1 Formal proof

Lau (1982) assumes that both demand and marginal cost functions are twice continuously differentiable.

Claim 1. Assume that the inverse demand function satisfies twice continuously differentiability and the demand rotation IV and the aggregate quantity are non-separable, and the marginal cost function is linear. The conduct parameter and the marginal cost parameters are identified.

Let $P = f(Q, z_1)$ be a twice continuously differentiable function and $MC = \gamma_0 + \gamma_1 Q + \gamma_2 z_2$. For the sake of a contradiction, suppose that the parameters are not identified. This implies that for $(\theta, \gamma_0, \gamma_1, \gamma_2) \neq (\theta', \gamma_0', \gamma_1', \gamma_2')$, two first-order conditions

$$f(Q, z_1) = -\theta \frac{\partial f}{\partial Q}(Q, z_1) Q + \gamma_0 + \gamma_1 Q + \gamma_2 z_2 \tag{1}$$

$$f(Q, z_1) = -\theta' \frac{\partial f}{\partial Q}(Q, z_1) Q + \gamma_0' + \gamma_1' Q + \gamma_2' z_2, \tag{2}$$

where the reduced form function $Q = h_2(z_1, z_2)$ and $Q = h_2'(z_1, z_2)$ from (1) and (2) respectively are identical.

This in turn implies

$$\frac{\partial h_2}{\partial z_1} = \frac{\frac{\partial f}{\partial z_1} + \theta \frac{\partial^2 f}{\partial z_1 \partial Q} Q}{(1+\theta)\frac{\partial f}{\partial Q} + \theta \frac{\partial^2 f}{\partial Q^2} Q - \gamma_0} = \frac{\partial h_2'}{\partial z_1} = \frac{\frac{\partial f}{\partial z_1} + \theta' \frac{\partial^2 f}{\partial z_1 \partial Q} Q}{(1+\theta')\frac{\partial f}{\partial Q} + \theta' \frac{\partial^2 f}{\partial Q^2} Q - \gamma_0'} \tag{3}$$

and

$$\frac{\partial h_2}{\partial z_2} = -\frac{\gamma_2}{(1+\theta)\frac{\partial f}{\partial Q} + \theta \frac{\partial^2 f}{\partial Q^2} Q - \gamma_0} = \frac{\partial h_2'}{\partial z_2} = -\frac{\gamma_2'}{(1+\theta')\frac{\partial f}{\partial Q} + \theta' \frac{\partial^2 f}{\partial Q^2} Q - \gamma_0'} \tag{4}$$

From (3) and (4), we have

$$\frac{\frac{\partial f}{\partial z_1} + \theta \frac{\partial^2 f}{\partial z_1 \partial Q} Q}{\frac{\partial f}{\partial z_1} + \theta' \frac{\partial^2 f}{\partial z_1 \partial Q} Q} = \frac{\gamma_2}{\gamma_2'}$$

$$\implies \frac{\theta \gamma_2' - \theta' \gamma_2}{\gamma_2' - \gamma_2} \frac{\partial^2 f}{\partial z_1 \partial Q} Q + \frac{\partial f}{\partial z_1} = 0 \tag{5}$$

Recall that we assumed that $\gamma_2 \neq \gamma_2'$ and $\theta \neq \theta'$. When $\theta \gamma_2' - \theta' \gamma_2 = 0$, the left hand side of (5) implies that $\frac{\partial f}{\partial z_1} = 0$ for all z_1, which contradicts to the fact that f is twice continuously differentiable.

When $\theta\gamma_2' - \theta'\gamma_2 \neq 0$, (5) implies that for $i \neq j$,

$$\frac{\partial}{\partial Q}\left(\frac{\partial f}{\partial z_{1i}} / \frac{\partial f}{\partial z_{1j}}\right) = 0 \tag{6}$$

which implies when Q and z are separable. However this also contradicts to the assumption. Therefore, we can conclude that the parameters are identified.

2 Identification under log-log demand

2.1 log-log demand and linear marginal cost

Let Q^*_{mt} be a new exogenous variable. The supply relationship can be written as

$$P_{mt}(1 - \theta Q^*_{mt}) = \gamma_0 + \gamma_1 Q_{mt} + \varepsilon_{mt}.$$

Take any two different combinations of parameters, $(\theta, \gamma_0, \gamma_1) \neq (\theta, \gamma_0, \gamma_1)$. Without loss of generality, assume that $\theta' > \theta$. Observable equivalence implies that the following two equations

$$\begin{aligned} P_{mt}(1 - \theta Q^*_{mt}) &= \gamma_0 + \gamma_1 Q_{mt} + \varepsilon_{mt}, \\ P_{mt}(1 - \theta' Q^*_{mt}) &= \gamma_0' + \gamma_1' Q_{mt} + \varepsilon_{mt} \end{aligned}$$

are equivalent for all (Q^*_{mt}, Q_{mt}). By taking expectation for both sides of equations and combining the equations, we have

$$\begin{aligned} P_{mt}(1 - \theta Q^*_{mt}) &= \gamma_0 + \gamma_1 Q_{mt}, \\ P_{mt}(1 - \theta' Q^*_{mt}) &= \gamma_0' + \gamma_1' Q_{mt}, \\ \Longrightarrow \log\left(\frac{1 - \theta Q^*_{mt}}{1 - \theta' Q^*_{mt}}\right) &= \log\left(\frac{\gamma_0 + \gamma_1 Q_{mt}}{\gamma_0' + \gamma_1' Q_{mt}}\right). \end{aligned} \tag{7}$$

By the assumption, the left hand side is concave and increasing in Q^*_{mt}. For the left hand side, the derivative with respect to Q_{mt} is

$$\begin{aligned} \frac{d}{dQ_{mt}} \log\left(\frac{\gamma_0 + \gamma_1 Q_{mt}}{\gamma_0' + \gamma_1' Q_{mt}}\right) &= \frac{1}{\frac{\gamma_0 + \gamma_1 Q_{mt}}{\gamma_0' + \gamma_1' Q_{mt}}} \frac{\gamma_1(\gamma_0' + \gamma_1' Q_{mt}) - \gamma_1'(\gamma_0 + \gamma_1 Q_{mt})}{(\gamma_0' + \gamma_1' Q_{mt})^2} \\ &= \frac{\gamma_1\gamma_0' - \gamma_1'\gamma_0}{(\gamma_0 + \gamma_1 Q_{mt})(\gamma_0' + \gamma_1' Q_{mt})}. \end{aligned} \tag{8}$$

Since $\left(\frac{\gamma_0+\gamma_1 Q_{mt}}{\gamma_0'+\gamma_1' Q_{mt}}\right)$ can not be negative, $\gamma_0 + \gamma_1 Q_{mt}$ and $\gamma_0' + \gamma_1' Q_{mt}$ should have same sign, which implies that the denominator of (8) is positive. When $\gamma_1\gamma_0' - \gamma_1'\gamma_0 = 0$, the right hand side is constant, and both sides cross at most once. When $\gamma_1\gamma_0' - \gamma_1'\gamma_0 < 0$, the right hand side is decreasing and concave in Q_{mt}, and hence both sides cross at most once. When $\gamma_1\gamma_0' - \gamma_1'\gamma_0 > 0$, the right hand side is increasing and concave in Q_{mt}. In this case, when (Q_{mt}, Q_{mt}^*) satisfies

$$Q_{mt}^* = \frac{(\gamma_0 - \gamma_0') + (\gamma_1 - \gamma_1')Q_{mt}}{(\theta'\gamma_0 - \theta\gamma_0') + (\theta'\gamma_1 - \theta\gamma_1')Q_{mt}},$$

both sides coincide, and hence we can not identify the parameters. If not, both sides cross at most twice. Therefore, for all cases, when (Q_{mt}, Q_{mt}^*) has more than three values, we can identify the parameters.

2.2 log-log demand and log marginal cost

Take any two different combinations of parameters, $(\theta, \gamma_0, \gamma_1) \neq (\theta', \gamma_0', \gamma_1')$. Without loss of generality, assume that $\theta' > \theta$. Observable equivalence implies that given two equations,

$$\log P_{mt} = -\log\left(1 - \theta Q_{mt}^*\right) + \gamma_0 + \gamma_1 Q_{mt} + \varepsilon_{mt}, \tag{9}$$

$$\log P_{mt}' = -\log\left(1 - \theta' Q_{mt}^*\right) + \gamma_0' + \gamma_1' Q_{mt} + \varepsilon_{mt}, \tag{10}$$

$\log P_{mt}$ and $\log P_{mt}'$ are equivalent for all (Q_{mt}, Q_{mt}^*). From these two equations, we have

$$0 = -\log\left(\frac{1 - \theta Q_{mt}^*}{1 - \theta' Q_{mt}^*}\right) + (\gamma_0 - \gamma_0') + (\gamma_1 - \gamma_1')Q_{mt} \tag{11}$$

By solving the equation with respect to Q_{mt}^*, we have

$$Q_{mt}^* \equiv f(Q_{mt}) = \frac{\exp((\gamma_0 - \gamma_0') + (\gamma_1 - \gamma_1')Q_{mt})}{\theta' \exp((\gamma_0 - \gamma_0') + (\gamma_1 - \gamma_1')Q_{mt}) - \theta}.$$

Denote the graph of f as $G = \{(Q_{mt}, f(Q_{mt}) \in \mathbb{R}^2 \mid Q_{mt} \in [0, \bar{Q}]\}$. This implies that given Q_{mt}, the value of Q_{mt}^* satisfying (11) is uniquely determined and observable equivalence holds only for the combinations of Q_{mt} and Q_{mt}^* that are in G. In other words, any combination of Q_{mt} and Q_{mt}^* not included in G violates observable equivalence. Q: Does G have zero measure? it seems yes. https://math.stackexchange.com/questions/920101/a-circle-is-a-set-of-measure-zero-generalizations

Claim 2. If f is non-decreasing and concave and g is concave, a composite function $h(x) = f(g(x))$ is also a concave function. If g is increasing, h is increasing, and if g is decreasing, h is decreasing.

References

Lau, Lawrence J, "On identifying the degree of competitiveness from industry price and output data," *Economics Letters*, 1982, *10* (1-2), 93–99. 1