Appendix for Online Publication

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A Model with optimal outcomes

Here, I provide analytic solutions to the limit distribution of housing wealth in a modified version of the model with overlapping generations and no sub-optimal transactions. The main theoretical result is that heterogeneity in the ownership of the common asset is only possible through exogenous sources of variation.

A.1 Two Identical Agents

First, I focus on the simplest version of the model: two agents iteratively buying and selling A. I assume that these agents are identical and that they meet every period to engage in a transaction. I also assume overlapping generations with a constant life expectation of $1/(1-\delta_i) = T_i = T$. Because the trading space does not exist for identical agents, this version of the model is static when $A_i(0) = A_j(0)$. However, one can induce dynamics as long as the endowments are different. For the purpose of analytic compactness, and without loss of generality, I collapse most parameters from the utility-maximization problem (equation 2 from the main text) by assuming $w_i = 1$, $\tau_i = 1$, $B_i = 0$, $s_i = 0$, $h_i = 1$ and $\delta_i = 1$. Now, $\hat{U}_i$ becomes homogeneous and the optimal equilibrium quantity is

$$q_A^* = \frac{A_j - A_i}{4}, \tag{1}$$

where $A_i < A_j$, *i.e.* i is the buyer and j is the seller.

Generally speaking, the purpose of assuming full time-consistency is to make the budget constraint binding. In this context, it means to bind housing choices, so all purchases (if the transaction space exist) are optimal. Full time-consistency, however, still carries complications for analytic tractability. Nevertheless, it is possible to bind housing decisions to the budget constraint by assuming $T \leq \alpha$. In this way, I can circumvent the complications of the inter-temporal optimization problem, which allows me to study the evolution of A_i through standard mathematical tools from dynamical system.

Now, assume that $A_i(0) < A_j(0)$, so i is the buyer and j is the seller. According to equation 1, purchasing does not change the roles of the agents. Hence, as long as the transaction occurs in the optimal equilibrium, $A_i(t) < A_j(t)$ for any period t. Then, the dynamic equations describing the evolution of the agents' shares of the asset are

$$A_i(t+1) = A_i(t) + q_A^* \tag{2}$$

$$A_j(t+1) = A_j(t) - q_A^*, \tag{3}$$

which can be rewritten as

$$A_i(t+1) = \frac{3}{4}A_i(t) + \frac{1}{4}A_j(t) \tag{4}$$

$$A_j(t+1) = \frac{3}{4}A_j(t) + \frac{1}{4}A_i(t). \tag{5}$$

Next, let me define inequality in the distribution of the common asset as the absolute difference $|A_i(t+1) - A_j(t+1)|$. Then, replacing equations 4 and 5 in this expression yields

$$|A_i(t+1) - A_j(t+1)| = \frac{|A_i(t) - A_j(t)|}{2}. \tag{6}$$

This means that, with every trade, the difference in assets between agents shrinks by half, independently of who is the buyer and seller. As time goes by, inequality in this two-agent society vanishes, so the model converges to a stable egalitarian distribution. Next I generalize this result for a population of N homogeneous agents with different endowments.

A.2 N Identical Agents

Suppose that the assumptions in the previous section hold for a population of N agents with different endowments of the common asset. Every period, agents are randomly paired[1]. This generates two sources of uncertainty: (1) the probability $1/(N-1)$ of agent i being matched to j, and (2) the probability θ that i is the buyer and j is the seller. Considering these elements, the evolution equation of i's share of the common asset is given by

$$\begin{aligned} A_i(t+1) = \frac{1}{N-1}\left[\theta_{i1}(A_i(t)+q^*_{i1}) + (1-\theta_{i1})(A_i(t)-q^*_{i1})\right] + \dots \\ + \frac{1}{N-1}\left[\theta_{i(i-1)}(A_i(t)+q^*_{i(i-1)}) + (1-\theta_{i(i-1)})(A_i(t)-q^*_{i(i-1)})\right] \\ + \frac{1}{N-1}\left[\theta_{i(i+1)}(A_i(t)+q^*_{i(i+1)}) + (1-\theta_{i(i+1)})(A_i(t)-q^*_{i(i+1)})\right] + \dots \\ + \frac{1}{N-1}\left[\theta_{iN}(A_i(t)+q^*_{iN}) + (1-\theta_{iN})(A_i(t)-q^*_{iN})\right], \end{aligned} \tag{7}$$

which simplifies into

$$A_i(t+1) = \frac{3}{4}A_i(t) + \frac{1}{4}\left(\frac{1}{N-1}\sum_{j\neq i} A_j(t)\right). \tag{8}$$

Note that equation 8 is the sum of the ith column of the stochastic matrix

[1]This assumes an even N, although the case of an odd population can be easily derived through the same logic.

$$\begin{bmatrix} \frac{3}{4} & \frac{1}{4(N-1)} & \frac{1}{4(N-1)} & \cdots \\ \frac{1}{4(N-1)} & \frac{3}{4} & \frac{1}{4(N-1)} & \cdots \\ \frac{1}{4(N-1)} & \frac{1}{4(N-1)} & \frac{3}{4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{9}$$

Therefore, one can treat this problem as a Markov chain where the probability π_i of being in state i in the steady state represents the share of A held by agent i. Then, the stationary distribution $[\pi_1, \pi_2, \ldots, \pi_N]$ can be obtained by solving the linear system

$$\begin{bmatrix} 0 & \pi_2 & \pi_3 & \cdots \\ \pi_1 & 0 & \pi_3 & \cdots \\ \pi_1 & \pi_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} \frac{N-1}{N} \\ \frac{N-1}{N} \\ \frac{N-1}{N} \\ \vdots \end{bmatrix}, \tag{10}$$

which yields

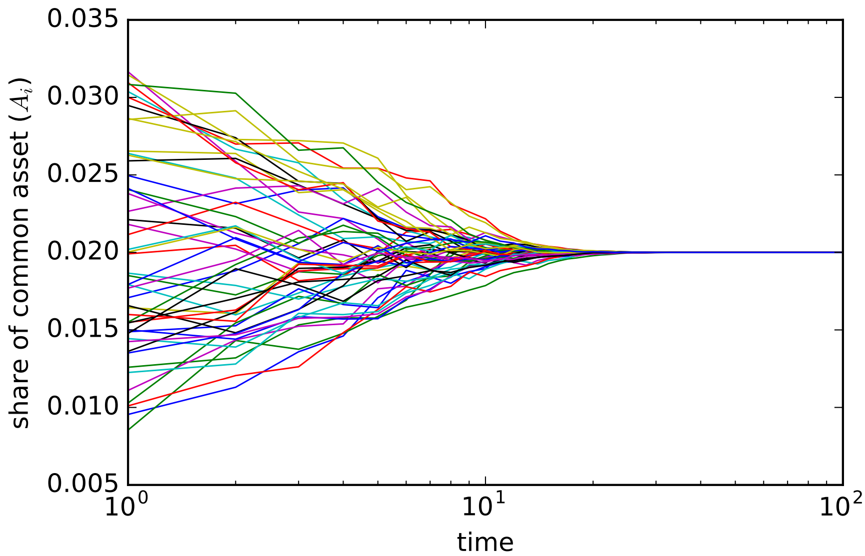
$$\pi_i = \frac{1}{N} \Rightarrow A_i = \frac{1}{N} \sum_j A_j \forall i. \tag{11}$$

Therefore, the population with N agents also reaches an egalitarian distribution of the common asset. Figure 1 shows a simulation for a population of 50 agents with randomly chosen initial endowments of A. Regardless of the initial endowments and of the population size, all agents invariably end up with the same amount of common asset. Next, I extend this Markov analysis to a population of N heterogeneous agents.

A.3 Heterogeneous Agents

First, allow me to homogenize the agents' endowments in order to specify a population of heterogeneous agents with an identical $A_i(0)$. Heterogeneity can be induced through any parameter. However, in order to guarantee optimal equilibrium outcomes, I need to make sure that the heterogeneous parameter δ_i

Figure 1: Individual trajectories to egalitarian distribution



is such that $T_i \leq \alpha_i$. Once more, for compactness and without loss of generality, let me focus on the case in which the only parameter is δ. In this case, the optimal equilibrium quantity is

$$q_A^* = \frac{\delta_i A_j - \delta_j A_i}{2(\delta_i + \delta_j)}, \tag{12}$$

and the evolution of agent i's share is given by

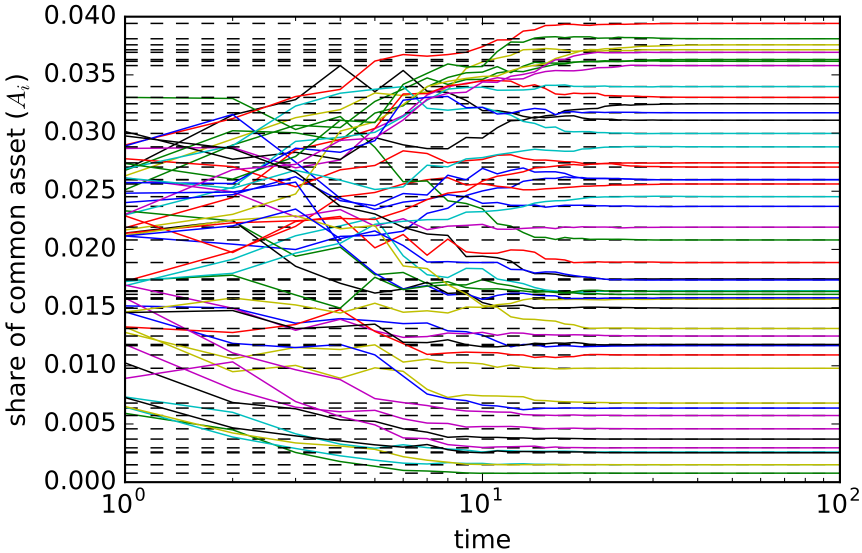
$$A_i(t+1) = \left[1 - \frac{\sum_{j \neq i} \delta_j}{2 \sum_j \delta_j}\right] A_i(t) + \frac{\delta_i}{2 \sum_j \delta_j} \sum_{j \neq i} A_j(t). \tag{13}$$

It follows that, in the steady state, the share of agent i is

$$A_i = \frac{\delta_i}{\sum_j \delta_j} \sum_j A_j. \tag{14}$$

This means that i's share is proportional to its life expectation, so inequality depends on the distribution of expectations. Figure 2 confirms this result for 50 agents and randomly chosen δ_is. Independently of the population size, the economy invariably reaches a distribution of the common asset that is dictated by the distribution of δ_i.

Figure 2: Individual trajectories to predicted distribution



The dashed lines indicate analytic predictions.

The same result emerges when introducing additional parameters, i.e. the distribution of housing wealth is always a function of the exogenous heterogeneity. Of course, the solutions for those specifications will be more complicated than this one. The important takeaway is that, in order to produce heterogeneity in the ownership of A under optimal equilibria, exogenous sources of variation are necessary. This is so because (1) a population of identical agents will not transact at all or (2) homogeneous agents with different endowments will converge to an egalitarian society. Thus, time-consistency runs the risk of neglecting effects of the *tatonement*.

On a final note, the optimally-induced version of the model presented here does not have any stochastic element other than the random matching process (because agents do not die anymore). In fact, the distributions are entirely deterministic as they are a one-to-one map from the model parameters. This implies that, regardless of the particular order of interaction, all agents eventually arrive to their corresponding shares of common asset. In the model presented in the main text, the dynamics are richer since the birth/death induces flows of the common asset across the population. In fact, although the aggregate dynamics reach a steady state, the individual trajectories of the agents do not settle in a fixed point. Such dynamics are destroyed in a fully rational overlapping generations approach.

B Connection to the one parameter inequality process

An interesting result is that the evolution equation 7 has a direct connection to the one parameter inequality process (OPIP) created by Angle (1986). OPIP consists of individuals randomly matched, and with each encounter translating into i transferring wealth to j or vice versa with a probability D_{ij}. Marching my notation to Angle's one, $D_{ij} = \theta_{ij}$, so the OPIP dynamic equations of two agents are

$$\begin{array}{c} A_i(t+1) = A_i(t) + \theta_{ij}\omega_{ij}A_j(t) - (1-\theta ij)\omega_{ij}A_i(t) \\ A_j(t+1) = A_j(t) + \theta_{ij}\omega_{ij}A_i(t) - (1-\theta ij)\omega_{ij}A_j(t) \end{array}, \tag{15}$$

where ω_{ij} is the fraction of wealth to be transferred.[2].

Note that the term $\theta_{ij}(A_i(t) + q^*_{ij}) + (1-\theta_{ij})(A_i(t) - q^*_{ij})$ from equation 7 can be rewritten as $A_i(t) + \theta_{ij}q^*_{ij} - (1-\theta_{ij}q^*_{ij})$, which has the same form as the OPIP equations. Therefore, by replacing the previous expression in one of the OPIP ones I get that parameter ω_{ij} from the OPIP is

[2]This particular specification with a differentiated ω for each pair of agents corresponds to a latter extension of the OPIP model called the inequality process with distributed omega (Angle, 1986).

$$\omega_{ij} = \frac{q^*_{ij}}{A_i(t)} = \frac{1}{4}\left(\frac{A_j(t)}{A_i(t)} - 1\right). \tag{16}$$

Angle (1986) shows that ω is responsible for the shape of the wealth distribution (he derives Gamma and Pareto distributions). Thus, I have found a regime under which the OPIP model generates a delta distribution. Interestingly, such regime corresponds to optimal equilibrium outcomes derived from economic interactions in a homogeneous population.[3]

For the case with heterogeneous expectations, the OPIP parameter is

$$\omega_{ij} = \frac{1}{2(\delta_i + \delta_j)}\left(\delta_i \frac{A_j(t)}{A_i(t)} - \delta_j\right). \tag{17}$$

This result can also be obtained from heterogeneity in other parameters (but with more complicated solutions). From this, I can say that the OPIP main parameter can be derived from the agents' characteristics and their interactions in an optimal equilibrium setting.

C Ergodic-like aggregate behavior

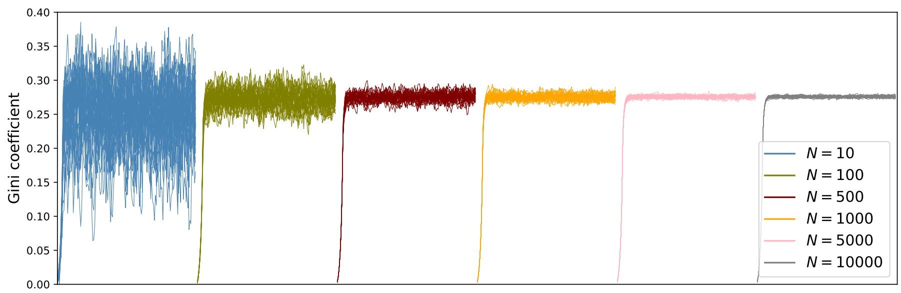
From simulation experiments, the model seems to be stable at the aggregate level once it has passed the transient. However, it is not clear whether the outcome of a simulation would be, on average, the same from another independent run. Furthermore, it is not obvious that these outcomes are insensitive to scale. Because of the discrete nature of the transaction outcomes, it is not straightforward to establish the properties of the Markov chain behind the full model, although one would expect that it has similar qualities as the one presented in Appendix A. For this reason, I provide evidence of ergodic-like behavior through simulation.

Figure 3 shows different time series corresponding to independent simulations grouped by population size. In there, I run the model under a hypothetical

[3] Note that most of these econophysics models assume homogeneous agents since they are inspired on the interactions of indistinguishable particles.

paremeterization presented in Table 1 of the main text. I vary the population in sizes of 10, 100, 500, 1000, 5000 and 10000 agents. This graphic provides two important insight. First, the steady-state Gini coefficient shows variance that is sensitive to scale. Second, the larger the population, the smaller the inter-temporal variance becomes. Hence, for large populations one would expect the Gini to exhibit ergodic-like behavior. Thus, running the model at a one-to-one scale with Great Britain should provide robust estimates.

Figure 3: Individual trajectories to predicted distribution



References

Angle, J. (1986). The Surplus Theory of Social Stratification and the Size Distribution of Personal Wealth. *Social Forces*, 65(2):293–326.