trait impl PSRN for TulapPSRN

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This proof resides in **"contrib"** because it has not completed the vetting process.

Proves soundness of `TulapPSRN`.

`edge` accepts parameter `self`, containing the state of the Tulap sampler and `R` specifying the rounding mode.

This implementation is susceptible to floating-point vulnerabilities.

Warning 1 (Code is not constant-time)**.** The implementation of `edge` uses procedures that are vulnerable to timing attacks.

PR History

- Pull Request #1126

1 Hoare Triple

Preconditions

- Variable `self` is of type `TulapPSRN`.
- Generic `R` denotes the rounding mode, one of "up" or "down".

Pseudocode

```python

class TulapPSRN(object):
    def __init__(self, shift, epsilon, delta) -> None:
        self.shift = shift
        self.exp_eps = Fraction(epsilon.neg_inf_exp())
        self.exp_neg_eps = Fraction((-epsilon).inf_exp())
        self.c = (1 - delta) / (1 + self.exp_eps)
        self.delta = delta
        self.uniform = UniformPSRN()

        if c >= 0.5:
            raise ValueError("c must be less than 1/2")

    def q_cnd(self, unif) -> Fraction | None:  # CND quantile function for f
        if unif < c:
            return self.q_cnd(1 - self.f(unif)) - 1
```

```
        elif unif <= 1 - self.c:  # the linear function
            num = unif - 1 / 2
            den = 1 - 2 * self.c
            if den.is_zero():
                return
            return num / den
        else:
            return self.q_cnd(self.f(1 - unif)) + 1

    def f(self, unif):
        t1 = 1 - self.delta - self.exp_eps * unif
        t2 = self.exp_neg_eps * (1 - self.delta - unif)
        return max(t1, t2, 0)

    def edge(self, R):
        return self.q_cnd(self.uniform.edge(R)) + self.shift

    def refine(self):
        self.uniform.refine()

    def refinements(self):
        return self.uniform.refinements()
```

Postcondition

`edge` returns an estimate of the true Tulap sample, a distribution with CDF defined in `make_tulap`.

2 Proof

Proof. The cdf of Tulap$(0, b, q)$ is

$$F_N(x) = \begin{cases} 0 & F_{N_0}(x) < q/2 \\ \frac{F_{N_0}(x) - q/2}{1-q} & q/2 \leq F_{N_0}(x) \leq 1 - q/2 \\ 1 & F_{N_0}(x) > 1 - q/2. \end{cases}$$

By inspection, the fixed point of $f_{\epsilon,\delta}$ is $c = \frac{1-\delta}{1+e^{\epsilon}}$. It is easy to verify that $F_N(x) = c(1/2 - x) + (1-c)(x + 1/2)$ for $x \in (-1/2, 1/2)$.

The function then uses the inverse transform of a sample of a uniform RV to sample a Tulap RV centered at zero. The function then returns the value, shifted by `self.shift`.

□