SDDP.jl

Stochastic dual dynamic programming in Julia

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Northwestern

October 23, 2018

Outline

Some background

Risk

SDDP.jl tutorial

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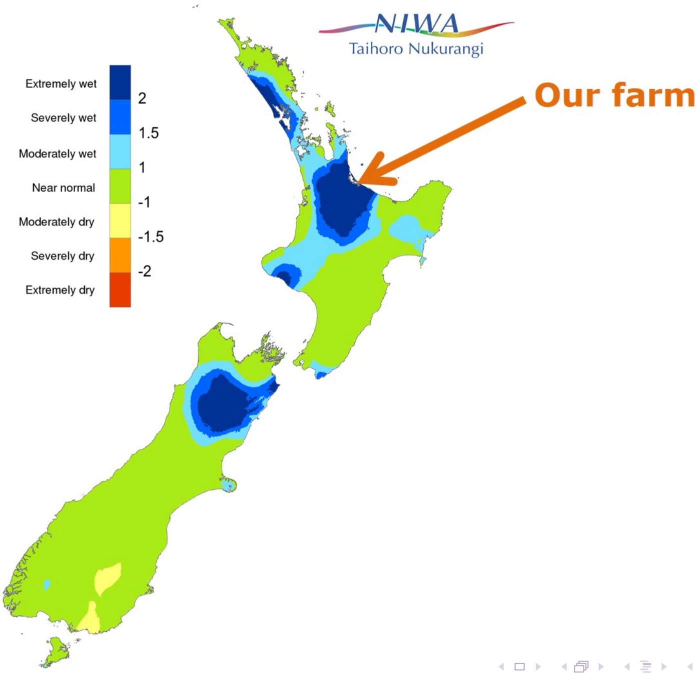
Some background

Risk

SDDP.jl tutorial


In my opinion, all palm oil should be banned.

SPI Drought Index for 9am 27/08/2017 to 9am 26/09/2017



What went wrong?

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The dairy farmer problem

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maximise:	revenue from milk production less operating costs
by deciding:	the number of cows to farm
	the quantity of grass to feed
	the quantity of supplement to feed
	when to dry-off the herd
subject to:	obtaining a high Body Condition Score at the end of the season
	uncertainty in grass growth
	uncertainty in the milk price

To learn more about this, come to my talk

Wednesday, October 31, 2018 @ 2:00 p.m. Room 274 Animal Sciences Bldg.

Outline

Why care about risk?

If the tail matters more than the average.

- For a farmer, bad years mean cows starve or you go bankrupt and lose your farm
- For electricity generators, bad years mean blackouts in cities
- In finance, bad years mean losing all your money

Definition

A *risk measure* $\mathbb{F}$ is a function that maps a random variable to a real number.

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Math

We restrict our attention to random variables with a finite sample space $\Omega := \{z_1, z_2, \ldots, z_K\}$ equipped with a sigma algebra of all subsets of Ω and respective (strictly positive) probabilities $\{p_1, p_2, \ldots, p_K\}$.

We denote the random variable with the uppercase Z.

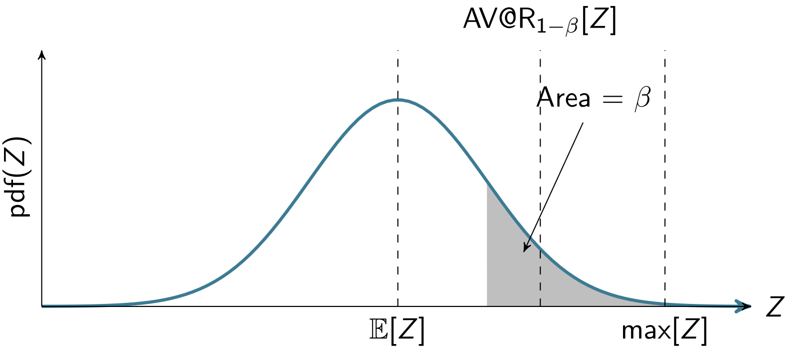
Definition

The *Average Value-at-Risk* at the β quantile ($\mathrm{AV@R}_{1-\beta}$) is:

$$\mathrm{AV@R}_{1-\beta}[Z] = \inf_{\zeta} \left\{ \zeta + \frac{1}{\beta} \sum_{k=1}^{K} p_k (z_k - \zeta)_+ \right\},$$

where $(x)_+ = \max\{0, x\}$. (Rockafellar and Uryasev 2002)

Note that when $\beta = 1$, $\mathrm{AV@R}_{1-\beta}[Z] = \mathbb{E}[Z]$, and $\lim_{\beta \to 0} \mathrm{AV@R}_{1-\beta}[Z] = \max[Z]$.



$\mathrm{AV@R}_{1-\beta}[Z]$
Area $= \beta$
pdf(Z)
$\mathbb{E}[Z]$
$\max[Z]$
Z

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Definition

A *coherent* risk measure is a risk measure $\mathbb{F}$ that satisfies the axioms of Artzner et al. 1999. For two discrete random variables Z_1 and Z_2, each with drawn from a sample space with K elements, the axioms are:

- **Monotonicity**: If $Z_1 \leq Z_2$, then $\mathbb{F}[Z_1] \leq \mathbb{F}[Z_2]$.
- **Sub-additivity**: For Z_1, Z_2, then $\mathbb{F}[Z_1 + Z_2] \leq \mathbb{F}[Z_1] + \mathbb{F}[Z_2]$.
- **Positive homogeneity**: If $\lambda \geq 0$ then $\mathbb{F}[\lambda Z] = \lambda \mathbb{F}[Z]$.
- **Translation equivariance**: If $a \in \mathbb{R}$ then $\mathbb{F}[Z + a] = \mathbb{F}[Z] + a$.

Positive homogeneity and sub-additivity give:

- **Convexity**: For $\lambda \in [0, 1]$,
  $\mathbb{F}[\lambda Z_1 + (1 - \lambda) Z_2] \leq \lambda \mathbb{F}[Z_1] + (1 - \lambda) \mathbb{F}[Z_2]$.

We can also define coherent risk measures in terms of *risk sets*. That is, a coherent risk measure $\mathbb{F}$ has a dual representation that can be viewed as taking the expectation of the random variable with respect to the worst probability distribution within some set $\mathfrak{A}$ of possible distributions:

$$\mathbb{F}[Z] = \sup_{\xi \in \mathfrak{A}} \mathbb{E}_{\xi}[Z] = \sup_{\xi \in \mathfrak{A}} \sum_{k=1}^{K} \xi_k z_k, \tag{1}$$

where $\mathfrak{A}$ is a convex subset of:

$$\mathfrak{P} = \left\{ \xi \in \mathbb{R}^K \ : \ \sum_{k=1}^{K} \xi_k = 1, \ \xi \geq 0 \right\}.$$

Static Risk Measures

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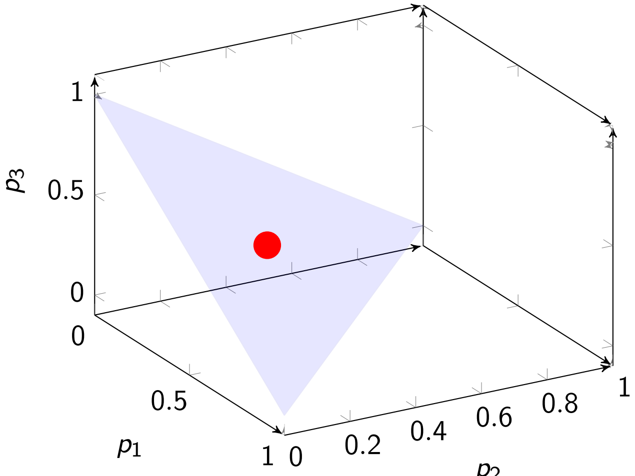
If $\mathfrak{A}$ is a singleton, containing only the original probability distribution, then the risk measure $\mathbb{F}$ is equivalent to the expectation operator.



Static Risk Measures

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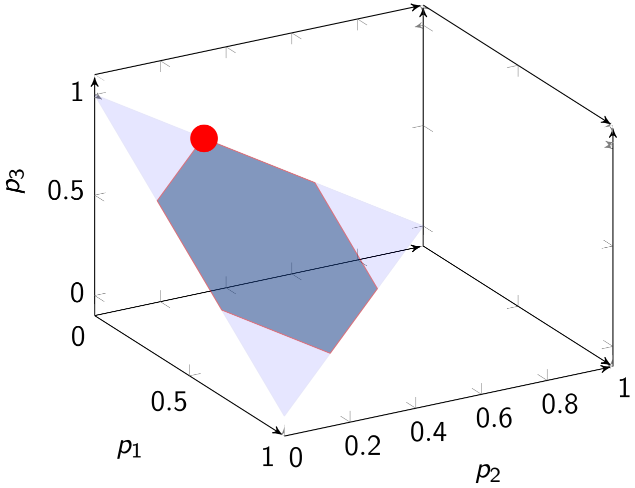
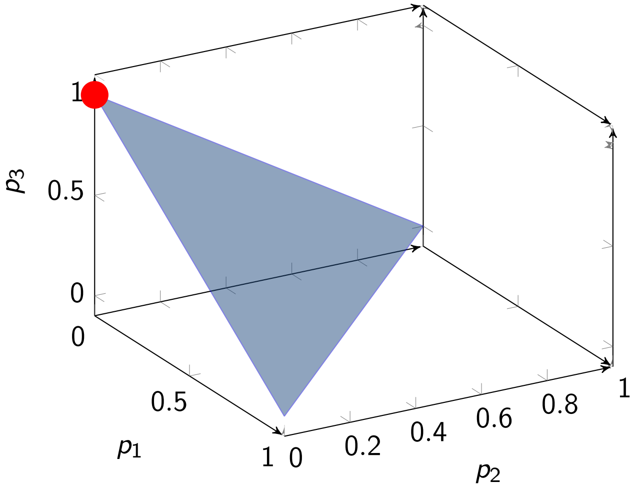
If $\mathfrak{A} = \left\{ \xi \in \mathfrak{P} \mid \xi_k \leq \frac{p_k}{\beta},\ k = 1, 2, \ldots, K \right\}$, then the risk measure $\mathbb{F}$ is equivalent to AV@R$_{1-\beta}$.



Static Risk Measures

If $\mathfrak{A} = \mathfrak{P}$, then $\mathbb{F}$ is the Worst-case risk measure.



Dynamic Risk Measures

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From static to dynamic risk measures. There are three main views:

End-of-horizon: most natural

See, e.g., Pflug and Pichler 2016; Baucke, Downward, and Zakeri 2018

$$\mathbb{F}[Z_1, Z_2, Z_3] = \mathbb{F}_{\omega_1,\omega_2,\omega_3}[Z_1 + Z_2 + Z_3]$$

Nested: easiest to compute

See, e.g., Ruszczyński 2010; Philpott, de Matos, and Finardi 2013

$$\mathbb{F}[Z_1, Z_2, Z_3] = \mathbb{F}_{\omega_1}[Z_1 + \mathbb{F}_{\omega_2|\omega_1}[Z_2 + \mathbb{F}_{\omega_3|\omega_1,\omega_2}[Z_3]\,]\,]$$

Expected conditional: a compromise

See, e.g., Homem-de-Mello and Pagnoncelli 2016

$$\mathbb{F}[Z_1, Z_2, Z_3] = \mathbb{F}_{\omega_1}[Z_1] + \mathbb{E}_{\omega_1}[\mathbb{F}_{\omega_2|\omega_1}[Z_2] + \mathbb{E}_{\omega_2|\omega_1}[\mathbb{F}_{\omega_3|\omega_1,\omega_2}[Z_3]]]$$

End-of-horizon risk measure

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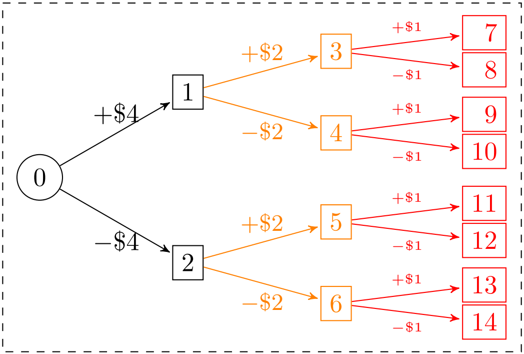

Dynamic Risk Measures

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Nested risk measure

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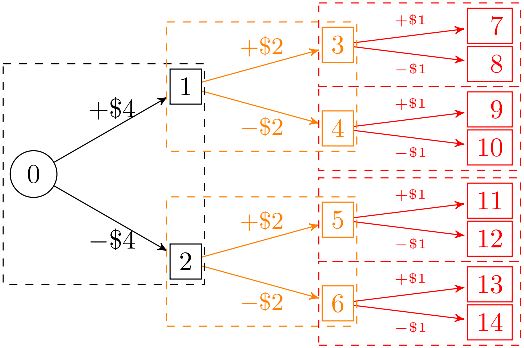

Risk in SDDP

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Recall our favourite dynamic programming recursion:

$$\begin{aligned} V_t(x_t, \omega_t) = \min_{u_t} \quad & C_t(x_t, u_t, \omega_t) + \mathop{\mathbb{E}}_{\omega_{t+1} \in \Omega_{t+1}} \left[V_{t+1}(x_{t+1}, \omega_{t+1}) \right] \\ \text{s.t.} \quad & x_{t+1} = T_t(x_t, u_t, \omega_t) \\ & u_t \in U_t(x_t, \omega_t), \end{aligned}$$

where the decision-rule $\pi_t(x_t, \omega_t)$ takes the value of u_t in the optimal solution.

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Recall

Given an original probability distribution $\{p_1, p_2, \ldots, p_K\}$ and a coherent risk measure $\mathbb{F}$, there exists a *changed* probability distribution $\{\xi_1, \xi_2, \ldots, \xi_K\}$ such that $\mathbb{F}[Z] = \mathbb{E}_\xi[Z]$.

The meat of the matter. Consider the following (re-stated) proposition from Philpott, de Matos, and Finardi 2013:

Proposition

Suppose for each $\omega \in \Omega$, that $\lambda(\bar{x}, \omega)$ is a subgradient of $V(x, \omega)$ at $\bar{x}$. Then, given $\mathbb{F}[V(\bar{x}, \omega)] = \mathbb{E}_{\xi}[V(\bar{x}, \omega)]$, $\mathbb{E}_{\xi}[\lambda(\bar{x}, \omega)]$ is a subgradient of $\mathbb{F}[V(x, \omega)]$ at $\bar{x}$.

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So what is this saying?

To obtain a cut for $\mathbb{F}_{\omega_{t+1}\in\Omega_{t+1}}[V_{t+1}(x_{t+1},\omega_{t+1})]$

- We can go and solve the $t+1$ stage problems to obtain an objective value $\bar{\theta}_{\omega_{t+1}}$ and a dual vector $\lambda_{\omega_{t+1}}$ for each realization of ω_{t+1}.
- Normally, we take the expectation of these to get the cut

$$\theta_{t+1} \geq \mathbb{E}[\bar{\theta}_{\omega_{t+1}}] + \mathbb{E}[\lambda_{\omega_{t+1}}]^\top(x_{t+1} - \bar{x}_{t+1})$$

- Instead, we compute ξ according to $\bar{\theta}_{\omega_{t+1}}$ and then take the risk-adjusted expectation to get the cut

$$\theta_{t+1} \geq \mathbb{E}_{\xi}[\bar{\theta}_{\omega_{t+1}}] + \mathbb{E}_{\xi}[\lambda_{\omega_{t+1}}]^\top(x_{t+1} - \bar{x}_{t+1})$$

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POWDER

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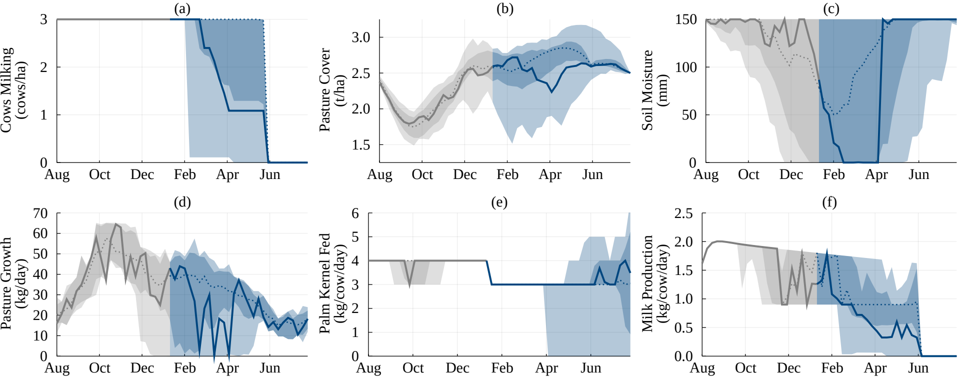
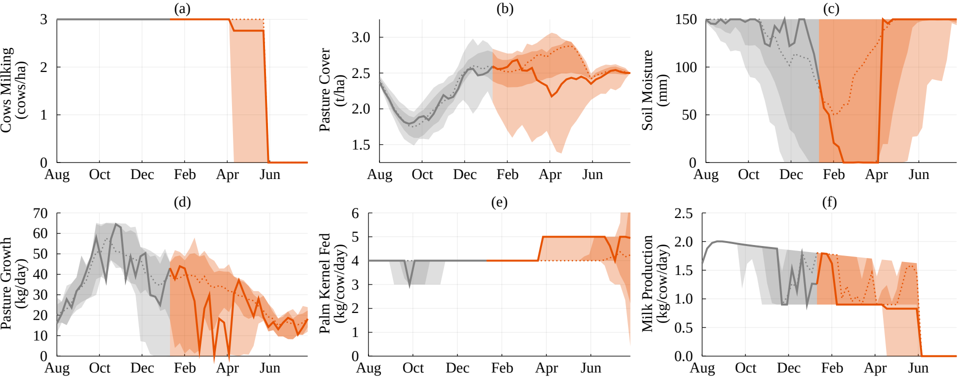


POWDER



(a)
Cows Milking (cows/ha)
3
2
1
0
Aug Oct Dec Feb Apr Jun
(b)
Pasture Cover (t/ha)
3.0
2.5
2.0
1.5
Aug Oct Dec Feb Apr Jun
(c)
Soil Moisture (mm)
150
100
50
0
Aug Oct Dec Feb Apr Jun
(d)
Pasture Growth (kg/day)
70
60
50
40
30
20
10
0
Aug Oct Dec Feb Apr Jun
(e)
Palm Kernel Fed (kg/cow/day)
6
5
4
3
2
1
0
Aug Oct Dec Feb Apr Jun
(f)
Milk Production (kg/cow/day)
2.5
2.0
1.5
1.0
0.5
0.0
Aug Oct Dec Feb Apr Jun

But Wait! Danger ahead!

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Do these nested risk measures make sense?

Remember how the *end-of-horizon* risk measure made the most sense:

$$\mathbb{F}[X_1, X_2, X_3] = \mathbb{F}[X_1 + X_2 + X_3]$$

But we actually used the *nested* risk measure:

$$\mathbb{F}[X_1, X_2, X_3] = \mathbb{F}[X_1 + \mathbb{F}[X_2 + \mathbb{F}[X_3 \mid X_1, X_2] \mid X_1]]$$

What is the interpretation of a nested risk measure? This can lead to perverse, counter-intuitive results!

POWDER

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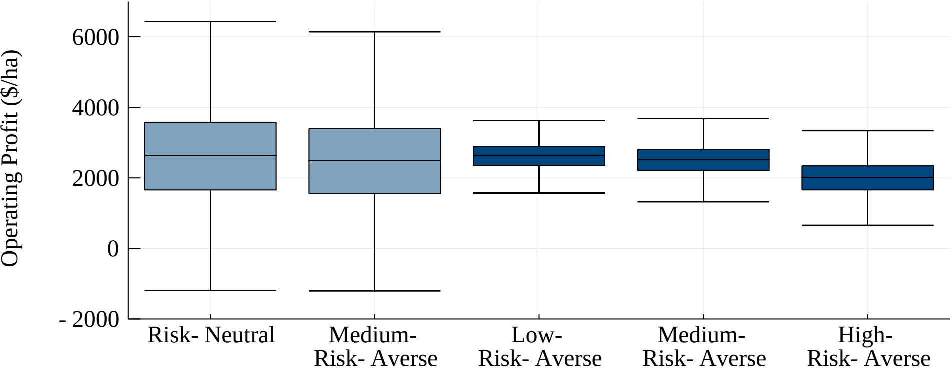

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References II

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