Lecture 12

Group Velocity and Phase Velocity

Many modern ideas on wave propagation originated in the famous works of Lord Raleigh, and the problems we intend to discuss are no exception to this rule. The distinction between phase velocity and group velocity appears very early in Rayleigh's papers. The problem is discussed in particular in connection with measurements of the velocity of light and this is a place where a curious error was introduced regarding the angle of aberration.

Let us recall that the usual velocity of waves is defined as giving the phase difference between the vibrations observed at two different points in a free plane wave. A wave

$$\psi = A \cos\left(\mathbf{k}\bullet\mathbf{x} - \omega t\right) ,$$

travels with a phase velocity $\mathbf{v}_p = \omega \frac{\hat{\mathrm{k}}}{|\mathbf{k}|}$.

Another velocity can be defined, if we consider the propagation of a peculiarity (to use Rayleigh's term), that is, of a change of amplitude impressed on a train of waves. This is what we now call a *modulation* impressed on a carrier. The modulation results in the building up of some "groups" of large amplitude (Rayleigh) which move along with the *group velocity* U. In wave mechanics, Schrodinger called these groups "wave packets." A simple combination of groups obtains when we superimpose on one plane wave another plane wave with slightly different phase velocity and frequency such that we have

$$
\begin{aligned}
&\cos(\mathbf{k}.\mathbf{x}-\omega t)+\cos\left[(\mathbf{k}+\delta\,\mathbf{k}).\mathbf{x}-(\omega+\delta\,\omega)t\right] \\
&=2\cos\left[\frac{(\mathbf{k}.\mathbf{x}-\omega t)+\{(\mathbf{k}+\delta\,\mathbf{k}).\mathbf{x}-(\omega+\delta\,\omega)t\}}{2}\right] \\
&\quad\cos\left[\frac{(\mathbf{k}.\mathbf{x}-\omega t)-\{(\mathbf{k}+\delta\,\mathbf{k}).\mathbf{x}-(\omega+\delta\,\omega)t\}}{2}\right] \\
&=2\cos\left[\left(\mathbf{k}+\frac{\delta\,\mathbf{k}}{2}\right).\mathbf{x}-\left(\omega+\frac{\delta\omega}{2}\right)t\right]\cos\left(\frac{\delta\,\mathbf{k}}{2}.\mathbf{x}-\frac{\delta\omega}{2}t\right) ,
\end{aligned}
$$

where the first term corresponds to a carrier wave and the second term corresponds to the modulation envelope. Thus the superposition results in a carrier wave propagating with a phase velocity

$$\mathbf{v}_p=\left(\omega+\frac{\delta\omega}{2}\right)\frac{\hat{k}'}{\left|\mathbf{k}+\dfrac{\delta\,\mathbf{k}}{2}\right|} ,$$

where

$$\hat{k}'=\frac{\left(\mathbf{k}+\dfrac{\delta\,\mathbf{k}}{2}\right)}{\left|\mathbf{k}+\dfrac{\delta\,\mathbf{k}}{2}\right|} .$$

and a modulation envelope propagating with a group velocity

$$\mathbf{v}_g=\frac{\delta\,\omega}{|\delta\,\mathbf{k}|}\hat{k}'' ,$$

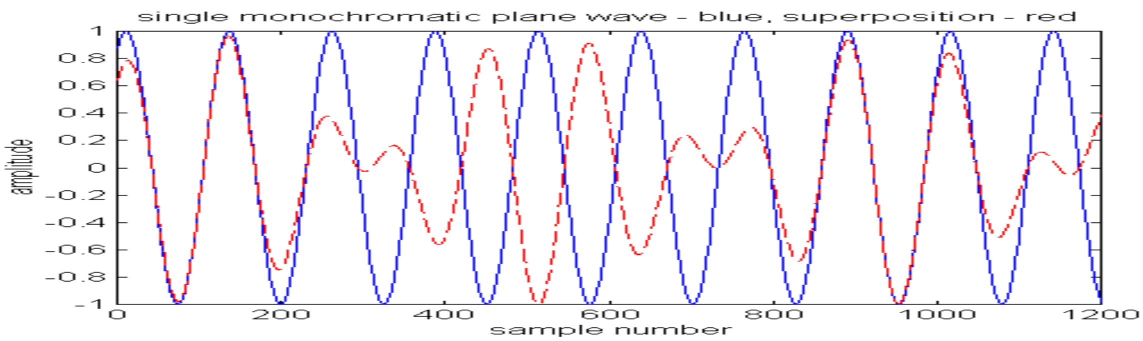
where

$$\hat{k}''=\frac{\delta\,\mathbf{k}}{|\delta\,\mathbf{k}|} .$$

In three dimensions the group velocity therefore becomes

$$\mathbf{v}_g = \hat{x}_1 \frac{\delta \omega}{\delta k_1} + \hat{x}_2 \frac{\delta \omega}{\delta k_2} + \hat{x}_1 \frac{\delta \omega}{\delta k_3} \; .$$

This expression is convenient to use if the dispersion relation is given explicitly as $\omega = f(k_1, k_2, k_3)$.



However from Christoffel equation we have seen that the dispersion relation can be obtained in the following implicit form

$$\left| \boldsymbol{k}^2 \Gamma_{\boldsymbol{ij}}\left(\hat{\boldsymbol{l}}\right) - \boldsymbol{\rho}\boldsymbol{\omega}^2 \right| = \Omega\left(\boldsymbol{\omega}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3\right) = 0 \; .$$

where $\hat{l}$ is the propagation direction $\left(\hat{l} \equiv \hat{k}\right)$. The above equation cannot always be transformed to get an explicit form for ω as a function of wave number. In such cases the required derivatives are obtained by implicit differentiation. For example, from the above equation, we have

$$\left(\frac{\partial \Omega}{\partial \omega} \delta\omega + \frac{\partial \Omega}{\partial k_1} \delta k_1 \right)_{k_2 k_3} = 0 \; ,$$

$$\left(\frac{\partial \omega}{\partial k_1} \right)_{k_2 k_3} = -\frac{\partial \Omega / \partial k_1}{\partial \Omega / \partial \omega} \; , \ldots etc.$$

Thus the group velocity can always be evaluated as

$$\mathbf{v}_g = -\frac{\nabla_k \Omega}{\partial\Omega/\partial\omega} \ ,$$

where the gradient of Ω is taken with respect to **k.**

In loss less medium it can be shown that $\mathbf{v}_e = \mathbf{v}_g$.

This provides us with an alternative and simpler way to compute ray velocity/group velocity surfaces, which we describe below.

Recall that

$$\Omega = \Omega(\mathbf{k}, \omega) = \Omega(\omega\mathbf{p}, \omega) = F(\mathbf{p}, \omega) \ ,$$
$$\nabla_k \Omega = \nabla_k F = \nabla_p F\left(\frac{\partial \mathbf{p}}{\partial \mathbf{k}}\right) = \frac{1}{\omega}\nabla_p F \ .$$

Also we can write

$$\frac{\partial\Omega}{\partial\boldsymbol{\omega}} = \nabla_{\boldsymbol{p}}\boldsymbol{F}.\frac{\partial\mathbf{p}}{\partial\boldsymbol{\omega}} = \nabla_{\boldsymbol{p}}\boldsymbol{F}.\left(-\frac{\mathbf{k}}{\boldsymbol{\omega}^2}\right) = -\frac{1}{\boldsymbol{\omega}}\mathbf{p}.\nabla_{\boldsymbol{p}}\boldsymbol{F} \ .$$

Therefore,

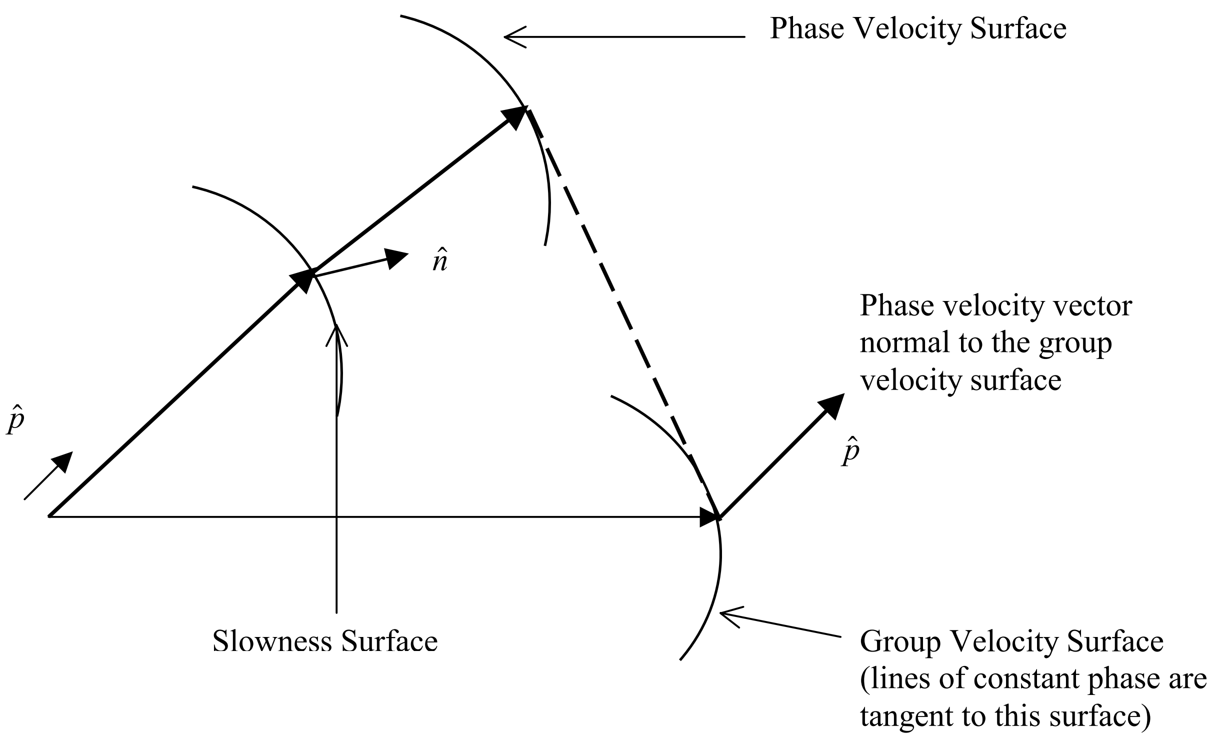
$$\mathbf{v}_g = -\frac{\frac{1}{\omega}\nabla_p F}{-\frac{1}{\omega}\mathbf{p}.\nabla_p F} = \frac{\nabla_p F}{\mathbf{p}.\nabla_p F} \ .$$

or, in terms of the slowness vector, the group velocity can be written as

$$\mathbf{v}_g = \frac{\nabla_{\mathbf{p}} F}{\mathbf{p}.\nabla_{\mathbf{p}} F} = \frac{1}{p} \frac{\hat{n}}{\cos(\mathbf{p}, \hat{n})} = \frac{\hat{n}}{\mathbf{p}.\hat{n}} \quad .$$

where $\hat{n}$ is the unit normal to the slowness surface. Thus the group velocity has the same meaning as the energy velocity (normal to the slowness surface).

The relation between group velocity, phase velocity and the slowness surface can be shown as follows:



Note

$$\boxed{\begin{aligned} \mathbf{v}_p &= \frac{1}{|\mathbf{p}|}\hat{p} \ , \\ \mathbf{v}_g &= \frac{\hat{n}}{\mathbf{p}.\hat{n}} \ . \end{aligned}}$$