fn get_rounding_distance

Michael Shoemate

> This proof resides in **"contrib"** because it has not completed the vetting process.

Proves soundness of the implementation of `get_rounding_distance` in `mod.rs` at commit f5bb719 (outdated[1]).

1 Hoare Triple

Precondition

Compiler-Verified

- Generic `T` implements trait `Float`

- Type `i32` implements the trait `ExactIntCast`<T.Bits>, where `T.Bits` is the type of the native bit representation of `T`.

User-Verified

None

Pseudocode

```
1  def get_rounding_distance() -> RBig:
2      k_min = get_min_k(T) #
3      if k < k_min: #
4          raise f"k ({k}) must not be smaller than {k_min}"
5
6      # input has granularity 2^{k_min} (subnormal float precision)
7      input_gran = x_mul_2k(RBig.ONE, k_min) #
8
9      # discretization rounds to the nearest 2^k
10     output_gran = x_mul_2k(RBig.ONE, k) #
11
12     # the worst-case increase in sensitivity due to discretization is
13     #     the range, minus the smallest step in the range
14     distance = output_gran - input_gran #
15
16     # rounding may occur on all vector elements
17     if not distance.is_zero(): #
18         if size is None: #
19             raise "domain size must be known if discretization is not exact"
20
21         match P:
22             case 1:
23                 distance *= RBig.from_(size)
```

---

[1] See new changes with `git diff f5bb719..7b685d3 rust/src/measurements/noise/nature/float/utilities/mod.rs`

```
24              case 2:
25                  distance *= RBig.try_from(f64.inf_cast(size).inf_sqrt())
26              case _:
27                  raise f"norm ({P}) must be one or two"
28
29      return distance
```

Postcondition

Theorem 1.1. Let D denote the space of `size`-dimensional vectors whose elements are in $\mathbb{Z}2^{k_{min}}$, where $2^{k_{min}}$ is the smallest distance between adjacent non-equal values in `T`. Let round_k be a function that rounds each element to the nearest multiple of 2^k, with ties rounding down. Return $\max_{x,x' \in D} ||\text{round}_k(x) - \text{round}_k(x')||_P - ||x - x'||_P$, the increase in the sensitivity due to rounding.

Proof. We first consider the increase in sensitivity due to rounding one element. The greatest increase in sensitivity occurs when one value x is rounded down, and x' rounded up. The greatest round down occurs when x is offset from the output grid by 2^{k-1}, and the greatest round up occurs when x' is offset from the output grid by $2^{k-1} + 2^{k_{min}}$.

$$\max_{x,x'} |\text{round}_k(x) - \text{round}_k(x')| \tag{1}$$

$$\leq \max_{x,x'} |(x - 2^{k-1}) - (x' + 2^{k-1} - 2^{k_{min}})| \tag{2}$$

$$= \max_{x,x'} |(x - x') - 2 \cdot 2^{k-1} + 2^{k_{min}}| \tag{3}$$

$$= \max_{x,x'} |x - x'| + 2^k - 2^{k_{min}} \tag{4}$$

$$\tag{5}$$

Now apply this same logic to a vector of rounded values.

$$\max_{x,x'} ||\text{round}_k(x) - \text{round}_k(x')||_P \tag{6}$$

$$\leq \max_{x,x'} ||x - x' + r||_P \qquad \text{where } r \text{ is a vector of } 2^k - 2^{k_{min}} \tag{7}$$

$$\leq \max_{x,x'} ||x, x'||_P + ||r||_P \qquad \text{triangle inequality} \tag{8}$$

$$= \max_{x,x'} ||x, x'||_P + n^{1/P} \cdot (2^k - 2^{k_{min}}) \tag{9}$$

$$\tag{10}$$

Substituting into the return criteria of the postcondition, the return value is:

$$\max_{x,x'} ||\text{round}_k(x) - \text{round}_k(x')||_P - ||x - x'||_p \tag{11}$$

$$\leq \max_{x,x'} ||x, x'||_P + n^{1/P} \cdot (2^k - 2^{k_{min}}) + \max_{x,x'} ||x, x'||_P \tag{12}$$

$$= n^{1/P} \cdot (2^k - 2^{k_{min}}) \tag{13}$$

We now focus on showing correctness of the implementation. By the postcondition of `get_min_k` on line 2, `min_k` is the k such that the smallest distance between adjacent non-equal values of type `T` is 2^k (the distance between subnormals).

Line 3 ensures that k is not too small, as any smaller k would result in unused precision in the noise sample due to the output being rounded to the nearest `T`. This check is not necessary for privacy; it prevents wasted performance.

Since the precondition for `x_mul_2k` that $k \neq$ `i32.MIN` is satisfied on line 3, then by the postcondition of `x_mul_2k` on line 7, `input_gran` is the distance between subnormals, 2^k. Similarly, `output_gran` on line 10 is the distance between adjacent values in the rounded space.

The greatest possible increase in distances between rounded values is thus $2^k - 2^{k_{min}}$, as defined on line 14.

When $k = k_{min}$, the rounding is a no-op and $2^k - 2^{k_{min}}$ is zero, so line 17 skips the vector calculations. Otherwise line 18 ensures the vector size n is known, and the following lines increase the distance by a factor of $n^{1/P}$, resulting in a conservative upper estimate of the expected bound. $\qquad\square$