SDDP.jl

Stochastic dual dynamic programming in Julia

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Northwestern

October 22, 2018

Outline

Some background

Risk

SDDP.jl tutorial

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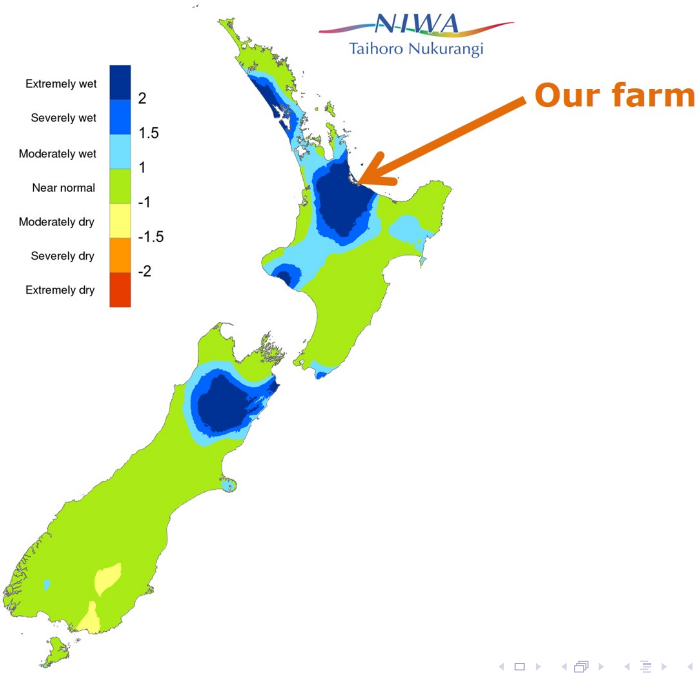
SDDP.jl tutorial


In my opinion, all palm oil should be banned.

SPI Drought Index for 9am 27/08/2017 to 9am 26/09/2017



What went wrong?

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- and they had plenty of feed so they didn't buy more.

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The dairy farmer problem

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maximise:	revenue from milk production less operating costs
by deciding:	the number of cows to farm
	the quantity of grass to feed
	the quantity of supplement to feed
	when to dry-off the herd
subject to:	obtaining a high Body Condition Score at the end of the season
	uncertainty in grass growth
	uncertainty in the milk price

To learn more about this, come to my talk

Wednesday, October 31, 2018 @ 2:00 p.m. Room 274 Animal Sciences Bldg.

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Why care about risk?

If the tail matters more than the average.

- For a farmer, bad years mean cows starve or you go bankrupt and lose your farm
- For electricity generators, bad years mean blackouts in cities
- In finance, bad years mean losing all your money

Definition

Risk measure A *risk measure* $\mathbb{F}$ is a function that maps a random variable to a real number.

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Math

We restrict our attention to random variables with a finite sample space $\Omega := \{z_1, z_2, \ldots, z_K\}$ equipped with a sigma algebra of all subsets of Ω and respective (strictly positive) probabilities $\{p_1, p_2, \ldots, p_K\}$.

We denote the random variable with the uppercase Z.

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Two simple examples that you already know

Given $\Omega = \{1, 2, 3\}$ with equal probability.

- Expectation: $\mathbb{F}[Z] = \mathbb{E}[Z] = (1 + 2 + 3)/3 = 2$
- Worst-cast: $\mathbb{F}[Z] = \max\{1, 2, 3\} = 3$

Definition

The *Average Value-at-Risk* at the β quantile ($\text{AV@R}_{1-\beta}$) is:

$$\text{AV@R}_{1-\beta}[Z] = \inf_{\zeta} \left\{ \zeta + \frac{1}{\beta} \sum_{k=1}^{K} p_k (z_k - \zeta)_+ \right\},$$

where $(x)_+ = \max\{0, x\}$.

Note that when $\beta = 1$, $\text{AV@R}_{1-\beta}[Z] = \mathbb{E}[Z]$, and $\lim_{\beta \to 0} \text{AV@R}_{1-\beta}[Z] = \max[Z]$.

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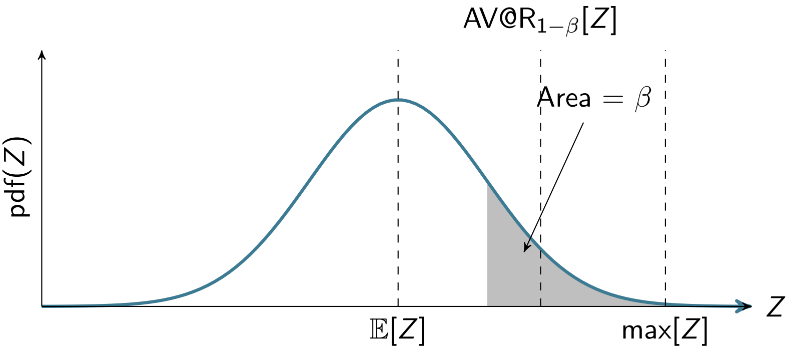

The primal formulation of AV@R

$$\min_{\xi,x} \quad \xi + \frac{1}{\beta}\frac{1}{K}\sum_{k=1}^{K} x_k \tag{1}$$

$$\text{s.t.} \quad x_k \geq z_k - \xi \tag{2}$$

$$x_k \geq 0 \tag{3}$$

AV@R Examples

Given $\Omega = \{1, 2, 3\}$ with equal probability.

- $\text{AV@R}_{1-{}^{2}/_{3}}$: $\mathbb{F}[Z] = 0.5 \times 3 + 0.5 \times 2 = 2.5$
- $\text{AV@R}_{1-0.5}$: $\mathbb{F}[Z] = 2/3 \times 3 + 1/3 \times 2 = 2\,{}^{2}/_{3}$

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Definition

A *coherent* risk measure is a risk measure $\mathbb{F}$ that satisfies the axioms of Artzner (1999). For two discrete random variables Z_1 and Z_2, each with drawn from a sample space with K elements, the axioms are:

- **Monotonicity**: If $Z_1 \leq Z_2$, then $\mathbb{F}[Z_1] \leq \mathbb{F}[Z_2]$.
- **Sub-additivity**: For Z_1, Z_2, then $\mathbb{F}[Z_1 + Z_2] \leq \mathbb{F}[Z_1] + \mathbb{F}[Z_2]$.
- **Positive homogeneity**: If $\lambda \geq 0$ then $\mathbb{F}[\lambda Z] = \lambda \mathbb{F}[Z]$.
- **Translation equivariance**: If $a \in \mathbb{R}$ then $\mathbb{F}[Z + a] = \mathbb{F}[Z] + a$.

Positive homogeneity and sub-additivity give:

- **Convexity**: For $\lambda \in [0, 1]$,
  $\mathbb{F}[\lambda Z_1 + (1 - \lambda) Z_2] \leq \lambda \mathbb{F}[Z_1] + (1 - \lambda) \mathbb{F}[Z_2]$.

We can also define coherent risk measures in terms of *risk sets*. That is, a coherent risk measure $\mathbb{F}$ has a dual representation that can be viewed as taking the expectation of the random variable with respect to the worst probability distribution within some set $\mathfrak{A}$ of possible distributions:

$$\mathbb{F}[Z] = \sup_{\xi \in \mathfrak{A}} \mathbb{E}_{\xi}[Z] = \sup_{\xi \in \mathfrak{A}} \sum_{k=1}^{K} \xi_k z_k, \tag{4}$$

where $\mathfrak{A}$ is a convex subset of:

$$\mathfrak{P} = \left\{ \xi \in \mathbb{R}^K \ : \ \sum_{k=1}^{K} \xi_k = 1, \ \xi \geq 0 \right\}.$$

z	$\mathbb{E}$	max	$\text{AV@R}_{0.5}$
1	$1/3$	0	0
2	$1/3$	0	$1/3$
3	$1/3$	1	$2/3$
$\mathbb{F}[Z]$	2	3	2.67

Table: Caption

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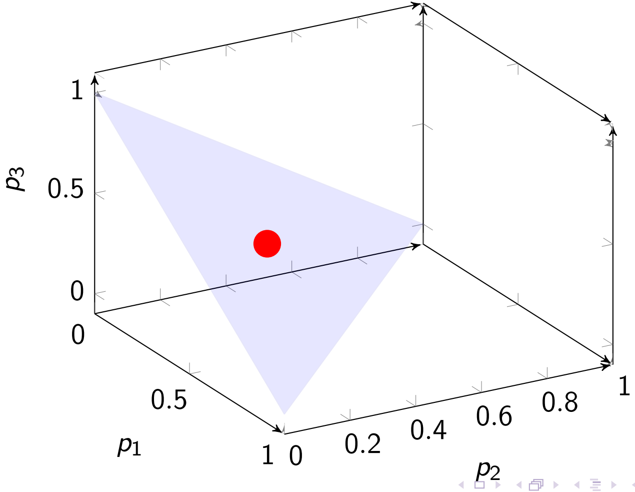
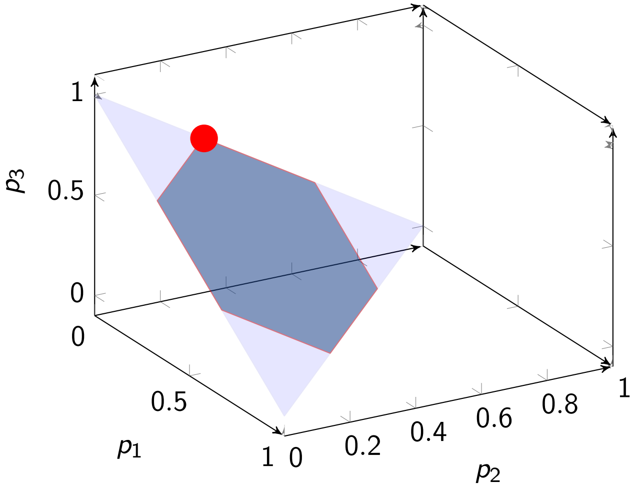
If $\mathfrak{A}$ is a singleton, containing only the original probability distribution, then the risk measure $\mathbb{F}$ is equivalent to the expectation operator.

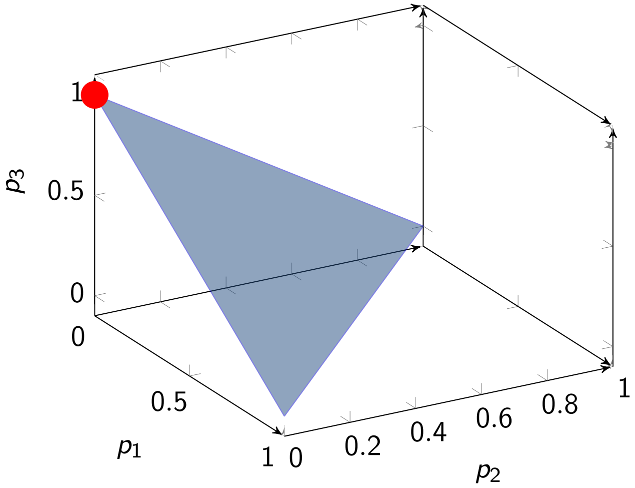


If $\mathfrak{A} = \left\{ \xi \in \mathfrak{P} \mid \xi_k \leq \frac{p_k}{\beta},\ k = 1, 2, \ldots, K \right\}$, then the risk measure $\mathbb{F}$ is equivalent to $\mathsf{AV@R}_{1-\beta}$.



Risk

If $\mathfrak{A} = \mathfrak{P}$, then $\mathbb{F}$ is the Worst-case risk measure.



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$$
\begin{aligned}
V_t(x_t, \omega_t) = \min_{u_t} \quad & C_t(x_t, u_t, \omega_t) + \underset{\omega_{t+1} \in \Omega_{t+1}}{\mathbb{E}} [V_{t+1}(x_{t+1}, \omega_{t+1})] \\
\text{s.t.} \quad & x_{t+1} = T_t(x_t, u_t, \omega_t) \\
& u_t \in U_t(x_t, \omega_t),
\end{aligned}
$$

where the decision-rule $\pi_t(x_t, \omega_t)$ takes the value of u_t in the optimal solution.

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Definition

Given an original probability distribution $\{p_1, p_2, \ldots, p_K\}$ and a coherent risk measure $\mathbb{F}$, there exists a *changed* probability distribution $\{\xi_1, \xi_2, \ldots, \xi_K\}$ such that $\mathbb{F}[Z] = \mathbb{E}_\xi[Z]$.

Definition

Suppose for each $\omega \in \Omega$, that $g(\hat{x}, \omega)$ is a subgradient of $Z(x, \omega)$ at $\hat{x}$. Then, given $\mathbb{F}[Z(x, \omega)] = \mathbb{E}_\xi[Z(x, \omega)]$, $\mathbb{E}_\xi[g(\hat{x}, \omega)]$ is a subgradient of $\mathbb{F}[Z(x, \omega)]$ at $\hat{x}$.

So what is this saying?

- We can go and solve the next stage problems to obtain an objective value $\bar{\theta}_{\omega_{t+1}}$ and a dual vector $\lambda_{\omega_{t+1}}$ for each realization of ω_{t+1}.
- Normally, we take the expectation of these to get the cut

$$\theta_{t+1} \geq \mathbb{E}[\bar{\theta}_{\omega_{t+1}}] + \mathbb{E}[\lambda_{\omega_{t+1}}]^\top (x_{t+1} - \bar{x}_{t+1})$$

- Instead, we compute ξ and then take the risk-adjusted expectation to get the cut

$$\theta_{t+1} \geq \mathbb{E}_\xi[\bar{\theta}_{\omega_{t+1}}] + \mathbb{E}_\xi[\lambda_{\omega_{t+1}}]^\top (x_{t+1} - \bar{x}_{t+1})$$

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POWDER

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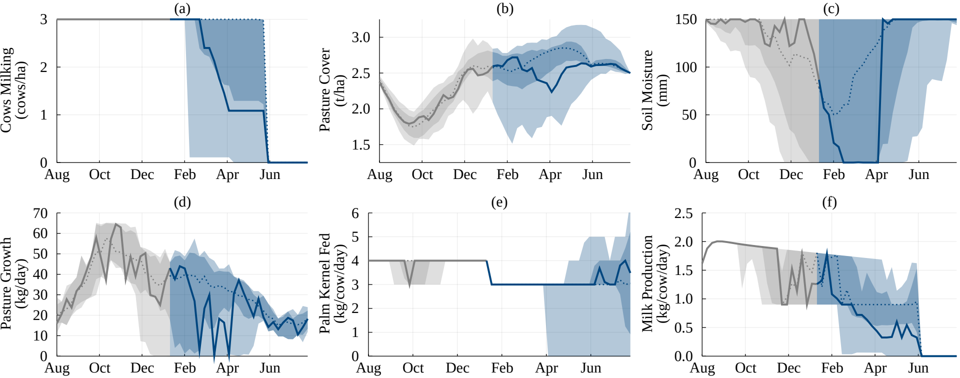

POWDER

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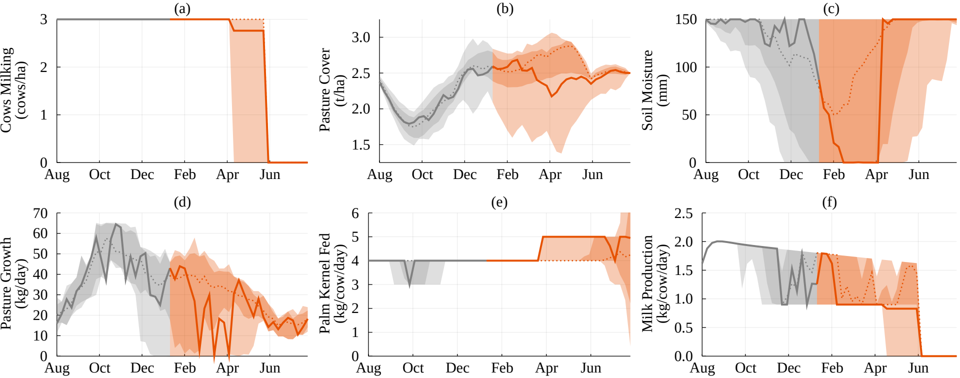

But Wait! Danger ahead!

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Do these nested risk measures make sense?

Often, we want to calculate the *end-of-horizon* value:

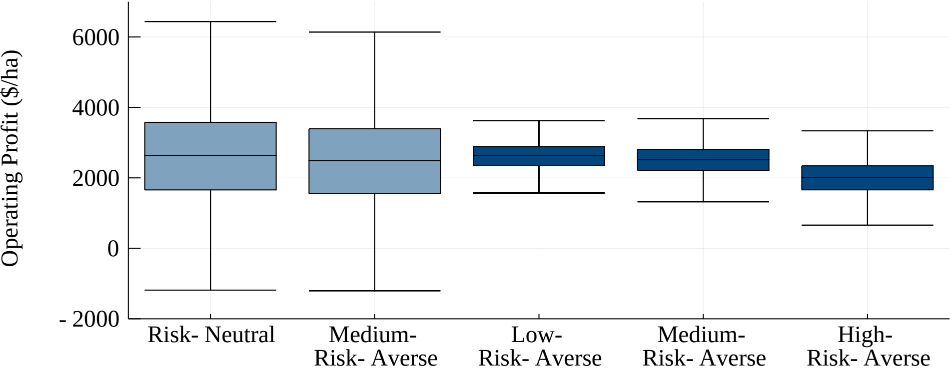
$$\mathbb{F}[X_1, X_2, X_3] = \mathbb{F}[X_1 + X_2 + X_3]$$

But we are actually calculating the *nested* value:

$$\mathbb{F}[X_1, X_2, X_3] = \mathbb{F}[X_1 + \mathbb{F}[X_2 + \mathbb{F}[X_3 \mid X_1, X_2] \mid X_1]]$$

This can lead to perverse, counter-intuitive results!

POWDER



Operating Profit ($/ha)
6000
4000
2000
0
- 2000
Risk- Neutral
Medium-
Risk- Averse
Low-
Risk- Averse
Medium-
Risk- Averse
High-
Risk- Averse