

# Introduction to Aerial Robotics

## Lecture 2

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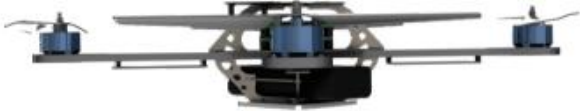
# Outline

- Rigid Body Transformations
- Rotational Motions
- Rotation Representations
- Rigid Body Motions
- Rigid Body Velocities
- Quadrotor Dynamics

# Rigid Body Transformations

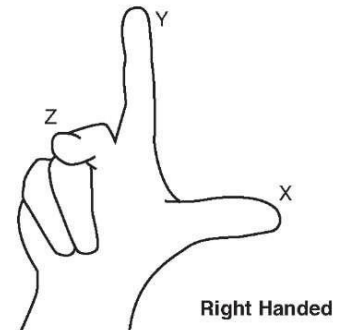
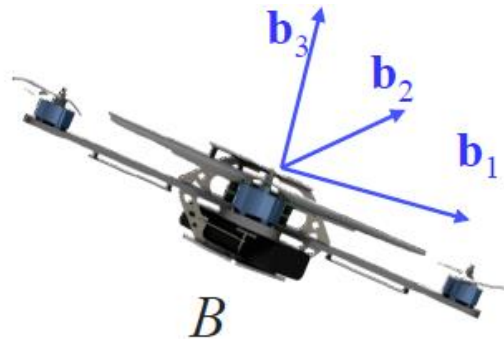
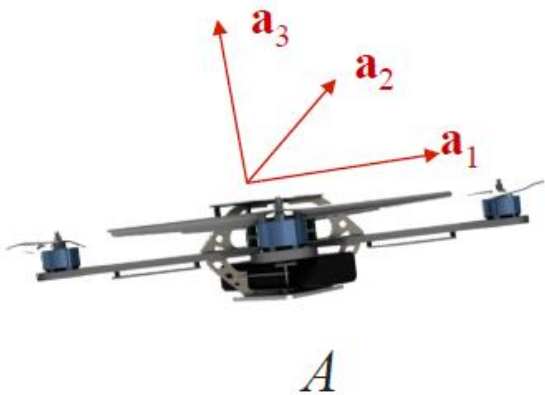
# Rigid Body

- Two distinct positions and orientations of the same rigid body
  - Let  $\mathbf{p}$  and  $\mathbf{q}$  be two points on a rigid body
  - $\|\mathbf{p}(t) - \mathbf{q}(t)\| = \|\mathbf{p}(0) - \mathbf{q}(0)\| = \text{constant}$



# Reference Frames

- We associate any position and orientation with a reference frame
  - We use **right-handed** coordinate frames
  - We can find three linearly independent vectors  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  that are basis vectors for reference frame  $A$
  - We can write any vector as a linear combination of basis vectors in either frame  $\mathbf{v} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3$



# Notation

- Vectors
  - $x, y, a, \dots$
- Matrices
  - $A, B, C, \dots$
- Reference frames
  - $A, B, C, \dots$
  - $a, b, c, \dots$
- Transformations
  - ${}^A\mathbf{A}_B, {}^A\mathbf{R}_B \dots$
  - $\mathbf{A}_{ab}, \mathbf{R}_{ab} \dots$
  - $g_{ab}(\cdot), h_{ab}(\cdot) \dots$

Be Aware of **Potential Confusion!!!**

# Rigid Body Displacement

- Object:

$$O \in \mathbb{R}^3$$

- Rigid body displacement

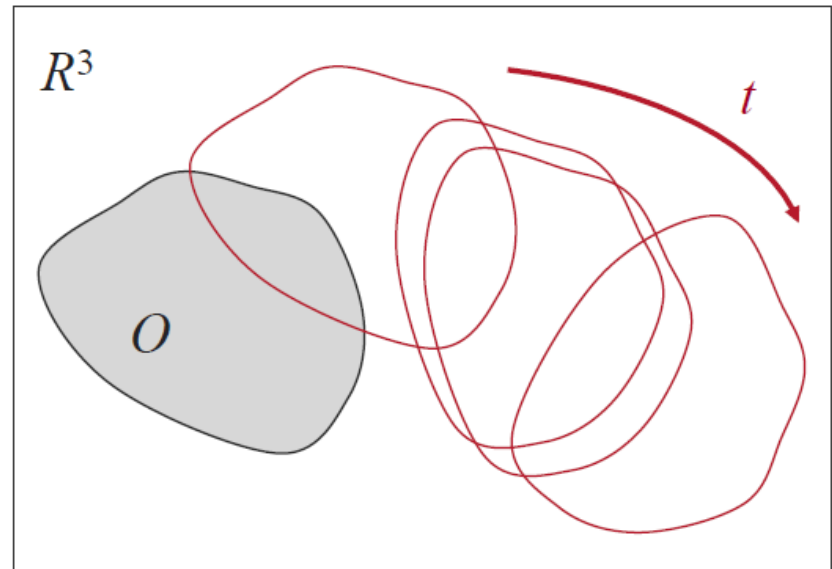
– Map

$$g: O \rightarrow \mathbb{R}^3$$

- Rigid body motion

– Continuous family of maps

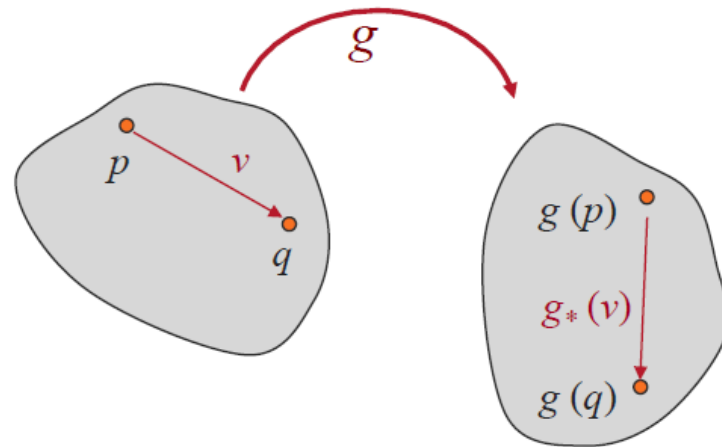
$$g(t) : O \rightarrow \mathbb{R}^3$$



# Rigid Body Displacement

- A displacement of a transformation of points
  - Transformation ( $g$ ) of points induces an action ( $g^*$ ) on vectors

$$g_*(\mathbf{v}) = g(\mathbf{q}) - g(\mathbf{p})$$





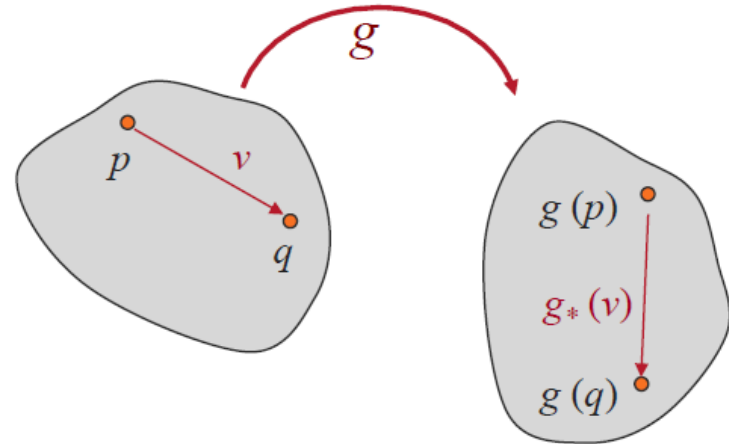
# Definition of Rigid Body Displacement

- Lengths are preserved

$$\|g(\mathbf{q}) - g(\mathbf{p})\| = \|\mathbf{q} - \mathbf{p}\|$$

- Cross products are preserved

$$g_*(\mathbf{v}) \times g_*(\mathbf{w}) = g_*(\mathbf{v} \times \mathbf{w})$$



**Why?**

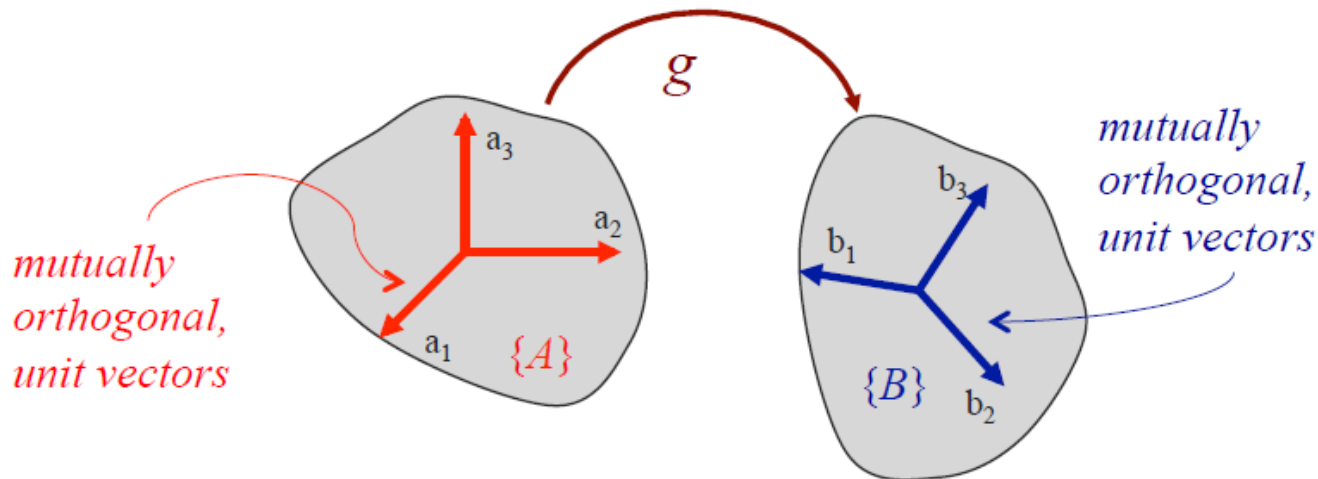
Eliminate internal reflection:  $(x, y, z) \rightarrow (x, y, -z)$

# Properties of Rigid Body Displacement

- Inner products are also preserved

$$g_*(\mathbf{v}) \cdot g_*(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$$

- Orthogonal vectors are mapped to orthogonal vectors





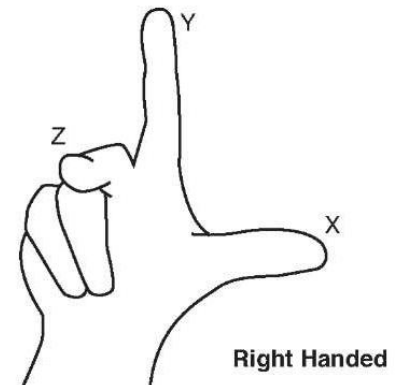
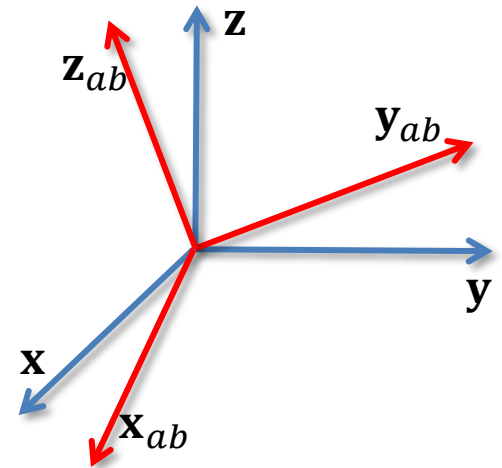
# Rigid Body Displacement

- Rigid body displacements are transformations that satisfy two important properties:
  1. Lengths are preserved
  2. Cross products are preserved
- Rigid body transformations and rigid body displacements are often used interchangeably
  - Transformations generally used to describe relationship between reference frames attached to different rigid bodies.
  - Displacements describe relationships between two positions and orientation of a frame attached to a displaced rigid body

# Rotational Motions

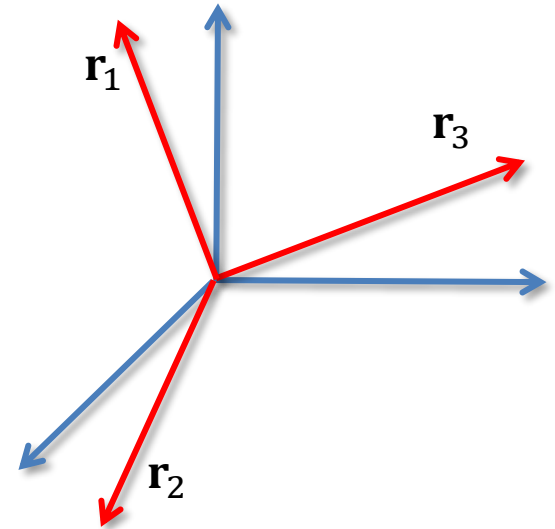
# Rotation

- Coordinate frames are right-handed
- Principle axes of frame A:
  - $\mathbf{x} = [1 \ 0 \ 0]^T$
  - $\mathbf{y} = [0 \ 1 \ 0]^T$
  - $\mathbf{z} = [0 \ 0 \ 1]^T$
- Principle axes of frame B:
  - $\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab} \in \mathbb{R}^3$
- The Rotation Matrix:
  - $\mathbf{R}_{ab} = [\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}]$
  - Coordinates of principle axes of B related to A



# Properties of a Rotation Matrix

- Let  $\mathbf{R} = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3]$  be a rotation matrix
- Orthogonal:
  - $\mathbf{r}_i^T \cdot \mathbf{r}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$
  - $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$
- Special orthogonal:
  - $\det \mathbf{R} = \mathbf{r}_1^T \cdot (\mathbf{r}_2 \times \mathbf{r}_3) = \mathbf{r}_1^T \cdot \mathbf{r}_1 = 1$
- The set of all rotations forms the Special Orthogonal Group
  - Special orthogonal group
  - 3D rotations:  $SO(3)$
  - 2D rotations:  $SO(2)$
  - $SO(n) = \{\mathbf{R} \in \mathbb{R}^{n \times n} | \mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}, \det \mathbf{R} = 1\}$



# Properties of a Rotation Matrix

- $SO(3) = \{\mathbf{R} \in \mathbb{R}^{3 \times 3} | \mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}, \det \mathbf{R} = 1\}$
- $SO(3)$  is a group under the operation of matrix multiplication
  1. Closure: If  $\mathbf{R}_1, \mathbf{R}_2 \in SO(3)$ , then  $\mathbf{R}_1 \cdot \mathbf{R}_2 \in SO(3)$
  2. Identity: The identity matrix is the identity element
  3. Inverse: If  $\mathbf{R} \in SO(3)$ , then  $\mathbf{R}^{-1} \in SO(3)$
  4. Associativity:  $\mathbf{R}_1 \cdot (\mathbf{R}_2 \cdot \mathbf{R}_3) = (\mathbf{R}_1 \cdot \mathbf{R}_2) \cdot \mathbf{R}_3$

$(G, \cdot)$  is a group if:

- 1)  $g_1, g_2 \in G \Rightarrow g_1 \cdot g_2 \in G$
- 2)  $\exists! e \in G, \text{ s.t. } g \cdot e = e \cdot g = g, \forall g \in G$
- 3)  $\forall g \in G, \exists! g^{-1} \in G, \text{ s.t. } g \cdot g^{-1} = g^{-1} \cdot g = e$
- 4)  $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

Group examples:

1. The set of all integers with addition operation
2. The set of all real numbers with arithmetic operations

# Properties of a Rotation Matrix

- A transformation that rotates the coordinates of a point from frame  $B$  to frame  $A$

- Let  $\mathbf{q}_b = [x_b, y_b, z_b]^T \in \mathbb{R}^3$  be coordinate of point  $\mathbf{q}$  in frame  $B$

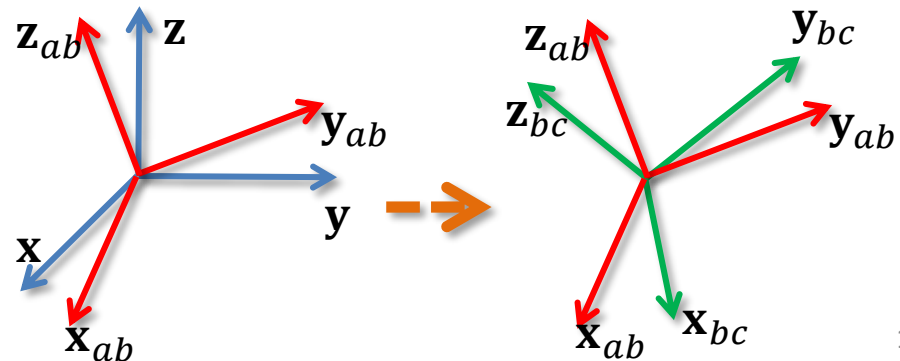
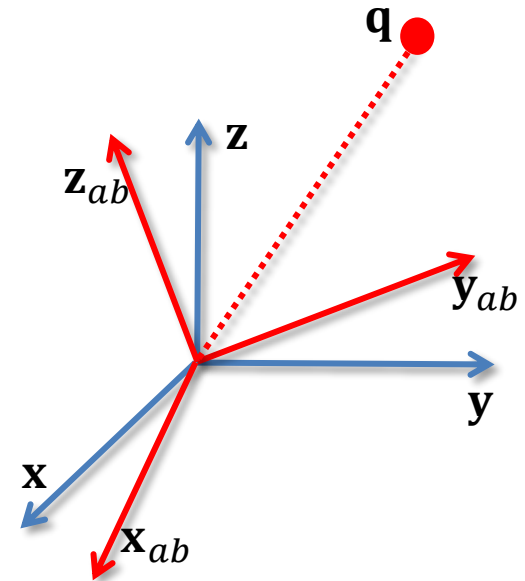
- Let  $\mathbf{q}_a$  be coordinate of point  $\mathbf{q}$  in frame  $A$

- $\mathbf{q}_a = x_b \cdot \mathbf{x}_{ab} + y_b \cdot \mathbf{y}_{ab} + z_b \cdot \mathbf{z}_{ab} =$

$$[\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}] \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \mathbf{R}_{ab} \cdot \mathbf{q}_b$$

- Composition Rule

- $\mathbf{R}_{ac} = \mathbf{R}_{ab} \cdot \mathbf{R}_{bc}$





# Rotation is Rigid Body Transformation

$\mathbf{R}_{ab} = [\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}]$  preserves:

○ Length:

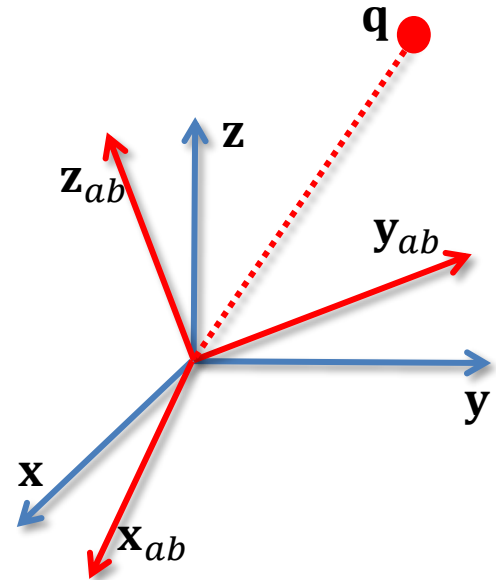
$$- \|\mathbf{R}_{ab}(\mathbf{p}_b - \mathbf{q}_b)\| = \|\mathbf{p}_b - \mathbf{q}_b\|$$

○ Cross product:

$$- \mathbf{R}_{ab}(\mathbf{v} \times \mathbf{w}) = (\mathbf{R}_{ab}\mathbf{v}) \times (\mathbf{R}_{ab}\mathbf{w})$$

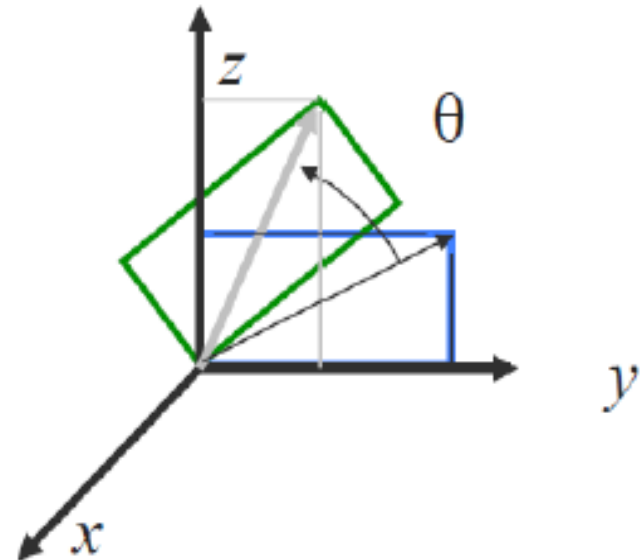
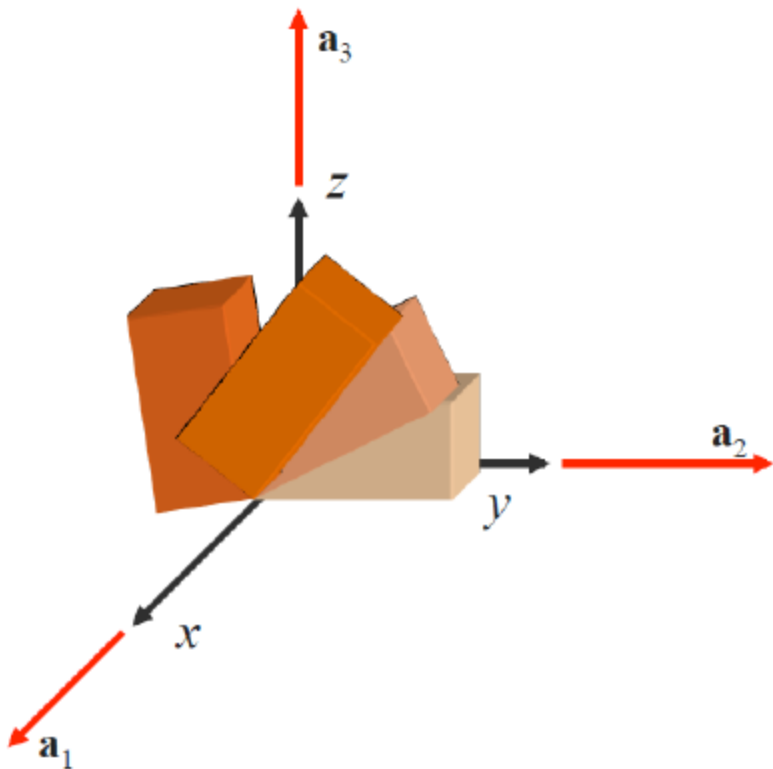
- Use the fact  $\mathbf{R}(\mathbf{v})^\wedge \mathbf{R}^T = (\mathbf{R}\mathbf{v})^\wedge$  to prove, where

$$(\mathbf{a})^\wedge = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \text{ is the skew-symmetric matrix, and } \mathbf{a} \times \mathbf{b} = (\mathbf{a})^\wedge \mathbf{b}$$



# Example - Rotation

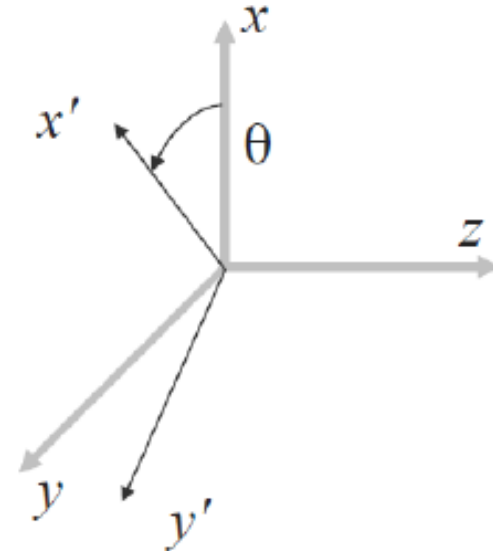
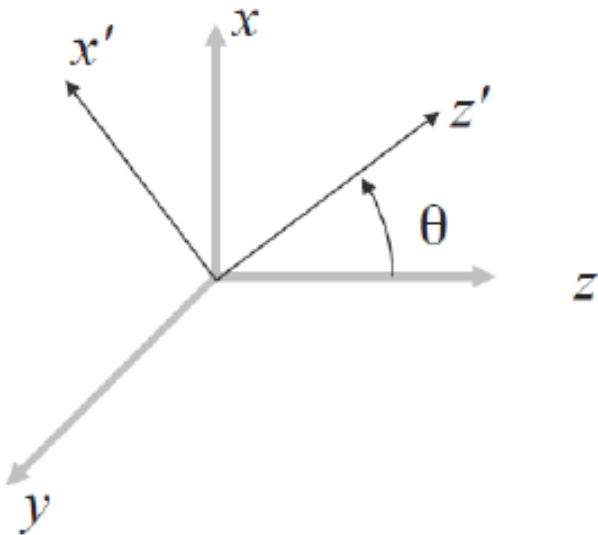
$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



# Example - Rotation

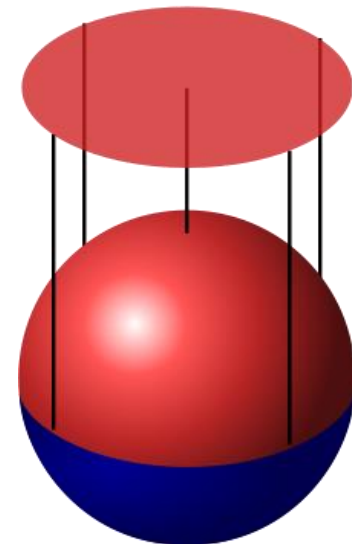
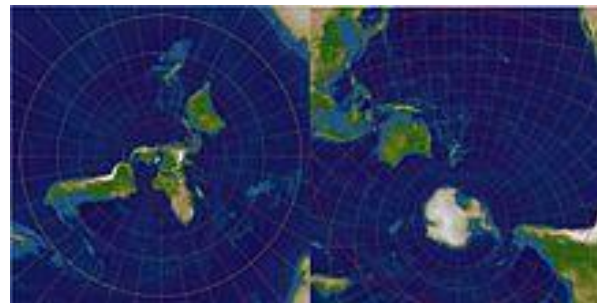
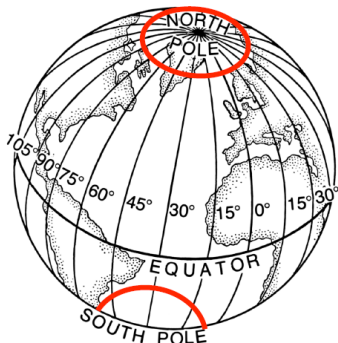
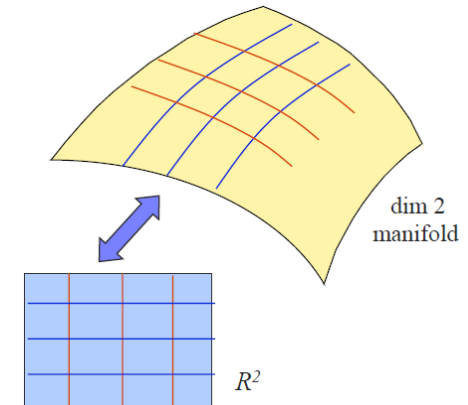
$$\mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Properties of Rotation

- $SO(3)$  is a continuous group
  - The multiplication operation is a continuous operation
  - The inverse is a continuous function
- $SO(3)$  is a smooth manifold
  - A manifold of dimension  $n$  is a set  $M$  which is locally resembled to Euclidean space  $\mathbb{R}^n$  near each point
  - Example: sphere is a differentiable manifold that is locally resembled to  $\mathbb{R}^2$



# Rotation Representations

# Rotation Representations

- Rotation matrices
- Euler angles
- Exponential coordinates
- Quaternions
  - Will be discussed later in the semester
  - Slides are provided as appendix for this lecture

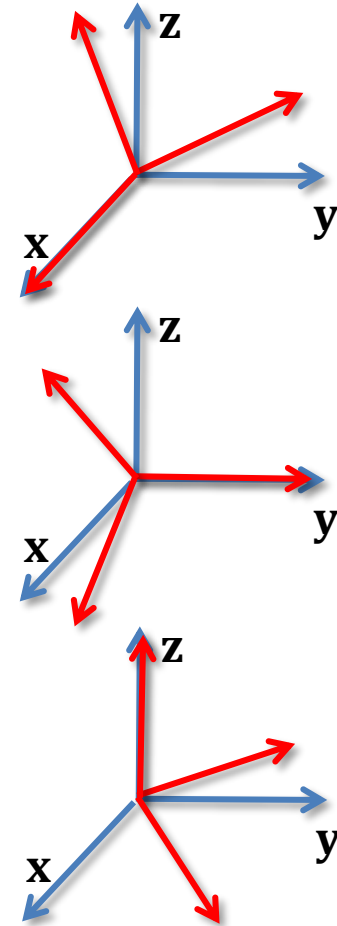
# Euler Angles

- Elementary rotations:

$$- R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix}$$

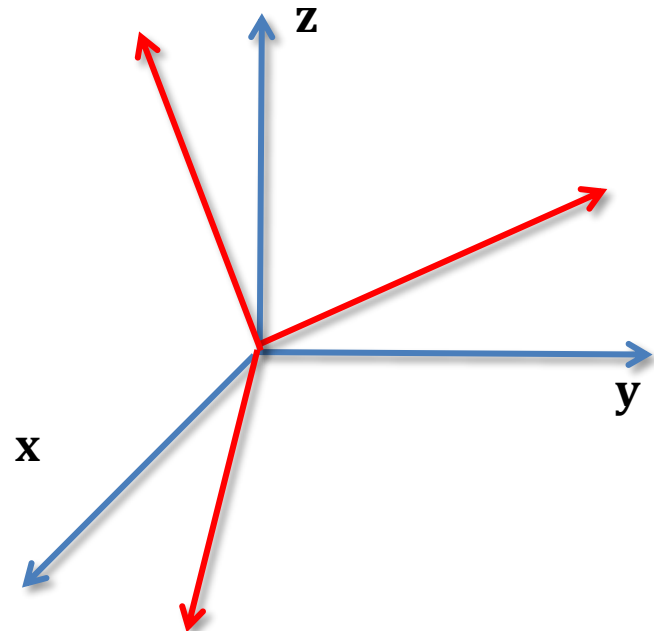
$$- R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$- R_z(\psi) = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



# Euler Angles

- Any rotation can be described by three successive rotations about linear independent axes
- However, this is an almost 1-1 transform with singularities:
  - $R_z(\psi) \cdot R_x(\phi) \cdot R_y(\theta) \Rightarrow R$
  - $R_z(\psi) \cdot R_x(\phi) \cdot R_y(\theta) \nLeftarrow R$





# Euler Angles

- Different Euler angle conversions:

Proper Euler angles	Tait-Bryan angles
$X_1 Z_2 X_3 = \begin{bmatrix} c_2 & -c_3 s_2 & s_2 s_3 \\ c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 \\ s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$	$X_1 Z_2 Y_3 = \begin{bmatrix} c_2 c_3 & -s_2 & c_2 s_3 \\ s_1 s_3 + c_1 c_3 s_2 & c_1 c_2 & c_1 s_2 s_3 - c_3 s_1 \\ c_3 s_1 s_2 - c_1 s_3 & c_2 s_1 & c_1 c_3 + s_1 s_2 s_3 \end{bmatrix}$
$X_1 Y_2 X_3 = \begin{bmatrix} c_2 & s_2 s_3 & c_3 s_2 \\ s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 \\ -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$	$X_1 Y_2 Z_3 = \begin{bmatrix} c_2 c_3 & -c_2 s_3 & s_2 \\ c_1 s_3 + c_3 s_1 s_2 & c_1 c_3 - s_1 s_2 s_3 & -c_2 s_1 \\ s_1 s_3 - c_1 c_3 s_2 & c_3 s_1 + c_1 s_2 s_3 & c_1 c_2 \end{bmatrix}$
$Y_1 X_2 Y_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 \\ s_2 s_3 & c_2 & -c_3 s_2 \\ -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$	$Y_1 X_2 Z_3 = \begin{bmatrix} c_1 c_3 + s_1 s_2 s_3 & c_3 s_1 s_2 - c_1 s_3 & c_2 s_1 \\ c_2 s_3 & c_2 c_3 & -s_2 \\ c_1 s_2 s_3 - c_3 s_1 & c_1 c_3 s_2 + s_1 s_3 & c_1 c_2 \end{bmatrix}$
$Y_1 Z_2 Y_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 \\ c_3 s_2 & c_2 & s_2 s_3 \\ -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$	$Y_1 Z_2 X_3 = \begin{bmatrix} c_1 c_2 & s_1 s_3 - c_1 c_3 s_2 & c_3 s_1 + c_1 s_2 s_3 \\ s_2 & c_2 c_3 & -c_2 s_3 \\ -c_2 s_1 & c_1 s_3 + c_3 s_1 s_2 & c_1 c_3 - s_1 s_2 s_3 \end{bmatrix}$
$Z_1 Y_2 Z_3 = \begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 \\ c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 \\ -c_3 s_2 & s_2 s_3 & c_2 \end{bmatrix}$	$Z_1 Y_2 X_3 = \begin{bmatrix} c_1 c_2 & c_1 s_2 s_3 - c_3 s_1 & s_1 s_3 + c_1 c_3 s_2 \\ c_2 s_1 & c_1 c_3 + s_1 s_2 s_3 & c_3 s_1 s_2 - c_1 s_3 \\ -s_2 & c_2 s_3 & c_2 c_3 \end{bmatrix}$
$Z_1 X_2 Z_3 = \begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 \\ c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 \\ s_2 s_3 & c_3 s_2 & c_2 \end{bmatrix}$	$Z_1 X_2 Y_3 = \begin{bmatrix} c_1 c_3 - s_1 s_2 s_3 & -c_2 s_1 & c_1 s_3 + c_3 s_1 s_2 \\ c_3 s_1 + c_1 s_2 s_3 & c_1 c_2 & s_1 s_3 - c_1 c_3 s_2 \\ -c_2 s_3 & s_2 & c_2 c_3 \end{bmatrix}$

# Euler Angles

- Example: Z-Y-Z Euler angles:
  - Sequence of three rotations about body-fixed axes

- $\mathbf{R} = \mathbf{R}_z(\phi) \cdot \mathbf{R}_y(\theta) \cdot \mathbf{R}_z(\psi)$

- $$\mathbf{R} = \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- If  $\sin \theta \neq 0$ :
  - $\theta = \arccos(r_{33})$
  - $\psi = \operatorname{atan2}\left(\frac{r_{32}}{\sin \theta}, -\frac{r_{31}}{\sin \theta}\right)$
  - $\phi = \operatorname{atan2}\left(\frac{r_{23}}{\sin \theta}, \frac{r_{13}}{\sin \theta}\right)$

# Euler Angles

- Example: Z-Y-Z Euler angles:

- Sequence of three rotations about body-fixed axes

- $\mathbf{R} = \mathbf{R}_z(\phi) \cdot \mathbf{R}_y(\theta) \cdot \mathbf{R}_z(\psi)$

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- If  $\sin \theta = 0$ :

- $$\mathbf{R} = \begin{bmatrix} c\phi c\psi - s\phi s\psi & -c\phi s\psi - s\phi c\psi & 0 \\ c\phi s\psi + s\phi c\psi & -s\phi s\psi + c\phi c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{R}_z(\phi + \psi)$$

- As long as  $\phi + \psi$  is preserved, we have infinite set of Euler angles!

# Exponential Coordinates

- Scalar differential equation:

$$- \begin{cases} \dot{x}(t) = ax(t) \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = e^{at}x_0$$

- Matrix differential equation:

$$- \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \Rightarrow \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

$$- e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{1}{2}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \cdots + \frac{1}{n!}\mathbf{A}^n + \cdots$$

# Exponential Coordinates

- Degree-of-freedom of  $SO(3)$ :

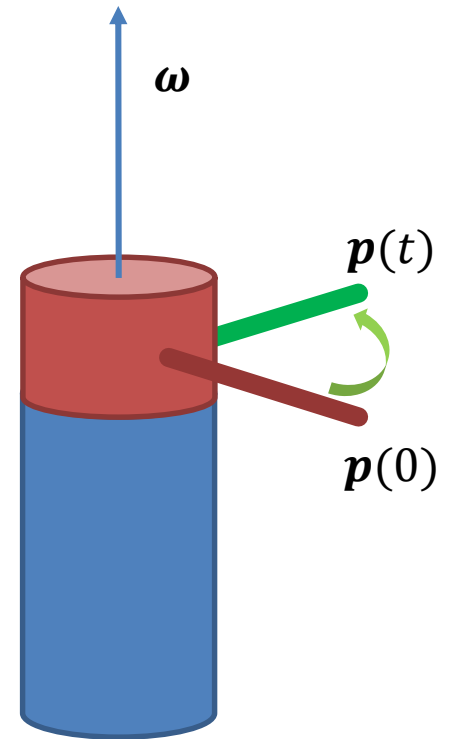
$$- \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$- \mathbf{r}_i^T \cdot \mathbf{r}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \Rightarrow 6 \text{ constraints}$$

- $\mathbf{R}$  has only 3 independent parameters

- Consider the motion of a point about a unit length rotating link  $\boldsymbol{\omega}$  at constant unit velocity:

$$- \begin{cases} \dot{\mathbf{p}}(t) = \boldsymbol{\omega} \times \mathbf{p}(t) = \hat{\boldsymbol{\omega}} \cdot \mathbf{p}(t) \\ \mathbf{p}(0) = \mathbf{p}_0 \end{cases} \Rightarrow \mathbf{p}(t) = e^{\hat{\boldsymbol{\omega}} t} \mathbf{p}_0$$



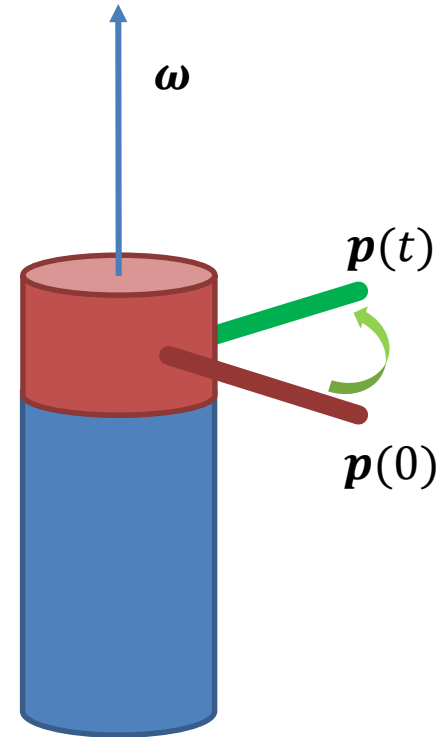
# Exponential Coordinates

- Consider the motion of a point about a unit length rotating link  $\omega$  at constant unit velocity:

$$- \begin{cases} \dot{\mathbf{p}}(t) = \omega \times \mathbf{p}(t) = \hat{\omega} \cdot \mathbf{p}(t) \\ \mathbf{p}(0) = \mathbf{p}_0 \end{cases} \Rightarrow \mathbf{p}(t) = e^{\hat{\omega}t} \mathbf{p}_0$$

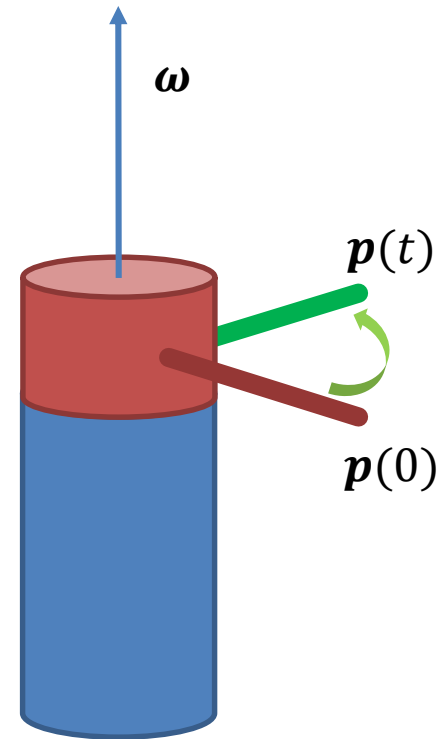
$$- \hat{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

- Rotating about  $\omega$  at unit velocity for  $\theta$  units:
  - $R(\omega, \theta) = e^{\hat{\omega}\theta}$



# Exponential Coordinates

- The vector space of all  $3 \times 3$  skew-symmetric matrices is denoted as  $so(3)$ :
  - $so(3) = \{\mathbf{S} \in \mathbb{R}^{3 \times 3} : \mathbf{S}^T = -\mathbf{S}\}$
- The exponential map:
  - $\mathbf{R}(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{\omega}}\theta} = \mathbf{I} + \hat{\boldsymbol{\omega}} \sin \theta + \hat{\boldsymbol{\omega}}^2 (1 - \cos \theta)$
- $e^{\hat{\boldsymbol{\omega}}\theta} \in SO(3)$ 
  - $[e^{\hat{\boldsymbol{\omega}}\theta}]^{-1} = e^{-\hat{\boldsymbol{\omega}}\theta} = e^{\hat{\boldsymbol{\omega}}^T\theta} = [e^{\hat{\boldsymbol{\omega}}\theta}]^T$
  - Since  $\det e^0 = 1$ , and both determinant and exponential map are continuous functions, we know  $\det e^{\hat{\boldsymbol{\omega}}\theta} = 1$
- The exponential map is onto (many to one)
  - $\theta = 0 \Rightarrow \boldsymbol{\omega}$  can be chosen arbitrary

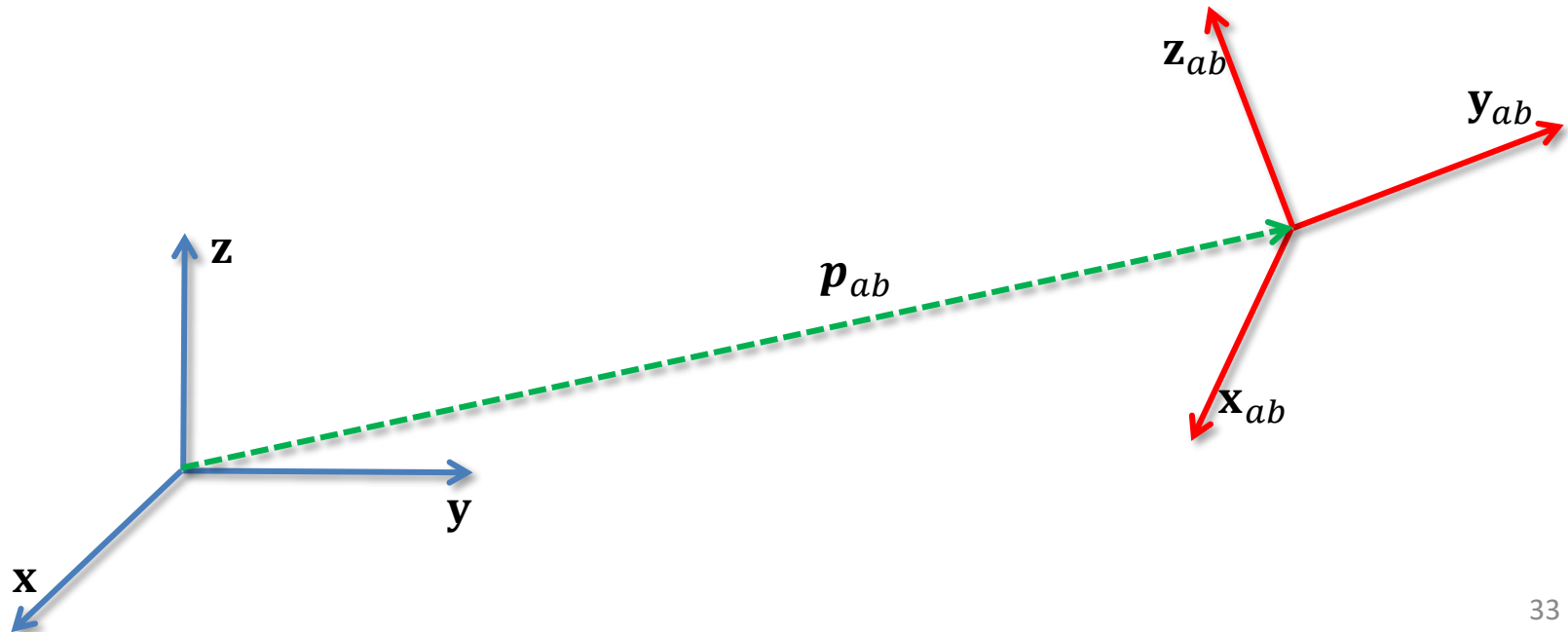
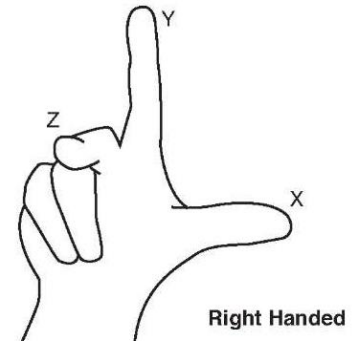


# Rigid Body Motions



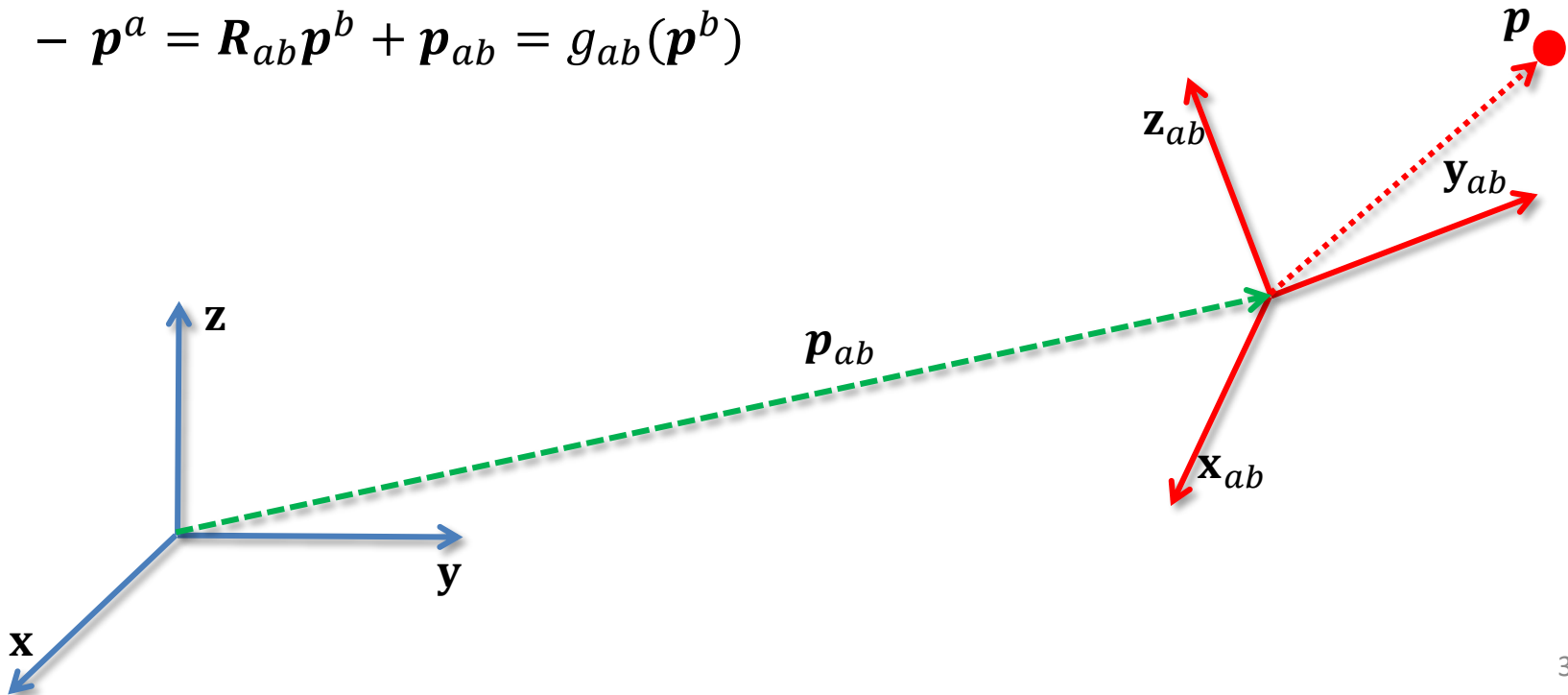
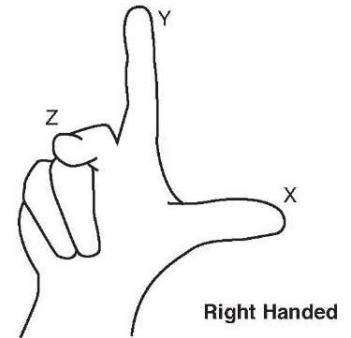
# Rigid Body Motion

- General rigid body motions that includes both translation and rotation forms the product space of  $\mathbb{R}^3$  and  $SO(3)$ . Denoted as  $SE(3)$  – Special Euclidean group.
  - $SE(3) = \{(\mathbf{p}, \mathbf{R}): \mathbf{p} \in \mathbb{R}^3, \mathbf{R} \in SO(3)\} = \mathbb{R}^3 \times SO(3)$



# Rigid Body Motion

- Special Euclidean group:
  - $SE(3) = \{(\mathbf{p}, \mathbf{R}): \mathbf{p} \in \mathbb{R}^3, \mathbf{R} \in SO(3)\} = \mathbb{R}^3 \times SO(3)$
- Transformation of a point between different coordinate frames:
  - $\mathbf{p}^a = \mathbf{R}_{ab}\mathbf{p}^b + \mathbf{p}_{ab} = \mathbf{g}_{ab}(\mathbf{p}^b)$



# Rigid Body Motion

- Homogeneous coordinates of a point:

$$- \bar{\mathbf{p}} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

- Homogeneous coordinates of a vector:

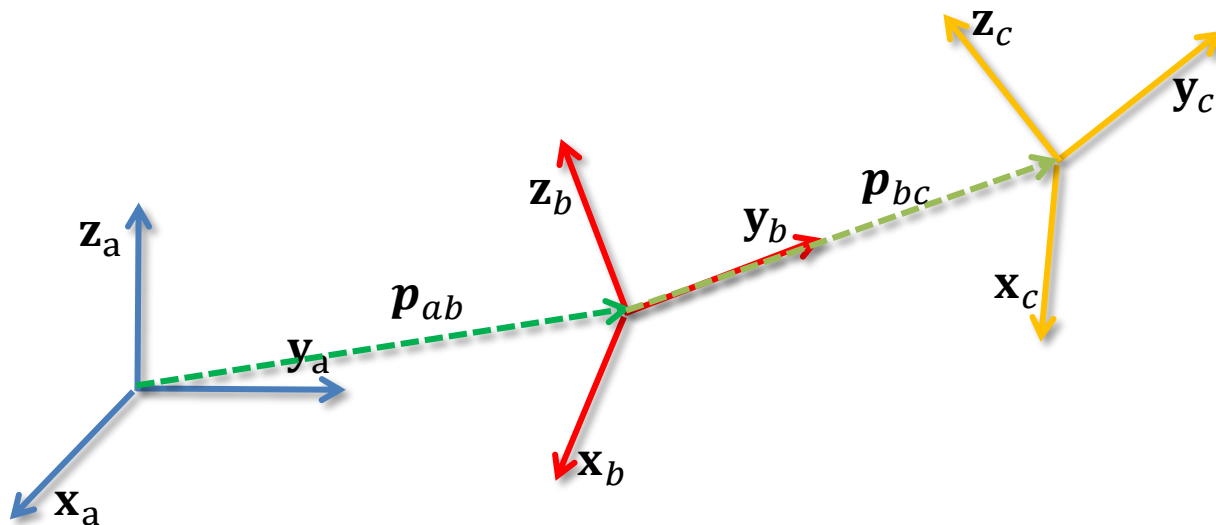
$$- \bar{\mathbf{v}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

- Homogeneous representation of rigid body motion:

$$- \bar{\mathbf{p}}^a = \begin{bmatrix} \mathbf{p}^a \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{ab} & \mathbf{p}_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}^b \\ 1 \end{bmatrix} = \bar{\mathbf{g}}_{ab} \bar{\mathbf{p}}^b$$

# Rigid Body Motion

- Homogeneous representation of rigid body motion:
  - $\bar{g}_{ab} = \begin{bmatrix} \mathbf{R}_{ab} & \mathbf{p}_{ab} \\ 0 & 1 \end{bmatrix}$
- Composition rule for rigid body motions:
  - $\bar{g}_{ac} = \bar{g}_{ab} \cdot \bar{g}_{bc} = \begin{bmatrix} \mathbf{R}_{ab}\mathbf{R}_{bc} & \mathbf{R}_{ab}\mathbf{p}_{bc} + \mathbf{p}_{ab} \\ 0 & 1 \end{bmatrix}$
  - Compare with composition of rotational motion:  $\mathbf{R}_{ac} = \mathbf{R}_{ab} \cdot \mathbf{R}_{bc}$



# Properties of Rigid Body Motion

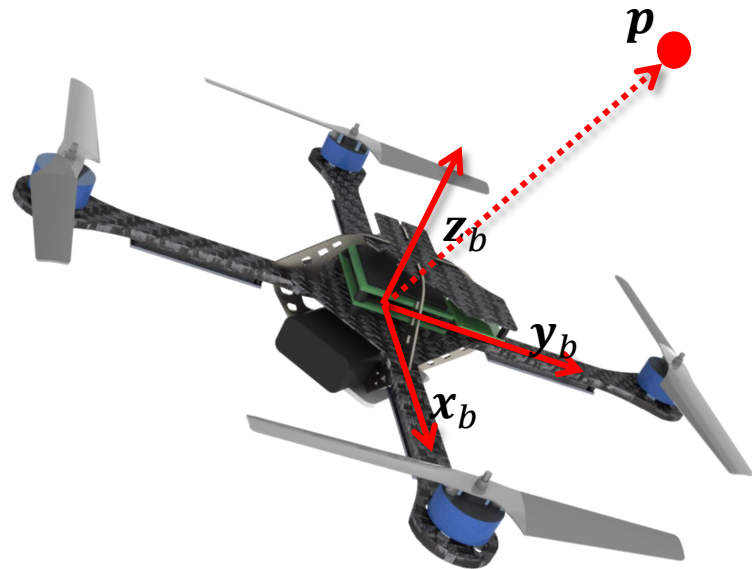
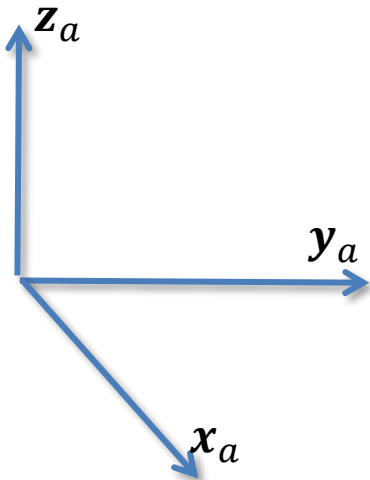
- $SE(3) = \{(\mathbf{p}, \mathbf{R}): \mathbf{p} \in \mathbb{R}^3, \mathbf{R} \in SO(3)\} = \mathbb{R}^3 \times SO(3)$
- $SE(3)$  is a group under the operation of matrix multiplication
  - Closure
  - Identity
  - Inverse
  - Associativity
- $g \in SE(3)$  is a rigid body transformation
  - Lengths are preserved
  - Cross products are preserved

**Prove it yourself!**

# Rigid Body Velocities

# Angular Velocity

- Coordinate frames:
  - Frame  $A$ : spatial frame
  - Frame  $B$ : body frame
- A point attached to the body follows a rotational path in spatial frame:
  - $\mathbf{p}^a(t) = \mathbf{R}_{ab}(t)\mathbf{p}^b$



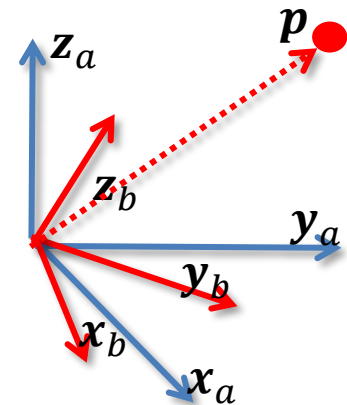
# Angular Velocity

- Coordinate frames:
  - Frame  $A$ : spatial frame
  - Frame  $B$ : body frame
- A point attached to the body follows a rotational path in spatial frame:
  - $\mathbf{p}^a(t) = \mathbf{R}_{ab}(t)\mathbf{p}^b$
- The velocity of the point in spatial frame:
  - $\mathbf{v}_p^a(t) = \frac{d}{dt}\mathbf{p}^a(t) = \dot{\mathbf{R}}_{ab}(t)\mathbf{p}^b$
- This can be rewritten as:
  - $\mathbf{v}_p^a(t) = \boxed{\dot{\mathbf{R}}_{ab}(t)\mathbf{R}_{ab}^{-1}(t)}\mathbf{R}_{ab}(t)\mathbf{p}^b$

Skew-symmetric matrix.

Why?

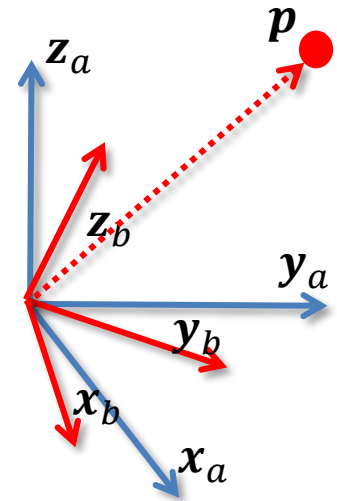
Prove  $\mathbf{R}\mathbf{R}^T = \mathbf{I} \implies \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = \mathbf{0}$





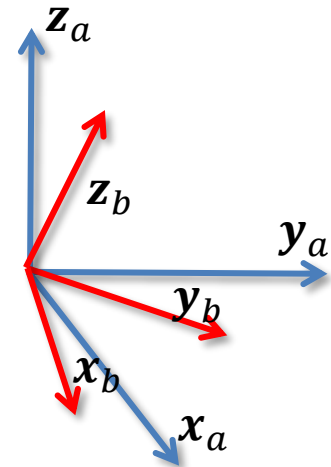
# Angular Velocity

- The instantaneous spatial angular velocity  $\omega_{ab}^a$ 
  - $\hat{\omega}_{ab}^a = \dot{R}_{ab} \cdot R_{ab}^{-1}$
- The instantaneous body angular velocity  $\omega_{ab}^b$ 
  - $\hat{\omega}_{ab}^b = R_{ab}^{-1} \cdot \dot{R}_{ab}$
- Conversion:
  - $\hat{\omega}_{ab}^b = R_{ab}^{-1} \cdot \hat{\omega}_{ab}^a \cdot R_{ab}$
  - $\omega_{ab}^b = R_{ab}^{-1} \cdot \omega_{ab}^a$
- Velocity induced by rotational motion:
  - $v_p^a = \hat{\omega}_{ab}^a \cdot R_{ab} \cdot p^b = \omega_{ab}^a \times p^a$
  - $v_p^b = R_{ab}^T \cdot v_p^a = \omega_{ab}^b \times p^b$



# Angular Velocity

- Numerical Integration
  - $\dot{\mathbf{R}} = \mathbf{R}\hat{\boldsymbol{\omega}}^b \Rightarrow \mathbf{R}(t + \Delta t) \sim \mathbf{R}(t) + \Delta t \cdot \mathbf{R}(t)\hat{\boldsymbol{\omega}}^b$
  - $\dot{\mathbf{R}} = \hat{\boldsymbol{\omega}}^a \mathbf{R} \Rightarrow \mathbf{R}(t + \Delta t) \sim \mathbf{R}(t) + \Delta t \cdot \hat{\boldsymbol{\omega}}^a \mathbf{R}(t)$
- Constant speed rotation
  - $\mathbf{R}(t) = \mathbf{R}_0 \cdot \exp(\hat{\boldsymbol{\omega}}_0^b \cdot t)$
  - $\mathbf{R}(t) = \exp(\hat{\boldsymbol{\omega}}_0^a \cdot t) \cdot \mathbf{R}_0$



# Angular Velocity

- Simple example

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \dot{\mathbf{R}} = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

# Angular Velocity

- Simple example

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\widehat{\boldsymbol{\omega}}_{ab}^b = \mathbf{R}^T \dot{\mathbf{R}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} = \begin{bmatrix} \widehat{0} \\ 0 \\ 1 \end{bmatrix} \dot{\theta}$$

$$\widehat{\boldsymbol{\omega}}_{ab}^a = \dot{\mathbf{R}} \mathbf{R}^T = \dot{\theta} \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} = \begin{bmatrix} \widehat{0} \\ 0 \\ 1 \end{bmatrix} \dot{\theta}$$

# Angular Velocity

- Two rotations

$$\mathbf{R} = \mathbf{R}_z(\theta) \mathbf{R}_x(\varphi)$$

$$\hat{\boldsymbol{\omega}}^b = \mathbf{R}^T \dot{\mathbf{R}} = (\mathbf{R}_z \mathbf{R}_x)^T (\dot{\mathbf{R}}_z \mathbf{R}_x + \mathbf{R}_z \dot{\mathbf{R}}_x)$$

$$= \mathbf{R}_x^T \mathbf{R}_z^T \dot{\mathbf{R}}_z \mathbf{R}_x + \mathbf{R}_x^T \dot{\mathbf{R}}_x$$

$$\hat{\boldsymbol{\omega}}^a = \dot{\mathbf{R}} \mathbf{R}^T = (\dot{\mathbf{R}}_z \mathbf{R}_x + \mathbf{R}_z \dot{\mathbf{R}}_x) (\mathbf{R}_z \mathbf{R}_x)^T$$

$$= \dot{\mathbf{R}}_z \mathbf{R}_z^T + \mathbf{R}_z \dot{\mathbf{R}}_x \mathbf{R}_x^T \mathbf{R}_z^T$$

# Rigid Body Velocity

- General rigid body transformation:

$$g_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$$

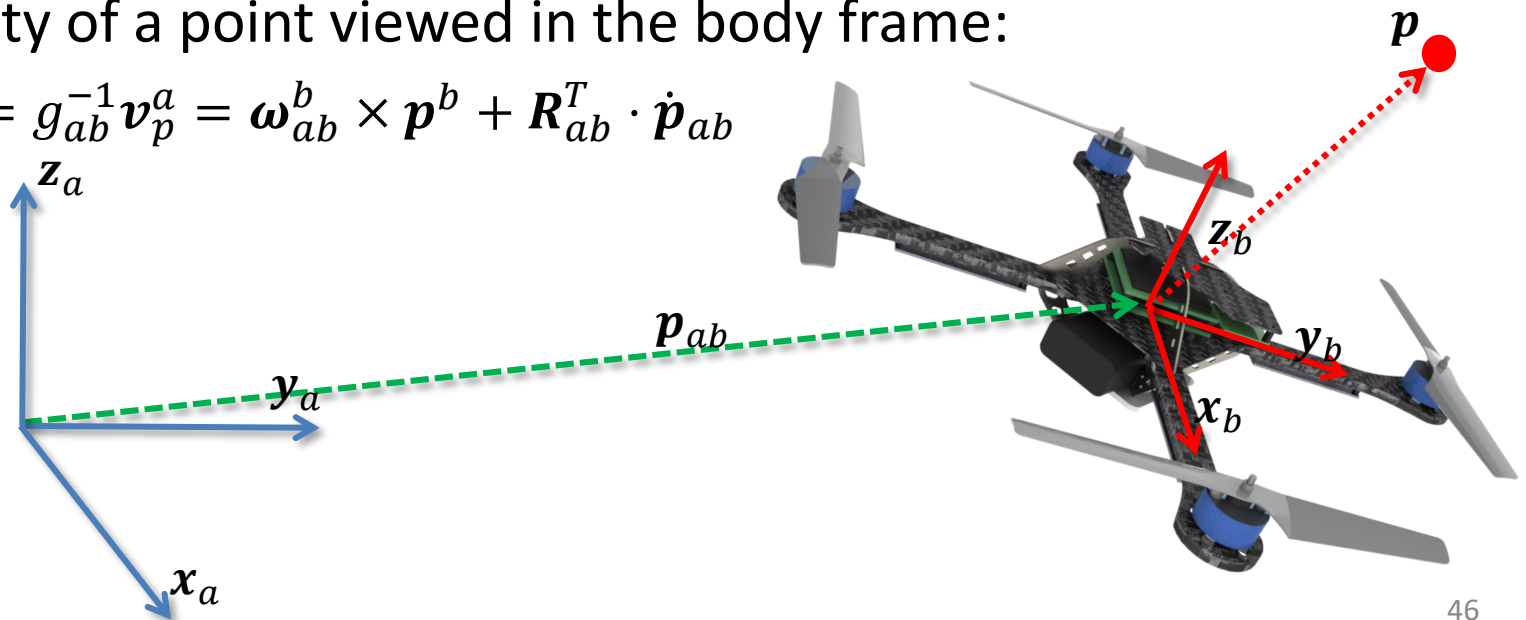
For detailed interpretation, refer to Chapter 2.4 of “A Mathematical Introduction to Robotic Manipulation”

- Velocity of a point viewed in the spatial frame:

$$v_p^a = \dot{g}_{ab} g_{ab}^{-1} p^a = \omega_{ab}^a \times p^a - \omega_{ab}^a \times p_{ab} + \dot{p}_{ab}$$

- Velocity of a point viewed in the body frame:

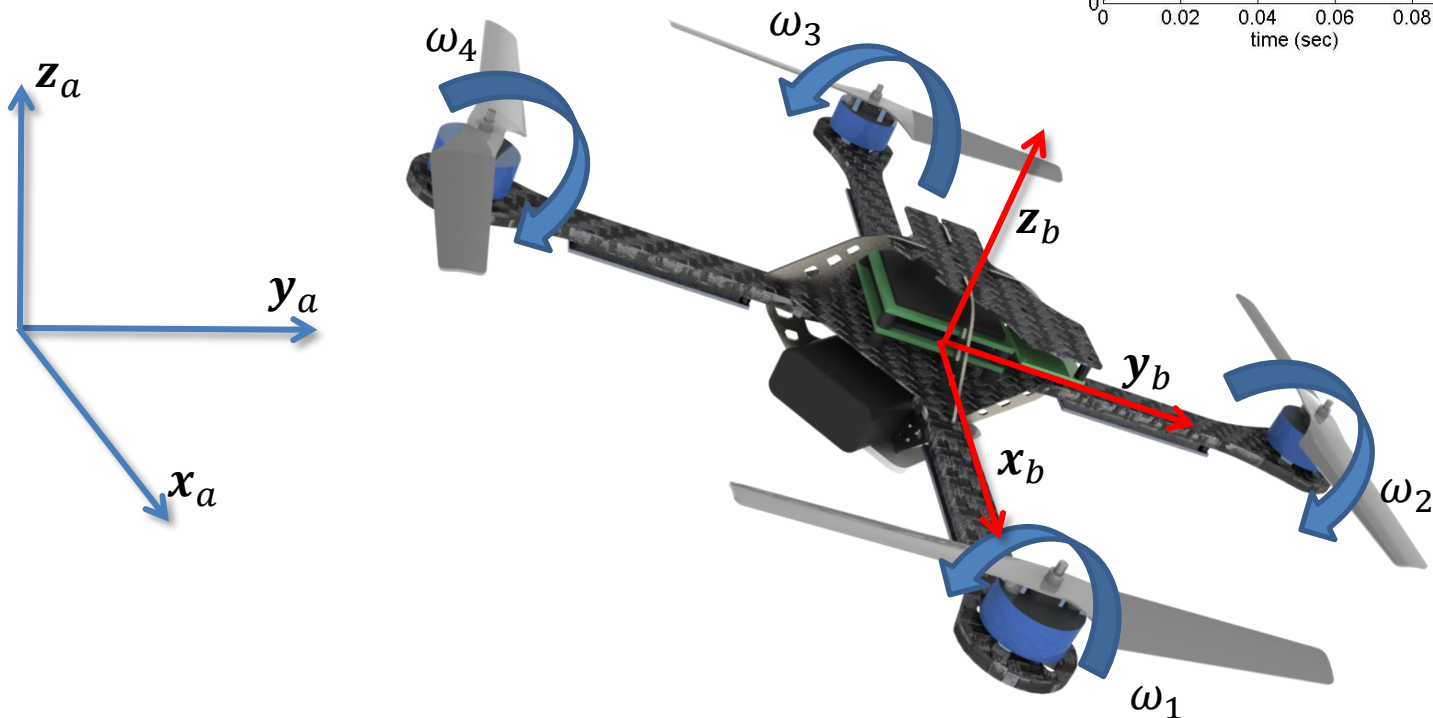
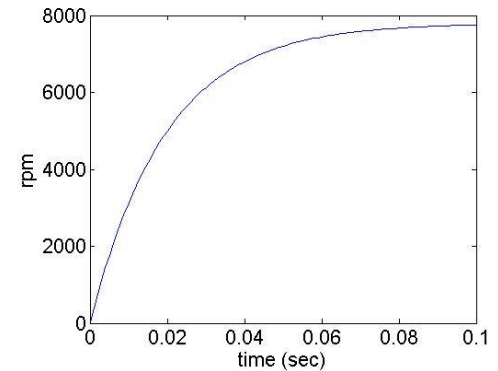
$$v_p^b = g_{ab}^{-1} v_p^a = \omega_{ab}^b \times p^b + R_{ab}^T \cdot \dot{p}_{ab}$$



# Quadrotor Dynamics

# Quadrotor Dynamics

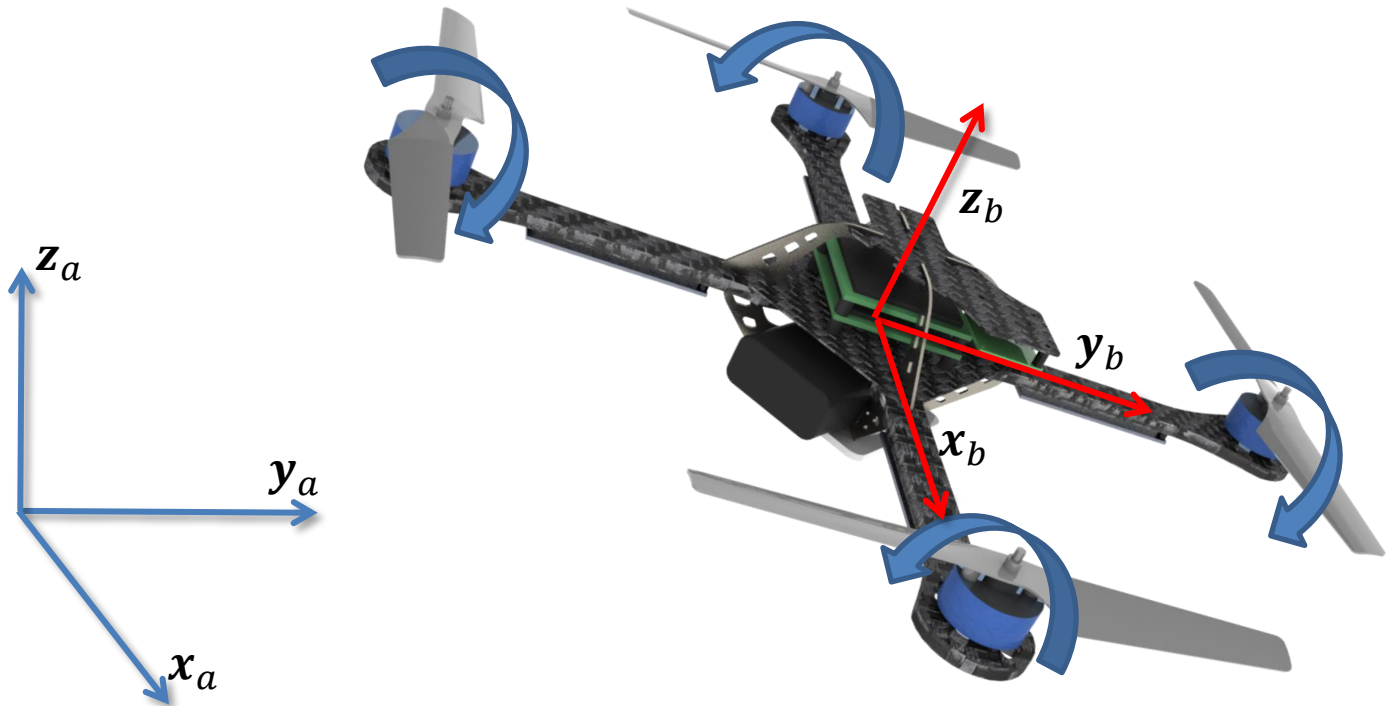
- Motor model:  $\dot{\omega}_i = k_m(\omega_i^{des} - \omega_i)$
- Thrust from individual motor:  $F_i = k_F \omega_i^2$
- Moment from individual motor:  $M_i = k_M \omega_i^2$





# Quadrotor Dynamics

- Z-X-Y Euler Angles:  $\mathbf{R}_{ab} = \mathbf{R}_z(\psi) \cdot \mathbf{R}_x(\phi) \cdot \mathbf{R}_y(\theta)$
- Sequence of three rotations about body-fixed axes
- What are the singularities?

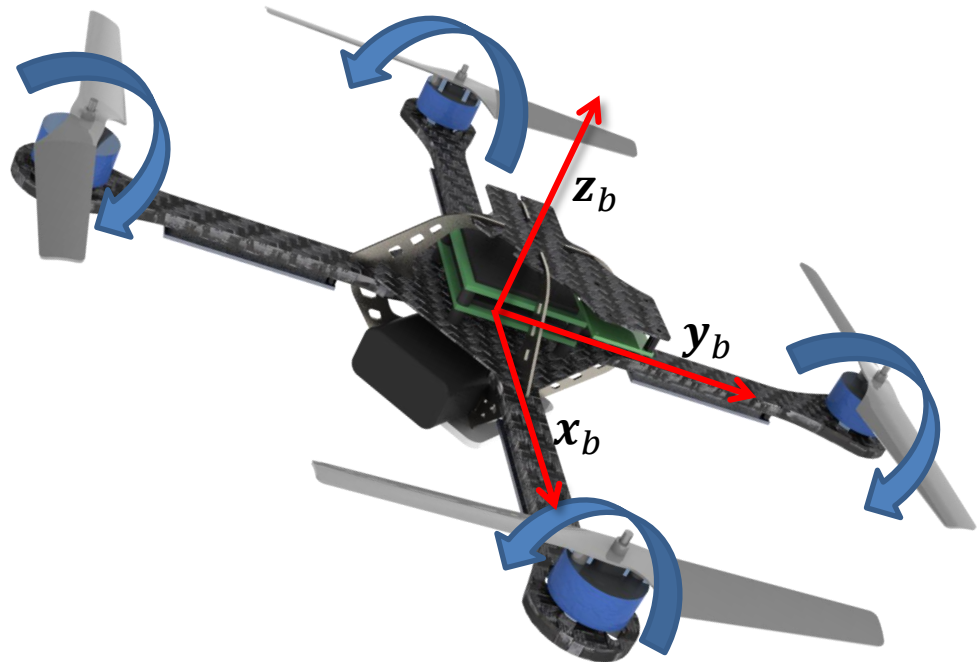
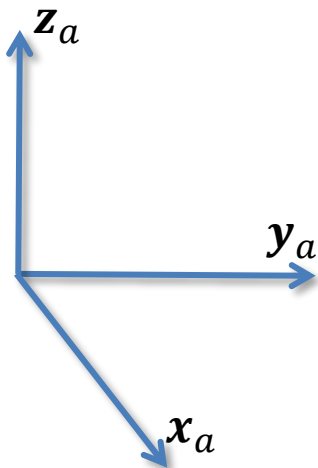


# Quadrotor Dynamics

$$\bullet \mathbf{R}_{ab} = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi + c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\psi c\theta s\phi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix}$$

$$\bullet \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -c\phi s\theta \\ 0 & 1 & s\phi \\ s\theta & 0 & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

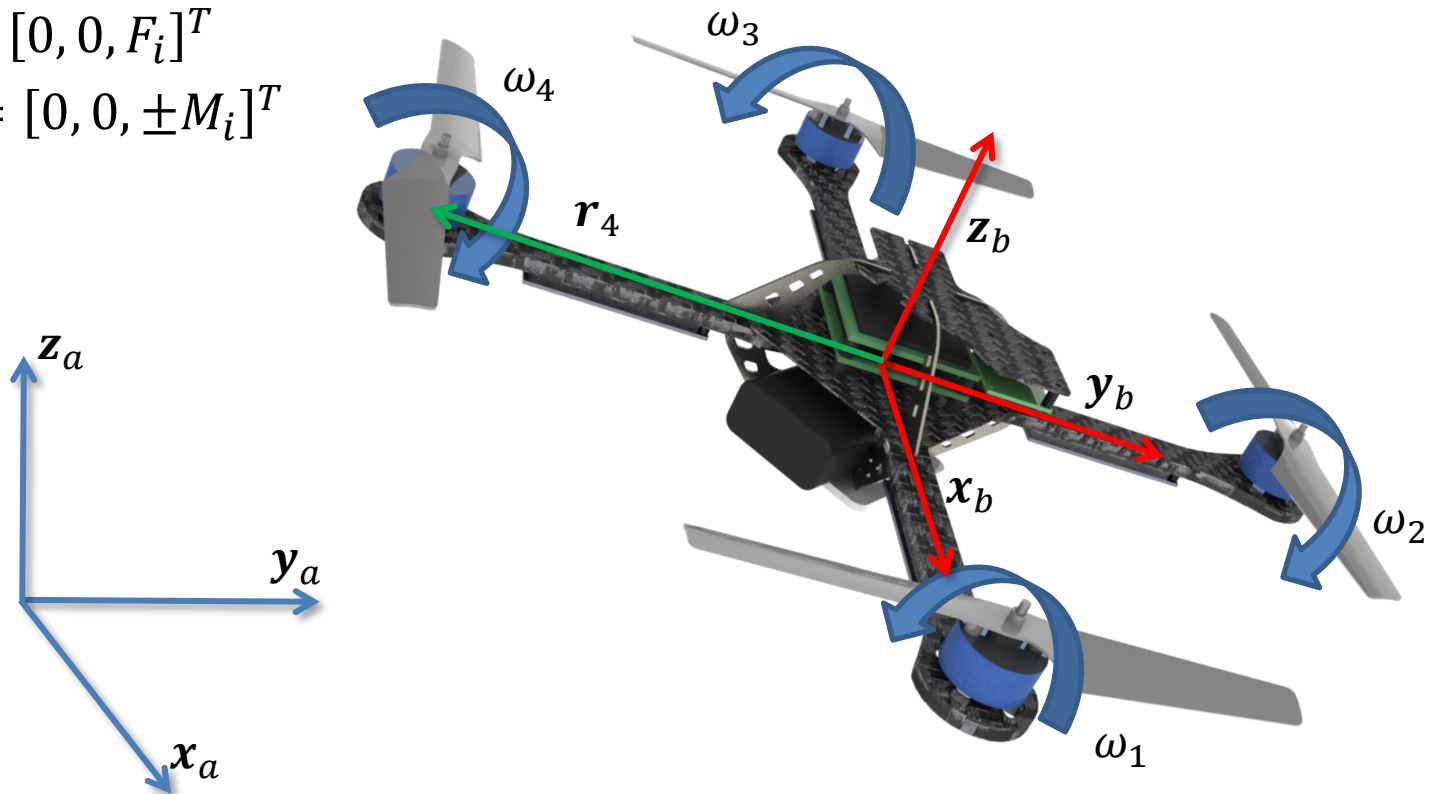
Instantaneous body  
angular velocity.



# Quadrotor Dynamics

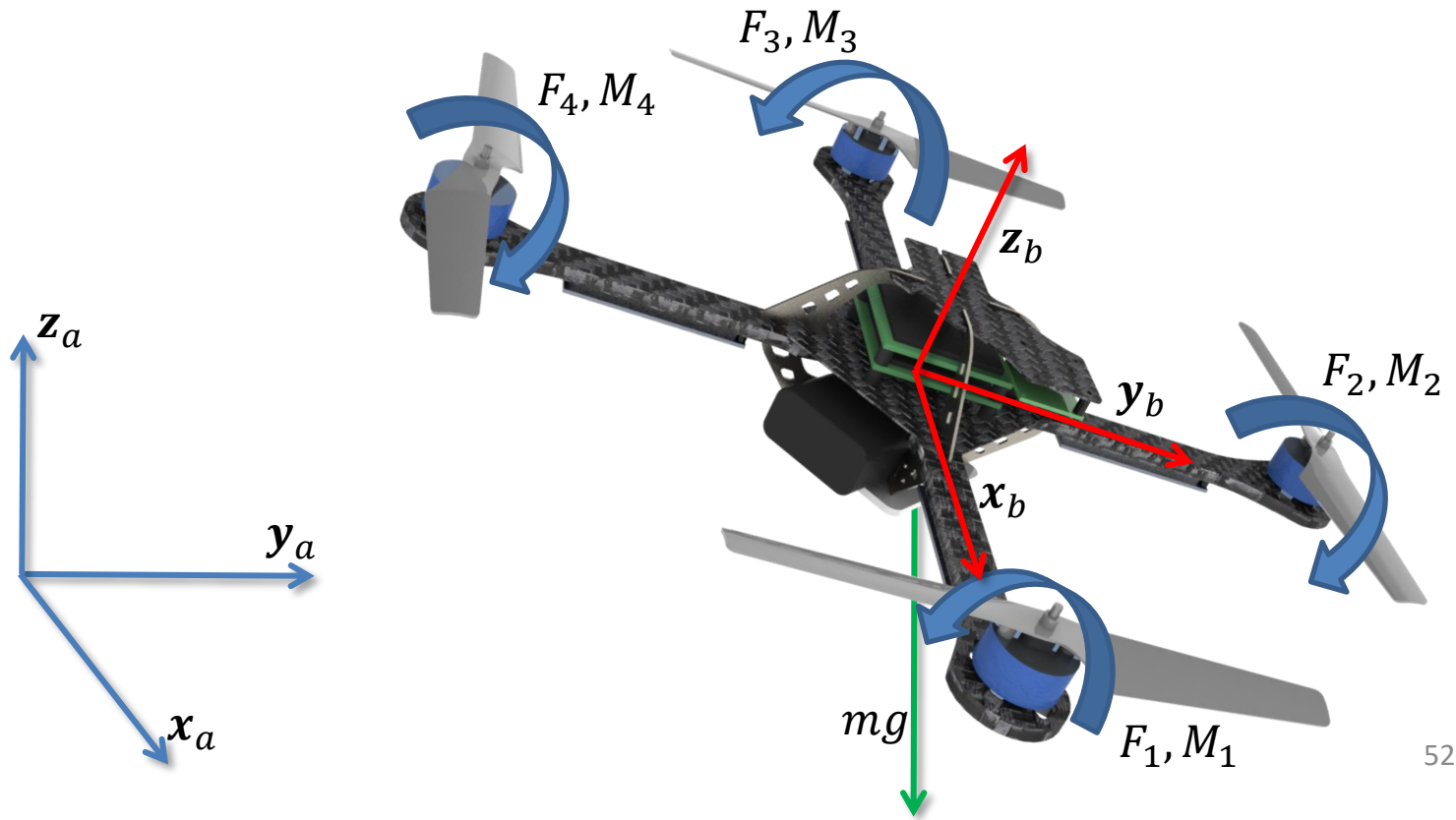
- Consider body frame

- $$- \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 - \mathbf{R}_{ab}^T [0, 0, mg]^T$$
- $$- \mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \mathbf{r}_4 \times \mathbf{F}_4 + \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_4$$
- $$- \mathbf{F}_i = [0, 0, F_i]^T$$
- $$- \mathbf{M}_i = [0, 0, \pm M_i]^T$$

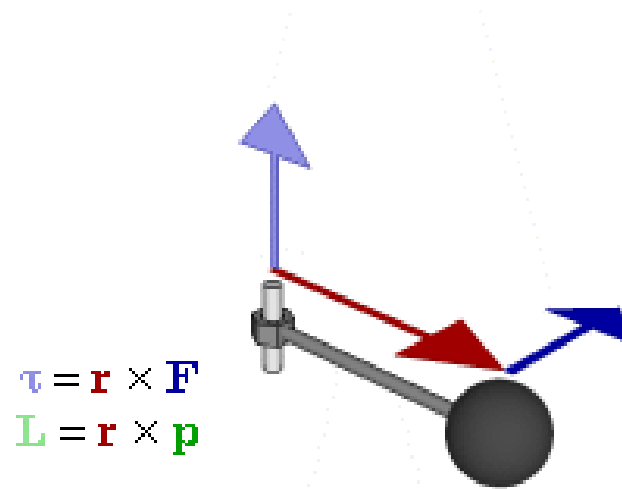


# Newton-Euler Equations

- Newton Equation:  $m\ddot{\mathbf{p}}^a = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \mathbf{R}_{ab} \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$



# Newton-Euler Equations



Relationship between force ( $F$ ), torque/moment of force( $\tau$ ), momentum ( $p$ ), and angular momentum ( $L$ ) vectors in a rotating system.  $r$  is the position vector.

# Newton-Euler Equations

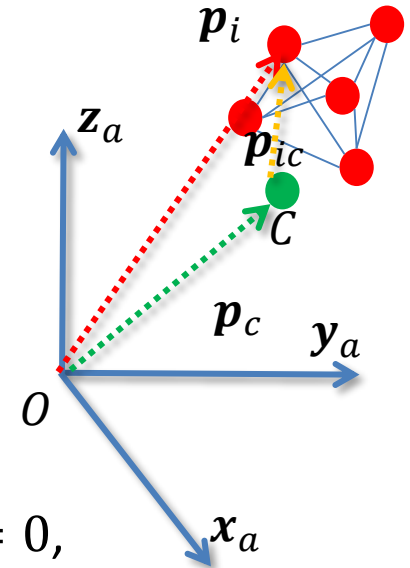
- The rigid body as a collection of particles

- Center of mass (CoM):  $\mathbf{p}_c$
- Position of the i-th particle to CoM:  $\mathbf{p}_{ic} = \mathbf{p}_i - \mathbf{p}_c$
- Velocity of the i-th particle to CoM:  $\mathbf{v}_{ic} = \dot{\mathbf{p}}_i - \dot{\mathbf{p}}_c = \mathbf{v}_i - \mathbf{v}_c$
- Angular momentum of the i-th particle:

$$\mathbf{H}_i = \mathbf{p}_{ic} \times m_i \mathbf{v}_i$$

- Angular momentum of the rigid body:

- $\mathbf{H} = \sum \mathbf{p}_{ic} \times m_i \mathbf{v}_i$
- Since:  $\sum m_i \mathbf{p}_{ic} = \sum m_i (\mathbf{p}_i - \mathbf{p}_c) = \sum m_i \mathbf{p}_i - \mathbf{p}_c \sum m_i = 0$ ,
- We have:  $\sum \mathbf{p}_{ic} \times m_i \mathbf{v}_c = (\sum m_i \mathbf{p}_{ic}) \times \mathbf{v}_c = 0$
- Therefore:  $\mathbf{H} = \sum \mathbf{p}_{ic} \times m_i \mathbf{v}_i - \sum \mathbf{p}_{ic} \times m_i \mathbf{v}_c = \sum \mathbf{p}_{ic} \times m_i \mathbf{v}_{ic}$
- Since:  $\mathbf{v}_{ic} = \boldsymbol{\omega} \times \mathbf{p}_{ic}$ ,
- We have:  $\mathbf{H} = \sum \mathbf{p}_{ic} \times (\boldsymbol{\omega} \times m_i \mathbf{p}_{ic}) = -\sum \mathbf{p}_{ic} \times (m_i \mathbf{p}_{ic} \times \boldsymbol{\omega})$

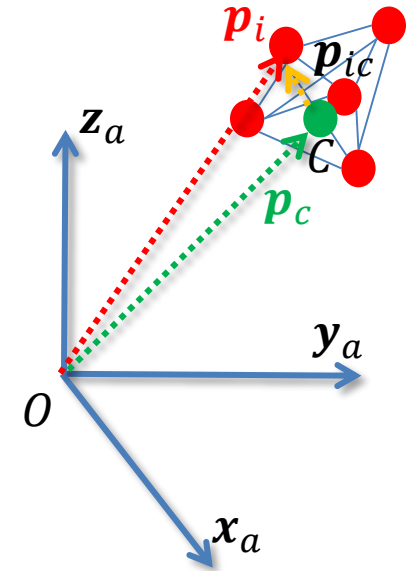




# Newton-Euler Equations

- Rotational dynamics

- Angular momentum:  $\mathbf{H} = \sum \mathbf{p}_{ic} \times m_i \mathbf{v}_i$
- Take the derivative:  $\dot{\mathbf{H}} = \sum \dot{\mathbf{p}}_{ic} \times m_i \mathbf{v}_i + \sum \mathbf{p}_{ic} \times m_i \dot{\mathbf{v}}_i$
- Since  $\sum \dot{\mathbf{p}}_{ic} \times m_i \mathbf{v}_i = \sum \mathbf{v}_i \times m_i \mathbf{v}_i - \mathbf{v}_c \times m_i \mathbf{v}_i =$   
 $\sum -\mathbf{v}_c \times m_i \mathbf{v}_i = -\mathbf{v}_c \times \frac{d}{dt} \sum m_i \mathbf{p}_i = -\mathbf{v}_c \times$   
 $\frac{d}{dt} \mathbf{p}_c \sum m_i = 0$
- We have  $\dot{\mathbf{H}} = \sum \mathbf{p}_{ic} \times m_i \dot{\mathbf{v}}_i$
- Referring to Newton's second law:  $\mathbf{F}_i + \sum_{i \neq j} \mathbf{F}_{ij} =$   
 $m_i \dot{\mathbf{v}}_i$
- $\dot{\mathbf{H}} = \sum \mathbf{p}_{ic} \times m_i \dot{\mathbf{v}}_i = \sum \mathbf{p}_{ic} \times (\mathbf{F}_i + \sum_{i \neq j} \mathbf{F}_{ij}) =$   
 $\sum \mathbf{p}_{ic} \times \mathbf{F}_i$
- We also know that the external moment:  $\mathbf{M} =$   
 $\sum \mathbf{p}_{ic} \times \mathbf{F}_i$
- We have the rotational dynamics:  $\mathbf{M} = \dot{\mathbf{H}}$

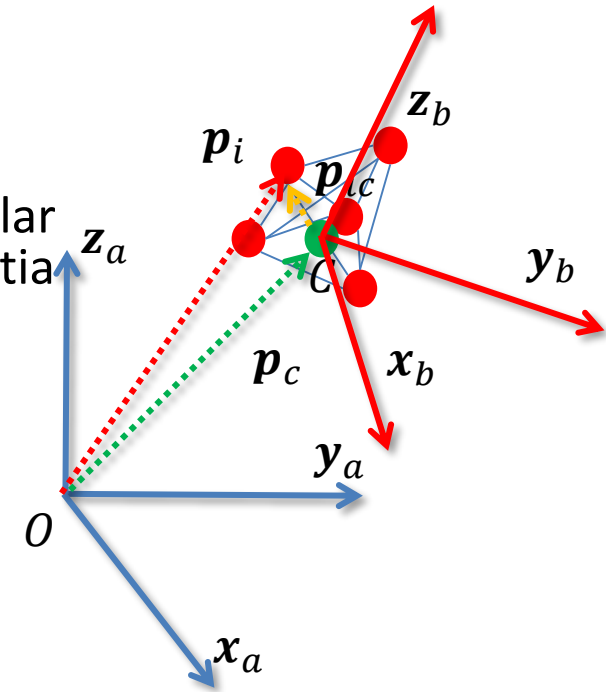


# Newton-Euler Equations

- Finishing the work on rotational dynamics

- Given:  $\mathbf{H} = -\sum \mathbf{p}_{ic} \times (m_i \mathbf{p}_{ic} \times \boldsymbol{\omega})$
- And using the fact:  $(\mathbf{R}\mathbf{a}) \times (\mathbf{R}\mathbf{b}) = \mathbf{R}(\mathbf{a} \times \mathbf{b})$ 
  - $\mathbf{R}$ : rotation matrix
  - $\mathbf{a}, \mathbf{b}$ : vectors
- We can transform the representation of the angular momentum to the body frame with constant inertia matrix:

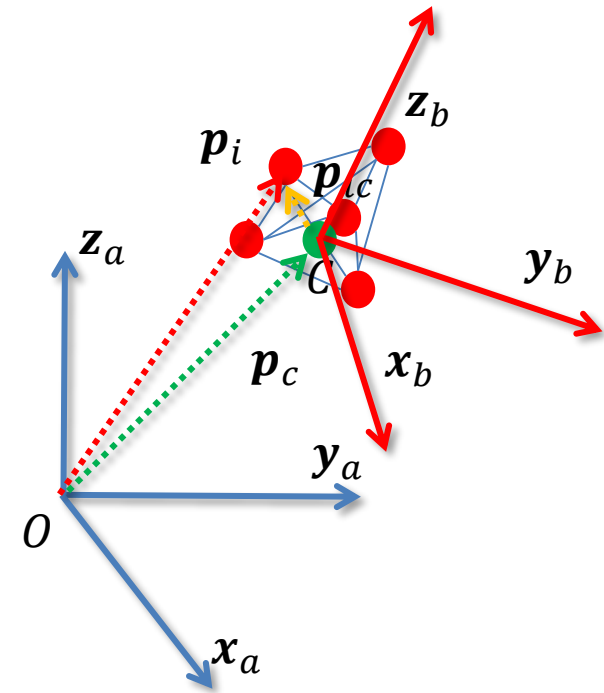
$$\begin{aligned}
 \mathbf{H} &= -\sum \mathbf{p}_{ic} \times (m_i \mathbf{p}_{ic} \times \boldsymbol{\omega}) \\
 &= -\sum \mathbf{R}_{ab} \mathbf{p}_{ic}^b \times (m_i \mathbf{R}_{ab} \mathbf{p}_{ic}^b \times \mathbf{R}_{ab} \boldsymbol{\omega}^b) \\
 &= -\mathbf{R}_{ab} \sum \mathbf{p}_{ic}^b \times (m_i \mathbf{p}_{ic}^b \times \boldsymbol{\omega}^b) \\
 &= -\mathbf{R}_{ab} \sum m_i \cdot \mathbf{p}_{ic}^b \times (\hat{\mathbf{p}}_{ic}^b \cdot \boldsymbol{\omega}^b) \\
 &= \mathbf{R}_{ab} \left( -\sum m_i \cdot \hat{\mathbf{p}}_{ic}^b \cdot \hat{\mathbf{p}}_{ic}^b \right) \cdot \boldsymbol{\omega}^b = \mathbf{R}_{ab} (\mathbf{I}^b \boldsymbol{\omega}^b)
 \end{aligned}$$





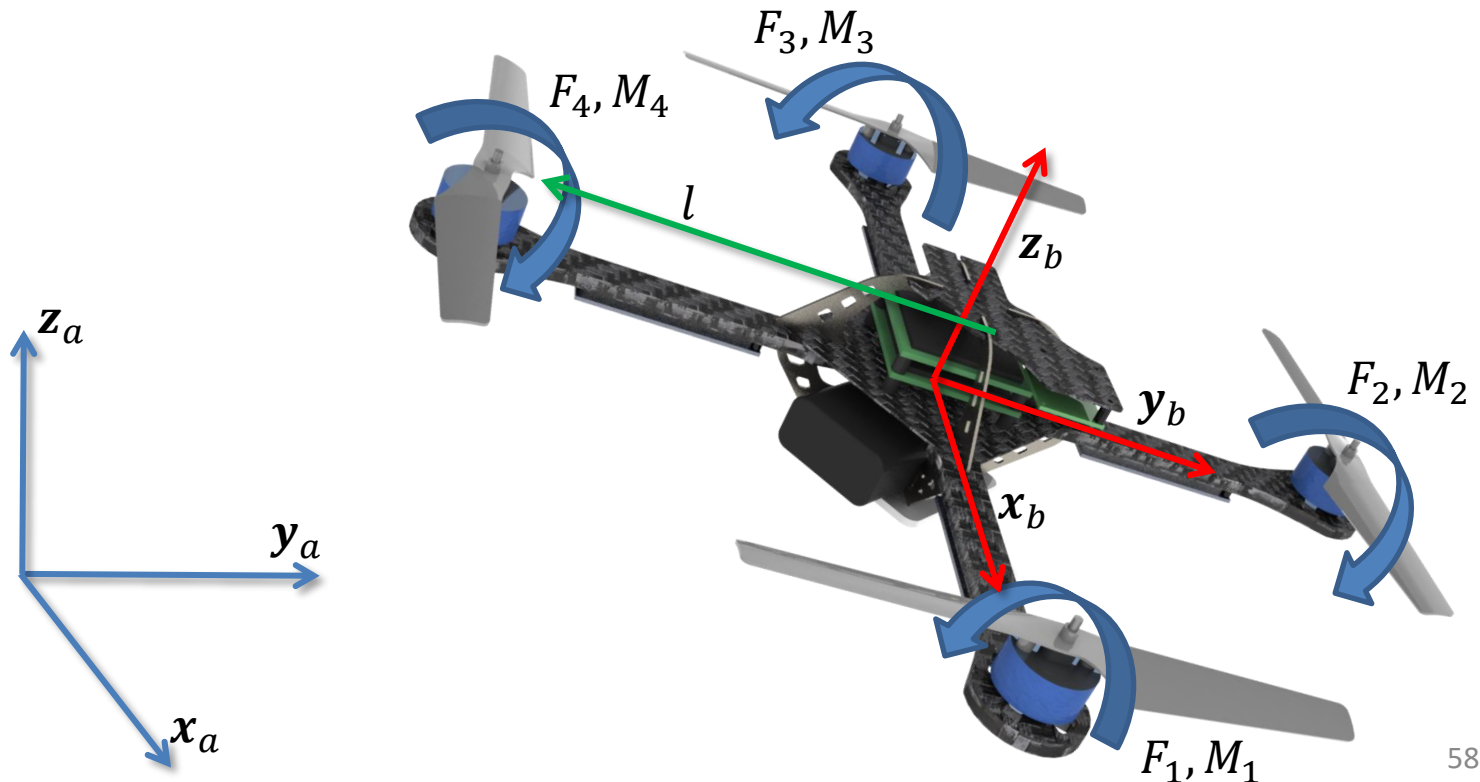
# Newton-Euler Equations

- Finishing the work on rotational dynamics
  - Given  $\mathbf{H} = \mathbf{R}_{ab}(\mathbf{I}^b \boldsymbol{\omega}^b)$
  - Take the derivative:  $\dot{\mathbf{H}} = \dot{\mathbf{R}}_{ab} \mathbf{I}^b \boldsymbol{\omega}^b + \mathbf{R}_{ab} \mathbf{I}^b \dot{\boldsymbol{\omega}}^b = \mathbf{R}_{ab} \hat{\boldsymbol{\omega}}^b \mathbf{I}^b \boldsymbol{\omega}^b + \mathbf{R}_{ab} \mathbf{I}^b \dot{\boldsymbol{\omega}}^b = \mathbf{R}_{ab} (\boldsymbol{\omega}^b \times (\mathbf{I}^b \boldsymbol{\omega}^b) + \mathbf{I}^b \dot{\boldsymbol{\omega}}^b)$
  - Also transform the moment into body frame:  $\mathbf{M} = \mathbf{R}_{ab} \mathbf{M}^b$
  - Finally:  $\mathbf{M}^b = \boldsymbol{\omega}^b \times (\mathbf{I}^b \boldsymbol{\omega}^b) + \mathbf{I}^b \dot{\boldsymbol{\omega}}^b$



# Newton-Euler Equations

- Euler Equation: 
$$\mathbf{I} \cdot \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \mathbf{I} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} l(F_2 - F_4) \\ l(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}$$





# Quadrotor Dynamics

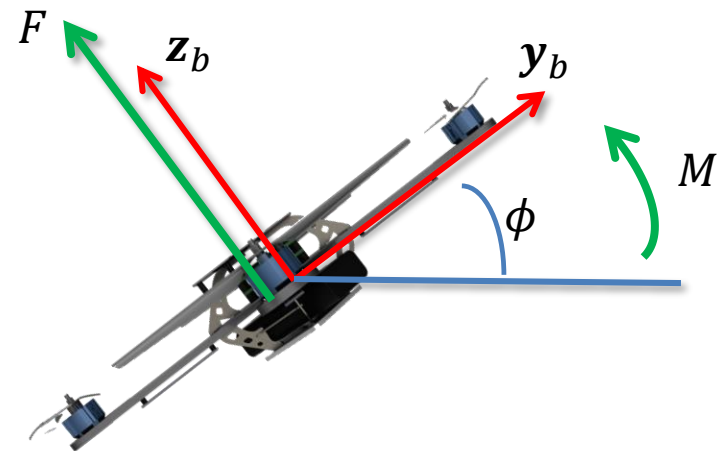
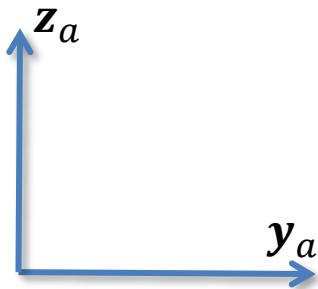
- Motor model:  $\dot{\omega}_i = k_m(\omega_i^{des} - \omega_i)$
- Thrust from individual motor:  $F_i = k_F \omega_i^2$
- Moment from individual motor:  $M_i = k_M \omega_i^2$

- Newton Equation:  $m\ddot{\mathbf{p}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \mathbf{R} \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$

- Euler Equation:  $\mathbf{I} \cdot \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \mathbf{I} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} l(F_2 - F_4) \\ l(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}$

# A Planar Quadrotor

$$\bullet \begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{m} \sin \phi & 0 \\ \frac{1}{m} \cos \phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} F \\ M \end{bmatrix}$$



$\omega_1$

# Assignment

- Chapter 2.1-2.4 of “A Mathematical Introduction to Robotic Manipulation”

# Next Lecture...

- Control basics
- Quadrotor control
- Trajectory generation

# Introducing Quaternion

- Rotation matrix  $\mathbf{R} \in SO(3)$ 
  - No singularity
  - Redundant parameters
  - Kinematics:  $\dot{\mathbf{R}} = \mathbf{R}\hat{\boldsymbol{\omega}}$ ,  $\boldsymbol{\omega}$  is the body angular rate
- ZYX Euler angle
  - Singular at roll angle of 90 degrees
  - Minimum number of parameters

$$\text{– Kinematics: } \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \frac{\cos(\psi)}{\cos(\theta)} & \frac{\sin(\psi)}{\cos(\theta)} & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ \cos(\psi)\tan(\theta) & \sin(\psi)\tan(\theta) & 1 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

- Is there a singularity free parameterization that with reduced parameters? **YES → quaternion**

# Introducing Quaternion

- Recall a rotation matrix  $\mathbf{R}$  can be decomposed as a rotation axis  $\mathbf{u}$  of angle  $\theta$  by  $\mathbf{R} = e^{\hat{\mathbf{u}}\theta} = \mathbf{I} + \hat{\mathbf{u}} \sin \theta + \hat{\mathbf{u}}^2(1 - \cos \theta)$
- Any vector  $\mathbf{p}$ , after rotation  $\mathbf{R}$ , is  $\mathbf{p}' = \mathbf{R}\mathbf{p}$

$$\Rightarrow \begin{bmatrix} 0 \\ \mathbf{p}' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix}$$

A representation of rotation

$$\Rightarrow \begin{bmatrix} 0 \\ \mathbf{p}' \end{bmatrix} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \mathbf{u} \end{bmatrix} \circ \begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix} \circ \begin{bmatrix} \cos \frac{\theta}{2} \\ -\sin \frac{\theta}{2} \mathbf{u} \end{bmatrix}; \forall \mathbf{p}, \mathbf{u}, \theta$$

A new binary operation

where

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \circ \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$



# Definitions of Quaternion

- Quaternion

$$\boldsymbol{Q} \stackrel{\text{def}}{=} q_0 + q_1 \boldsymbol{i} + q_2 \boldsymbol{j} + q_3 \boldsymbol{k} \stackrel{\text{def}}{=} q_0 + \boldsymbol{q} \stackrel{\text{def}}{=} \begin{bmatrix} q_0 \\ \boldsymbol{q} \end{bmatrix}$$

where  $\boldsymbol{i} \circ \boldsymbol{i} = \boldsymbol{j} \circ \boldsymbol{j} = \boldsymbol{k} \circ \boldsymbol{k} \stackrel{\text{def}}{=} -1; \boldsymbol{i} \circ \boldsymbol{j} \stackrel{\text{def}}{=} \boldsymbol{k}; \boldsymbol{j} \circ \boldsymbol{k} \stackrel{\text{def}}{=} \boldsymbol{i}; \boldsymbol{k} \circ \boldsymbol{i} \stackrel{\text{def}}{=} \boldsymbol{j}$

- Conjugate

$$\bar{\boldsymbol{Q}} \stackrel{\text{def}}{=} q_0 - q_1 \boldsymbol{i} - q_2 \boldsymbol{j} - q_3 \boldsymbol{k} \stackrel{\text{def}}{=} q_0 - \boldsymbol{q} \stackrel{\text{def}}{=} \begin{bmatrix} q_0 \\ -\boldsymbol{q} \end{bmatrix}$$

- Norm

$$\|\boldsymbol{Q}\| \stackrel{\text{def}}{=} \boldsymbol{Q} \circ \bar{\boldsymbol{Q}}$$

- Inverse

$$\boldsymbol{Q}^{-1} \stackrel{\text{def}}{=} \frac{\bar{\boldsymbol{Q}}}{\|\boldsymbol{Q}\|}$$

# Properties of Quaternion

- Property 1

$$\mathbf{A} = a_0 + \mathbf{a}, \quad \mathbf{B} = b_0 + \mathbf{b} \quad \text{particularly}$$

$$\mathbf{A} \circ \mathbf{B} = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) + (a_0 \mathbf{b} + \mathbf{a} b_0 + \mathbf{a} \times \mathbf{b})$$

$$(0 + \mathbf{a}) \circ (0 + \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

- Property 2  $(\mathbf{A} \circ \mathbf{B}) \circ \mathbf{C} = \mathbf{A} \circ (\mathbf{B} \circ \mathbf{C})$

- Property 3  $\overline{\mathbf{A} \circ \mathbf{B}} = \overline{\mathbf{B}} \circ \overline{\mathbf{A}}$

- Property 4  $\mathbf{A} = a_0 + a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$   
 $\mathbf{B} = b_0 + b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$

Exactly what  
we need for  
representing  
rotations!!

$$\mathbf{A} \circ \mathbf{B} = \underbrace{\begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}}_{\mathcal{L}(\mathbf{A})} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \underbrace{\begin{bmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{bmatrix}}_{\mathcal{R}(\mathbf{B})} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

# Properties of Quaternion

- Product of Quaternions  $\mathbf{A} \circ \mathbf{B} = \mathcal{L}(\mathbf{A}) \begin{bmatrix} b_0 \\ \mathbf{b} \end{bmatrix} = \mathcal{R}(\mathbf{B}) \begin{bmatrix} a_0 \\ \mathbf{a} \end{bmatrix}$

$$\mathcal{L}(\mathbf{A}) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} = \begin{bmatrix} a_0 & -\mathbf{a}^T \\ \mathbf{a} & a_0 \mathbf{I}_3 + \hat{\mathbf{a}} \end{bmatrix}$$

$$\mathcal{R}(\mathbf{B}) = \begin{bmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{bmatrix} = \begin{bmatrix} b_0 & -\mathbf{b}^T \\ \mathbf{b} & b_0 \mathbf{I}_3 - \hat{\mathbf{a}} \end{bmatrix}$$

# Unit Quaternion and Rotation

Recall

$$\underbrace{\begin{bmatrix} 0 \\ \mathbf{p}' \end{bmatrix}}_{\mathbf{P}'} = \underbrace{\begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \mathbf{u} \end{bmatrix}}_{\mathbf{Q}} \circ \underbrace{\begin{bmatrix} 0 \\ \mathbf{p} \end{bmatrix}}_{\mathbf{P}} \circ \underbrace{\begin{bmatrix} \cos \frac{\theta}{2} \\ -\sin \frac{\theta}{2} \mathbf{u} \end{bmatrix}}_{\bar{\mathbf{Q}}}$$

$$\mathbf{P}' \stackrel{\text{def}}{=} 0 + \mathbf{p}'; \mathbf{P} \stackrel{\text{def}}{=} 0 + \mathbf{p};$$

$$\mathbf{Q} \stackrel{\text{def}}{=} \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \mathbf{u}$$

If using quaternion  $Q$  to represent a rotation:

Basic principle:  $\mathbf{P}' = \mathbf{Q} \circ \mathbf{P} \circ \bar{\mathbf{Q}}$

- $\mathbf{P}', \mathbf{P}$  are respectively the coordinates of the post-rotation vector and prior-rotation vector, both in the **same** frame.
- $\mathbf{Q}$  is the rotation quaternion associated with the rotation

# Unit Quaternion and Rotation

Basic principle:  $P' = Q \circ P \circ \overline{Q}$

Property 1 Constant vector:  $P^b = \overline{Q} \circ P \circ R$

- $P^b, P$  are respectively the coordinates in the body frame and the world frame, of the **same** vector
- Can be interpreted as that the vector is rotating in the opposite direction, then call the first result

# Unit Quaternion and Rotation

Basic principle:  $P' = Q \circ P \circ \bar{Q}$

Property 2 Two sequent rotations

- Case 1

$Q_1, Q_2$  are the two rotation quaternions where the rotation axes are both represented in the **initial** frame

$$\left. \begin{aligned} P' &= Q_1 \circ P \circ \bar{Q}_1 \\ P'' &= Q_2 \circ P' \circ \bar{Q}_2 = \underbrace{Q_2 \circ Q_1}_{Q} \circ P \circ \bar{Q}_1 \circ \bar{Q}_2 \end{aligned} \right\} \Rightarrow Q = Q_2 \circ Q_1$$

- Case 2

$Q_1, Q_2$  are the two rotation quaternions where the rotation axis of  $Q_2$  is represented in the frame obtained by performing  $Q_1$ .

$$P' = Q_1 \circ P \circ \bar{Q}_1$$

$$\begin{aligned} P'' &= (\underbrace{Q_1 \circ Q_2 \circ \bar{Q}_1}_{Q}) \circ P' \circ \overline{Q_1 \circ Q_2 \circ \bar{Q}_1} = (Q_1 \circ Q_2 \circ \bar{Q}_1) \circ Q_1 \circ P \circ \bar{Q}_1 \circ \overline{Q_1 \circ Q_2 \circ \bar{Q}_1} = \underbrace{Q_1 \circ Q_2}_{Q} \circ P \circ \bar{Q}_1 \circ Q_2 \\ &\Rightarrow Q = Q_1 \circ Q_2 \end{aligned}$$

# Quaternion Kinematics

Recall

$$P' = Q \circ P \circ \bar{Q}$$

Taking derivative yields

$$\begin{aligned}\dot{P}' &= \dot{Q} \circ P \circ \bar{Q} + Q \circ P \circ \dot{\bar{Q}} \\ &= \dot{Q} \circ \underbrace{\bar{Q} \circ P' \circ Q}_{P} \circ \bar{Q} + Q \circ \underbrace{\bar{Q} \circ P' \circ Q}_{P} \circ \dot{\bar{Q}} \\ &= \dot{Q} \circ \bar{Q} \circ P' + P' \circ Q \circ \dot{\bar{Q}}\end{aligned}$$

As  $Q \circ \bar{Q} = 1$

Taking derivative yields

$$\dot{Q} \circ \bar{Q} + Q \circ \dot{\bar{Q}} = 0 \quad \Rightarrow \quad \left. \begin{aligned} Q \circ \dot{\bar{Q}} &= -\dot{Q} \circ \bar{Q} \\ Q \circ \dot{\bar{Q}} &= \overline{\dot{Q} \circ \bar{Q}} \end{aligned} \right\} \dot{Q} \circ \bar{Q} = 0 + a; Q \circ \dot{\bar{Q}} = 0 - a$$

Notice

$$\dot{P}' = (0 + a) \circ P' + P' \circ (0 - a)$$

# Quaternion Kinematics

$$\begin{aligned}
 \dot{\mathbf{P}}' &= (0 + \mathbf{a}) \circ \mathbf{P}' + \mathbf{P}' \circ (0 - \mathbf{a}) \\
 &= \begin{bmatrix} 0 \\ \mathbf{a} \end{bmatrix} \circ \begin{bmatrix} 0 \\ \mathbf{p}' \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{p}' \end{bmatrix} \circ \begin{bmatrix} 0 \\ -\mathbf{a} \end{bmatrix} \\
 &= \begin{bmatrix} 0 \\ \mathbf{a} \times \mathbf{p}' \end{bmatrix} - \mathbf{a} \cdot \mathbf{p}' + \begin{bmatrix} 0 \\ \mathbf{p}' \times (-\mathbf{a}) \end{bmatrix} - \mathbf{p}' \cdot (-\mathbf{a})
 \end{aligned}$$

$$\begin{bmatrix} 0 \\ \dot{\mathbf{p}}' \end{bmatrix} = \begin{bmatrix} 0 \\ 2\mathbf{a} \times \mathbf{p}' \end{bmatrix}$$

Recall  $\dot{\mathbf{p}}' = \boldsymbol{\omega} \times \mathbf{p}'$ ,  $\boldsymbol{\omega}$  is angular velocity vector represented in the **world** frame

Then  $2\mathbf{a} = \boldsymbol{\omega} \Rightarrow \mathbf{a} = \frac{1}{2}\boldsymbol{\omega}$

$$\Rightarrow \dot{\mathbf{Q}} \circ \bar{\mathbf{Q}} = \begin{bmatrix} 0 \\ \mathbf{a} \end{bmatrix} = \frac{1}{2} \underbrace{\begin{bmatrix} 0 \\ \boldsymbol{\omega} \end{bmatrix}}_{\boldsymbol{\Omega}} \Rightarrow \dot{\mathbf{Q}} = \frac{1}{2} \boldsymbol{\Omega} \circ \mathbf{Q}$$

$$\mathbf{Q} = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} \Rightarrow \dot{\mathbf{Q}} = \frac{1}{2} \boldsymbol{\Omega} \circ \mathbf{Q} = \mathcal{L}(\boldsymbol{\Omega}) \mathbf{Q} = \frac{1}{2} \begin{bmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & \hat{\boldsymbol{\omega}} \end{bmatrix} \mathbf{Q}$$



# Quaternion Kinematics

$$\dot{Q} = \frac{1}{2} \Omega \circ Q$$

$$\Omega = \begin{bmatrix} 0 \\ \omega \end{bmatrix}$$

$\omega$  is angular velocity represented in the **world** frame

– What if  $\omega$  is represented in the **body** frame (i.e.  $\omega^b$ )?

$$\Omega^b = \begin{bmatrix} 0 \\ \omega^b \end{bmatrix} \text{ is represented in body frame}$$

$$\Rightarrow \Omega^b = \bar{Q} \circ \Omega \circ Q$$

$$\Rightarrow \boxed{\dot{Q} = \frac{1}{2} Q \circ \Omega^b}$$

$$Q = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix}$$

$$\Rightarrow \dot{Q} = \frac{1}{2} Q \circ \Omega^b = \mathcal{R}(\Omega^b) Q = \frac{1}{2} \begin{bmatrix} 0 & -\omega^{bT} \\ \omega^b & -\widehat{\omega^b} \end{bmatrix} Q$$

# Unit Quaternion and Rotation

- Rotation to quaternion:  $\mathbf{Q} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \mathbf{u} \end{bmatrix}$ , where  $\mathbf{R} = e^{\hat{\mathbf{u}}\theta}$
- Quaternion to rotation
- $\mathbf{R} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_0q_2 + q_1q_3) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(-q_0q_1 + q_2q_3) \\ 2(-q_0q_2 + q_1q_3) & 2(q_0q_1 + q_2q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$
- Rotating a vector:  $\mathbf{P}' = \mathbf{Q} \circ \mathbf{P} \circ \overline{\mathbf{Q}}$
- Rotating a frame:  $\mathbf{P}^b = \overline{\mathbf{Q}} \circ \mathbf{P} \circ \mathbf{Q}$
- Sequential rotation (extrinsic):  $\mathbf{Q} = \mathbf{Q}_n \dots \circ \mathbf{Q}_2 \circ \mathbf{Q}_1$
- Sequential rotation (intrinsic):  $\mathbf{Q} = \mathbf{Q}_1 \circ \mathbf{Q}_2 \dots \circ \mathbf{Q}_n$
- Kinematics under spatial frame:  $\dot{\mathbf{Q}} = \frac{1}{2} \boldsymbol{\Omega} \circ \mathbf{Q} = \frac{1}{2} \begin{bmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & \hat{\boldsymbol{\omega}} \end{bmatrix} \mathbf{Q}$
- Kinematics under body frame:  $\dot{\mathbf{Q}} = \frac{1}{2} \mathbf{Q} \circ \boldsymbol{\Omega} = \frac{1}{2} \begin{bmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & -\hat{\boldsymbol{\omega}} \end{bmatrix} \mathbf{Q}$