

# impl TopKMeasure for MaxDivergence

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This proof resides in “**contrib**” because it has not completed the vetting process.

This document proves soundness of `permute_and_flip` [2] in `mod.rs` at commit `e62b0aa2` (outdated<sup>1</sup>). `permute_and_flip` noisily selects the index of the greatest score from a vector of input scores.

Permute and flip is equivalent to report noisy max with exponential noise [1]. Report noisy max exponential is implemented via permute and flip because of its discrete nature. Implementation-wise, we will follow permute-and-flip, yet prove the correctness of the algorithm via this equivalence.

## 1 Hoare Triple

### Precondition

#### Compiler-verified

- Method `noisy_top_k` *Types consistent with pseudocode.*
- Method `privacy_map` *Types consistent with pseudocode.*

#### Caller-verified

- Method `noisy_top_k`
  - `x` elements are non-null.
  - `scale` is finite and non-negative.
- Method `privacy_map`
  - `d_in` is non-null and positive.
  - `scale` is non-null and positive.

### Pseudocode

```
1 # MaxDivergence
2 def noisy_top_k(x: list[TIA], scale: f64, k: usize, negate: bool) -> list[usize]:
3     return exponential_top_k(x, scale, k, negate)
4
5 def privacy_map(d_in: f64, scale: f64) -> f64:
6     return d_in.inf_div(scale)
```

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<sup>1</sup>See new changes with `git diff e62b0aa2..0abb9d rust/src/measurements/noisy_top_k/mod.rs`

## Postcondition

**Theorem 1.1.** The implementation is consistent with all associated items in the **TopKMeasure** trait.

1. Method **noisy\_top\_k**:

- Returns the index of the top element  $z_i$ , where each  $z_i \sim \text{DISTRIBUTION}(\text{shift} = y_i, \text{scale} = \text{scale})$ , and each  $y_i = -x_i$  if **negate**, else  $y_i = x_i$ ,  $k$  times with removal.
- Errors are data-independent, except for exhaustion of entropy.

2. Method **privacy\_map**: For any  $x, x'$  where  $d_{\text{in}} \geq d_{\text{Range}}(x, x')$ , return  $d_{\text{out}} \geq D_{\text{self}}(f(x), f(x'))$ , where  $f(x) = \text{noisy\_top\_k}(x = x, k = 1, \text{scale} = \text{scale})$ .

**Definition 1.2.** A random variable follows the Exponential distribution if it has density

$$f(x) = \frac{1}{\beta} e^{-z} \quad (1)$$

where  $z = \frac{x-\mu}{\beta}$ ,  $\mu$  is the shift (location) parameter and  $\beta$  is the scale parameter.

*Proof of postcondition: **noisy\_top\_k**.* The preconditions of **exponential\_noisy\_max** are met, therefore by the postcondition of **exponential\_top\_k**, the postcondition of **noisy\_top\_k** is satisfied.  $\square$

Before proving the privacy guarantees, we state a few required definitions and lemmas:

**Definition 1.3.** Report noisy max with exponential noise computes the index of the maximum element from a set of candidates  $u \in \mathbf{d\_in}$ , adds isotropic exponential noise  $Z_i \sim \text{Exp}(1/\lambda)$  to each element in the candidate set  $u$  and returns the maximum index as follows:

$$\text{RNM-Exp}(s) = \text{argmax}_i(s_i + Z_i), Z_i \sim \text{Exp}(\lambda) \quad (2)$$

**Lemma 1.4.** The permute-and-flip mechanism is equivalent to the report-noisy-max with exponential noise mechanism.

See [1] for proof of Lemma 1.4.

**Lemma 1.5.** Let  $X_1, X_2 \sim \text{Exp}(\lambda)$ ,  $\Delta \geq 0$ , then

$$\Pr[X_1 - X_2 \geq \Delta] = e^{-\Delta\lambda} \Pr[X_1 - X_2 \geq 0] \quad (3)$$

*Proof of Lemma 1.5.*

$$\Pr[X_1 - X_2 \geq \Delta] \quad (4)$$

$$= 1 - \Pr[X_1 \leq \Delta + X_2] \quad (5)$$

$$= 1 - \int_0^\infty \Pr[X_1 \leq \Delta + X_2 | X_2 = x] \Pr[X_2 = x] dx \quad \text{by Law of Total Probability} \quad (6)$$

$$= 1 - \int_0^\infty \Pr[X_1 \leq \Delta + x] \Pr[X_2 = x] dx \quad \text{by the fact that } \Delta > 0 \quad (7)$$

$$= 1 - \int_0^\infty \lambda(1 - e^{-(x+\Delta)\lambda}) e^{-x\lambda} dx \quad (8)$$

$$= 1 - \lambda \int_0^\infty e^{-x\lambda} dx + \lambda e^{-\Delta\lambda} \int_0^\infty e^{-2x\lambda} dx \quad (9)$$

$$= 1 - 1 + e^{-\Delta\lambda}/2 \quad \Pr[X_1 - X_2 \leq 0] = \Pr[X_1 - X_2 \geq 0] = 1/2 \quad (10)$$

$$= e^{-\Delta\lambda} \Pr[X_1 - X_2 \geq 0] \quad (11)$$

$\square$

**Lemma 1.6.** Let  $u, v \in \text{input\_domain}$  be two vectors of scores. Assume  $u, v$  in  $\text{input\_domain}$  are  $\mathbf{d\_in}$ -close under **LInfDistance** and  $\text{privacy\_map}(\mathbf{d\_in}) \leq \mathbf{d\_out}$ . Let  $Z^* = \min_{Z_i} \{u_i + Z_i \geq u_j + Z_j\}, \forall i \neq j$ . Then

$$\ln \left( \frac{\Pr[\text{function}(u) = i]}{\Pr[\text{function}(v) = i]} \right) = \ln \left( \frac{\Pr[Z_i \geq Z^*]}{\Pr[Z_i \geq Z^* + \mathbf{d\_in}]} \right) \quad (12)$$

*Proof.*

$$\ln \left( \frac{\Pr[\text{function}(u) = i]}{\Pr[\text{function}(v) = i]} \right) \quad (13)$$

$$= \ln \left( \frac{\Pr[\text{RNM-Exp}(u) = i]}{\Pr[\text{RNM-Exp}(v) = i]} \right) \quad \text{by Lemma 1.4} \quad (14)$$

$$= \ln \left( \frac{\Pr[\arg\max_k (u_k + Z_k) = i]}{\Pr[\arg\max_k (v_k + Z_k) = i]} \right) \quad \text{by Definition 1.3} \quad (15)$$

Observe that for a fixed  $i$ , report noisy max outputs  $i$  if:

$$u_i + Z^* \geq u_j + Z_j, \forall i \neq j \quad \Longleftrightarrow \quad (16)$$

$$u_i + (v_i - v_i) + Z^* \geq u_j + (v_j - v_j) + Z_j \quad \Longleftrightarrow \quad (17)$$

$$v_i + (u_i - v_i) + Z^* \geq v_j + (u_j - v_j) + Z_j \quad \Longleftrightarrow \quad (18)$$

$$v_i + ((u_i - v_i) - (u_j - v_j) + Z^*) \geq v_j + Z_j \quad \Longleftrightarrow \quad (19)$$

$$v_i + (\Delta + Z^*) \geq v_j + Z_j \quad (20)$$

In other words, if  $Z_i \geq (\Delta + Z^*)$ , then  $\text{function}(u) = \text{function}(v) = i$ . This yields us:

$$\ln \left( \frac{\Pr[\arg\max_k (u_k + Z_k) = i]}{\Pr[\arg\max_k (v_k + Z_k) = i]} \right) = \ln \left( \frac{\Pr[Z_i \geq Z^*]}{\Pr[Z_i \geq \Delta + Z^*]} \right) \quad (21)$$

□

*Proof of postcondition: **privacy\_map**.*

$$\max_{u \sim v} D_\infty(M(u) | M(v)) \quad (22)$$

$$= \max_{u \sim v} \max_i \ln \left( \frac{\Pr[\text{function}(u) = i]}{\Pr[\text{function}(v) = i]} \right) \quad (23)$$

$$= \max_{u \sim v} \max_i \ln \left( \frac{\Pr[\text{RNM-Exp}(u) = i]}{\Pr[\text{RNM-Exp}(v) = i]} \right) \quad \text{by Lemma 1.4} \quad (24)$$

$$= \max_{u \sim v} \max_i \ln \left( \frac{\Pr[\arg\max_k (u_k + Z_k) = i]}{\Pr[\arg\max_k (v_k + Z_k) = i]} \right) \quad \text{by Definition 1.3} \quad (25)$$

$$= \max_{u \sim v} \max_i \ln \left( \frac{\Pr[Z_i \geq Z^*]}{\Pr[Z_i \geq Z^* + \mathbf{d\_in}]} \right) \quad \text{by Lemma 1.6} \quad (26)$$

$$\leq \frac{\mathbf{d\_in}}{\text{scale}} \quad \text{by Lemma 1.5} \quad (27)$$

□

## References

- [1] Zeyu Ding, Daniel Kifer, Thomas Steinke, Yuxin Wang, Yingtai Xiao, Danfeng Zhang, et al. The permute-and-flip mechanism is identical to report-noisy-max with exponential noise. *arXiv preprint arXiv:2105.07260*, 2021.

- [2] Ryan McKenna and Daniel Sheldon. Permute-and-flip: A new mechanism for differentially private selection, 2020.