

# fn get\_rounding\_distance

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This proof resides in “**contrib**” because it has not completed the vetting process.

Proves soundness of the implementation of `get_rounding_distance` in `mod.rs` at commit f5bb719 (outdated<sup>1</sup>).

## 1 Hoare Triple

### Precondition

#### Compiler-Verified

- Generic T implements trait `Float`
- Type `i32` implements the trait `ExactIntCast<T.Bits>`, where `T.Bits` is the type of the native bit representation of T.

#### User-Verified

None

### Pseudocode

```
1 def get_rounding_distance() -> RBig:
2     k_min = get_min_k(T) #
3     if k < k_min: #
4         raise f"k ({k}) must not be smaller than {k_min}"
5
6     # input has granularity 2^{k_min} (subnormal float precision)
7     input_gran = x_mul_2k(RBig.ONE, k_min) #
8
9     # discretization rounds to the nearest 2^k
10    output_gran = x_mul_2k(RBig.ONE, k) #
11
12    # the worst-case increase in sensitivity due to discretization is
13    #   the range, minus the smallest step in the range
14    distance = output_gran - input_gran #
15
16    # rounding may occur on all vector elements
17    if not distance.is_zero(): #
18        if size is None: #
19            raise "domain size must be known if discretization is not exact"
20
21    match P:
22        case 1:
23            distance *= RBig.from_(size)
```

<sup>1</sup>See new changes with git diff f5bb719..6b27cb3 rust/src/measurements/noise/nature/float/utilities/mod.rs

```

24     case 2:
25         distance *= RBig.try_from(f64.inf_cast(size).inf_sqrt())
26     case _:
27         raise f"norm ({P}) must be one or two"
28
29     return distance

```

## Postcondition

**Theorem 1.1.** Let  $D$  denote the space of `size`-dimensional vectors whose elements are in  $\mathbb{Z}2^{k_{min}}$ , where  $2^{k_{min}}$  is the smallest distance between adjacent non-equal values in  $T$ . Let  $\text{round}_k$  be a function that rounds each element to the nearest multiple of  $2^k$ , with ties rounding down. Return  $\max_{x,x' \in D} ||\text{round}_k(x) - \text{round}_k(x')||_P - ||x - x'||_P$ , the increase in the sensitivity due to rounding.

*Proof.* We first consider the increase in sensitivity due to rounding one element. The greatest increase in sensitivity occurs when one value  $x$  is rounded down, and  $x'$  rounded up. The greatest round down occurs when  $x$  is offset from the output grid by  $2^{k-1}$ , and the greatest round up occurs when  $x'$  is offset from the output grid by  $2^{k-1} + 2^{k_{min}}$ .

$$\max_{x,x'} |\text{round}_k(x) - \text{round}_k(x')| \quad (1)$$

$$\leq \max_{x,x'} |(x - 2^{k-1}) - (x' + 2^{k-1} - 2^{k_{min}})| \quad (2)$$

$$= \max_{x,x'} |(x - x') - 2 \cdot 2^{k-1} + 2^{k_{min}}| \quad (3)$$

$$= \max_{x,x'} |x - x'| + 2^k - 2^{k_{min}} \quad (4)$$

$$(5)$$

Now apply this same logic to a vector of rounded values.

$$\max_{x,x'} ||\text{round}_k(x) - \text{round}_k(x')||_P \quad (6)$$

$$\leq \max_{x,x'} ||x - x' + r||_P \quad \text{where } r \text{ is a vector of } 2^k - 2^{k_{min}} \quad (7)$$

$$\leq \max_{x,x'} ||x, x'||_P + ||r||_P \quad \text{triangle inequality} \quad (8)$$

$$= \max_{x,x'} ||x, x'||_P + n^{1/P} \cdot (2^k - 2^{k_{min}}) \quad (9)$$

$$(10)$$

Substituting into the return criteria of the postcondition, the return value is:

$$\max_{x,x'} ||\text{round}_k(x) - \text{round}_k(x')||_P - ||x - x'||_P \quad (11)$$

$$\leq \max_{x,x'} ||x, x'||_P + n^{1/P} \cdot (2^k - 2^{k_{min}}) + \max_{x,x'} ||x, x'||_P \quad (12)$$

$$= n^{1/P} \cdot (2^k - 2^{k_{min}}) \quad (13)$$

We now focus on showing correctness of the implementation. By the postcondition of `get_min_k` on line 2, `min_k` is the  $k$  such that the smallest distance between adjacent non-equal values of type  $T$  is  $2^k$  (the distance between subnormals).

Line 3 ensures that  $k$  is not too small, as any smaller  $k$  would result in unused precision in the noise sample due to the output being rounded to the nearest  $T$ . This check is not necessary for privacy; it prevents wasted performance.

Since the precondition for `x_mul_2k` that  $k \neq \text{i32.MIN}$  is satisfied on line 3, then by the postcondition of `x_mul_2k` on line 7, `input_gran` is the distance between subnormals,  $2^k$ . Similarly, `output_gran` on line 10 is the distance between adjacent values in the rounded space.

The greatest possible increase in distances between rounded values is thus  $2^k - 2^{k_{min}}$ , as defined on line 14.

When  $k = k_{min}$ , the rounding is a no-op and  $2^k - 2^{k_{min}}$  is zero, so line 17 skips the vector calculations. Otherwise line 18 ensures the vector size  $n$  is known, and the following lines increase the distance by a factor of  $n^{1/P}$ , resulting in a conservative upper estimate of the expected bound.  $\square$