fn make_bounded_range_to_zCDP

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This proof resides in **"contrib"** because it has not completed the vetting process.

Proves soundness of `bounded_range_to_zCDP` in `mod.rs` at commit 0b8f4222 (outdated[1]). The conversion between bounded range [DR19] and zCDP comes from Lemma 3.2 in [CR20]. The proof in this document is an adaptation of Theorem 5 here.

1 Hoare Triple

Preconditions

Compiler-verified

- Variable `meas` is a valid measurement of type `Measurement<DI, TO, MI, RangeDivergence>`
- Generic `DI` (input domain) is a type with trait `Domain`.
- Generic `MI` (input metric) is a type with trait `Metric`.
- `MetricSpace` is implemented for `(DI, MI)`. Therefore `MI` is a valid metric on `DI`.

Human-verified

None

Pseudocode

```
def make_bounded_range_to_zCDP(meas: Measurement) -> Measurement:
    def privacy_map(d_in: f64) -> f64:
        return meas.map(d_in).inf_powi(ibig(2)).inf_div(8.0)

    return meas.with_map( #
        meas.input_metric,
        ZeroConcentratedDivergence,
        PrivacyMap.new_fallible(privacy_map),
    )
```

Postcondition

Theorem 1.1 (Postcondition)**.** For every setting of the input parameters `(meas, DI, TO, MI)` to `make_bounded_range_to_zCDP` such that the given preconditions hold, `make_bounded_range_to_zCDP` raises an exception (at compile time or run time) or returns a valid measurement. A valid measurement has the following property:

[1]See new changes with `git diff 0b8f4222..e1e35809 rust/src/combinators/measure_cast/bounded_range_to_zCDP/mod.rs`

1. (Privacy guarantee). For every pair of elements x, x' in `input_domain` and for every pair (`d_in`, `d_out`), where `d_in` has the associated type for `input_metric` and `d_out` has the associated type for `output_measure`, if x, x' are `d_in`-close under `input_metric`, `privacy_map(d_in)` does not raise an exception, and `privacy_map(d_in)` $\leq$ `d_out`, then `function`(x), `function`(x') are `d_out`-close under `output_measure`.

By the precondition, `meas` is a valid measurement with `RangeDivergence` privacy measure.

Definition 1.2 (Range Divergence)**.** For any two distributions Y, Y' and any non-negative d, Y, Y' are d-close under the bounded-range privacy measure whenever

$$D_{\mathrm{BR}}(Y, Y') = \sup_{y_0, y_1 \in \mathrm{Supp}(Y)} \mathcal{L}_{Y,Y'}(y_0) - \mathcal{L}_{Y,Y'}(y_1) \tag{1}$$

Definition 1.3 (Privacy Loss)**.** The *privacy loss* of an outcome y with respect to random variables Y and Y' is defined as

$$\mathcal{L}_{Y,Y'}(y) = \ln\left(\frac{\mathbb{P}[Y = y]}{\mathbb{P}[Y' = y]}\right). \tag{2}$$

If y is not in the support of Y' then we define the privacy loss as infinite. The *privacy loss random variable* Z is distributed according to $\mathcal{L}_{Y,Y'}(y)$ where y is obtained by sampling $y \sim Y$.

Lemma 1.4. Given a privacy loss random variable Z with respect to random variables Y and Y', if $\mathrm{supp}(Y) = \mathrm{supp}(Y')$, then

$$\mathop{\mathbb{E}}_{y \sim Y}[\exp(-Z)] = 1. \tag{3}$$

Proof.

$$\begin{aligned}
&\mathop{\mathbb{E}}_{y \sim Y}[\exp(-Z)] && (4)\\
&= \int_{\mathrm{supp}(Y)} \exp(-Z) dY && (5)\\
&= \int_{\mathrm{supp}(Y)} \mathbb{P}[Y = y] \cdot \exp(-\mathcal{L}_{Y,Y'}(y)) dy && \text{by Definition 1.3} \quad (6)\\
&= \int_{\mathrm{supp}(Y)} \mathbb{P}[Y = y] \cdot \frac{\mathbb{P}[Y' = y]}{\mathbb{P}[Y = y]} dy && (7)\\
&= \int_{\mathrm{supp}(Y')} \mathbb{P}[Y' = y] dy && \text{since } Y \text{ and } Y' \text{ have the same support} \quad (8)\\
&= 1 && (9)
\end{aligned}$$

□

Definition 1.5 (Hoeffding's Lemma)**.** Let X be a random variable supported on $[a, b]$. Then for any $\lambda \in \mathbb{R}$,

$$\mathbb{E}[\exp(\lambda X)] \leq \exp\left(\mathbb{E}[X] \cdot \lambda + \frac{(b-a)^2}{8} \cdot \lambda^2\right) \tag{10}$$

Lemma 1.6. Given two distributions Y and Y' that are η-close under range divergence, then the associated privacy loss random variable satisfies

$$\mathbb{E}[Z] \leq \frac{1}{8}\eta^2 \tag{11}$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[\exp(-Z)\right] &\leq \exp\left(-\mathbb{E}[Z] + \frac{\eta^2}{8}\right) && \text{by 1.5 since } Z \in [-t, \eta - t], \text{ let } \lambda = -1 && (12)\\
\implies \mathbb{E}\left[Z\right] &\leq \frac{\eta^2}{8} - \log \mathbb{E}[\exp(-Z)] && \text{rearrange terms} && (13)\\
&= \frac{\eta^2}{8} && \text{by Lemma 1.4} && (14)
\end{aligned}$$

Lemma 1.4 can only be applied when Y and Y' have the same support. This requirement is satisfied via a proof by contradiction: if Y and Y' have different supports, then the privacy loss would be infinite, meaning that Y and Y' are not η-close under range divergence. □

Definition 1.7 (zero-Concentrated Divergence Privacy Loss Random Variable)**.** For a privacy loss random variable Z with respect to two distributions Y, Y' and any non-negative d, Y, Y' are d-close under the zero-concentrated divergence measure if, for every possible choice of $\alpha \in (1, \infty)$,

$$\mathbb{E}[\exp(\alpha Z)] \leq \exp(\alpha(\alpha + 1)d) \tag{15}$$

Theorem 1.8 (Range Divergence implies zero-Concentrated Divergence)**.** If two random variables Y and Y' are η-close under range divergence, then they are also $\frac{1}{8}\eta^2$-close under zero-concentrated divergence.

Proof.

$$\begin{aligned}
&\mathbb{E}\left[\exp(\alpha Z)\right] && \text{starting from Definition 1.7} && (16)\\
&\leq \exp\left(\mathbb{E}[Z]\alpha + \frac{\eta^2}{8}\alpha^2\right) && \text{by 1.5 since } Z \in [-t, \eta - t], \text{ let } \lambda = \alpha && (17)\\
&\leq \exp\left(\frac{\eta^2}{8}\alpha + \frac{\eta^2}{8}\alpha^2\right) && \text{by Lemma 1.6} && (18)\\
&= \exp\left(\frac{\eta^2}{8}\alpha\left(\alpha + 1\right)\right) && && (19)\\
&= \exp(\alpha\left(\alpha + 1\right)\rho) && \text{where } \rho = \frac{\eta^2}{8} && (20)
\end{aligned}$$

□

Proof of Theorem 1.1. By the postcondition of `Measurement.with_map`, on line 5, `make_bounded_range_to_zCDP` returns a measurement with the same input metric, output metric `ZeroConcentratedDivergence`, and a privacy map that computes $\eta^2/8$ with conservative rounding. The privacy guarantee holds by the precondition that `meas` is valid measurement, together with Theorem 1.8. Therefore the returned measurement is a valid measurement. □

References

[CR20] Mark Cesar and Ryan Rogers. Unifying privacy loss composition for data analytics. *CoRR*, abs/2004.07223, 2020.

[DR19] David Durfee and Ryan Rogers. Practical differentially private top-k selection with pay-what-you-get composition. *CoRR*, abs/1905.04273, 2019.