

# fn score\_candidates\_map

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This proof resides in “**contrib**” because it has not completed the vetting process.

Proves soundness of `score_candidates_map` in `mod.rs` at commit `f5bb719` (outdated<sup>1</sup>). `score_candidates_map` returns a specific function that can be used to prove stability of the quantile scoring transformation.

## 1 Hoare Triple

### Precondition

`alpha_den > alpha_num`.

### Function

```
1 def score_candidates_map(alpha_num, alpha_den, known_size) -> Callable[[int], int]:
2     def stability_map(d_in):
3         if known_size:
4             return T.inf_cast(d_in // 2).inf_mul(alpha_den)
5         else:
6             abs_dist_const: u64 = max(alpha_num, alpha_den - alpha_num) #
7             return T.exact_int_cast(d_in).alerting_mul(abs_dist_const)
8
9     return stability_map
```

### Postcondition

**Theorem 1.1.** Define function as follows:

$$\text{function}(x) = \text{score\_candidates}_i(x, \text{candidates}, \text{alpha\_num}, \text{alpha\_den}, \text{size\_limit})$$

The function calls `score_candidates` with fixed choices of `candidates`, `alpha_num`, `alpha_den` and `size_limit`.

If the input domain is the set of all vectors with non-null elements of type `TI`, input metric is either `SymmetricDistance` or `InsertDeleteDistance`, and the postcondition of function is satisfied, then function is stable.

This means that, for any two elements  $x, x'$  in the `input_domain`, where  $x, x'$  share the same length if `known_length` is true, and any pair  $(d\_in, d\_out)$ , where  $d\_in$  has the associated type for `input_metric` and  $d\_out$  has the associated type for `output_metric`, then when  $x, x'$  are  $d\_in$ -close under `input_metric` and `stability_map(d_in) ≤ d_out`, `function(x)`, `function(x')` are  $d\_out$ -close under `output_metric`.

The sensitivity of this function differs depending on whether the size of the input vector is known. First, consider the case where the size is unknown.

<sup>1</sup>See new changes with `git diff f5bb719..396608b rust/src/transformations/quantile_score_candidates/mod.rs`

**Lemma 1.2.** If  $d_{Sym}(x, x') = 1$ , then  $d_{\infty}(\text{function}(x), \text{function}(x')) \leq \max(\alpha_{num}, \alpha_{den} - \alpha_{num})$ .

*Proof.* Assume  $d_{Sym}(x, x') = 1$ .

$$\begin{aligned}
& d_{\infty}(\text{function}(x)_i, \text{function}(x')_i) \\
&= \max_i |\text{function}(x)_i - \text{function}(x')_i| && \text{by definition of } d_{\infty} \\
&= \max_i |\text{abs\_diff}(\alpha_{den} \cdot \min(\#(x < C_i), l), \alpha_{num} \cdot \min(|x| - \#(x = C_i), l)) && \text{by definition of } \text{function} \\
&\quad \text{abs\_diff}(\alpha_{den} \cdot \min(\#(x' < C_i), l), \alpha_{num} \cdot \min(|x'| - \#(x' = C_i), l))| \\
&= \alpha_{den} \cdot \max_i |\min(\#(x < C_i), l) - \alpha \cdot \min(|x| - \#(x = C_i), l)| \\
&\quad |\min(\#(x' < C_i), l) - \alpha \cdot \min(|x'| - \#(x' = C_i), l)| \\
&\leq \alpha_{den} \cdot \max_i |\#(x < C_i) - \alpha \cdot (|x| - \#(x = C_i))| \\
&\quad |\#(x' < C_i) - \alpha \cdot (|x'| - \#(x' = C_i))|
\end{aligned}$$

Consider each of the three cases of adding or removing an element in  $x$ .

Case 1. Assume  $x'$  is equal to  $x$ , but with some  $x_j < C_i$  added or removed.

$$\begin{aligned}
&= \alpha_{den} \cdot \max_i |(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot \#(x > C_i)| \\
&\quad - |(1 - \alpha) \cdot (\#(x < C_i) \pm 1) - \alpha \cdot \#(x > C_i)| \\
&\leq \alpha_{den} \cdot \max_i |(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot \#(x > C_i)| \\
&\quad - (|(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot \#(x > C_i)| + |\pm (1 - \alpha)|) && \text{by triangle inequality} \\
&= \alpha_{den} \cdot \max_i |1 - \alpha| && \text{scores cancel} \\
&= \alpha_{den} - \alpha_{num} && \text{since } \alpha \leq 1
\end{aligned}$$

Case 2. Assume  $x'$  is equal to  $x$ , but with some  $x_j > C_i$  added or removed.

$$\begin{aligned}
&= \alpha_{den} \cdot \max_i |(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot \#(x > C_i)| \\
&\quad - |(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot (\#(x > C_i) \pm 1)| \\
&\leq \alpha_{den} \cdot \max_i |(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot \#(x > C_i)| \\
&\quad - (|(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot \#(x > C_i)| + |\pm \alpha|) && \text{by triangle inequality} \\
&= \alpha_{den} \cdot \max_i |\alpha| && \text{scores cancel} \\
&= \alpha_{num} && \text{since } \alpha \geq 0
\end{aligned}$$

Case 3. Assume  $x'$  is equal to  $x$ , but with some  $x_j = C_i$  added or removed.

$$\begin{aligned}
&= \alpha_{den} \cdot \max_i |(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot \#(x > C_i)| \\
&\quad - |(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot \#(x > C_i)| \\
&= 0 && \text{no change in score}
\end{aligned}$$

Take the union bound over all cases.

$$\leq \max(\alpha_{num}, \alpha_{den} - \alpha_{num})$$

□

Now consider the case where the dataset size is known.

**Lemma 1.3.** If  $d_{CO}(x, x') \leq 1$ , then  $d_\infty(\text{function}(x), \text{function}(x')) \leq \alpha_{den}$ .

*Proof.* Assume  $d_{CO}(x, x') \leq 1$ .

$$\begin{aligned}
& d_\infty(\text{function}(x), \text{function}(x')) \\
&= \max_i |\text{function}(x)_i - \text{function}(x')_i| && \text{by definition of } d_\infty \\
&= \max_i |\text{abs\_diff}(\alpha_{den} \cdot \min(\#(x < C_i), l), \alpha_{num} \cdot \min(|x| - \#(x = C_i), l)) \\
&\quad - \text{abs\_diff}(\alpha_{den} \cdot \min(\#(x' < C_i), l), \alpha_{num} \cdot \min(|x'| - \#(x' = C_i), l))| && \text{by def. of function} \\
&= \alpha_{den} \cdot \max_i ||\min(\#(x < C_i), l) - \alpha \cdot \min(|x| - \#(x = C_i), l)| \\
&\quad - |\min(\#(x' < C_i), l) - \alpha \cdot \min(|x'| - \#(x' = C_i), l)|| \\
&= \alpha_{den} \cdot \max_i ||\#(x < C_i) - \alpha \cdot (|x| - \#(x = C_i))| \\
&\quad - |\#(x' < C_i) - \alpha \cdot (|x'| - \#(x' = C_i))||
\end{aligned}$$

Consider each of the four cases of changing a row in  $x$ .

Case 1. Assume  $x'$  is equal to  $x$ , but with some  $x_j < C_i$  replaced with  $x'_j > C_i$ .

$$\begin{aligned}
&= \alpha_{den} \cdot \max_i |(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot \#(x > C_i)| \\
&\quad - |(1 - \alpha) \cdot (\#(x < C_i) - 1) - \alpha \cdot (\#(x > C_i) + 1)| && \text{by definition of function} \\
&\leq \alpha_{den} \cdot \max_i |(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot \#(x > C_i)| \\
&\quad - (|(1 - \alpha) \cdot \#(x < C_i) - \alpha \cdot \#(x > C_i)| + |1|) && \text{by triangle inequality} \\
&= \alpha_{den} \cdot \max_i |1| && \text{scores cancel} \\
&= \alpha_{den}
\end{aligned}$$

Case 2. Assume  $x'$  is equal to  $x$ , but with some  $x_j > C_i$  replaced with  $x'_j < C_i$ .

$$= \alpha_{den}$$

by symmetry, follows from Case 1.

Case 3. Assume  $x'$  is equal to  $x$ , but with some  $x_j \neq C_i$  replaced with  $C_i$ .

$$\leq \max(\alpha_{num}, \alpha_{den} - \alpha_{num})$$

equivalent to one removal (see `make_quantile_score_candidates`)

Case 4. Assume  $x'$  is equal to  $x$ , but with some  $x_j = C_i$  replaced with  $x'_j \neq C_i$ .

$$\leq \max(\alpha_{num}, \alpha_{den} - \alpha_{num})$$

equivalent to one addition (see `make_quantile_score_candidates`)

Take the union bound over all cases.

$$d_\infty(x_i, x'_i) \leq \max(\alpha_{den}, \max(\alpha_{num}, \alpha_{den} - \alpha_{num})) = \alpha_{den}$$

$$\text{since } \max(\alpha, 1 - \alpha) \leq 1$$

□

*Proof of postcondition.* Assume the input domain is the set of all vectors with non-null elements of type TI, input metric is either `SymmetricDistance` or `InsertDeleteDistance`, and the postcondition of `function` is satisfied.

First, consider the case where the size is unknown. Take any two members  $s, s'$  in the `input_domain` and any pair  $(d\_in, d\_out)$ , where `d_in` has the associated type for `input_metric` and `d_out` has the associated type for `output_metric`. Assume  $s, s'$  are `d_in`-close under `input_metric` and that `stability_map(d_in) ≤ d_out`.

$$\begin{aligned} d\_out &= \max_{s \sim s'} d_\infty(s, s') && \text{where } s = \text{function}(x) \\ &= \max_{s \sim s'} \max_i |s_i - s'_i| && \text{by definition of } \text{LInfDistance}, \text{ without monotonicity} \\ &\leq \sum_j^{d\_in} \max_{Z_j \sim Z_{j+1}} \max_i |s_{i,j} - s_{i,j+1}| && \text{by path property } d_{Sym}(Z_i, Z_{i+1}) = 1, x = Z_0 \text{ and } x' = Z_{d\_in} \\ &\leq \sum_j^{d\_in} \max(\alpha_{num}, \alpha_{den} - \alpha_{num}) && \text{by 1.2} \\ &\leq d\_in \cdot \max(\alpha_{num}, \alpha_{den} - \alpha_{num}) \end{aligned}$$

This formula matches the stability map in the case where the dataset size is unknown.

Now, consider the case where the size is known. Take any two elements  $s, s'$  in the `input_domain` and any pair  $(d\_in, d\_out)$ , where `d_in` has the associated type for `input_metric` and `d_out` has the associated type for `output_metric`. Assume  $s, s'$  are `d_in`-close under `input_metric` and that `stability_map(d_in) ≤ d_out`.

$$\begin{aligned}
\mathbf{d\_out} &= \max_{s \sim s'} d_{\infty}(s, s') \\
&= \max_{s \sim s'} \max_i |s_i - s'_i| && \text{by definition of \texttt{LInfDistance}, without monotonicity} \\
&\leq \sum_j^{\mathbf{d\_in}/2} \max_{Z_j \sim Z_{j+1}} \max_i |s_{i,j} - s_{i,j+1}| && \text{by path property } d_{CO}(Z_i, Z_{i+1}) = 1, x = Z_0 \text{ and } Z_{\mathbf{d\_in}} = x' \\
&\leq \sum_j^{\mathbf{d\_in}/2} \alpha_{den} && \text{by 1.3} \\
&\leq (\mathbf{d\_in}/2) \cdot \alpha_{den}
\end{aligned}$$

This formula matches the stability map in the case where the dataset size is known.

It is shown that  $\mathbf{function}(x)$ ,  $\mathbf{function}(x')$  are  $\mathbf{d\_out}$ -close under  $\mathbf{output\_metric}$  for any choice of input arguments.  $\square$