

# Paréntesis de Poisson

**Luis A. Núñez**

*Escuela de Física, Facultad de Ciencias,  
Universidad Industrial de Santander, Santander, Colombia*



3 de diciembre de 2024

- Definición
- Propiedades
- Dos ejemplos
- Paréntesis de Poisson y Transformaciones Canónicas

- Sea una función  $f(q_i, p_i, t)$  en el espacio de fase  $(q_i, p_i)$ ,  $i = 1, \dots, s$ , de un sistema mecánico con un Hamiltoniano  $\mathcal{H}(q_i, p_i, t)$ .

- Sea una función  $f(q_i, p_i, t)$  en el espacio de fase  $(q_i, p_i)$ ,  $i = 1, \dots, s$ , de un sistema mecánico con un Hamiltoniano  $\mathcal{H}(q_i, p_i, t)$ .
- La derivada total de  $f$  es  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$ .

- Sea una función  $f(q_i, p_i, t)$  en el espacio de fase  $(q_i, p_i)$ ,  $i = 1, \dots, s$ , de un sistema mecánico con un Hamiltoniano  $\mathcal{H}(q_i, p_i, t)$ .
- La derivada total de  $f$  es  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$ .
- Sus ecuaciones de Hamilton son  $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$ ,  $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$

- Sea una función  $f(q_i, p_i, t)$  en el espacio de fase  $(q_i, p_i)$ ,  $i = 1, \dots, s$ , de un sistema mecánico con un Hamiltoniano  $\mathcal{H}(q_i, p_i, t)$ .
- La derivada total de  $f$  es  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$ .
- Sus ecuaciones de Hamilton son  $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$ ,  $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$
- Sustituyendo, tenemos  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) + \frac{\partial f}{\partial t}$

- Sea una función  $f(q_i, p_i, t)$  en el espacio de fase  $(q_i, p_i)$ ,  $i = 1, \dots, s$ , de un sistema mecánico con un Hamiltoniano  $\mathcal{H}(q_i, p_i, t)$ .
- La derivada total de  $f$  es  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$ .
- Sus ecuaciones de Hamilton son  $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$ ,  $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$ .
- Sustituyendo, tenemos  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) + \frac{\partial f}{\partial t}$ .
- Entonces el paréntesis de Poisson de la función  $f$  para un sistema con hamiltoniano  $\mathcal{H}$  es  $\{f, \mathcal{H}\} \equiv \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right)$ .

- Sea una función  $f(q_i, p_i, t)$  en el espacio de fase  $(q_i, p_i)$ ,  $i = 1, \dots, s$ , de un sistema mecánico con un Hamiltoniano  $\mathcal{H}(q_i, p_i, t)$ .
- La derivada total de  $f$  es  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$ .
- Sus ecuaciones de Hamilton son  $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$ ,  $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$
- Sustituyendo, tenemos  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) + \frac{\partial f}{\partial t}$
- Entonces el paréntesis de Poisson de la función  $f$  para un sistema con hamiltoniano  $\mathcal{H}$  es  $\{f, \mathcal{H}\} \equiv \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right)$ .
- Con lo cual  $\frac{df}{dt} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$



- Sea una función  $f(q_i, p_i, t)$  en el espacio de fase  $(q_i, p_i)$ ,  $i = 1, \dots, s$ , de un sistema mecánico con un Hamiltoniano  $\mathcal{H}(q_i, p_i, t)$ .
- La derivada total de  $f$  es  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$ .
- Sus ecuaciones de Hamilton son  $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$ ,  $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$
- Sustituyendo, tenemos  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) + \frac{\partial f}{\partial t}$
- Entonces el paréntesis de Poisson de la función  $f$  para un sistema con hamiltoniano  $\mathcal{H}$  es  $\{f, \mathcal{H}\} \equiv \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right)$ .
- Con lo cual  $\frac{df}{dt} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$
- Si  $f$  no depende explícitamente del tiempo (cantidad conservada), tenemos  $\{f, \mathcal{H}\} = 0$

- Sea una función  $f(q_i, p_i, t)$  en el espacio de fase  $(q_i, p_i)$ ,  $i = 1, \dots, s$ , de un sistema mecánico con un Hamiltoniano  $\mathcal{H}(q_i, p_i, t)$ .
- La derivada total de  $f$  es  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t}$ .
- Sus ecuaciones de Hamilton son  $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$ ,  $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$ .
- Sustituyendo, tenemos  $\frac{df}{dt} = \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right) + \frac{\partial f}{\partial t}$ .
- Entonces el paréntesis de Poisson de la función  $f$  para un sistema con hamiltoniano  $\mathcal{H}$  es  $\{f, \mathcal{H}\} \equiv \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} \right)$ .
- Con lo cual  $\frac{df}{dt} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}$ .
- Si  $f$  no depende explícitamente del tiempo (cantidad conservada), tenemos  $\{f, \mathcal{H}\} = 0$ .
- Para  $f(q_i, p_i, t)$  y  $g(q_i, p_i, t)$  podemos definir el paréntesis de Poisson de  $f$  y  $g$  como  $\{f, g\} \equiv \sum_{i=1}^s \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$ .

- El paréntesis de Poisson puede ser considerado como una operación entre dos funciones definidas en un espacio algebraico que asigna otra función en ese espacio.

- El paréntesis de Poisson puede ser considerado como una operación entre dos funciones definidas en un espacio algebraico que asigna otra función en ese espacio.
- El paréntesis de Poisson, como operación algebraica, posee las siguientes propiedades (características de un álgebra de Lie):
  - $\{f, g\} = -\{g, f\}$ ,  $\{f, f\} = 0$  (antisimetría).
  - $\{f, c\} = 0$ , si  $c = \text{cte}$ .
  - $\{af_1 + bf_2, g\} = a\{f_1, g\} + b\{f_2, g\}$ ,  $a, b = \text{ctes}$ , un operador lineal.
  - $\{f_1 f_2, g\} = f_1 \{f_2, g\} + f_2 \{f_1, g\}$ , (no asociativo).
  - $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$  la identidad de Jacobi.

- Como  $p_i$  y  $q_i$  representan coordenadas independientes tenemos

$$\{q_i, f\} = \sum_{k=1}^s \left( \frac{\partial q_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = \sum_{k=1}^s \delta_{ik} \frac{\partial f}{\partial p_k} = \frac{\partial f}{\partial p_i}$$

$$\{p_i, f\} = \sum_{k=1}^s \left( \frac{\partial p_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = - \sum_{k=1}^s \delta_{ik} \frac{\partial f}{\partial q_k} = - \frac{\partial f}{\partial q_i}$$

- Como  $p_i$  y  $q_i$  representan coordenadas independientes tenemos

$$\{q_i, f\} = \sum_{k=1}^s \left( \frac{\partial q_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \cancel{\frac{\partial q_i}{\partial p_k} \frac{\partial f}{\partial q_k}}^0 \right) = \sum_{k=1}^s \delta_{ik} \frac{\partial f}{\partial p_k} = \frac{\partial f}{\partial p_i}$$

$$\{p_i, f\} = \sum_{k=1}^s \left( \cancel{\frac{\partial p_i}{\partial q_k} \frac{\partial f}{\partial p_k}}^0 - \frac{\partial p_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = - \sum_{k=1}^s \delta_{ik} \frac{\partial f}{\partial q_k} = - \frac{\partial f}{\partial q_i}.$$

- Si  $f = p_j$ , ó  $f = q_j$ ,  $\Rightarrow \{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}$

- Como  $p_i$  y  $q_i$  representan coordenadas independientes tenemos

$$\{q_i, f\} = \sum_{k=1}^s \left( \frac{\partial q_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = \sum_{k=1}^s \delta_{ik} \frac{\partial f}{\partial p_k} = \frac{\partial f}{\partial p_i}$$

$$\{p_i, f\} = \sum_{k=1}^s \left( \frac{\partial p_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = - \sum_{k=1}^s \delta_{ik} \frac{\partial f}{\partial q_k} = - \frac{\partial f}{\partial q_i}.$$

- Si  $f = p_j$ , ó  $f = q_j$ ,  $\Rightarrow \{q_i, q_j\} = 0$ ,  $\{p_i, p_j\} = 0$ ,  $\{q_i, p_j\} = \delta_{ij}$
- Las ecuaciones de Hamilton pueden escribirse como

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \{q_i, \mathcal{H}\} \quad \dot{p}_i = - \frac{\partial \mathcal{H}}{\partial q_i} = \{p_i, \mathcal{H}\}$$

- Como  $p_i$  y  $q_i$  representan coordenadas independientes tenemos

$$\{q_i, f\} = \sum_{k=1}^s \left( \frac{\partial q_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = \sum_{k=1}^s \delta_{ik} \frac{\partial f}{\partial p_k} = \frac{\partial f}{\partial p_i}$$

$$\{p_i, f\} = \sum_{k=1}^s \left( \frac{\partial p_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = - \sum_{k=1}^s \delta_{ik} \frac{\partial f}{\partial q_k} = - \frac{\partial f}{\partial q_i}.$$

- Si  $f = p_j$ , ó  $f = q_j$ ,  $\Rightarrow \{q_i, q_j\} = 0$ ,  $\{p_i, p_j\} = 0$ ,  $\{q_i, p_j\} = \delta_{ij}$
- Las ecuaciones de Hamilton pueden escribirse como
$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \{q_i, \mathcal{H}\} \quad \dot{p}_i = - \frac{\partial \mathcal{H}}{\partial q_i} = \{p_i, \mathcal{H}\}$$
- En Mecánica Cuántica, la operación  $[A, B] = AB - BA$  es el conmutador de los operadores u observables  $A$  y  $B$ .



- Como  $p_i$  y  $q_i$  representan coordenadas independientes tenemos

$$\{q_i, f\} = \sum_{k=1}^s \left( \frac{\partial q_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = \sum_{k=1}^s \delta_{ik} \frac{\partial f}{\partial p_k} = \frac{\partial f}{\partial p_i}$$

$$\{p_i, f\} = \sum_{k=1}^s \left( \frac{\partial p_i}{\partial q_k} \frac{\partial f}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial f}{\partial q_k} \right) = - \sum_{k=1}^s \delta_{ik} \frac{\partial f}{\partial q_k} = - \frac{\partial f}{\partial q_i}.$$

- Si  $f = p_j$ , ó  $f = q_j$ ,  $\Rightarrow \{q_i, q_j\} = 0$ ,  $\{p_i, p_j\} = 0$ ,  $\{q_i, p_j\} = \delta_{ij}$
- Las ecuaciones de Hamilton pueden escribirse como
$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \{q_i, \mathcal{H}\} \quad \dot{p}_i = - \frac{\partial \mathcal{H}}{\partial q_i} = \{p_i, \mathcal{H}\}$$
- En Mecánica Cuántica, la operación  $[A, B] = AB - BA$  es el conmutador de los operadores u observables  $A$  y  $B$ .
- La estructura algebraica de la Mecánica Clásica, expresada en las propiedades de los paréntesis de Poisson, se preserva en la Mecánica Cuántica. En particular,  $\{q_i, p_j\} = i\hbar\delta_{ij}$ , donde  $\hbar$  es la constante de Planck (dividida por  $2\pi$ ).

- Calcular  $\{r, \mathbf{p}\}$ , donde  $r = (x^2 + y^2 + z^2)^{1/2}$ 
  - $\{r, \mathbf{p}\} = \{r, p_x\} \hat{\mathbf{x}} + \{r, p_y\} \hat{\mathbf{y}} + \{r, p_z\} \hat{\mathbf{z}}$
  - $\{r, p_x\} = \sum_{i=1}^3 \left( \frac{\partial r}{\partial q_i} \frac{\partial p_x}{\partial p_i} - \frac{\partial r}{\partial p_i} \frac{\partial p_x}{\partial q_i} \right) = \frac{\partial r}{\partial x} \frac{\partial p_x}{\partial p_x} = \frac{x}{r}$
  - $\{r, p_y\} = \frac{y}{r}, \quad \{r, p_z\} = \frac{z}{r}$
  - luego  $\{r, \mathbf{p}\} = \frac{x}{r} \hat{\mathbf{x}} + \frac{y}{r} \hat{\mathbf{y}} + \frac{z}{r} \hat{\mathbf{z}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$

- Calcular  $\{r, \mathbf{p}\}$ , donde  $r = (x^2 + y^2 + z^2)^{1/2}$

- $\{r, \mathbf{p}\} = \{r, p_x\} \hat{\mathbf{x}} + \{r, p_y\} \hat{\mathbf{y}} + \{r, p_z\} \hat{\mathbf{z}}$
- $\{r, p_x\} = \sum_{i=1}^3 \left( \frac{\partial r}{\partial q_i} \frac{\partial p_x}{\partial p_i} - \frac{\partial r}{\partial p_i} \frac{\partial p_x}{\partial q_i} \right) = \frac{\partial r}{\partial x} \frac{\partial p_x}{\partial p_x} = \frac{x}{r}$
- $\{r, p_y\} = \frac{y}{r}, \quad \{r, p_z\} = \frac{z}{r}$
- luego  $\{r, \mathbf{p}\} = \frac{x}{r} \hat{\mathbf{x}} + \frac{y}{r} \hat{\mathbf{y}} + \frac{z}{r} \hat{\mathbf{z}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$

- Dadas las componentes del momento angular  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ :

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x$$

Calcular los paréntesis de Poisson para las componentes de  $\mathbf{p}$  y  $\mathbf{L}$ :

$$\{p_y, L_x\} = -\frac{\partial L_x}{\partial y} = -p_z; \quad \{p_x, L_x\} = -\frac{\partial L_x}{\partial x} = 0;$$

$$\{p_z, L_y\} = -\frac{\partial L_y}{\partial z} = -p_x$$

$$\{L_x, L_y\} = \sum_{i=1}^3 \left( \frac{\partial L_x}{\partial q_i} \frac{\partial L_y}{\partial p_i} - \frac{\partial L_x}{\partial p_i} \frac{\partial L_y}{\partial q_i} \right)$$

$$\{L_x, L_y\} =$$

$$\left( \frac{\partial L_x}{\partial x} \frac{\partial L_y}{\partial p_x} - \frac{\partial L_x}{\partial p_x} \frac{\partial L_y}{\partial x} \right) + \left( \frac{\partial L_x}{\partial y} \frac{\partial L_y}{\partial p_y} - \frac{\partial L_x}{\partial p_y} \frac{\partial L_y}{\partial y} \right) + \left( \frac{\partial L_x}{\partial z} \frac{\partial L_y}{\partial p_z} - \frac{\partial L_x}{\partial p_z} \frac{\partial L_y}{\partial z} \right)$$

$$\{L_x, L_y\} = xp_y - yp_x = L_z; \quad \{L_y, L_z\} = L_x \quad \{L_z, L_x\} = L_y$$

- Calcular  $\{r, \mathbf{p}\}$ , donde  $r = (x^2 + y^2 + z^2)^{1/2}$

- $\{r, \mathbf{p}\} = \{r, p_x\} \hat{\mathbf{x}} + \{r, p_y\} \hat{\mathbf{y}} + \{r, p_z\} \hat{\mathbf{z}}$

- $\{r, p_x\} = \sum_{i=1}^3 \left( \frac{\partial r}{\partial q_i} \frac{\partial p_x}{\partial p_i} - \frac{\partial r}{\partial p_i} \frac{\partial p_x}{\partial q_i} \right) = \frac{\partial r}{\partial x} \frac{\partial p_x}{\partial p_x} = \frac{x}{r}$

- $\{r, p_y\} = \frac{y}{r}, \quad \{r, p_z\} = \frac{z}{r}$

- luego  $\{r, \mathbf{p}\} = \frac{x}{r} \hat{\mathbf{x}} + \frac{y}{r} \hat{\mathbf{y}} + \frac{z}{r} \hat{\mathbf{z}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$

- Dadas las componentes del momento angular  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ :

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x$$

Calcular los paréntesis de Poisson para las componentes de  $\mathbf{p}$  y  $\mathbf{L}$ :

$$\{p_y, L_x\} = -\frac{\partial L_x}{\partial y} = -p_z; \quad \{p_x, L_x\} = -\frac{\partial L_x}{\partial x} = 0;$$

$$\{p_z, L_y\} = -\frac{\partial L_y}{\partial z} = -p_x$$

$$\{L_x, L_y\} = \sum_{i=1}^3 \left( \frac{\partial L_x}{\partial q_i} \frac{\partial L_y}{\partial p_i} - \frac{\partial L_x}{\partial p_i} \frac{\partial L_y}{\partial q_i} \right)$$

$$\{L_x, L_y\} =$$

$$\left( \frac{\partial L_x}{\partial x} \frac{\partial L_y}{\partial p_x} - \frac{\partial L_x}{\partial p_x} \frac{\partial L_y}{\partial x} \right) + \left( \frac{\partial L_x}{\partial y} \frac{\partial L_y}{\partial p_y} - \frac{\partial L_x}{\partial p_y} \frac{\partial L_y}{\partial y} \right) + \left( \frac{\partial L_x}{\partial z} \frac{\partial L_y}{\partial p_z} - \frac{\partial L_x}{\partial p_z} \frac{\partial L_y}{\partial z} \right)$$

$$\{L_x, L_y\} = xp_y - yp_x = L_z; \quad \{L_y, L_z\} = L_x \quad \{L_z, L_x\} = L_y$$

- Entonces,  $\{L_i, L_j\} = \epsilon_{ijk} L_k$ . En Mecánica Cuántica,  $[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$ .

Para una transformación canónica  $Q_j = Q_j(q_k, p_k, t)$  y  $P_i = P_i(q_k, p_k, t)$  se cumple que  $\{Q_i, Q_j\} = 0$ ,  $\{P_i, P_j\} = 0$ ,  $\{Q_i, P_j\} = \delta_{ij}$

- Sea  $Q = q + e^P$  &  $P = p$

Para una transformación canónica  $Q_j = Q_j(q_k, p_k, t)$  y  $P_i = P_i(q_k, p_k, t)$  se cumple que  $\{Q_i, Q_j\} = 0$ ,  $\{P_i, P_j\} = 0$ ,  $\{Q_i, P_j\} = \delta_{ij}$

- Sea  $Q = q + e^p$  &  $P = p$
- $\{Q, Q\} = \frac{\partial Q}{\partial q} \frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial Q}{\partial q} = (1)(e^p) - (e^p)(1) = 0$ ;

Para una transformación canónica  $Q_j = Q_j(q_k, p_k, t)$  y  $P_i = P_i(q_k, p_k, t)$  se cumple que  $\{Q_i, Q_j\} = 0$ ,  $\{P_i, P_j\} = 0$ ,  $\{Q_i, P_j\} = \delta_{ij}$

- Sea  $Q = q + e^p$  &  $P = p$
- $\{Q, Q\} = \frac{\partial Q}{\partial q} \frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial Q}{\partial q} = (1)(e^p) - (e^p)(1) = 0$ ;
- $\{P, P\} = \frac{\partial P}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial P}{\partial p} \frac{\partial P}{\partial q} = (0)(1) - (1)(0) = 0$ ; y finalmente

Para una transformación canónica  $Q_j = Q_j(q_k, p_k, t)$  y  $P_i = P_i(q_k, p_k, t)$  se cumple que  $\{Q_i, Q_j\} = 0$ ,  $\{P_i, P_j\} = 0$ ,  $\{Q_i, P_j\} = \delta_{ij}$

- Sea  $Q = q + e^p$  &  $P = p$
- $\{Q, Q\} = \frac{\partial Q}{\partial q} \frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial Q}{\partial q} = (1)(e^p) - (e^p)(1) = 0$ ;
- $\{P, P\} = \frac{\partial P}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial P}{\partial p} \frac{\partial P}{\partial q} = (0)(1) - (1)(0) = 0$ ; y finalmente
- $\{Q, P\} = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q}$ . Calculando cada una  $\frac{\partial Q}{\partial q} = 1$ ,  $\frac{\partial Q}{\partial p} = e^p$ ,  $\frac{\partial P}{\partial q} = 0$ , y  $\frac{\partial P}{\partial p} = 1$ , obtenemos  $\{Q, P\} = (1)(1) - (e^p)(0) = 1$ .



Para una transformación canónica  $Q_j = Q_j(q_k, p_k, t)$  y  $P_i = P_i(q_k, p_k, t)$  se cumple que  $\{Q_i, Q_j\} = 0$ ,  $\{P_i, P_j\} = 0$ ,  $\{Q_i, P_j\} = \delta_{ij}$

- Sea  $Q = q + e^P$  &  $P = p$
- $\{Q, Q\} = \frac{\partial Q}{\partial q} \frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial Q}{\partial q} = (1)(e^P) - (e^P)(1) = 0$ ;
- $\{P, P\} = \frac{\partial P}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial P}{\partial p} \frac{\partial P}{\partial q} = (0)(1) - (1)(0) = 0$ ; y finalmente
- $\{Q, P\} = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q}$ . Calculando cada una  $\frac{\partial Q}{\partial q} = 1$ ,  $\frac{\partial Q}{\partial p} = e^P$ ,  $\frac{\partial P}{\partial q} = 0$ , y  $\frac{\partial P}{\partial p} = 1$ , obtenemos  $\{Q, P\} = (1)(1) - (e^P)(0) = 1$ .
- Por lo tanto, como la transformación cumple con  $\{Q, Q\} = 0$ ,  $\{P, P\} = 0$ , y  $\{Q, P\} = 1$ . **Es canónica**

Consideremos la siguiente transformación

$$P_1 = p_1 - 2p_2, \quad Q_1 = q_1 \text{ y } P_2 = -2q_1 - q_2, \quad Q_2 = p_2$$

- $\bullet \{Q_1, P_1\} = \sum_{i=1}^s \left( \frac{\partial Q_1}{\partial q_i} \frac{\partial P_1}{\partial p_i} - \frac{\partial Q_1}{\partial p_i} \frac{\partial P_1}{\partial q_i} \right)$ 
$$\{Q_1, P_1\} = \left( \frac{\partial Q_1}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left( \frac{\partial Q_1}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 1$$

Consideremos la siguiente transformación

$$P_1 = p_1 - 2p_2, \quad Q_1 = q_1 \text{ y } P_2 = -2q_1 - q_2, \quad Q_2 = p_2$$

- $\bullet \{Q_1, P_1\} = \sum_{i=1}^s \left( \frac{\partial Q_1}{\partial q_i} \frac{\partial P_1}{\partial p_i} - \frac{\partial Q_1}{\partial p_i} \frac{\partial P_1}{\partial q_i} \right)$ 
$$\{Q_1, P_1\} = \left( \frac{\partial Q_1}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left( \frac{\partial Q_1}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 1$$
- $\bullet \{Q_2, P_2\} = \left( \frac{\partial Q_2}{\partial q_1} \frac{\partial P_2}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_2}{\partial q_1} \right) + \left( \frac{\partial Q_2}{\partial q_2} \frac{\partial P_2}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_2}{\partial q_2} \right) = 1$

Consideremos la siguiente transformación

$$P_1 = p_1 - 2p_2, \quad Q_1 = q_1 \text{ y } P_2 = -2q_1 - q_2, \quad Q_2 = p_2$$

- $\bullet \{Q_1, P_1\} = \sum_{i=1}^s \left( \frac{\partial Q_1}{\partial q_i} \frac{\partial P_1}{\partial p_i} - \frac{\partial Q_1}{\partial p_i} \frac{\partial P_1}{\partial q_i} \right)$ 
$$\{Q_1, P_1\} = \left( \frac{\partial Q_1}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left( \frac{\partial Q_1}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 1$$
- $\bullet \{Q_2, P_2\} = \left( \frac{\partial Q_2}{\partial q_1} \frac{\partial P_2}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_2}{\partial q_1} \right) + \left( \frac{\partial Q_2}{\partial q_2} \frac{\partial P_2}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_2}{\partial q_2} \right) = 1$
- $\bullet \{Q_2, P_1\} = \left( \frac{\partial Q_2}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left( \frac{\partial Q_2}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 0$

Consideremos la siguiente transformación

$$P_1 = p_1 - 2p_2, \quad Q_1 = q_1 \text{ y } P_2 = -2q_1 - q_2, \quad Q_2 = p_2$$

- $\bullet \{Q_1, P_1\} = \sum_{i=1}^s \left( \frac{\partial Q_1}{\partial q_i} \frac{\partial P_1}{\partial p_i} - \frac{\partial Q_1}{\partial p_i} \frac{\partial P_1}{\partial q_i} \right)$ 
$$\{Q_1, P_1\} = \left( \frac{\partial Q_1}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left( \frac{\partial Q_1}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 1$$
- $\bullet \{Q_2, P_2\} = \left( \frac{\partial Q_2}{\partial q_1} \frac{\partial P_2}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_2}{\partial q_1} \right) + \left( \frac{\partial Q_2}{\partial q_2} \frac{\partial P_2}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_2}{\partial q_2} \right) = 1$
- $\bullet \{Q_2, P_1\} = \left( \frac{\partial Q_2}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left( \frac{\partial Q_2}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 0$
- $\bullet \{Q_1, P_2\} = \left( \frac{\partial Q_1}{\partial q_1} \frac{\partial P_2}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_2}{\partial q_1} \right) + \left( \frac{\partial Q_1}{\partial q_2} \frac{\partial P_2}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_2}{\partial q_2} \right) = 0$

Consideremos la siguiente transformación

$$P_1 = p_1 - 2p_2, \quad Q_1 = q_1 \text{ y } P_2 = -2q_1 - q_2, \quad Q_2 = p_2$$

- $\bullet \{Q_1, P_1\} = \sum_{i=1}^s \left( \frac{\partial Q_1}{\partial q_i} \frac{\partial P_1}{\partial p_i} - \frac{\partial Q_1}{\partial p_i} \frac{\partial P_1}{\partial q_i} \right)$ 
$$\{Q_1, P_1\} = \left( \frac{\partial Q_1}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left( \frac{\partial Q_1}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 1$$
- $\bullet \{Q_2, P_2\} = \left( \frac{\partial Q_2}{\partial q_1} \frac{\partial P_2}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_2}{\partial q_1} \right) + \left( \frac{\partial Q_2}{\partial q_2} \frac{\partial P_2}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_2}{\partial q_2} \right) = 1$
- $\bullet \{Q_2, P_1\} = \left( \frac{\partial Q_2}{\partial q_1} \frac{\partial P_1}{\partial p_1} - \frac{\partial Q_2}{\partial p_1} \frac{\partial P_1}{\partial q_1} \right) + \left( \frac{\partial Q_2}{\partial q_2} \frac{\partial P_1}{\partial p_2} - \frac{\partial Q_2}{\partial p_2} \frac{\partial P_1}{\partial q_2} \right) = 0$
- $\bullet \{Q_1, P_2\} = \left( \frac{\partial Q_1}{\partial q_1} \frac{\partial P_2}{\partial p_1} - \frac{\partial Q_1}{\partial p_1} \frac{\partial P_2}{\partial q_1} \right) + \left( \frac{\partial Q_1}{\partial q_2} \frac{\partial P_2}{\partial p_2} - \frac{\partial Q_1}{\partial p_2} \frac{\partial P_2}{\partial q_2} \right) = 0$
- $\bullet$  Por lo tanto, la transformación **Es canónica**