

# CompositionMeasure for ZeroConcentratedDivergence

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This proof resides in “**contrib**” because it has not completed the vetting process.

Proves soundness of the implementation of `CompositionMeasure` for `ZeroConcentratedDivergence` in `mod.rs` at commit `f5bb719` (outdated<sup>1</sup>).

## 1 Hoare Triple

### Precondition

#### Compiler-Verified

Types matching pseudocode.

#### Caller-Verified

None

### Pseudocode

```
1 class CompositionMeasure(ZeroConcentratedDivergence):
2     def composability( # 
3         self, adaptivity: Adaptivity
4     ) -> Composability:
5         match adaptivity:
6             case Adaptivity.FullyAdaptive:
7                 return Composability.Sequential
8             case _:
9                 return Composability.Concurrent
10
11     def compose(self, d_mids: Vec[Self_Distance]) -> Self_Distance:
12         d_out = 0.0
13         for d_mid in d_mids:
14             d_out = d_out.inf_add(d_mid)
15         return d_out
```

### Postcondition

**Theorem 1.1.** `composability` returns `Ok(out)` if the composition of a vector of privacy parameters `d_mids` is bounded above by `self.compose(d_mids)` under `adaptivity` adaptivity and `out`-composability. Otherwise returns an error.

**Definition 1.2** (Definition of `ZeroConcentratedDivergence`). For any two distributions  $Y, Y'$  and any non-negative  $d$ ,  $Y, Y'$  are  $d$ -close under the zero-concentrated divergence measure if, for every possible choice of  $\alpha \in (1, \infty)$ ,

<sup>1</sup>See new changes with `git diff f5bb719..b67b63c rust/src/combinators/sequential_composition/mod.rs`

$$D_\alpha(Y, Y') = \frac{1}{1-\alpha} \mathbb{E}_{x \sim Y'} \left[ \ln \left( \frac{\Pr[Y=x]}{\Pr[Y'=x]} \right)^\alpha \right] \leq d \cdot \alpha. \quad (1)$$

**Lemma 1.3.** 1.  $\mathcal{F}(\rho_1, \rho_2, \dots; \rho) = \mathbb{I}(\sum_i \rho_i \leq \rho)$  is a valid  $\rho$ -zCDP IM-filter.

2.  $\mathcal{G}(\rho_1, \dots, \rho_k) = \sum_{i=1}^k \rho_i$  is a valid  $\rho$ -zCDP IM-privacy loss accumulator.

*Proof.* For any fixed choice of  $\alpha > 1$ , we have that  $\mathcal{F}(\rho_1, \rho_2, \dots; \rho) = \mathbb{I}(\sum_i \epsilon_i/\alpha \leq \epsilon/\alpha)$ , by 1.2, which by Theorem 1.22[VW21] is a valid  $(\alpha, \epsilon)$ -RDP IM-filter.

Similarly, for any fixed choice of  $\alpha > 1$ , we have that  $\mathcal{G}(\rho_1, \dots, \rho_k) = \sum_i \epsilon_i/\alpha \leq \epsilon/\alpha$ , by 1.2, which by Theorem 1.22[VW21] is a valid  $(\alpha, \epsilon)$ -RDP IM-privacy loss accumulator.  $\square$

*Proof.* By the postcondition of [InfAdd](#) we have that  $\sum_i \mathbf{d\_mids}_i \leq \mathbf{compose}(\mathbf{d\_mids})$ .

Adaptivity	Sequential	Concurrent
Non-Adaptive	Lemma 1.7[BS16]	Corollary 1[Lyu22]
Adaptive	Lemma 1.7[BS16]	Corollary 1[Lyu22]
Fully-Adaptive	Remark 4.4[FZ22]	<a href="#">1.3</a>

This table is reflected in the implementation of `composability` on line 2.

$\square$

## References

- [BS16] Mark Bun and Thomas Steinke. Concentrated differential privacy: Simplifications, extensions, and lower bounds, 2016.
- [FZ22] Vitaly Feldman and Tijana Zrnic. Individual privacy accounting via a renyi filter, 2022.
- [Lyu22] Xin Lyu. Composition theorems for interactive differential privacy, 2022.
- [VW21] Salil Vadhan and Tianhao Wang. Concurrent composition of differential privacy, 2021.