

fn cdp_delta

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March 24, 2025

Proves soundness of `fn cdp_delta` in `cdp_delta.rs` at commit `0b8f4222` (outdated¹). This proof is an adaptation of [subsection 2.3](#) of [\[CKS20\]](#).

1 Bound Derivation

Definition 1.1. (Privacy Loss Random Variable). Let $M : \mathcal{X}^n \rightarrow \mathbb{Y}$ be a randomized algorithm. Let $x, x' \in \mathcal{X}^n$ be neighboring inputs. Define $f : \mathcal{Y} \rightarrow \mathbb{R}$ by $f(y) = \log \left(\frac{\mathbb{P}[M(x)=y]}{\mathbb{P}[M(x')=y]} \right)$. Let $Z = f(M(x))$, the privacy loss random variable, denoted $Z \leftarrow \text{PrivLoss}(M(x)||M(x'))$.

Lemma 1.2. [\[CKS20\]](#) Let $\epsilon, \delta \geq 0$. Let $M : \mathcal{X}^n \rightarrow \mathcal{Y}$ be a randomized algorithm. Then M satisfies (ϵ, δ) -differential privacy if and only if

$$\delta \geq \mathbb{E}_{Z \leftarrow \text{PrivLoss}(M(x)||M(x'))} [\max(0, 1 - e^{\epsilon - Z})] \quad (1)$$

(2)

for all $x, x' \in \mathcal{X}^n$ differing on a single element.

Proof. Fix neighboring inputs $x, x' \in \mathcal{X}^n$. Let $f : \mathcal{Y} \rightarrow \mathbb{R}$ be as in [1.1](#). For notational simplicity, let $Y = M(x)$, $Y' = M(x')$, $Z = f(Y)$ and $Z' = -f(Y')$. This is equivalent to $Z \leftarrow \text{PrivLoss}(M(x)||M(x'))$. Our first goal is to prove that

$$\sup_{E \subset \mathcal{Y}} \mathbb{P}[Y \in E] - e^\epsilon \mathbb{P}[Y' \in E] = \mathbb{E}[\max\{0, 1 - e^{\epsilon - Z}\}]. \quad (3)$$

For any $E \subset \mathcal{Y}$, we have

$$\mathbb{P}[Y' \in E] = \mathbb{E}[\mathbb{I}[Y' \in E]] = \mathbb{E}[\mathbb{I}[Y \in E] e^{-f(Y)}]. \quad (4)$$

This is because $e^{-f(y)} = \frac{\mathbb{P}[Y'=y]}{\mathbb{P}[Y=y]}$.

Thus, for all $E \subset \mathcal{Y}$, we have

$$\mathbb{P}[Y \in E] - e^\epsilon \mathbb{P}[Y' \in E] = \mathbb{E} \left[\mathbb{I}[Y \in E] (1 - e^{\epsilon - f(Y)}) \right] \quad (5)$$

Now it is easy to identify the worst event as $E = \{y \in \mathcal{Y} : 1 - e^{\epsilon - f(y)} > 0\}$. Thus

$$\sup_{E \subset \mathcal{Y}} \mathbb{P}[Y \in E] - e^\epsilon \mathbb{P}[Y' \in E] = \mathbb{E} \left[\mathbb{I}[1 - e^{\epsilon - f(Y)} > 0] (1 - e^{\epsilon - f(Y)}) \right] = \mathbb{E}[\max\{0, 1 - e^{\epsilon - Z}\}] \quad (6)$$

□

¹See new changes with `git diff 0b8f4222..rust/src/combinators/measure_cast/zCDP_to_approxDP/cdp_delta/cdp_delta.rs`

Theorem 1.3. [CKS20] Let $M : \mathcal{X}^n \rightarrow \mathcal{Y}$ be a randomized algorithm. Let $\alpha \in (1, \infty)$ and $\epsilon \geq 0$. Suppose $D_\alpha(M(x)||M(x')) \leq \tau$ for all $x, x' \in \mathcal{X}^n$ differing in a single entry.² Then M is (ϵ, δ) -differentially private for

$$\delta = \frac{e^{(\alpha-1)(\tau-\epsilon)}}{\alpha-1} \left(1 - \frac{1}{\alpha}\right)^\alpha \quad (7)$$

Proof. Fix neighboring $x, x' \in \mathcal{X}^n$ and let $Z \leftarrow \text{PrivLoss}(M(x)||M(x'))$. We have

$$\mathbb{E}[e^{(\alpha-1)Z}] = e^{(\alpha-1)D_\alpha(M(x)||M(x'))} \leq e^{(\alpha-1)\tau} \quad (8)$$

By 1.2, our goal is to prove that $\delta \geq \mathbb{E}[\max\{0, 1 - e^{\epsilon-Z}\}]$. Our approach is to pick $c > 0$ such that $\max\{0, 1 - e^{\epsilon-Z}\} \leq ce^{(\alpha-1)z}$ for all $z \in \mathbb{R}$. Then

$$\mathbb{E}[\max\{0, 1 - e^{\epsilon-Z}\}] \leq \mathbb{E}[ce^{(\alpha-1)z}] \leq ce^{(\alpha-1)\tau}. \quad (9)$$

We identify the smallest possible value of c :

$$c = \sup_{z \in \mathbb{R}} \frac{\max\{0, 1 - e^{\epsilon-z}\}}{e^{(\alpha-1)z}} = \sup_{z \in \mathbb{R}} e^{z-\alpha z} - e^{\epsilon-\alpha z} = \sup_{z \in \mathbb{R}} f(z) \quad (10)$$

where $f(z) = e^{z-\alpha z} - e^{\epsilon-\alpha z}$. We have

$$f'(z) = e^{z-\alpha z}(1-\alpha) - e^{\epsilon-\alpha z}(-\alpha) = e^{-\alpha z}(\alpha e^z - (\alpha-1)e^\epsilon) \quad (11)$$

Clearly $f'(z) = 0 \iff e^z = \frac{\alpha-1}{\alpha}e^\epsilon \iff z = \epsilon - \log(1 - 1/\alpha)$. Thus

$$c = f(\epsilon - \log(1 - 1/\alpha)) \quad (12)$$

$$= \left(\frac{\alpha}{\alpha-1}e^\epsilon\right)^{1-\alpha} - e^\epsilon \left(\frac{\alpha}{\alpha-1}e^\epsilon\right)^{-\alpha} \quad (13)$$

$$= \left(\frac{\alpha}{\alpha-1}e^\epsilon - e^\epsilon\right) \left(\frac{\alpha}{\alpha-1}e^{-\epsilon}\right)^\alpha \quad (14)$$

$$= \frac{e^\epsilon}{\alpha-1} \left(1 - \frac{1}{\alpha}\right)^\alpha e^{-\alpha\epsilon}. \quad (15)$$

Thus

$$\mathbb{E}[\max\{0, 1 - e^{\epsilon-Z}\}] \leq \frac{e^\epsilon}{\alpha-1} \left(1 - \frac{1}{\alpha}\right)^\alpha e^{-\alpha\epsilon} e^{(\alpha-1)\tau} = \frac{e^{(\alpha-1)(\tau-\epsilon)}}{\alpha-1} \left(1 - \frac{1}{\alpha}\right)^\alpha = \delta \quad (16)$$

□

Corollary 1. [CKS20] Let $M : \mathcal{X}^n \rightarrow \mathcal{Y}$ be a randomized algorithm. Let $\alpha \in (1, \infty)$ and $\epsilon \geq 0$. Suppose $D_\alpha(M(x)||M(x')) \leq \tau$ for all $x, x' \in \mathcal{X}^n$ differing in a single entry. Then M is (ϵ, δ) -differentially private for

$$\epsilon = \tau + \frac{\ln(1/\delta) + (\alpha-1)\ln(1-1/\alpha) - \ln(\alpha)}{\alpha-1} \quad (17)$$

Proof. This follows by rearranging 1.3. □

Corollary 2. Let $M : \mathcal{X}^n \rightarrow \mathcal{Y}$ be a randomized algorithm satisfying ρ -concentrated differential privacy. Then M is (ϵ, δ) -differentially private for any $0 < \delta \leq 1$ and

$$\epsilon = \inf_{\alpha \in (1, \infty)} \alpha\rho + \frac{\ln(1/\delta) + (\alpha-1)\ln(1-1/\alpha) - \ln(\alpha)}{\alpha-1} \quad (18)$$

Proof. This follows from 1 by taking the infimum over all divergence parameters α . □

²This is the definition of (α, τ) -Rényi differential privacy.

2 Pseudocode

Precondition

None.

Implementation

```
1 def cdp_delta(rho: float, eps: float) -> float:
2     """The Rust code may be easier to follow due to more commenting."""
3     if rho.is_sign_negative():
4         raise ValueError(f"rho ({rho}) must be non-negative")
5
6     if eps.is_sign_negative():
7         raise ValueError(f"epsilon ({eps}) must be non-negative")
8
9     if rho.is_zero() or eps.is_infinite():
10        return 0.0
11
12    if rho.is_infinite():
13        return 1.0
14
15    a_max = eps.inf_add(1.0).inf_div((2.0).neg_inf_mul(rho)).inf_add(2.0)
16
17    a_min = 1.01
18
19    while True:
20        diff = a_max - a_min
21
22        a_mid = a_min + diff / 2.0
23
24        if a_mid == a_max or a_mid == a_min:
25            break
26
27        # calculate derivative
28        deriv = (2.0 * a_mid - 1.0) * rho - eps + a_mid.recip().neg().ln_1p()
29
30        if deriv.is_sign_negative():
31            a_min = a_mid
32        else:
33            a_max = a_mid
34
35    # calculate delta
36    a_1 = a_max.inf_sub(1.0)
37    ar_e = a_max.inf_mul(rho).inf_sub(eps)
38
39    try:
40        t1 = a_1.inf_mul(ar_e)
41
42    except OpenDPException:
43
44        # if t1 is negative, then handle negative overflow by making t1 larger: the most
45        # negative finite float
46        # making t1 larger makes delta larger, so it's still a valid upper bound
47        if a_1.is_sign_negative() != ar_e.is_sign_negative():
48            t1 = 1.7976931348623157e308 # f64::MIN
49        else:
50            raise
51
52    t2 = a_max.inf_mul(a_max.recip().neg().inf_ln_1p())
53
54    delta = t1.inf_add(t2).inf_exp().inf_div((a_max.inf_sub(1.0)))
```

```

55 # delta is always <= 1
56 delta.min(1.0)

```

Postcondition

Theorem 2.1. For any possible setting of ρ and ϵ , `cdp_delta` either returns an error, or a δ such that any ρ -differentially private measurement is also (ϵ, δ) -differentially private.

3 Proof

Proof. The code always finds an $\alpha_* \approx \mathbf{a_max} \geq 1.01$. Since $\mathbf{a_max} \in (1, \infty)$, then by 1, any ρ -differentially private measurement is also $(\epsilon(\mathbf{a_max}), \delta)$ -differentially private. Define $\delta_{cons}(\alpha)$ as a “conservative” function for computing $\delta(\epsilon)$, where floating-point arithmetic is computed with conservative rounding such that $\delta_{cons}(\alpha) \geq \delta(\alpha)$ for $\forall \alpha \in (1, \infty)$. Since $\mathbf{delta} = \delta_{cons}(\mathbf{a_max}) \geq \delta(\mathbf{a_max})$, then any $(\epsilon, \delta(\mathbf{a_max}))$ -differentially private measurement is also $(\epsilon, \mathbf{delta})$ -differentially private. \square

References

[CKS20] Clément L. Canonne, Gautam Kamath, and Thomas Steinke. The discrete gaussian for differential privacy. *CoRR*, abs/2004.00010, 2020.