# fn make_select_private_candidate

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> This proof resides in **"contrib"** because it has not completed the vetting process.

Proves soundness of make_select_private_candidate in mod.rs at commit f5bb719 (outdated[1]).

make_select_private_candidate returns a Measurement that returns a release from measurement whose score is above threshold, or may fail and return nothing.

# 1 Hoare Triple

## Precondition

### Compiler-verified

- Generic DI (input domain) is a type with trait Domain.

- Generic MI (input metric) is a type with trait Metric.

- MetricSpace is implemented for (DI, MI). Therefore MI is a valid metric on DI.

- Argument measurement is a measurement whose output metric is MaxDivergence, and releases a tuple (f64, TO), where TO is some arbitrary type.

- Argument stop_probability is of type f64.

- Argument threshold is of type f64.

### User-verified

None

## Pseudocode

```
1  def make_select_private_candidate (
2      measurement : Measurement ,
3      stop_probability : float ,
4      threshold : float ,
5  ) -> Measurement :
6      if not 0 <= stop_probability < 1: #
7          raise "stop_probability must be in [0, 1)"
8
9      if not threshold.is_finite ():
10         raise "threshold must be finite"
11
12     scale = None
13     if stop_probability > 0.0:
```

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[1]See new changes with `git diff f5bb719..76ab1a9 rust/src/combinators/select_private_candidate/mod.rs`

```
14        ln_cp = (1.0).neg_inf_sub(stop_probability).inf_ln()
15        scale = ln_cp.recip().neg().into_rational() #
16
17    def function(arg):
18        remaining_iterations = None
19        if scale is not None:
20            remaining_iterations = UBig.ONE + sample_geometric_exp_fast(scale) #
21
22        while True:
23            score, output = measurement(arg)
24
25            if score >= threshold:
26                return score, output
27
28            if remaining_iterations is not None:
29                remaining_iterations -= UBig.ONE
30                if remaining_iterations == UBig.ZERO:
31                    return None
32
33    return Measurement(
34        input_domain=measurement.input_domain,
35        input_metric=measurement.input_metric,
36        output_measure=measurement.output_measure,
37        function=function,
38        privacy_map=lambda d_in: measurement.map(d_in).inf_mul(2),
39    )
```

## Postcondition

For every setting of the input parameters (`measurement, stop_probability, threshold, DI, MI, TO`)
to `make_select_private_candidate` such that the given preconditions hold, `make_select_private_candidate`
raises an error (at compile time or run time) or returns a valid measurement. A valid measurement has the
following properties:

1. (Data-independent runtime errors). For every pair of members $x$ and $x'$ in `input_domain`, `invoke`($x$)
   and `invoke`($x'$) either both return the same error or neither return an error.

2. (Privacy guarantee). For every pair of members $x$ and $x'$ in `input_domain` and for every pair
   (`d_in, d_out`), where `d_in` has the associated type for `input_metric` and `d_out` has the associated
   type for
   `output_measure`, if $x, x'$ are `d_in`-close under `input_metric`, `privacy_map(d_in)` does not raise
   an error, and `privacy_map(d_in)` = `d_out`, then `function`($x$), `function`($x'$) are `d_out`-close under
   `output_measure`.

# 2   Proofs

This section shows that the pseudocode implements Theorem 3.1 in [1], where `stop_probability` is $\gamma$ and
`threshold` is $\tau$.

**Theorem 2.1.** The function pseudocode is equivalent to Algorithm 1 in [1].

*Proof.* `scale` is $-1/\ln(1-\gamma)$, with arithmetic conservatively rounded down. The rounding down corresponds to higher effective stop probability. A higher effective stop probability results in a lower effective
privacy usage $\epsilon_0$. This means the advertised privacy loss is a conservative overestimate.

Since $\gamma$ is restricted to $[0, 1)$ by 6, scale is non-negative. Therefore in 20, the precondition of `sample_geometric_exp_fast`
is satisfied. In Algorithm 1, the mechanism is invoked once before potentially terminating, so one is added
to the sample. `remaining_iterations`, a sample from the shifted geometric distribution, is then equivalent
to a count of the number of coin flips made until sampling one heads, as is used in Algorithm 1.

We then only run as many iterations as has been sampled, as in Algorithm 1. In the case where $T$ is infinity and $\gamma$ is zero, then the algorithm only terminates when score exceeds threshold.

If the measurement releases a score of NaN, then this is effectively treated as a score below negative infinity, as NaN will never compare greater than the threshold.

Otherwise, the rest of the algorithm is evidently equivalent. $\qquad\square$

**Theorem 2.2.** `make_select_private_candidate` is consistent with Theorem 3.1 of [1].

*Proof.* Since there is no limit on iterations, $\epsilon_0$ is zero. By 2.1, the function is equivalent to Algorithm 1, so we can claim parts a-e of Theorem 3.1. Since `measurement` is a valid measurement, then we know that it satisfies $\epsilon_1$-DP. The privacy map returns $2\epsilon_1 + \epsilon_0$ with arithmetic rounded conservatively up, which is consistent with part b in Theorem 3.1. $\qquad\square$

**(Privacy guarantee.)** By 2.2, assuming correctness of Theorem 3.1 part b in [1], then for every pair of elements $x, x' \in$ `input_domain` and every $d_{MI}(x, x') \leq$ `d_in` with `d_in` $\geq 0$, if $x, x'$ are `d_in`-close then `function`$(x)$, `function`$(x')$ are `privacy_map(d_in)`-close under `output_measure` (the Max-Divergence).

# References

[1] Jingcheng Liu and Kunal Talwar. Private selection from private candidates, 2018.