fn conservative_discrete_laplacian_tail_to_alpha

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Proof for `conservative_discrete_laplacian_tail_to_alpha`.

1 Hoare Triple

Precondition

Compiler-verified

- Argument `scale` is of type `f64`.
- Argument `tail` is of type `u32`.

User-verified

- Argument `scale` > 0.
- Argument `tail` > 0.

Pseudocode

```python
def conservative_discrete_laplacian_tail_to_alpha(scale: f64, tail: u32) -> f64:
    t = f64.neg_inf_cast(tail)
    numer = t.neg_inf_div(-scale).inf_exp()
    denom = scale.recip().neg_inf_exp().neg_inf_add(1.0)
    return numer.inf_div(denom)
```

Postcondition

Definition 1.1. Define $Y \sim \mathcal{L}_{\mathbb{Z}}(0, s)$, a random variable following the discrete laplace distribution:

$$\forall x \in \mathbb{Z} \qquad \Pr[X = x] = \frac{e^{1/s} - 1}{e^{1/s} + 1} e^{-|x|/s} \tag{1}$$

Theorem 1.2. Assume $X \sim \mathcal{L}_{\mathbb{Z}}(0, s)$, and $t > 0$.

$$\alpha = P[X > t] = \frac{e^{-t/s}}{e^{-1/s} + 1} \tag{2}$$

2 Proof

Proof.

$$\begin{aligned}
\alpha &= P[X > t] \\
&= \sum_{x=t+1}^{\infty} \frac{e^{1/s}-1}{e^{1/s}+1} e^{-|x|/s} \\
&= \frac{e^{1/s}-1}{e^{1/s}+1} \sum_{x=t+1}^{\infty} e^{-x/s} && \text{since } t > 0 \\
&= \frac{e^{1/s}-1}{e^{1/s}+1} \frac{e^{(1-(t+1))/s}}{e^{1/s}-1} && \text{since } \sum_{x=t}^{\infty} p^x = \frac{p^t}{1-p} \text{ if } |p| < 1, \text{ let } p = e^{1/s} \\
&= \frac{e^{-t/s}}{e^{1/s}+1}
\end{aligned}$$

□