

# fn sample\_discrete\_gaussian

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This proof resides in “**contrib**” because it has not completed the vetting process.

Proves soundness of `fn sample_discrete_gaussian` in `mod.rs` at commit `0be3ab3e6` (outdated<sup>1</sup>). This proof is an adaptation of subsection 5.3 of [CKS20].

## Vetting history

- Pull Request #519

## 1 Hoare Triple

### Precondition

$\text{scale} \in \mathbb{Q} \wedge \text{scale} \geq 0$

### Implementation

```
1 def sample_discrete_gaussian(scale) -> int:
2     if scale == 0:
3         return 0
4
5     t = floor(scale) + 1
6     sigma2 = scale**2
7
8     while True:
9         candidate = sample_discrete_laplace(t)
10        x = abs(candidate) - sigma2 / t
11        bias = x**2 / (2 * sigma2)
12        if sample_bernoulli_exp(bias):
13            return candidate
```

### Postcondition

For any setting of the input parameter `scale` such that the given preconditions hold, `sample_discrete_gaussian` either returns `Err(e)` due to a lack of system entropy, or `Ok(out)`, where `out` is distributed as  $\mathcal{N}_{\mathbb{Z}}(0, \text{scale}^2)$ .

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<sup>1</sup>See new changes with `git diff 0be3ab3e6..916ddb4a rust/src/traits/samplers/cks20/mod.rs`

## 2 Proof

**Definition 2.1.** (Discrete Gaussian). [CKS20] Let  $\mu, \sigma \in \mathbb{R}$  with  $\sigma > 0$ . The discrete gaussian distribution with location  $\mu$  and scale  $\sigma$  is denoted  $\mathcal{N}_{\mathbb{Z}}(\mu, \sigma^2)$ . It is a probability distribution supported on the integers and defined by

$$\forall x \in \mathbb{Z} \quad P[X = x] = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sum_{y \in \mathbb{Z}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}} \quad \text{where } X \sim \mathcal{N}_{\mathbb{Z}}(\mu, \sigma^2)$$

**Lemma 2.2.** `sample_discrete_gaussian` only returns `Err(e)` when there is a lack of system entropy.

*Proof.* By the non-negativity precondition on `scale`, `t` on line 5 is non-negative, so the precondition on `sample_discrete_laplace` is met. Similarly, since `bias` on line 11 is non-negative, the preconditions on `sample_bernoulli_exp` are met. By the definitions of `sample_discrete_laplace` and `sample_bernoulli_exp`, an error is only returned when there is a lack of system entropy. The only source of errors in `sample_discrete_gaussian` is from the invocation of these functions, therefore `sample_discrete_gaussian` only returns `Err(e)` when there is a lack of system entropy.  $\square$

We now condition on not returning an error. Let  $t = \lfloor \sigma \rfloor + 1$ , and fix any iteration of the loop.

**Lemma 2.3.** [CKS20] If  $y$  is a realization of  $Y \sim \mathcal{L}_{\mathbb{Z}}(0, t)$ , and  $c$  is a realization of  $C \sim \text{Bernoulli}(\exp(-(|y| - \sigma^2/t)^2/(2\sigma^2)))$ , then  $E[C] = \frac{1 - e^{-1/\sigma}}{1 + e^{-1/\sigma}} e^{-\frac{\sigma^2}{2t^2}} \sum_{y \in \mathbb{Z}} e^{-\frac{y^2}{2\sigma^2}}$ .

*Proof.*

$$\begin{aligned} E[C] &= E[E[C|Y]] \\ &= E\left[e^{-\frac{(|Y| - \sigma^2/t)^2}{2\sigma^2}}\right] && \text{since } E[\text{Bernoulli}(p)] = p \\ &= \frac{1 - e^{-1/\sigma}}{1 + e^{-1/\sigma}} \sum_{y \in \mathbb{Z}} e^{-\frac{(|y| - \sigma^2/t)^2}{2\sigma^2} - |y|/t} && \text{expectation over } Y \sim \mathcal{L}_{\mathbb{Z}}(0, \sigma) \\ &= \frac{1 - e^{-1/\sigma}}{1 + e^{-1/\sigma}} e^{-\frac{\sigma^2}{2t^2}} \sum_{y \in \mathbb{Z}} e^{-\frac{y^2}{2\sigma^2}} \end{aligned}$$

$\square$

We now show that conditioning  $Y$  on the success of  $C$  gives the desired output distribution.

**Theorem 2.4.** [CKS20] If  $y$  is a realization of  $Y \sim \mathcal{L}_{\mathbb{Z}}(0, t)$  and  $c$  is a realization of  $C \sim \text{Bernoulli}(\exp(-(|y| - \sigma^2/t)^2/(2\sigma^2)))$ , then  $P[Y = y | C = \top] = \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sum_{y' \in \mathbb{Z}} e^{-\frac{y'^2}{2\sigma^2}}}$ . That is,  $Y|_{C=\top} \sim \mathcal{N}_{\mathbb{Z}}(0, \sigma^2)$ .

*Proof.*

$$\begin{aligned} P[Y = y | C = \top] &= \frac{P[C = \top | Y = y] P[Y = y]}{P[C = \top]} && \text{Bayes' Theorem} \\ &= \frac{e^{-\frac{(|y| - \sigma^2/t)^2}{2\sigma^2}} \frac{1 - e^{-1/t}}{1 + e^{-1/t}} e^{-|y|/t}}{E[C]} && \text{by definition of } \mathcal{L}_{\mathbb{Z}}(0, \sigma) \\ &= \frac{e^{-\frac{(|y| - \sigma^2/t)^2}{2\sigma^2}} e^{-|y|/t}}{e^{-(\sigma/t)^2/2} \sum_{y' \in \mathbb{Z}} e^{-\frac{y'^2}{2\sigma^2}}} && \text{by 2.3} \\ &= \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sum_{y' \in \mathbb{Z}} e^{-\frac{y'^2}{2\sigma^2}}} \end{aligned}$$

$\square$

**Lemma 2.5.** If the outcome of `sample_discrete_gaussian` is `Ok(out)`, then `out` is distributed as  $\mathcal{N}_{\mathbb{Z}}(0, scale^2)$ .

*Proof.* In the 2.2 proof, it was established that the preconditions on `sample_discrete_laplace` are met, so `candidate` on line 9 is distributed as  $\mathcal{L}_{\mathbb{Z}}(0, t)$ . Similarly, by the definition of `sample_bernoulli_exp`, the outcome is distributed according to  $Bernoulli(\exp(-(|y| - \sigma^2/t)^2/(2\sigma^2)))$ . Since on line 12, we condition returning `candidate` on a `⊤` sample, the conditions to apply 2.4 are met. Therefore `out` is distributed as  $\mathcal{N}_{\mathbb{Z}}(0, scale^2)$ .  $\square$

*Proof.* 1 holds by 2.2 and 2.5.  $\square$

## References

- [CKS20] Clément L. Canonne, Gautam Kamath, and Thomas Steinke. The discrete gaussian for differential privacy. *CoRR*, abs/2004.00010, 2020.