fn permute_and_flip

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> This proof resides in **"contrib"** because it has not completed the vetting process.

This document proves soundness of permute_and_flip [1] in mod.rs at commit e62b0aa2 (outdated[1]). permute_and_flip noisily selects the index of the greatest score from a vector of input scores.

1 Hoare Triple

Preconditions

Types consistent with pseudocode.

Pseudocode

```
1  def permute_and_flip(x: list[RBig], scale: RBig, replacement: bool):
2      if scale.is_zero():   #
3          return max(range(x.len()), key=lambda i: x[i])
4
5      # begin nonzero scale
6      x_max = max(x)
7      permutation = list(range(x.len()))
8
9      sequence = range(0, len(x)) if replacement else repeat(0)
10
11     for left in sequence:
12         right = left + sample_uniform_uint_below(x.len() - left)
13         # fisher-yates shuffle up to left
14         permutation.swap(left, right)
15
16         candidate = permutation[left]
17         if sample_bernoulli_exp((x_max - x[candidate]) / scale):
18             return candidate
19
20     raise "at least one x[candidate] is equal to x_max"
```

Postcondition

Theorem 1.1.   • If replacement is set, returns a sample from $\mathcal{M}_{EM}$ (as defined in MS2023 Definition 4), otherwise returns a sample from $\mathcal{M}_{PF}$ (as defined in MS2023 Lemma 1), where $\texttt{scale} = \frac{2 \cdot \Delta}{\epsilon}$.

• Errors are data-independent, except for exhaustion of entropy.

Proof. By swapping elements on line 13, an online Fisher-Yates shuffle is applied up to and including index left.

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[1] See new changes with `git diff e62b0aa2..9b02cf48 rust/src/measurements/noisy_top_k/mod.rs`

Substituting $\texttt{scale} = \frac{2\Delta}{\epsilon}$, the argument to $\texttt{sample\_bernoulli\_exp}$ is then $\frac{\epsilon}{2\Delta}(q_* - q_r)$, which is non-negative, satisfying the precondition of $\texttt{sample\_bernoulli\_exp}$. Therefore by the postcondition of $\texttt{sample\_bernoulli\_exp}$, the response is a sample from $\mathrm{Bern}(\exp(-x))$, where $x = \frac{\epsilon}{2\Delta}(q_* - q_r)$. Therefore the response is a sample from $\mathrm{Bern}(\exp\left(\frac{\epsilon}{2\Delta}(q_r - q_*)\right))$, which is equivalent to Algorithm 1 in [1].

The only source of error is due to entropy exhaustion. $\qquad\square$

References

[1] Ryan McKenna and Daniel Sheldon. Permute-and-flip: A new mechanism for differentially private selection, 2020.