_i are also group invariant, time-independent and satisfy the Maurer-Cartan structure equation

$$\mathrm{d}\,\omega^k = -\frac{1}{2}\,\mathcal{C}^k_{\ ij}\,\omega^i \wedge \omega^j. \tag{3.13}$$

Here, d represents the exterior derivative and $\wedge$ denotes the exterior product. In essence, constructing a Bianchi-type cosmological model requires selecting the appropriate structure constants corresponding to its Lie algebra and solving equation (3.13) to determine the specific basis of one-forms. This procedure enables the explicit construction of the metric components, allowing the spacetime line element to be expressed as

$$ds^2 = -dt^2 + g_{ij}(t)\,\omega^i \otimes \omega^j. \tag{3.14}$$

In general, this metric will exhibit the symmetries associated with the corresponding Bianchi group, ensuring consistency with the underlying algebraic structure of the model. Additionally, in Bianchi models with conformal expansion, we can always choose coordinates so that the line-element takes the form [95]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\tilde{h}_{ab}\,(x^c)\,\mathrm{d}x^a\ \mathrm{d}x^b, \tag{3.15}$$

where $a(t)$ is the scalar factor that controls the distances in Σ_t[33], $\tilde{h}_{ab}(x^c)$ is the spatial part of the induced metric defined in (2.18) and is related

[32] In the context of Bianchi models, this approach is quite useful when we want to introduce the Lagrangian and Hamiltonian formalism [63].

[33] Essentially, it is the scalar factor analogue to the usual FLRW cosmologies.

with them by means of a conformal transformation $h_{ab}(x^\mu) = a^2(t)\,\tilde{h}_{ab}(x^c)$. We shall study this important particular case in chapter 5 (cf. theorem 6.3).

Below, for each Bianchi model, referred to as $\mathcal{B}$(I-IX), we present the corresponding structure constants alongside the basis 1-forms that satisfy equation (3.13), based on chapter 11 in [96] and chapter 2 in [1].

- ⋆ $\mathcal{B}$(I): Class A. The tensor $\hat{n}^{ij}$ has three zero eigenvalues, indicating that the group is Abelian. Consequently, the structure constants vanish, i.e., $\mathcal{C}^k{}_{ij} = 0$. A suitable set of basis 1-forms is given by

$$\{\omega^i\} = \{\mathrm{dx}, \quad \mathrm{dy}, \quad \mathrm{dz}\}.$$

- ⋆ $\mathcal{B}$(II): Class A. The tensor $\hat{n}^{ij}$ has two zero eigenvalues, with the only non-zero structure constant given by $\mathcal{C}^3_{\ 12} = -1$. A suitable set of basis 1-forms is given by

$$\{\omega^i\} = \{\mathrm{dx}, \quad \mathrm{dy}, \quad \mathrm{dz} - x\,\mathrm{dy}\}.$$

- ⋆ $\mathcal{B}$(III): Class B. The tensor $\hat{n}^{ij}$ possesses a single zero eigenvalue, while the remaining two eigenvalues exhibit opposite signs. Type III can be decomposed into the direct sum of Lie algebras of dimensions 1 and 2. Additionally, it can be regarded as a special case of type VI_h with the parameter $h = -1$. The non-zero structure constant is $\mathcal{C}^3_{\ 13} = 1$ and a set of 1-form basis is

$$\{\omega^i\} = \left\{\mathrm{d}\mathbf{x}, \quad \mathrm{d}\mathbf{y}, \quad e^{-x}\ \mathrm{d}\mathbf{z}\right\}.$$

- ⋆ $\mathcal{B}$(IV): Class B. The symmetric tensor $\hat{n}^{ij}$ possesses two zero eigenvalues, with the nonzero structure constants given by $\mathcal{C}^3{}_{13} = \mathcal{C}^3{}_{12} = \mathcal{C}^2{}_{12} = 1$. A corresponding set of left-invariant one-forms satisfying these relations is given by

$$\{\omega^i\} = \{\ \mathrm{d}\mathbf{x}, \quad e^{-x}\ \mathrm{d}\mathbf{y}, \quad e^{-x}\,(\ \mathrm{d}\mathbf{z} - x\ \ \mathrm{d}\mathbf{y})\}.$$

* $\mathcal{B}$(V): Class B. The symmetric tensor $\hat{n}^{ij}$ possesses three zero eigenvalues, with the nonzero structure constants given by ${\mathcal{C}^3}_{13} = 1$ and ${\mathcal{C}^2}_{12} = 1$. A corresponding set of left-invariant one-forms satisfying these relations is given by

$$\{\omega^i\} = \{\ \mathrm{d}\mathbf{x}, \quad e^{-x}\ \mathrm{d}\mathbf{y}, \quad e^{-x}\ \mathrm{d}\mathbf{z}\}.$$

* $\mathcal{B}$(VI$_0$): Class B. The symmetric tensor $\hat{n}^{ij}$ has one zero eigenvalue. The non-vanishing structure constants are ${\mathcal{C}^2}_{12} = -1$ and ${\mathcal{C}^3}_{13} = 1$. A corresponding set of one-forms basis satisfying these relations is given by

$$\{\omega^i\} = \{\ \mathrm{d}\mathbf{x}, \quad e^{x}\ \mathrm{d}\mathbf{y}, \quad e^{-x}\ \mathrm{d}\mathbf{z}\}.$$

* $\mathcal{B}$(VI$_h$): Class B. The symmetric tensor $\hat{n}^{ij}$ has one zero eigenvalue. This is a one-parameter family of invariant sets. The non-vanishing structure constants are ${\mathcal{C}^2}_{12} = p$ and ${\mathcal{C}^3}_{13} = 1$. A corresponding set of one-forms basis satisfying these relations is given by

$$\{\omega^i\} = \{\ \mathrm{d}\mathbf{x}, \quad e^{-px}\ \mathrm{d}\mathbf{y}, \quad e^{-x}\ \mathrm{d}\mathbf{z}\}.$$

* $\mathcal{B}$(VII$_0$): Class A. The symmetric tensor $\hat{n}^{ij}$ has one zero eigenvalue. The non-vanishing structure constants are ${\mathcal{C}^2}_{13} = -1$ and ${\mathcal{C}^3}_{12} = 1$. A set of one-form basis satisfying these relations is

$$\{\omega^i\} = \{\ \mathrm{d}\mathbf{x}, \quad \sin(x)\ \mathrm{d}\mathbf{z} - \cos(x)\ \mathrm{d}\mathbf{y}, \quad \cos(x)\ \mathrm{d}\mathbf{z} + \sin(x)\ \mathrm{d}\mathbf{y}\}.$$

* $\mathcal{B}$(VII$_h$): Class B. The symmetric tensor $\hat{n}^{ij}$ has one zero eigenvalue. This is a one-parameter family of one-form basis. The non-vanishing structure constants are ${\mathcal{C}^2}_{12} = {\mathcal{C}^3}_{13} = q$, ${\mathcal{C}^2}_{13} = -1$ and ${\mathcal{C}^3}_{12} = 1$. A set of one-form basis satisfying these relations is

$$\{\omega^i\} = \{\ \mathrm{d}\mathbf{x}, \quad e^{-qx}\,(\sin(x)\ \mathrm{d}\mathbf{z} - \cos(x)\ \mathrm{d}\mathbf{y}), \quad e^{-qx}\,(\cos(x)\ \mathrm{d}\mathbf{z} + \sin(x)\ \mathrm{d}\mathbf{y})\}.$$

* $\mathcal{B}$(VIII): Class A. The symmetric tensor $\hat{n}^{ij}$ does not have zero eigenvalues. The non-vanishing structure constants are ${\mathcal{C}^1}_{23} = {\mathcal{C}^2}_{31} = {\mathcal{C}^3}_{21} = -1$. A set of one-form basis satisfying these relations is

$$\{\omega^i\} = \{\cosh(y)\cos(z)\ \mathrm{d}\mathbf{x} - \sin(z)\ \mathrm{d}\mathbf{y}, \quad \cosh(y)\sin(z)\ \mathrm{d}\mathbf{x} + \cos(z)\ \mathrm{d}\mathbf{y},$$
$$\sinh(y)\ \mathrm{d}\mathbf{x} + \ \mathrm{d}\mathbf{z}\}.$$

⋆ $\mathcal{B}$(IX): Class A. The symmetric tensor $\hat{n}^{ij}$ does not have zero eigenvalues. The non-vanishing structure constants are $\mathcal{C}^1_{\ 23} = \mathcal{C}^2_{\ 31} = \mathcal{C}^3_{\ 12} = 1$. A set of one-form basis satisfying these relations is

$$\{\omega^i\} = \{(\cos(y)\cos(z) - \sin(z))\ \mathrm{d}\mathbf{x}, \quad \cos(y)\sin(z)\ \mathrm{d}\mathbf{x} + \cos(z)\ \mathrm{d}\mathbf{y},$$
$$-\sin(y)\ \mathrm{d}\mathbf{x} + \ \mathrm{d}\mathbf{z}\}.$$

Chapter 4

Abelian and Non-Abelian gauge theories

The power of science is acquired through a kind of pact with the devil: at the expense of a progressive evanescence of the everyday world. Science becomes monarch, but when it does, its kingdom is hardly more than a realm of ghosts.

—*Ernesto Sabato. One and the universe.* (1945).

This chapter builds extensively on chapter 8 in *The quantum theory of fields volume 1* [97], chapter 15 in *The quantum theory of fields volume 2* [54], *A new pedagogical way of finding out the gauge field strength tensor in Abelian and non-Abelian local gauge field theories* [98], *Bianchi cosmologies with p-form gauge fields* [36] and *Balancing anisotropic curvature with gauge fields in a class of shear-free cosmological models* [95].

We are interested in gauge theories. Concretely, we want to find out a specific configuration of non-Abelian gauge fields in addition to the usual matter content of the universe, as a candidate to sustain the anisotropies present in the extended FLRW shear-free models presented briefly in chapter 1.

A gauge theory is a type of field theory in which the Lagrangian and, furthermore, the dynamics of the system itself are invariant under local transformations according to certain Lie groups. The term gauge, in turn, refers to any specific mathematical formalism to regulate redundant degrees of freedom in the Lagrangian of a physical system. The transformations between possible gauges, called *gauge transformations*, form a Lie group, referred to as the symmetry group or the gauge group of the theory.

Associated with any Lie group is the Lie algebra of group generators. For each group generator, a corresponding field, called the *gauge field*, is included in the Lagrangian to ensure the invariance of the action under the local group transformations, known as *gauge invariance*. If the symmetry group is non-commutative, then the gauge theory is referred to as non-Abelian gauge theory [54]. To study this kind of matter configuration, we shall describe the backbone behind the Abelian and non-Abelian gauge theories for their implementations in the following chapters.

4.1 Abelian gauge field theories

Abelian gauge theories are those in which the gauge group is commutative, meaning the generators of the group commute with each other. In Abelian gauge field theories, the transformations acting on a fermion field commute. We focus here on the case of the $U(1)$ gauge group, under which the fermion field ψ transforms according to

$$\psi' = e^{ig\epsilon(\vec{x})}\,\psi. \tag{4.1}$$

In this context, the prime symbol denotes the transformed quantity, g is the coupling constant[34], and $\epsilon(\vec{x})$ is a spatially dependent scalar field specifying the magnitude of the transformation.

The guiding principle is that the complete Lagrangian describing a fundamental interaction should remain invariant under transformations of certain chosen groups. The first component of this complete Lagrangian is Dirac's Lagrangian, which encapsulates the mass and kinetic properties of a fermion field:

$$\mathcal{L}_D = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi. \tag{4.2}$$

[34] Also referred to as a gauge coupling parameter, a numerical value indicating the strength of the interaction [54].

Here, $\bar{\psi}$ is the conjugate spinor associated with the fermion field ψ, γ^μ are the Dirac matrices, and m is the fermion mass. If ϵ were independent of $\vec{x}$ (i.e., the transformations were global), then $\partial_\mu\psi$ would transform just like ψ, keeping the Dirac Lagrangian gauge-invariant. However, under gauge transformations –where ϵ varies with $\vec{x}$–$\partial_\mu\psi$ transforms according to

$$(\partial_\mu\psi)' = ig\left(\partial_\mu\epsilon(\vec{x})\right)e^{ig\epsilon(\vec{x})}\psi + e^{ig\epsilon(\vec{x})}\partial_\mu\psi, \tag{4.3}$$

which breaks the gauge invariance of $\mathcal{L}_D$. To eliminate the troublesome $\partial_\mu\epsilon(\vec{x})$ term, one must introduce a gauge field A_μ. This field, in conjunction with ∂_μ, replaces $\partial_\mu\psi$ when acting on ψ. The resulting construction, known as the *gauge covariant derivative* of the fermion field, is an operator akin to an ordinary derivative but transforms in the same manner as the fermion field [98]:

$$D_\mu\psi := \partial_\mu\psi - igA_\mu\psi. \tag{4.4}$$

In this way, the Lagrangian (4.2) can be written as

$$\mathcal{L}_D = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi. \tag{4.5}$$

The gauge field A_μ must obey an appropriate transformation law to eliminate the $\partial_\mu\epsilon(\vec{x})$ term and restore gauge invariance in $\mathcal{L}_D$. Specifically, we require:

$$(D_\mu\psi)' = e^{ig\epsilon(\vec{x})}D_\mu\psi, \tag{4.6}$$

so the gauge field must transform as

$$A'_\mu = A_\mu + \partial_\mu\epsilon(\vec{x}). \tag{4.7}$$

Remark 4.1. A key point is that A_μ mediates interactions between the fermion field and its antiparticle, acting as the field messenger for the fundamental interaction characterised by the chosen group, which in this case is $U(1)$ [97].

We can conclude that in the construction of the whole gauge-invariant Lagrangian, only ψ and $D_\mu \psi$ can be used: the only way in which the gauge field A_μ can be implemented is through its gauge covariant derivatives. Nevertheless, a free term quadratic in $\partial_\mu A_\nu$ —constituting the gauge field's kinetic term— must be embedded in a gauge-invariant expression within the Lagrangian. Consequently, we must incorporate such terms. Thus, it could be interesting to find out directly how it transforms:

$$(\partial_\mu A_\nu)' = \partial_\mu \left(A_\nu + \partial_\nu \epsilon(\vec{x}) \right) = \partial_\mu A_\nu + \partial_\mu \partial_\nu \epsilon(\vec{x}). \tag{4.8}$$

Thus, we can automatically identify the annoying term $\partial_\mu \partial_\nu \epsilon(\vec{x})$. Maybe the antisymmetrisation could be the proper way to eliminate this unwanted term:

$$\begin{aligned}(\partial_\mu A_\nu - \partial_\nu A_\mu)' &= \partial_\mu A_\nu + \partial_\mu \partial_\nu \epsilon(\vec{x}) - \partial_\nu A_\mu - \partial_\nu \partial_\mu \epsilon(\vec{x}) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{4.9}$$

Indeed, we have identified a gauge-invariant quantity, composed solely of derivatives of the gauge field, which we designate as the gauge field strength tensor $F_{\mu\nu}$, defined as [97]

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.10}$$

The goal is to construct a gauge-invariant term in the Lagrangian that includes a kinetic term for the gauge field, namely a term quadratic in $\partial_\mu A_\nu$. The most straightforward approach is to form a Lorentz-invariant contraction of the field strength tensor with itself:

$$\mathcal{L}_{K-A} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{4.11}$$

Indeed, the field strength tensor (4.10) could be rewritten as

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= \partial_\mu A_\nu - igA_\mu A_\nu - \partial_\nu A_\mu + igA_\nu A_\mu \\
&= D_\mu A_\nu - D_\nu A_\mu \\
&= D_{[\mu} A_{\nu]},
\end{aligned}
\tag{4.12}
$$

where the operator D_μ is the same as defined in (4.4); however, $D_\mu A_\nu$ is not a covariant derivative, since it does not transform as a field that belongs to a gauge group $U(1)$ representation [98]:

$$
\begin{aligned}
(D_\mu A_\nu)' =&\partial_\mu A'_\nu - igA'_\mu A'_\nu \\
=&\partial_\mu \left(A_\nu + \partial_\nu \epsilon(\vec{x})\right) \\
&- ig\left(A_\mu + \partial_\mu \epsilon(\vec{x})\right)\left(A_\nu + \partial_\nu \epsilon(\vec{x})\right) \\
=&\partial_\mu A_\nu + \partial_\mu \partial_\nu \epsilon(\vec{x}) - ig\left[A_\mu A_\nu + A_\mu\left(\partial_\nu \epsilon(\vec{x})\right)\right. \\
&\left. + \left(\partial_\mu \epsilon(\vec{x})\right) A_\nu + \partial_\mu \epsilon(\vec{x}) \partial_\nu \epsilon(\vec{x})\right].
\end{aligned}
\tag{4.13}
$$

Initially, one may regard the representation of the field strength tensor in (4.12) as superfluous; nonetheless, as will become evident later, this formulation is indispensable for generalising the framework towards the non-Abelian gauge theories.

4.2 Non-Abelian gauge field theories

In the non-Abelian gauge field theories, the transformations over a fermion field do not commute. Thus, we will consider the transformations under the $SU(N)$ gauge group. Let Ψ be a fermion field, such that its N-dimensional spinor transformation has the form

$$
\Psi' = e^{i\,g\,\vec{\epsilon}(\vec{x})\cdot\vec{T}}\,\Psi. \tag{4.14}
$$

Here, $\vec{\epsilon}(\vec{x})$ is an $(N^2 - 1)$ dimensional vector that denotes the amount of the transformation which, in turn, depends on the spatial location, and $\vec{T}$ denotes the "vector" built with the $N^2 - 1$ generators of the $SU(N)$ group. The latter satisfies the following Lie algebra:

$$\Big[T_a, T_b\Big] = i\, f^c{}_{ab}\, T_c, \tag{4.15}$$

with $f^c{}_{ab}$ being the totally antisymmetric structure constants of the group and a, b, c runs from 1 to $N^2 - 1$. In this case, Dirac's Lagrangian takes the form

$$\mathcal{L}_D = \bar{\Psi}\left[i\,\gamma^\mu\,(\partial_\mu \hat{1}) - m\hat{1}\right]\Psi, \tag{4.16}$$

where $\hat{1}$ is the unit matrix. Given that we are interested in local gauge transformations, the term $(\partial_\mu \hat{1})\Psi$ transform as

$$\left[\left(\partial_\mu \hat{1}\right)\psi\right]' = \left(\partial_\mu e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}\right)\psi + e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}\left(\partial_\mu \hat{1}\right)\psi, \tag{4.17}$$

which ruins the gauge invariance of (4.16). Therefore, to get rid of the term $\left(\partial_\mu e^{ig\,\vec{\epsilon}(\vec{x})\cdot\vec{T}}\right)$ in (4.17), we need introduce $N^2 - 1$ gauge fields $A^a{}_\mu$ that are grouped into a single matrix by using the group generators T_a:

$$A_\mu = A^a{}_\mu\, T_a. \tag{4.18}$$

Such a matrix-gauge field, together with $\partial_\mu \hat{1}$ operating on Ψ, defines the covariant derivative of the fermion field and replaces $\left(\partial_\mu \hat{1}\right)\Psi$:

$$D_\mu \Psi := \left(\partial_\mu \hat{1}\right)\Psi - i\, g\, A_\mu \Psi, \tag{4.19}$$

such that it transforms as the fermion itself:

$$\left(D_\mu \Psi\right)' = e^{ig\,\vec{\epsilon}(\vec{x})\cdot\vec{T}} D_\mu \Psi. \tag{4.20}$$

As a consequence, the new Dirac's Lagrangian $\tilde{\mathcal{L}}_D$ takes the form

$$\tilde{\mathcal{L}}_D = \bar{\Psi}\left[i\,\gamma^\mu\, D_\mu - m\hat{1}\right]\Psi, \tag{4.21}$$

which is gauge invariant as long as the matrix-gauge field A_μ complies with a suitable transformation rule so the $\partial_\mu e^{ig\,\vec{\epsilon}(\vec{x})\cdot\vec{T}}$ factor disappears:

$$A'_\mu = e^{ig\,\vec{\epsilon}(\vec{x})\cdot\vec{T}} A_\mu e^{-ig\,\vec{\epsilon}(\vec{x})\cdot\vec{T}} - \frac{i}{g}\Big(\partial_\mu e^{ig\,\vec{\epsilon}(\vec{x})\cdot\vec{T}}\Big) e^{-ig\,\vec{\epsilon}(\vec{x})\cdot\vec{T}}. \tag{4.22}$$

Remark 4.2. Similar to A_μ in the Abelian case, the non-Abelian gauge fields A_μ^a introduce interactions among different fermion fields and are the field messengers of the fundamental interaction described by the $SU(N)$ group [98].

From the development made in the Abelian case, it follows that only ψ and $D_\mu\psi$ can appear explicitly when constructing a fully gauge-invariant Lagrangian; the gauge field A_μ must enter exclusively via covariant derivatives. Nevertheless, one must still accommodate free-particle terms quadratic in $\partial_\mu A_\nu^a$ to provide the kinetic terms for the gauge fields, which must arise from gauge-invariant expressions. Consequently, terms involving $\partial_\mu A_\nu$ become the central elements of our construction. Similar to the Abelian case, we shall study directly how this object transforms:

$$
\begin{aligned}
(\partial_\mu A_\nu)' =& \partial_\mu \Big[e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}} A_\nu e^{-ig\vec{\epsilon}(\vec{x})\cdot\vec{T}} \\
& -\frac{i}{g}\left(\partial_\nu e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}\right) e^{-ig\vec{\epsilon}(\vec{x})\cdot\vec{T}} \Big] \\
=& \left[\partial_\mu\left(e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}\right)\right] A_\nu e^{-ig\vec{\epsilon}(\vec{x})\cdot\vec{T}} \\
& + e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}} \left(\partial_\mu A_\nu\right) e^{-ig\vec{\epsilon}(\vec{x})\cdot\vec{T}} \\
& + e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}} A_\nu \left[\partial_\mu\left(e^{-ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}\right)\right] \\
& - \frac{i}{g}\left[\partial_\mu\partial_\nu\left(e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}\right)\right] e^{-ig\vec{\epsilon}(\vec{x})\cdot\vec{T}} \\
& - \frac{i}{g}\left[\partial_\nu\left(e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}\right)\right]\left[\partial_\mu\left(e^{-ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}\right)\right].
\end{aligned}
\tag{4.23}
$$

Analogously to Dirac's Lagrangian case, the troublesome term is the partial derivative of $\vec{\epsilon}(\vec{x})$. As a first attempt to avoid it, we could try to antisymmetrise (4.23), in a similar way as we did in (4.9); however, it will be useless, as it can be checked in [98]. Although we can use the interesting curiosity found in (4.12): the strength tensor can be written in terms of $D_{[\mu}A_{\nu]}$. We can try building an object in non-Abelian gauge field theories following this apparent curiosity:

$$\begin{aligned} D_{[\mu}A_{\nu]} &= \left[\left(\partial_\mu \hat{1}\right) - igA_\mu\right] A_\nu - \left[\left(\partial_\nu \hat{1}\right) - igA_\nu\right] A_\mu \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right], \end{aligned} \tag{4.24}$$

and then defines this object as the matrix-gauge field strength tensor [54]:

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]. \tag{4.25}$$

The next step consists of studying how this object transforms. By means of the transformation law of the gauge field in (4.22), we can conclude that

$$F'_{\mu\nu} = e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}} F_{\mu\nu} e^{-ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}, \tag{4.26}$$

so that we can eliminate the troublesome term and, additionally, $F_{\mu\nu}$ turns out to transform into the adjoint representation of the $SU(N)$ group [98]. Furthermore, (4.25) can be considered as a covariant derivative of the gauge field itself.

We aim to form a gauge-invariant term from the matrix-valued gauge field strength tensor. Since the Lagrangian contains only scalars, we will use the trace of (4.26) and verify its gauge invariance:

$$\begin{aligned} \mathrm{Tr}\left(F'_{\mu\nu}\right) &= \mathrm{Tr}\left(e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}} F_{\mu\nu} e^{-ig\epsilon(\vec{x})\cdot\vec{T}}\right) \\ &= \mathrm{Tr}\left(F_{\mu\nu} e^{-ig\epsilon(\vec{x})\cdot\vec{T}} e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}\right) \\ &= \mathrm{Tr}\left(F_{\mu\nu}\right). \end{aligned} \tag{4.27}$$

The trace of the Lorentz-invariant matrix quantity $-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ is also gauge invariant:

$$\begin{aligned} \left[\mathrm{Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right)\right]' &= \mathrm{Tr}\left[e^{ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right)e^{-ig\vec{\epsilon}(\vec{x})\cdot\vec{T}}\right] \\ &= \mathrm{Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right). \end{aligned} \tag{4.28}$$

Hence, we arrive at the sought-after Lorentz-invariant Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{K-A} = \mathrm{Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right) &= -\frac{1}{2}g_{ab}F^{a}_{\mu\nu}F^{b\mu\nu} \\
&= -\frac{1}{4}\delta_{ab}F^{a}_{\mu\nu}F^{b\mu\nu},
\end{aligned}
\tag{4.29}
$$

where $\mathrm{Tr}\left(T_a T_b\right) = g_{ab}$ corresponds to the induced metric on the group $g_{ab} = \frac{\delta_{ab}}{2}$[35] and

$$
F^{a}_{\mu\nu} = \partial_\mu A^{a}_{\nu} - \partial_\nu A^{a}_{\mu} + g f^{a}_{bc} A^{b}_{\mu} A^{c}_{\nu}. \tag{4.30}
$$

The kinetic terms for the gauge fields –i.e., free-particle terms quadratic in $\partial_\mu A^{a}_{\nu}$–appear in the Lagrangian given by equation (4.29). However, due to the $-ig\left[A_\mu, A_\nu\right]$ term in equation (4.25), additional self-interaction terms among different gauge fields also arise, in contrast to Abelian gauge field theories where no such self-interactions occur.

4.3 Free p-form gauge theories

To explore the possibility of anisotropic hairs, it is essential to identify a matter source that can sustain anisotropies. A natural candidate for this role is the p-form field, which provides a general framework for modelling anisotropic matter sectors. In particular, the construction of shear-free Bianchi cosmologies necessitates a source capable of counterbalancing the anisotropic spatial curvature that arises in the shear propagation equation[36].

For a Lorentzian manifold of dimension n, the canonical volume form η is given by the relation $\eta = \star 1$[37], and hence any Lorentz scalar (function) f defines a volume form $\mathcal{V} = \star f$. Since the volume form is a top-form, integrating it will again give a scalar. A functional S may therefore be constructed in a coordinate-invariant manner as $S = \int \star f$. The question is

[35] This is also usually known as the Cartan metric tensor. For more details see the Appendix F.

[36] For a detailed overview of p-forms, the reader is referred to Appendix A.1.

[37] Here,$\star 1$ represents a ”canonical“ 0-form.

now which volume form $\mathcal{V}$ we take to define our theory. In constructing a gauge theory, there is a natural choice. In particular, we take $\mathcal{V} = -\frac{1}{2}\mathcal{F}\wedge\star\mathcal{F}$, where $\mathcal{F}$ is a $(p+1)$-form constructed by the exterior derivative of a p-form $\mathcal{A}$. The action now reads

$$S = -\frac{1}{2}\int \mathcal{F}\wedge\star\mathcal{F}\,, \tag{4.31}$$

where

$$\mathcal{F} = \mathbf{d}\mathcal{A} = \frac{1}{p!}\nabla_\nu \mathcal{A}_{\mu_1\ldots\mu_p}\, \omega^\nu \wedge \omega^{\mu_1}\wedge\cdots\wedge\omega^{\mu_p},$$

such that, $(d\mathcal{A})_{\nu\mu_1\ldots\mu_p} = \nabla_{[\nu}\mathcal{A}_{\mu_1\ldots\mu_p]}$. Thus,

$$\mathcal{F} = d\mathcal{A} = \frac{1}{(p+1)!}\nabla_{[\nu}\mathcal{A}_{\mu_1\ldots\mu_p]}\, \omega^\nu \wedge \omega^{\mu_1}\wedge\cdots\wedge\omega^{\mu_p}. \tag{4.32}$$

The Hodge dual is given by

$$\star\mathcal{F} = \frac{1}{(p+1)!(n-p-1)!}\mathcal{F}^{\mu_1\ldots\mu_{p+1}}\,\eta_{\mu_1\ldots\mu_{p+1}\nu_1\ldots\nu_{n-p-1}}\,\omega^{\nu_1}\wedge\cdots\wedge\omega^{\nu_{n-p-1}}. \tag{4.33}$$

From (4.32) and (4.33) the explicit expression for the $\mathcal{F}$ and $\star\mathcal{F}$ components are

$$\begin{aligned}\mathcal{F}_{\mu_1\ldots\mu_{p+1}} &= (p+1)\,\nabla_{[\mu_1}\mathcal{A}_{\mu_2\ldots\mu_{p+1}]}\\ \star\,\mathcal{F}_{\nu_1\ldots\nu_{n-p-1}} &= \frac{1}{(p+1)!}\mathcal{F}^{\mu_1\ldots\mu_{p+1}}\,\eta_{\mu_1\ldots\mu_{p+1}\nu_1\ldots\nu_{n-p-1}}.\end{aligned} \tag{4.34}$$

Thus, the action (4.31) in components can be written as

$$S = -\frac{1}{2(p+1)!}\int \mathrm{d}^4x\,\sqrt{-g}\,\mathcal{F}^{\mu_1\cdots\mu_{p+1}}\,\mathcal{F}_{\mu_1\ldots\mu_{p+1}}. \tag{4.35}$$

Furthermore, the components of the energy-momentum tensor can be written as

$$\begin{aligned}\mathcal{T}_{\mu\nu} &\equiv -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}\\ &= \frac{1}{p!}\mathcal{F}_\mu^{\alpha_1\ldots\alpha_p}\mathcal{F}_{\nu\alpha_1\ldots\alpha_p} - \frac{1}{2(p+1)!}g_{\mu\nu}\mathcal{F}^{\alpha_1\ldots\alpha_{p+1}}\mathcal{F}_{\alpha_1\ldots\alpha_{p+1}},\end{aligned} \tag{4.36}$$

where $\mathcal{L} = -\frac{1}{2(p+1)!}\mathcal{F}^{\mu_1\cdots\mu_{p+1}}\mathcal{F}_{\mu_1\cdots\mu_{p+1}}$ is the Lagrangian density.

The contracted Bianchi identity, which expresses the conservation of energy-momentum, is given by $\mathcal{T}^{\mu\nu}{}_{;\nu} = 0$. However, within the framework of exterior calculus, this condition can be formulated in a more elegant and geometrically insightful manner as

$$d\mathcal{F} = d(d\mathcal{A}) = 0 \;\longrightarrow\; \nabla_{[\nu}\mathcal{F}_{\mu_1\cdots\mu_{p+1}]}. \tag{4.37}$$

Furthermore, under the assumption of the absence of sources, the Hodge dual $\star\mathcal{F}$ must also be closed [63]. Mathematically, this condition is expressed as

$$d\star\mathcal{F} = 0 \;\longrightarrow\; \nabla_{\mu_1}\mathcal{F}^{\mu_1\cdots\mu_{p+1}} = 0, \tag{4.38}$$

which corresponds to the equations of motion derived by varying the Lagrangian density with respect to the p-form $\mathcal{A}$.

> Remark 4.3. This symmetry under the Hodge dual transformation $\mathcal{F} \to \star\mathcal{F}$ is also present in the energy-momentum tensor, as we shall demonstrate below. Indeed, the theory of a canonical p-form with action (4.31) is physically equivalent to that of a $(2-p)$-form theory through the Hodge dual at the field strength $p+1$ level [99]. The duality between a 2-form gauge field and a 0-form gauge field, corresponding to a canonical massless scalar, will be examined in detail below. In the specific case of Maxwell theory ($p = 1$), this duality manifests as the well-known self-duality, wherein electric and magnetic components transform into each other [100].

For each p, we shall decompose the energy-momentum tensor (4.34) following the standard $1+3$ covariant decomposition framework[38]

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \pi_{\mu\nu}, \tag{4.39}$$

[38]For further details, see the appendix D.

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the local metric of the instantaneous rest space orthogonal to the timelike unit norm vector field u^μ (Cf. (2.20)). As we studied in chapter 2, we shall identify the vector u^μ as the four-velocity of a comoving observer, so the metric $h_{\mu\nu}$ will correspond to the induced metric on homogeneity hypersurfaces Σ_t. Besides, the observer sees the following energy density ρ, pressure P, energy flux q^γ and anisotropic stresses $\pi_{\mu\nu}$ [64]:

$$\rho = u^\mu u^\nu T_{\mu\nu}, \qquad P = \frac{1}{3} h^{\mu\nu} T_{\mu\nu}, \qquad q^\gamma = -h^{\gamma\mu} u^\nu T_{\mu\nu},$$
$$\pi_{\mu\nu} = \left(h^\alpha_{(\mu} h^\beta_{\nu)} - \frac{1}{3} h_{\mu\nu} h^{\alpha\beta} \right) T_{\alpha\beta}. \tag{4.40}$$

Remark 4.4. The last two observables lie in the space orthogonal to the observer, satisfying $u^\gamma q_\gamma = 0$ and $u^\mu \pi_{\mu\gamma} = 0$. Given that the anisotropic stress is traceless ($\pi^\mu{}_\mu = 0$), it is often termed anisotropic pressure, while the trace component contributes to the isotropic pressure P [4].

Remark 4.5. The theories formulated from the general $(p+1)$-form action (4.31) exhibit the following key properties [36]:

1) There exists gauge invariance $\mathcal{L} \mapsto \tilde{\mathcal{L}}$ under $\mathcal{A} \to \mathcal{A} + d\mathcal{U}$, where $\mathcal{U}$ is a $p-1$-form.
2) The equations of motion contain derivatives up to second order only.
3) The Lagrangian involves, at most, second-order terms in the field strength $\mathcal{F}$.
4) The theory is constructed exclusively from exterior derivatives of a p-form.
5) The fields involved are minimally coupled to gravity[a].

[a]The principle of minimal gravitational coupling requires that the total Lagrangian for the field equations of general relativity consists of two additive parts, one corresponding to the free gravitational Lagrangian and the other corresponding to external source fields in curved spacetime [101].

4.3.1 p-form classification

From now on, we denote by $\mathcal{A}$ the p-form gauge field, and refer to the associated $(p+1)$-form $\mathcal{F} = d\mathcal{A}$ as its field strength. Moreover, we shall assume spatial homogeneity at the field-strength level, formalised by the following definition:

Definition 7 (Spatially homogeneous gauge field [36]). At the level of the field strength $\mathcal{F}$, a gauge field is considered spatially homogeneous if it satisfies the condition

$$\mathcal{F}\,(t, \mathbf{x}) \implies \mathcal{F}\,(t). \tag{4.41}$$

Note, however, that since we choose to build the $(p+1)$-form field from an underlying gauge field, we have

$$\mathcal{F}\,(t) = d\mathcal{A}\,(t, \mathbf{x}). \tag{4.42}$$

That is, the gauge field $\mathcal{A}(t, \mathbf{x})$ may exhibit both spatial and temporal dependence[a].

[a]Consequently, this definition is more general compared to [102], where only time-dependence was allowed.

To organize the different scenarios of p-form matter fields constructed by taking the exterior derivative of a p-form, we shall use the notation introduced below: $\{a, b\}$ where a denotes the rank of the $(p+1)$-form $\mathcal{F}$ and b the rank of its Hodge dual $\star\mathcal{F}$.

In four-dimensional space-time where $a + b = 4$, there are three distinct cases to consider at the field strength level:

i.) $\{2, 2\}$ ii.) $\{3, 1\}$ or $\{1, 3\}$ iii.) $\{4, 0\}$.

The degeneracy observed in case **ii.)** arises due to the symmetry inherent in equations (4.32). Conversely, in case **iii.)**, such degeneracy does not occur, as the condition $\mathcal{F} \neq d\mathcal{A}$ contradicts equation (4.32), leaving only the configuration $\{4,0\}$ [36].

The $\{4,0\}$ case

This particular case corresponds precisely to a cosmological constant scenario [103]. From equation (4.34), it follows that a 4-form $\mathcal{D}$ may be derived from a 3-form. By introducing the definition $\star\mathcal{D} = c$, it results that

$$\mathcal{L}_{4-f} = -\frac{1}{48}\mathcal{D}_{\mu_1\cdots\mu_4}\mathcal{D}^{\mu_1\cdots\mu_4} = \frac{1}{2}c^2 \qquad \Longrightarrow \qquad T^{4-f}_{\mu\nu} = \frac{1}{2}g_{\mu\nu}\,c^2. \quad (4.43)$$

Additionally, from (4.37) and (4.38), we have that

$$\begin{aligned} d\mathcal{D} = 0 \quad &\longrightarrow \quad 0 = 0, \\ d\star\mathcal{D} = 0 \quad &\longrightarrow \quad \nabla_\mu c = 0 \to \partial_\mu c = 0. \end{aligned} \quad (4.44)$$

The $\{1,3\}$ and $\{3,1\}$ cases

Case $\{1,3\}$: Following (4.34), it is possible to build a 1-form $\mathcal{J}$ starting from a scalar field $\phi(t,\mathbf{x})$. Thus,

$$\mathcal{L}_{1-f} = -\frac{1}{2}\,\mathcal{J}_\mu\mathcal{J}^\mu \qquad \Longrightarrow \qquad T^{1-f}_{\mu\nu} = \mathcal{J}_\mu\mathcal{J}_\nu - \frac{1}{2}\,g_{\mu\nu}\,\mathcal{J}_\gamma\mathcal{J}^\gamma. \quad (4.45)$$

Moreover, equations (4.37) and (4.38) take the form

$$\begin{aligned} d\mathcal{J} = 0 \quad &\longrightarrow \quad \nabla_{[\mu}\mathcal{J}_{\nu]} = 0, \\ d\star\mathcal{J} = 0 \quad &\longrightarrow \quad \nabla_\mu\mathcal{J}^\mu = 0. \end{aligned} \quad (4.46)$$

The equations described above describe a massless scalar field [99].

Case $\{3,1\}$: From equation (4.34), one can construct a 3-form $\mathcal{C}$ starting from a 2-form $\mathcal{B}$. By means of its Hodge dual, which is a 1−form, it is possible to express the components as $\star\mathcal{C}_\mu = \frac{1}{6}\eta_{\alpha\beta\gamma\delta}\,\mathcal{C}^{\alpha\beta\gamma}$. Furthermore,

$$\mathcal{L}_{3-f} = -\frac{1}{12}\mathcal{C}_{\mu\nu\gamma}\,\mathcal{C}^{\mu\nu\gamma} \quad \Longrightarrow \quad T^{3-f}_{\mu\nu} = \star\mathcal{C}_\mu\star\mathcal{C}_\nu - \frac{1}{2}g_{\mu\nu}\star\mathcal{C}_\gamma\star\mathcal{C}^\gamma. \quad (4.47)$$

Moreover, equations (4.37) and (4.38) take the form

$$\begin{aligned} d\mathcal{C} = 0 \quad &\longrightarrow \quad \nabla_\mu \star \mathcal{C}^\mu = 0, \\ d\star\mathcal{C} = 0 \quad &\longrightarrow \quad \nabla_{[\mu} * \mathcal{C}_{\nu]} = 0. \end{aligned} \tag{4.48}$$

It is worth emphasising that the $\{1,3\}$ and $\{3,1\}$ configurations are equivalent, and these two cases constitute the primary focus of the present research work.

The $\{2,2\}$ case

The remaining possibility involves a 2-form $\mathcal{E}$ obtained from a 1-form $\mathcal{B}$, in accordance with (4.34). This yields

$$\mathcal{L}_{2-f} = -\frac{1}{4}\mathcal{E}_{\mu\nu}\mathcal{E}^{\mu\nu} \quad \Longrightarrow \quad T^{2-f}_{\mu\nu} = -\mathcal{E}_{\mu\gamma}\mathcal{E}^{\gamma}{}_{\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{E}_{\gamma\delta}\mathcal{E}^{\gamma\delta}, \tag{4.49}$$

which corresponds precisely to the source-free electromagnetic Lagrangian [100]. Equations (4.37) and (4.38) become

$$\begin{aligned} d\mathcal{E} = 0 \quad &\longrightarrow \quad 3\nabla_{[\mu}\mathcal{E}_{\nu\lambda]} = 0, \\ d\star\mathcal{E} = 0 \quad &\longrightarrow \quad \nabla_\mu \mathcal{E}^{\mu\nu} = 0. \end{aligned} \tag{4.50}$$

These are the known Maxwell's equations.

Chapter 5

How to balance the anisotropic curvature with a comoving gauge field?

This chapter builds extensively on chapter 1 in *Cosmological models from a geometric point of view* [6], chapter 3 in *3+1 Formalism in General Relativity* [60], chapter 4 in *Relativity on Curved Manifolds* [61], chapter 4 in *Relativistic Cosmology* [4], chapter 3 in *Lecture notes in Lie groups and Lie algebras* [62], chapter 2 in *Tales from Wonderland* [1] and chapter 15 in *Einstein's General Theory of Relativity* [63].

The Bianchi cosmological models are characterised by metrics supporting a three-dimensional isometry group, which acts transitively on spacelike hypersurfaces, known as surfaces of homogeneity. Under this structure and the application of Frobenius' theorem, these spacetimes allow for a foliation into homogeneous surfaces, denoted as Σ_t, each identified by the time coordinate $t = x^0$, and possessing constant curvature scalars.

Let u^μ represent the unique timelike vector field associated with the 4-velocity of a comoving observer, which is orthogonal to these homogeneous surfaces. Consequently, we shall focus on the non-tilted Bianchi models. Hence, the vector field u^μ also represents all matter fields' flow. In these models, the congruence of fundamental observers is characterised by being non-accelerated and irrotational, with the primary kinematics quantities of interest being the Hubble expansion scalar H and the shear tensor $\sigma_{\mu\nu}$, given by

$$3H = \nabla_\mu u^\mu \qquad \text{and} \qquad \sigma_{\mu\nu} = \nabla_{(\nu} u_{\mu)} - H h_{\mu\nu}, \tag{5.1}$$

respectively.

Besides, we shall consider two categories of matter fields: the constituents of the ΛCDM model, represented as a collection of comoving perfect fluids, and a set of p-form gauge fields, denoted as $\mathcal{A}_{\mu_1 \dots \mu_p}$. To avoid inducing anisotropy in the time direction, we impose that the heat flux generated by the gauge fields vanishes. Consequently, the energy-momentum tensors for the ΛCDM constituents, $\mathcal{T}_{\mu\nu}$, and for the gauge fields, $\mathcal{T}^{(\mathcal{A})}_{\mu\nu}$, assume the following form

$$\mathcal{T}_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu} : \quad \rho = \sum_\ell \rho_\ell, \quad P = \sum_\ell P_\ell, \tag{5.2}$$

$$\mathcal{T}^{(\mathcal{A})}_{\mu\nu} = \rho_{\mathcal{A}} u_\mu u_\nu + P_{\mathcal{A}} h_{\mu\nu} + \pi_{\mu\nu}, \tag{5.3}$$

where the index ℓ runs over all ΛCDM constituents fluids. Thus, the total energy-momentum tensor takes the form

$$T_{\mu\nu} = (\rho_{\mathcal{A}} + \rho)\, u_\mu u_\nu + (P_{\mathcal{A}} + P)\, h_{\mu\nu} + \pi_{\mu\nu}. \tag{5.4}$$

Additionally, the matter fields satisfy the evolution equations

$$\dot{\rho} + 3H(\rho + P) = 0, \tag{5.5}$$

$$\dot{\rho}_{\mathcal{A}} + 3H\left(\rho_{\mathcal{A}} + P_{\mathcal{A}}\right) = -\pi^{\mu\nu}\sigma_{\mu\nu}, \tag{5.6}$$

whereas the congruence evolves according to the Raychudhuri equation and the shear propagation equation, in addition to the Friedman equation that constrains the variables:

$$\dot{H} + H^2 = -\frac{1}{6}(\rho + 3P) - \frac{1}{6}\left(\rho_{\mathcal{A}} + 3P_{\mathcal{A}}\right) - \frac{2}{3}\sigma^2, \tag{5.7}$$

$$\dot{\sigma}_{\mu\nu} + 3H\sigma_{\mu\nu} = \pi_{\mu\nu} - {}^3S_{\mu\nu}, \tag{5.8}$$

$$3H^2 - \sigma^2 + \frac{{}^3R}{2} = \rho + \rho_{\mathcal{A}}. \tag{5.9}$$

Here, an overdot denotes a time derivative along the congruence –such as $\dot{\rho} = u^\mu \nabla_\mu \rho$–, the shear scalar σ is defined by $\sigma^2 = \sigma^{\mu\nu}\,\sigma_{\mu\nu}$, and

$$ {}^3S_{\mu\nu} = {}^3R_{\mu\nu} - \frac{{}^3R}{3} h_{\mu\nu}, \tag{5.10} $$

represents the trace-free part of the three-dimensional Ricci tensor defined on Σ_t.

> Remark 5.1. The full set of propagation equations, equivalent to Einstein's equations, typically includes terms for vorticity $\omega_{\mu\nu}$, 4-acceleration $\dot{u}^\mu$, and their derivatives [4]. However, in the non-tilted models studied here, the matter moves geodesically, eliminating both 4-acceleration and vorticity, and thus simplifying the propagation equations.

In general, equations (5.7)-(5.9) have more unknowns than equations, unless an equation of state is specified. Therefore, we require a matter model with a suitable equation of state that can sustain the inherent anisotropy in Bianchi models.

Thus, considering orthogonal shear-free models provides a very convenient scenario for acquiring new insights. Let us therefore consider a cosmological model with a vanishing shear tensor during the whole evolution. Consequently, the vanishing of the shear and of its time derivative reduces the shear propagation equation (5.8) to

$$ {}^3S_{\mu\nu} = \pi_{\mu\nu}, \tag{5.11} $$

known as the shear-free condition [55].

If the anisotropic stresses are zero, the geometry must have constant curvature, as the absence of such stresses aligns with the assumption that these spacetimes correspond to FLRW universes. However, if we permit $\pi_{\mu\nu} \neq 0$ despite the absence of shear, the model inevitably cannot have constant curvature. Similarly, the anisotropic stresses cannot vanish if the geometry lacks constant curvature. This behaviour is observed in cosmological models based on Bianchi types that do not include FLRW as a special case or in the Kantowski-Sachs model, where vanishing shear does not lead to spatial isotropy [88]. Therefore, in shear-free orthogonal models, a clear interplay exists between anisotropic stresses and curvature anisotropy. Deviations from spatial isotropy lead to anisotropic pressures, while anisotropic stresses quantify how much the homogeneous hypersurfaces deviate from spatial isotropy [55]. The main goal of this research consists in investigating systematically under which conditions non-abelian gauge fields are capable of balancing the anisotropic curvature in this way, and what role the non-abelian character of the fields plays, in contrast to the abelian case investigated in [95].

Chapter 6
Attempts toward balancing anisotropic curvature via comoving gauge fields

Principle of Sufficient Reason: No fact can hold or be real, and no proposition can be true, unless there is a sufficient reason why it is so and not otherwise

—*Gottfried Wilhelm Leibniz. Monadology* (1714).

We shall first turn to a previous study on the Abelian case presented by Mikjel Thorsrud [95], which systematically analyses shear-free cosmological models supported by p-form gauge fields. These models feature anisotropic spatial sections that expand isotropically, mimicking the expansion history of standard FLRW cosmologies. The work presents a full classification of shear-free solutions in anisotropic, spatially homogeneous, and orthogonal spacetimes containing n independent p-form fields ($p \in \{0, 1, 2, 3\}$) alongside ΛCDM fluids. In that setting, the gauge fields counterbalance anisotropic curvature in the shear propagation equation, yielding solutions dynamically equivalent to FLRW models with an effective curvature constant K_{eff} determined by both spatial curvature and gauge field energy density.

The principal aim of the present investigation is to generalise the framework proposed by Thorsrud by studying the behaviour of non-Abelian p-form gauge fields, as formulated in chapter 4. Specifically, the study seeks to determine, if possible, the configurations of n non-Abelian p-form gauge fields capable of counterbalancing the intrinsic anisotropic spatial curvature of Bianchi cosmologies. This endeavour is directed toward establishing a complete classification of exact shear-free solutions, consistent with the criteria developed in chapter 5. Accordingly, we begin by presenting the principal results obtained by Thorsrud, followed by the contributions and findings of the current research.

6.1 Known results: Abelian gauge field configurations

We shall consider all non-trivial p-form gauge fields in spacetime governed by the action (4.31), namely those with $p \in \{0, 1, 2, 3\}$. However, in light of the objective to construct a comprehensive classification of exact shear-free solutions, as outlined in Chapter 5, the case $p = 3$—being physically equivalent to a cosmological constant Λ [103]—can be excluded[39]. Consequently, the analysis will henceforth be restricted to the cases $p \in \{0, 1, 2\}$. To avoid ambiguity, we adopt the following convention: when referring to the gauge field $\mathcal{A}$, we shall use Arabic numerals $\{0, 1, 2, \ldots\}$; whereas, when referring to the corresponding field strength $\mathcal{F} = d\mathcal{A}$, we shall employ Roman numerals $\{I, II, III, IV, \ldots\}$.

6.1.1 0-form

For a 0-form gauge field ϕ, the Lagrangian and the energy momentum tensor, taking into account (4.45), can be written as

[39] For further detail, refer to subsection 6.1.4 below.

$$\mathcal{L} = -\frac{1}{2}\mathcal{J}^{\gamma}\mathcal{J}_{\gamma} = -\frac{1}{2}\nabla_{\gamma}\phi\nabla^{\gamma}\phi \implies T_{\mu\nu} = \nabla_{\mu}\phi\nabla_{\nu}\phi - \frac{1}{2}g_{\mu\nu}\nabla_{\gamma}\phi\nabla^{\gamma}\phi, \tag{6.1}$$

which corresponds to the massless scalar field Lagrangian, and its energy-momentum tensor.

To obtain its associated dynamic quantities, it is mandatory to make a 1+3 covariant decomposition. In the I-form at the strength level, its decomposition can be written in components as

$$J_{\mu} = -\varphi u_{\mu} + v_{\mu}, \tag{6.2}$$

where v^{α} is a spacelike vector orthogonal to u^{α}, meaning that $v^{\gamma}v_{\gamma} > 0$ and $u^{\gamma}v_{\gamma} = 0$. Under these conditions, the energy density, pressure, energy flux, and anisotropic stress corresponding to the energy-momentum tensor (6.1) can be expressed as follows:

$$\rho = \frac{1}{2}\left(v^2 + \varphi^2\right), \quad P = \frac{1}{2}\left(-\frac{1}{3}v^2 + \varphi^2\right), \quad q^{\mu} = -\varphi v^{\mu}, \tag{6.3}$$
$$\pi_{\mu\nu} = v_{\mu}v_{\nu} - \frac{1}{3}v^2 h_{\mu\nu},$$

where we have used the definitions presented in (4.40).

6.1.2 1-form

In the case of a 1-form gauge field $\mathcal{A}_{\mu}$, the Lagrangian and the associated energy-momentum tensor, as dictated by equation (4.49), can be expressed as

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \quad \implies \quad T_{\mu\nu} = \mathcal{F}_{\mu\gamma}\mathcal{F}^{\gamma}{}_{\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{F}_{\gamma\delta}\mathcal{F}^{\gamma\delta}, \tag{6.4}$$

where $\mathcal{F} = \mathbf{d}\mathcal{A}$. To derive the corresponding dynamical quantities, it is essential to perform a $1+3$ covariant decomposition. For the II-form at the field strength level, this decomposition can be expressed in components as

$$E_\mu = F_{\mu\nu}u^\nu, \quad B_\mu = *F_{\mu\nu}u^\nu = \frac{1}{2}\eta_{\mu\alpha\beta}F^{\alpha\beta} \Longleftrightarrow F_{\mu\nu} = 2u_{[\mu}E_{\nu]} + \eta_{\mu\nu\alpha}B^\alpha. \tag{6.5}$$

Hence, the components of the energy-momentum tensor (6.4) can be expressed as

$$\begin{aligned} T_{\mu\nu} = -E_\mu E_\nu - B_\mu B_\nu + u_\mu u_\nu \left(E^2 + B^2\right) + \frac{1}{2}g_{\mu\nu}\left(E^2 + B^2\right) \\ + 2u_{(\mu}\eta_{\nu)\alpha\beta}E^\alpha B^\beta. \end{aligned} \tag{6.6}$$

This decomposition is, as a consistency check, trace-free. Here, $\eta_{\alpha\beta\gamma} \equiv \eta_{\alpha\beta\gamma\delta}u^\delta$ denotes the three-dimensional Levi-Civita form defined on the spatial hypersurface orthogonal to the observer, satisfying $u^\alpha\eta_{\alpha\beta\gamma} = 0$[40]. From this, the corresponding expressions for the energy density, pressure, energy flux, and anisotropic stress are

$$\begin{gathered} \rho = \frac{1}{2}\left(E^2 + B^2\right), \quad P = \frac{1}{6}\left(E^2 + B^2\right), \quad q^\mu = \eta^{\mu\alpha\beta}E_\alpha B_\beta, \\ \pi_{\mu\nu} = -E_\mu E_\nu + \frac{1}{3}E^2 h_{\mu\nu} - B_\mu B_\nu + \frac{1}{3}B^2 h_{\mu\nu}. \end{gathered} \tag{6.7}$$

Remark 6.1. It is important to recognise that the scalars ρ and P are invariant only in the covariant sense: although they do not depend on the choice of coordinate frame, they are determined by the observer's four-velocity u^μ. In contrast, the equation of state $P/\rho = 1/3$ holds universally for all electromagnetic fields and all observers. This implies that the equation of state is invariant in a stronger sense, being independent of both the electromagnetic field tensor $F_{\mu\nu}$ and the observer u^μ; it is therefore a Lorentz-invariant quantity.

6.1.3 2-form

For a 2-form gauge field $\mathcal{N}_{\mu\nu}$, the Lagrangian and the associated energy-momentum tensor, as specified by equation (4.47), are expressed by

[40] For further details, refer to Appendix E.

$$\mathcal{L} = -\frac{1}{12} C^{\alpha\beta\gamma} C_{\alpha\beta\gamma} \implies T_{\mu\nu} = \frac{1}{2} C^{\alpha\beta}_{\mu} C_{\nu\alpha\beta} - \frac{1}{12} g_{\mu\nu} C_{\alpha\beta\gamma} C^{\alpha\beta\gamma}. \tag{6.8}$$

As discussed in Subsection 4.3.1, to make the physical equivalence with a 0-form massless scalar field explicit, equation (6.8) can be reformulated using the Hodge dual $\star\mathbf{C}$, which is a 1-form whose components are

$$\star C_\delta = \frac{1}{6} \eta_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma} \iff C_{\alpha\beta\gamma} = -\eta_{\alpha\beta\gamma\delta} \star C^\delta. \tag{6.9}$$

By comparing the field strengths in equations (6.1) and (6.8), one observes the duality $\nabla_\mu \phi \leftrightarrow \star\mathcal{C}_\mu$, under which the corresponding energy-momentum tensors map into one another. Given that, in the free theories governed by the action (4.35), the field strength is the sole physical degree of freedom, this duality suffices to establish the physical equivalence between the $p = 0$ and $p = 2$ cases. Therefore, both cases exhibit essentially identical $1 + 3$ decompositions:

$$\star C_\mu = -\varphi u_\mu + v_\mu \iff C_{\alpha\beta\gamma} = \varphi \eta_{\alpha\beta\gamma} - \eta_{\alpha\beta\gamma\delta} v^\delta, \tag{6.10}$$

and share the same dynamical quantities as given by (6.3).

Remark 6.2. From (6.3), it is important to note that the energy flux q^μ vanishes only when the Hodge dual $\star\boldsymbol{C}$ is either purely spacelike, with $\star C_\mu = v_\mu$, or purely timelike, with $\star C_\mu = -\varphi u_\mu$, relative to the observer. In contrast to the Maxwell case for $p = 1$ discussed above, the equation of state P/ρ is not fixed but is instead a dynamical quantity that depends on both the field $\star\boldsymbol{C}$ and the observer's four-velocity $\mathbf{u}$. Specifically, when the Hodge dual $\star\boldsymbol{C}$ is orthogonal to the observer, the 0-form and the 2-form field yields an equation of state $P/\rho = -1/3$.

6.1.4 3-form

Given the aim of establishing a comprehensive classification of exact shear-free solutions, as discussed in Chapter 5, the 3-form case can be excluded from the analysis. This configuration is physically equivalent to a cosmological constant Λ [103], and therefore does not produce any anisotropic stresses that counterbalance the curvature anisotropy inherent in Bianchi models.

6.2 Known theorems

Once analysed the various p-form cases and their corresponding dynamical quantities, we are now in a position to determine which type of p-form, as governed by the action (4.31), is suitable for counterbalancing anisotropic spatial curvature in the shear-free limit.

Theorem 6.3 (Spatial curvature decay in Bianchi models [95]). *The components of the spatial Ricci tensor* ${}^3S_{\mu\nu}$ *relative to an orthonormal basis exhibit a decay proportional to* $1/a^2$ *in the shear-free limit, where* $a(t)$ *denotes the scale factor governing distances in the homogeneous spatial hypersurfaces* Σ_t.

Proof

Previously, in equation (3.15), we derived the general form of the line element for a Bianchi-type cosmological model undergoing conformal expansion, given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\,\tilde{h}_{ab}\,(x^c)\,\mathrm{d}x^a\;\mathrm{d}x^b.$$

As $\tilde{h}_{ab}(x^c)$ is a function of the spatial coordinates x^c alone, it follows that the corresponding Ricci tensor ${}^3\tilde{R}_{ab}$ will also depend exclusively on these spatial coordinates. In addition, since $\partial_c\, a(t) = 0$, the Ricci tensor remains invariant under this conformal transformation, i.e. ${}^3R_{ab} = {}^3\tilde{R}_{ab}\,(x^c)$. Therefore, relative to a coordinate basis with the line element given by (3.15), the components of ${}^3R_{ab}$ and ${}^3\tilde{R}_{ab}$ are equal and time-independent.

In the following, we shall use hats to denote the components of these tensors relative to an orthonormal basis; for instance, ${}^3R_{\hat{a}\hat{b}}$ and ${}^3\tilde{R}_{\hat{a}\hat{b}}$. Let $\Lambda^a_{\hat{a}}\,(x^\mu)$ represent the corresponding basis transformation matrix, and $\tilde{\Lambda}^a_{\hat{a}}\,(x^c)$ its inverse, where the latter depends solely on the spatial coordinates. Thus, by definition

$$\underbrace{\tilde{\Lambda}^a_{\hat{a}}(x^c)}_{(\square)}\,\tilde{h}_{ab}(x^c)\,\tilde{\Lambda}^b_{\hat{b}}(x^c) = \delta_{\hat{a}\hat{b}}. \tag{6.11}$$

Considering the conformal transformation between the induced metric $\mathbf{h}$ and its spatial part $\tilde{\mathbf{h}}$ given by $\mathbf{h} = a^2(t)\,\tilde{\mathbf{h}}$, we can rewrite (6.11) as

$$\underbrace{\left(\frac{1}{a(t)}\tilde{\Lambda}^a_{\hat{a}}\,(x^c)\right)}_{(\square)}\,h_{ab}\,(x^\mu)\,\left(\frac{1}{a(t)}\tilde{\Lambda}^b_{\hat{b}}\,(x^c)\right) = \delta_{\hat{a}\hat{b}}. \tag{6.12}$$

A comparison between the $(\square)$ term in equations (6.11) and (6.12) reveals that the transformation matrices satisfy the following relation:

$$\Lambda^b_{\hat{b}}\,(x^\mu) = \frac{1}{a(t)}\tilde{\Lambda}^b_{\hat{b}}\,(x^c)\,. \tag{6.13}$$

Taking into account (5.10) and (6.13), it follows that

$${}^3R_{\hat{a}\hat{b}}\,(x^\mu) = \frac{{}^3R_{ab}\,(x^c)}{a^2(t)}. \tag{6.14}$$

Thus, we have established that the components of ${}^3R_{ab}$ relative to an orthonormal basis exhibit a decay proportional to $1/a^2(t)$. Consequently, the same behaviour holds for the trace-free part ${}^3S_{ab}$. □

Remark 6.4. In FLRW models, it is well known that the spatial curvature scales as ${}^3R \propto 1/a^2(t)$ [4]. Similarly, all curvature tensors expressed in an orthonormal basis exhibit the same dependence. Here, we demonstrated that in the shear-free limit, the spatial curvature in all Bianchi models also decays as $1/a^2(t)$. This behaviour is expected, given that the transformation $a^2(t) \mapsto a^2(t + \Delta t)$ corresponds to a conformal rescaling of the three-dimensional hypersurfaces.

Remark 6.5. The relation ${}^3S_{ab} \propto 1/a^2$, in conjunction with the shear-free condition given by equation (4.11), rigorously entails that the anisotropic stress tensor $\pi_{\mu\nu}$ must exhibit the same asymptotic behaviour, decaying proportionally to $1/a^2$.

Building on this result, one may inquire whether a connection exists between the equation of state corresponding to each p-form case and the decay behaviour of the anisotropic stress tensor $\pi_{\mu\nu}$ in the shear-free limit. The theorem that follows is devoted to addressing this question.

Theorem 6.6 (Decay relation between ρ and $\pi_{\mu\nu}$ [95]). *The energy momentum tensor of the gauge field is homogeneous and quadratic in the field strength, hence this implies that*

$$\pi_{\mu\nu} \propto 1/a^2 \Longrightarrow \rho_{\mathcal{A}} \propto 1/a^2. \tag{6.15}$$

Proof

We shall start rewriting the energy density ρ in terms of the stress tensor $\pi_{\mu\nu}$. Taking into account the 1+3 decomposition of the field strength given in (6.2), (6.5) and (6.10) for the p-form gauge fields with $p \in \{0, 1, 2\}$, we note that in the zero flux energy case $q^\mu = 0$ the dynamic quantities obey the following structure:

$$\rho_{\mathcal{A}} = \frac{1}{2}X^2, \quad P_{\mathcal{A}} = \frac{(-1)^{p+1}}{6}X^2, \quad \pi_{\mu\nu} = (-1)^p\left(X_\mu X_\nu - \frac{1}{3}X^2 h_{\mu\nu}\right), \tag{6.16}$$

where,

$$X_\mu = \begin{cases} \nabla_\mu \phi, & \text{if } p = 0 \\ B_\mu \text{ or } E_\mu, & \text{if } p = 1 \\ \star C_\mu, & \text{if } p = 2 \end{cases}$$

It is important to observe that the only distinction between the $p = 0$ and $p = 2$ cases lies in the formal definition of X_μ, consistent with their physical equivalence via Hodge duality. For the Maxwell case ($p = 1$), we have assumed either $E_\mu = 0$ or $B_\mu = 0$ to ensure that the Poynting vector $q^\mu = \eta^{\mu\alpha\beta} E_\alpha B_\beta$, vanishes. As before, we employ an orthonormal frame $\{\mathbf{e}_\mu\}$, where $\mathbf{e}_0 = \partial_t$ is orthogonal to the hypersurfaces Σ_t, though in this case, we omit the use of hats on indices. In this frame, we obtain:

$$X_\mu = (0, X_1, X_2, X_3)\,, \quad X^2 = X_1 X_1 + X_2 X_2 + X_3 X_3. \tag{6.17}$$

When X_μ has only a single non-zero component, denoted $X_{♪}$ with $♪ \in \{1, 2, 3\}$, both the energy density ρ and the anisotropic stress tensor $\pi_{\mu\nu}$ are quadratic in $X_{♪}$, and are related through the expression:

$$\rho = \frac{3(-1)^p}{4} \pi_{♪♪}, \tag{6.18}$$

and the implication (6.15) follows. If X_μ has two independent non-zero components, denoted by $X_{♫}$ and $X_{♬}$ with $(♫, ♬) \in \{1, 2, 3\}$, then we obtain

$$\pi_{♫♫} = \frac{(-1)^p}{3} \left(2X_{♫}X_{♫} - X_{♬}X_{♬} \right), \quad \pi_{♬♬} = \frac{(-1)^p}{3} \left(2X_{♬}X_{♬} - X_{♫}X_{♫} \right), \tag{6.19}$$

such that it can be inverted to obtain

$$\rho = \frac{3(-1)^p}{2} \left(\pi_{♫♫} + \pi_{♬♬} \right), \tag{6.20}$$

and again the implication (6.15) follows. In the final case, where X_μ contains three independent non-zero components, we examine the off-diagonal spatial components of the anisotropic stress tensor $\pi_{\mu\nu}$, all of which are non-zero. These components can be inverted to yield:

$$\rho = \frac{(-1)^p}{2} \left(\frac{\pi_{12}\pi_{13}}{\pi_{23}} + \frac{\pi_{21}\pi_{23}}{\pi_{13}} + \frac{\pi_{31}\pi_{32}}{\pi_{12}} \right). \tag{6.21}$$

We note that $\pi_{\mu\nu} \propto 1/a^2$ implies $\rho \propto 1/a^2$.

Hence, implication (6.15) has been demonstrated for comoving p-form gauge fields with $p \in \{0, 1, 2\}$. Moreover, this implication may equivalently be reformulated as:

$$\rho \not\propto 1/a^2 \Longrightarrow \pi_{\mu\nu} \not\propto 1/a^2. \qquad \square \tag{6.22}$$

Remark 6.7. In the shear-free limit, where $\sigma_{\mu\nu} \to 0$, the matter evolution equation (4.6) implies that the decay $\rho \propto 1/a^2$ is only compatible with an equation of state $P/\rho = -1/3$. However, for a II-form field at the strength level, the equation of state is $+1/3$, which, in light of equation (4.6), necessitates an energy density decay of $1/a^4$ in the shear-free regime. Consequently, condition (6.22) decisively rules out the free Maxwell field as a viable source to counterbalance the anisotropic curvature.

Remark 6.8. An equation of state $P/\rho = -1/3$ can only be realised in the I-form case at the field strength level—or equivalently, in the III-form case via its Hodge dual, as shown in equation (6.9)—under the restriction of vanishing energy flux. Therefore, within the framework of Abelian free p-form gauge theories, the only viable candidates capable of supporting anisotropic spatial curvature are the $p = 0$ case and the $p = 2$ case via Hodge duality.

6.3 Non-Abelian gauge field configurations

Building upon the Abelian case discussed in the previous section, we now proceed to examine all non-trivial non-Abelian p-form gauge fields in spacetime for $p \in \{0, 1, 2, 3\}$, with an associated $SU(2)$ Lie group, as introduced in Chapter 4. The aim is to contrast these results with those obtained by Thorsrud in the Abelian scenario and to assess whether the inclusion of non-Abelian structures leads to novel dynamical behaviours or solutions.

6.3.1 0-form

We start with the action for the gauge field sector, defined by

$$S = -\frac{1}{2}\int \mathrm{d}x^4 \sqrt{-|g|}\, \mathcal{J}^2, \tag{6.23}$$

where $\mathcal{J}^2 \equiv \mathcal{J}^a \mathcal{J}_a$, and $\mathcal{J}^a_\mu = D_\mu \mathcal{X}^a$ with $\mathcal{X}^a$ being a scalar field. Furthermore, the covariant derivative D of a vector $\mathcal{X}$ is in this context is defined by means of (4.19), hence,

$$D_\mu \mathcal{X}^a \equiv \nabla_\mu \mathcal{X}^a + g\varepsilon^a{}_{bc} A^b_\mu \mathcal{X}^c.$$

Thus,the 1-form $\mathcal{J}$ can be decomposed as[41]

$$\mathcal{J} = -\omega\, \underline{u} + \underline{v},$$

or written in components,

$$\mathcal{J}^a_\mu = -\,\omega^a\, u_\mu + v^a_\mu. \tag{6.24}$$

Here, v^a_μ is a spacelike vector orthogonal to u_μ and ω^a is a scalar field. Hence, the energy-momentum tensor associated to (6.23) takes the form

$$T_{\mu\nu} = \omega^2\, h_{\mu\nu} - \omega^a\, (u_\mu v_{\nu\, a} + u_\nu v_{\mu\, a}) + v_{\mu\, a} v^a_\nu - \frac{1}{2}\, (\omega^2 + v^2)\, g_{\mu\nu}, \tag{6.25}$$

where $\omega^2 = \omega_a\, \omega^a$ and $v^2 = v^a_\mu\, v^\mu_a$. Given these conditions, the energy density, pressure, energy flux, and anisotropic stress associated with the energy-momentum tensor (6.25) take the following form:

$$\rho = \frac{1}{2}\left(v^2 + \omega^2\right), \qquad P = \frac{1}{2}\left(-\frac{1}{3}v^2 + \omega^2\right), \tag{6.26}$$
$$q^\mu = -\omega^a\, v^\mu_a, \qquad \pi_{\mu\nu} = v^a_\mu v_{\nu\, a} - \frac{1}{3} v^2 h_{\mu\nu},$$

[41]For further details regarding this 1+3 decomposition, refer to Appendix B.

where the quantities are obtained using the definitions provided in equation (4.40).

In essence, we have obtained results analogous to those presented by Thorsrud in Subsection 6.1.1, recovering the same equation of state $P/\rho = -1/3$. Consequently, remark 6.8 can be extended to include the non-Abelian 0-form case. This indicates that such a configuration is capable of sustaining anisotropic spatial curvature in the shear-free limit. Therefore, both Abelian and non-Abelian 0-form gauge fields yield valid shear-free solutions. Nonetheless, the non-Abelian nature of the gauge field does not introduce any significant physical distinctions or novel dynamical effects when compared to the Abelian case.

6.3.2 1-form

For the 1-form non-Abelian configuration, we start with the action for the gauge field sector, defined by

$$S = -\frac{1}{4}\int \mathrm{d}x^4 \sqrt{-|g|}\, \mathcal{F}^2. \tag{6.27}$$

In the above, $\mathcal{F}$ denotes the SU(2) gauge field-strength tensor, defined as $\mathcal{F} = D\mathcal{A}$. Accordingly, when expressed in a general basis, the field strength tensor can be written in terms of the SU(2) generators T_a as follows:

$$\mathcal{F} = \frac{1}{2}\mathcal{F}^a{}_{\mu\nu}\,\mathrm{T}_a\,\boldsymbol{\omega}^\mu \wedge \boldsymbol{\omega}^\nu \quad \to \quad \mathcal{F}^a{}_{\mu\nu} = 2\nabla_{[\mu}\mathcal{A}^a{}_{\nu]} + g\varepsilon^a_{b\,bc}\mathcal{A}^b{}_\mu\mathcal{A}^c{}_\nu, \tag{6.28}$$

or by means of the 1+3 decomposition[42], we arrive at

$$\mathcal{F}^a = \underline{u} \wedge \underline{\mathcal{E}}^a + \star(\underline{u} \wedge \underline{\mathcal{B}}^a),$$

or written in components,

$$\mathcal{F}^a = \left(u_{[\mu}\mathcal{E}^a{}_{\nu]} + \frac{1}{2}\eta_{\lambda\mu\nu}\mathcal{B}^{a\lambda}\right)\omega^\mu \wedge \omega^\nu \to \mathcal{F}^a{}_{\mu\nu} = 2u_{[\mu}\mathcal{E}^a{}_{\nu]} + \eta_{\lambda\mu\nu}\mathcal{B}^{a\lambda}, \tag{6.29}$$

[42] For further details regarding this decomposition, refer to Appendix B.

where,

$$\mathcal{B}^a \equiv \star_3 \mathcal{F}^a = \frac{1}{2}\eta_{ijk}\mathcal{F}^{aij}\omega^k \to \mathcal{B}^a_k = \frac{1}{2}\eta_{ijk}\mathcal{F}^{aij} \quad \text{and} \quad \mathcal{E}^a_\mu \equiv \mathcal{F}^a_{\mu\nu}u^\nu. \tag{6.30}$$

The energy-momentum tensor associated with the action (6.27) takes the form

$$\begin{aligned} T_{\mu\nu} = &- \mathcal{E}^a_\mu \mathcal{E}_{\nu\, a} - \mathcal{B}^a_\mu \mathcal{B}_{\nu\, a} + u_\mu u_\nu \left(E^2 + B^2\right) + \frac{1}{2} g_{\mu\nu}\left(\mathcal{E}^2 + \mathcal{B}^2\right) \\ &+ 2u_{(\mu}\eta_{\nu)\alpha\beta}E^\alpha_a B^{\beta\, a}. \end{aligned} \tag{6.31}$$

In the above, $\mathcal{E}^2 = \mathcal{E}^a\mathcal{E}_a = \mathcal{E}^a\mathcal{E}^b g_{ab} = -2\delta_{ab}\mathcal{E}^a\mathcal{E}^b$, and an analogous relation holds for $\mathcal{B}^2$. Up to this point, everything is consistent with the standard irreducible decomposition given in Appendix B. It is now appropriate to recall that these components are ultimately functions of the gauge field variables A^m_μ. In particular, using the definition $\mathcal{F} = D\mathcal{A}$, we obtain:

$$\begin{aligned} \mathcal{E}^a_i &= 2\nabla_{[i}\mathcal{A}^a{}_{0]} + g\varepsilon^a{}_{bc}\mathcal{A}^b{}_0\mathcal{A}^c_i, \\ \mathcal{B}^a_i &= \eta_{ijk}\nabla^{[j}\mathcal{A}^{ak]} + \frac{1}{2}g\eta_{ijk}\varepsilon^a{}_{bc}\mathcal{A}^{bj}\mathcal{A}^{ck}. \end{aligned} \tag{6.32}$$

Then we get the following energy, pressure, energy flux and anisotropic stress via (4.40) given by:

$$\begin{aligned} \rho^{\mathcal{F}} &= \frac{1}{2}\left(\mathcal{E}^2 + \mathcal{B}^2\right), \\ p^{\mathcal{F}} &= \frac{1}{6}\left(\mathcal{E}^2 + \mathcal{B}^2\right), \\ q^{\mathcal{F}}_\lambda &= \eta_{\lambda\gamma\beta}\mathcal{E}_a{}^\gamma\mathcal{B}^{a\beta}, \\ \pi^{\mathcal{F}}_{\mu\nu} &= -\left(\mathcal{B}^a{}_\mu\mathcal{B}_{a\nu} + \mathcal{E}^a{}_\mu\mathcal{E}_{a\nu}\right) + \frac{1}{3}h_{\mu\nu}\left(\mathcal{E}^2 + \mathcal{B}^2\right). \end{aligned} \tag{6.33}$$

The analysis reveals that the results obtained in the non-Abelian 1-form case closely parallel those previously established by Thorsrud in Subsection 6.1.2, leading once again to an equation of state of the form $P/\rho = +1/3$. This correspondence justifies the generalisation of Remark 6.7 to include the non-Abelian scenario. However, such an equation of state is incompatible with the decay condition required to support anisotropic curvature in the shear-free regime. It follows that neither Abelian nor non-Abelian 1-form gauge fields can serve as suitable sources for sustaining anisotropic shear-free cosmologies. Furthermore, the introduction of the non-Abelian structure does not substantially modify the system's physical behaviour when contrasted with its Abelian counterpart.

6.3.3 0-form plus 1-form

Given that the previous two cases yielded no substantial deviations from the results obtained by Thorsrud in the Abelian framework, it is natural to explore whether novel features arise when these configurations are combined within a non-Abelian setting. To this end, we begin by considering the action governing the gauge field sector, defined as follows:

$$S = -\int \mathrm{d}x^4 \sqrt{-|g|}\left(\frac{f}{4}\,\mathcal{F}^2 + \frac{1}{2}\,\mathcal{J}^2 + V\right), \tag{6.34}$$

where f is a scalar function that considers the interaction between the 0-form and 1-form non-Abelian gauge fields, and V is a potential.

The energy-momentum associated with the action (6.34) can be written as

$$T_{\mu\nu} = f\,\mathcal{F}_{\mu}^{\lambda\,a}\,\mathcal{F}_{\lambda\nu\,a} - \frac{f}{4}\,g_{\mu\nu}\,\mathcal{F}^2 + \mathcal{J}_{\mu}^{a}\,\mathcal{J}_{\nu}^{a} - \frac{1}{2}\,g_{\mu\nu}\,\mathcal{J}^2 - g_{\mu\nu}\,V. \tag{6.35}$$

The matter fields satisfy the evolution equation

$$\dot{\rho} + 3H\,(\rho + p) = -\pi_{\mu\nu}\,\sigma^{\mu\nu}, \tag{6.36}$$

whereas the congruence evolves according to the Raychudhuri equation and the shear propagation equation

$$\dot{H} + H^2 = -\frac{1}{6}\,(\rho + 3p) - \frac{2}{3}\sigma^2, \tag{6.37}$$

$$\dot{\sigma}_{\mu\nu} + 3H\sigma_{\mu\nu} = \pi_{\mu\nu} - {}^{3}S_{\mu\nu}, \tag{6.38}$$

in agreement with saw in chapter 5. Here, the energy density ρ corresponds to the total energy density, i.e. $\rho = \rho^{\mathcal{J}} + \rho^{\mathcal{F}} + \rho^{V}$. The same applies to the total isotropic pressure p and the total anisotropic stresses $\pi_{\mu\nu}$. Additionally,

$$^{3}S_{\mu\nu} = {}^{3}R_{\mu\nu} - \frac{^{3}R}{3} h_{\mu\nu},$$

is the trace-free three-dimensional Ricci tensor on the hypersurfaces Σ_t. Besides, there is one Hamiltonian constraint among the variables:

$$3H^2 - \sigma^2 + \frac{^{3}R}{2} = \rho + \rho, \tag{6.39}$$

that we shall sometimes refer to as the "Friedmann equation". The energy density, isotropic pressure, flux energy and stress tensor for each kind of $\{0, I, II, III\}$-form are given by

$$\text{0-form } V \begin{cases} \rho^V = V, & q^v_\mu = 0, \\ p^V = -V, & \pi^V_{\mu\nu} = 0. \end{cases} \tag{6.40}$$

$$I\text{-form } \mathcal{J}^a_\mu \begin{cases} \rho^{\mathcal{J}} = \frac{1}{2}\left(\omega^2 + v^2\right), & q^{\mathcal{J}}_\mu = -\omega^a\, v_{\mu\, a}, \\ p^{\mathcal{J}} = \frac{1}{2}\Big(\omega^2 - \frac{1}{3}\, v^2\Big), & \pi^{\mathcal{J}}_{\mu\nu} = v^a_\mu\, v_{\nu\, a} - \frac{1}{2}\, h_{\mu\nu}\, v^2. \end{cases} \tag{6.41}$$

$$II\text{-form } \mathcal{F}^a_{\mu\nu} \begin{cases} \rho^{\mathcal{F}} = \frac{f}{2}\left(\mathcal{E}^2 + \mathcal{B}^2\right), & q^{\mathcal{F}}_\mu = f\, \eta^{\mu\alpha\beta}\, \mathcal{E}^a_\alpha\, \mathcal{B}_{\beta\, a}, \\ p^{\mathcal{F}} = \frac{f}{6}\left(\mathcal{E}^2 + \mathcal{B}^2\right), & \pi^{\mathcal{F}}_{\mu\nu} = -f\left(\mathcal{E}^a_\mu\, \mathcal{E}_{\nu\, a} + \mathcal{B}^a_\mu\, \mathcal{B}_{\nu\, a}\right) \\ & \qquad +\frac{f}{3}\, h_{\mu\nu}\left(\mathcal{E}^2 + \mathcal{B}^2\right). \end{cases} \tag{6.42}$$

To guarantee non-tilted fluids, we have to impose that the total flux q vanish, so we have to impose both that $\mathcal{E}^a_\mu = 0$ or $\mathcal{B}^a_\mu = 0$, and $\omega^a = 0$. Thus,

$$\text{0-form } V \begin{cases} \rho^V = V, & q^v_\mu = 0, \\ p^V = -V, & \pi^V_{\mu\nu} = 0. \end{cases} \tag{6.43}$$

$$I\text{-form } \mathcal{J}^a_\mu \begin{cases} \rho^{\mathcal{J}} = \frac{1}{2}\, v^2, & q^{\mathcal{J}}_\mu = 0, \\ p^{\mathcal{J}} = -\frac{1}{6}\, v^2, & \pi^{\mathcal{J}}_{\mu\nu} = v^a_\mu\, v_{\nu\, a} - \frac{1}{2}\, h_{\mu\nu}\, v^2. \end{cases} \tag{6.44}$$

$$II\text{-form } \mathcal{F}^a_{\mu\nu} \begin{cases} \rho^{\mathcal{F}} = \frac{f}{2}\, \mathcal{B}^2, & q^{\mathcal{F}}_\mu = 0, \\ p^{\mathcal{F}} = \frac{f}{6}\, \mathcal{B}^2, & \pi^{\mathcal{F}}_{\mu\nu} = -f\, \mathcal{B}^a_\mu\, \mathcal{B}_{\nu\, a} + \frac{f}{3}\, h_{\mu\nu}\, \mathcal{B}^2. \end{cases} \tag{6.45}$$

Moreover, we choose an orthonormal frame, where

$$\mathbf{e}_0 = \partial_t \quad \text{and} \quad \mathbf{u} = \mathbf{e}_0 \quad \Rightarrow \quad u^0 = 1 : \qquad u^\mu\, u_\mu = -1.$$

Considering that our choice of frame matches the local Lorentz frame at the observer, we have that

$$\mathcal{E}^\mu = (0, \vec{\mathcal{E}}) \;\rightarrow\; \mathcal{E}^i \qquad \text{and} \qquad \mathcal{B}^\mu = (0, \vec{\mathcal{B}}) \;\rightarrow\; \mathcal{B}^i. \tag{6.46}$$

Hence, given the choice of an observer whose local rest space coincides with the hypersurface of simultaneity[43], and considering the conformal expansion inherent to Bianchi models—where, according to equation (3.15), the induced metric takes the form $h_{ab}(x^\mu) = a^2(t)\, \tilde{h}_{ab}(x^c)$—it follows from equation (6.46) that it is appropriate to work with the induced metric $\tilde{h}_{ij}$ rather than h_{ij}, since the observer perceives no temporal evolution within the hypersurfaces Σ_t[44]. Consequently, equations (6.43)–(6.45) may be rewritten as follows:

$$0\text{-form } V \begin{cases} \rho^V = V, & q^v_i = 0, \\ p^V = -V, & \pi^V_{ij} = 0. \end{cases} \tag{6.47}$$

$$I\text{-form } \mathcal{J}^a_i \begin{cases} \rho^{\mathcal{J}} = \frac{1}{2}\, v^2, & q^{\mathcal{J}}_i = 0, \\ p^{\mathcal{J}} = -\frac{1}{6}\, v^2, & \pi^{\mathcal{J}}_{ij} = v^a_i\, v_{j\, a} - \frac{1}{2}\, \tilde{h}_{ij}\, v^2. \end{cases} \tag{6.48}$$

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[43] For a detailed proof of this statement, the reader is referred to Chapter 12 of [100].

[44] Essentially, this means that $\tilde{h}_{ij}$ evolves as a constant.

$$II\text{-form } \mathcal{F}^a_{ij} \begin{cases} \rho^{\mathcal{F}} = \frac{f}{2}\,\mathcal{B}^2, \quad q^{\mathcal{F}}_i = 0, \\ p^{\mathcal{F}} = \frac{f}{6}\,\mathcal{B}^2, \quad \pi^{\mathcal{F}}_{ij} = -f\,\mathcal{B}^a_i\,\mathcal{B}_{j\,a} + \frac{f}{3}\,\tilde{h}_{ij}\,\mathcal{B}^2. \end{cases} \tag{6.49}$$

Thus, the key results presented by Thorsrud in sections 6.1 and 6.2 can be suitably extended to encompass the case under consideration, as follows:

1) Taking the energy density and isotropic pressure from equation (6.48), it follows that the equation of state $\rho^{\mathcal{J}}/p^{\mathcal{J}} = -1/3$ remains unchanged from the Abelian case. Consequently, in the shear-free limit, the matter evolution equation (6.36) indicates that the energy density decays as $\rho^{\mathcal{J}} \propto 1/a^2$, which in turn implies that:

$$\begin{aligned} \rho^{\mathcal{J}} &= \frac{1}{2}\nu^2 \propto \frac{1}{a^2} \\ &= \frac{1}{2}\tilde{h}_{ij}\nu^i_a\,\nu^{j\,a} \propto \frac{1}{a^2} : \quad \tilde{h}_{ij} \propto cte \\ \therefore\; \nu^i_a &\propto \frac{1}{a} \quad \text{and} \quad \nu_{i\,a} \propto \frac{1}{a}. \end{aligned}$$

2) Hence, the stress tensor of the 1-form at the strength level must evolve as

$$\pi^{\mathcal{J}}_{ij} = \underbrace{v^a_i\,v_{j\,a}}_{\propto 1/a^2} - \frac{1}{2}\,\underbrace{\tilde{h}_{ij}\,v^2}_{\propto 1/a^2} \quad \therefore\; \pi^{\mathcal{J}}_{ij} \propto 1/a^2,$$

which confirms and extends the results obtained in theorem 6.6.

3) Regarding the II-form case at the field strength level, and similarly to the I-form case, the equation of state derived from (6.49) remains identical to that of the Abelian scenario, as the function f does not influence at all. Therefore, we have $\rho^{\mathcal{F}}/p^{\mathcal{F}} = -1/3$. In the shear-free limit, the matter evolution equation (3.15) implies that the energy density decays as $\rho^{\mathcal{F}} \propto 1/a^4$, which in turn leads to:

$$\begin{aligned} \rho^{\mathcal{F}} &= \frac{1}{2}f\,\mathcal{B}^2 \propto \frac{1}{a^2} \\ &= \frac{1}{2}f\,\tilde{h}_{ij}\,\mathcal{B}^i_a\,\mathcal{B}^{j\,a} : \quad \tilde{h}_{ij} \propto cte \\ \therefore\; f\mathcal{B}^2 &\propto \frac{1}{a^4},\; f\mathcal{B}^i_a\,\mathcal{B}^{j\,a} \propto \frac{1}{a^4} \;\text{ and }\; f\mathcal{B}^a_i\,\mathcal{B}_{j\,a} \propto \frac{1}{a^4}. \end{aligned}$$

4) The latter implies that the anisotropic stress tensor corresponding to the II-form at the strength level must decay as

$$\pi^{\mathcal{F}}_{ij} = -\underbrace{f\mathcal{B}^a_i\mathcal{B}_{ja}}_{\propto 1/a^4} + \frac{1}{3}\underbrace{f\,\tilde{h}_{ij}\mathcal{B}^2}_{\propto 1/a^4} \quad \therefore \pi^{\mathcal{F}}_{ij} \propto 1/a^4,$$

what confirms and extends the results obtained in remark 6.7.

The results can be summarised as follows:

$$\begin{cases} H \propto 1/a \quad \Longrightarrow \dot{H} \propto 1/a^2, \\ \rho^{\mathcal{J}} \propto 1/a^2 \Longrightarrow \nu^a_i \propto 1/a \quad \text{and} \quad \nu^i_a \propto 1/a, \\ \rho^{\mathcal{J}} \propto 1/a^2 \Longrightarrow \pi^{\mathcal{J}}_{ij} \propto 1/a^2, \\ \rho^{\mathcal{F}} \propto 1/a^4 \Longrightarrow f\,\mathcal{B}^a_i\,\mathcal{B}_{j\,a} \propto 1/a^4 \quad \text{and} \quad f\,\mathcal{B}^i_a\,\mathcal{B}^{j\,a} \propto 1/a^4, \\ \rho^{\mathcal{F}} \propto 1/a^4 \Longrightarrow \pi^{\mathcal{F}}_{ij} \propto 1/a^4, \\ \tilde{h}_{ij} \propto cte, \quad {}^3R \propto 1/a^2 \quad \text{and} \quad {}^3S_{ij} \propto 1/a^2. \end{cases} \tag{6.50}$$

To determine whether this non-Abelian gauge field configuration can sustain the anisotropic spatial curvature—and thereby satisfy the shear-free condition (4.11)—it is essential to examine the decay behaviour of each term in the dynamical equations (6.37)–(6.41) within the shear-free regime, to evaluate the validity of the condition rigorously. If it is not inherently satisfied, one may further investigate whether specific decay properties of the interaction function f—which governs the coupling between the gauge fields—can ensure compliance with the shear-free constraint.

In the context of non-tilted fluids and under the shear-free regime, the Raychaudhuri equation (6.37), the shear propagation equation (6.38), and the Friedmann equation (6.39), corresponding to an interacting non-Abelian gauge field comprising both 1-form and 2-form components, together with a potential term, assume the following respective forms:

$$\underbrace{\dot{H}}_{\propto 1/a^2} + \underbrace{H^2}_{\propto 1/a^2} = \frac{1}{3}V - \frac{1}{6}\underbrace{f\,\mathcal{B}^2}_{\propto 1/a^4}, \tag{6.51}$$

$$\underbrace{{}^3S_{ij}}_{\propto 1/a^2} = \underbrace{\nu_i^a\,\nu_{j\,a}}_{\propto 1/a^2} - \frac{1}{3}\underbrace{\tilde{h}_{ij}\,\nu^2}_{\propto 1/a^2} - \underbrace{f\,\mathcal{B}_i^a\,\mathcal{B}_{j\,a}}_{\propto 1/a^4} + \frac{1}{3}\underbrace{f\,\tilde{h}_{ij}\,\mathcal{B}^2}_{\propto 1/a^2}, \tag{6.52}$$

$$3\underbrace{H^2}_{\propto 1/a^2} + \frac{1}{3}\underbrace{{}^3R}_{\propto 1/a^2} = \frac{1}{3}\underbrace{\nu^2}_{\propto 1/a^2} + \frac{2}{3}\underbrace{f\,\mathcal{B}^2}_{\propto 1/a^2}. \tag{6.53}$$

Irrespective of the evolution of the potential, a consistent pattern is evident across the three equations: the contribution of the II-form at the field strength level makes it impossible to achieve matching decay rates on both sides of each equation. Moreover, it appears unfeasible to isolate the interaction function f in a manner that allows for the determination of its required decay behaviour to sustain the anisotropic curvature. In conclusion, this class of interacting non-Abelian gauge fields fails to counterbalance the anisotropic spatial curvature and, therefore, cannot yield shear-free solutions.

6.3.4 Another alternative: $q_\mu^{\mathcal{J}} + q_\mu^{\mathcal{F}} = 0$

In the preceding case, we assumed, aiming to ensure non-tilted fluids, that each energy flux individually vanishes. However, an alternative approach yielding the same condition requires that the total energy flux vanishes; that is, the combined contribution of the 1-form and 2-form energy fluxes is equal to zero. Hence,

$$q_\mu^{\mathcal{J}} + q_\mu^{\mathcal{F}} = 0 \quad \Longrightarrow \quad f\,\eta_{\mu\alpha\beta}\,\mathcal{E}_a^\alpha\,\mathcal{B}^{\beta\,a} = \omega^a\,v_\mu, \tag{6.54}$$

which implies that

$$\begin{aligned}\nu^2 = \nu_\mu^a\nu_a^\mu &= f^2\,\omega^{-2}\,\eta^{\mu\alpha\beta\Lambda}\eta_{\mu\omega\Gamma\theta}\,\mathcal{E}_\alpha^a\,\mathcal{B}_\alpha^b\,\mathcal{E}_b^\omega\,\mathcal{B}_a^\Gamma u_\Lambda u^\theta \\ &= f^2\,\omega^{-2}\Big(\mathcal{E}^2\,\mathcal{B}^2 + \mathcal{E}_\Gamma^a\,\mathcal{B}_a^\Gamma\,\mathcal{B}_\omega^b\,\mathcal{E}_b^\omega\Big),\end{aligned} \tag{6.55}$$

where the identities involving the Levi-Civita symbol, as outlined in Appendix E, have been employed.

Hence, the expressions for the energy density, isotropic pressure, energy flux, and anisotropic stress tensor differ from those given in equations (6.40)–(6.42) in the following manner:

$$\text{0-form } V \begin{cases} \rho^V = V, & q_i^v = 0, \\ p^V = -V, & \pi_{ij}^V = 0. \end{cases} \tag{6.56}$$

$$I\text{-form } \mathcal{J}_i^a \begin{cases} \rho^{\mathcal{J}} = \frac{1}{2}\left[\omega^2 + f^2\,\omega^{-2}\Big(\mathcal{E}^2\mathcal{B}^2 + \mathcal{E}_i^a\,\mathcal{B}_a^i\,\mathcal{E}_j^b\,\mathcal{B}_b^j\Big)\right], \\ p^{\mathcal{J}} = \frac{1}{2}\Big[\omega^2 - \frac{1}{3}f^2\,\omega^{-2}\Big(\mathcal{E}^2\mathcal{B}^2 + \mathcal{E}_i^a\,\mathcal{B}_a^i\,\mathcal{E}_j^b\,\mathcal{B}_b^j\Big)\Big], \\ q_i^{\mathcal{J}} = -\omega^a\,v_{i\,a}, \\ \pi_{ij}^{\mathcal{J}} = f^2\,\omega^{-2}\Big[\eta_{ilm}\eta_{jnp}\,\mathcal{E}_a^l\,\mathcal{B}^{ma}\,\mathcal{E}_b^n\,\mathcal{B}^{pb} \\ \qquad -\frac{1}{2}\tilde{h}_{ij}\Big(\mathcal{E}^2\mathcal{B}^2 + \mathcal{E}_h^a\,\mathcal{B}_a^h\,\mathcal{E}_k^b\,\mathcal{B}_b^k\Big)\Big]. \end{cases} \tag{6.57}$$

$$II\text{-form } \mathcal{F}_{ij}^a \begin{cases} \rho^{\mathcal{F}} = \frac{f}{2}\,(\mathcal{E}^2 + \mathcal{B}^2), & q_i^{\mathcal{F}} = \omega^a\,\nu_{i\,a}, \\ p^{\mathcal{F}} = \frac{f}{6}\,(\mathcal{E}^2 + \mathcal{B}^2), & \pi_{ij}^{\mathcal{F}} = -f(\mathcal{E}_i^a\,\mathcal{E}_{j\,a} + \mathcal{B}_i^a\,\mathcal{B}_{j\,a}) \\ & \qquad + \frac{f}{3}\,\tilde{h}_{ij}(\mathcal{E}^2 + \mathcal{B}^2). \end{cases} \tag{6.58}$$

Although the equation of state cannot be explicitly determined for the 1-form case, since the ratio $\rho^{\mathcal{J}}/p^{\mathcal{J}}$ remains expressed in terms of the electric and magnetic components, for the 2-form contribution, the equation of state remains unchanged, yielding $\rho^{\mathcal{F}}/p^{\mathcal{F}} = +1/3$. This, in turn, implies that:

$$\rho^{\mathcal{F}} \propto 1/a^4 \implies f\,\mathcal{E}^2 \propto 1/a^4 \quad \text{and} \quad f\,\mathcal{B}^2 \propto 1/a^4. \tag{6.59}$$

By substituting expressions (6.56)–(6.58) into the Raychaudhuri equation (6.37), the shear propagation equation (6.38), and the Friedmann equation (6.39), one finds that all resulting terms remain expressed in forms analogous to (6.59). This leads to the same drawback encountered in the previous section. Consequently, even when accounting for the constraint (6.54), this configuration of interacting gauge fields fails to counterbalance the anisotropic spatial curvature and thus cannot support shear-free solutions.

6.3.5 2-form

In subsection 6.1.3, the analysis relied on Hodge duality with the 0-form case to explore the behaviour of the 2-form field. However, in the non-Abelian context, this correspondence does not necessarily hold, making it essential to examine it independently.

We start with the action for the gauge field sector, defined by

$$S = -\frac{1}{12}\int \mathrm{d}x^4\sqrt{-|g|}\,\mathcal{H}^2, \tag{6.60}$$

where $\mathcal{H}^2 \equiv \mathcal{H}^a\mathcal{H}_a$, and $\mathcal{H}^a_{\mu\nu\alpha} = D_\mu\,\mathcal{C}^a_{\nu\alpha}$, with $\mathcal{C}^a_{\nu\alpha}$ denoting a non-Abelian two-form gauge field. In this setting, the III-form $\mathcal{H}^a$ admits the following decomposition[45]:

$$\mathcal{H}^a = \underline{u} \wedge \epsilon(\vec{u}, \vec{b}^a, \cdot, \cdot) + \omega^a\,\epsilon(\vec{u}, \cdot, \cdot, \cdot),$$

or written in components:

$$\mathcal{H}^a_{\mu\nu\alpha} = -\left(3\,u_{[\mu}\,\eta_{\nu\alpha]\,\rho}b^{\rho\,a} + \omega^a\,\eta_{\mu\nu\alpha}\right), \tag{6.61}$$

such that, $\eta_{\mu\nu\alpha} \equiv \eta_{\mu\nu\alpha\Gamma}\,u^\Gamma$. Thus, the energy-momentum tensor associated to (6.60) takes the form

$$T_{\mu\nu} = \omega^2\,h_{\mu\nu} - \omega^a(u_\mu b_{\nu\,a} + u_\nu b_{\mu\,a}) + b^a_\mu b_{\nu\,a} - \frac{1}{2}(\omega^2 + b^2)g_{\mu\nu}. \tag{6.62}$$

This expression precisely reproduces the energy-momentum tensor corresponding to the 0-form case presented in (6.25). Consequently, the associated energy density, isotropic pressure, energy flux, and anisotropic stress coincide with those given in (6.26). It follows that the conclusions drawn in subsection 6.3.1 for the non-Abelian 0-form case remain applicable: both Abelian and non-Abelian 2-form gauge fields give rise to viable shear-free solutions. However, the non-Abelian character does not yield any substantial physical differences or introduce new dynamical features relative to the Abelian scenario.

[45] For additional details on this decomposition, see Appendix B.

6.3.6 3-form

As a last attempt, we shall study the 3-form non-Abelian gauge field configuration. The action for this sector is defined by

$$S = -\frac{1}{48}\int \mathrm{d}x^4\sqrt{-|g|}\,\mathcal{K}^2. \tag{6.63}$$

Here, $\mathcal{K}^2 \equiv \mathcal{K}^a\mathcal{K}_a$, with the field strength given by $\mathcal{K}^a_{\mu\nu\alpha\omega} = D_\mu\mathcal{I}^a_{\nu\alpha\omega}$, where $\mathcal{I}^a_{\nu\alpha\omega}$ represents a non-Abelian 3-form gauge field. In the context of a four-dimensional spacetime manifold, the field strength–being a IV-form–is necessarily a top-form and, as such, must be proportional to the canonical volume form η, as discussed in [100]. Therefore,

$$\mathcal{K}^a_{\mu\nu\alpha\omega} = \varphi^a\,\eta_{\mu\nu\alpha\omega}.$$

Thus, the components of the energy-momentum tensor related to the action (6.63) are

$$T_{\mu\nu} = \frac{1}{6}\mathcal{K}^{a\,\omega\alpha\beta}_{\mu}\,\mathcal{K}_{a\,\nu\omega\alpha\beta} - \frac{1}{48}g_{\mu\nu}\mathcal{K}^2 = -\frac{1}{2}\varphi^2\,g_{\mu\nu}. \tag{6.64}$$

Employing equation (4.40), the resulting expressions for the energy density, isotropic pressure, energy flux, and anisotropic stress are as follows:

$$\rho^{\mathcal{K}} = -p^{\mathcal{K}} = \frac{1}{2}\varphi^2 \quad \text{and} \quad q^{\mathcal{K}}_\mu = \pi^{\mathcal{K}}_{\mu\nu} = 0. \tag{6.65}$$

Essentially, this configuration does not contribute to counterbalancing the anisotropic spatial curvature in the shear-free limit and is thus ruled out as a viable candidate for producing shear-free solutions, mirroring the conclusions reached for the analogous Abelian case examined in Subsection 6.1.4.

Chapter 7

Summary

7.1 Research question and assumptions

Starting from the Copernican principle and adopting the weak cosmological principle as a reasonable consequence—while setting aside the strict assumption of isotropy given what observations tell us, as discussed in Chapter 1—we are naturally led to ask: how likely is the universe we observe? This brings us to the guiding question of our research:

> Question 1. Can the spatially anisotropic solutions to general relativity be dynamically distinguished from FLRW cosmologies?

A rigorous framework for addressing this question is provided by the analysis of shear-free cosmological solutions—spacetimes in which the shear tensor identically vanishes throughout cosmic evolution. In such models, the vanishing of anisotropic stresses necessarily implies that the spatial sections possess constant curvature, thus rendering the geometry dynamically equivalent to that of the standard FLRW models. Conversely, allowing for non-zero anisotropic stresses precludes the possibility of constant curvature, indicating that any deviation from isotropy in the stress-energy tensor must be compensated by corresponding deviations in the geometry. This correlation is a characteristic feature of spacetimes based on Bianchi-type universes.

Considering cosmological models as the quadruple $(\mathcal{M}, \mathbf{g}, \mathbf{u}, \mathbf{\Gamma})$, in agreement with definition 1, the present research work proceed under the following assumptions:

Assumption 7.1 (Philosophy). We assume the weak cosmological principle. Furthermore, we assume that the manifold $\mathcal{M}$ of the model is spatially homogeneous and spatially anisotropic.

Assumption 7.2 (Geometry). We take as the geometry model the Bianchi cosmologies $(\mathcal{M}, \mathbf{g}, \mathbf{u}, \mathbf{\Gamma})$, defined as a spacetime in which the metric $\mathbf{g}$ admits a three-dimensional Lie group of isometries acting simply transitively on spacelike hypersurfaces Σ_t, identified as the homogeneous spatial sections of the spacetime manifold[a].

[a]See definition 6.

In previous work, Mikel Thorsrud examined the question 1 by considering matter sources composed of comoving perfect fluids, such as those described by the ΛCDM model, together with a set of Abelian p-form gauge fields with vanishing energy flux. Under these conditions, he showed that certain anisotropic cosmological models, including Bianchi types II, III, VI_0 and Kantowski–Sachs spacetimes, admit shear-free solutions that are dynamically equivalent to standard FLRW cosmologies [95]. The primary aim of the present research is to extend Thorsrud's results by implementing a collection of non-Abelian p-form gauge fields, thereby enabling a systematic comparison between the Abelian and non-Abelian frameworks. Hence,

Assumption 7.3 (Matter). We consider two classes of matter fields: the standard ΛCDM components, modelled as a collection of comoving perfect fluids, and a set of p-form gauge fields $\mathcal{A}$ with vanishing energy flux.

Assumption 7.4 (Theory). Einstein's General Theory of Relativity is assumed to be the correct theory of gravity.

7.2 Method

The question 1, along with its subsidiary inquiries, will be examined within the theoretical framework defined by assumptions 7.1–7.4, which specify the class of cosmological models under investigation. The subsequent analysis will proceed according to the following key steps.

Step 1: Construct a Bianchi model by applying the method outlined in Section 3.2, yielding the metric components in expression (3.15) that manifest the full symmetry structure specified in assumption 7.2.

Step 2: Once the matter content is specified according to assumption 7.3, and its associated energy-momentum tensor is given by equation (5.4), impose the shear-free condition (5.11) to ensure the vanishing of the shear and thus identify shear-free solutions.

Step 3: In light of theorem 6.6 and the shear-free condition (5.11), restrict attention to matter configurations whose anisotropic stress decays as $1/a^2$. Accordingly, examine all non-trivial non-Abelian p-form gauge fields as potential candidates. In our case $p \in \{0, 1, 2, 3\}$.

7.3 Main results

This work has systematically examined non-Abelian p-form gauge fields with $p \in \{0, 1, 2, 3\}$, along with certain interacting configurations, to assess their ability to sustain anisotropic spatial curvature in shear-free cosmologies. The analysis confirms that both Abelian and non-Abelian 0-form and 2-form fields lead to shear-free solutions, with an equation of state $P/\rho = -1/3$. These cases successfully satisfy the shear-free condition, but the non-Abelian structure does not introduce any new physical or dynamical features beyond those already present in the Abelian setting.

By contrast, the 1-form and 3-form cases fail to support anisotropic curvature under the shear-free condition. In the 1-form scenario, the resulting equation of state $P/\rho = +1/3$ leads to an energy decay incompatible with the necessary conditions for preserving shear-free anisotropic geometry. The 3-form configuration, on the other hand, contributes no effective anisotropic stress and is therefore automatically excluded as a viable candidate. In both cases, the non-Abelian character of the fields does not alter the conclusions reached in the Abelian context.

Interacting configurations involving combinations of 0-form and 1-form gauge fields were also investigated. However, in all such cases, the resulting dynamics fail to produce compatible decay rates across the evolution equations, even when additional constraints are imposed. The inability to isolate and appropriately tune the interaction function f further hinders the viability of these models. Accordingly, interacting non-Abelian gauge fields of this type cannot counterbalance anisotropic curvature and are not viable sources for shear-free Bianchi cosmologies.

Appendix A

A Brief Guide to Differential Geometry

> *Manifolds crop up everywhere in mathematics. These generalisations of curves and surfaces to arbitrarily many dimensions provide the mathematical context for understanding "space" in all of its manifestations.*
>
> —*John M. Lee. Introduction to Smooth Manifolds* (2013).

This appendix builds extensively on chapters 14, 15 and 16 in *Special Relativity in General Frames* [100], chapter 3 in *3+1 Formalism in General Relativity* [60] and chapter 2 in *An introduction to theory of rotating stars* [104].

A.1 Manifold and its tangent and cotangent spaces

Spacetime requires four coordinates to identify an event, which in classical and special relativity is taken to be globally true (i.e., events map to $\mathbb{R}^4$). In general relativity, however, we do not assume any global properties from the outset. This is analogous to describing Earth's surface with two coordinates locally but failing to capture its global geometry by simply extending those coordinates worldwide. Thus, we use the notion of a manifold—a set where each point has a neighbourhood resembling $\mathbb{R}^n$[46], yet the overall structure may differ globally.

[46]Basically, a manifold is a set made up of pieces that "look like" open subsets of $\mathbb{R}^n$ such that these pieces can be "sewn together" smoothly [105].

Definition 8 (Manifold [105]). An n-dimensional, $\mathcal{C}^\infty$, real manifold $\mathcal{M}$ is a set together with a collection of subsets $\{\mathcal{O}_\alpha\}$ satisfying the following properties:

1) Each $p \in \mathcal{M}$ lies in at least one $\mathcal{O}_\alpha$, i.e., the $\{O_\alpha\}$ cover M.
2) For each α, there is a one-to-one, onto, map $\psi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha$, where $\mathcal{U}_\alpha$ is an open subset of $\mathbb{R}^n$.
3) If any two sets $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ overlap, $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$ (where $\emptyset$ denotes the empty set), we can consider the map $\psi_\beta \circ {\psi_\alpha}^{-1}$ (where $\circ$ denotes composition) which takes points in $\psi_\alpha (\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \subset \mathcal{U}_\alpha \subset \mathbb{R}^n$ to points in $\psi_\beta (\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \subset \mathcal{U}_\beta \subset \mathbb{R}^n$.

We require these subsets of $\mathbb{R}^n$ to be open and this map to be C^∞, i.e., infinitely continuously differentiable. (Since we are dealing here with maps of $\mathbb{R}^n$ into $\mathbb{R}^n$, the advanced calculus notion of C^∞ functions applies.)

One can define vectors at each point of a manifold by considering the tangent space $T_p\mathcal{M}$. For example, a river flow on Earth's surface can be modeled as a vector field, assigning a vector in $T_p\mathcal{M}$ to each point p. Intuitively, this arises from collecting all tangent vectors to every possible curve passing through p.

In an analogous way, the co-tangent space $T_p^*\mathcal{M}$ is where 1-forms (with $p = 1$ in (A.7)) reside. Let $\{\mathbf{e}_\mu\}$ be a basis for $T_p\mathcal{M}$. Then a corresponding basis $\{\boldsymbol{\omega}^\nu\}$ for $T_p^*\mathcal{M}$ is defined by the dual relation

$$\boldsymbol{\omega}^\nu (\mathbf{e}_\mu) = \delta^\nu_\mu. \tag{A.1}$$

A.2 Group action on spacetime

Spacetime symmetries are described in a coordinate-independent way by introducing a (symmetry) group G that acts on the manifold $\mathcal{M}$. Each transformation in G shifts points in $\mathcal{M}$, and the metric $\boldsymbol{g}$ remains invariant under these displacements. Formally, a group action of G on $\mathcal{M}$ is defined by the mapping[47]

$$\begin{array}{rccc} \Phi: & G \times \mathcal{M} & \longrightarrow & \mathcal{M} \\ & (g,p) & \longmapsto & \Phi(g,p) := g(p), \end{array} \tag{A.2}$$

such that

- $\star$ $\forall\,(g,h) \in G^2, \forall\, p \in \mathcal{M}$, $g(h(p)) = gh(p)$, where gh denotes the product of g by h according to the group law of G (see Fig. A.1).
- $\star$ If e is the identity element of the group G, then $\forall\, p \in \mathcal{M}$, $e(p) = p$.

> Definition 9 (The orbit of a point). It is defined as the set $\{g(p), g \in G\} \subset \mathcal{M}$, i.e., the set of points which are connected to p by some group transformation[a].
>
> [a]In fact, the set of all orbits of $p \in \mathcal{M}$ must be a submanifold of $\mathcal{M}$ [6].

An important subclass of group actions arises when G is a one-dimensional *Lie group* (i.e., a continuous group). In a neighbourhood of the identity element e, each group element can be labelled by a real parameter t such that $g_{t=0} = e$. The orbit of a point $p \in \mathcal{M}$ under this action is then either $\{p\}$ (if p is a fixed point) or a one-dimensional curve in $\mathcal{M}$. In the latter case, t serves as a natural parameter along the curve (see Fig. A.2). The tangent vector associated with this parameter is termed the *generator of the symmetry group with respect to the t parametrisation*, given by

$$\vec{\boldsymbol{\xi}} = \frac{d\vec{\boldsymbol{x}}}{dt}. \tag{A.3}$$

Here, $d\vec{\boldsymbol{x}}$ is the infinitesimal displacement taking p to $g_{dt}(p)$ (see Fig. A.2). Consequently, in any infinitesimal neighborhood of p, the action of G amounts to translations along the vector $dt\,\vec{\boldsymbol{\xi}}$.

[47]Do not confuse the generic element g of the group G with the metric tensor $\mathbf{g}$.

Definition 10 (Transitively action on a group [6]). A group G is said to act transitively on a manifold $\mathcal{M}$ if given any two points $p, q \in \mathcal{M}$ there exist a group element $g \in G$ that connects them; i.e.,

$$g(p) = p. \tag{A.4}$$

Clearly, a group is transitive on each of its orbits.

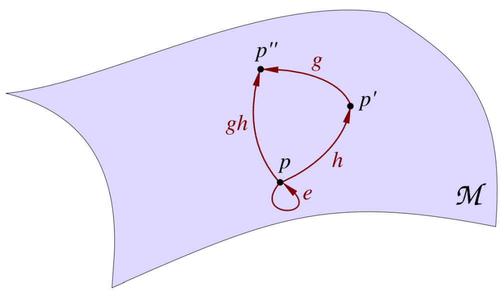

Figure A.1: Group action on the spacetime manifold $\mathcal{M}$. *Taken from* [60].

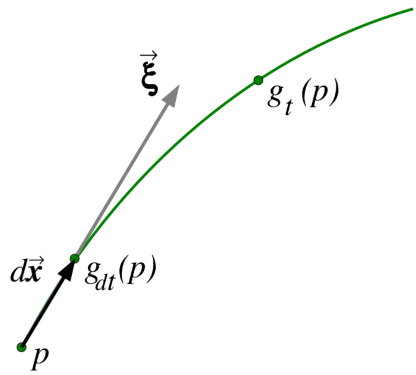

Figure A.2: Under the action of a one-dimensional Lie group labelled by $t \in \mathbb{R}$, the orbit of a point p forms a curve, and the vector $\vec{\xi} = \frac{d\vec{x}}{dt}$ is the generator corresponding to the parameter t. *Taken from* [60].

Corollary A.1. *If the element $g \in G$ that enters in (A.4) is unique, the group is said to be simply-transitive*[a]*. If g in (A.4) is not unique, the group is said to be multiply-transitive on each orbit [6].*

[a]Colloquially we could say that there is only one way to get from p to q.

A.3 Differential forms and their operations

In what follows, let us consider an n-dimensional space E. For any integer $p \in \mathbb{N}$, a *differential p-form* $\mathcal{A}$ is defined to be a smooth field of p forms –i.e., a smooth field of alternating tensors of valence p– as described below [100].

Within the space of all tensors, a notable subset consists of those multilinear forms that are completely antisymmetric –namely, the $(0, p)$– type tensors which change sign upon interchange of any two arguments:

$$
\begin{aligned}
p = 2: &\qquad \mathcal{A}(\vec{v}_1, \vec{v}_2) = -\mathcal{A}(\vec{v}_2, \vec{v}_1),\\
p = 3: &\qquad \mathcal{A}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = -\mathcal{A}(\vec{v}_2, \vec{v}_1, \vec{v}_3),\\
\vdots &\\
p = n: &\qquad \mathcal{A}(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n) = -\mathcal{A}(\vec{v}_2, \vec{v}_1, \cdots, \vec{v}_n).
\end{aligned}
$$

Such multilinear forms are termed *alternate forms* because they vanish whenever two of their arguments coincide. For any integer $p \geq 2$, a *p-form* is defined as an alternate form of valence p. We denote the set of all p-forms on E by $\mathscr{A}_p(E)$.

Definition 11 (Wedge product [100]). Let $\mathscr{A}_p(E)$ and $\mathscr{A}_q(E)$, the spaces of p-forms and q-forms on an n-dimensional vector space E, respectively. We define the wedge product as the mapping

$$
\begin{aligned}
\wedge : \mathscr{A}_p(E) \times \mathscr{A}_q(E) &\longrightarrow \mathscr{A}_{p+q}(E) \\
(\mathcal{A}, \mathcal{B}) \qquad &\longmapsto \mathcal{A} \wedge \mathcal{B},
\end{aligned}
\tag{A.5}
$$

such that

$$
\mathcal{A} \wedge \mathcal{B}\,(\vec{v}_1, \cdots, \vec{v}_{p+q}) := \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{k(\sigma)} \mathcal{A}(\vec{v}_{\sigma(1)}, \cdots, \vec{v}_{\sigma(p)}) \times \mathcal{B}(\vec{v}_{\sigma(p+1)}, \cdots, \vec{v}_{\sigma(p+q)}).
\tag{A.6}
$$

Here, $(\vec{v}_1, \cdots, \vec{v}_{p+q})$ is any element of E^{p+q}, $\mathfrak{S}_{p+q}$ denotes the group of all permutations of $p+q$ elements, and $k(\sigma)$ is the number of transpositions into which the permutation σ can be factorized. In the above definition, $\mathcal{A} \wedge \mathcal{B}$ is an alternate form of valence $p+q$, ensuring that the map (A.5) is well defined.

By means of the wedge product, any p-form can be written as

$$
\mathcal{P} = \frac{1}{p!}\, \mathcal{P}_{\mu_1 \cdots \mu_p}\, \boldsymbol{\omega}^{\mu_1} \wedge \cdots \wedge \boldsymbol{\omega}^{\mu_p}.
\tag{A.7}
$$

In an n-dimensional space, it is necessary that $p \leq n$. Consequently, the n-form is called the *top-form*. Because a top form possesses only one component, all such forms are necessarily proportional. The *volume form*, η, is an example of a top-form, defined as [63]

$$
\eta = \frac{1}{n!} \sqrt{|g|}\, \varepsilon_{\mu_1 \cdots \mu_n}\, \boldsymbol{\omega}^{\mu_1} \wedge \cdots \wedge \boldsymbol{\omega}^{\mu_n}.
\tag{A.8}
$$

Here, $|g|$ denotes the absolute value of the determinant of the metric tensor $\mathbf{g}$, and $\varepsilon_{\mu_1 \cdots \mu_n}$ is the standard antisymmetric symbol of rank n. The Hodge dual $\star\mathcal{P}$ of a p-form $\mathcal{P}$ is defined as an $(n-p)$-form obtained by contracting $\mathcal{P}$ with the volume form. Formally, this is expressed as

$$
\begin{aligned}
\star\mathcal{P} &= \frac{1}{p!(n-p)!} \eta_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_{n-p}} \mathcal{P}^{\mu_1 \cdots \mu_p} \boldsymbol{\omega}^{\nu_1} \wedge \cdots \wedge \boldsymbol{\omega}^{\nu_{n-p}} \\
&= \frac{1}{(n-p)!} * \mathcal{P}_{\mu_1 \cdots \mu_{n-p}} \boldsymbol{\omega}^{\mu_1} \wedge \cdots \wedge \boldsymbol{\omega}^{\mu_{n-p}}.
\end{aligned}
\tag{A.9}
$$

On the other hand, the exterior derivative $\mathbf{d}$ is a linear operator that maps $(p-1)$-forms to p-forms, i.e., $\mathbf{d} : \mathscr{A}_{p-1}(E) \to \mathscr{A}_p(E)$. Letting ∇ denote the covariant derivative, one finds for a ($p-$ 1)-form $\mathcal{K}$ that

$$\mathbf{d}\mathcal{K} = \frac{1}{(p-1)!} \nabla_{\mu_1} \mathcal{K}_{\mu_2 \cdots \mu_p} \boldsymbol{\omega}^{\mu_1} \wedge \cdots \wedge \boldsymbol{\omega}^{\mu_p}. \tag{A.10}$$

As both $\mathbf{d}\mathcal{K}$ and $\mathcal{P}$ are p-forms, one might have $\mathcal{P} = \mathbf{d}\mathcal{K}$. When this condition holds, $\mathcal{P}$ is said to be *exact*. Furthermore, for any p form $\mathcal{P}$, the exterior derivative satisfies the following property:

$$\mathbf{d}^2 \mathcal{P} = 0. \tag{A.11}$$

However, it does not generally extend to vector-valued p forms[48]. A p-form $\mathcal{P}$ that satisfies $\mathbf{d}\mathcal{P} = 0$ is called *closed*. Hence, by (A.11), all exact p-forms are necessarily closed, but the converse is not always true. This is where Poincaré's lemma becomes relevant.

Lemma A.2 (Poincare's lemma [63]). *For any star-shaped*[a] *open set U there will, for any closed p-form $\mathcal{P}$, exist a $(p-1)$-form $\mathcal{K}$ such that $\mathcal{P} = \mathbf{d}\mathcal{K}$.*

[a]If V is a finite-dimensional vector space, a subset $U \subseteq V$ is said to be star-shaped if there is a point $c \in U$ such that for every $x \in U$, the line segment from c to x is entirely contained in U [84].

[48]For further details and a discussion on applying exterior calculus to general relativity, see Chapter 6 in [63].

Appendix B

$1 + 3$ p-form gauge field decomposition

This appendix builds extensively on chapters $3, 14, 15$ and 16 in *Special Relativity in General Frames* [100].

Theorem B.1 (1-form). *Let $\mathcal{J}$ a one-form. Given a unit timelike vector $\vec{u}$ (in practice, it will be the 4-velocity of some observer), there exists a one-form $\underline{v}$ and a unique vector scalar field φ such that*

$$\mathcal{J} = -\phi\, \underline{u} + \underline{v} \quad \Longrightarrow \quad \mathcal{J}_\mu = -\phi u_\mu + v_\mu. \tag{B.1}$$

Theorem B.2 (2-form). *Let $\mathcal{A}$ be a 2-form:*

$$\forall (\vec{v}, \vec{w}) \in E^2, \quad \mathcal{A}(\vec{v}, \vec{w}) = -\mathcal{A}(\vec{w}, \vec{v}).$$

Given a unit timelike vector $\vec{u}$ (in practice, it will be the 4 -velocity of some observer), there exists a unique linear form $\underline{q} \in E^*$ and a unique vector $\vec{b} \in E$ such that

$$\mathcal{A} = \underline{u} \otimes \underline{q} - \underline{q} \otimes \underline{u} + \epsilon(\vec{u}, \vec{b}, \cdot, \cdot) : \quad \langle \underline{q}, \vec{u} \rangle = 0 \quad \textit{and} \quad \vec{u} \cdot \vec{b} = 0, \tag{B.2}$$

where $\underline{u}$ is the dual form associated to $\vec{u}$.

Proof

We shall start studying the action of two arbitrary vectors $(\vec{v}, \vec{w}) \in E^2$ on the 2-form $\mathcal{A}$:

$$\begin{aligned}\mathcal{A}\left(\vec{v},\vec{w}\right) &= \left(\underline{u}\otimes q - q\otimes \underline{u} + \epsilon(\vec{u},\vec{b},\cdot,\cdot)\right)\left(\vec{v},\vec{w}\right)\\ &= \left(\underline{u}\otimes \underline{q}\right)\left(\vec{v},\vec{w}\right) - \left(\underline{q}\otimes \underline{u}\right)\left(\vec{v},\vec{w}\right) + \epsilon(\vec{u},\vec{b},\vec{v},\vec{w})\\ &= \langle \underline{u},\vec{v}\rangle\langle \underline{q},\vec{w}\rangle - \langle \underline{q},\vec{v}\rangle\langle \underline{u},\vec{v}\rangle + \epsilon(\vec{u},\vec{b},\vec{v},\vec{b}\rangle,\end{aligned} \tag{B.3}$$

where $\langle \underline{u},\vec{v}\rangle = \vec{u}\cdot\vec{v}$.

Let us build the expression (B.2). From there, we can infer the action of the unit timelike vector $\vec{u}$ on $\mathcal{A}$:

$$\begin{aligned}\mathcal{A}\left(\cdot,\vec{u}\right) &= \left(\underline{u}\otimes \underline{q} - \underline{q}\otimes \underline{u} + \epsilon(\vec{u},\vec{b},\cdot,\cdot)\right)\left(\cdot,\vec{u}\right)\\ &= \left(\underline{u}\otimes \underline{q}\right)\left(\cdot,\vec{u}\right) - \left(\underline{q}\otimes \underline{u}\right)\left(\cdot,\vec{u}\right) + \cancelto{0}{\epsilon(\vec{u},\vec{b},\cdot,\vec{u})}\\ &= \underline{u}\,\cancelto{0}{\langle \underline{q},\vec{u}\rangle} - \underline{q}\,\cancelto{-1}{\langle \underline{u},\vec{u}\rangle}.\end{aligned}$$

Then, we set

$$\underline{q} = \mathcal{A}\left(\cdot,\vec{u}\right). \tag{B.4}$$

Thus, $\underline{q}$ is the linear form defined by $\forall\vec{v}\in E$, such that, $\langle \underline{q},\vec{v}\rangle = \mathcal{A}\left(\vec{v},\vec{u}\right)$. Taking into account the antisymmetry of $\mathcal{A}$, we can infer that $\langle \underline{q},\vec{u}\rangle = 0$. Therefore, the second expression in (B.2) is fulfilled. Now, from (B.2) let us define a new 2-form as

$$\mathcal{B} := \mathcal{A} - \underline{u}\otimes \underline{q} + \underline{q}\otimes \underline{u}, \tag{B.5}$$

such that

$$\mathcal{B}\left(\cdot,\vec{u}\right) = \cancelto{\underline{q}}{\mathcal{A}\left(\cdot,\vec{u}\right)} - \underline{u}\,\cancelto{0}{\langle \underline{q},\vec{u}\rangle} + \underline{q}\,\cancelto{-1}{\langle \underline{u},\vec{u}\rangle} = \underline{q} - \underline{q} = 0.$$

With this in mind, we shall determine the action of $\mathcal{B}$ on the hyperplane E_u normal to $\vec{u}$, where $(E_u,\mathbf{g})$ is a Euclidean space[49]. Let us choose an orthonormal basis in $(E_u,\mathbf{g})$, denoted by $(\vec{e}_i) = (\vec{e}_1,\vec{e}_2,\vec{e}_3)$. If $\vec{u}$ is the 4-velocity of an observer, one may choose $(\vec{e}_i)$ as the three spatial vectors of the observer's local frame in such a way that we can define the following three numbers:

[49]For details see chapter 3, section 2 in [100].

$$b^1 \coloneqq \mathcal{B}\,(\vec{e}_2, \vec{e}_3), \qquad b^2 \coloneqq \mathcal{B}\,(\vec{e}_3, \vec{e}_1), \qquad b^3 \coloneqq \mathcal{B}\,(\vec{e}_1, \vec{e}_2), \tag{B.6}$$

and construct the vector

$$\vec{b} \coloneqq b^i\,\vec{e}_i \;\in E_u. \tag{B.7}$$

We can see that $\vec{b}$ satisfies the third expression in (B.2): $\vec{u}\cdot\vec{b} = 0$. Besides, for any pair of vectors $\vec{v}, \vec{w} \in E_u$, thanks to the antisymmetry of $\mathcal{B}$, it follows that

$$\begin{aligned}\mathcal{B}\,(\vec{v}, \vec{w}) &= \mathcal{B}\,(v^i\,\vec{e}_i, w^j\,\vec{e}_j) = v^i\,w^j\,\mathcal{B}\,(\vec{e}_i, \vec{e}_j) \\ &= v^1w^2b^3 - v^2w^1b^3 - v^1w^3b^2 + v^3w^1b^2 + v^2w^3b^1 - v^3w^2b^1 \\ &= \begin{vmatrix} b^1 & v^1 & w^1 \\ b^2 & v^2 & w^2 \\ b^3 & v^3 & w^3 \end{vmatrix}. \end{aligned} \tag{B.8}$$

This corresponds to the usual mixed product of three vectors in the Euclidean space $(E_u, \mathbf{g})$, assuming that an orientation has been selected such that $(\vec{e}_i)$ forms a right-handed basis. However, it is important to recall that the choice of an orientation of a vector space of dimension n is equivalent to the choice of a fully antisymmetric n-linear form [84]. For $n = 4$, the antisymmetric four-linear form is the standard Levi-Civita tensor ϵ. However, in this context, we focus on the $n = 3$ case, where it is natural to choose the antisymmetric trilinear form ϵ_u, which is defined from ϵ by [60]

$$\forall \vec{v}_1, \vec{v}_2, \vec{v}_3 \;\in E_u, \qquad \epsilon_u(\vec{v}_1, \vec{v}_2, \vec{v}_3) \coloneqq \epsilon(\vec{u}, \vec{v}_1, \vec{v}_2, \vec{v}_3). \tag{B.9}$$

It represents a trilinear form in E_u and satisfies $\epsilon_u\,(\vec{e}_1, \vec{e}_2, \vec{e}_3) = 1$. Through metric duality, ϵ_u defines the cross product of two vectors in E_u as

$$\forall\;\vec{v}, \vec{w} \;\in E_u^2, \qquad \vec{v} \times_u \vec{w} \coloneqq \epsilon_u\,(\vec{v}, \vec{w}, \cdot) = \epsilon\,(\vec{u}, \vec{v}, \vec{w}, \cdot). \tag{B.10}$$

In other words, the expression (B.10) is nothing more than the mixed product of three vectors $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ in E_u:

$$\epsilon_u\left(\vec{v}_1,\vec{v}_2,\vec{v}_3\right)=\left(\vec{v}_1\times_u\vec{v}_2\right)\cdot\vec{v}_3=\left(\vec{v}_2\times_u\vec{v}_3\right)\cdot\vec{v}_1=\left(\vec{v}_3\times_u\vec{v}_1\right)\cdot\vec{v}_2. \tag{B.11}$$

Since $\epsilon_u\left(\vec{e}_1,\vec{e}_2,\vec{e}_3\right)=1$ and thanks to the expression in (B.10), (B.8) can be rewritten in terms of (B.11) as

$$\begin{aligned}\forall\,\vec{v},\vec{w}\in E_u^2,\quad \mathcal{B}\left(\vec{v},\vec{w}\right)&=\epsilon_u\left(\vec{b},\vec{v},\vec{w}\right)=b^i\,v^j\,w^k\,\epsilon_u\left(\vec{e}_i,\vec{e}_j,\vec{e}_k\right)\\&=\begin{vmatrix}b^1 & v^1 & w^1\\ b^2 & v^2 & w^2\\ b^3 & v^3 & w^3\end{vmatrix}\\&=\epsilon\left(\vec{u},\vec{b},\vec{v},\vec{w}\right).\end{aligned} \tag{B.12}$$

Considering the definition of the 2-form $\mathcal{B}$ in (B.5), this confirms the decomposition (B.2) of the 2 -form $\mathcal{A}$. However, the proof of the uniqueness of the linear form $\underline{q}$ and the vector $\vec{b}$ is still required:

1) *Uniqueness of $\underline{q}$:* If $\langle\underline{q},\vec{u}\rangle=0$, then

$$\begin{aligned}\forall\,\vec{v}\in E,\quad \mathcal{A}\left(\vec{v},\vec{u}\right)&=\langle\underline{u},\vec{v}\rangle\,\cancelto{0}{\langle\underline{q},\vec{u}\rangle}-\langle\underline{q},\vec{v}\rangle\,\cancelto{-1}{\langle\underline{u},\vec{u}\rangle}+\cancelto{0}{\epsilon\left(\vec{u},\vec{b},\vec{v},\vec{u}\right)}\\&=\langle\underline{q},\vec{v}\rangle,\end{aligned}$$

so $\underline{q}\coloneqq\mathcal{A}\left(\cdot,\vec{u}\right)$ is the only possible choice.

2) *Uniqueness of $\vec{b}$:* If we restrict (B.3) to the hyperplane E_u, we have that

$$\forall(\vec{u},\vec{w})\in E_u^2,\qquad \mathcal{A}\left(\vec{v},\vec{w}\right)=\epsilon\left(\vec{u},\vec{b},\vec{v},\vec{w}\right)=\epsilon_u\left(\vec{b},\vec{v},\vec{w}\right). \tag{B.13}$$

Let $\vec{b}'\in E_u$ be another vector that satisfies the decomposition (B.2). Consequently, from (B.13), it follows that

$$\epsilon_u\left(\vec{b}',\vec{v},\vec{w}\right)=\epsilon_u\left(\vec{b},\vec{v},\vec{w}\right)\implies\epsilon_u\left(\vec{b}-\vec{b}',\vec{v},\vec{w}\right)=0.$$

Since ϵ_u is non-degenerate on E_u, we can conclude that $\vec{b}'-\vec{b}=\vec{0}$, which shows the unicity of $\vec{b}$. $\square$

Remark B.3. There exists a more elegant and compact manner to express the decomposition (B.2), given by:

$$\mathcal{A} = \underline{u} \wedge \underline{q} + \star(\underline{u} \wedge \underline{b}) : \quad \langle \underline{q}, \vec{u} \rangle = 0 \quad \text{and} \quad \vec{u} \cdot \vec{b} = 0. \tag{B.14}$$

The Hodge star operator permit us to express the vector $\vec{b}$ in terms of $\mathcal{A}$ and $\vec{u}$, as we already expressed $\underline{q}$ in terms of $\mathcal{A}$ and $\vec{u}$ in (B.4). Let us take the Hodge star of (B.14):

$$\star\mathcal{A} = \star(\underline{u} \wedge \underline{q}) + \star\star\,(\underline{u} \wedge \underline{b}) = \epsilon(\vec{u}, \vec{q}, \cdot, \cdot) - \underline{u} \wedge \underline{b}. \tag{B.15}$$

Setting the first argument of this 2-form to $\vec{u}$, we obtain the linear form

$$\star\mathcal{A}(\vec{u}, \cdot) = \underbrace{\epsilon(\vec{u}, \vec{q}, \vec{u}, \cdot)}_{0} - \underbrace{\langle \underline{u}, \vec{u} \rangle}_{1}\, \underline{b} + \underbrace{\langle \underline{b}, \vec{u} \rangle}_{\vec{b}\cdot\vec{u}=0}\, \underline{u} = \underline{b}. \tag{B.16}$$

We have thus

$$\underline{b} = \star\mathcal{A}(\vec{u}, \cdot), \tag{B.17}$$

which contrasts with the expression obtained in (B.4). Hence, in the decomposition (B.14), the 1-form $\underline{q}$ is obtained directly from $\mathcal{A}$, whereas the vector $\vec{b}$ is obtained from the Hodge dual of $\mathcal{A}$.

Theorem B.4 (3-form). *Let $\mathcal{J}$ be a 3-form:*

$$\forall(\vec{v}, \vec{w}, \vec{f}) \in E^3, \quad \mathcal{J}(\vec{v}, \vec{w}, \vec{f}) = -\mathcal{J}(\vec{w}, \vec{v}, \vec{f}) = -\mathcal{J}(\vec{v}, \vec{f}, \vec{w}).$$

Given a unit timelike vector $\vec{\boldsymbol{u}}$ (in practice, it will be the 4-velocity of some observer), there exists a unique scalar field φ and a unique vector $\vec{\boldsymbol{b}} \in E$ such that

$$\mathcal{J} = \underline{\mathbf{u}} \wedge \epsilon(\vec{\mathbf{u}}, \vec{\mathbf{b}}, \ldots,) + \varphi\, \epsilon(\vec{\mathbf{u}}, \vec{\mathbf{b}}, \ldots,) : \quad \vec{\mathbf{u}} \cdot \vec{\mathbf{b}} = 0. \tag{B.18}$$

Proof

Let us define a 3-form $\mathcal{J} \in E$ by

$$\begin{aligned}\mathcal{J} &= \underline{u} \wedge \mathcal{A} + \underline{u} \wedge \epsilon\,(\vec{u}, \vec{b}, \cdot, \cdot) \\ &= \underline{u} \otimes \mathcal{A} - \mathcal{A} \otimes \underline{u} + \underline{u} \otimes \epsilon\,(\vec{u}, \vec{b}, \cdot, \cdot) - \epsilon\,(\vec{u}, \vec{b}, \cdot, \cdot) \otimes \underline{u},\end{aligned} \tag{B.19}$$

where $\mathcal{A}$ is a 2-form and ϵ is the Levi-Civita tensor. Similarly to what we did in the proof of the above theorem, we set

$$\mathcal{A} \coloneqq \mathcal{J}\,(\cdot, \cdot, \vec{u}), \tag{B.20}$$

such that,

$$\begin{aligned}\mathcal{A}\,(\cdot, \vec{u}) &= \mathcal{J}\,(\cdot, \vec{u}, \vec{u}) \\ &= \cancelto{-1}{\langle \underline{u}, \vec{u} \rangle}\,\mathcal{A}\,(\cdot, \vec{u}) - \mathcal{A}\,(\cdot, \vec{u})\,\cancelto{-1}{\langle \underline{u}, \vec{u} \rangle} + \cancelto{-1}{\langle \underline{u}, \vec{u} \rangle}\,\cancelto{0}{\epsilon\,(\vec{u}, \vec{b}, \cdot, \vec{u})} \\ &= 0.\end{aligned} \tag{B.21}$$

Now, we define a 3-form as

$$\mathcal{D} \coloneqq \mathcal{J} - \underline{u} \otimes \mathcal{A} + \mathcal{A} \otimes \underline{u}, \tag{B.22}$$

in such manner that

$$\begin{aligned}\mathcal{D}\,(\cdot, \cdot, \vec{u}) &= \cancelto{\mathcal{A}}{\mathcal{J}\,(\cdot, \cdot, \vec{u})} - \underline{u}\,\cancelto{0}{\mathcal{A}\,(\cdot, \vec{u})} + \mathcal{A}\,\cancelto{-1}{\langle \underline{u}, \vec{u} \rangle} \\ &= \mathcal{A} - \mathcal{A} = 0.\end{aligned} \tag{B.23}$$

Besides, we shall define the action of $\mathcal{D}$ in the hyperplane E_u normal to $\vec{u}$: let $(\vec{e}_i) \in E_u$ be the orthonormal vector basis associated to the observer's local frame, where each vector $\vec{v} \in E_u$ can be written as

$$\vec{v} = v^i\,\vec{e}_i \in E_u : \qquad \vec{u} \cdot \vec{v} = 0.$$

Regarding to the action of any three vectors $(\vec{b}, \vec{v}, \vec{w}) \in E_u$ on $\mathcal{D}$, it follows that

$$\mathcal{D}\,(\vec{b}, \vec{v}, \vec{w}) = b^i\,v^j\,w^k\,\mathcal{D}\,(\vec{e}_i, \vec{e}_j, \vec{e}_k), \tag{B.24}$$

such that any 3-form defined on E_u will be proportional to the volume-form [84], so

$$
\begin{aligned}
&\mathcal{D}\,(\vec{e}_i, \vec{e}_j, \vec{e}_k) \propto \epsilon_u\,(\vec{e}_i, \vec{e}_j, \vec{e}_k) \\
\therefore\;\; &\mathcal{D}\,(\vec{e}_i, \vec{e}_j, \vec{e}_k) = \varphi\, \epsilon_u\,(\vec{e}_i, \vec{e}_j, \vec{e}_k): \quad \varphi \in \mathbb{R}.
\end{aligned}
\tag{B.25}
$$

Hence, from (B.12) we can conclude that

$$
\mathcal{D}\,(\vec{b}, \vec{v}, \vec{w}) = \varphi \begin{vmatrix} b^1 & v^1 & w^1 \\ b^2 & v^2 & w^2 \\ b^3 & v^3 & w^3 \end{vmatrix}.
$$

Thus,

$$
\mathcal{D}\,(\vec{b}, \vec{v}, \vec{w}) = \varphi\,[(\vec{b} \times_{\vec{u}} \vec{v}) \cdot \vec{w}] = \varphi\, \epsilon_u\,(\vec{b}, \vec{v}, \vec{w}). \tag{B.26}
$$

Previously, we could demonstrate that any 2-form $\mathcal{A}$ admits a unique orthonormal decomposition given by (B.2). Thus, using (B.26) we can rewrite the 3-form orthogonal decomposition (B.18) as follows

$$
\begin{aligned}
\mathcal{J} &= \underline{u} \wedge (\underline{u} \wedge \underline{q} + \epsilon(\vec{u}, \vec{b}, \cdot, \cdot)) + \varphi\, \epsilon(\vec{u}, \cdot, \cdot, \cdot) \\
&= \cancelto{0}{\underline{u} \wedge \underline{u} \wedge \underline{q}} + \underline{u} \wedge \epsilon(\vec{u}, \vec{b}, \cdot, \cdot) + \varphi\, \epsilon(\vec{u}, \cdot, \cdot, \cdot).
\end{aligned}
$$

Therefore,

$$
\mathcal{J} = \underline{u} \wedge \epsilon(\vec{u}, \vec{b}, \cdot, \cdot) + \varphi\, \epsilon(\vec{u}, \cdot, \cdot, \cdot), \tag{B.27}
$$

that corresponds with the decomposition (B.18). However, there remains to show the uniqueness of the 2-form $\mathcal{A}$ and the scalar φ:

1) *Uniqueness of $\mathcal{A}$:* If (B.21) holds, then $\forall(\vec{d}, \vec{f}) \in E$

$$
\begin{aligned}
\mathcal{J}\,(\vec{d}, \vec{f}, \vec{u}) &= (\underline{u} \otimes \mathcal{A} - \mathcal{A} \otimes \underline{u} + \epsilon\,(\vec{u}, \cdot, \cdot, \cdot))\,(\vec{d}, \vec{f}, \vec{u}) \\
&= \langle \underline{u}, \vec{d} \rangle \cancelto{0}{\mathcal{A}(\vec{f}, \vec{u})} - \mathcal{A}(\vec{d}, \vec{f}) \cancelto{-1}{\langle \underline{u}, \vec{u} \rangle} + \cancelto{0}{\epsilon(\vec{u}, \vec{d}, \vec{f}, \vec{u})} \\
&= \mathcal{A}(\vec{d}, \vec{f}) \qquad \therefore\, \mathcal{A} = \mathcal{J}(\cdot, \cdot, \vec{u}),
\end{aligned}
$$

which is in agreement with (B.20).

2) *Uniqueness of φ:* We suppose that there exists another $\varphi' \in \mathbb{R}$ such that (B.26) holds, so

$$\varphi\, \epsilon_u(\vec{b}, \vec{d}, \vec{f}) = \varphi'\, \epsilon_u(\vec{b}, \vec{d}, \vec{f}) \implies (\varphi - \varphi')\, \epsilon_u(\vec{b}, \vec{d}, \vec{f}) = 0.$$

Since ϵ_u is non-degenerate on E_u, we can conclude that $\varphi - \varphi' = 0$, which shows the uniqueness of φ. $\square$

Appendix C

Velocity field decomposition

This appendix builds on chapters 11 in *Einstein's General Theory of Relativity* [63] and *Tales from Wonderland* [1].

Let $\mathbf{u}$ be the four-velocity field and define the four-acceleration as

$$\mathbf{a} = \frac{d\mathbf{u}}{d\tau}, \tag{C.1}$$

with τ being the proper time. Employing a semicolon (;) for the covariant derivative and using a dot ($\dot{}$) to denote differentiation with respect to τ, we can express the components of (C.1) as

$$\dot{u}_\alpha = a_\alpha = u_{\alpha;\mu}\, u^\mu. \tag{C.2}$$

The projector $h_{\mu\nu}$ maps tensors onto the simultaneity slice orthogonal to the four-velocity $\mathbf{u}$. Consequently, the covariant derivative of $\mathbf{u}$ can be expressed as

$$u_{\alpha;\beta} = \frac{1}{3}\,\theta\, h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \dot{u}_\alpha u_\beta, \tag{C.3}$$

where θ is the expansion scalar, $\sigma_{\alpha\beta}$ denotes the shear tensor, and $\omega_{\alpha\beta}$ represents the vorticity tensor. They are defined as follows [4]:

$$\begin{aligned}
\theta &= u^\mu{}_{;\mu}, \\
\sigma_{\alpha\beta} &= u_{(\alpha;\beta)} - \frac{1}{3} u^\mu{}_{;\mu}\, h_{\alpha\beta} + \dot{u}_{(\alpha} u_{\beta)}, \\
\omega_{\alpha\beta} &= u_{[\alpha;\beta]} + \dot{u}_{[\alpha} u_{\beta]}.
\end{aligned} \tag{C.4}$$

Here, square brackets around indices indicate an antisymmetric combination, while parentheses denote a symmetric combination.

Freely moving particles follow geodesics, which are the analogue of straight lines in curved spacetime. Consequently, in the absence of external forces, we have

$$\mathbf{a} = 0. \tag{C.5}$$

Throughout this thesis, we assume that the fundamental observers move freely. In addition, we require co-motion, implying that

$$\mathbf{u} = \partial_\tau. \tag{C.6}$$

Combining this result with (C.5) and noting that $\theta = 3H$ (where H is the Hubble parameter), we arrive at

$$\begin{aligned} \theta &= 3H, \\ u_{(\alpha;\beta)} &= \sigma_{\alpha\beta} + H\, h_{\alpha\beta}, \\ \omega_{\alpha\beta} &= 0. \end{aligned} \tag{C.7}$$

Finally, the expansion tensor $\theta_{\mu\nu}$ takes the form

$$\theta_{\alpha\beta} = \sigma_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}. \tag{C.8}$$

Appendix D

Energy-momentum tensor standard irreducible decomposition

This appendix builds on chapters 3 in *Relativistic Hydrodynamics* [106] and *Tales from Wonderland* [1].

Let $\mathcal{T}^{\mu\nu}$ be the components of a rank (2,0) tensor and let $h_{\mu\nu}$ be the projection onto the hypersurfaces orthogonal to the 4-velocity u^μ. Thus, we decompose the full metric $g_{\mu\nu}$ according to

$$g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu. \tag{D.1}$$

Since u_μ is time-like, $h_{\mu\nu}$ will always represent spatial sections. As usual, we define the lower components according to

$$\mathcal{T}_{\mu\nu} = T^{\alpha\beta} g_{\alpha\mu} g_{\beta\nu}. \tag{D.2}$$

Then, using (D.1), we find

$$T_{\mu\nu} = T^{\alpha\beta}(h_{\alpha\mu} - u_\alpha u_\mu)(h_{\beta\nu} - u_\beta u_\nu). \tag{D.3}$$

Expanding the brackets, we obtain

$$T_{\mu\nu} = h^\alpha{}_\mu h^\beta{}_\nu T_{\alpha\beta} + u_\mu u_\nu u^\alpha u^\beta T_{\alpha\beta} - u_\mu h^\beta{}_\nu u^\alpha T_{\alpha\beta} - u_\nu u^\beta h^\alpha{}_\mu T_{\alpha\beta}. \tag{D.4}$$

Projecting onto $u^\mu u^\nu$: The energy density ρ is the scalar quantity we observe in the comoving frame. It is defined as

$$\rho \equiv u^\alpha u^\beta T_{\alpha\beta}. \tag{D.5}$$

Projecting one index onto u^μ and one onto $h_{\alpha\mu}$: To get the energy flow q_ν we project one 'leg' on each side. We find

$$q_\nu \;\equiv\; -h_\nu{}^\alpha u^\beta T_{\alpha\beta}. \tag{D.6}$$

The spatial part: The part of $T^{\mu\nu}$ projected onto spatial sections is

$$h^\alpha{}_\mu h^\beta{}_\nu T_{\alpha\beta}. \tag{D.7}$$

These will therefore give the purely space-like components of $T_{\mu\nu}$. More specifically, the isotropic pressure p is now given as the trace, whereas the rest, $\pi_{\mu\nu}$, represents the shear. Hence

$$p = \frac{1}{3} h^{\mu\nu} T_{\mu\nu} \tag{D.8}$$

and

$$\pi_{\mu\nu} \;\equiv\; h^\alpha{}_\mu h^\beta{}_\nu T_{\alpha\beta} - p\, h_{\mu\nu}. \tag{D.9}$$

With these definitions we may rewrite $T_{\mu\nu}$.

Symmetric $T^{\mu\nu}$: Any symmetric tensor $T^{\mu\nu}$ may now be decomposed such that

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\, h_{\mu\nu} + 2q_{(\mu} u_{\nu)} + \pi_{\mu\nu}. \tag{D.10}$$

Appendix E

The volume-form

Given an n-dimensional pseudo-Riemannian manifold $\mathcal{M}$ we may define the components of the Levi-Civita n-form as

$$\eta_{\alpha\beta\gamma\delta} = \sqrt{|g|}\varepsilon_{\alpha\beta\gamma\delta}, \tag{E.1}$$

where $|g|$ is the absolute value of the determinant g of the metric tensor with components $g_{\mu\nu}$. Also, $\varepsilon_{\alpha\beta\gamma\delta}$ is the totally skew symbol, and we define

$$\varepsilon_{0123} = 1. \tag{E.2}$$

As usual, we use upper and lower indices with the metric. Consequently, one may show that for an n-dimensional space-time, the following relation holds [63]:

$$\eta^{\mu_1\cdots\mu_{n-p}\nu_1\cdots\nu_p}\eta_{\mu_1\cdots\mu_{n-p}\sigma_1\cdots\sigma_p} = -(n-p)!\,p!\,\delta^{\nu_1\cdots}_{[\mu_1\cdots}\delta^{\nu_p}_{\mu_p]}. \tag{E.3}$$

For the particular case $n = 4$, one finds

$$\eta^{\nu_1\nu_2\nu_3\nu_4}\eta_{\mu_1\mu_2\mu_3\mu_4} = -4!\delta^{\nu_1\cdots}_{[\mu_1\cdots}\delta^{\nu_4}_{\mu_4]}, \tag{E.4}$$

$$\eta^{\mu_1\nu_2\nu_3\nu_4}\eta_{\mu_1\mu_2\mu_3\mu_4} = -3!\delta^{\nu_2\cdots}_{[\mu_2\cdots}\delta^{\nu_4}_{\mu_4]}, \tag{E.5}$$

$$\eta^{\mu_1\mu_2\nu_3\nu_4}\eta_{\mu_1\mu_2\mu_3\mu_4} = -4!\delta^{\nu_3}_{[\mu_3}\delta^{\nu_4}_{\mu_4]}, \tag{E.6}$$

$$\eta^{\mu_1\mu_2\mu_3\nu_4}\eta_{\mu_1\mu_2\mu_3\mu_4} = -3!\delta^{\nu_4}_{\mu_4}, \tag{E.7}$$

$$\eta^{\mu_1\mu_2\mu_3\mu_4}\eta_{\mu_1\mu_2\mu_3\mu_4} = -4!. \tag{E.8}$$

We now use the Hodge dual to define a covariant antisymmetric symbol in the $(n-1)$-dimensional hypersurface orthogonal to the observers with velocity $\boldsymbol{u} = u^\mu \mathbf{e}_\mu$. We define

$$\star\boldsymbol{u} := {}_{(n-1)}\boldsymbol{\eta}, \tag{E.9}$$

where $\star$ is the Hodge-star operator and ${}_{(n-1)}$ is there to show that this is the skew $(n-1)$-form inherited from $\boldsymbol{\eta}$ in the pseudo-Riemannian manifold $\mathcal{M}$. Spelling it all out, we have

$$\star\boldsymbol{u} = \frac{1}{(n-1)!}\eta_{\alpha\nu_1\cdots\nu_{n-1}}u^{\alpha}\boldsymbol{\omega}^{\nu_1}\wedge\cdots\wedge\boldsymbol{\omega}^{\nu_{n-1}}. \tag{E.10}$$

Thus the components of ${}_{(n-1)}\boldsymbol{\eta}$ are given by

$$\eta_{\nu_1\cdots\nu_{(n-1)}} = \eta_{\alpha\nu_1\cdots\nu_{(n-1)}}u^{\alpha}. \tag{E.11}$$

We do not need the subscript in front, as the number of indices reveals where the form lives. Using this, and the fact that $u^{\alpha}u_{\alpha} = -1$, one obtains the general result

$$\eta^{\mu_1\cdots\mu_{n-1-p}\nu_1\cdots\nu_p}\eta_{\mu_1\cdots\mu_{n-1-p}\sigma_1\cdots\sigma_p} = (n-1-p)!\,p!\,\delta^{\nu_1\cdots}_{[\mu_1\cdots}\delta^{\nu_p}_{\mu_p]}. \tag{E.12}$$

We observe the sign difference relative to the corresponding expression in the manifold $\mathcal{M}$. Specifying to $n = 4$ again, the components of ${}_3\boldsymbol{\eta}$ are given by

$$\eta_{\lambda\mu\nu} = \eta_{\alpha\lambda\mu\nu}u^{\alpha}. \tag{E.13}$$

Finally, as useful particular cases of (E.3), we find that

$$\eta^{\nu_1\nu_2\nu_3}\eta_{\mu_1\mu_2\mu_3} = 3!\delta^{\nu_1\cdots}_{[\mu_1\cdots}\delta^{\nu_3}_{\mu_3]}, \tag{E.14}$$

$$\eta^{\mu_1\nu_2\nu_3}\eta_{\mu_1\mu_2\mu_3} = 2!\delta^{\nu_2}_{[\mu_2}\delta^{\nu_3}_{\mu_3]}, \tag{E.15}$$

$$\eta^{\mu_1\mu_2\nu_3}\eta_{\mu_1\mu_2\mu_3} = 2!\delta^{\nu_3}_{\mu_3}, \tag{E.16}$$

$$\eta^{\mu_1\mu_2\mu_3}\eta_{\mu_1\mu_2\mu_3} = 3!. \tag{E.17}$$

Appendix F

The Cartan metric tensor

For the Lie-algebra valued 1-forms $\mathcal{A}^a$, one may use the Cartan metric tensor to raise and lower indices. The Cartan metric tensor is defined such that in a specific basis, its components g_{ij} are given as

$$g_{ij} = f^k{}_{il} f^l{}_{jk} \,, \tag{F.1}$$

where $f^j{}_{lk}$ are the structure coefficients of the Lie algrebra. For the $su(2)$ algebra,

$$f^j{}_{lk} = \varepsilon^j{}_{lk}, \tag{F.2}$$

where $\varepsilon^j{}_{lk}$ is the Levi-Civita antisymmetric symbol, defined such that $\varepsilon^a{}_{bc} = 1$. Note that we may now use the Cartan-Killing metric tensor to raise and lower indices, so that, for instance

$$\varepsilon_{abc} = g_{am} \varepsilon^m{}_{bc}. \tag{F.3}$$

Using one of the relations in (E.12), we readily prove the explicit expression for the Cartan-Killing metric tensor in our case:

$$g_{ij} = -2\delta_{ij}. \tag{F.4}$$

Its inverse, denoted by g^{jk} and fulfilling the defining relation $g_{ij} g^{jk} = \delta^i{}_k$ is given by

$$g^{jk} = -\frac{1}{2}\delta^{jk}. \tag{F.5}$$

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