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Mathematical Methods for Engineers and Scientists 3

Fourier Analysis, Partial Differential Equations and Variational Methods

With 79 Figures and 4 Tables

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5.1 One-Dimensional Wave Equations

5.1.1 The Governing Equation of a Vibrating String

As an example, we will derive the equation that governs the small vibrations of an elastic string of length L, fixed at both endpoints. The dependent variable $u(x, t)$ represents, at time t, the displacement of the point of the string that is at distance x away from the first end point 0.

We shall assume that the string is homogeneous, that is, the mass of the string per unit length, denoted as ρ, is a constant. We also shall assume that the string undergoes only small vertical displacements from its equilibrium position. (The displacements do not have to be in vertical direction, but for the sake of discussion, we assume it is.)

Let us consider the segment of the string between x and $x + \Delta x$, where Δx is a small increment, as shown in Fig. 5.1. The quantities T_1 and T_2 in the figure are the tensions at the points P and Q of the string. Both T_1 and T_2 are tangential to the curve of the spring. Because there is no horizontal motion of the string, the net horizontal force exerted on the segment must be zero. In other words, the horizontal components of the tensions at P and Q must be equal and opposite. That is

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = T, \tag{5.1}$$

where T is a constant equal to the horizontal force with which the string is stretched. If the amplitude is small, we can regard T as the tension of the string.

There is a net force in the vertical direction, F_u, that causes the vertical motion of the string. Clearly

$$F_u = T_2 \sin\theta_2 - T_1 \sin\theta_1.$$

By Newton's second law, this force is equal to the mass of the segment, $\rho\,\Delta x$, times the acceleration which is the second derivative of the displacement with respect to time. That is

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\,\Delta x \frac{\partial^2 u}{\partial t^2}.$$

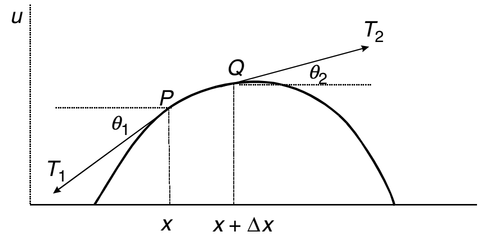

Fig. 5.1. A vibrating string at time t

Dividing this equation by T and using (5.1), we have

$$\frac{T_2 \sin\theta_2}{T_2 \cos\theta_2} - \frac{T_1 \sin\theta_1}{T_1 \cos\theta_1} = \frac{\rho\,\Delta x}{T}\frac{\partial^2 u}{\partial t^2}.$$

Thus

$$\tan\theta_2 - \tan\theta_1 = \frac{\rho\,\Delta x}{T}\frac{\partial^2 u}{\partial t^2}. \tag{5.2}$$

But $\tan\theta_2$ and $\tan\theta_1$ are the slopes of the curve of the string at $x + \Delta x$ and x, respectively,

$$\tan\theta_2 = \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x},$$
$$\tan\theta_1 = \left(\frac{\partial u}{\partial x}\right)_{x}.$$

Hence (5.2) can be written as

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\rho\,\Delta x}{T}\frac{\partial^2 u}{\partial t^2}. \tag{5.3}$$

Recall the definition of a derivative

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \lim_{\Delta x\to 0}\frac{F(x+\Delta x) - F(x)}{\Delta x}.$$

If it is understood that Δx is approaching zero even without the limit sign, then we can write

$$F(x+\Delta x) - F(x) = \frac{\mathrm{d}F}{\mathrm{d}x}\Delta x.$$

Thus it is clear that

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\Delta x = \frac{\partial^2 u}{\partial x^2}\Delta x.$$

Therefore (5.3) becomes

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}. \tag{5.4}$$

This is the so called one-dimensional wave equation. We see that it is linear, homogeneous, and of second-order.

If the string is fixed at both ends, we have two boundary conditions

$$\text{B.C.:}\ \ u(0,t) = 0;\ u(L,t) = 0.$$

Furthermore, if the string is initially displaced into a position $u = f(x)$ and released at rest from that position, then we have the following initial conditions:

$$\text{I.C.}: \; u(x,0) = f(x); \; u_t(x,0) = 0,$$

where $u_t(x,0)$ denotes the first partial derivative of $u(x,t)$ with respect to t and then evaluated at $t=0$

$$u_t(x,0) = \left. \frac{\partial u(x,t)}{\partial t} \right|_{t=0}.$$

The first condition says that the initial shape of the string is $f(x)$, the second condition simply says that at $t=0$, the velocity everywhere in the string is zero.

Of course, it is possible that the string also has initial velocity. In that case, the initial conditions become

$$u(x,0) = f(x); \; u_t(x,0) = g(x).$$

5.1.2 Separation of Variables

To describe the motion of the string, one must solve the differential equation and the solution must satisfy the boundary and initial conditions. Specifically let us find a formula for the transverse displacement $u(x,t)$ of the stretched string which satisfies (5.4). For simplicity of writing, let us first define

$$a^2 = \frac{T}{\rho}. \tag{5.5}$$

It turns out a has a physical meaning which will become clear later.

A powerful and classical method of solving linear boundary value problems in partial differential equations is the method of separation of variables which reduces a partial differential equation into ordinary differential equations. Although not all problems can be solved by this method and there are other methods, generally separation of variables is the first method one should try.

Let us solve the mathematical problem consisting of the following:

$$\text{D.E.}: \; \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2}, \tag{5.6}$$

$$\text{B.C.}: \; u(0,t) = 0; \qquad u(L,t) = 0, \tag{5.7}$$

$$\text{I.C.}: \; u(x,0) = f(x); \quad u_t(x,0) = 0. \tag{5.8}$$

The assumption of separation of variables is that we can write $u(x,t)$ as

$$u(x,t) = X(x)T(t),$$

where X is a function of x alone and T is a function of t alone. The justification of the assumption of this method is that it works. It follows from this assumption, that:

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x)\right) T(t) = X''(x)T(t),$$

$$\frac{\partial^2 u}{\partial t^2} = X(x)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} T(t)\right) = X(x)T^{''}(t).$$

Thus (5.6) becomes

$$X''(x)T(t) = \frac{1}{a^2} X(x)T''(t).$$

Dividing both sides of the equation by $X(x)T(t)$

$$\frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{a^2}\frac{X(x)T''(t)}{X(x)T(t)},$$

we obtain

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)}.$$

The left-hand side of this equation is a function of x alone, it cannot vary with t. However, it is equal to a function of t which cannot vary with x. This is possible if and only if both sides are equal to the same common constant α. This leads to

$$\frac{X''(x)}{X(x)} = \alpha,$$

$$\frac{1}{a^2}\frac{T''(t)}{T(t)} = \alpha.$$

It follows that:

$$X''(x) = \alpha X(x) \tag{5.9}$$

$$T''(t) = \alpha a^2 T(t). \tag{5.10}$$

The partial differential equation is now decomposed into two ordinary differential equations.

Eigenvalues and Eigenfunctions. If $u(x,t)$ is to satisfy the first boundary condition, then

$$u(0,t) = X(0)T(t) = 0$$

for all t. Since $T(t)$ is changing with t, the only possibility that this can be true is that

$$X(0) = 0.$$

Similarly, the condition $u(L,t) = 0$ leads to

$$X(L) = 0.$$

So far we have not specified the value of the separation constant α, it could be less than zero, equal to zero, or greater than zero. It is easy to show that if $\alpha \geq 0$, there is no solution that can satisfy these boundary conditions.

First, if $\alpha = 0$, the solution of (5.9) is $X(x) = Ax+B$. In this case $X(0) = 0$ requires $B = 0$. Thus, $X(L) = AL$. Since $X(L) = 0$, therefore $A = 0$. This leads to $X(x) = 0$ which is a trivial solution for the case that u is identically equal to zero for all x and t.

When $\alpha > 0$, let us write $\alpha = \mu^2$ with μ being real. Then the solution of $X''(x) = \mu^2 X(x)$ is $X(x) = C \cosh \mu x + D \sinh \mu x$. With $X(0) = 0, C$ must be equal to zero. Thus $X(x) = D \sinh \mu x$. Since $\sinh \mu L \neq 0$, $X(L) = 0$ requires $D = 0$. Again this gives only the trivial solution.

Therefore α must be less than zero. Let us write $\alpha = -\mu^2$, so (5.9) becomes

$$X''(x) = -\mu^2 X(x).$$

The general solution of this equation is

$$X(x) = A \cos \mu x + B \sin \mu x.$$

Thus $X(0) = A$ and the condition $X(0) = 0$ means $A = 0$. Hence we are left with

$$X(x) = B \sin \mu x.$$

To satisfy the condition $X(L) = 0$, μ must be chosen to be

$$\mu = \frac{n\pi}{L}, \quad n = 1,\ 2,\ 3, \ldots.$$

Therefore, for each n, there is a solution $X_n(x)$

$$X_n(x) = B_n \sin \frac{n\pi}{L} x, \quad n = 1, 2, \ldots \tag{5.11}$$

where B_n is an arbitrary constant. The numbers $\alpha = -n^2\pi^2/L^2$ for which this problem has nontrivial solutions are called eigenvalues, and the corresponding functions (5.11) are called eigenfunctions.

Solution of the Problem. It is important to keep in mind that α in (5.9) and in (5.10) must be the same. When $\alpha = -n^2\pi^2/L^2$, (5.9) is a distinct problem for each different positive integer n. For a fixed integer n, (5.10) becomes

$$T_n''(t) = -\frac{n^2\pi^2}{L^2} a^2 T_n(t).$$

The solution of this equation is

$$T_n(t) = C_n \cos \frac{n\pi a}{L} t + D_n \sin \frac{n\pi a}{L} t.$$

Thus, each

$$u_n(x,t) = X_n(x) T_n(t)$$

is a solution of the differential equation. An important theorem of the linear homogeneous partial differential equation is the principle of superposition. If u_1 and u_2 are solutions of a linear homogeneous differential equation, then

$$\boxed{u = c_1 u_1 + c_2 u_2,}$$

where c_1 and c_2 are arbitrary constants, is also a solution of that equation. This theorem can be easily proved by showing the equation is satisfied with this combination as the solution.

Therefore the general solution is given by

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n X_n(x) T_n(t) \\ &= \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi a}{L} t + b_n \sin \frac{n\pi a}{L} t \right) \sin \frac{n\pi}{L} x. \qquad (5.12) \end{aligned}}$$

where we have combined three arbitrary constants $c_n C_n B_n$ into a single constant a_n, and $c_n D_n B_n$ into b_n. Now the coefficients a_n and b_n can be chosen to satisfy the initial conditions.

One of the initial condition is

$$u_t(x,0) = \sum_{n=1}^{\infty} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(a_n \cos \frac{n\pi a}{L} t + b_n \sin \frac{n\pi a}{L} t \right) \right]_{t=0} \sin \frac{n\pi}{L} x = 0,$$

which leads to

$$\sum_{n=1}^{\infty} b_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} x = 0.$$

Since $\sin \frac{n\pi}{L} x$ is a complete set in the interval of $0 \le x \le L$, all coefficients must be zero. Another way to see that all b_n are zero is the following. This equation is a Fourier sine series of zero. The coefficients are integrals of zero times some sine function. Obviously they are zero. Therefore

$$b_n = 0.$$

Thus we are left with

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t, \qquad (5.13)$$

where the coefficients a_n can be chosen to satisfy the other initial condition.

Since $u(x,0) = f(x)$, it follows from the last equation

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x = f(x).$$

This is the half-range Fourier sine series of $f(x)$ between 0 and L. Therefore a_n is given by the Fourier coefficient

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x.$$

Thus, the solution of this problem is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(x')\sin\frac{n\pi x'}{L}\mathrm{d}x'\right]\sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t. \tag{5.14}$$

Example 5.1.1. A guitar string of length L is pulled upward at the middle so it reaches height h. What is the subsequent motion of the string if it is released from the rest?

Solution 5.1.1. To find the subsequent motion means to find the displacement of the string as a function of t. That is, we have to solve for $u(x,t)$ from the equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2}.$$

Since the two ends of the guitar string are fixed, so we have to satisfy the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0.$$

Furthermore, it can be readily shown that the initial shape of the string is given by

$$f(x) = \begin{cases} \dfrac{2h}{L}x & \text{for } \ 0 \le x \le \dfrac{L}{2}, \\ \dfrac{2h}{L}(L-x) & \text{for } \ \dfrac{L}{2} \le x \le L. \end{cases}$$

Since it is released from rest, so the initial velocity everywhere in the string is zero. This means the derivative of $u(x,t)$ with respect to time evaluated at $t = 0$ is zero. Thus the initial conditions are

$$u(x,0) = f(x), \qquad u_t(x,0) = 0.$$

According to (5.14), $u(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t,$$

where

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x \\ &= \frac{2}{L}\int_0^{L/2}\frac{2h}{L}x\sin\frac{n\pi x}{L}\mathrm{d}x + \frac{2}{L}\int_{L/2}^{L}\frac{2h}{L}(L-x)\sin\frac{n\pi x}{L}\mathrm{d}x \\ &= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}. \end{aligned}$$

Therefore

$$
\begin{aligned}
u(x,t) &= \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t \\
&= \frac{8h}{\pi^2}\left(\sin\frac{\pi}{L}x\cos\frac{\pi a}{L}t-\frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi a}{L}t\right. \\
&\quad\left.+\frac{1}{5^2}\sin\frac{5\pi}{L}x\cos\frac{5\pi a}{2}t-\frac{1}{7^2}\sin\frac{7\pi}{L}x\cos\frac{7\pi a}{L}t+\cdots\right). \qquad (5.15)
\end{aligned}
$$

It is interesting to see the time development of the displacements. The shapes of the string at various times are shown in the left-hand side column of Fig. 5.2. The individual components are shown in the right-hand side column of the same figure. The string oscillates up and down as expected. We have shown the positions of the string within half of a cycle. After that it will go back

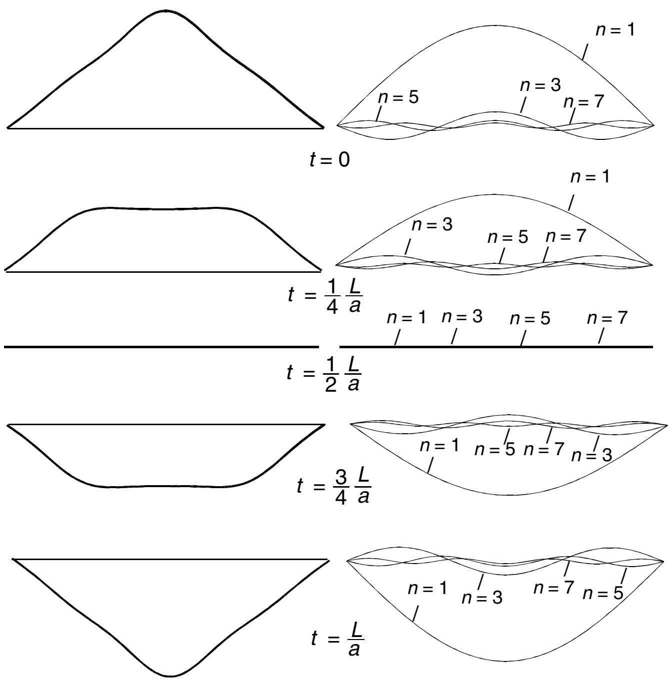

Fig. 5.2. The time development of the displacements of a string after its middle point is pulled up a distance h and released from that position. The left-hand side column is the shape of the string at various times obtained by summing up the first four nonzero terms of the series (5.15). The right-hand side column are the positions of the four individual terms of the series at the correponding times. Although different components oscillate at different frequencies, they sum up to a string going up and down as expected. It is seen that the fundamental (the first term) dominates the motion

to its original position and then repeat the motion. During this time interval of half a cycle, the fundamental (the first term in the series, $\sin \frac{\pi}{L} x \cos \frac{\pi a}{L} t$) also completes half of its cycle. The third harmonic (the second nonzero term, $\sin \frac{3\pi}{L} x \cos \frac{3\pi a}{L} t$) actually completes one and half of its cycle. Various components oscillate at various different frequencies, yet together they sum up to the oscillation shown in the left-hand side column. In fact we have summed up only four nonzero terms, so the lines for the shape of the string are somewhat curved and the corners are somewhat rounded. If we use the computer to plot

$$\frac{8h}{\pi^2} \sum_{n=1}^{N} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t,$$

with $N = 50$, then all the lines in the left-hand side column will be straight and all corners sharply pointed. The amplitudes of higher components are very small, but they do make the sum converge to the exact value. It is seen that in this case the fundamental dominates the motion.

5.1.3 Standing Wave

For the physical interpretation of the series (5.14), let us suppose that the string is suddenly released from the position $u(x, 0) = \sin \frac{2\pi}{L} x$. In this case, the coefficient a_n is given by

$$a_n = \frac{2}{L} \int_0^L \sin \frac{2\pi x}{L} \sin \frac{n\pi x}{L} \mathrm{d}x = \begin{cases} 1 & n = 2, \\ 0 & n \neq 2. \end{cases}$$

Therefore the subsequent displacement of the string is

$$u(x, t) = \sin \frac{2\pi}{L} x \cos \frac{2\pi a}{L} t.$$

This motion is shown in Fig. 5.3. At any instant of time, $u(x, t)$ is a pure sine curve

$$u(x, t) = A_2(t) \sin \frac{2\pi}{L} x,$$

where $A_2(t)$ is the amplitude of the sine wave and $A_2(t) = \cos \dfrac{2\pi a}{L} t$. Note that the points at $x = 0$, $x = L/2$, and $x = L$ are fixed in time. They are called nodes. Between the nodes, the string oscillates up and down. This kind of motion is known as standing wave.

In general (5.13) can be regarded as

$$u(x, t) = \sum_{n=1}^{\infty} a_n u_n(x, t), \tag{5.16}$$

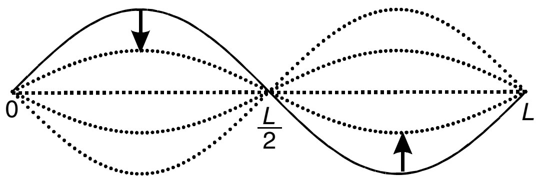

Fig. 5.3. The standing wave of $\sin \frac{2\pi}{L} x \cos \frac{2\pi a}{L} t$

where

$$u_n(x,t) = \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t$$

is known as the nth normal mode. One characteristic of a normal mode is that once the string is vibrating in the standing wave of that mode, it will continue to vibrate in that mode forever. Of course, if there is damping, its amplitude will eventually die down.

The time dependence of each normal mode is given by $\cos \frac{n\pi a t}{L}$ which is a periodic function. The period is defined as the time interval after which the function will return to its original value. Let P_n be the period, so that

$$\cos \frac{n\pi a}{L}(t + P_n) = \cos \frac{n\pi a}{L} t. \tag{5.17}$$

Since

$$\cos \frac{n\pi a}{L}(t + P_n) = \cos \left(\frac{n\pi a}{L} t + \frac{n\pi a}{L} P_n \right)$$

clearly

$$\frac{n\pi a}{L} P_n = 2\pi.$$

Therefore

$$P_n = \frac{2L}{na}.$$

Frequency ν_n is defined as the number of oscillations in one second (the unit of frequency is called Hertz, Hz), that is

$$\nu_n = \frac{1}{P_n} = \frac{na}{2L}.$$

Therefore the series (5.16) represents the motion of a string (of violin or guitar) as a superposition of infinitely many normal modes, each vibrating with a different frequency. The lowest of these frequencies

$$\nu_1 = \frac{a}{2L} = \frac{1}{2L} \sqrt{\frac{T}{\rho}}$$

is called the fundamental frequency. Here we have used the definition of a given in (5.5). The fundamental frequency usually predominates in the sound we hear. The frequency $\nu_n = n\nu_1$, of the nth overtone or harmonic is an integral multiple of ν_1.

Note that once L, T, ρ are chosen, the fundamental frequency is fixed. The initial conditions do not affect ν_1; instead, they determine the coefficients in (5.14) and hence the extent to which the higher harmonics contribute to the sound produced. Therefore the initial conditions affect the overall frequency mixture (known as timbre), rather than the fundamental frequency. For example, if the string of a violin is bowed at some other point than its center, the amplitudes of higher harmonics would have been different than shown in Fig. 5.2. By choosing the point properly any desired harmonic may be emphasized or diminished, a fact well known to musicians.

Once a musical instrument is constructed, the length of string L and the density ρ cannot be changed. Therefore tuning is usually done by changing the tension T.

The spatial dependence of the first few normal modes is shown in Fig. 5.4. The first mode ($n = 1$) is called the fundamental mode, represents a harmonic time dependence of frequency $a/2L$. The second harmonic or first overtone ($n = 2$) vibrates harmonically with frequency a/L, twice as fast as the fundamental mode. Its motion is also shown in Fig. 5.4. Note that, in addition to the two end points, the midpoint of this harmonic is a node. Similarly, the third ($n = 3$) and fourth ($n = 4$) harmonics have two and three nodes, respectively, in addition to the two end points.

In describing the frequency of the oscillation, the angular frequency ω_n (radians per second) in often used

$$\omega_n = 2\pi\nu_n = \frac{\pi n a}{L}.$$

Another quantity associated with wave motion is the wavelength. The wavelength λ_n is defined such that $u_n(x,t)$ will return to its original value if x is increased by λ_n, that is

$$u_n(x + \lambda_n, t_0) = u_n(x, t_0). \tag{5.18}$$

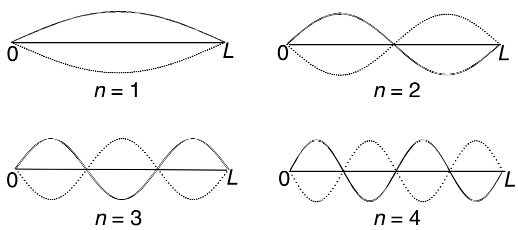

Fig. 5.4. The first four normal modes of a vibrating string. Each normal mode is a standing wave. The nth normal mode has $n - 1$ nodes, excluding the nodes at the two end points

Since

$$u_n(x+\lambda_n, t_0) = \sin\left(\frac{n\pi}{L}x + \frac{n\pi}{L}\lambda_n\right)\cos\frac{n\pi a}{L}t_0;$$
$$u_n(x, t_0) = \sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t_0$$

it is clear that (5.18) will be satisfied if

$$\frac{n\pi}{L}\lambda_n = 2\pi.$$

Therefore

$$\lambda_n = \frac{2L}{n}. \tag{5.19}$$

Thus, for $n = 1,\ L = \frac{1}{2}\lambda;\ n = 2,\ L = \lambda;\ n = 3,\ L = \frac{3}{2}\lambda,\ n = 4,\ L = 2\lambda.$ These relations are clearly demonstrated in Fig. 5.4. Therefore the distance between two nodes of a standing wave is half of a wavelength.

Often a quantity known as wave number k_n (number of wavelengths in the interval of 2π)

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L} \tag{5.20}$$

is used to describe the wave form. In this notation, the normal mode $u_n(x,t)$ is written as

$$u_n(x,t) = \sin k_n x \cos\omega_n t. \tag{5.21}$$

A very important relationship between the frequency and the wavelength is

$$\nu_n\lambda_n = \frac{na}{2L}\frac{2L}{n} = a = \sqrt{\frac{T}{\rho}}. \tag{5.22}$$

This relation says that the frequency is proportional to the inverse of the wavelength and the proportionality constant is equal to the square root of tension over density. A standard physics experiment is shown in Fig. 5.5. A string of density ρ and tension T is connected to a vibrator, the frequency of which can be varied. Standing wave patterns will occur for certain discrete values of the frequency. The wavelength of each standing wave can then be measured. After several standing waves of different wavelength are measured,

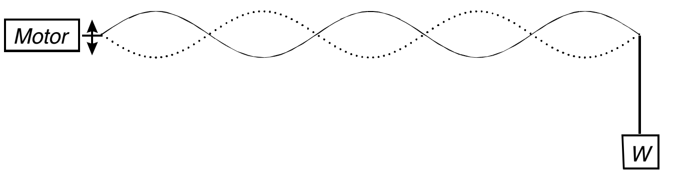

Fig. 5.5. A standing wave experiment to verify the relationship between the frequency and the wavelength

we can plot the frequency against the inverse of the wavelength. The curve is indeed a straight line and the slope of the line is indeed equal to $\sqrt{T/\rho}$.

This is not only a demonstration of a physical principle, but also a demonstration of the power of analysis. We have applied the Newton's law, which relates the force to the acceleration of the particle, to the motion of a string through the use of calculus and concluded that the frequency and the wavelength must satisfy the relation shown in (5.22). This can then be verified in the laboratory.

If the wave is traveling down on an infinite line, one may think that the frequency is the number of wave cycles generated per second and each extends a distance of one wavelength, therefore $\nu_n \lambda_n = a$ is the distance the wave travels in one second. In other words, a is the velocity of a traveling wave. This is indeed the case, as we will clearly see in Sect. 5.1.4.

5.1.4 Traveling Wave

In Sect. 5.1.3, we have shown that each normal mode is a standing wave. Now we wish to show that the same normal mode can also be regarded as a superposition of two traveling waves in the opposite direction.

Using the trigonometric identity

$$\sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right],$$

we can write (5.21) as

$$\begin{aligned} u_n(x,t) &= \frac{1}{2}\sin(k_n x + \omega_n t) + \frac{1}{2}\sin(k_n x - \omega_n t) \\ &= \frac{1}{2}\sin[k_n(x+at)] + \frac{1}{2}\sin[k_n(x-at)], \end{aligned} \tag{5.23}$$

where we have used the fact

$$\frac{\omega_n}{k_n} = \nu_n \lambda_n = a.$$

Before we discuss the interpretation of (5.23), let us first examine the behavior of the function $f(x-ct)$. In this function, the variables x and t are combined in the particular way of $x - ct$. Suppose at $t = 0$, the function $f(x)$ looks like the solid curve in Fig. 5.6. If the maximum value of the function $f(x_{\mathrm{m}})$ is at $x = x_{\mathrm{m}}$, then at a later time t, the function $f(x-ct)$ will have the same maximum value $f(x_{\mathrm{m}})$ at $x = x_{\mathrm{m}} + ct$. This means that the maximum point has moved a distance ct in the time interval of t. In fact, it is not difficult to see that the whole function has moved a distance ct to the right in the time interval t, as shown by the dotted curve in Fig. 5.6. Therefore $f(x-ct)$ represents the function bodily moving (without changing the shape of the function) to the right with velocity c. Similarly, $f(x+ct)$ represents the function traveling to the left with velocity c.

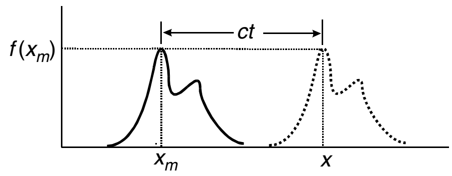

Fig. 5.6. Traveling wave. The *solid curve* shows what the function $f(x - ct)$ might be like at $t = 0$, the *dashed curve* shows what the function is at a later time t

It is now clear that $\sin[k_n(x + at)]$ and $\sin[k_n(x - at)]$ in the normal mode of (5.23) are two sine waves traveling in opposite directions with the same speed a. It is interesting to write (5.13) in terms of traveling waves

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} a_n \left[\sin k_n(x + at) + \sin k_n(x - at)\right]. \tag{5.24}$$

Since initially at $t = 0$ the string is displaced into the form $f(x)$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin k_n x \tag{5.25}$$

clearly

$$f(x + at) = \sum_{n=1}^{\infty} a_n \sin k_n(x + at),$$

$$f(x - at) = \sum_{n=1}^{\infty} a_n \sin k_n(x - at).$$

Thus

$$u(x,t) = \frac{1}{2}f(x + at) + \frac{1}{2}f(x - at). \tag{5.26}$$

In other words, when the string is released at $t = 0$ from the displaced position $f(x)$, it will split into two equal parts, one traveling to the right and the other to the left with the same speed a.

However, there is a question about the range over which $f(x)$ is defined. The initial displacement $f(x)$ is defined between 0 and L. But now the argument is $x + at$ or $x - at$. Since t can take any value, the argument certainly exceeds the range between 0 and L. In order to have (5.26) valid for all t, we must extend the argument of the function beyond this range. Since (5.26) is obtained from (5.25) and $\sin k_n x = \sin \frac{n\pi}{L} x$, which is an odd periodic function with period $2L$, the functions in (5.26) must also have this property. So if we denote f^* to be the odd periodic extension of f with period $2L$, then

$$u(x,t) = \frac{1}{2}f^*(x + at) + \frac{1}{2}f^*(x - at) \tag{5.27}$$

is valid for all t.

Example 5.1.2. With the traveling wave interpretation, solve the problem of the previous example of a string pulled at the middle.

Solution 5.1.2. With the initial displacement of the string

$$u(x,0) = f(x) = \begin{cases} \dfrac{2h}{L}x & \text{if} \quad 0 < x < \dfrac{L}{2} \\ \dfrac{2h}{L}(L-x) & \text{if} \quad \dfrac{L}{2} < x < L \end{cases},$$

the subsequent displacement $u(x,t)$ is given by

$$u(x,t) = \frac{1}{2}f^*(x+at) + \frac{1}{2}f^*(x-at).$$

To interpret this expression, first we imagine the function $f(x)$ is antisymmetrically extended from 0 to $-L$, and then periodically extended from $-\infty$ to ∞ with a period $2L$. Then half of this extended function is moving to the right with velocity a and the other half moving to the left with the same velocity as shown in Fig. 5.7. The sum of these two traveling waves in the region $0 \leq x \leq L$ is the displacement $u(x,t)$ of the string.

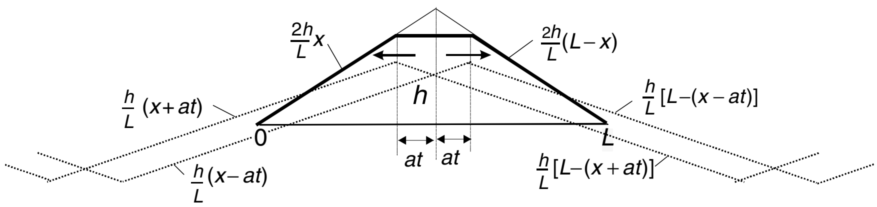

Fig. 5.7. The traveling wave interpretation of the solution of the wave equation with initial and boundary conditions. The displacement $u(x,t)$ is the sum of half of the extended initial function traveling to the left and half traveling to the right with the same velocity a

As a consequence, we see that at any time $t = T$, for $T \leq \frac{L}{2a}$, the displacements are

$$\begin{aligned} u(x,T) &= \frac{1}{2}\left\{\frac{2h}{L}(x+aT) + \frac{2h}{L}(x-aT)\right\} \\ &= \frac{2h}{L}x \qquad \text{if } 0 \leq x \leq \left(\frac{L}{2} - aT\right), \end{aligned}$$

$$u(x,T) = \frac{1}{2}\left\{\frac{2h}{L}[L-(x+aT)] + \frac{2h}{L}(x-aT)\right\}$$
$$= \frac{2h}{L}\left(\frac{L}{2} - aT\right) \quad \text{if } \left(\frac{L}{2} - aT\right) \le x \le \left(\frac{L}{2} + aT\right),$$

$$u(x,T) = \frac{1}{2}\left\{\frac{2h}{L}[L-(x+aT)] + \frac{2h}{L}[L-(x-aT)]\right\}$$
$$= \frac{2h}{L}(L-x) \qquad \text{if } (\frac{L}{2} + aT) \le x \le L.$$

These results are shown as the thick line in Fig. 5.7.

The displacements $u(x,t)$ as a function of time are shown in Fig. 5.8. In the left-hand side column, the positions of the string are shown at various times t. Each case is a superposition of two traveling waves, one to the left and one to the right, shown in the right-hand side column. Both of them are traveling with the same speed a. The sum of these two traveling waves describes the exact up and down motion of the string. It is interesting to compare Fig. 5.8 with Fig. 5.2. They describe the same motion but with two different interpretations.

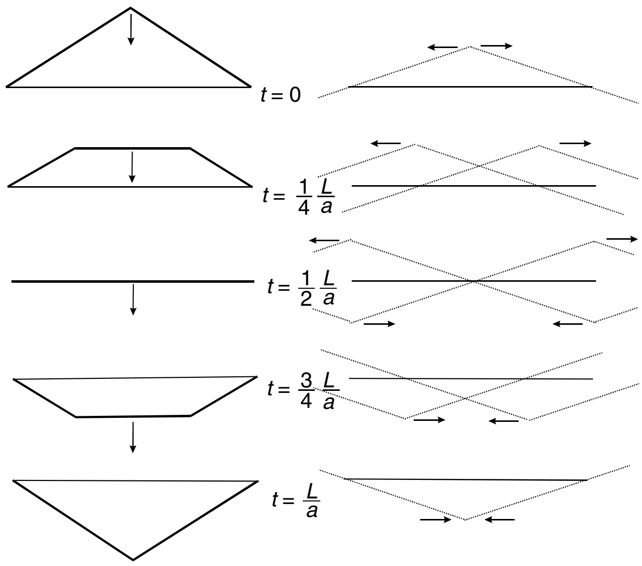

Fig. 5.8. The graph of the solution of the vibrating string with initial displacement $u(x,0)$ shown on the top of the left-hand side column. At various time t, the string will assume such positions as indicated in the left-hand side column. The positions are obtained as the superposition of a wave traveling to the right and a wave traveling to the left shown in the right-hand side column

Problems with Initial Velocity. Let us consider the case that the string is initially at rest but with a initial velocity of $g(x)$. The displacements of the string is the solution of the following problem:

$$\begin{aligned}
\text{D.E.:}\quad & \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2}, \\
\text{B.C.:}\quad & u(0,t) = 0;\ u(L,t) = 0, \\
\text{I.C.:}\quad & u(x,0) = 0;\ u_t(x,0) = g(x).
\end{aligned}$$

With separation of variables, we will obtain (5.12) just as before, since the differential equation and the boundary conditions are the same

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi a}{L}t + b_n \sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x.$$

The initial condition $u(x,0) = 0$ means that

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x = 0.$$

Therefore all a_n must be equal to zero. Thus

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi a}{L}t \sin\frac{n\pi}{L}x$$

and

$$\frac{\partial}{\partial t}u(x,t) = \sum_{n=1}^{\infty} b_n \frac{n\pi a}{L}\cos\frac{n\pi a}{L}t\, \sin\frac{n\pi}{L}x. \tag{5.28}$$

It follows from the other initial condition $u_t(x,0) = g(x)$ that:

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \frac{n\pi a}{L}\sin\frac{n\pi}{L}x = g(x). \tag{5.29}$$

This is a Fourier sine series, therefore

$$b_n \frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,\mathrm{d}x.$$

Thus the solution $u(x,t)$ is given by the infinite series

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{n\pi a}\int_0^L g(x')\sin\frac{n\pi}{L}x'\,\mathrm{d}x'\right]\sin\frac{n\pi a}{L}t\, \sin\frac{n\pi}{L}x.$$

This solution is expressed in terms of a summation of infinite standing waves. We can also express it in terms of the sum of two traveling waves. With trigonometric identity

$$\sin a \, \cos b = \frac{1}{2} \left[\sin(a+b) + \sin(a-b) \right],$$

we can write (5.28) as

$$\begin{aligned}
\frac{\partial}{\partial t} u(x,t) &= \sum_{n=1}^{\infty} b_n \frac{n\pi a}{L} \cos \frac{n\pi a}{L} t \, \sin \frac{n\pi}{L} x \\
&= \frac{1}{2} \sum_{n=1}^{\infty} b_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} (x+at) + \frac{1}{2} \sum_{n=1}^{\infty} b_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} (x-at).
\end{aligned}$$

Using (5.29,) we can write this expression as

$$\frac{\partial}{\partial t} u(x,t) = \frac{1}{2} g^* (x+at) + \frac{1}{2} g^* (x-at),$$

where g^* is the odd periodic extension of g with period of $2L$, for the same reason as f^* is the odd periodic extension of f with a period of $2L$.

An integration of $\frac{\partial}{\partial t} u(x,t)$ will yield $u(x,t)$. The constant of integration is determined by the initial condition $u(x,0) = 0$. This condition is satisfied by the following integral:

$$u(x,t) = \int_0^t \frac{\partial u(x,t')}{\partial t'} \mathrm{d}t' = \frac{1}{2} \int_0^t g^* (x+at') \, \mathrm{d}t' + \frac{1}{2} \int_0^t g^* (x-at') \, \mathrm{d}t'.$$

With a change of variable

$$\tau = x + at', \qquad \mathrm{d}t' = \frac{1}{a} \mathrm{d}\tau,$$

the first integral on the right-hand side can be written as

$$\frac{1}{2} \int_0^t g^* (x+at') \, \mathrm{d}t' = \frac{1}{2a} \int_x^{x+at} g^*(\tau) \mathrm{d}\tau,$$

since at $t' = 0$, $\tau = x$ and at $t' = t$, $\tau = x + at$.

Similarly, the second integral can be written as

$$\frac{1}{2} \int_0^t g^* (x-at') \, \mathrm{d}t' = -\frac{1}{2a} \int_x^{x-at} g^*(\tau) \mathrm{d}\tau.$$

It follows that:

$$\begin{aligned}
u(x,t) &= \frac{1}{2a} \int_x^{x+at} g^*(\tau) \mathrm{d}\tau - \frac{1}{2a} \int_x^{x-at} g^*(\tau) \mathrm{d}\tau \\
&= \frac{1}{2a} \int_{x-at}^{x+at} g^*(\tau) \mathrm{d}\tau. \qquad (5.30)
\end{aligned}$$

This is the solution for the case that the string has no initial displacement but is given an initial velocity $g(x)$.

Superposition of Solutions. If the string is given both an initial displacement and an initial velocity,

$$u(x,0)=f(x), \quad u_t(x,0)=g(x), \tag{5.31}$$

then the subsequent displacements can be written as the superposition of (5.27) and (5.30,) namely

$$u(x,t)=\frac{1}{2}\left[f^*(x-at)+f^*(x+at)\right]+\frac{1}{2a}\int_{x-at}^{x+at} g^*(\tau)\mathrm{d}\tau. \tag{5.32}$$

Note that both terms satisfy the homogeneous differential equation and the boundary conditions, while their sum clearly satisfies the initial conditions of (5.31).

In general the solution of a linear problem containing more than one nonhomogeneous conditions can be written as a sum of the solutions of problems each of which contains only one nonhomogeneous condition. The resolution of the original problem in this way, although not necessary, often simplifies the process of solving the problem.

5.1.5 Nonhomogeneous Wave Equations

Vibrating String with External Force. If there is an external force acting on the stretched string, then there will be an extra term in the governing differential equation. For example, if the weight of the string is not negligible, then in the derivation of (5.4), we must add to the equation the downward gravitational force, $-\rho\,\Delta x g$, where g is the constant gravitational acceleration. As a consequence, (5.6) becomes

$$\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{g}{a^2}=\frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2}. \tag{5.33}$$

Let us solve this equation with the same boundary and initial conditions as the previous problem:

$$u(0,t)=0, \qquad u(L,t)=0,$$

$$u(x,0)=f(x), \quad u_t(x,0)=0.$$

Since (5.33) is an nonhomogeneous equation, a straightforward application of separation of variables will not work. However, the following device will reduce this nonhomogeneous partial differential equation into a homogeneous partial differential equation plus an ordinary differential equation which we can solve. Let

$$u(x,t)=U(x,t)+\phi(x),$$

then

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 U(x,t)}{\partial x^2} + \frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2},$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 U(x,t)}{\partial t^2},$$

so the problem becomes

$$\text{D.E.:} \quad \frac{\partial^2 U(x,t)}{\partial x^2} + \frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} - \frac{g}{a^2} = \frac{1}{a^2}\frac{\partial^2 U(x,t)}{\partial t^2},$$

$$\text{B.C.:} \quad u(0,t) = U(0,t) + \phi(0) = 0, \; u(L,t) = U(L,t) + \phi(L) = 0,$$

$$\text{I.C.:} \quad u(x,0) = U(x,0) + \phi(x) = f(x), \quad u_t(x,0) = U_t(x,0) = 0.$$

Now we require

$$\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} - \frac{g}{a^2} = 0,$$

$$\phi(0) = 0, \quad \phi(L) = 0.$$

This is a second-order ordinary differential equation with two boundary conditions, which can be readily solved to give

$$\phi(x) = \frac{g}{2a^2}\left(x^2 - Lx\right).$$

With $\phi(x)$ so chosen, what we are left with are the differential equation and boundary and initial conditions for $U(x,t)$

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 U(x,t)}{\partial t^2},$$

$$U(0,t) = 0, \quad U(L,t) = 0,$$

$$U(x,0) = f(x) - \phi(x), \; U_t(x,0) = 0.$$

Note that other than the modification of one of the initial conditions, this is the same equation we solved before. Therefore we can write down its solutions immediately,

$$U(x,t) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi a}{L}t \, \sin\frac{n\pi}{L}x,$$

$$b_n = \frac{2}{L}\int_0^L \left[f\left(x'\right) - \frac{g}{2a^2}\left(x'^2 - Lx'\right)\right] \sin\frac{n\pi}{L}x' \, \mathrm{d}x'.$$

It follows that the displacements of the string, including the effect of its own weight, are given by:

$$u(x,t) = \frac{g}{2a^2}\left(x^2 - Lx\right)$$
$$+\sum_{n=1}^{\infty}\left\{\frac{2}{L}\int_0^L\left[f(x) - \frac{g}{2a^2}\left(x^2 - Lx\right)\right]\sin\frac{n\pi}{L}x\,\mathrm{d}x\right\}\cos\frac{n\pi a}{L}t\,\sin\frac{n\pi}{L}x.$$

Forced Vibration and Resonance. Now suppose that the string fixed at both ends is influenced by a periodic external force per unit length $F(t) = F_1\cos\omega t$. In this case, the string will satisfy the nonhomogeneous partial differential equation

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} + F_0\cos\omega t = \frac{\partial^2 u(x,t)}{\partial t^2}, \tag{5.34}$$

where $F_0 = F_1/\rho$. The boundary conditions remain the same

$$u(0,t) = 0, \qquad u(L,t) = 0.$$

If the string is initially at rest in equilibrium when the external force begins to act, then the displacement $u(x,t)$ will also have to satisfy the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0.$$

Since the external force is purely sinusoidal, it is relatively easy to find a solution to satisfy the differential equation and the boundary conditions. Just like solving ordinary differential equation, we know that the particular solution will have to oscillate with $\cos\omega t$. Therefore, let us take the trial solution

$$v(x,t) = X(x)\cos\omega t.$$

Replace $u(x,t)$ with $v(x,t)$ in (5.34), we have

$$a^2X''(x)\cos\omega t + F_0\cos\omega t = -\omega^2X(x)\cos\omega t$$

or

$$X''(x) = -\frac{\omega^2}{a^2}X(t) - \frac{F_0}{a^2},$$

which yields the solution

$$X(x) = A\cos\frac{\omega x}{a} + B\sin\frac{\omega x}{a} - \frac{F_0}{\omega^2}.$$

The boundary conditions require

$$X(0) = X(L) = 0.$$

Thus

$$X\,(0) = A - \frac{F_0}{\omega^2} = 0, \qquad A = \frac{F_0}{\omega^2}.$$

Furthermore

$$X\,(L) = \frac{F_0}{\omega^2}\cos\frac{\omega L}{a} + B\sin\frac{\omega L}{a} - \frac{F_0}{\omega^2} = 0$$

or

$$B = \frac{F_0}{\omega^2}\frac{\left(1-\cos\frac{\omega L}{a}\right)}{\sin\frac{\omega L}{a}},$$

except for $\omega = n\pi a/L$ with even n, in that case $B = 0$. Therefore, in general

$$X(x) = \frac{F_0}{\omega^2}\cos\frac{\omega x}{a} + \frac{F_0}{\omega^2}\frac{\left(1-\cos\frac{\omega L}{a}\right)}{\sin\frac{\omega L}{a}}\sin\frac{\omega x}{a} - \frac{F_0}{\omega^2}, \tag{5.35}$$

$$v(x,t) = X(x)\cos\omega t.$$

But this solution does not satisfy the initial conditions. Therefore we resort to the method of splitting the solution into two parts

$$u(x,t) = v(x,t) + U(x,t).$$

In terms of v and U, the original equation and the boundary of initial conditions become

$$a^2\frac{\partial^2 v(x,t)}{\partial x^2} + a^2\frac{\partial^2 U(x,t)}{\partial x^2} + F_0\cos\omega t = \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2 U(x,t)}{\partial t^2},$$

$$\begin{aligned} u(0,t) &= v(0,t) + U(0,t) = 0, \\ u(L,t) &= v(L,t) + U(L,t) = 0, \\ u(x,0) &= v(x,0) + U(x,0) = 0, \\ u_t(x,0) &= v_t(x,0) + U_t(x,0) = 0. \end{aligned}$$

Since

$$a^2\frac{\partial^2 v(x,t)}{\partial x^2} + F_0\cos\omega t = \frac{\partial^2 v(x,t)}{\partial t^2},$$
$$v(0,t) = 0, \qquad v(L,t) = 0,$$

and

$$v(x,0) = X(x), \qquad v_t(x,0) = -\omega X(x)\sin 0 = 0.$$

Hence

$$a^2\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{\partial^2 U(x,t)}{\partial t^2},$$
$$U(0,t) = 0, \quad U(L,t) = 0,$$
$$U(x,0) = -X(x), \quad U_t(x,0) = 0.$$

This is the homogeneous differential equation we solved before

$$U(x,t)=\sum_{n=1}^{\infty}\left(-\frac{2}{L}\int_0^L X(x')\sin\frac{n\pi}{L}x'\,\mathrm{d}x'\right)\sin\frac{n\pi}{L}x\,\cos\frac{n\pi a}{L}t. \tag{5.36}$$

Therefore the solution $u(x,t)$ is given by

$$u(x,t)=X(x)\cos\omega t+U(x,t)$$

with $X(x)$ given by (5.35) and $U(x,t)$ given by (5.36).

This solution is valid for any ω. However, if ω approach $\omega_m=\frac{m\pi a}{L}$ with an odd integer m, then $X\,(x)$ in (5.35) approaches ∞, thus resonance occurs. But if $\omega=\frac{m\pi a}{L}$ with an even integer m, then

$$X(x)=\frac{F_0}{\omega^2}\cos\frac{m\pi x}{L}-\frac{F_0}{\omega^2}$$

and resonance does not occur in this case.

5.1.6 D'Alembert's Solution of Wave Equations

Using the separation of variables, we have solved the vibrating string problem by first finding the eigenvalues and eigenfunctions dictated by the boundary conditions. In the next step, we used the initial conditions to determine the constants in the Fourier series of the solution. Now we will introduce a method of doing just the opposite. We will first solve the initial values problem, and then find the solution to satisfy the boundary conditions.

Let us solve the following pure initial values problem:

$$\begin{aligned}\text{D.E.:}\quad &\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2}\\ \text{I.C.:}\quad &u(x,0)=f(x),\ u_t(x,0)=g(x),\end{aligned}$$

for $0<t<\infty$ and $-\infty<x<\infty$. It turns out the general solution of this equation can be found by a change of variables:

$$\begin{aligned}\zeta&=x+at\\ \eta&=x-at.\end{aligned}$$

According to the chain rule

$$\begin{aligned}\frac{\partial}{\partial x}&=\frac{\partial\zeta}{\partial x}\frac{\partial}{\partial\zeta}+\frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}=\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\eta},\\ \frac{\partial}{\partial t}&=\frac{\partial\zeta}{\partial t}\frac{\partial}{\partial\zeta}+\frac{\partial\eta}{\partial t}\frac{\partial}{\partial\eta}=a\frac{\partial}{\partial\zeta}-a\frac{\partial}{\partial\eta},\end{aligned}$$

so the differential equation becomes

$$\left(\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\eta}\right)u=\frac{1}{a^2}\left(a\frac{\partial}{\partial\zeta}-a\frac{\partial}{\partial\eta}\right)\left(a\frac{\partial}{\partial\zeta}-a\frac{\partial}{\partial\eta}\right)u$$

or

$$\left(\frac{\partial^2}{\partial\zeta^2}+2\frac{\partial^2}{\partial\zeta\partial\eta}+\frac{\partial^2}{\partial\eta^2}\right)u=\left(\frac{\partial^2}{\partial\zeta^2}-2\frac{\partial^2}{\partial\zeta\partial\eta}+\frac{\partial^2}{\partial\eta^2}\right)u.$$

Clearly

$$\frac{\partial^2}{\partial\zeta\partial\eta}u=0.$$

This new equation can be solved easily by two straightforward integrations. Integration with respect to ζ gives an arbitrary function $A(\eta)$ of η, that is

$$\frac{\partial}{\partial\eta}u=A(\eta),$$

since

$$\frac{\partial}{\partial\zeta}\left(\frac{\partial}{\partial\eta}u\right)=\frac{\partial}{\partial\zeta}A(\eta)=0.$$

The second integration with respect to η gives

$$u=\int A(\eta)\,\mathrm{d}\eta+G(\zeta),$$

where $G(\zeta)$ is an arbitrary function of ζ. Since $A(\eta)$ is arbitrary, we might as well write $F(\eta)$ in place of $\int A(\eta)\,\mathrm{d}\eta$. Thus

$$u(\zeta,\eta)=F(\eta)+G(\zeta).$$

Substituting back the original variables, we have

$$u(x,t)=F(x-at)+G(x+at). \tag{5.37}$$

Thus the general solution of the wave equation is a sum of two arbitrary moving waves, each moving in opposite direction with velocity a.

It is easy to see that (5.37) is indeed the solution of the wave equation. We can use the chain rule

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t}&=\frac{\mathrm{d}F(x-at)}{\mathrm{d}(x-at)}\frac{\partial(x-at)}{\partial t}+\frac{\mathrm{d}G(x+at)}{\mathrm{d}(x+at)}\frac{\partial(x+at)}{\partial t}\\&=-aF'(x-at)+aG'(x+at),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u(x,t)}{\partial t^2}&=-a\frac{\mathrm{d}F'(x-at)}{\mathrm{d}(x-at)}\frac{\partial(x-at)}{\partial t}+a\frac{\mathrm{d}G'(x+at)}{\mathrm{d}(x+at)}\frac{\partial(x+at)}{\partial t}\\&=a^2F''(x-at)+a^2G''(x+at).\end{aligned}$$

Similarly, we obtain

$$\frac{\partial^2 u(x,t)}{\partial x^2} = F''(x-at) + G''(x+at).$$

It is readily seen that the differential equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

is satisfied.

Now if we impose the initial conditions

$$u(x,0) = f(x),$$
$$u_t(x,0) = g(x),$$

we have

$$u(x,0) = F(x) + G(x) = f(x), \tag{5.38}$$

$$u_t(x,0) = -aF'(x) + aG'(x) = g(x). \tag{5.39}$$

Integrating (5.39) from any fixed point, say 0, to x gives

$$-aF(x) + aG(x) + aF(0) - aG(0) = \int_0^x g(x')\mathrm{d}x'$$

or

$$-F(x) + G(x) = \frac{1}{a}\int_0^x g(x')\mathrm{d}x' - F(0) + G(0). \tag{5.40}$$

Solving for $F(x)$ and $G(x)$ from (5.38) and (5.40), we get

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2a}\int_0^x g(x')\mathrm{d}x' + \frac{1}{2}\left[F(0) - G(0)\right], \tag{5.41}$$

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2a}\int_0^x g(x')\mathrm{d}x' - \frac{1}{2}\left[F(0) - G(0)\right]. \tag{5.42}$$

If we replace the argument x by $x - at$ on both sides of (5.41), we can write

$$F(x-at) = \frac{1}{2}f(x-at) - \frac{1}{2a}\int_0^{x-at} g(x')\mathrm{d}x' + \frac{1}{2}\left[F(0) - G(0)\right].$$

Similarly, replacing the argument x by $x + at$ on both sides of (5.42), we have

$$G(x+at) = \frac{1}{2}f(x+at) + \frac{1}{2a}\int_0^{x+at} g(x')\mathrm{d}x' - \frac{1}{2}\left[F(0) - G(0)\right].$$

Thus

$$u(x,t) = F(x-at) + G(x+at) = \frac{1}{2}\left[f(x-at) + f(x+at)\right]$$
$$+\frac{1}{2a}\int_0^{x+at} g(x')\mathrm{d}x' - \frac{1}{2a}\int_0^{x-at} g(x')\mathrm{d}x'.$$

Reversing the upper and lower limits of the last integral and combining it with the preceding one, we obtain

$$u(x,t) = \frac{1}{2}\left[f(x-at) + f(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(x')\mathrm{d}x'. \tag{5.43}$$

This is the solution for $-\infty < x < \infty$ without any boundary condition.

Now suppose that the string is of finite length from 0 to L, and both ends are fixed, so we have the following boundary conditions:

$$u(0,t) = 0, \qquad u(L,t) = 0.$$

In this case, $f(x)$ and $g(x)$ are, of course, defined only in the range of $0 \leq x \leq L$. We want to find the solution that satisfy these additional conditions.

First, if $u(0,t) = 0$, then according to (5.43)

$$u(0,t) = \frac{1}{2}\left[f(-at) + f(at)\right] + \frac{1}{2a}\int_{-at}^{+at} g(x')\mathrm{d}x' = 0.$$

Since $f(x)$ and $g(x)$ are two unrelated functions, to satisfy this condition we must have

$$f(-at) + f(at) = 0, \tag{5.44}$$

$$\int_{-at}^{+at} g(x')\mathrm{d}x' = 0. \tag{5.45}$$

It is clear from (5.44) that

$$f(x) = -f(-x).$$

Therefore $f(x)$ must be an odd function. In other words, $f(x)$ should be antisymmetrically extended to negative x.

The integral in (5.45) can be written as

$$\int_{-at}^{0} g(x')\mathrm{d}x' + \int_0^{at} g(x)\mathrm{d}x = 0.$$

With a change of variable $x' = -x$ in the first integral, we can write it as

$$\int_{-at}^{0} g(x')\mathrm{d}x' = \int_0^{at} g(-x)\mathrm{d}x.$$

For

$$\int_0^{at} g(-x)\mathrm{d}x + \int_0^{at} g(x)\mathrm{d}x = 0,$$

we must have

$$g(-x) = -g(x).$$

Therefore, $g(x)$ also is an odd function.

Similarly, to satisfy the other boundary condition

$$u(L,t) = \frac{1}{2}\left[f(L-at) + f(L+at)\right] + \frac{1}{2a}\int_{L-at}^{L+at} g(x')\mathrm{d}x' = 0,$$

we require

$$f(L-at) + f(L+at) = 0, \tag{5.46}$$

$$\int_{L-at}^{L+at} g(x')\mathrm{d}x' = 0. \tag{5.47}$$

Thus

$$f(L-ct) = -f(L+ct).$$

Since $f(x)$ is an odd function, so

$$f(L-ct) = -f(-L+ct).$$

Therefore

$$f(-L+ct) = f(L+ct).$$

It follows that:

$$f(-L+ct+2L) = f(L+ct) = f(-L+ct)$$

This shows that $f(x)$ should be a periodic function of period $2L$.

Now (5.47) can be written as

$$\int_{L-at}^{L} g(x')\mathrm{d}x' + \int_{L}^{L+at} g(x')\mathrm{d}x' = 0.$$

With a change of variable, $x' = L - x$ in the first integral, it becomes

$$\int_{L-at}^{L} g(x')\mathrm{d}x' = \int_0^{ct} g(L-x)\mathrm{d}x.$$

Change the variable x' to $L + x$ in the second integral, we can write

$$\int_{L}^{L+at} g(x')\mathrm{d}x' = \int_0^{ct} g\left(L+x\right)\mathrm{d}x.$$

So for
$$\int_0^{ct} g(L-x)\mathrm{d}x + \int_0^{ct} g(L+x)\mathrm{d}x = 0,$$
we must have
$$g(L-x) = -g(L+x).$$
Since $g(L-x)$ is an odd function, so
$$g(L-x) = -g(-L+x).$$
Thus
$$g(-L+x) = g(L+x).$$
It follows that:
$$g(-L+x+2L) = g(L+x) = g(-L+x)$$
Therefore $g(x)$ should also be a periodic function of period $2L$.

Thus, if we define $f^*(x)$ and $g^*(x)$ as the odd periodic functions of period $2L$, whose definitions in $0 \le x \le L$ are, respectively, $f(x)$ and $g(x)$, then the solution of the problem is given by
$$u(x,t) = \frac{1}{2}\left[f^*(x-at) + f^*(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g^*(\tau)\mathrm{d}\tau, \tag{5.48}$$
which is exactly the same as (5.32).

This is known as D'Alembert's solution. The French mathematician Jean le Rond D'Alembert (1717–1783) first discovered it around 1750s. This method is very elegant but unfortunately it is limited to the solution of this type of equations. The method of separation of variable which, as we have seen, can also give the same solution, is more general. We shall use it to solve many other types of partial differential equations.

Example 5.1.3. A string of length L with tension T and density ρ, fixed at both ends, is initially given the displacement of a small triangular pulse at $x = L/4$, shown on the top line of Fig. 5.9, and is released from rest. Determine its subsequent motion.

Solution 5.1.3. Let the initial triangular pulse be $f(x)$, the subsequent displacement is simply given by
$$u(x,t) = \frac{1}{2}\left[f^*(x-at) + f^*(x+at)\right]$$
where f^* is equal to $f(x)$ in the range of $0 \le x \le L$, outside this range f^* is the antisymmetrical periodic extension of $f(x)$ with a period of $2L$. This extension is shown in Fig. 5.9 as the dotted lines. The actual displacements of

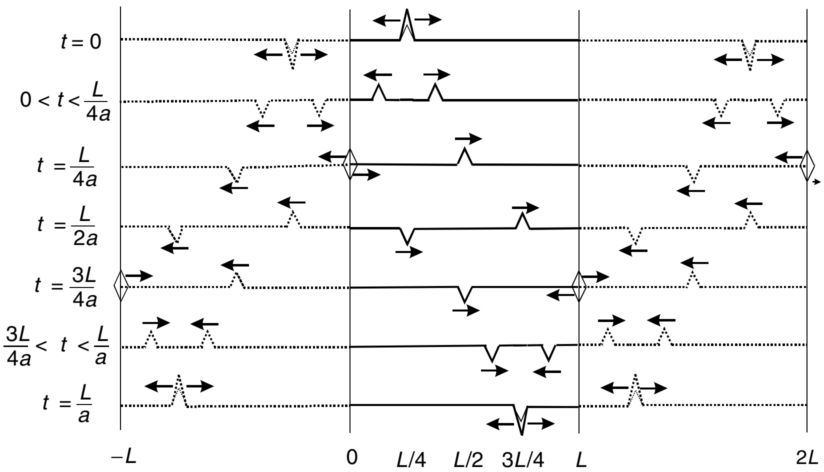

Fig. 5.9. Time development of an initial triangular pulse at $x = L/4$ on a string of length L, fixed at both ends

the string are shown in Fig. 5.9 as the thick dark lines in the "physical" space between 0 and L. However, after $t > L/4a$, what happens in the physical space is the result of some dotted pulse moving from the "mathematical" space into the "physical" space. In other words, the finite string of length L can be regarded as if it is only a segment of an infinitely long line. The pulses are moving on this infinite line. What happens in the segment between 0 and L is the displacements of the real string that we can see. Other parts of this infinitely long line are only mathematical constructs which are used to predict what will happen in the real string.

Soon after the pulse is released, it splits into two equal parts and moving in the opposite direction with the same velocity a which is equal to $\sqrt{T/\rho}$. This motion is shown in the second line of Fig. 5.9. At $t = L/4a$, the left pulse reaches the end point at $x = 0$. It will gradually disappear as shown in the third line of the figure. Then an identical pulse will reappear but upside down. In the time interval $L/4a < t < 3L/4a$, there are two pulses $L/2$ distance apart, one up and one down, both moving to the right as shown in the fourth line. At $t = 3L/4a$, the right pulse reaches the end point at $x = L$ and gradually disappears. This is shown in the fifth line. Soon after an identical pulse reappears upside down, and moves to the left. In the time interval $3L/4a < t < L/a$, two pulses, both upside down, are moving toward each other. This is shown in the sixth line. The end of the first half cycle is at $t = L/a$. At that time, the original pulse appears upside down at $x = 3L/4$. This is shown in the last line of Fig. 5.9. After that the motion will repeat itself in reverse direction until $t = 2L/a$. That is the end of the first cycle and the string will return to its original shape. These are a well-known facts which can be easily checked.

Example 5.1.4. When a piano wire is struck by a narrow hammer at x_0, a localized velocity is imparted to that point. At that moment, the wire is still at rest, but it will start to vibrate thereafter. Describe the motion assuming $a = \sqrt{T/\rho}$ and $0 < x_0 < L/4$.

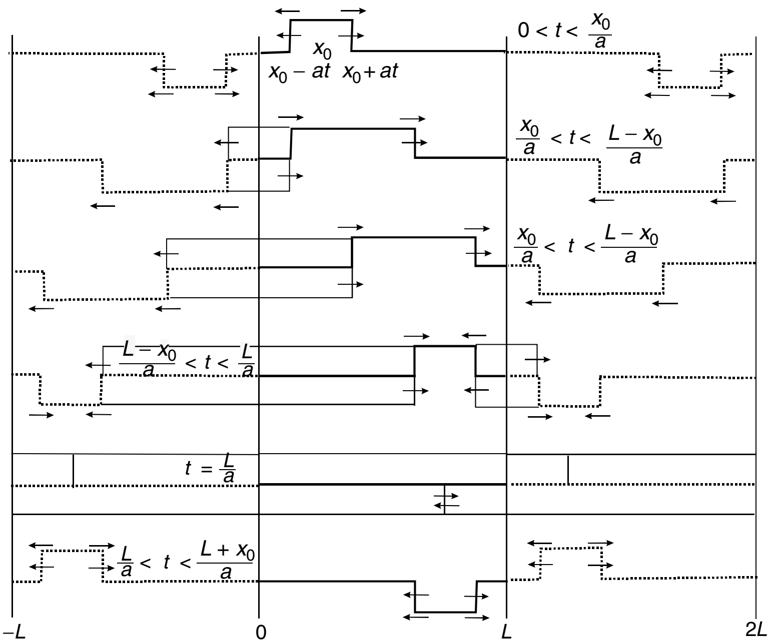

Fig. 5.10. The motion of wire of length L after a locialized velocity is imparted at x_0

Solution 5.1.4. To find the displacements of the wire, we have to solve the boundary value problem of the vibrating string with the following initial conditions

$$u(x,0) = 0 \qquad \text{and} \qquad u_t(x,0) = \delta\,(x - x_0),$$

where $\delta\,(x - x_0)$ is a delta function. According to (5.48),

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} \delta^*\,(x' - x_0)\,\mathrm{d}x',$$

where $\delta^*\,(x - x_0)$ is the odd periodic function of period $2L$, whose definition between 0 and L is the delta function $\delta\,(x - x_0)$. By the definition of delta function

$$\int_{x-at}^{x+at} \delta\,(x' - x_0)\,\mathrm{d}x' = \begin{cases} 1 & \text{if} \quad x - at < x_0 < x + at \\ 0 & \qquad \text{otherwise} \end{cases}.$$

By adding at to both sides, we see that the condition $x-at < x_0$ is equivalent to $x < x_0 + at$. Similarly, $x_0 < x + at$ is equivalent to $x_0 - at < x$. That is, between $x_0 - at$ and $x_0 + at$, the integral is equal to 1, outside this range, it is equal to zero.

This result can also be obtained by using

$$\delta\left(x-x_0\right) = \frac{\mathrm{d}}{\mathrm{d}x} U\left(x-x_0\right),$$

where $U\left(x-x_0\right)$ is the step function, and

$$\begin{aligned}\int_{x-at}^{x+at} \delta\left(x'-x_0\right) \mathrm{d}x' &= U\left(x+at-x_0\right) - U(x-at-x_0) \\ &= U\left(x-\left(x_0-at\right)\right) - U\left(x-\left(x_0+at\right)\right).\end{aligned}$$

This gives the same result. This means that soon after the wire is struck, a rectangular pulse centered at x_0 with a height of $1/2a$ will appear as shown in Fig. 5.10.

The actual displacements of the wire between 0 and L (which we call the "physical" space) are shown as the thick dark lines in Fig. 5.10. The images due to antisymmetrical and periodic extensions in $-L < x < 0$ and $0 < x < 2L$ (which are part of the "mathematical" space) are shown as dotted lines in the figure. These rectangular pulses will continue to expand at a constant rate on an infinite line without limit. After a while, some images are coming from the mathematical space into the physical space. They will then cancel part of the expanding rectangular pulse originally in the physical space to give the actual displacements of the wire. As a result, the motion of the wire appears as follows.

First the width of the rectangular pulse will expand at a constant rate as shown in the first line of Fig. 5.10. At $t = x_0/a$, the left edge of this rectangular pulse reaches the end point at $x = 0$. Immediately it bounces back and moves toward the right. In the time interval $x_0/a < t < (L-x_0)/a$, a rectangular pulse of constant width of $2x_0$ is moving to the right with velocity of a. This motion is shown in the second and the third line of Fig. 5.10. At $t = (L-x_0)/a$, the right edge of the rectangular pulse reaches the end point at $x = L$ and bounces back. The width of the rectangular pulse starts to shrink as shown in the fourth line of Fig. 5.10. At $t = L/a$, the pulse disappears. What happened is that the two negative rectangular pulses have moved from the mathematical space so far into the physical space so that they touch each other. As a consequence, they completely cancel the positive rectangular pulse. This is shown in line five. Soon after that, the two negative rectangular pulses overlap and over compensate the positive rectangular pulse, as a result another reactangular pulse reappears up side down. This is shown in the last line of Fig. 5.10. After that the motion will repeat itself in reverse order. The