

# Bayesian statistics with R

## 2. The likelihood

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## Likelihood

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- Usually, when talking about probability distributions, we assume that we know the parameter values.

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- In the real world, it is usually the other way around.

## A question of interest might be for example:

*We have observed 3 births by a female during her 10 breeding attempts. What does this tell us about the true probability of getting a successful breeding attempt for this female? For the population?*

- We don't know what the probability of a birth is.
- But we can see what the probability of getting our data would be for different values:

```
dbinom(x=3,size=10,prob=0.1)
```

```
#> [1] 0.05739563
```

- We don't know what the probability of a birth is.
- But we can see what the probability of getting our data would be for different values:

```
dbinom(x=3,size=10,prob=0.9)
```

```
#> [1] 8.748e-06
```

- We don't know what the probability of a birth is.
- But we can see what the probability of getting our data would be for different values:

```
dbinom(x=3,size=10,prob=0.25)
```

```
#> [1] 0.2502823
```



```
dbinom(x=3,size=10,prob=0.1)
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dbinom(x=3,size=10,prob=0.9)
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dbinom(x=3,size=10,prob=0.25)
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So we would be more likely to observe 3 births if the probability is 0.25 than 0.1 or 0.9.

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- This reasoning is so common in statistics that it has a special name.
- **The likelihood** is the probability of observing the data under a certain model.
- The data are known, we usually consider the likelihood as a function of the model parameters  $\theta_1, \theta_2, \dots, \theta_p$

$$L = P(\theta_1, \theta_2, \dots, \theta_p \mid \text{data})$$

## Likelihood functions

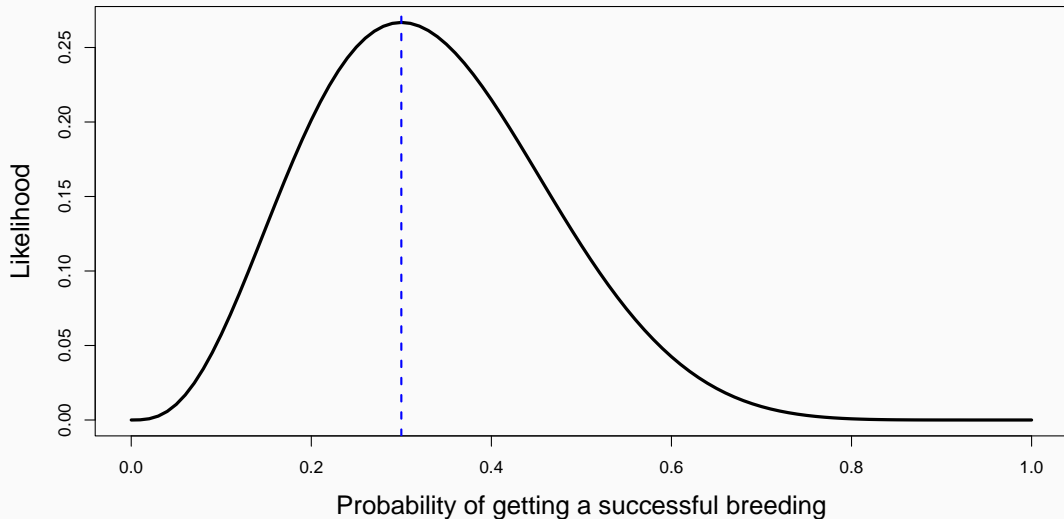
We may create a function to calculate a likelihood:

```
lik.fun <- function(parameter){  
  ll <- dbinom(x=3, size=10, prob=parameter)  
  return(ll)  
}
```

```
lik.fun(0.3)  
#> [1] 0.2668279
```

```
lik.fun(0.6)  
#> [1] 0.04246733
```

## Maximize the likelihood (3 successes out of 10 attempts)



The *maximum* of the likelihood is at value 0.3

# Maximum likelihood estimation

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- There is always a set of parameters that gives you the highest likelihood of observing the data, and this is the MLE.
- These can be calculated using:
  - Trial and error (not efficient!).
  - Compute the maximum of a function by hand (rarely doable in practice).
  - An iterative optimization algorithm: `?optim` in R.



**By hand: compute MLE of  $p$  from  $Y \sim \text{Bin}(N = 10, p)$  with  $k = 3$  successes**

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- $\log(L(p)) = \text{cte} + k \log(p) + (N - k) \log(1 - p).$

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- Compute derivate w.r.t.  $p$ :  $\frac{d \log(L)}{dp} = \frac{k}{p} - \frac{(N - k)}{(1 - p)}$ .
- Then solve  $\frac{d \log(L)}{dp} = 0$ ; the MLE is  $\hat{p} = \frac{k}{N} = \frac{3}{10} = 0.3$ .
- Here, the MLE is the proportion of observed successes.

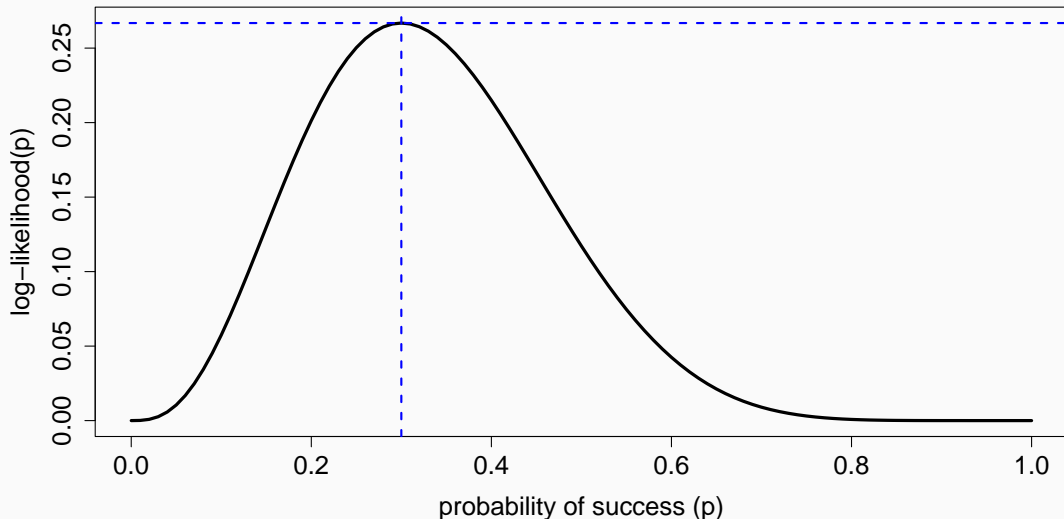
## Using a computer: MLE of $p$ from $Y \sim \text{Bin}(N = 10, p)$ with $k = 3$ successes

```
lik.fun <- function(parameter) dbinom(x=3, size=10, prob=parameter)
# ?optimize
optimize(lik.fun, c(0,1), maximum=TRUE)
#> $maximum
#> [1] 0.3000157
#>
#> $objective
#> [1] 0.2668279
```

Use `optim` when the number of parameters is  $> 1$ .

## Using a computer: MLE of $p$ from $Y \sim \text{Bin}(N = 10, p)$ with $k = 3$ successes

Binomial likelihood with 3 successes out of 10 attempts





## Your turn: Practical 2

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