fn integrate_discrete_laplace_tail

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Proof for `integrate_discrete_laplace_tail`.

Definition 0.1. Define $Y \sim \mathcal{L}_{\mathbb{Z}}(0, s)$, a random variable following the discrete laplace distribution:

$$\forall x \in \mathbb{Z} \qquad \Pr[X = x] = \frac{e^{1/s} - 1}{e^{1/s} + 1} e^{-|x|/s} \tag{1}$$

Theorem 0.2. Assume $X \sim \mathcal{L}_{\mathbb{Z}}(0, s)$.

$$\alpha = P[X \geq t] = \frac{e^{-t/s}}{1 + e^{-1/s}} \tag{2}$$

Proof. First, assume $t > 0$.

$$\begin{aligned}
\alpha &= P[X \geq t] \\
&= \sum_{x=t}^{\infty} \frac{e^{1/s} - 1}{e^{1/s} + 1} e^{-|x|/s} \\
&= \frac{e^{1/s} - 1}{e^{1/s} + 1} \sum_{x=t}^{\infty} e^{-x/s} && \text{since } t > 0 \\
&= \frac{e^{1/s} - 1}{e^{1/s} + 1} \frac{e^{(1-t)/s}}{e^{1/s} - 1} && \text{since } \sum_{x=t}^{\infty} p^x = \frac{p^x}{1 - p} \text{ if } |p| < 1, \text{ let } p = e^{1/s} \\
&= \frac{e^{(1-t)/s}}{e^{1/s} + 1} \\
&= \frac{e^{-t/s}}{1 + e^{-1/s}} && \text{for numerical stability}
\end{aligned}$$

□