Matrix of a rotation in the 3D space

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A rotation in a three-dimensional vector space is defined by a unitary vector $\vec{u}$ that indicates the direction of the rotation axis and by a rotation angle θ. A base of this space $B = (\vec{\imath}, \vec{\jmath}, \vec{k})$ constitutes a set of unitary and pairwise orthogonal vectors. This document shows how to write the matrix $\boldsymbol{R}_{\vec{u},\theta}$ in the base B.

The projection $\vec{M}_{\|}$ of any vector $\vec{M}$ on $\vec{u}$ is invariant by rotation around $\vec{u}$.

$$\vec{M}_{\|} = (\vec{M} \cdot \vec{u}).\vec{u}$$

The vector $\vec{M}_{\perp}$ defined by:

$$\vec{M}_{\perp} = \vec{M} - \vec{M}_{\|} = \vec{M} - (\vec{M} \cdot \vec{u}).\vec{u}$$

is orthogonal to $\vec{u}$ and therefore to $\vec{M}_{\|}$:

$$(\vec{M} - (\vec{M} \cdot \vec{u}).\vec{u}) \cdot \vec{u} = \vec{M} \cdot \vec{u} - (\vec{M} \cdot \vec{u}).\vec{u} \cdot \vec{u} = 0$$

because $\vec{u} \cdot \vec{u} = 1$. $\vec{M}_{\perp}$ rotates in the plane that is orthogonal to the rotation axis. The vector $\vec{M}_{\otimes}$ defined by:

$$\vec{M}_{\otimes} = \vec{u} \wedge \vec{M}_{\perp}$$

it orthogonal to $\vec{M}_{\|}$ and to $\vec{M}_{\perp}$. It can be also written as

$$\vec{M}_{\otimes} = \vec{u} \wedge \vec{M}$$

because

$$(\vec{M} \cdot \vec{u}).\vec{u} \wedge \vec{u} = 0.$$

From the definitions of $\vec{M}_{\|}$, $\vec{M}_{\perp}$, and $\vec{M}_{\otimes}$:

$$\begin{aligned} \vec{M}_{\perp} \cdot \vec{u} &= 0 \\ \vec{M}_{\otimes} \cdot \vec{u} &= 0 \\ \vec{M}_{\perp} \cdot \vec{M}_{\otimes} &= 0 \\ ||\vec{M}_{\perp}|| &= ||\vec{M}_{\otimes}|| \end{aligned}$$

Therefore, the vector

$$\vec{M} = \vec{M}_{\|} + \vec{M}_{\perp}$$

is transformed by the rotation into

$$\begin{aligned}\vec{M}_{\text{rot}} &= \vec{M}_{\|} + \cos\theta.\vec{M}_{\perp} + \sin\theta.\vec{M}_{\otimes} \\ &= \cos\theta.\vec{M} + (1-\cos\theta).(\vec{M}\cdot\vec{u}).\vec{u} + \sin\theta.\vec{u}\wedge\vec{M}\end{aligned}$$

because $\vec{M}_{\|}$ is invariant by rotation around $\vec{u}$ and $\vec{M}_{\perp}$ is rotated by θ in the vector plane defined by $\vec{M}_{\perp}$ and $\vec{M}_{\otimes}$, two orthogonal vectors of the same norm. A rotation by $\theta = \pi/2$ transforms $\vec{M}_{\perp}$ into $\vec{M}_{\otimes}$ as expected.

The substitution of $\vec{M}$ by $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ provides the content of the columns of the three terms in $\boldsymbol{R}_{\vec{u},\theta}$ as a function of θ and of u_x, u_y, u_z (with $u_x^2+u_y^2+u_z^2 = 1$), the coordinates of $\vec{u}$.

For example, $\vec{\imath}\cdot\vec{u} = u_x$ so that

$$(\vec{\imath}\cdot\vec{u}).\vec{u} = \begin{pmatrix} u_x^2 \\ u_x u_y \\ u_x u_z \end{pmatrix}$$

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As well,

$$\vec{u}\wedge\vec{\imath} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ u_z \\ -u_y \end{pmatrix}$$

Finally,

$$\begin{aligned}\boldsymbol{R}_{\vec{u},\theta} &= \cos\theta\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (1-\cos\theta)\begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{pmatrix} \\ &\quad + \sin\theta\begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}\end{aligned}$$

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