PERSPECTIVE PROJECTION

In a great deal of graphical work (for animation and simulation as well as for virtual reality) the prime intention is to present an image that is as realistic as possible, so the viewer's sense of depth perception must be enhanced. The most easily managed type of depth cues for this purpose are those that depend on the techniques of <u>perspective</u>. When perspective has been used, lines that are parallel appear to converge to <u>vanishing points</u> and as a consequence the size of object appears to be reduced according to its distance from the viewing point, and foreshortening occurs according to orientation and distance.

Projection and perspective are separate processes: projection acts on a three-dimensional object and produces a two-dimensional image, while perspective, like rotation, reflection, translation or scaling, is a transformation that acts on a three-dimensional object to give us a three-dimensional image. However, unlike these others, perspective is a transformation that always distorts the shape of objects.

The rotation group R: deaaling with local scaling, rotation about an axis or reflection in a plane

$$\begin{bmatrix} a & d & g & h \\ b & e & i & k \\ c & f & j & l \\ p & q & r & s \end{bmatrix}$$

The translation elements T: for shifts in the directions of the axes.

For overall scaling, S

perspective elements, P

Thus when we have a matrix

$$\begin{bmatrix} a & d & g & h \\ b & e & i & k \\ c & f & j & l \\ p & q & r & s \end{bmatrix}$$

and not all of the values p, q, r are zero, then this matrix will perform a transformation involving perspective. We now proceed to investigate the simplest form of perspective, and then to combine it with an orthographic projection to give us a <u>perspective-projection</u> which may be viewed on a two-dimensional screen.

Single point Perspective:

When the transforming matrix has just one non-zero entry among the down three positions of its final row then it produces a <u>single point perspective</u>. The following matrix, where $r \neq 0$, is the simplest one we can write; it produces a single point perspective with the viewing point on the z-axis:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 \end{bmatrix}$$

Of course there are similar matrices that will produce single point perspectives with viewing points on the x-axis (when $q = r = 0$, but $p \neq 0$), or on the y-axis (when $p = r = 0$, but $q \neq 0$).

When we apply the above matrix to a general homogeneous vector we get

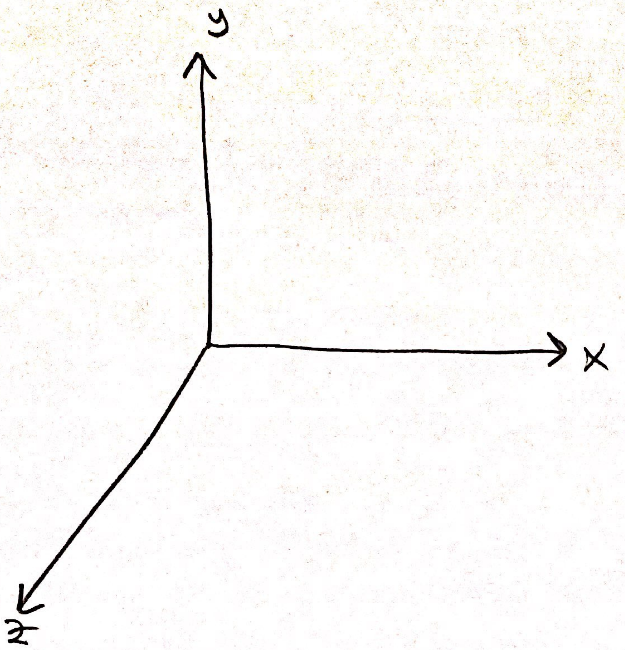
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ rz+1 \end{pmatrix} = \begin{pmatrix} \frac{x}{rz+1} \\ \frac{y}{rz+1} \\ \frac{z}{rz+1} \\ 1 \end{pmatrix}$$

so we see that the physical coordinates of the image point are $\left(\frac{x}{rz+1}, \frac{y}{rz+1}, \frac{z}{rz+1}\right)$. These coordinates give the transformed point in three dimensions.

We take the homogeneous vector corresponding to the coordinates of the point transformed by perspective as above and project them onto the plane $z=0$. We get

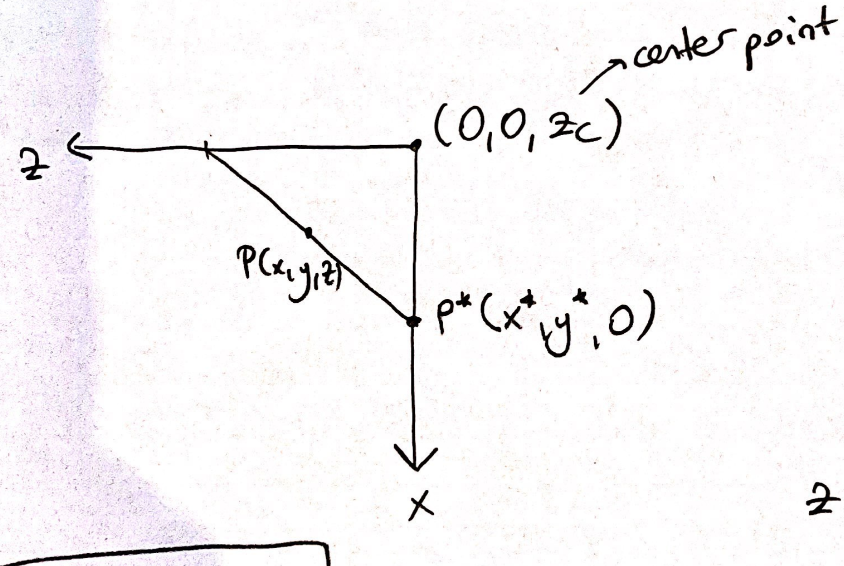
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{x}{rz+1} \\ \frac{y}{rz+1} \\ \frac{z}{rz+1} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{x}{rz+1} \\ \frac{y}{rz+1} \\ 0 \\ 1 \end{pmatrix}$$

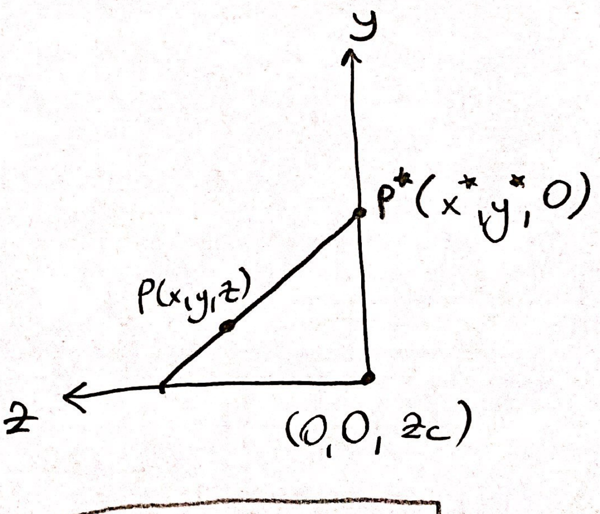
thus the physical coordinates, in three dimensions, of the transformed point are $\left(\frac{x}{rz+1}, \frac{y}{rz+1}, 0\right)$. When the image point is viewed on the viewing plane $z=0$, then the only relevant coordinates are first two, namely $\left(\frac{x}{rz+1}, \frac{y}{rz+1}\right)$.



A typical point on an object is $P(x, y, z)$ which has its perspective-projection $P^*(x^*, y^*, 0)$ on the viewing plane $z=0$.

For the orthographic projections we arranged the three-dimensional object so that all its vertices had positive coordinates, in x, y and z. When the perspective projection is applied to an object in this position: the viewing point is positioned at z_c on the positive part of the z-axis and the number r is negative.





$$\boxed{\frac{x^*}{x} = \frac{z_c}{z_c - z}}$$

$$x^* = x \cdot \frac{z_c}{z_c - z}$$

$$\boxed{\frac{y^*}{y} = \frac{z_c}{z_c - z}}$$

$$y^* = y \cdot \frac{z_c}{z_c - z}$$

$$\begin{pmatrix} x^* \\ y^* \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \cdot \frac{z_c}{z_c - z} \\ y \cdot \frac{z_c}{z_c - z} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \\ \frac{z_c - z}{z_c} \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \\ 1 - \frac{z}{z_c} \end{pmatrix}$$

If you look from x, then x = 0
" " " from y, then y = 0
" " " " ", then z = 0

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{z_c} & 1 \end{pmatrix}$$

$\to z$

Perspective projection with respect to point $\begin{pmatrix} 0 \\ 0 \\ z_c \end{pmatrix}$, then

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{z_c} & 1 \end{pmatrix}$$

$\to z$

Perspective projection w.r.t. the point $\begin{pmatrix} 0 \\ y_c \\ 0 \end{pmatrix}$, then

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{y_c} & 0 & 1 \end{pmatrix}$$

Perspective projection w.r.t the point $\begin{pmatrix} x_c \\ 0 \\ 0 \end{pmatrix}$, then

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{x_c} & 0 & 0 & 1 \end{pmatrix}$$

x

Vanishing Points

In two dimensions, all lines that have the same gradient and so go in the same direction have the same point at infinity; for the lines parallel to the z-axis this is the point given by $(0\ 0\ 1\ 0)$. We apply the perspective transformation to this point:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & r & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ r \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \\ 1 \end{pmatrix}$$

$h = r = 1$

Thus the phsical coordinates of the image of this point at infinity are $\begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \end{pmatrix}$, and this is known as VP_z, the

principal vanishing point of lines parallel to the z-axis.
It is rather surprising that a point at infinity, by definition unattainable, has an ordinary, finite image point. All the lines parallel to the z-axis, approaching from the positive z-direction, have the same point at infinity; thus all of them when transformed pass through this same principal vanishing point. VP_z is the point to which these parallel lines appear to converge when we apply perspective.
Geometrically, this vanishing point lies on the z-axis as far 'behind' the viewing plane as the viewing point is in front of it: it the reflection of the viewing point in the viewing plane.