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Chandrasekhar limit: an elementary approach based on classical physics and quantum theory

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Abstract
In a brief article published in 1931, Subrahmanyan Chandrasekhar made public an important astronomical discovery. In his article, the then young Indian astrophysicist introduced what is now known as the *Chandrasekhar limit*. This limit establishes the maximum mass of a stellar remnant beyond which the repulsion force between electrons due to the exclusion principle can no longer stop the gravitational collapse. In the present article, we create an elemental approximation to the Chandrasekhar limit, accessible to non-graduate science and engineering students. The article focuses especially on clarifying the origins of Chandrasekhar's discovery and the underlying physical concepts. Throughout the article, only basic algebra is used as well as some general notions of classical physics and quantum theory.

1. Introduction

In a brief article published in 1931, the then young Indian astrophysicist Subrahmanyan Chandrasekhar (1910–1995) made public an important astronomical discovery. The article established the maximum mass of a stellar remnant beyond which the repulsion between electrons due to the exclusion principle was unable to hold back the gravitational collapse [1]. It can be shown that this limit has a value of ~1.4 $M_\odot$, where $M_\odot$ is the mass of the sun. When the mass of the remnant is below this value, known as the *Chandrasekhar limit*, the repulsion between electrons is able to stop the gravitational collapse, leading to the formation of a dense and stable star, known as *white dwarf* [3].

Typically, the use of differential and integral calculus is required to perform the mathematical derivation of the Chandrasekhar limit [2–9]. This reduces considerably the audience to which the subject may be accessible. The goal of this article is to develop an original approximation to the Chandrasekhar limit that only requires elemental algebra and some notions of Classical Physics

[3] The term *white dwarf* was introduced by the American astronomer Willem Jacob Luyten (1899–1994) in 1922. However, apparently the term was later popularized by the English astrophysicist Sir Arthur Eddington (1882–1944).

and Quantum Theory. The article is aimed at undergraduate students of sciences and engineering. The exposition emphasizes on clarifying the genesis of Chandrasekhar's discovery, as well as the conceptual aspects of the matter.

The article is organized as follows. First, we present a wide view of the subject, introducing the notions of *white dwarf* and the *Chandrasekhar limit*. Next, we analyze the repulsion between electrons produced by Pauli's exclusion principle and introduce the concept of *degeneracy pressure*. Later we develop a non-relativistic approach to study the equilibrium state in a white dwarf. Then we introduce a relativistic approach to obtain an approximate formula for the Chandrasekhar limit. Finally, we briefly discuss the importance of the Indian astrophysicist work.

2. White dwarfs and the Chandrasekhar limit: an overview

A star is a giant sphere of incandescent plasma that shines because of the thermonuclear fusion reactions in its core, where atomic nuclei join to form heavier nuclei, releasing enormous amounts on energy in the process. For approximately 90% of their lives, while stars are said to be in the *main sequence*, the compressive force exerted by gravity is stopped by expansive thermal pressure. The later has its origin in the stellar core, where nuclear fusion takes place [2, 7, 8].

Throughout their evolution, stars loose a considerable part of their mass, and when nuclear fuel runs out, all that is left is a stellar remnant corresponding to the core of the original star. Under these conditions, thermonuclear reactions stop and temperature drops. At some point thermal pressure is unable to counteract the star's remnant own gravitational force, which leads to the remnant's collapse [2, 7, 8].

During the gravitational collapse, electrons and nuclei are forced to occupy an ever-shrinking volume. When density gets high enough, thermal pressure becomes negligible and a new form of pressure appears. This pressure, of quantum origin, is called *degeneracy pressure* [10, 11]. As we will discuss later, degeneracy pressure manifests itself as a repulsion between electrons opposing gravitational collapse. When degeneracy pressure exactly matches the pressure due to gravity, the stellar remnant reaches a new state of equilibrium

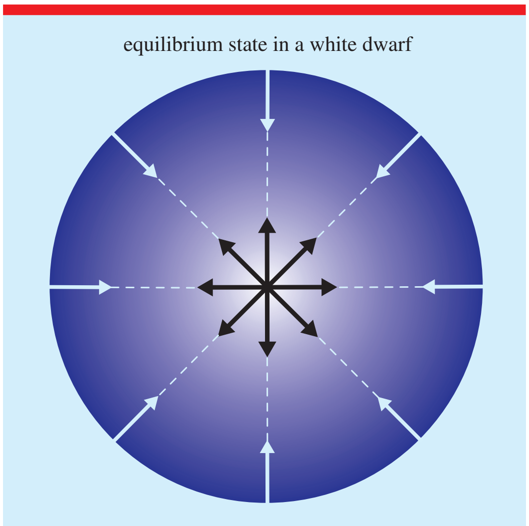

Figure 1. In a white dwarf, the gravity force (white arrows) is exactly matched by the repulsive force between electrons (black arrows) due to the degeneracy pressure.

called *white dwarf*. White dwarfs have densities of the order of 10^9 kg m^{-3}, at these high densities electrons and atomic nuclei behave as free particles (they possess only kinetic energy), and degeneracy pressure is produced mainly due to electrons because the pressure associated with nuclei is very low and can be ignored [2, 7, 8] (see figure 1).

Can degeneracy pressure always stop gravitational collapse? In other words, is the fate of every star to become a white dwarf? Using Newton's Gravitational Theory, Quantum Mechanics and Special Relativity Theory, Chandrasekhar showed that degeneracy pressure can stop gravitational collapse only when the mass of the stellar remnant does not exceed this value:

$$M_{\mathrm{Ch}} = 3.15\left(\frac{Z}{A}\right)^2\left(\frac{\hbar c}{Gm_{\mathrm{p}}^2}\right)^{3/2} m_{\mathrm{p}} \qquad (1)$$

This formula establishes *Chandrasekhar mass limit* M_{Ch}, more widely known as the *Chandrasekhar limit* [6, 8], where $m_p = 1.67 \times 10^{-27}$ kg is the mass of the proton, $\hbar = 1.05 \times 10^{-34}$ J · s is the reduced Planck constant, $G = 6.67 \times 10^{-11}$ N · m^2 · kg^{-2} is the universal gravitational constant, $c = 3.00 \times 10^8$ $m \cdot \mathrm{s}^{-1}$ is the speed of light in vacuum, Z is the atomic number of the nuclei composing the star and A is the mass number. A typical white dwarf is composed of helium, carbon and oxygen, so $Z/A \approx 0.5$. Introducing these values into equation (1), it is found that $M_{\mathrm{Ch}} \approx 1.4\, M_{\odot}$, where $M_{\odot} = 1.98 \times 10^{30}$ kg.

3. Exclusion principle and degeneracy pressure

Fundamental particles have a property called *spin* that parametrizes their intrinsic angular momentum, and is measured in terms of the reduced Plank constant $\hbar = h/2\pi$. Elemental particles with integer spin value ($0\hbar$, $1\hbar$, $2\hbar$, etc) are called bosons, whereas particles with fractional values of spin ($\hbar/2$, $3\hbar/2$, $5\hbar/2$, etc) are called fermions [10, 11]. The exclusion principle applies only to fermions, which includes the proton, the neutron and the electron, which have possible spin values of magnitude $\hbar/2$. Applied to these class of particles, the exclusion principle establishes that *a given region of space can contain a maximum of two electrons with the same energy at the same time, one with a spin of* $+\hbar/2$ *and the other one with a spin of* $-\hbar/2$ [10, 11].

Hence, the exclusion principle acts as a repulsive force between identical fermions when the volume where they are confined is reduced. This repulsive force can be interpreted as an expansive pressure called *degeneracy pressure*. Degeneracy pressure is a phenomenon very much unlike thermal pressure, which maintains equilibrium in stars in the main sequence. As we will soon see, an important aspect of the degeneracy pressure is its independence from temperature as it depends only on density [8].

Fermions can be forced to occupy small volumes due to a compressive force such as the one generated by a gravitational collapse. Using Heisenberg's Uncertainty Principle, it is possible to obtain an approximate mathematical relation for the minimum volume that a group of identical fermions can occupy. One of the forms adopted by this principle is [11]:

$$\Delta p \Delta x \geqslant \frac{\hbar}{2} \tag{2}$$

where Δp is the uncertainty in the linear momentum of the particle and Δx is the uncertainty in its position. For a given value of Δp, the smallest volume cell compatible with the uncertainty principle is of the order $\Delta x^3 \approx \hbar^3/(2\Delta p)^3$. Let us consider a volume V composed of a large number of cells of size Δx^3, and let's assume V contains N_{f} identical fermions. According to the uncertainty principle, the smallest value of V is attained when each cell contains a maximum of two identical fermions with spins $+\hbar/2$ and $-\hbar/2$ respectively. Thus, the minimum value of V will be[4]:

$$V_{\min} \approx \frac{1}{2} N_{\mathrm{f}} \Delta x^3 \tag{3}$$

The physical state of fermions confined in $V_{\min}$ is called degenerate matter, and is the state adopted by electrons in a white dwarf.

4. White dwarfs with a non-relativistic approach: Eddington's paradox

Some important physicists had analyzed the equilibrium conditions and gravitational collapse of a stellar remnant before Chandrasekhar's discovery. These analyses had led to a paradox unveiled by the English astrophysicist Sir Arthur Eddington (1882–1944). According to *Eddington's paradox*, white dwarfs existence was impossible because no known force could stop the gravitational collapse [12]. However, astronomical observations clearly showed otherwise.

Solving this paradox required the acceptance that the laws of physics used by Eddington had failed. Another English astrophysicist, Ralph Fowler (1889–1944) was the first to solve the paradox. Fowler used quantum mechanics to study the conditions for the equilibrium of a white dwarf [13]. His calculations showed that for any value of the remnant mass, electron degeneracy pressure is always able to stop gravitational collapse. In the following lines, we develop a simple and plausible argument that allows us to understand Fowler's conclusion. The argument is constructed around a very simplified semi-classic model of a stellar remnant's interior. Concretely, our strategy is to model degeneracy pressure as if it were thermal pressure, where moving electrons transfer momentum to the walls of the cell in which they are contained. When the cell shrinks to the minimum size Δx^3 compatible with the uncertainty principle, the pressure exerted by the electrons approaches degeneracy pressure, which as mentioned above is independent from temperature.

Let us consider a stellar remnant of mass M and volume $V_{\min}$, made up of N_e electrons. According to equation (3) it must be that:

[4] A rigorous mathematical treatment of this requires the use of a 6 degrees of freedom phase-space of the particles.

$$\Delta x^3 \approx 2\frac{V_{\min}}{N_e} \tag{4}$$

Let us consider one of these cells inside the stellar remnant and let us imagine two electrons moving freely inside it, undergoing elastic collisions. If Δp is the uncertainty in the each electron's linear momentum and Δt is the uncertainty in time, the net force F_e applied by the two electron over the walls of the cell is:

$$F_e \approx 2\frac{\Delta p}{\Delta t} \tag{5}$$

Let v_e be the electron's velocity, which we will assume to be non-relativistic ($v_e \ll c$). As $\Delta t \approx \Delta x / v_e$, F_e can be re-written as:

$$F_e \approx 2\frac{\Delta p v_e}{\Delta x} \tag{6}$$

If m_e is the electron's mass, then $\Delta p \approx m_e v_e$. Therefore, the pressure generated by the electron is of the order:

$$P_e \approx \frac{F_e}{\Delta x^2} \approx 2\frac{m_e v_e^2}{\Delta x^3} \tag{7}$$

Replacing in this equation the value of Δx^3 in equation (4):

$$P_e \approx m_e v_e^2 \frac{N_e}{V_{\min}} \tag{8}$$

From equations (2) and (4), v_e can be written as:

$$v_e \approx \frac{\hbar}{2m_e}\frac{1}{\Delta x} = \frac{\hbar}{2m_e}\left(\frac{N_e}{2V_{\min}}\right)^{1/3} \tag{9}$$

Replacing this value of v_e in equation (8):

$$P_e \approx \frac{(1/256)^{1/3}\hbar^2}{m_e}\left(\frac{N_e}{V_{\min}}\right)^{5/3} \tag{10}$$

We define the amount of electrons per unit-mass as:

$$N = \frac{N_e}{M} \tag{11}$$

Using this definition in equation (10) we have:

$$P_e \approx \frac{(1/256)^{1/3}\hbar^2}{m_e}\left(\frac{NM}{V_{\min}}\right)^{5/3} = \frac{(1/256)^{1/3}\hbar^2}{m_e} N^{5/3}\left(\frac{M}{V_{\min}}\right)^{5/3} \tag{12}$$

where $M/V_{\min}$ is the mean density of the stellar remnant. An atomic nucleus is composed by A nucleons and Z protons. The mass of a nucleon is approximately 2000 times greater than the mass of an electron. If we neglect the mass of the electron, the number on nuclei N_{nuc} contained in the volume $V_{\min}$ of the stellar remnant is equal to the quotient between M and the mass of the nuclei. Because the mass of a nucleon is close to the mass of a proton m_p, the mass of each nuclei is around Am_p. Thus $N_{\text{nuc}} = M/Am_p$. If there are Z protons per nuclei, the total number of protons will be $ZN_{\text{nuc}} = ZM/Am_p$. Assuming that the stellar remnant has no net electrical charge, the number of electrons must be equal to the number of protons:

$$N_e = \frac{ZM}{Am_p} \tag{13}$$

Combining equations (11) and (13):

$$N = \frac{Z}{Am_p} \tag{14}$$

Introducing this value of N into equation (12):

$$P_e \approx \frac{(1/256)^{1/3}\hbar^2}{m_e}\left(\frac{Z}{Am_p}\right)^{5/3}\left(\frac{M}{V_{\min}}\right)^{5/3} \tag{15}$$

Apart from a dimensionless factor ~12, expression (15) corresponds to the *equation of state of a degenerate non-relativistic electron gas*, and can be derived rigorously in a Quantum Mechanics framework [8, 14]. The degeneracy pressure P_e given by equation(15) is valid when $v_e \ll c$. This equation shows that P_e is independent from temperature and only depends on density, as stated in the previous section. Assuming a stellar remnant of radius R, and given that $V_{\min} \approx R^3$, equation (15) can be re-written as:

$$P_e \approx \frac{(1/256)^{1/3}\hbar^2}{m_e m_p^{5/3}}\left(\frac{Z}{A}\right)^{5/3}\frac{M^{5/3}}{R^5} \tag{16}$$

On the other hand, the force of gravity F_g between different parts of the stellar remnant is of the order GM^2/R^2, and given that the surface area is proportional to R^2, the pressure due to gravity is of the order of:

$$P_g \approx \frac{F_g}{R^2} \approx \frac{GM^2}{R^4} \tag{17}$$

In order for the stellar remnant to reach a new equilibrium and form a white dwarf, P_e must match P_g:

$$\frac{GM^2}{R^4} \approx \frac{(1/256)^{1/3}\hbar^2}{m_e m_p^{5/3}}\left(\frac{Z}{A}\right)^{5/3}\frac{M^{5/3}}{R^5} \tag{18}$$

Solving for M:

$$M \approx \frac{\hbar^6}{256 G^3 m_e^3 m_p^5}\left(\frac{Z}{A}\right)^5 R^{-3} \propto R^{-3} \tag{19}$$

This equation shows that for any value of M, there will always exist a value of R for which the degeneracy pressure can stop the gravitational collapse. Therefore, contrary to what the Chandrasekhar limit establishes, equation (19) suggests that all stars will eventually become a white dwarf. How can we explain this discrepancy? The answer is that our calculations so far have not imposed any limit on the electrons velocity. In other words, equation (19) makes an implicit assumption that electrons can always move fast enough to stop the collapse. We will further discuss this issue on the following section.

5. White dwarfs with a relativistic approach: the Chandrasekhar limit

After studying Eddington's paradox and reviewing Fowlers calculations, Chandrasekhar noted that the latter had not taken into account the restrictions imposed by the Special Theory of Relativity. According to this theory, no particle with mass, particularly the electron, can achieve the speed of light c. Thereupon, Chandrasekhar started to analyze the equilibrium conditions of a white dwarf taking into account relativistic physics [1, 15].

In the lines below, we develop a simple and plausible argument that leads to an approximate formula for the Chandrasekhar limit. For this, we will once more use the simplified model of the previous section, resuming the calculations from equation (6). However, this time we will take into account relativity, limiting the speed of the electron to c. Under these conditions, taking $v_e \approx c$ in equation (6) yields:

$$F_e \approx 2\frac{\Delta pc}{\Delta x} \tag{20}$$

From this equation, the pressure generated by the electron is:

$$P_e \approx \frac{F_e}{\Delta x^2} \approx 2\frac{\Delta pc}{\Delta x^3} \tag{21}$$

Introducing the value of Δx^3 given by equation (4):

$$P_e \approx \Delta pc\left(\frac{N_e}{V_{min}}\right) \tag{22}$$

From equations (2) and (4), Δp can be written as:

$$\Delta p \approx \frac{\hbar}{2\Delta x} = \frac{\hbar}{2}\left(\frac{N_e}{2V_{min}}\right)^{1/3} \tag{23}$$

Using this value of Δp in equation (22):

$$P_e \approx \left(\frac{1}{16}\right)^{1/3}\hbar c\left(\frac{N_e}{V_{min}}\right)^{4/3} \tag{24}$$

Replacing in this equation the value of N_e from equation (11):

$$P_e \approx \left(\frac{1}{16}\right)^{1/3}\hbar c N^{4/3}\left(\frac{M}{V_{min}}\right)^{4/3} \tag{25}$$

Using the value of N given by equation (14):

$$P_e \approx \left(\frac{1}{16}\right)^{1/3}\hbar c\left(\frac{Z}{Am_p}\right)^{4/3}\left(\frac{M}{V_{min}}\right)^{4/3} \tag{26}$$

Except for a dimensionless factor ~0.2, this last equation corresponds to the *equation of state for a relativistic degenerate spin-1/2 fermion gas*, and can be rigorously derived in the frame of Quantum Mechanics and Special Relativity [8, 14]. Equation (26) provides the degeneracy pressure when $v_e \approx c$. Once again, it can be observed that P_e does not depend on temperature and only depends on the average density M/V_{min}. Considering that $V_{min} \approx R^3$ equation (26) takes the form:

$$P_e \approx \left(\frac{1}{16}\right)^{1/3}\frac{\hbar c}{m_p^{4/3}}\left(\frac{Z}{A}\right)^{4/3}\frac{M^{4/3}}{R^4} \tag{27}$$

For the white dwarf to form and remain in equilibrium P_e must be equal to the gravitational pressure P_g. Equating equations (17) and (27) and solving for M, we obtain an approximate expression for the Chandrasekhar limit[5]:

$$M \approx 0.25\left(\frac{Z}{A}\right)^2\left(\frac{\hbar c}{Gm_p^2}\right)^{3/2} m_p \tag{28}$$

[5] The Chandrasekhar limit can be written in an interesting form as a function of Panck's mass, defined as $M_{Pl} = (\hbar c/G)^{1/2} = 2.18 \times 10{-}8$ kg.

Upon comparing this equation with the Chandrasekhar's limit (equation (1)), we note that they differ only in the dimensionless constants, and thus we have an excellent approximation. We can also observe that M does not depend on the stellar remnant radius. Therefore, unlike equation (19), equation (28) states that there is a maximum value for the mass, beyond which electron degeneracy pressure is unable to stop the gravitational collapse.

From a relativistic point of view, we interpret equation (28) as follows. As M increases, so does the gravitational force, which leads to a decrease in R. According to the uncertainty principle, a reduction in R makes the available volume for the electron's movement smaller, which in turn increases linear momentum and velocity v_e. However, v_e cannot exceed c, but in principle there is no physical law that prohibits M to increase indefinitely, for instance by means of accretion. We then conclude that once $v_e \approx c$, any increase in M will lead to a gravitational collapse that cannot be stopped by the electron degeneracy pressure.

What is the final destiny of a stellar remnant with a mass greater than Chandrasekhar limit? To answer this question properly, we need to know the equation of state of high dense hadronic matter. However, current knowledge of this equation is insufficient [8, 16]. Nevertheless, there is consensus that the stellar remnant will collapse into a *neutron star* or a *black hole* [3].

6. Chandrasekhar's legacy

Few scientific events are more awe-inspiring than the fact that the stars follow the plans of a simple mortal. Armed with a profound knowledge of physics at his time, Chandrasekhar was able to foretell an astronomical phenomenon that was later confirmed by countless observations. Today, after 83 years since Chandrasekhar published his findings, not a single white dwarf has been observed with a mass greater than ~1.4 $M_\odot$ [8]. Chandrasekhar's significant scientific contribution was awarded with the Nobel Prize in Physics in 1983 by the Sweden Science Academy.

After his work on white dwarfs, Chandrasekhar kept an extraordinary and prolific research work in many subjects. Most notably, he made important contributions in the fields of radiative transfer in stars, stellar structure and mathematical theory of black holes [17–20]. Not for nothing is Chandrasekhar considered one of the biggest astrophysicist of our time.

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