Basics of Spin-$\frac{1}{2}$ Systems

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1. In a sequential Stern-Gerlach experiment, where the incident beam is first split in the $\hat{i}$ direction, the $|\uparrow\rangle_x$ is selected. This beam is now sent through another Stern-Gerlach apparatus, which splits the beam in $\hat{k}$ direction. The beam is then found to be split evenly in two discrete bands. From this description, which among the following is a possible representation of $|\uparrow\rangle_x$ in the $\hat{S}_z$ basis?

I. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

II. $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

III. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

IV. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Ans. The beam is split evenly, which means the probability of finding a particle in the beam with $|\uparrow\rangle_z$ or $|\downarrow\rangle_z$ is $\frac{1}{2}$. But, since the particle was in the $|\uparrow\rangle_x$ beam, the initial state of the particle was $|\uparrow\rangle_x$. Hence, we can conclude,

$$|\langle\uparrow_x \mid \uparrow_z\rangle|^2 = |\langle\uparrow_x \mid \downarrow_z\rangle|^2 = \frac{1}{2}.$$

From this, we can guess the following possible representation:

$$\begin{aligned} |\uparrow\rangle_x &= \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_z + |\downarrow\rangle_z \right) \\ &= \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned}$$

which is **IV**.

2. In the basis of $\hat{S}_z$. how do you express $|\downarrow\rangle_x$?

I. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

II. $\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

III. $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

IV. $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Ans. Since $|\downarrow\rangle_x$ is orthonormal to $|\uparrow\rangle_x$, and $|\uparrow\rangle_x$ is given by $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we conclude that $|\downarrow\rangle_x$ is $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which is **III**.

3. In the previous sequential Stern-Gerlach experiment, suppose we replace the X-Stern-Gerlach apparatus with an Y-Stern-Gerlach apparatus, select the $|\uparrow\rangle_y$ beam instead, and send it through the the Z-Stern-Gerlach apparatus, only to find the same even splitting. Then, which among the following is a possible representation of $|\uparrow\rangle_y$ in the $\hat{S}_z$ basis?

I. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$.

II. $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

III. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

IV. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Ans. Here also, following Problem **1**, we conclude that:

$$|\langle\uparrow_y \mid \uparrow_z\rangle|^2 = |\langle\uparrow_y \mid \downarrow_z\rangle|^2 = \frac{1}{2}.$$

So,

$$|\langle\uparrow_y \mid \uparrow_z\rangle| = |\langle\uparrow_y \mid \downarrow_z\rangle| = \frac{1}{\sqrt{2}}.$$

This is the only conclusion we can draw from the above experiment.
But certainly, the $|\uparrow\rangle_y$ state can be written as $\langle\uparrow_y \mid \uparrow_z\rangle |\uparrow\rangle_z + \langle\uparrow_y \mid \downarrow_z\rangle |\downarrow\rangle_z$, where the coefficients of $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$ can be complex. Since, in dealing with the $|\uparrow\rangle_x$ and $|\uparrow\rangle_y$ states, we used real coefficients, and we are forced to keep the absolute values of the coefficients $\frac{1}{\sqrt{2}}$, we are thus forced to write $\langle\uparrow_y \mid \uparrow_z\rangle$ and $\langle\uparrow_y \mid \downarrow_z\rangle$ as $\frac{1}{\sqrt{2}}e^{i\theta}$, where θ is an arbitrary phase, which never goes into the expectation values, the quantities of physical interest. That being the case, we have some freedom of choosing θ for the two coefficients, and we exploit this to write $\langle\uparrow_y \mid \uparrow_z\rangle = \frac{1}{\sqrt{2}}$ and $\langle\uparrow_y \mid \downarrow_z\rangle = \frac{1}{\sqrt{2}}e^{i\delta}$.
But, to determine this δ, we need to have some more information. And that information is actually embedded deep inside the experimental descriptions accompanying the question. A good physical theory does not depend on preferential directions, and indeed, the Stern-Gerlach apparatus have no way of realizing what axes we are labeling them with. So, had the $|\uparrow\rangle_y$ beam been passed through a Stern-Gerlach apparatus aligned in the X direction, we would have expected the experimental result to be the same: a two way splitting. In this light,

$$|\langle\uparrow_y \mid \uparrow_x\rangle|^2 = |\langle\uparrow_y \mid \downarrow_x\rangle|^2 = \frac{1}{2}.$$

We can replace $|\uparrow\rangle_x$ and $|\downarrow\rangle_x$ by their representations in the $\hat{S}_z$ basis to obtain the following:

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