Posterior Predictive Distribution

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Posterior Predictive (Formal Derivation)

We are trying to derive the posterior predictive distribution $P(Y|D)$ for a model with parameters θ, data D, and a prediction Y.
We need the following definitions:

Definition 1: Conditional Distribution

$$P(Y_1, Y_2, Y_3 | X_1, X_2, X_3) = \frac{P(Y_1, Y_2, Y_3, X_1, X_2, X_3)}{P(X_1, X_2, X_3)}$$

Definition 2: Marginal Distribution

$$P(Y_1, Y_2) = \int P(Y_1, Y_2, Y_3) \, dY_3$$

Derivation of Posterior Predictive Distribution $P(Y|D)$

$$P(Y|D) = \frac{P(Y, D)}{P(D)} \quad \text{(by definition)}$$

$$\Rightarrow P(Y|D) = \int \frac{P(Y, D, \theta)}{P(D)} \, d\theta \quad \text{(marginalization over } \theta\text{)}$$

$$= \int P(Y|D, \theta) \frac{P(D|\theta) P(\theta)}{P(D)} \, d\theta \quad \text{(chain rule: } P(Y, D, \theta) = P(Y|D, \theta) P(D|\theta) P(\theta))$$

$$= \int P(Y|D, \theta) P(\theta|D) \, d\theta \quad \text{(Bayes' rule for } P(\theta|D))$$

$$\text{(Assuming all data relevant to } Y \text{ is captured by } \theta, P(Y|D, \theta) = P(Y|\theta))$$

$$P(Y|D) = \int P(Y|\theta) P(\theta|D) \, d\theta \quad \text{(posterior predictive distribution)}$$