

fn sample_bernoulli_exp1

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This proof resides in “**contrib**” because it has not completed the vetting process.

Proves soundness of `fn sample_bernoulli_exp1` in `mod.rs` at commit `0be3ab3e6` (outdated¹).
`fn sample_bernoulli_exp1` returns a sample from the $Bernoulli(\exp(-x))$ distribution for some rational argument in $[0, 1]$. This proof is an adaptation of subsection 5.1 of [CKS20].

Vetting History

- Pull Request #519

1 Hoare Triple

Preconditions

- `x` is of type `Rational` and $x \in [0, 1]$
- `SampleBernoulli` is implemented for `Rational` probabilities

Pseudocode

```
1 def sample_bernoulli_exp1(x) -> bool:
2     k = 1
3     while True:
4         if bool.sample_bernoulli(x / k, false):
5             k += 1
6         else:
7             return is_odd(k)
```

Postcondition

For any setting of the input parameter `x` such that the given preconditions hold, `sample_bernoulli_exp1` either returns `Err(e)` due to a lack of system entropy, or `Ok(out)`, where `out` is distributed as $Bernoulli(\exp(-x))$.

2 Proof

Assume the preconditions are met.

Lemma 2.1. `sample_bernoulli_exp1` only returns `Err(e)` when there is a lack of system entropy.

¹See new changes with `git diff 0be3ab3e6..9a7b4dd61 rust/src/traits/samplers/cks20/mod.rs`

Proof. In all usages of `SampleBernoulli`, the argument passed satisfies its definition preconditions, by the preconditions on `x` and function logic. Thus, by its definition, `sample_bernoulli` only returns an error when there is a lack of system entropy. The only source of errors in `sample_bernoulli_exp1` is from the invocation of `sample_bernoulli`. Therefore `sample_bernoulli_exp1` only returns `Err(e)` when there is a lack of system entropy. \square

Lemma 2.2. Let k^* denote the final value of `k` on line 7. Then $P[K^* > n] = \frac{x^n}{n!}$ for any integer $n > 0$ [CKS20].

Proof. For $i \geq 0$, let a_i denote the i^{th} outcome of `sample_bernoulli` on line 4. By the definition of `sample_bernoulli`, under the established conditions and preconditions, each A_i is distributed as $Bernoulli(x/i)$.

$$\begin{aligned}
P[K^* > n] &= P[A_1 = A_2 = \dots = A_n = \top] && \text{since } K^* > n, \forall i \leq n, a_i = \top \\
&= \prod_{k=1}^n P[A_k = \top] && \text{all } A_i \text{ are independent} \\
&= \prod_{k=1}^n \frac{x}{k} && \text{since } A_k \sim Bernoulli(x/k) \\
&= \frac{x^n}{n!}
\end{aligned}$$

\square

Lemma 2.3. $is_odd(K^*) \sim Bernoulli(\exp(-x))$ [CKS20].

Proof.

$$\begin{aligned}
P[K^* \text{ odd}] &= \sum_{k=0}^{\infty} P[K^* = 2k + 1] \\
&= \sum_{k=0}^{\infty} (P[K^* > 2k] - P[K^* > 2k + 1]) \\
&= \sum_{k=0}^{\infty} \left(\frac{x^{2k}}{(2k)!} - \frac{x^{2k+1}}{(2k+1)!} \right) && \text{by 2.2} \\
&= \exp(-x)
\end{aligned}$$

Since `k` is distributed according to K^* , then `out` is distributed as $Bernoulli(\exp(-x))$. \square

Proof. 1 holds by 2.1 and 2.3. \square

References

[CKS20] Clément L. Canonne, Gautam Kamath, and Thomas Steinke. The discrete gaussian for differential privacy. *CoRR*, abs/2004.00010, 2020.