

fn make_bounded_range_to_pureDP

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This proof resides in “**contrib**” because it has not completed the vetting process.

Proves soundness of `fn make_bounded_range_to_pureDP`. This proof is an adaptation of Lemma 2 [here](#), which proves the conversion between bounded range [DR19] and pure DP.

1 Hoare Triple

Preconditions

Compiler-verified

- Variable `meas` is a valid measurement of type `Measurement<DI, TO, MI, RangeDivergence>`
- Generic DI (input domain) is a type with trait `Domain`.
- Generic MI (input metric) is a type with trait `Metric`.
- `MetricSpace` is implemented for (DI, MI). Therefore MI is a valid metric on DI.

User-verified

None

Pseudocode

```
1 def make_bounded_range_to_pureDP(meas: Measurement) -> Measurement:
2     return meas.with_map( #
3         meas.input_metric,
4         MaxDivergence,
5         PrivacyMap.new_fallible(lambda d_in: meas.privacy_map(d_in)),
6     )
```

Postcondition

Theorem 1.1 (Postcondition). For every setting of the input parameters (`meas`, DI, TO, MI) to `make_bounded_range_to_pureDP` such that the given preconditions hold, `make_bounded_range_to_pureDP` raises an exception (at compile time or run time) or returns a valid measurement. A valid measurement has the following property:

1. (Privacy guarantee). For every pair of elements x, x' in `input_domain` and for every pair (`d_in`, `d_out`), where `d_in` has the associated type for `input_metric` and `d_out` has the associated type for `output_measure`, if x, x' are `d_in`-close under `input_metric`, `privacy_map(d_in)` does not raise an exception, and `privacy_map(d_in) ≤ d_out`, then `function(x), function(x')` are `d_out`-close under `output_measure`.

To prove the postcondition, we will need to establish the following definitions and theorem.

Definition 1.2 (Range Divergence). For any two distributions Y, Y' and any non-negative d , Y, Y' are d -close under the bounded-range privacy measure whenever

$$D_{\text{BR}}(Y, Y') = \sup_{y_0, y_1 \in \text{Supp}(Y)} \mathcal{L}_{Y, Y'}(y_0) - \mathcal{L}_{Y, Y'}(y_1) \quad (1)$$

Definition 1.3 (Privacy Loss). The *privacy loss* of an outcome y with respect to random variables Y and Y' is defined as

$$\mathcal{L}_{Y, Y'}(y) = \ln \left(\frac{\mathbb{P}[Y = y]}{\mathbb{P}[Y' = y]} \right). \quad (2)$$

If y is not in the support of Y' then we define the privacy loss as infinite. The *privacy loss random variable* Z is distributed according to $\mathcal{L}_{Y, Y'}(y)$ where y is obtained by sampling $y \sim Y$.

Definition 1.4 (Max Divergence Privacy Loss Random Variable). For a privacy loss random variable Z with respect to two distributions Y, Y' and any non-negative d , Y, Y' are d -close under the max divergence measure if $|Z| \leq d$.

Theorem 1.5 (Range Divergence implies Max Divergence). If two random variables Y and Y' are η -close under range divergence, then they are also η -close under max divergence.

Proof. Claimed in Corollary 4.2 of [DR19]. A proof is provided here for completeness. Since the support must include zero, and the range divergence measures the width of the support, then $|Z| \leq \eta$. By Definition 1.4 this implies that Y, Y' are η -close under max divergence. \square

Proof of Theorem 1.1. By the postcondition of `Measurement.with_map`, on line 2, `make_bounded_range_to_pureDP` returns a measurement with the same input metric, output metric `MaxDivergence`, and logically equivalent privacy map. The privacy guarantee holds by the precondition that `meas` is valid measurement, together with Theorem 1.5. Therefore the returned measurement is a valid measurement. \square

References

[DR19] David Durfee and Ryan Rogers. Practical differentially private top-k selection with pay-what-you-get composition. *CoRR*, abs/1905.04273, 2019.