REGULAR OPERATIONS ON LANGUAGES

iACADEMY
SCHOOL OF COMPUTING • SCHOOL OF BUSINESS • SCHOOL OF DESIGN

SCHOOL OF COMPUTING
MITCH M. ANDAYA

UNION OPERATION

- Suppose there are two languages *X* and *Y*.

- The ***union*** of the languages *X* and *Y* (written as $X \cup Y$) is a language that is composed of all strings *w* such that *w* is a string from language *X* or *w* is a string from language *Y*.

- Mathematically,

$$X \cup Y = \{w \mid w \in X \text{ or } w \in Y\}$$

UNION OPERATION

- Example

  Language $X = \{aa, bb\}$
  Language $Y = \{bb, cc, dd\}$.

  The union of X and Y is:

  $$X \cup Y = \{aa, bb, cc, dd\}$$

iACADEMY
SCHOOL OF COMPUTING • SCHOOL OF BUSINESS • SCHOOL OF DESIGN

CONCATENATION OPERATION

- Suppose there are two languages *X* and *Y*.

- The concatenation of languages *X* and *Y* (written as $X \circ Y$) is a language that is composed of all strings $w = xy$ such that x is a string from language *X* and y is a string from language *Y*.

- Mathematically,

$$X \circ Y = \{xy \mid x \in X \text{ and } y \in Y\}$$

CONCATENATION OPERATION

- Example

Language X = {aa, bb}
Language Y = {bb, cc, dd}.

The concatenation of X and Y is:

$$X \circ Y = \{aabb, aacc, aadd, bbbb, bbcc, bbdd\}$$

STAR OPERATION

- Suppose there is a language *X*.
- The *star* of language *X* (written as *X**) is a language that is composed of all strings that are formed by concatenating 0 or more strings of *X*. The star operation is also called the ***Kleene Closure***.
- The star of any language includes the empty string *ε* and is always infinite.

STAR OPERATION

- Example

Language $X = \{aa, bb\}$

The star of X is:

$$X^* = \{\varepsilon, aa, bb, aaaa, aabb, aabbaa, bbbb, bbaabb, \ldots\}$$

iACADEMY
SCHOOL OF COMPUTING • SCHOOL OF BUSINESS • SCHOOL OF DESIGN

CLOSURE PROPERTY OF REGULAR LANGUAGES

- A set is said to be ***closed*** under a certain operation if performing that operation on the elements of the set produces an object that is still a member of that set.

- For example, the set of integers is closed under multiplication since multiplying integers always produces an integer.

- The set of integers is not closed under division since dividing two integers does not always produce another integer.

CLOSURE PROPERTY OF REGULAR LANGUAGES

- Closure Under The Regular Operations

  Is the family of regular languages closed under union, concatenation and star operations? Specifically,

  1. Is the union of regular languages also regular?
  2. Is the concatenation of regular languages also regular?
  3. Is the star of a regular language also regular?

CLOSURE PROPERTY OF UNION

- Is the union of regular languages also regular?

  Assume that L_1 is a regular language recognized by the NFA N_1:

  $$N_1 = \{Q_1, \Sigma_1, \delta_1, q_1, F_1\}$$

  Assume that L_2 is also a regular language recognized by the NFA N_2:

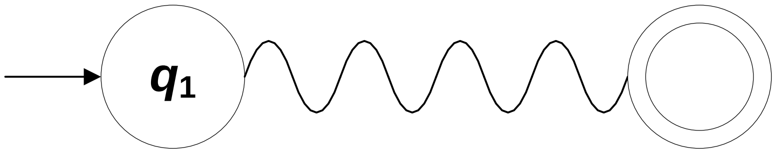
  $$N_2 = \{Q_2, \Sigma_2, \delta_2, q_2, F_2\}$$

  If the union $L_1 \cup L_2$ is regular, then there must be an NFA that recognizes it. Let this NFA be N_3:

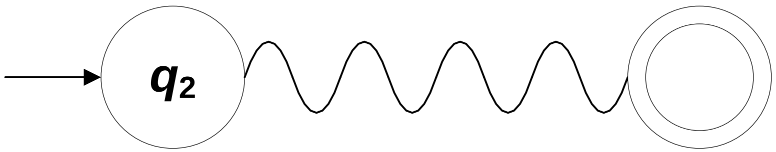
  $$N_3 = \{Q_3, \Sigma_3, \delta_3, q_3, F_3\}$$

CLOSURE PROPERTY OF UNION

- The following are the state diagrams for N_1 and N_2:



NFA N_1



NFA N_2

The NFA N_3 for the union of L_1 and L_2 should be able to accept a string if it is a member of either L_1 or L_2.

In other words, N_3 should accept an input string if it is accepted by either N_1 or N_2.

CLOSURE PROPERTY OF UNION

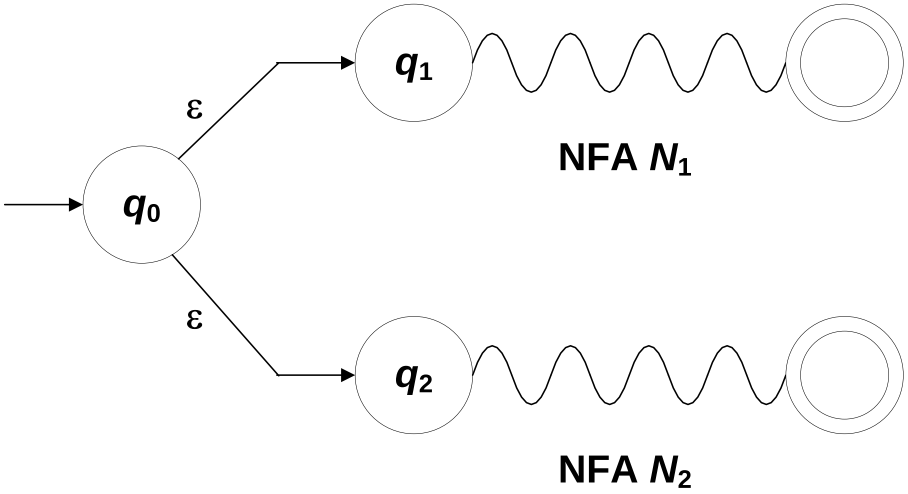
For every input string, NFA N_3 should be able to start two parallel computations.

One computation will try to see if the string belongs to L_1 (it is accepted by N_1).

The other computation will try to see if the string belongs to L_2 (it is accepted by N_2).

Therefore, N_3 will then be a combination of N_1 and N_2.

The state diagram for N_3 will be:



CLOSURE PROPERTY OF UNION

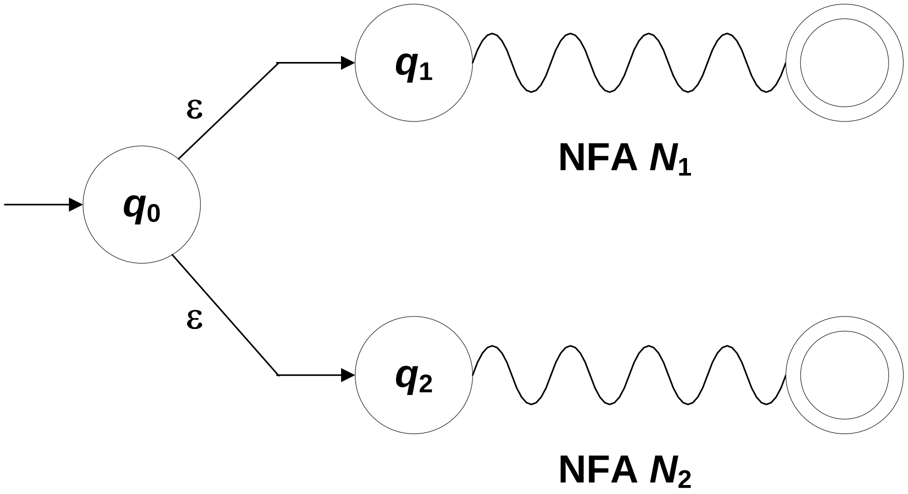
Because of the ε-transitions from the start state q_0 to state q_1 (the start state of N_1) and state q_2 (the start state of N_2), N_3 automatically starts two computations.

One computation "simulates" N_1 to determine if the input string is a member of L_1.

The second computation "simulates" N_2 to determine if the input string is a member of L_2.

If either computation ends up in a final state, then N_3 accepts the input string.

The state diagram for N_3 will be:

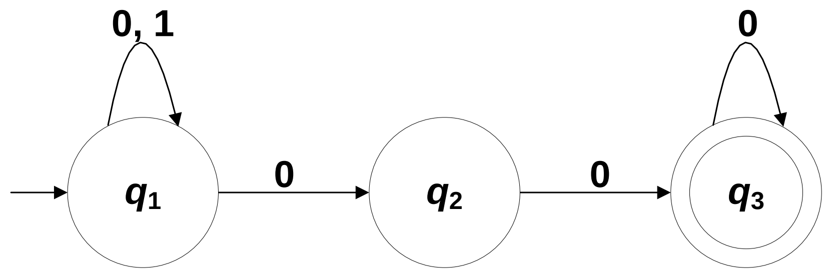


CLOSURE PROPERTY OF UNION

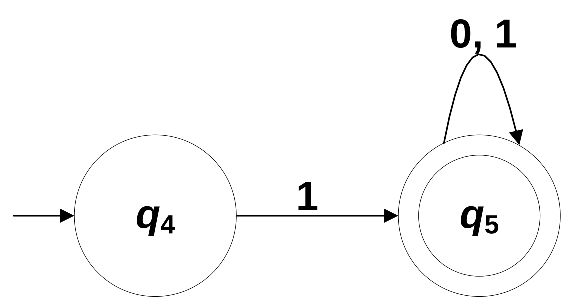
- Example:

  Let $L_1 = \{w | w \text{ ends with a 00}\}$ and $L_2 = \{w | w \text{ starts with a 1}\}$

NFA N_1 for L_1:

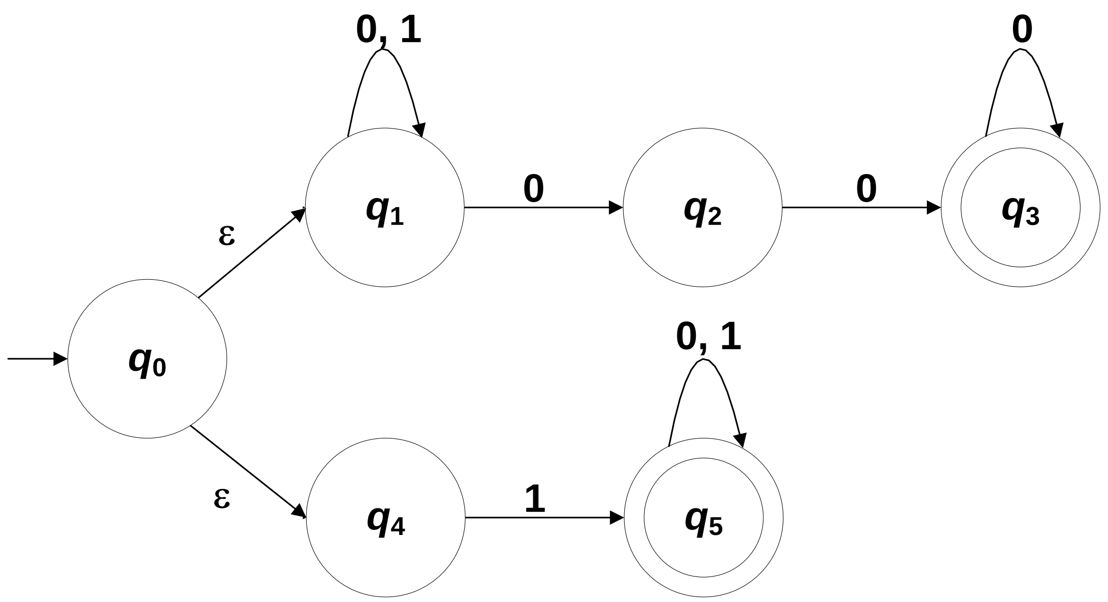


NFA N_2 for L_2:



CLOSURE PROPERTY OF UNION

NFA N_3 for L_3:



Therefore, $L_1 \cup L_2$ is a regular language.

CLOSURE PROPERTY OF CONCATENATION

- Is the concatenation of regular languages also regular?

Assume that L_1 is a regular language recognized by the NFA N_1:

$$N_1 = \{Q_1, \Sigma_1, \delta_1, q_1, F_1\}$$

Assume that L_2 is also a regular language recognized by the NFA N_2:

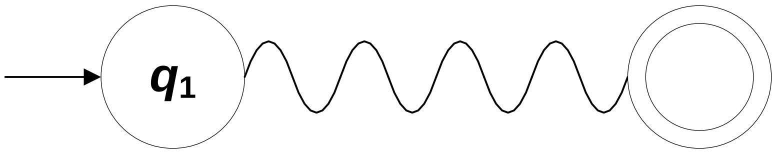
$$N_2 = \{Q_2, \Sigma_2, \delta_2, q_2, F_2\}$$

If the concatenation $L_1 \circ L_2$ is regular, then there must be an NFA that recognizes it. Let this NFA be N_3:

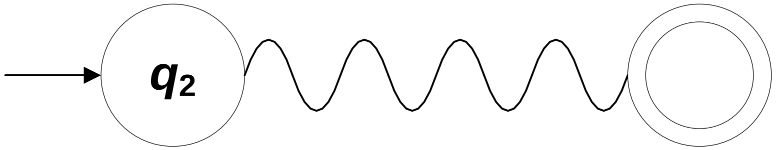
$$N_3 = \{Q_3, \Sigma_3, \delta_3, q_3, F_3\}$$

CLOSURE PROPERTY OF CONCATENATION

- The following are the state diagrams for N_1 and N_2:



NFA N_1



NFA N_2

The NFA N_3 for the concatenation of L_1 and L_2 should be able to accept a string if it is of the form xy where $x \in L_1$ and $y \in L_2$. In other words, N_3 should accept an input string if it can be divided into two parts where the first part is accepted by N_1 and the second part is accepted by N_2.

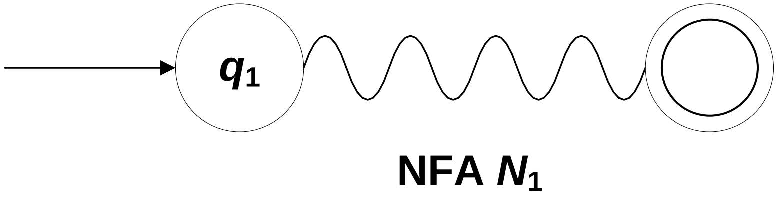
CLOSURE PROPERTY OF CONCATENATION

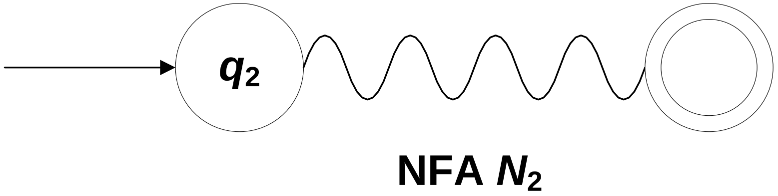
NFA N_3 will first try to see if the first part of the input string is accepted by N_1.

Once N_1 is in a final state, N_3 tries to see or guess if that is the point where the first part stops and the second part begins.

So NFA N_2 performs its computation.

The state diagram for N_3 will be



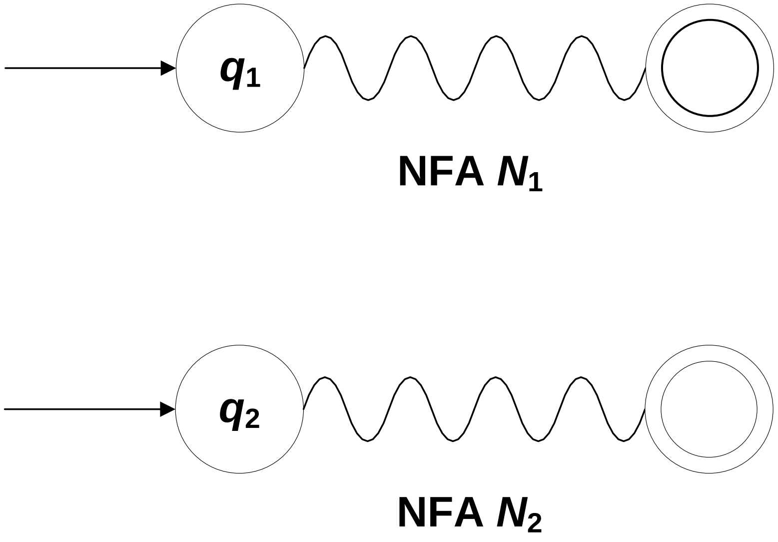


CLOSURE PROPERTY OF CONCATENATION

The start state of N_3 is the start state of N_1.

Upon arrival of the first symbol of the input string, N_3 starts "simulating" N_1 to determine if the first part of the string is a member of L_1 (it is accepted by N_1).

The state diagram for N_3 will be



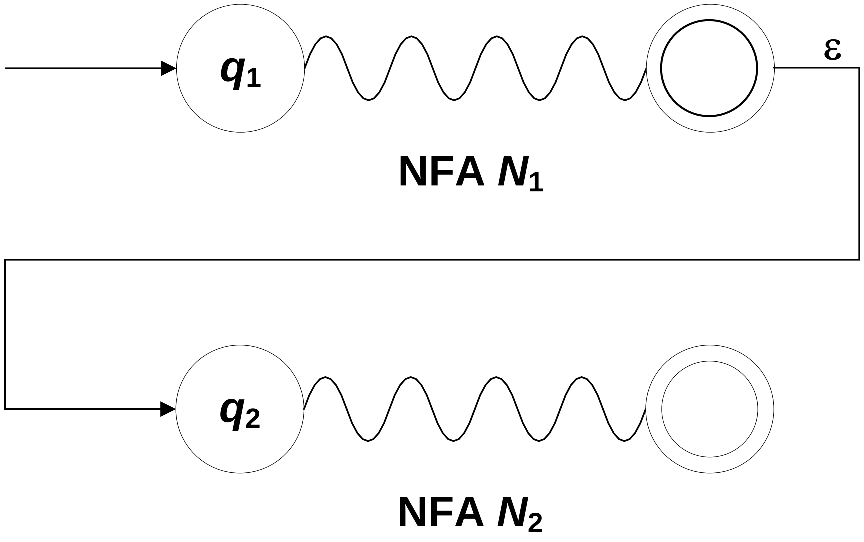
CLOSURE PROPERTY OF CONCATENATION

Every time the computation reaches a final state of N_1, N_3 assumes or guesses that this is the point where the first part ends and the second begins.

Hence, N_3 starts simulating N_2 to determine if the second part of the string is a member of L_2 (it is accepted by N_2).

The set of final states of N_3 is the set of final states of N_2.

The state diagram for N_3 will be

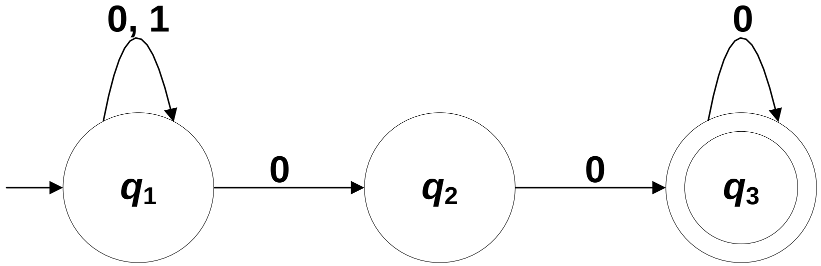


CLOSURE PROPERTY OF CONCATENATION

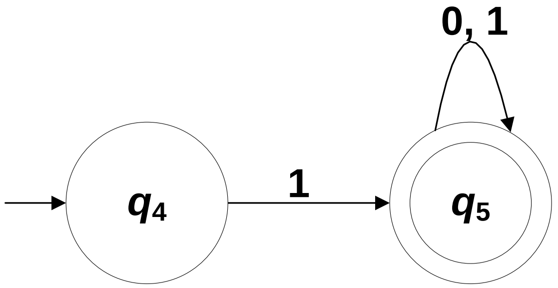
- Example:

  Let L_1 = {w|w ends with a 00} and L_2 = {w|w starts with a 1}

NFA N_1 for L_1:

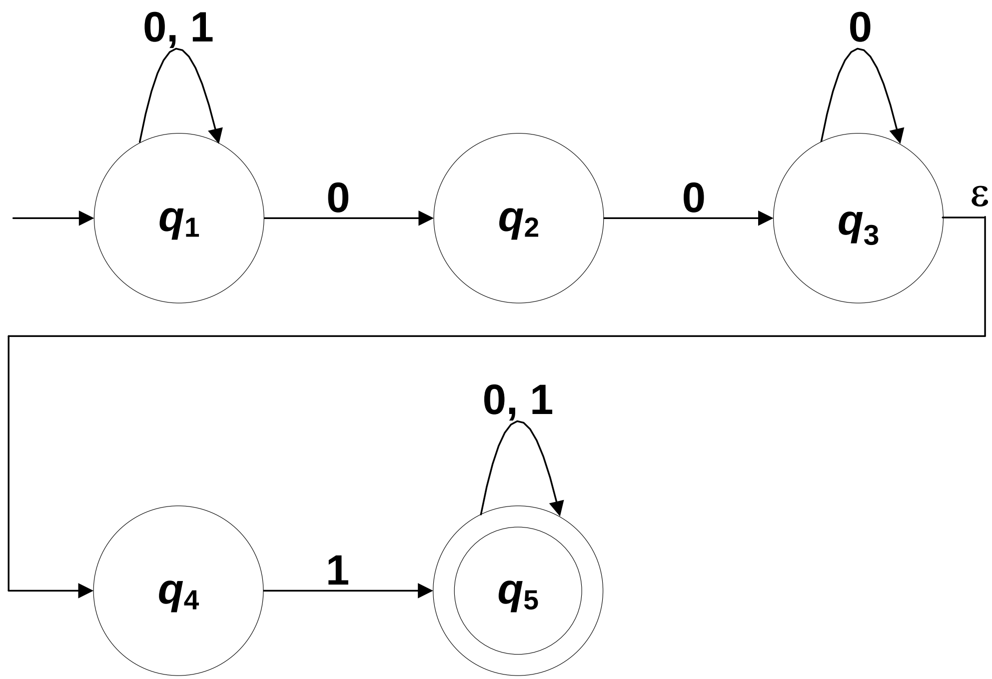


NFA N_2 for L_2:



CLOSURE PROPERTY OF CONCATENATION

NFA N_3 for L_3:



Therefore, $L_1 \circ L_2$ is a regular language.

CLOSURE PROPERTY OF STAR

- Is the star of regular languages also regular?

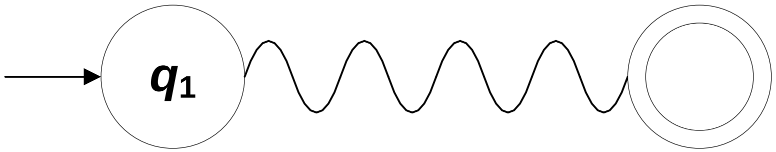
  Assume that L_1 is a regular language recognized by the NFA N_1:

$$N_1 = \{Q_1, \Sigma_1, \delta_1, q_1, F_1\}$$

  If L_1* (the star of L_1) is regular, then there must be an NFA that recognizes it. Let this NFA be N_2.

$$N_2 = \{Q_2, \Sigma_2, \delta_2, q_2, F_2\}$$

  The following is the state diagram for N_1:



NFA N_1

CLOSURE PROPERTY OF STAR

NFA N_2 should be able to accept a string if it is of the form $x_1x_2x_3$… where $x_i \in L_1$. In other words, N_2 should accept an input string if it can be divided into several parts where each part is accepted by N_1.

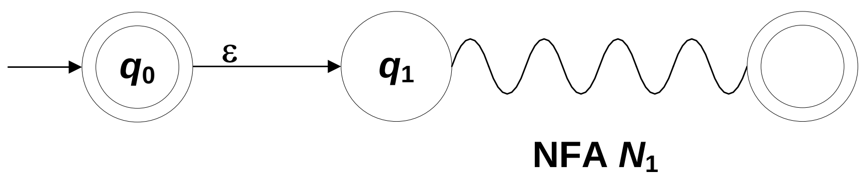
NFA N_2 will first try to see if the first part of the input string is accepted by N_1.

Once N_1 is in a final state, N_2 tries to see or guess if the second part is also accepted by N_1. So N_2 goes back to the start and begins computing again.

By definition of the star operation, N_2 should also be able to accept empty strings.

CLOSURE PROPERTY OF STAR

The state diagram for NFA N_2 will be

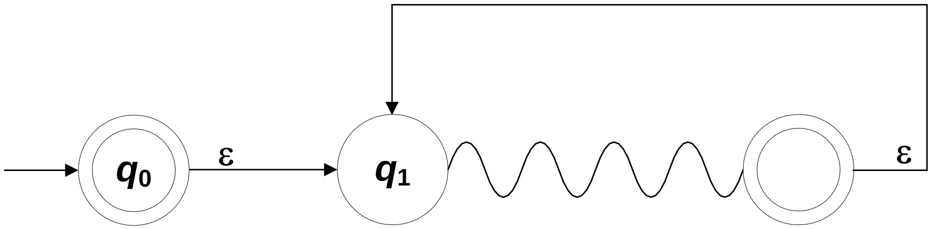


The start state of N_2 is a new state q_o which is also a final state. Adding this state ensures that the empty string is also accepted by N_2.

Upon arrival of the first symbol of the input string, N_2 starts "simulating" N_1 to determine if the first part of the string is a member of L_1.

CLOSURE PROPERTY OF STAR

The state diagram for NFA N_2 will be



NFA N_1

Every time the computation reaches a final state of N_1, N_2 assumes or guesses that this is the point where the first part ends and the second begins. Hence, N_2 goes back to the start and tries to see if the next part is also accepted by N_1.

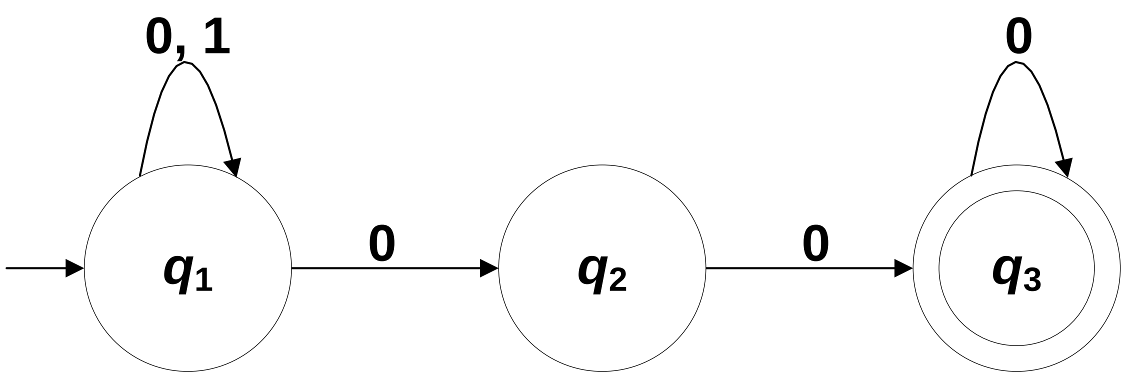
The set of final states of N_2 is the set of final states of N_1 plus state q_0.

CLOSURE PROPERTY OF STAR

- Example:

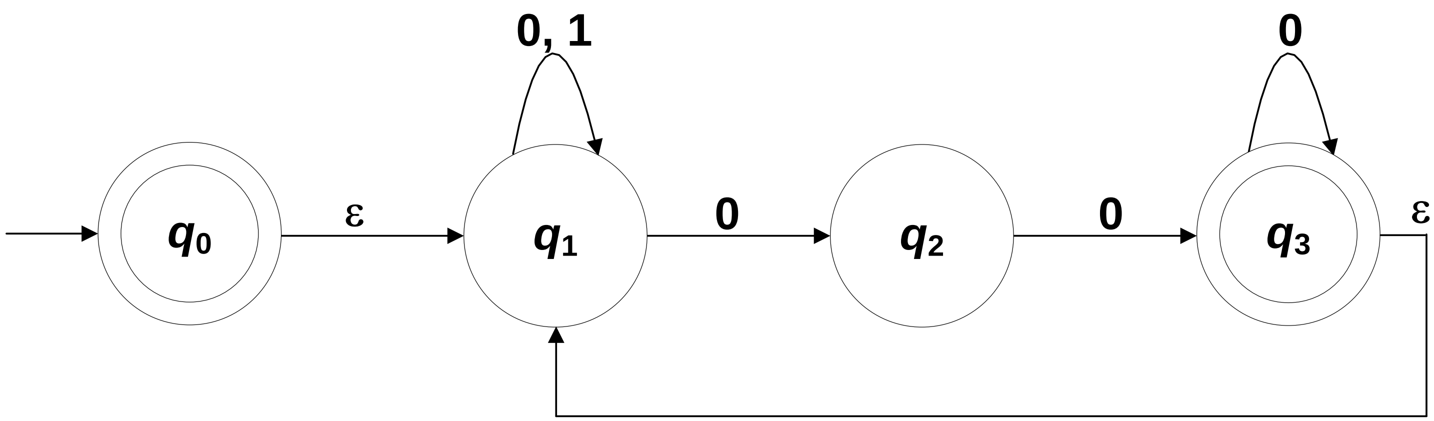
  Let L_1 = {w|w ends with a 00}

  NFA N_1 for L_1:



CLOSURE PROPERTY OF STAR

The state diagram for NFA N_2 for L_1*:



Therefore, L_1* is a regular language.

THEOREMS ON CLOSURE

- Theorem 2

  The family of regular languages is closed under the union operation.

- Theorem 3

  The family of regular languages is closed under the concatenation operation.

- Theorem 4

  The family of regular languages is closed under the star operation.

EXERCISES

- Given two languages L_1 and L_2 using the alphabet $\Sigma = \{0, 1\}$ described as:

  L_1 = All strings that contain the substring 010.
  L_2 = All strings that have even length.

  a. Give the state diagram for the NFA that accepts L_1.
  b. Give the state diagram for the NFA that accepts L_2.
  c. Give the state diagram for the NFA that accepts $L_1 \cup L_2$.
  d. Give the state diagram for the NFA that accepts $L_1 \bullet L_2$.
  e. Give the state diagram for the NFA that accepts L_1*.