`fn discrete_laplacian_scale_to_accuracy`

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May 18, 2025

This document contains materials associated with `discrete_laplacian_scale_to_accuracy`.

Definition 0.1. Let z be the true value of the statistic and X be the random variable the noisy release is drawn from. Define $Y = |X - z|$, the distribution of DP errors. Then for any statistical significance level `alpha`, denoted $\alpha \in [0, 1]$, and `accuracy`, denoted $a \geq 0$,

$$\alpha = P[Y \geq a] \tag{1}$$

Theorem 0.2. For any `scale` ≥ 0 denoted s, when $X \sim \mathcal{L}_{\mathbb{Z}}(z, s)$,

$$a = s \cdot ln(2/(\alpha(e^{1/s} + 1))) + 1 \tag{2}$$

Proof. Consider that the distribution of $(X - z) \sim \mathcal{L}_{\mathbb{Z}}(0, s)$. Then the PMF of Y is:

$$\forall y \geq 0 \qquad g(y) = (1 + 1[y \neq 0])\frac{1 - e^{-1/s}}{1 + e^{-1/s}}e^{-y/s} \tag{3}$$

The purpose of the indicator function is to avoid double-counting zero.

Now derive an expression for α:

$$\begin{aligned}
\alpha &= P[Y \geq a] \\
&= 1 - P[Y < a] \\
&= 1 - \sum_{y=0}^{a-1} g(y) && \text{where } g(y) \text{ is the distribution of Y} \\
&= 1 - \frac{1 - e^{-1/s}}{1 + e^{-1/s}}\left(1 + 2\sum_{y=0}^{a-1} e^{-y/s}\right) \\
&= 2\frac{e^{(1-a)/s}}{e^{1/s} + 1}
\end{aligned}$$

Invert to solve for a:

$$\begin{aligned}
2\frac{e^{(1-a)/s}}{e^{1/s} + 1} &= \alpha \\
e^{(1-a)/s} &= \alpha(e^{1/s} + 1)/2 \\
a &= 1 - s \cdot ln(\alpha(e^{1/s} + 1)/2) \\
a &= s \cdot ln(2/(\alpha(e^{1/s} + 1))) + 1
\end{aligned}$$

$\square$