Modelado de objetos autogravitantes anisótropos

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La Agenda

- Modelado Matemático
- Objetos autogravitantes anisótropos
- Desde Salamanca para el mundo
- El gradiente de presiones
- La ecuación de estado donde la Física está
  - Las ecuaciones de estado de siempre
  - Ecuaciones de estado barótropa y no barótropa
- La ecuación de Riccati
- Las ecuaciones de Estado para materia ultradensa realista
- La ecuación de Riccati realistas
- Adimensionalizar e integración numérica (Runge-Kutta)
- Varias estrategias para anisotropizar

https://compstar.uni-frankfurt.de/outreach/animations-cartoons/

Neutron Star Merger Gravitational Waves and Gamma Rays

https://www.youtube.com/watch?v=EAyk2OsKvtU

Modelado matemático

Planteamiento del problema

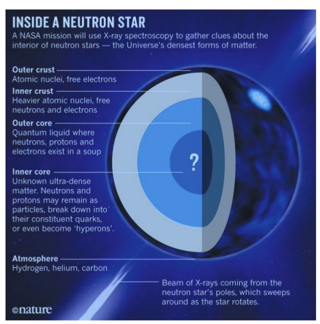
Suposiciones simplificadoras

Modelo matemático

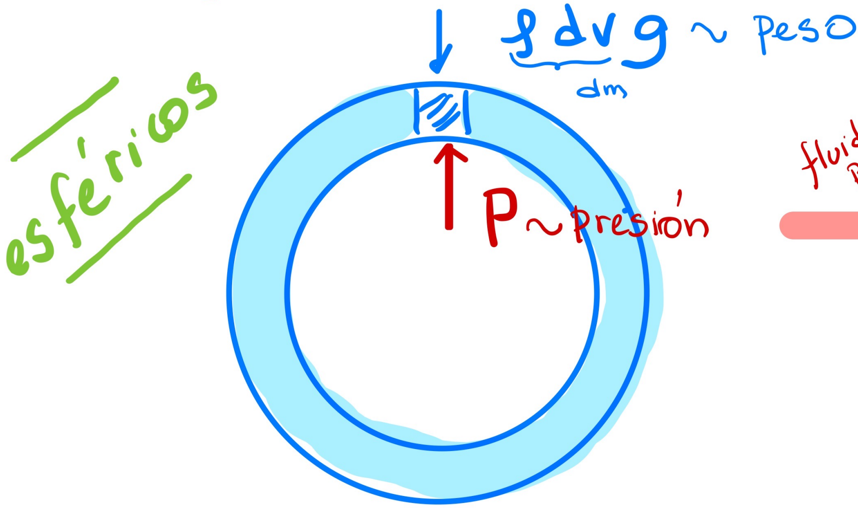
$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = \mathcal{F}\left(m(r), \rho(r), P(r), P_{\perp}(r), r\right),$$

Resultados y restricciones físicas

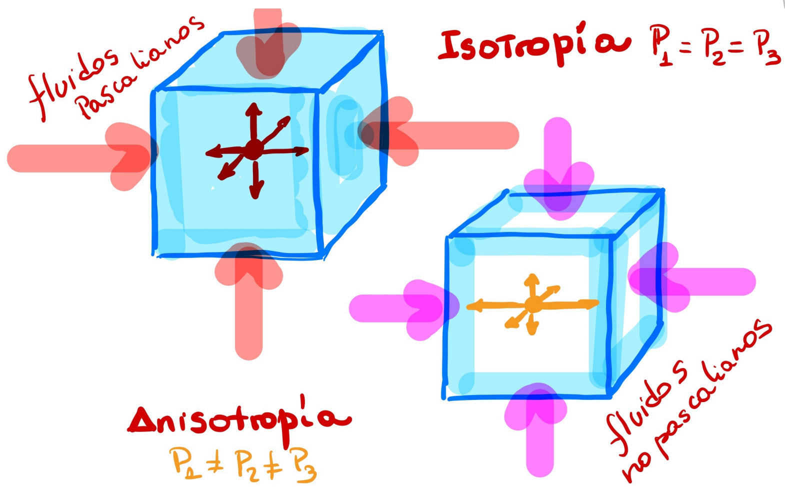
Realidad



Objetos autogravitantes



Isotropía *vs* anisotropía de presiones = Fluidos pascalianos *vs* no pascalianos



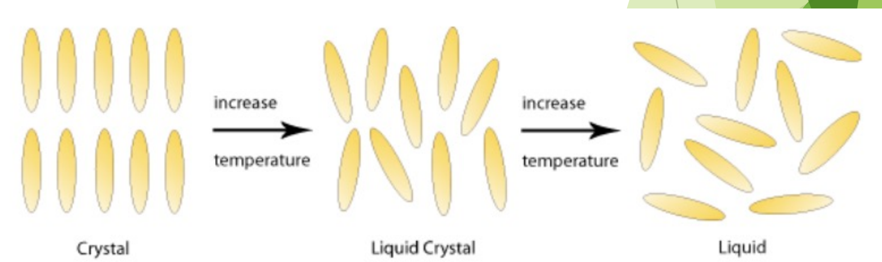
$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = \mathcal{F}\left(m(r), \rho(r), P(r), P_\perp(r), r\right),$$

$P(r)$: presión radial;

$P_\perp(r)$: presiones tangenciales;

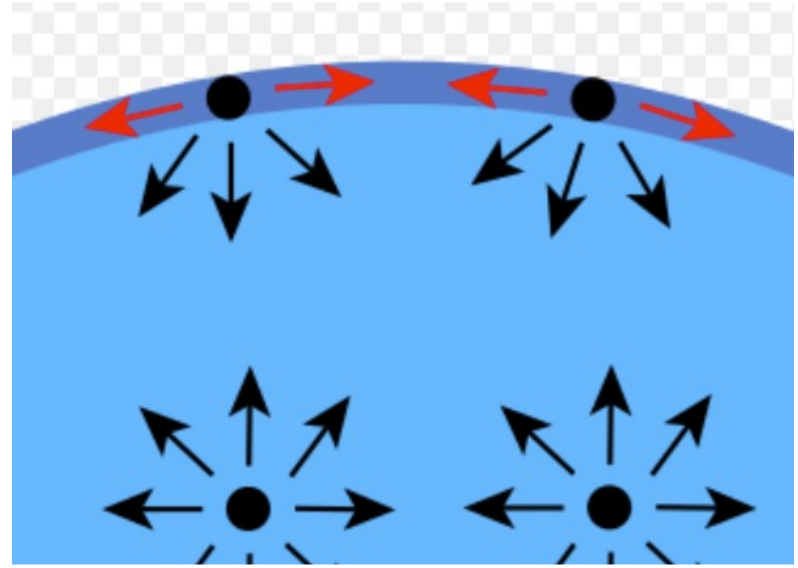
$\rho(r)$: densidad de masa;

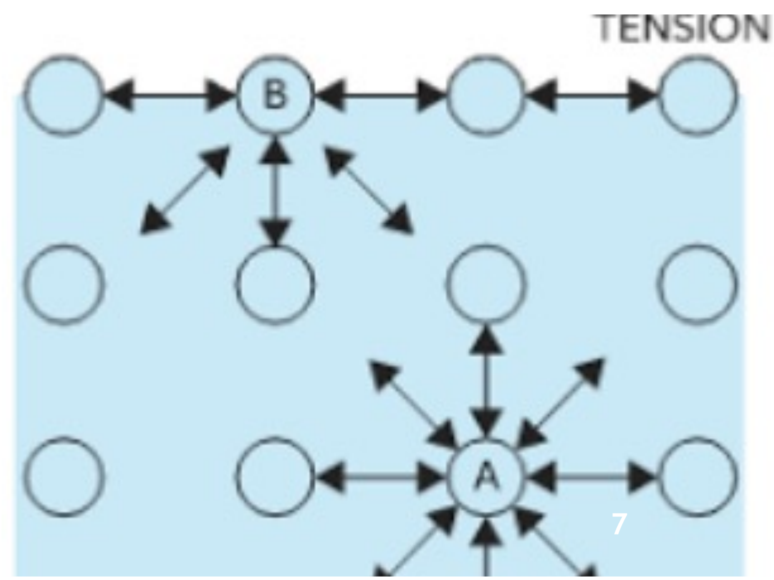
$m(r)$: masa contenida en una esfera de radio$r$;



Cristales líquidos

https://www.doitpoms.ac.uk/tlplib/liquid_crystals/intro.php




TENSION
B
A

Herrera, L. & Santos, N. O. Local anisotropy in self-gravitating systems. *Phys Reports* **286**, 53–130 (1997).

ELSEVIER

Physics Reports 286 (1997) 53–130

PHYSICS REPORTS

Local anisotropy in self-gravitating systems

L. Herrera[a,*,1], N.O. Santos[b]

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Contents

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Stability of the isotropic pressure condition.

L. Herrera[*]
Instituto Universitario de Física Fundamental y Matemáticas, Universidad de Salamanca, Salamanca 37007, Spain

(Dated: May 14, 2020)

We investigate the conditions for the (in)stability of the isotropic pressure condition in collapsing spherically symmetric, dissipative fluid distributions. It is found that dissipative fluxes, and/or energy density inhomogeneities and/or the appearance of shear in the fluid flow, force any initially isotropic configuration to abandon such a condition, generating anisotropy in the pressure. To reinforce this conclusion we also present some arguments concerning the axially symmetric case. The consequences ensuing our results are analyzed.

PACS numbers: 04.40.-b, 04.40.Nr, 04.40.Dg
Keywords: Relativistic Fluids, gravitational collapse, interior solutions.

I. INTRODUCTION

In theoretical physics it is usual to resort to different kinds of assumptions in order to solve (almost) any specific problem. Assumptions are restrictions imposed to simplify the problem under consideration, reflecting some of the essential aspects of the systems. Since all physical systems are subject to fluctuations, those essential aspects are as well. Accordingly the following questions naturally arise in the study of almost any physical problem, namely:

- Is any result obtained under the assumption A similar to that obtained under the "quasi–assumption" $A+\epsilon$ (where $\epsilon << 1$)? This question concerns the stability of the result.

- Under which conditions does assumption A remain valid all along the evolution of the system? This question concerns the stability of the assumption itself.

In this paper we endeavour to answer the questions above, in relation to the isotropic pressure condition.

For many years, both in the Newtonian and the relativistic regime, the isotropy of the pressure (the Pascal principle) has been a common (and a fundamental) assumption in the study of stellar structure and evolution. Therefore the two questions above deserve to be answered for the isotropic pressure condition.

Here, we shall focus on the question concerning the stability of the isotropic pressure condition, i.e. under which conditions such an assumption remains valid all along the evolution? More specifically, we endeavor to answer the following (related) questions:

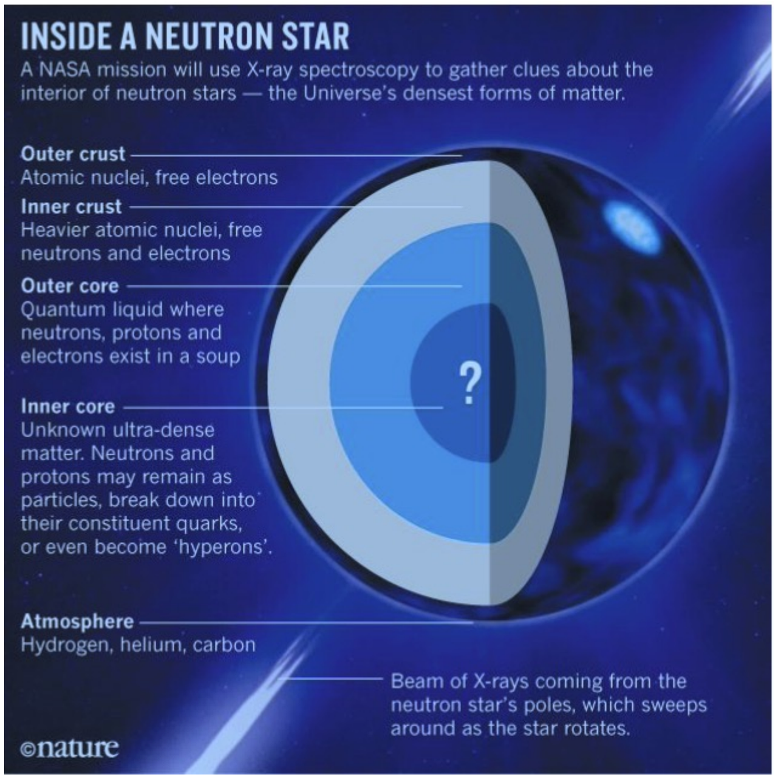
- What physical properties of the fluid distribution are related (and how) to the appearance of pressure anisotropy in an initially isotropic fluid?

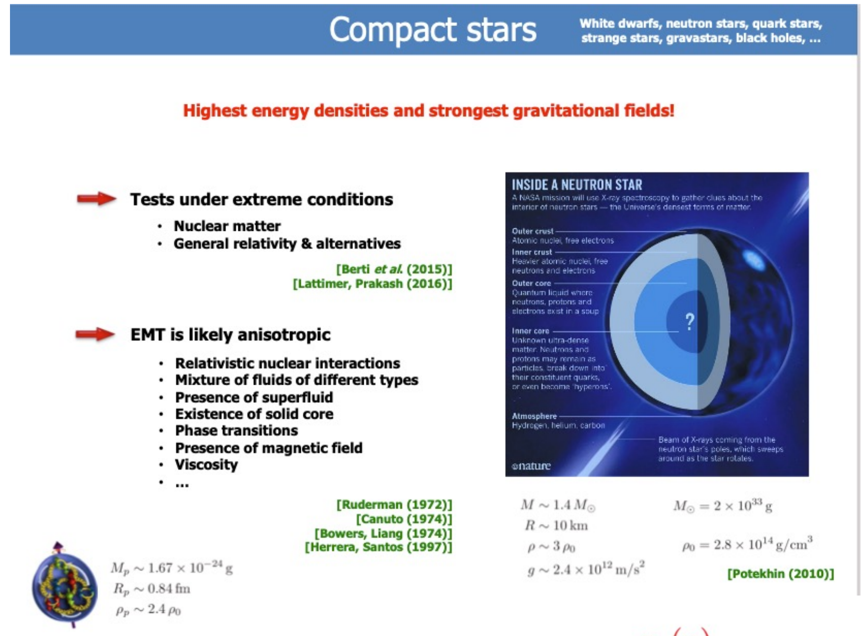
- Under which conditions does an initially isotropic configuration remain isotropic all along its evolution (stability problem)?

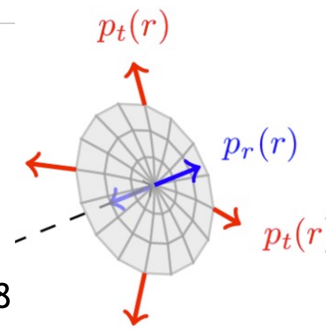
The relevance of the problem under consideration is illustrated, on the one hand, by the fact that many important results concerning relativistic fluids rely on the Pascal principle, and on the other hand, by the fact that pressure anisotropy is expected to be produced by physical processes usually present in very compact objects. This in turn explains the renewed interest in self-gravitating systems with anisotropic pressure observed in recent years. Indeed, the number of papers devoted to this issue is so large that we ask for the indulgence of the reader for not being exhaustive with the corresponding bibliography. Just as a small partial sample, let us mention the review paper [2] with a comprehensive bibliography until 1997, and some of the recent works that have appeared so far in the current (2020) year [3–38].

Our approach heavily relies on a differential equation

23 años no son nada...

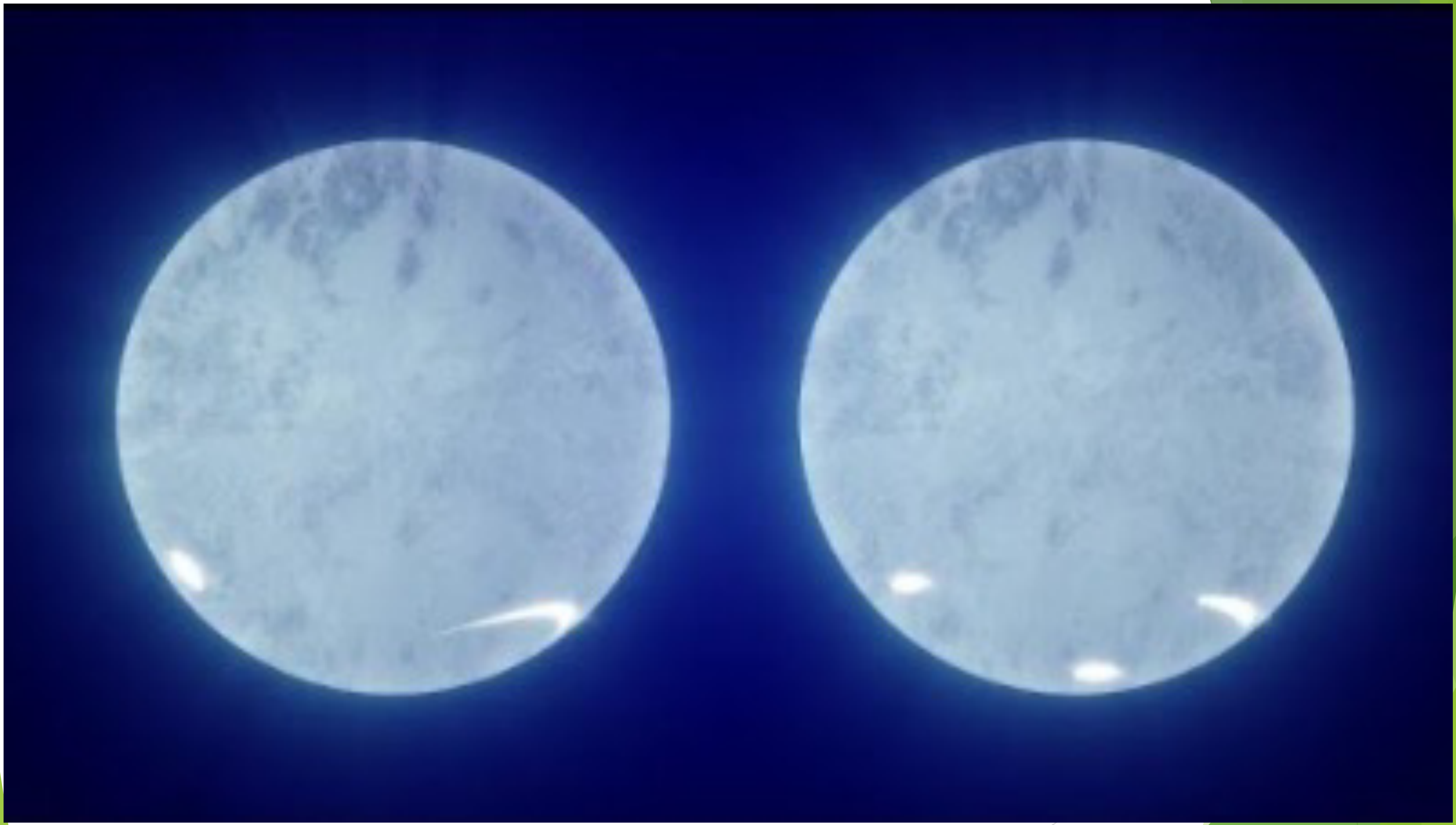






C. Lorcé (2018) Neutron stars and nucleons: Are they so different?
Correlations between partons in **nuclear** systems". **La Grande Motte**, **France**, 7-12 Oct 2018
https://ejc2018.sciencesconf.org/data/pages/trombino2018_09_24.pdf

https://www.nasa.gov/feature/goddard/2021/nasa-s-nicer-probes-the-squeezability-of-neutron-stars



El gradiente de presiones

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = \mathcal{F}\left(m(r), \rho(r), P(r), P_{\perp}(r), r\right)$$, En general

y distinguiremos dos casos particulares

Caso Newtoniano $$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2} + 2\frac{P_{\perp}(r) - P(r)}{r}$$

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}\left(1 + \frac{P(r)}{\rho(r)c^2}\right)\left(1 + \frac{4\pi r^3 P(r)}{m(r)c^2}\right)\left(1 - 2\frac{Gm(r)}{rc^2}\right)^{-1} + 2\frac{P_{\perp}(r) - P(r)}{r}$$

Caso relativista

Pero las ecuaciones de estructura estelar se completan con la definición de masa

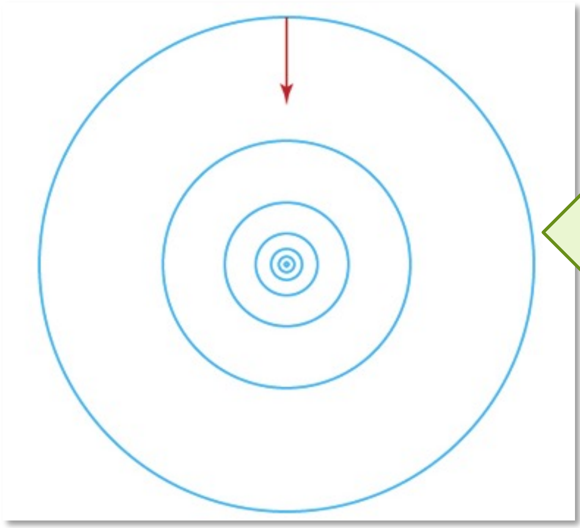
$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r)$$

Pero algo falta.....

Caso Newtoniano

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2} + 2\frac{P_\perp(r) - P(r)}{r}$$

$$P(r) = 0 \quad \Rightarrow 0 = -\frac{Gm(r)\rho(r)}{r^2} + 2\frac{P_\perp(r)}{r} \quad \Rightarrow P_\perp = \frac{Gm(r)\rho(r)}{2r}$$



Cáscaras esféricas sostenidas por tensiones tangenciales
Las cáscaras no pesan

Caso relativista

$$P(r) = 0 \quad \Rightarrow P_\perp = \frac{Gm(r)\rho(r)}{2r}\left(1 - 2\frac{Gm(r)}{rc^2}\right)^{-1}$$

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}\left(1 + \frac{P(r)}{\rho(r)c^2}\right)\left(1 + \frac{4\pi r^3 P(r)}{m(r)c^2}\right)\left(1 - 2\frac{Gm(r)}{rc^2}\right)^{-1} + 2\frac{P_\perp(r) - P(r)}{r}$$

Las ecuaciones de estado la otra parte de la Física

$$P(r) = \mathcal{W}(\rho(r), r)$$
$$P_\perp(r) = \mathcal{V}(\rho(r), P(r), r)$$

o, en su defecto

$$P(r) = \mathcal{W}(\rho(r), r)$$
$$P_\perp(r) = \mathcal{Y}(\rho(r), r)$$

Caso Newtoniano

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{m(r)\rho(r)}{r^2} + \frac{2(P_\perp(r) - P(r))}{r^2}$$

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r)$$

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{m(r)\rho(r)}{r^2}\left(1 + \frac{P(r)}{\rho(r)}\right)\left(1 + \frac{4\pi r^3 P(r)}{m(r)P(r)}\right)\left(1 - \frac{2m(r)}{r}\right)^{-1} + \frac{2(P_\perp(r) - P(r))}{r^2}$$

Caso relativista

Ecuación de estado

En física y química, una **ecuación de estado** es una ecuación constitutiva para sistemas hidrostáticos que describe el estado de agregación de la materia como una relación matemática entre la temperatura, la presión, el volumen, la densidad, la energía interna y posiblemente otras funciones de estado asociadas con la materia.[1]

Índice [ocultar]

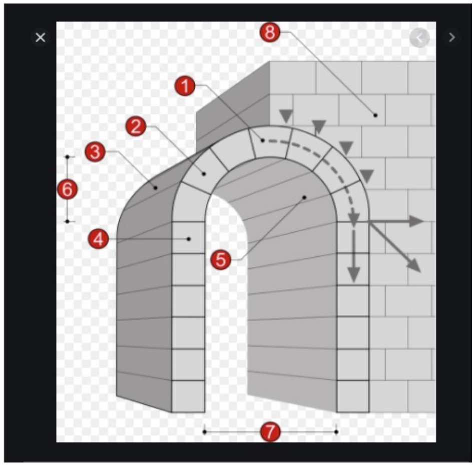
$$pV = nRT$$

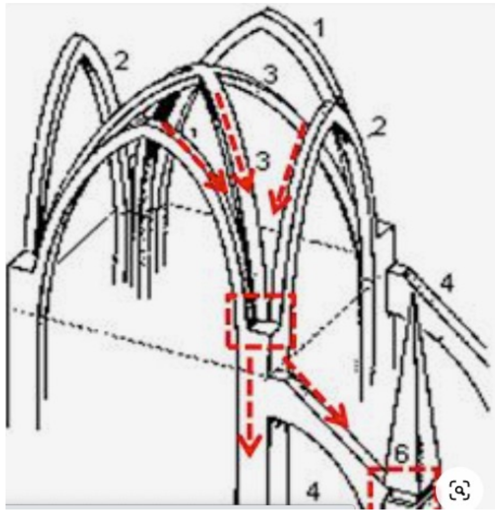
$$\left(p + \frac{a}{V_m^2}\right)(V_m - b) = RT$$

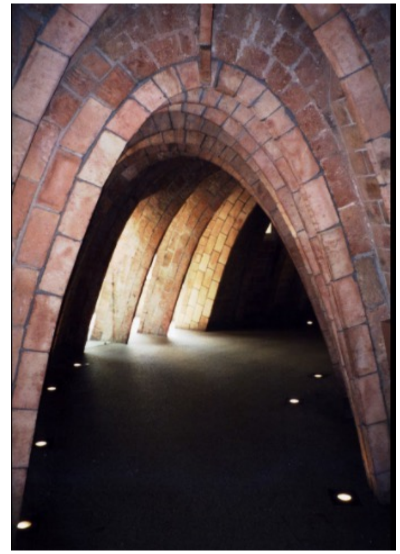
$$p = \frac{RT}{V_m - b} - \frac{a}{\sqrt{T}\,V_m\,(V_m + b)}$$

$$p = \frac{RT}{V_m - b} - \frac{a\,\alpha}{V_m\,(V_m + b)}$$

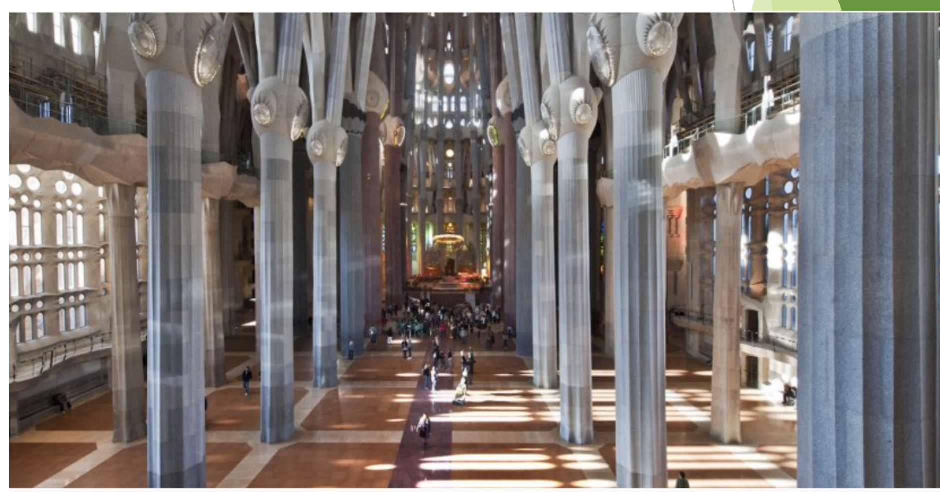
https://en.wikipedia.org/wiki/Equation_of_state







Tensiones Tangenciales en Arquitectura



Ecuaciones de estado barótropa

$$P(r) = \mathcal{W}(\rho(r), r)$$
$$P_\perp(r) = \mathcal{Y}(\rho(r), r)$$

Estrategia 1: despejar y sustituir

$$\left.\begin{array}{l} P_\perp(r) = \mathcal{Y}(\rho(r), r) \\ \\ \rho(r) = \mathcal{Z}(P(r), r) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \dfrac{\mathrm{d}P}{\mathrm{d}r} = \mathcal{G}\left(m(r), P(r), \mathcal{Z}(P(r), r), r\right), \\ \\ \dfrac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \mathcal{Z}(P(r), r) \end{array}\right.$$

$$\underbrace{P_0 = P(0) \quad \text{y} \quad m_0 = m(0) = 0}$$

Un ejemplo reciente

$$P(r) = \kappa_1 \rho^{\gamma_1}$$
$$P_\perp(r) = \kappa_2 \rho^{\gamma_2}$$

…y el sistema se integra, numéricamente

Abellán, G., Fuenmayor, E., & Herrera, L. (2020). The double polytrope for anisotropic matter: Newtonian case. *Physics of the Dark Universe*, *28*, 100549.

Abellán, G., Fuenmayor, E., Contreras, E., & Herrera, L. (2020). The general relativistic double polytrope for anisotropic matter. *Physics of the Dark Universe*, *30*, 100632.

Ecuaciones de estado barótropa

$$P(r) = \mathcal{W}(\rho(r), r)$$
$$P_\perp(r) = \mathcal{Y}(\rho(r), r)$$

Estrategia 2: derivar y sustituir

$$\left.\begin{aligned} \frac{\mathrm{d}P(\rho,r)}{\mathrm{d}r} &= \frac{\partial \mathcal{W}}{\partial \rho}\frac{\mathrm{d}\rho(r)}{\mathrm{d}r} \\ \frac{\mathrm{d}P_\perp(\rho,r)}{\mathrm{d}r} &= \frac{\partial \mathcal{Y}}{\partial \rho}\frac{\mathrm{d}\rho(r)}{\mathrm{d}r} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}r} &= \mathcal{F}(m(r), \rho(r), P(r), P_\perp(r), r), \\ \frac{\mathrm{d}m}{\mathrm{d}r} &= 4\pi r^2 \rho(r) \\ \frac{\mathrm{d}\rho}{\mathrm{d}r} &= \frac{1}{v_s}\frac{\mathrm{d}P}{\mathrm{d}r} \\ \frac{\mathrm{d}P_\perp(\rho,r)}{\mathrm{d}r} &= v_{s\perp}\frac{\mathrm{d}\rho(r)}{\mathrm{d}r} \end{aligned}\right.$$

$$\text{donde, } v_s = \frac{\partial \mathcal{W}}{\partial \rho} \quad \text{y} \quad v_{s\perp} = \frac{\partial \mathcal{Y}}{\partial \rho}$$

$$\overbrace{P_0 = P(0) = P_\perp(0), \quad \rho_0 = \rho(0) \text{y} \quad m_0 = m(0) = 0}$$

…y el sistema se integra, numéricamente

Ecuaciones de estado no barótropa

$$P(r) = \mathcal{W}(\rho(r), r)$$

$$P_\perp(r) = \mathcal{V}(\rho(r), P(r), r)$$

Estrategia 1: despejar y sustituir

$$\left.\begin{array}{l} P_\perp(r) = \mathcal{V}(\rho(r), P(r), r) \\ \\ \rho(r) = \mathcal{Z}(P(r), r) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \frac{\mathrm{d}P}{\mathrm{d}r} = \mathcal{G}\left(m(r), P(r), \mathcal{Z}(P(r), r), r\right), \\ \\ \underbrace{\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \mathcal{Z}(P(r), r)}_{P_0 = P(0) \quad \text{y} \quad m_0 = m(0) = 0} \end{array}\right.$$

…y el sistema se integra, numéricamente

Ecuaciones de estado no-barótropa

$$P(r) = \mathcal{W}(\rho(r), r)$$
$$P_\perp(r) = \mathcal{V}(\rho(r), P(r), r)$$

Estrategia 2: derivar y sustituir

$$\left.\begin{array}{l} \dfrac{\mathrm{d}P(\rho,r)}{\mathrm{d}r} = \dfrac{\partial \mathcal{W}}{\partial \rho}\dfrac{\mathrm{d}\rho(r)}{\mathrm{d}r} + \dfrac{\partial \mathcal{W}}{\partial r} \\ \\ \dfrac{\mathrm{d}P_\perp(\rho,r)}{\mathrm{d}r} = \dfrac{\partial \mathcal{V}}{\partial \rho}\dfrac{\mathrm{d}\rho(r)}{\mathrm{d}r} + \dfrac{\partial \mathcal{V}}{\partial P}\dfrac{\mathrm{d}P(r)}{\mathrm{d}r} + \dfrac{\partial \mathcal{V}}{\partial r} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \dfrac{\mathrm{d}P}{\mathrm{d}r} = \mathcal{F}(m(r), \rho(r), P(r), P_\perp(r), r), \\ \\ \dfrac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho(r), \\ \\ \dfrac{\mathrm{d}\rho}{\mathrm{d}r} = \dfrac{1}{v^2}\left(\dfrac{\mathrm{d}P}{\mathrm{d}r} - \dfrac{\partial \mathcal{W}}{\partial r}\right), \\ \\ \dfrac{\mathrm{d}P_\perp}{\mathrm{d}r} = v_\perp^2 \dfrac{\mathrm{d}\rho(r)}{\mathrm{d}r} + \dfrac{\partial \mathcal{V}}{\partial P}\dfrac{\mathrm{d}P(r)}{\mathrm{d}r} + \dfrac{\partial \mathcal{V}}{\partial r} \end{array}\right.$$

$$\text{donde, } v_s = \frac{\partial \mathcal{W}}{\partial \rho} \quad \text{y} \quad v_{s\perp} = \frac{\partial \mathcal{V}}{\partial \rho}$$

$$P_0 = P(0) = P_\perp(0), \quad \rho_0 = \rho(0) \text{y} \quad m_0 = m(0) = 0$$

…y el sistema se integra, numéricamente

…y el sistema se integra, numéricamente

Caso, newtoniano isótropo

$$\rho(r) = f(r) \Rightarrow m(r) = \int_0^r \mathrm{d}r\, 4\pi r^2 f(r) \Rightarrow \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2} \Rightarrow P(r)$$

Caso relativista isótropo

$$\rho(r) = f(r) \Rightarrow m(r) = \int_0^r \mathrm{d}r\, 4\pi r^2 f(r) \Rightarrow$$

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}\left(1 + \frac{P(r)}{\rho(r)c^2}\right)\left(1 + \frac{4\pi r^3 P(r)}{m(r)c^2}\right)\left(1 - 2\frac{Gm(r)}{rc^2}\right)^{-1} \Rightarrow$$

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -F(r) - G(r)P(r) + H(r)P^2(r)$$

Ecuación de Riccatti

This

Riccati equation

From Wikipedia, the free encyclopedia

In mathematics, a **Riccati equation** in the narrowest sense is any first-order ordinary differential equation that is quadratic in the unknown function. In other words, it is an equation of the form

$$y'(x) = q_0(x) + q_1(x)\,y(x) + q_2(x)\,y^2(x)$$

where $q_0(x) \neq 0$ and $q_2(x) \neq 0$. If $q_0(x) = 0$ the equation reduces to a Bernoulli equation, while if $q_2(x) = 0$ the equation becomes a first order linear ordinary differential equation.

The equation is named after Jacopo Riccati (1676–1754).[1]

More generally, the term **Riccati equation** is used to refer to matrix equations with an analogous quadratic term, which occur in both continuous-time and discrete-time linear-quadratic-Gaussian control. The steady-state (non-dynamic) version of these is referred to as the algebraic Riccati equation.

Contents [hide]

Obtaining solutions by quadrature [edit]

The correspondence between Riccati equations and second-order linear ODEs has other consequences. For example, if one solution of a 2nd order ODE is known, then it is known that another solution can be obtained by quadrature, i.e., a simple integration. The same holds true for the Riccati equation. In fact, if one particular solution y_1 can be found, the general solution is obtained as

$$y = y_1 + u$$

Substituting

$$y_1 + u$$

in the Riccati equation yields

$$y_1' + u' = q_0 + q_1 \cdot (y_1 + u) + q_2 \cdot (y_1 + u)^2,$$

and since

$$y_1' = q_0 + q_1\,y_1 + q_2\,y_1^2,$$

it follows that

$$u' = q_1\,u + 2\,q_2\,y_1\,u + q_2\,u^2$$

or

$$u' - (q_1 + 2\,q_2\,y_1)\,u = q_2\,u^2,$$

which is a Bernoulli equation. The substitution that is needed to solve this Bernoulli equation is

$$z = \frac{1}{u}$$

Substituting

$$y = y_1 + \frac{1}{z}$$

directly into the Riccati equation yields the linear equation

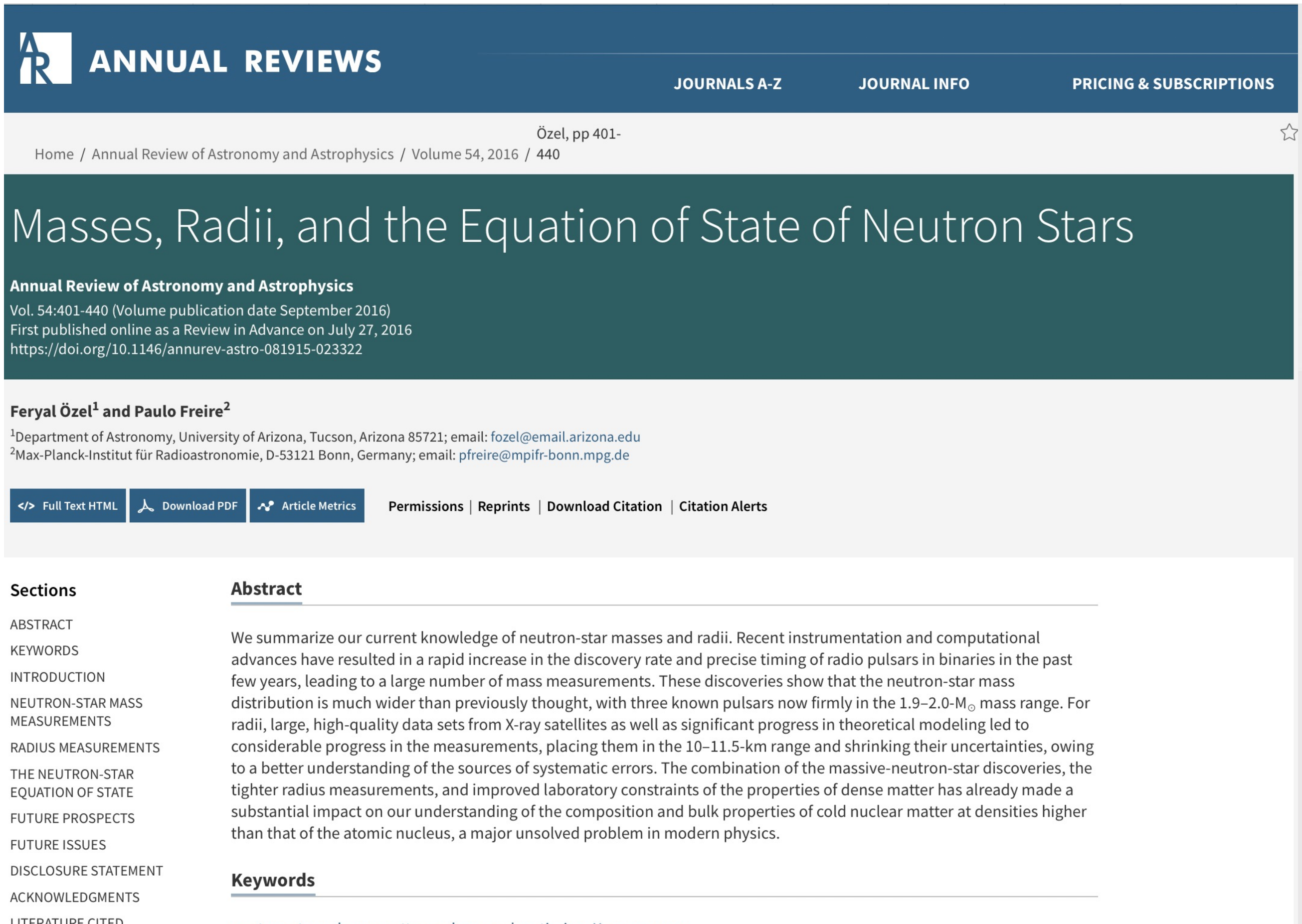
$$z' + (q_1 + 2\,q_2\,y_1)\,z = -q_2$$

A set of solutions to the Riccati equation is then given by

$$y = y_1 + \frac{1}{z}$$

where z is the general solution to the aforementioned linear equation.

See also [edit]



ANNUAL REVIEWS

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Home / Annual Review of Astronomy and Astrophysics / Volume 54, 2016 / Özel, pp 401-440

Masses, Radii, and the Equation of State of Neutron Stars

Annual Review of Astronomy and Astrophysics
Vol. 54:401-440 (Volume publication date September 2016)
First published online as a Review in Advance on July 27, 2016
https://doi.org/10.1146/annurev-astro-081915-023322

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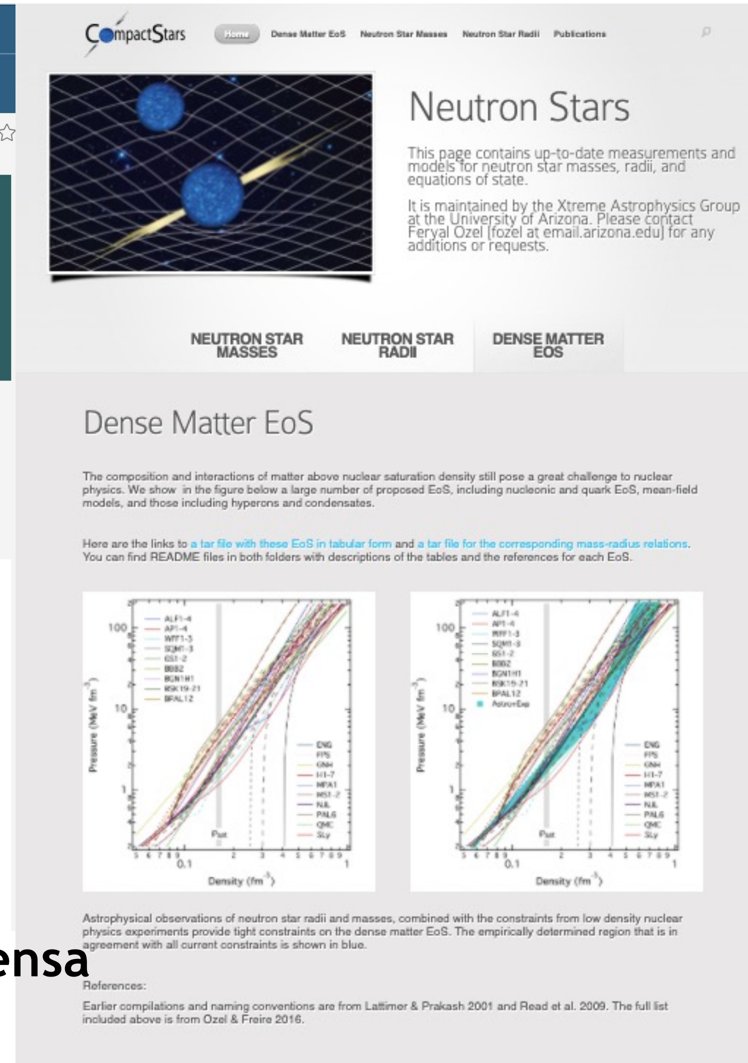
Sections

Abstract

We summarize our current knowledge of neutron-star masses and radii. Recent instrumentation and computational advances have resulted in a rapid increase in the discovery rate and precise timing of radio pulsars in binaries in the past few years, leading to a large number of mass measurements. These discoveries show that the neutron-star mass distribution is much wider than previously thought, with three known pulsars now firmly in the 1.9–2.0-$M_\odot$ mass range. For radii, large, high-quality data sets from X-ray satellites as well as significant progress in theoretical modeling led to considerable progress in the measurements, placing them in the 10–11.5-km range and shrinking their uncertainties, owing to a better understanding of the sources of systematic errors. The combination of the massive-neutron-star discoveries, the tighter radius measurements, and improved laboratory constraints of the properties of dense matter has already made a substantial impact on our understanding of the composition and bulk properties of cold nuclear matter at densities higher than that of the atomic nucleus, a major unsolved problem in modern physics.

Keywords



Ecuaciones de estado realistas para materia ultradensa

http://xtreme.as.arizona.edu/NeutronStars/

Ozel, F., & Freire, P. (2016). ***Masses, Radii, and Equation of State of Neutron Stars***. *ArXiv*. https://doi.org/10.1146/annurev-astro-081915-023322

Pero puede ser peor... No solo es Riccati

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -F(r) - G(r)P(r) + H(r)P^2(r)$$

Las ecuaciones de estado realistas imponen una Riccati numérica

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -F\begin{pmatrix} n(i) & \rho(i) & P(i) \\ n(1) & \rho(1) & P(1) \\ n(2) & \rho(2) & P(2) \\ n(3) & \rho(3) & P(3) \\ \vdots & \vdots & \vdots \end{pmatrix} - G\begin{pmatrix} n(i) & \rho(i) & P(i) \\ n(1) & \rho(1) & P(1) \\ n(2) & \rho(2) & P(2) \\ n(3) & \rho(3) & P(3) \\ \vdots & \vdots & \vdots \end{pmatrix} P(r) + H\begin{pmatrix} n(i) & \rho(i) & P(i) \\ n(1) & \rho(1) & P(1) \\ n(2) & \rho(2) & P(2) \\ n(3) & \rho(3) & P(3) \\ \vdots & \vdots & \vdots \end{pmatrix} P^2(r)$$

No hay salida...

Para ecuaciones realistas la integración debe ser numérica

Adimensionalizar, adimensionalizar 1/2

$$\frac{P}{P_0} \to \tilde{P}, \quad \frac{P_\perp}{P_0} \to \tilde{P}_\perp, \quad \frac{\rho}{\rho_0} \to \tilde{\rho}, \quad \frac{m}{M} \to \tilde{m}, \quad \text{y} \quad \frac{r}{R} \to \tilde{r}.$$

Con los siguientes parámetros

$$\mu = \frac{M}{R}\frac{G}{c^2}, \quad \kappa = \frac{P_0}{\rho_0 c^2} \quad \text{y} \quad \eta = \frac{\rho_0}{<\rho>}; \quad \text{con} \quad <\rho> = \frac{M}{\frac{4\pi}{3}R^3}$$

Entonces las ecuaciones

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}\left(1 + \frac{P(r)}{\rho(r)c^2}\right)\left(1 + \frac{4\pi r^3 P(r)}{m(r)c^2}\right)\left(1 - 2\frac{Gm(r)}{rc^2}\right)^{-1} + 2\frac{P_\perp(r) - P(r)}{r}$$

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r),$$

Se convierten en

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}} = -\frac{\mu}{\kappa}\frac{\tilde{m}\tilde{\rho}}{\tilde{r}^2}\left(1 + \kappa\frac{\tilde{P}}{\tilde{\rho}}\right)\left(1 + 3\eta\kappa\frac{\tilde{P}\tilde{r}^3}{\tilde{m}}\right)\left(1 - 2\mu\frac{\tilde{m}}{\tilde{r}}\right)^{-1} + 2\frac{\tilde{P}_\perp - \tilde{P}}{\tilde{r}}$$

$$\frac{\mathrm{d}\tilde{m}}{\mathrm{d}\tilde{r}} = 3\eta\tilde{r}^2\tilde{\rho},$$

Adimensionalizar, adimensionalizar 2/2

Pero podemos adimensionalizar de otra manera

$$\frac{P}{c^2\rho_0} \to \tilde{P} \quad \frac{P_\perp}{c^2\rho_0} \to \tilde{P}_\perp\,, \quad \frac{\rho}{\rho_0} \to \tilde{\rho}\,, \quad \frac{m}{M} \to \tilde{m}\,, \quad \text{y} \quad \frac{r}{R} \to \tilde{r}$$

Y entonces TOV nos queda como

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}} = -\mu\frac{\tilde{m}\tilde{\rho}}{\tilde{r}^2}\left(1+\frac{\tilde{P}}{\tilde{\rho}}\right)\left(1+3\xi\frac{\tilde{P}\tilde{r}^3}{\tilde{m}}\right)\left(1-2\mu\frac{\tilde{m}}{\tilde{r}}\right)^{-1} + 2\frac{\tilde{P}_\perp(\tilde{P},r)-\tilde{P}}{\tilde{r}}$$

$$\text{con} \quad \xi = \frac{\frac{4\pi R^3}{3}\rho_0}{M} = \frac{M_{\rho_0}}{M},$$

Y las ecuaciones de estructura se completan con la definición de masa

$$\frac{\mathrm{d}\tilde{m}}{\mathrm{d}\tilde{r}} = 3\xi\tilde{\rho}\tilde{r}^2$$

Este esquema tiene una particularidad.
Para convertirlo en un problema de *valores iniciales*,
la presión se debe integrar (numéricamente) “hacia atrás”.
Desde la superficie donde se anula la presión radial
hacia el centro donde adquiere su valor máximo

Adimensionalizamos y ahora integramos (numéricamente)

$$y'(x) = f(y(x), x) \Rightarrow \lim_{x_{k+1} \to x_k} \left(\frac{y_{k+1} - y_k}{x_{k+1} - x_k} \right) = f(y(x), x) \Rightarrow$$

$$y_{k+1} = y_k + \int_{x_k}^{x_{k+1}} \mathrm{d}\xi \ f\left(\xi, y(\xi)\right) \ ,$$

$$\text{Euler} \Rightarrow y_{k+1} = y_k + h \ f\left(x_k, y_k\right)$$

$$\text{Euler Mejorado o Heuns} \Rightarrow y_{k+1} = y_k + \frac{h}{2}\left[f(x_k, y_k) + f(x_{k+1}, y_{k+1})\right]$$

$$\Rightarrow y_{k+1} = y_k + \frac{h}{2}\left[f(x_k, y_k) + f(x_{k+1}, y_k + h \ f(x_k, y_k))\right]$$

Los métodos Runge-Kutta

$$f(\xi, y(\xi)) \approx \left[\alpha \ f(y_k, x_k) + \beta \ f(y_k + \delta \ f(y_k, x_k) h_k, x_k + \gamma \ h_k)\right] \quad \text{con} \quad h_k = x_{k+1} - x_k$$

scipy.integrate.solve_ivp

scipy.integrate.**solve_ivp**(*fun, t_span, y0, method='RK45', t_eval=None, dense_output=False, events=None, vectorized=False, args=None, **options*)

Solve an initial value problem for a system of ODEs.

This function numerically integrates a system of ordinary differential equations given an initial value:

```
dy / dt = f(t, y)
y(t0) = y0
```

Here t is a 1-D independent variable (time), y(t) is an N-D vector-valued function (state), and an N-D vector-valued functio determines the differential equations. The goal is to find y(t) approximately satisfying the differential equations, given an value y(t0)=y0.

Some of the solvers support integration in the complex domain, but note that for stiff ODE solvers, the right-hand side m complex-differentiable (satisfy Cauchy-Riemann equations [11]). To solve a problem in the complex domain, pass y0 with complex data type. Another option always available is to rewrite your problem for real and imaginary parts separately.

Parameters: **fun** : *callable*

Right-hand side of the system. The calling signature is `fun(t, y)`. Here *t* is a scalar, and there are t options for the ndarray *y*. It can either have shape (n,); then *fun* must return array_like with shape (n Alternatively, it can have shape (n, k); then *fun* must return an array_like with shape (n, k), i.e., each c corresponds to a single column in *y*. The choice between the two options is determined by *vectorize* argument (see below). The vectorized implementation allows a faster approximation of the Jacobian finite differences (required for stiff solvers).

t_span : *2-tuple of floats*

Interval of integration (t0, tf). The solver starts with t=t0 and integrates until it reaches t=tf.

y0 : *array_like, shape (n,)*

Initial state. For problems in the complex domain, pass *y0* with a complex data type (even if the initia is purely real).

method : *string or OdeSolver, optional*

Integration method to use:

- 'RK45' (default): Explicit Runge-Kutta method of order 5(4) [1]. The error is controlled assuming accuracy of the fourth-order method, but steps are taken using the fifth-order accurate formul extrapolation is done). A quartic interpolation polynomial is used for the dense output [2]. Can applied in the complex domain.
- 'RK23': Explicit Runge-Kutta method of order 3(2) [3]. The error is controlled assuming accuracy second-order method, but steps are taken using the third-order accurate formula (local extrap is done). A cubic Hermite polynomial is used for the dense output. Can be applied in the compl domain.
- 'DOP853': Explicit Runge-Kutta method of order 8 [13]. Python implementation of the "DOP853' algorithm originally written in Fortran [14]. A 7-th order interpolation polynomial accurate to 7- order is used for the dense output. Can be applied in the complex domain.
- 'Radau': Implicit Runge-Kutta method of the Radau IIA family of order 5 [4]. The error is controll with a third-order accurate embedded formula. A cubic polynomial which satisfies the collocati conditions is used for the dense output.
- 'BDF': Implicit multi-step variable-order (1 to 5) method based on a backward differentiation for for the derivative approximation [5]. The implementation follows the one described in [6]. A qu constant step scheme is used and accuracy is enhanced using the NDF modification. Can be ap in the complex domain.

Example: Solving Ordinary Differential Equations

In this notebook we will use Python to solve differential equations numerically.

In [1]:
```
# Import the required modules
import numpy as np
import matplotlib.pyplot as plt
# This makes the plots appear inside the notebook
%matplotlib inline
```

First-order equations

Let's try a first-order ordinary differential equation (ODE), say:

$$\frac{dy}{dx} + y = x, \qquad y(0) = 1.$$

This has a closed-form solution

$$y = x - 1 + 2e^{-x}$$

(Exercise: Show this, by first finding the integrating factor.)

We are going to solve this numerically.

First, let's import the "scipy" module and look at the help file for the relevant function, "integrate.odeint",

In [2]:
```
from scipy.integrate import odeint

# Define a function which calculates the derivative
def dy_dx(y, x):
    return x - y

xs = np.linspace(0,5,100)
y0 = 1.0  # the initial condition
ys = odeint(dy_dx, y0, xs)
ys = np.array(ys).flatten()
```

In [6]:
```
# Plot the numerical solution
plt.rcParams.update({'font.size': 14})  # increase the font size
plt.xlabel("x")
plt.ylabel("y")
plt.plot(xs, ys);
```

https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.odeint.html

Peroooo ¿ y la anisotropía?

Para el caso newtoniano $$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}} = -\frac{\mu}{\kappa}\frac{\tilde{m}\tilde{\rho}}{\tilde{r}^2} + 2\frac{\tilde{P}_\perp - \tilde{P}}{\tilde{r}}$$

Podemos suponer

$$\Delta_{Newton} = \tilde{P}_\perp - \tilde{P} = C_{Newton}\frac{\tilde{m}\tilde{\rho}}{r} \quad \Rightarrow \frac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}} = \left(2C_{Newton} - \frac{\mu}{\sigma}\right)\frac{\tilde{m}\tilde{\rho}}{\tilde{r}^2}$$

Pero también es posible

$$\Delta_{CH} = \tilde{P}_\perp - \tilde{P} = -\frac{\tilde{r}}{2}\left(\frac{1-h}{h}\right)\frac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}}$$

Cosenza, M., Herrera, L., Esculpi, M., & Witten, L. (1981). Some models of anisotropic spheres in general relativity. *Journal of Mathematical Physics*, *22*(1), 118-125.

o también

$$\Delta_{HIM} = \tilde{P}_\perp - \tilde{P} = C_{HIM}\frac{\tilde{P}}{\tilde{r}}$$

Horvat, D., Ilijić, S., & Marunović, A. (2010). Radial pulsations and stability of anisotropic stars with a quasi-local equation of state. *Classical and quantum gravity*, *28*(2), 025009.

Rahmansyah, A., Sulaksono, A., Wahidin, A. B., & Setiawan, A. M. (2020). Anisotropic neutron stars with hyperons: implication of the recent nuclear matter data and observations of neutron stars. *The European Physical Journal C*, *80*(8), 1-17.

Igual para el caso relativista

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}} = -\frac{\mu}{\kappa}\frac{\tilde{m}\tilde{\rho}}{\tilde{r}^2}\left(1+\kappa\frac{\tilde{P}}{\tilde{\rho}}\right)\left(1+3\eta\kappa\frac{\tilde{P}\tilde{r}^3}{\tilde{m}}\right)\left(1-2\mu\frac{\tilde{m}}{\tilde{r}}\right)^{-1} + 2\frac{\tilde{P}_\perp - \tilde{P}}{\tilde{r}}$$

Podemos suponer

$$\Delta_{JO} = \tilde{P}_\perp - \tilde{P} = C_{JO}\frac{\tilde{\rho}\tilde{m}}{2\tilde{r}^2(\tilde{r}-2\mu\tilde{m})}$$

Pero también es posible

$$\Delta_{CH} = \tilde{P}_\perp - \tilde{P} = -\frac{\tilde{r}}{2}\left(\frac{1-h}{h}\right)\frac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}}$$

Cosenza, M., Herrera, L., Esculpi, M., & Witten, L. (1981). Some models of anisotropic spheres in general relativity. *Journal of Mathematical Physics*, *22*(1), 118-125.

o también

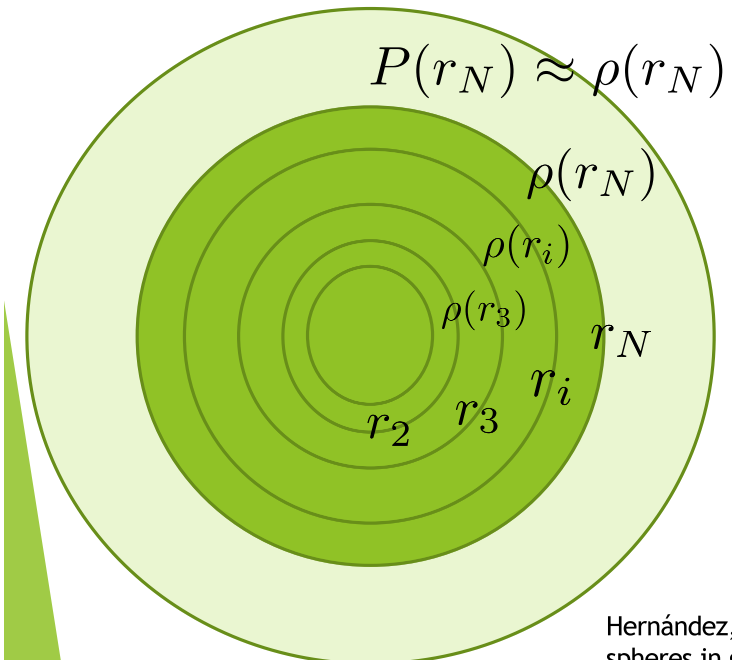
$$\Delta_{HIM} = \tilde{P}_\perp - \tilde{P} = C_{HIM}\frac{\tilde{P}}{\tilde{r}}$$

Horvat, D., Ilijić, S., & Marunović, A. (2010). Radial pulsations and stability of anisotropic stars with a quasi-local equation of state. *Classical and quantum gravity*, *28*(2), 025009.

A lo largo de varios años hemos propuesto algunas ecuaciones de estado

$$P(r) = \rho(r) - \frac{2}{r^3}\int_0^r \bar{r}^2 \rho(\bar{r})\,\mathrm{d}\bar{r} \approx \rho(r) - \frac{2}{r^3}\sum_i^N \rho(r_i)\,\Delta r_i$$

$$P(r_N) \approx \rho(r_N) - \frac{2}{r_N^3}\sum_i^N \rho(r_i)\,\Delta r_i$$



Una ecuación de estado no-local: La presión es función de las densidades sumadas
en todos los puntos interiores

Son las ecuaciones que modelan algunas grietas en las paredes

Estudiamos esferas con esta ecuación de estado para modelar posible
Objetos compactos relativistas

Hernández, H., & Núñez, L. A. (2004). Nonlocal equation of state in anisotropic static fluid spheres in general relativity. *Canadian journal of physics*, *82*(1), 29-51. Hernández, H., Nunez, L. A., & Percoco, U. (1999). Non-local equation of state in general relativistic radiating spheres. *Classical and Quantum Gravity*, *16*(3), 871.

¿Qué hemos hecho recientemente?

- Hernández, H., Suárez-Urango, D., & Núñez, L. A. (2021). Acceptability conditions and relativistic barotropic equations of state. *The European Physical Journal C*, *81*(3), 1-17.
- Suárez-Urango, D., Ospino, J., Hernández, H., & Núñez, L. A. (2022). Acceptability conditions and relativistic anisotropic generalized polytropes. *The European Physical Journal C*, *82*(2), 1-22.

Propusimos una ecuación barótropa $P = \kappa\rho^{1+\frac{1}{n}} + \alpha\rho - \beta$

con parámetros dependientes $\beta = \kappa{\rho_b}^{1+\frac{1}{n}} + \alpha\rho_b$

$$\kappa = \frac{\sigma - \alpha\left[1 - \varkappa\right]}{{\rho_c}^{\frac{1}{n}}\left[1 - \varkappa^{1+\frac{1}{n}}\right]} \qquad \text{con} \qquad \sigma = \frac{P_c}{\rho_c} \quad \text{y} \quad \varkappa = \frac{\rho_b}{\rho_c}$$

Las velocidades del sonido serán

$$v_s^2 = \frac{\partial P}{\partial \rho} = \kappa\left[1 + \frac{1}{n}\right]\rho^{\frac{1}{n}} + \alpha$$

$$v_{s_\perp}^2 = \frac{\partial P_\perp}{\partial \rho} = \frac{1}{2}\left[3 + r\frac{\rho''}{\rho'}\right]v_s^2 + \frac{1}{\rho'}\left[\frac{\mathrm{e}^{2\lambda}}{4}(\rho + P)\left(8\,\pi\,r^2 P - \mathrm{e}^{-2\lambda} + 1\right)\right]' + \frac{\kappa r(n+1)\rho^{\frac{1}{n}}}{2n^2\rho}\rho',$$

Propusimos dos perfiles de densidad

$$\textbf{Propuesta 1:} \qquad \mathrm{e}^{2\lambda} = \left[1 + Ar^2 + Br^4\right]^{-1} \;\Rightarrow\; \rho = -\frac{3A + 5Br^2}{8\pi}$$

y,

$$\textbf{Propuesta 2:} \qquad \mathrm{e}^{2\lambda} = \frac{K(1 + Ar^2)}{K + Br^2} \;\Rightarrow\; \rho = \frac{(KA - B)\left(3 + Ar^2\right)}{8\pi K\left(1 + Ar^2\right)^2}\,.$$

Integramos analíticamente dos modelos

$$\frac{\mathrm{d}P}{\mathrm{d}r} + (\rho + P)\frac{m + 4\pi r^3 P}{r(r - 2m)} - \frac{2}{r}\left(P_\perp - P\right) = 0\,,$$

Para una ecuación politropa generalizada

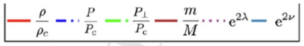
$$P = \kappa\rho^{1+\frac{1}{n}} + \alpha\rho - \beta$$

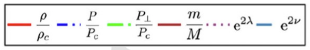
Propuesta 1

Propuesta 2

Table 1 Physical parameters for analytic polytropic solutions for EoS-1 and EoS-2, which fulfil the acceptability conditions. For EoS-1 the other α values correspond to: -0.10 ($n = 1.0$), -0.15 ($n = 1.5$) and -0.25 ($n = 2.0$). The parameter σ can take values between 0.10 and 0.18 approximately. The model EoS-1 could describe the mass of the millisecond pulsar in SR J1738+0333. The mass for the EoS-2 compact object is close to the lowest-mass-pulsar J0453+1559 companion ($1.174 \pm 0.004\ M_\odot$)

Input parameters	EoS-1	EoS-2
$\mu = 2M/r_b$	0.43	0.34
$\varkappa = \rho_b/\rho_c$	0.60	0.10
α	0.05 ($n = 0.5$)	0.05
Output parameters	**EoS-1**	**EoS-2**
$\rho_c \times 10^{15}$ (g/cm^3)	0.91	2.59
$\rho_b \times 10^{14}$ (g/cm^3)	5.46	2.59
M ($M_\odot$)	1.46	1.15





Y seguimos…

Suárez-Urango, D., Ospino, J., Hernández, H., & Núñez, L. A. (2022). Acceptability conditions and relativistic anisotropic generalized polytropes. *The European Physical Journal C*, *82*(2), 176.

Nos fuimos full numérico para la misma ecuación de estado barótropa

$$P = \kappa\rho^{1+\frac{1}{n}} + \alpha\rho - \beta$$

$$\dot{\Psi}(\xi) = -\frac{1}{\xi}\left[\frac{\left[\eta+\xi^3\mathcal{P}\right]\left[1+\mathcal{P}\Psi^{-n}\right]}{\xi-2\Upsilon(1+n)\eta} - \frac{2\Delta}{\rho_c\Upsilon(1+n)\Psi^n}\right]\left[1+\frac{\alpha n}{\Upsilon(1+n)\Psi}\right]^{-1}$$

$$\dot{\eta} = \xi^2\Psi^n$$

donde

$$r = a\xi\,, \quad \rho = \rho_c\Psi^n(\xi) \quad \text{y} \quad m = 4\pi a^3\rho_c\,\eta(\xi)\,,$$

C1 $2m/r < 1$ which implies

(a) that the metric potentials e^λ and e^ν are positive, finite and free from singularities within the matter distribution, satisfying $e^{\lambda_c} = 1$ and $e^{\nu_c} = \text{const}$ at the center of the configuration;

(b) the inner metric functions match to the exterior Schwarzschild solution at the boundary surface;

(c) the interior redshift should decrease with the increase of r [44, 45].

C2 Positive density and pressures, finite at the center of the configuration with $P_c = P_{\perp c}$ [45];

C3 $\rho' < 0$, $P' < 0$, $P'_\perp < 0$ with density and pressures having maximums at the center, thus $\rho'_c = P'_c = P'_{\perp c} = 0$, with $P_\perp \geq P$;

C4 The strong energy condition (SEC) for imperfect fluids, $\rho - P - 2P_\perp \geq 0$ [46, 47];

C5 The dynamic perturbation analysis restricts the adiabatic index [13, 48, 49, 50]

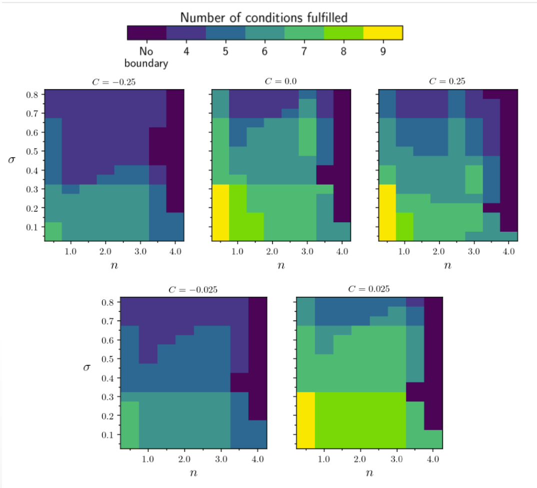
$$\Gamma = \frac{\rho + P}{P} v_s^2 \geq \frac{4}{3}\,.$$

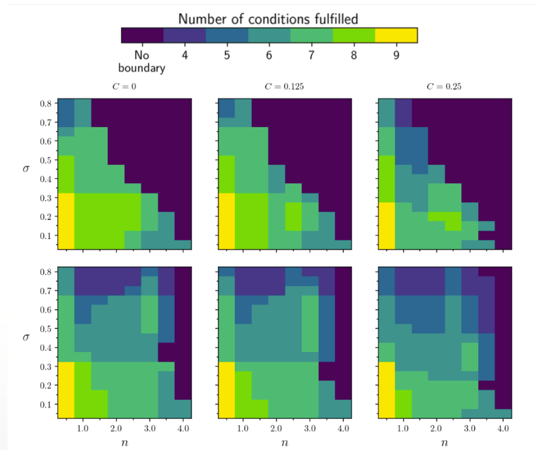
C6 Causality conditions on sound speeds: $0 < v_s^2 \leq 1$ and $0 < v_{s\perp}^2 \leq 1$;

C7 The Harrison-Zeldovich-Novikov stability condition: $\mathrm{d}M(\rho_c)/\mathrm{d}\rho_c > 0$ [51, 52].

C8 Cracking instability against local density perturbations, $\delta\rho = \delta\rho(r)$, briefly described in the next section and in references [37, 42, 53].

C9 The adiabatic convective stability condition $\rho'' \leq 0$, which is more restrictive than the outward decreasing density and pressure profiles [37].





Object				
Input parameters	J0737-3039 $n=0.50$	J1518+4904 $n=1.00$	GMn075 $n=0.75$	PMn075 $n=0.75$
C	0.09	0.125	0.05	0.05
α	-0.01	0.01	-0.01	0.0
$\varkappa$	0.05	0.15	0.17	0.0
σ	0.10	0.15	0.18	0.18
$\rho_c \times 10^{15}$ (g/cm^3)	0.66	1.79	1.41	1.41
Output parameters				
M ($M_\odot$)	1.33	1.56	1.50	1.56
R (km)	11.49	9.88	10.0	10.9
$2\mathcal{C}_\star$	0.34	0.47	0.44	0.42
$\rho_b \times 10^{14}$ (g/cm^3)	0.33	2.69	2.4	0.0
$k_{2\star}$	0.06	0.03	0.04	0.04
$\tilde{\Lambda}_{(1.4)*}$	165.30	50.80	63.60	70.40

Suárez-Urango, D., Ospino, J., Hernández, H., & Núñez, L. A. (2022). Acceptability conditions and relativistic anisotropic generalized polytropes. *The European Physical Journal C*, *82*(2), 176..

¿ Qué tienen que hacer Uds.?

Considere $\tilde{\rho} = (1 - \tilde{r}^2)$ e integre las ecuaciones de equilibrio hidrostático

Elija el caso newtoniano $\dfrac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}} = -\dfrac{\mu}{\kappa}\dfrac{\tilde{m}\tilde{\rho}}{\tilde{r}^2} + 2\dfrac{\tilde{P}_\perp - \tilde{P}}{\tilde{r}}$

Seleccione uno de los sabores de anisotropía $\Delta_{Newton},\ \Delta_{CH},\ \Delta_{HIM}$

Elija ahora el caso relativista $\dfrac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}} = -\dfrac{\mu}{\kappa}\dfrac{\tilde{m}\tilde{\rho}}{\tilde{r}^2}\left(1 + \kappa\dfrac{\tilde{P}}{\tilde{\rho}}\right)\left(1 + 3\eta\kappa\dfrac{\tilde{P}\tilde{r}^3}{\tilde{m}}\right)\left(1 - 2\mu\dfrac{\tilde{m}}{\tilde{r}}\right)^{-1} + 2\dfrac{\tilde{P}_\perp - \tilde{P}}{\tilde{r}}$

O la otra estrategia de adimensionalizar

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}\tilde{r}} = -\mu\frac{\tilde{m}\tilde{\rho}}{\tilde{r}^2}\left(1 + \frac{\tilde{P}}{\tilde{\rho}}\right)\left(1 + 3\xi\frac{\tilde{P}\tilde{r}^3}{\tilde{m}}\right)\left(1 - 2\mu\frac{\tilde{m}}{\tilde{r}}\right)^{-1} + 2\frac{\tilde{P}_\perp(\tilde{P}, r) - \tilde{P}}{\tilde{r}}$$

Seleccione alguno de los sabores de anisotropía relativista $\Delta_{JO},\ \Delta_{CH},\ \Delta_{HIM}$

Integre numéricamente para varios de los valores de los parámetros $\mu,\ \kappa,\ \eta$ o μ, ξ según el caso