Model and Statistical Analysis of the Motion of a Tired Random Walker in Continuum

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Abstract: The model of a tired random walker, whose jump-length decays exponentially in time, is proposed and the motion of such a tired random walker is studied systematically in one, two and three dimensional continuum. In all cases, the diffusive nature of walker, breaks down due to tiring which is quite obvious. In one dimension, the distribution of the displacement of a tired walker remains Gaussian (as observed in normal walker) with reduced width. In two and three dimensions, the probability distribution of displacement becomes nonmonotonic and unimodal. The most probable displacement and the deviation reduces as the tiring factor increases. The probability of return of a tired walker, decreases as the tiring factor increases in one and two dimensions. However, in three dimensions, it is found that the probability of return almost insensitive to the tiring factor. The probability distributions of first return time of a tired random walker do not show the scale invariance as observed for a normal walker in continuum. The exponents, of such power law distributions of first return time, in all three dimensions are estimated for normal walker. The exit probability and the probability distribution of first passage time are found in all three dimensions.

Keywords: Return probability, Exit probability, First passage time

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I. Introduction:

In statistical physics, polymerization[1, 2], diffusion[3] etc are some classic phenomena, which have drawn much attention of the researcher in last few decades. The basic mechanism of such phenomena are explained by random walk[4]. Various kinds of random walk are studied on the lattice in different dimensions by computer simulation. The absorbing phase transition in a conserved lattice gas with random neighbour particle hopping is studied[5]. Quenched averages for self avoiding walks on random lattices[6], asymptotic shape of the region visited by an Eulerian walker[7], linear and branched avalanches are studied in self avoiding random walks[8], effect of quenching is studied in quantum random walk[9]. The drift and trapping in biased diffusion on disordered lattices is also studied[10].

Some more interesting results on random walk were reported very recently. The average number of distinct sites visited by a random walker on the random graph[11], statistics of first passage time of the Browian motion conditioned by maximum value of area[12] are studied recently. It may be mentioned here that the first passage time in complex scale invariant media was studied very recently[13]. The thoery of mean first passage time for jump processes are developed [14] and verified by applying in Levy flights and fractional Brownian motion. The statistics of the gap and time interval between the highest positions of a Markovian one dimensional random walker[15], the universal statistics of longest lasting records random walks and Levy flights are also studied[16].

There are some studies of continuum random walks in real life problems. The exact solution of a Brownian inchworm model and self-propulsion was also studied[17], theory of continuum random walks and application in chemotaxis was developed[18]. Random walks in continuum was also studied for diffusion and reaction in catalyst[19]. Very recently, the random walk in continuum is studied with uniformly distributed random size of flight[20].

The living random walker in continuum gradually becomes tired as the time passes, in reality. This would reduce its energy, as a result the size of flight gets reduced gradually with time. This type of model is not yet found in the literatures. Here, in this paper, a model of tired random walker is proposed in continuum and its motion is studied systematically in one, two and three dimensional continuum. The numerical results of detailed statistical analysis of the motion of a tired random walker is reported here.

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II. Model and Results:

Generally, the motion of a random walker is studied by considering the time (t) independent size (R) of flight in each move. In this study, the model of a tired random walker is proposed in such a way that the size of flight of a walker decreases exponentially as $R(t) = e^{-\alpha t}$. A simple logic behind it may be stated as follows: if a living cell is moving continuously, its energy (basically kinetic energy) gradually decreases and hence the velocity, which in turn reduces its size of flight (i.e., jump-length per unit time). Here, α is tiring factor. The statistical behaviour of such a tired random walker is studied in one, two and three dimensional continuum. It may be noted here that such kind of behaviour of a tired random walker cannot be studied on the lattice.

In one dimension, the size of flight in each time step is $x = e^{-\alpha t}$. A walker starts its journey from the origin having the equal probability of choosing the left and right direction. In two dimensions (planar continuum), the tired walker starts its journey from the origin and it has a uniform probability of choosing any random direction (θ) distributed between 0 and 2π. Its motion can be represented mathematically as:

$$x(t+1) = x(t) + R(t)cos(\theta); y(t+1) = y(t) + R(t)sin(\theta) \qquad (1)$$

The displacement at time t is $r(t) = \sqrt{x^2(t) + y^2(t)}$. In planar continuum, the area of the region visited by a tired walker is obviously shorter than that visted by a normal walker, in a specified course of time. A typical such comparison is shown in Fig-1 with $\alpha = 0.001$. As a result, the mean square displacement does not show diffusive behaviour as shown by a normal walker. In long time, it gets saturated. A typical such comparison is shown in Fig-2 for $\alpha = 0.001$. The similar behaviours are also observed in one and three dimensions (not shown).

Now, let it be discussed systematically in one, two and three dimensions. In one dimensions, the probability distribution of the displacements of a walker are studied for $\alpha = 0.0$ (normal), 0.001(moderately tired) and 0.01(heavily tired). As usual, the distribution is normal (Gaussian) with zero mean in all the cases. However, as the tiring factor increases the distribution becomes sharper and sharper. These are depicted in Fig-3.

What will be the probability of return(P_R) in one dimension ? First of all, in continuum one should be careful in defining the probability of return.

In the lattice the probability of return is defined as the walker returns to its initial starting point. However, in continuum, it is quite unlikely that a tired walker returns to its initial starting point. Here, one may think that whether the tired walker returns within a linear zone (of size r_z) centered around the origin. Now put a large number (N_s) of walker at origin and allow them to walk (with different random sequence) upto a certain time (N_t) and then check how many walker returns within the preassigned returning zone (of size r_z). The calculated fraction is the probability of return (within time of observation N_t) in this particular model. In the lattice model this probability is 1, which can also be derived from exact calculations. In this model of tired walker, considering $r_z = 0.5$, $P_R = 0.992$ for $\alpha = 0.0$. This agrees well with exact calculation on lattice. As the tiring factor increases, P_R decreases. For moderately tired ($\alpha = 0.001$) walker, $P_R = 0.955$ and for heavily tired walker ($\alpha = 0.01$), $P_R = 0.874$. In Fig-4, the P_R is plotted against N_t for various values of α. Now, this probability of return (P_R) must depend on the size (r_z) of returning zone. To study the dependences of P_R on r_z, P_R is studied as a function of r_z for different values of α and shown in Fig-5. It shows that the P_R grows as r_z increases in the case of tired walker ($\alpha = 0.001, 0.01$), but P_R does not depend on r_z for normal walker. It is important to note here that even for heavily tired ($\alpha = 0.01$) random walker, the size of the flight, after $t = 10$ is larger than 0.90. So, the range of values of r_z, chosen here, does not have any chance that the walker remains in the returning zone immediately after starting its journey. So, the choice $r_z = 0.5$ is quite safe to study the probability of return.

How long a tired walker takes to return first time within returing zone ? How does the distribution of this first returning time (t_1) look like ? The probability distribution of first returning time (t_1) of a tired random walker is shown in Fig-6. A normal walker ($\alpha = 0.0$) shows a scale invariant ($P(t_1) \sim t_1^{-\beta}$) distribution of first returning time (t_1). The exponent $\beta \simeq 1.40$. However, this scale invariance nature of the distribution of first returning time, breaks down in the cases of tired walking (for $\alpha = 0.001, 0.01$) (see Fig-6).

How long a tired walker takes to exit through a point located at a certain distance (r_e) from the origin ? In one dimension, the distribution of first passage time (for a fixed distance $r_e = 25.0$), is studied for different values of α and shown in Fig-7. As the α increases, the most probable first passage time decreases.

What is the probability of exit of a tired walker one dimensional continuum ? The exit probability (for a fixed time of obsevation N_t) from a zone of distance r_e (measured from the origin) is also studied, in one dimension, as a function of r_e and shown in Fig-8. Here, the exit probability, of a tired walker, was found to decreases as the distance of zone (r_e) increases. However, it remains fixed (nearly 1) for a normal walker. It may also be noted that, the rate of fall of exit probability increases as the tiring factor (α) increases.

In two dimensions, the motion of a tired random walker is studied by using the rule given in equation -1. Here, the mean square displacement $< r^2 >$ is proportinal to the time t for $\alpha = 0.0$, reveals the diffusive behaviour. However, a moderately tired ($\alpha = 0.001$) walker does not show long time diffusive behaviour. This is quite obvious and already shown in Fig-2.

The distribution of absolute displacement is nonmonotonic unimodal function. It is shown in Fig-9. It is observed that the maximum probability of finding the walker at a distance (r_m) from the origin and the average distance ($\bar{r}$) both decreases as the tiring factor (α) increases. In this case, $r_m = 64.0$, $\bar{r} = 89.13$ for $\alpha = 0.0$, $r_m = 15.0$, $\bar{r} = 20.04$ for $\alpha = 0.001$ and $r_m = 5.0$, $\bar{r} = 6.42$ for $\alpha = 0.01$.

What will be the probability of return of a tired walker in planar continuum ? The probability of return within a circle of return having radius $r_z = 0.5$ is studied as a function of maximum time of observation N_t and shown in Fig-10. In planar continuum, a tired walker has a probability of return in a circle of radius $r_z = 0.5$ as follows: for $\alpha = 0.0$, $P_R = 0.737$, for $\alpha = 0.001$, $P_R = 0.620$ and for $\alpha = 0.01$, $P_R = 0.524$.

For a fixed value of N_t, the probability of return of a tired walker in planar contunuum, grows as the radius of returning zone increases. This is shown in Fig-11.

Here, like in one dimensional tired walker, the probability distribution of first returning time (t_1) shows a scale invariance ($P(t_1) \sim t_1^{-\beta}$) for $\alpha = 0.0$ with $\beta \simeq 1.10$. As the tiring factor α increases, the scale invariance nature of the distribution on first returning time breaks down. This is demonstrated in Fig-12.

In two dimensions, the distribution of first passage time (for a fixed distance $r_e = 25.0$), is studied for different values of α and shown in Fig-13. As the α increases, the most probable first passage time and mean first passage time decreases.

The exit probability (for a fixed time of obsevation N_t) from a circular zone of radius r_e (measured from the origin) is also studied, in two dimensions, as a function of r_e and shown in Fig-14. Here, the exit probability was found to decreases as the radius of circular zone (r_e) increases. Here also, the rate of fall of exit probability increases as the tiring factor (α) increases. However, the exit probability of a normal walker ($\alpha = 0$) remains unchanged (nearly 1) as r_e increases.

The tired walk in three dimensional continuum can also be generalized. The updating of coordinates obey the following rule:

$$x(t+1) = x(t) + R(t)sin(\theta)cos(\phi); \quad y(t+1) = y(t) + R(t)sin(\theta)sin(\phi)$$

$$z(t+1) = z(t) + R(t)cos(\theta) \tag{2}$$

where $R(t) = e^{-\alpha t}$, θ is uniformly distributed random angle between 0 and π and ϕ is uniformly distributed random angle between 0 and 2π. The displacement at time t is $r(t) = \sqrt{x^2(t) + y^2(t) + z^2(t)}$.

In 3D continuum, the motion of a tired random walker is studied by using the rule given in equation -2. Here, the mean square displacement $< r^2 >$ is proportinal to time t for $\alpha = 0.0$, reveals the diffusive behaviour (not shown). However, a tired ($\alpha \neq 0$) walker does not show long time diffusive behaviour (not shown).

The probability distribution of absolute displacement in 3D continuum is observed to be a nonmonotonic unimodal function. It is shown in Fig-15. It is observed that the maximum probability of finding the walker at a distance (r_m) from the origin and the mean displacement ($\bar{r}$) both decreases as the tiring factor (α) increases. In this case, $r_m = 72.0$, $\bar{r} = 91.59$ for $\alpha = 0.0$, $r_m = 16.0$, $\bar{r} = 20.65$ for $\alpha = 0.001$ and $r_m = 3.0$, $\bar{r} = 6.96$ for $\alpha = 0.01$.

What will be the probability of return in 3D continuum ? The probability of return within a sphere of return having radius $r_z = 0.5$ is studied as a function of maximum time of observation N_t and shown in Fig-16. In 3D continuum, unlike the cases in 1D and 2D continuum, a tired walker has a probability of return in a sphere of radius $r_z = 0.5$ is almost insensitive of the tiring factor α.

For a fixed value of N_t, the probability of return of a tired walker in 3D contunuum, grows as the radius of returning zone increases keeping the independence on tiring factor α. This is shown in Fig-17.

In 3D continuum, the probability distribution of first returning time (t_1) shows a scale invariance $(P(t_1) \sim t_1^{-\beta})$ for $\alpha = 0.0$. The exponent estmated $\beta \simeq 1.40$. As the tiring factor α increases, the scale invariance of the distribution on first returning time breaks down. This is demonstrated in Fig-18.

In three dimensions, the distribution of first passage time (for a fixed distance $r_e = 25.0$), is studied for different values of α and shown in Fig-19. As the α increases, the most probable first passage time and the mean first passage time decreases.

The exit probability (for a fixed time of obsevation N_t) from a spherical zone of radius r_e (measured from the origin) is also studied, in two dimensions, as a function of r_e and shown in Fig-20. Here, the exit probability, of a tired walker, was found to decreases as the radius of circular zone (r_e) increases. However, like the earlier cases, it reamins same (nearly 1) for all r_e. Here also the rate, of fall of the exit probability of a tired walker, increases as the tiring factor α increases.

III. Summary:

In this article, a model of tired random walker in continuum is proposed. Generally, a random walker moves with constant size of flight. However, as the time passes, if the walker gets tired, one should think of a time dependent size of flight. Here, this size of flight decays exponentially with time. The motion of such a tired walker is studied in one, two and three dimensional continuum. In this statistical investigation, the distribution of the absolute displacement, mean displacement, probability of return (within a specified zone), distribution of time of first return are studied systematically. In one and two dimensional continuum, the probability of return decreases as the tiring factor increases. However, in three dimensional continuum, this probability of return seems to be independent of the tiring factor. The distribution of first returning time in all dimensions (for normal walker with tiring factor $\alpha = 0$), shows power law behaviours. This scale invariance of the distribution of first returning time breaks down for $\alpha \neq 0$ in all dimensions.

The exit probability and the distribution of first passage time are studied. In all dimensions, the exit probability was found to decrease as the size of the zone (from where the tired walker exits out) increases. The rate of decrease of the exit probability was found to increase as the tiring factor α increases.

The first passage time is defined as the time required by a walker to exit from specified zone. This time has a distribution and this distribution is studied for various values of α. It was observed, in all dimensions, the most probable first passage time decreases as α increases.

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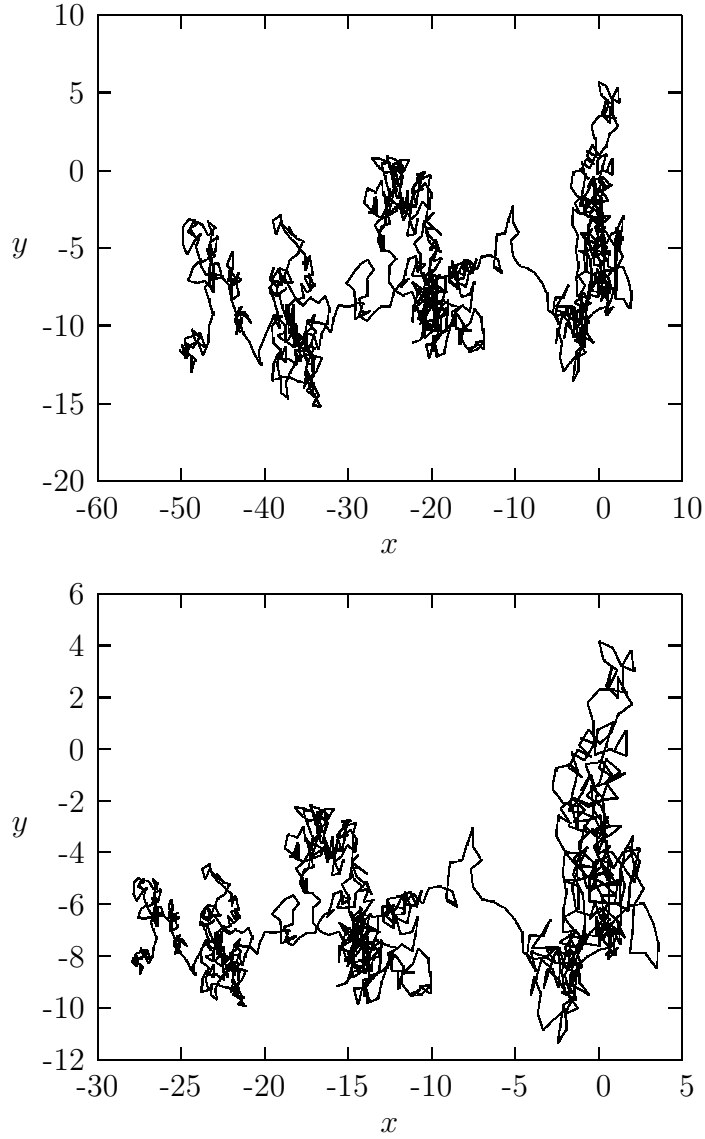
Fig-1. A sample random walk in 2D continuum. The top one shows a walk for $\alpha = 0.0$ and the bottom one shows a tired walk for $\alpha = 0.001$. Here, in both figures same sequence of random numbers are used. Note the scales of the axes of two figures.

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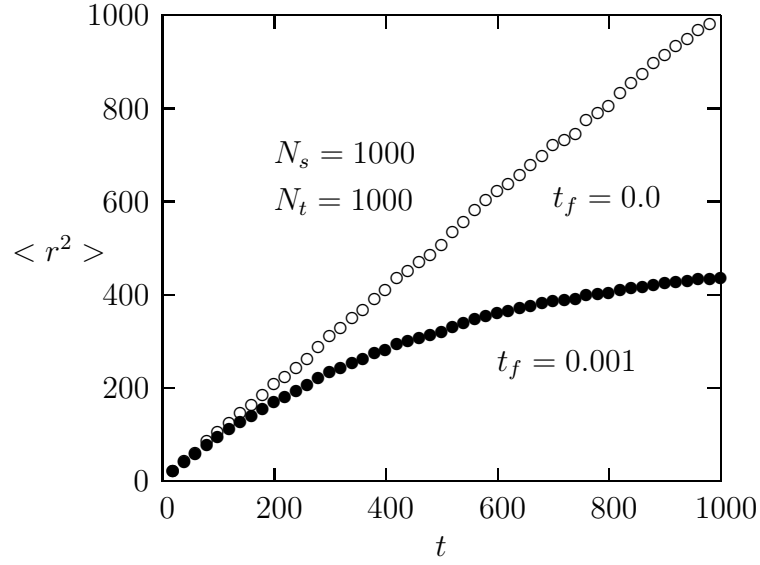
Fig-2. The $< r^2 >$ versus t in two dimensions. The tired walker does not show the diffusive behaviour.
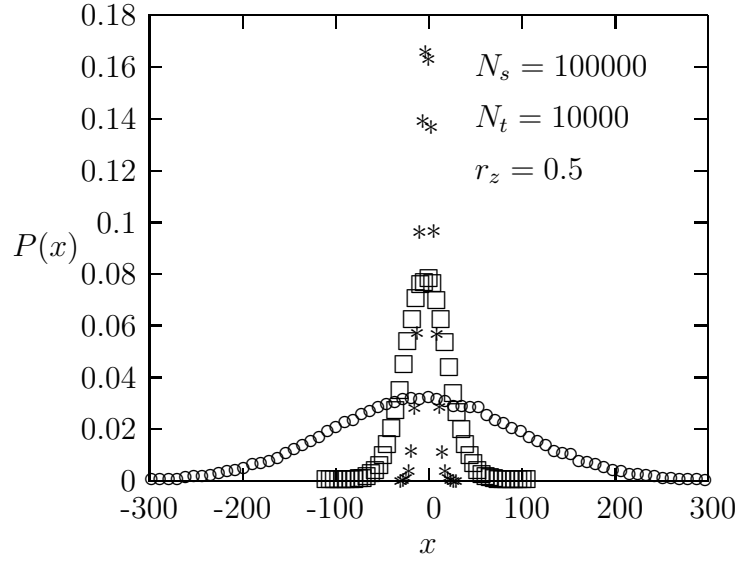
Fig-3. The distribution of the displacements of a tired random walker in one dimension. ($\circ$) represents $\alpha = 0.0$, ($\square$) represents $\alpha = 0.001$, and ($*$) represents $\alpha = 0.01$.

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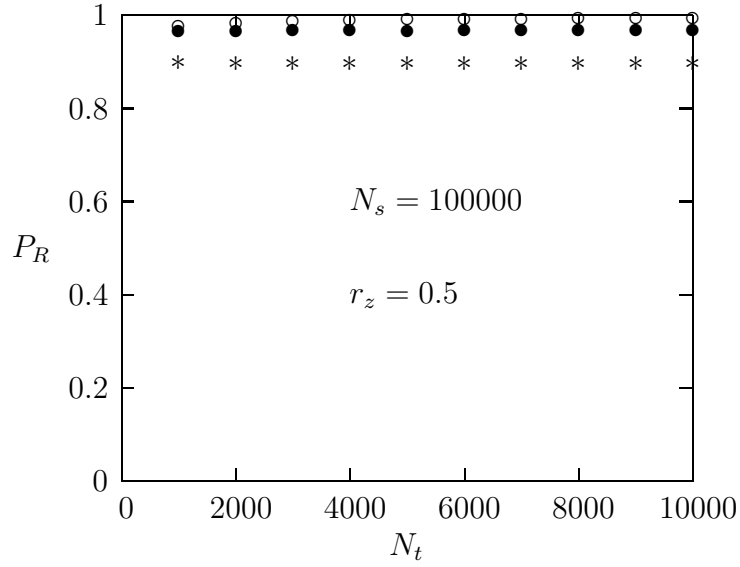
Fig-4. Probability of return (P_R) plotted against the maximum time N_t in one dimension. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\bullet)$ and $t_f = 0.01(*)$. Here in all cases $N_s = 100000$. The radius of returning zone is $r_z = 0.5$.
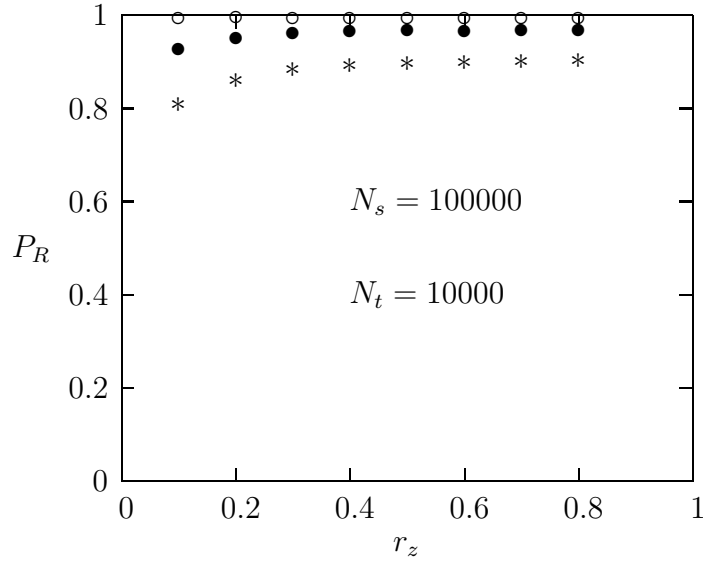
Fig-5. Probability of return (P_R) plotted against the radius of returning zone (r_z) in one dimension. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\bullet)$ and $\alpha = 0.01(*)$.
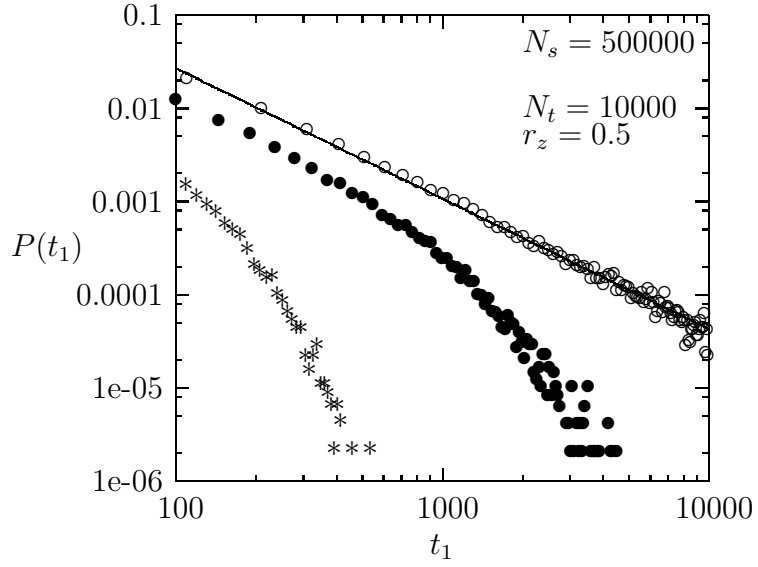
Fig-6. Probability distribution ($P(t_1)$) of first returning time (t_1) in one dimension. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\bullet)$ and $\alpha = 0.01(*)$. The solid line is $y = 17x^{-1.40}$.
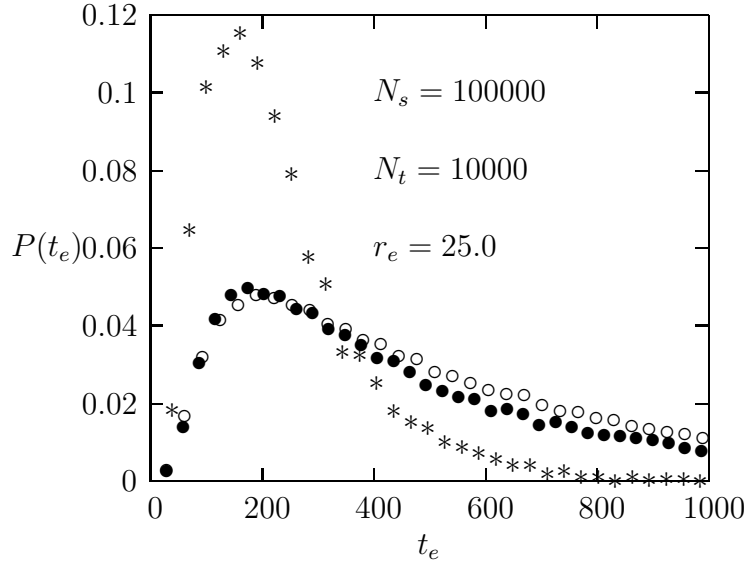
Fig-7. Probability distribution of first passage time in one dimension. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\bullet)$ and $\alpha = 0.004(*)$. Here, $N_s = 100000$, $N_t = 10000$ and $r_e = 25.0$.
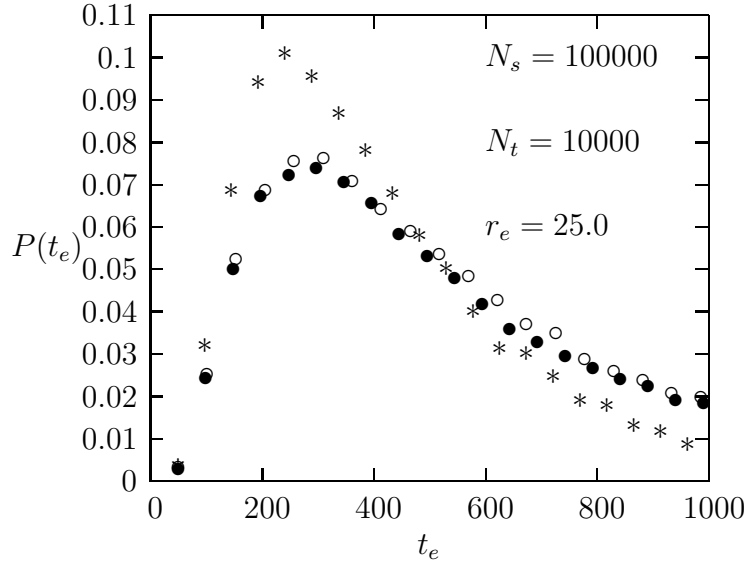
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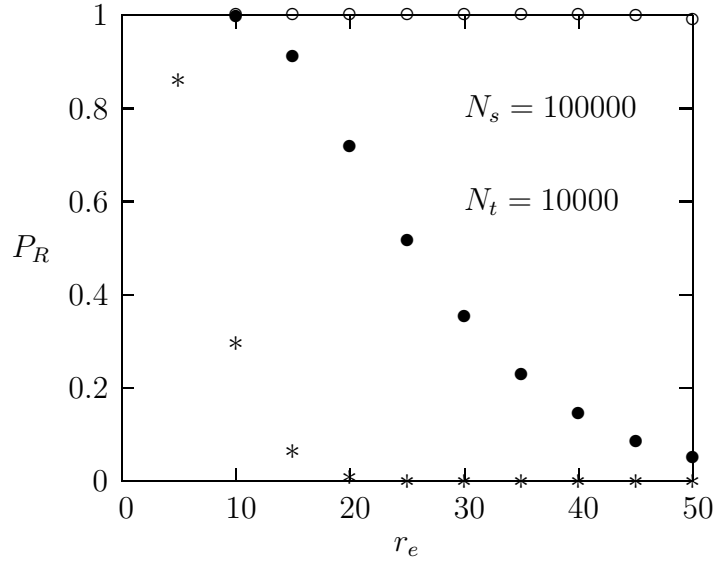
Fig-8. Exit probability P_e plotted against r_e in one dimension. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\bullet)$ and $\alpha = 0.01(*)$. Here, $N_s = 100000$ and $N_t = 10000$.

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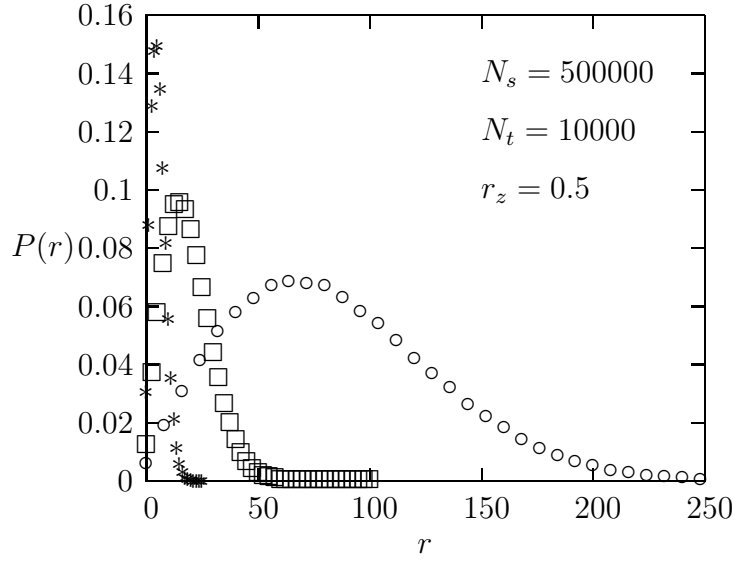
Fig-9. The distribution of displacements (r) of a tired random walker in two dimensions. ($\circ$) represents $\alpha = 0.0$, $r_m = 64.0$, $\bar{r}=89.13$, ($\square$) represents $\alpha = 0.001$, $r_m = 15.0$, $\bar{r}=20.04$ and ($*$) represents $\alpha = 0.01$, $r_m = 5.0$, $\bar{r}=6.42$.

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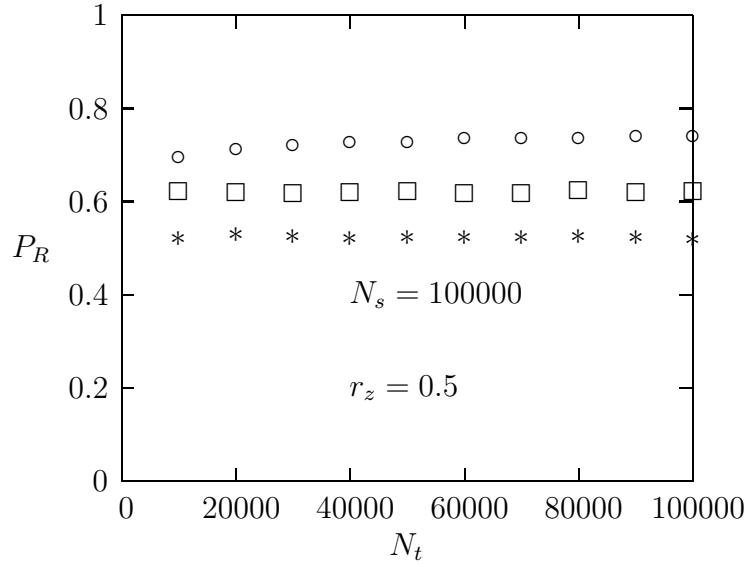
Fig-10. Probability of return (P_R) plotted against N_t in two dimensions. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\square)$ and $\alpha = 0.01(*)$. Here in all cases $N_s = 100000$. The radius of returning zone is $r_z = 0.5$.
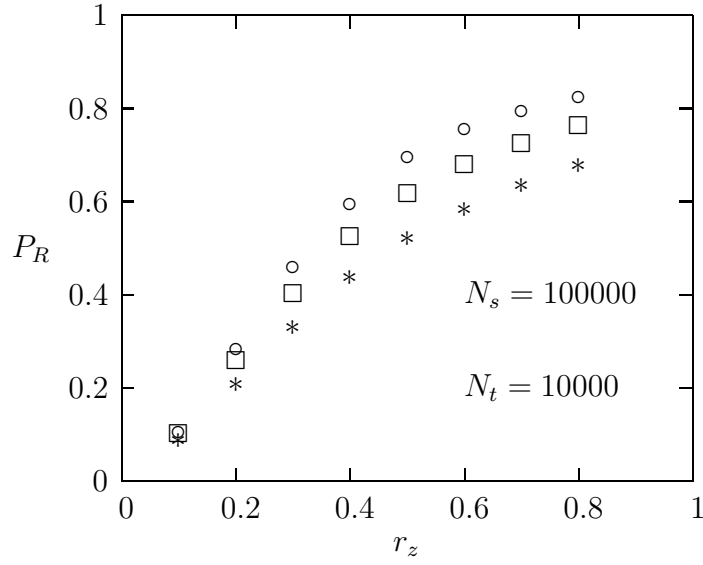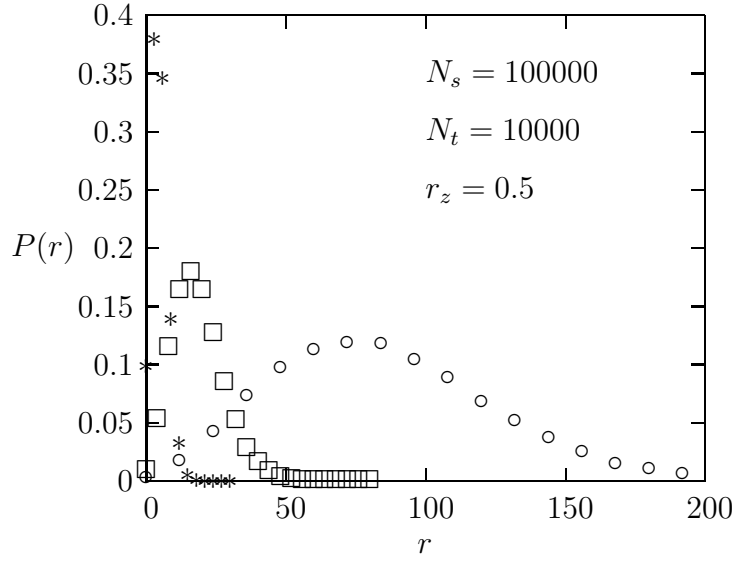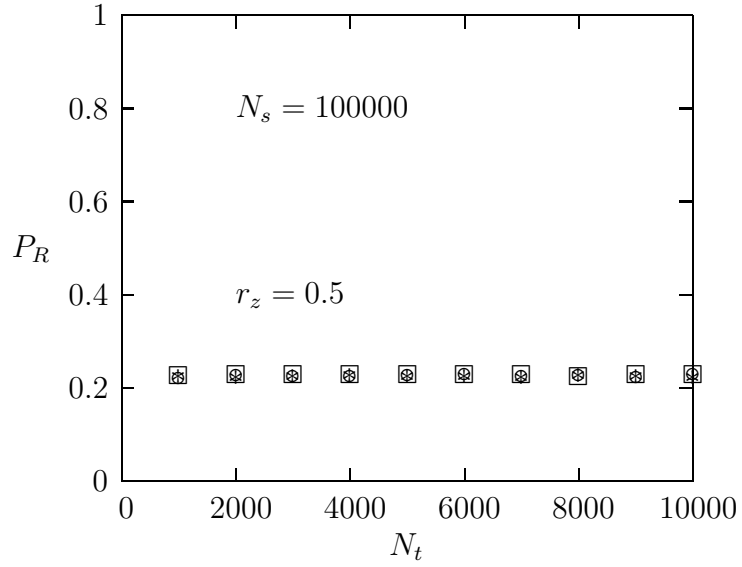
Fig-11. Probability of return (P_R) plotted against the radius of returning zone (r_z) in two dimensions. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\square)$ and $\alpha = 0.01(*)$.
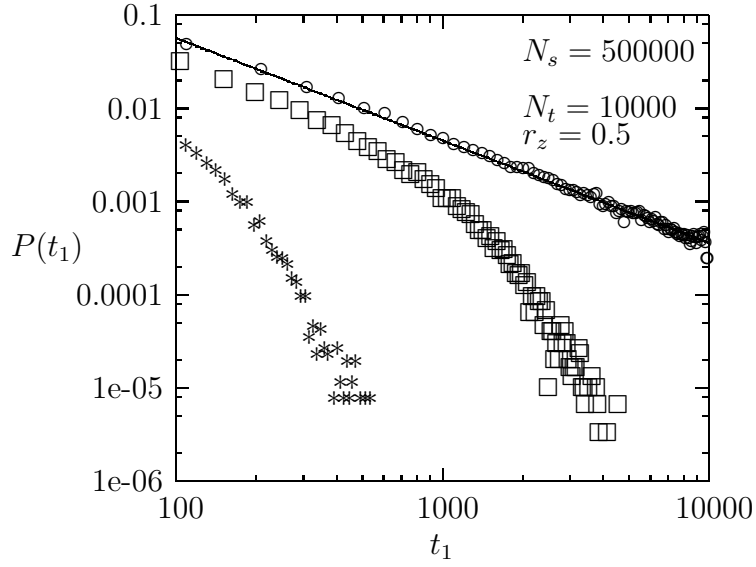
Fig-12. Probability distribution $(P(t_1))$of first returning time (t_1) of two dimensions. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\square)$ and $\alpha = 0.01(*)$. Solid line represents $y = 9x^{-1.10}$.

Fig-13. Probability distribution of first passage time in two dimensions. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\bullet)$ and $\alpha = 0.002(*)$. Here, $N_s = 100000$, $N_t = 10000$ and $r_e = 25.0$.
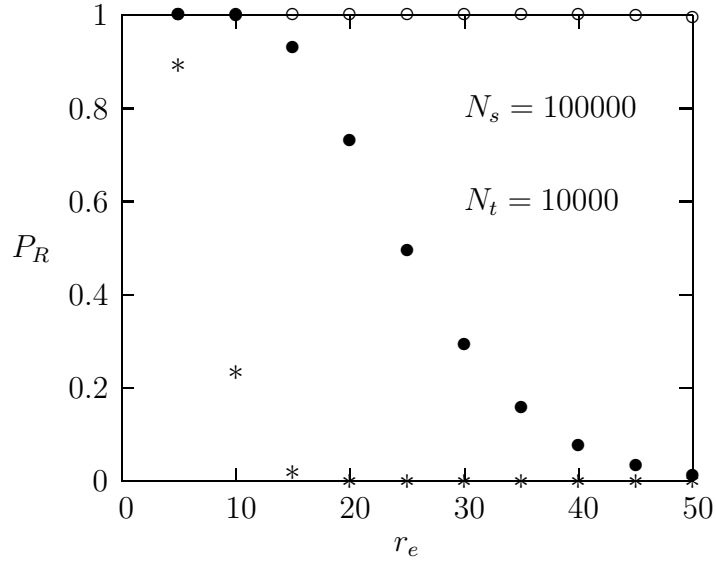
Fig-14. Exit probability P_e plotted against r_e in two dimensions. Different symbols correspond to different values of α. $\alpha = 0.0 (\circ)$, $\alpha = 0.001 (\bullet)$ and $\alpha = 0.01 (*)$. Here, $N_s = 100000$ and $N_t = 10000$.

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Fig-15. The distribution of the displacements of a tired random walker in three dimensions. ($\circ$) represents $\alpha = 0.0$, $r_m = 72.0$, $\bar{r}$=91.59, ($\square$) represents $\alpha = 0.001$, $r_m = 16.0$, $\bar{r}$=20.65 and ($*$) represents $\alpha = 0.01$, $r_m = 3.0$, $\bar{r}$=6.96.
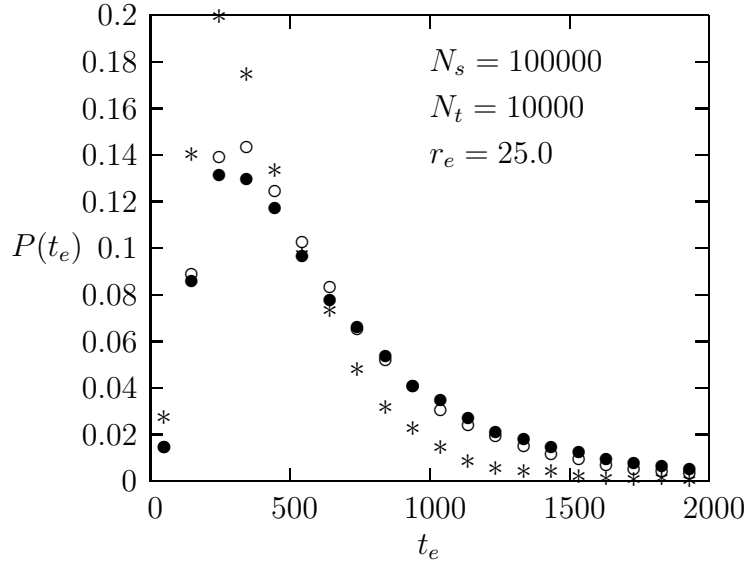
Fig-16. Probability of return (P_R) versus N_t in three dimensions. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\square)$ and $\alpha = 0.01(*)$. Here in all cases $N_s = 100000$. The radius of returning zone is $r_z = 0.5$.
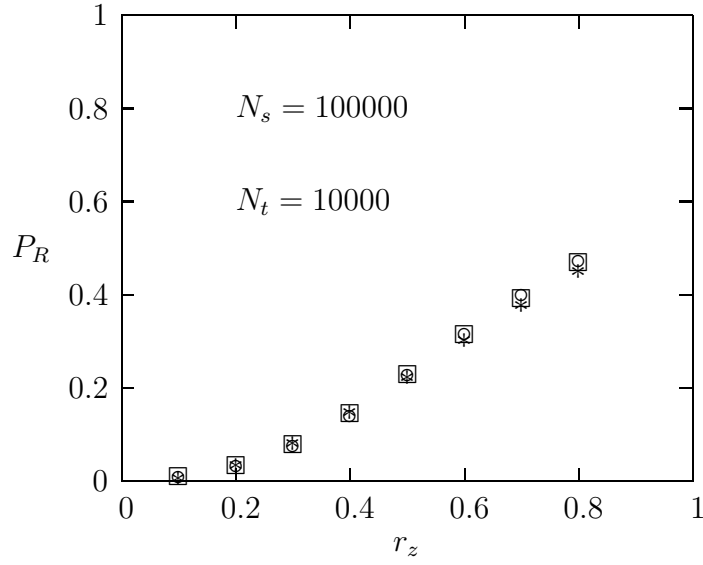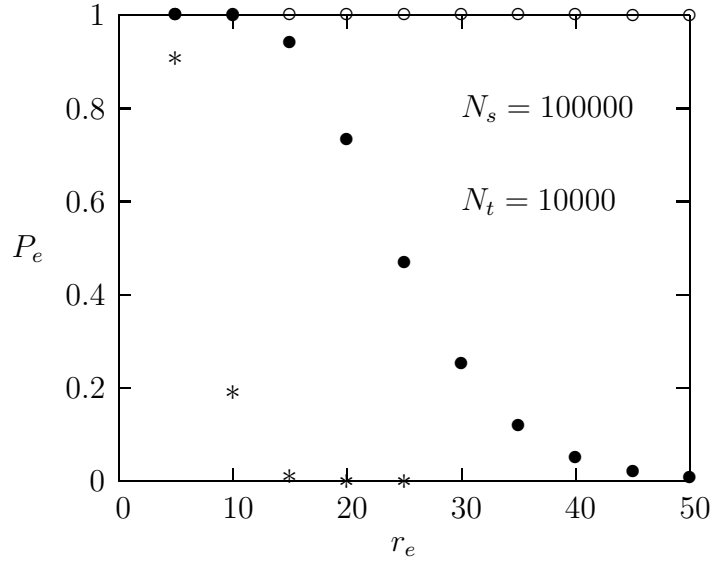
Fig-17. Probability of return (P_R) versus r_z in three dimensions. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\square)$ and $\alpha = 0.01(*)$.
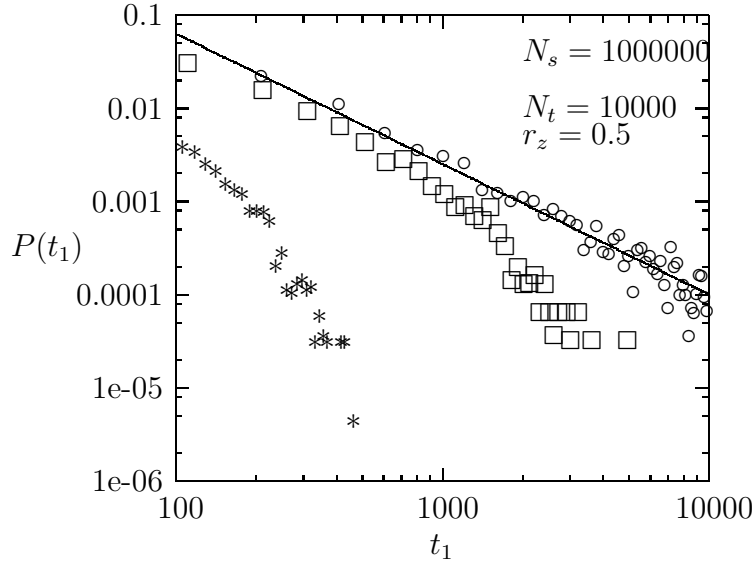
Fig-18. Probability distribution ($P(t_1)$) of first returning time (t_1) in three dimensions. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\square)$ and $\alpha = 0.01(*)$. Solid line represents $y = 40x^{-1.40}$.

Fig-19. Probability distribution of first passage time in three dimensions. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\bullet)$ and $\alpha = 0.002(*)$. Here, $N_s = 100000$, $N_t = 10000$ and $r_e = 25.0$.

Fig-20. Exit probability P_e plotted against r_e in three dimensions. Different symbols correspond to different values of α. $\alpha = 0.0(\circ)$, $\alpha = 0.001(\bullet)$ and $\alpha = 0.01(*)$. Here, $N_s = 100000$ and $N_t = 10000$.