

fn make_randomized_response_bool

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This proof resides in “**contrib**” because it has not completed the vetting process.

Proves soundness of `make_randomized_response_bool` in `mod.rs` at commit `f5bb719` (outdated¹).

`make_randomized_response_bool` accepts a parameter `prob` of type `f64` and a parameter `constant_time` of type `bool`. The function on the resulting measurement takes in a boolean data point `arg` and returns the truthful value `arg` with probability `prob`, or the complement `!arg` with probability `1 - prob`. The measurement function makes mitigations against timing channels if `constant_time` is set.

Warning 1 (Code is not constant-time). `make_randomized_response_bool` takes in a boolean `constant_time` parameter that protects against timing attacks on the Bernoulli sampling procedure. However, the current implementation does not guard against other types of timing side-channels that can break differential privacy, e.g., non-constant time code execution due to branching.

1 Hoare Triple

Preconditions

- Variable `prob` must be of type `f64`
- Variable `constant_time` must be of type `bool`

Pseudocode

```
1 def make_randomized_response_bool(prob: f64, constant_time: bool):
2     input_domain = AtomDomain(bool)
3     input_metric = DiscreteMetric()
4     output_measure = MaxDivergence()
5
6     if (prob < 0.5 or prob > 1): #
7         raise Exception("probability must be in [0.5, 1]")
8
9     if prob == 1.0:
10         c = float("inf")
11     else:
12         c = prob.inf_div((1).neg_inf_sub(prob)).inf_ln()
13
14     def privacy_map(d_in: u32) -> f64: #
15         if d_in == 0:
16             return 0
17         else:
```

¹See new changes with `git diff f5bb719..11d57a2 rust/src/measurements/randomized_response/mod.rs`

```

18         return c
19
20     def function(arg: bool) -> bool: #
21         sample = not sample_bernoulli_float(prob, constant_time)
22
23         return arg ^ sample
24
25     return Measurement(input_domain, function, input_metric, output_measure, privacy_map)

```

Postcondition

For every setting of the input parameters (`prob`, `constant_time`) to `make_randomized_response_bool` such that the given preconditions hold, `make_randomized_response_bool` raises an exception (at compile time or run time) or returns a valid measurement. A valid measurement has the following properties:

1. (Data-independent runtime errors). For every pair of elements x, x' in `input_domain`, `function(x)` returns an error if and only if `function(x')` returns an error.
2. (Privacy guarantee). For every pair of elements x, x' in `input_domain` and for every pair (d_{in}, d_{out}) , where d_{in} has the associated type for `input_metric` and d_{out} has the associated type for `output_measure`, if x, x' are d_{in} -close under `input_metric`, `privacy_map(d_in)` does not raise an exception, and `privacy_map(d_in) ≤ d_out`, then `function(x), function(x')` are d_{out} -close under `output_measure`.

2 Proof

Proof. (Privacy guarantee.)

Note 1. The following proof makes use of the following lemma that asserts the correctness of a Bernoulli sampler function.

Lemma 2.1. `sample_bernoulli_float` satisfies its postcondition.

`sample_bernoulli` can only fail due to lack of system entropy. This is usually related to the computer's physical environment and not the dataset. The rest of this proof is conditioned on the assumption that `sample_bernoulli` does not raise an exception.

Let x and x' be datasets in the input domain (either \top or \perp) that are d_{in} -close with respect to `input_metric`. Here, the metric is `DiscreteMetric` which enforces that $d_{in} \geq 1$ if $x \neq x'$ and $d_{in} = 0$ if $x = x'$. If $x = x'$, then the output distributions on x and x' are identical, and therefore the max-divergence is 0. Consider $x \neq x'$ and assume without loss of generality that $x = \top$ and $x' = \perp$. For shorthand, we let p represent `prob`, the probability that `sample_bernoulli_float` returns \top . Observe that $p = [0.5, 1.0]$ otherwise `make_randomized_response_bool` raises an error.

We now consider the max-divergence $D_{\infty}(Y||Y')$ over the random variables $Y = \text{function}(x)$ and $Y' = \text{function}(x')$.

$$\begin{aligned}
\max_{x \sim x'} D_\infty(Y || Y') &= \max_{x \sim x'} \max_{S \subseteq \text{Supp}(Y)} \ln \left(\frac{\Pr[Y \in S]}{\Pr[Y' \in S]} \right) \\
&\leq \max_{x \sim x'} \max_{y \in \text{Supp}(Y)} \ln \left(\frac{\Pr[Y = y]}{\Pr[Y' = y]} \right) && \text{Lemma 3.3 [1]} \\
&= \max_{x \sim x'} \max \left(\ln \left(\frac{\Pr[Y = \top]}{\Pr[Y' = \top]} \right), \ln \left(\frac{\Pr[Y = \perp]}{\Pr[Y' = \perp]} \right) \right) \\
&= \max \left(\ln \left(\frac{p}{1-p} \right), \ln \left(\frac{1-p}{p} \right) \right) \\
&= \ln \left(\frac{p}{1-p} \right)
\end{aligned}$$

We let $c = \text{privacy_map}(\text{d_in}) = \text{f64.inf_ln}(\text{prob.inf_div}(1.\text{neg_inf_sub}(\text{prob})))$. The computation of c rounds upward in the presence of floating point rounding errors. This is because $1.\text{neg_inf_sub}(\text{prob})$ appears in the denominator, and to ensure that the bound holds even in the presence of rounding errors, the conservative choice is to round down (so the quantity as a whole is bounded above). Similarly, inf_div and inf_ln round up.

When $\text{d_in} > 0$ and no exception is raised in computing $c = \text{privacy_map}(\text{d_in})$, then $\ln \left(\frac{p}{1-p} \right) \leq c$.

Therefore we've shown that for every pair of elements $x, x' \in \{\perp, \top\}$ and every $d_{DM}(x, x') \leq \text{d_in}$ with $\text{d_in} \geq 0$, if x, x' are d_in -close then $\text{function}(x), \text{function}(x') \in \{\perp, \top\}$ are $\text{privacy_map}(\text{d_in})$ -close under output_measure (the Max-Divergence). \square

References

- [1] Shiva P. Kasiviswanathan and Adam Smith. On the “semantics” of differential privacy: A bayesian formulation. *Journal of Privacy and Confidentiality*, 6(1), June 2014.