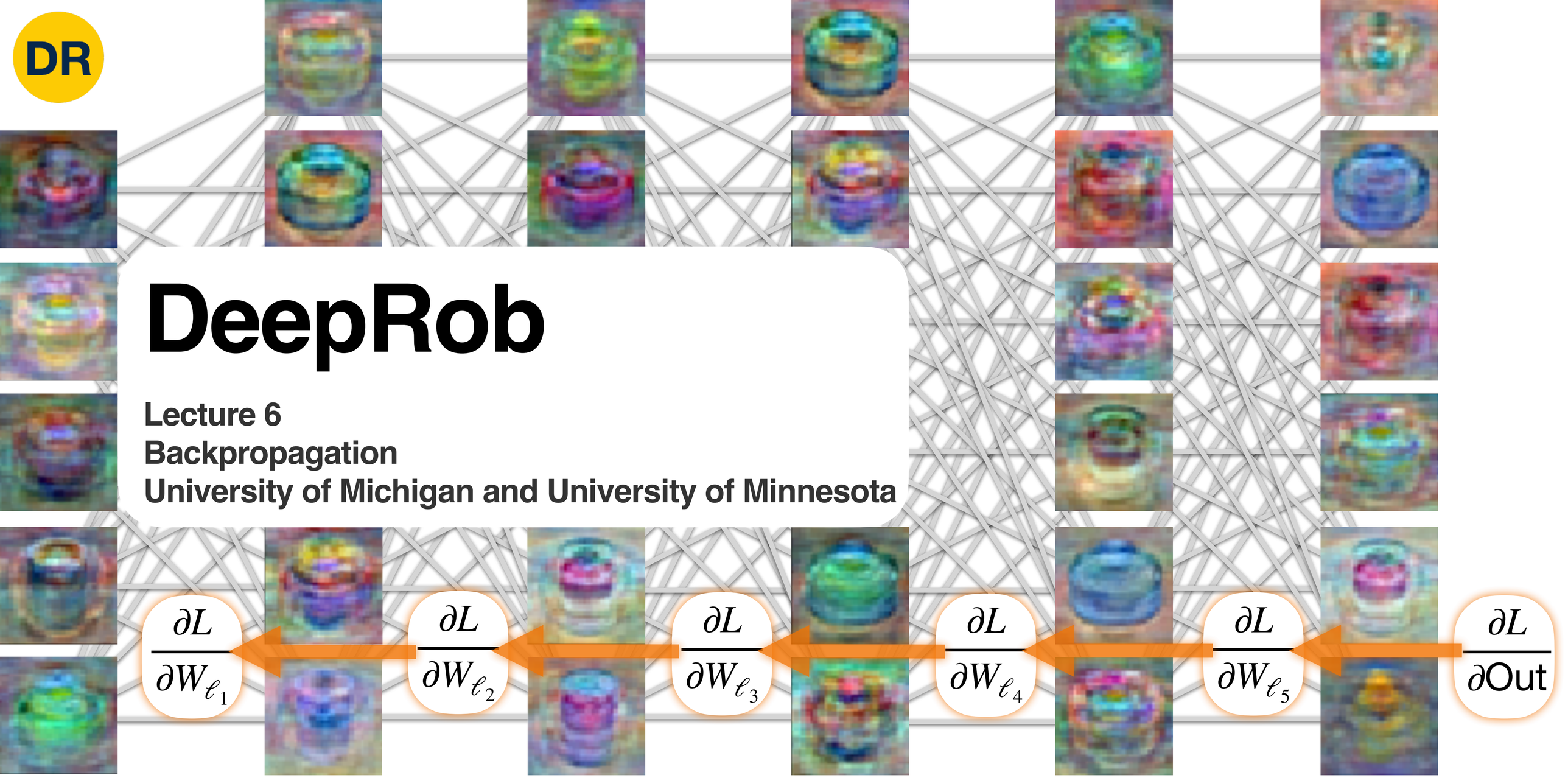
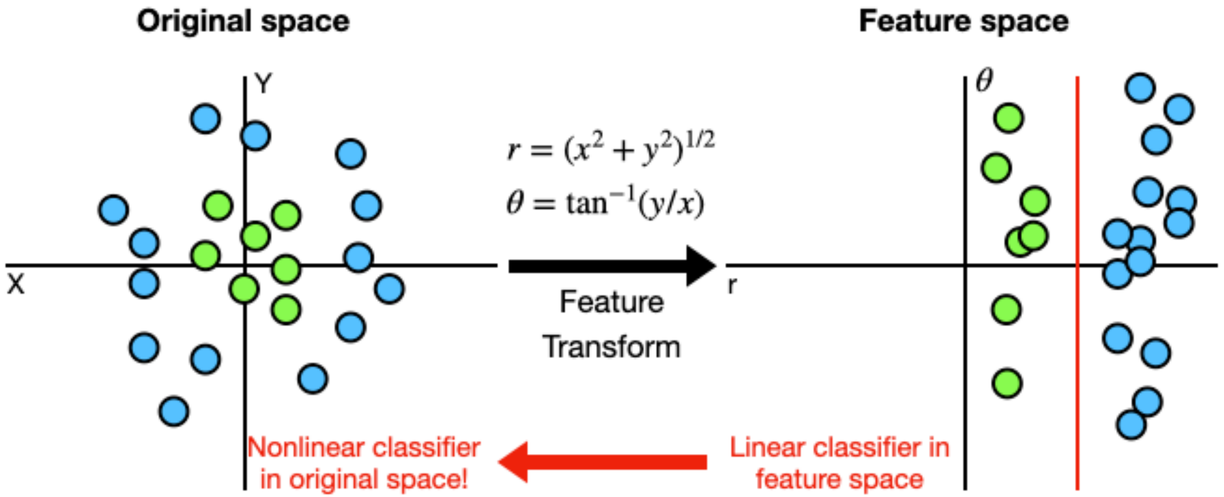

DR
DeepRob
Lecture 6
Backpropagation
University of Michigan and University of Minnesota
$\frac{\partial L}{\partial W_{\ell_1}}$
$\frac{\partial L}{\partial W_{\ell_2}}$
$\frac{\partial L}{\partial W_{\ell_3}}$
$\frac{\partial L}{\partial W_{\ell_4}}$
$\frac{\partial L}{\partial W_{\ell_5}}$
$\frac{\partial L}{\partial \text{Out}}$

Project 1 — Reminder

- Instructions and code available on the website
  - Here: deeprob.org/projects/project1/
- Uses Python, PyTorch and Google Colab
- Implement KNN, linear SVM, and linear softmax classifiers
- **Autograder is online and updated**
- **Due Thursday, January 26th 11:59 PM EST**

Recap from Previous Lecture

Feature transform + Linear classifier allows nonlinear decision boundaries

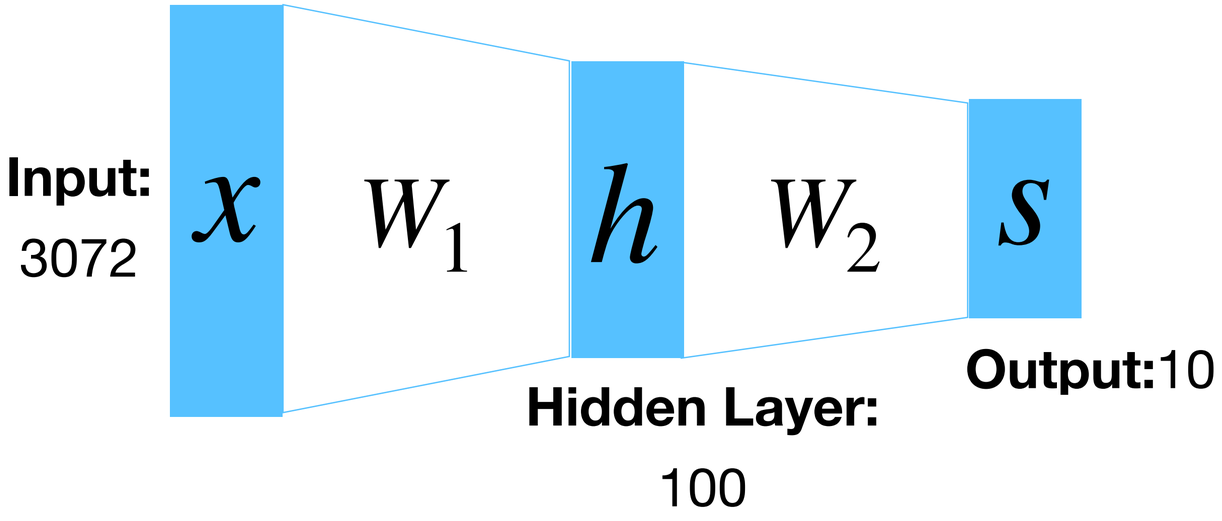


Neural Networks as learnable feature transforms

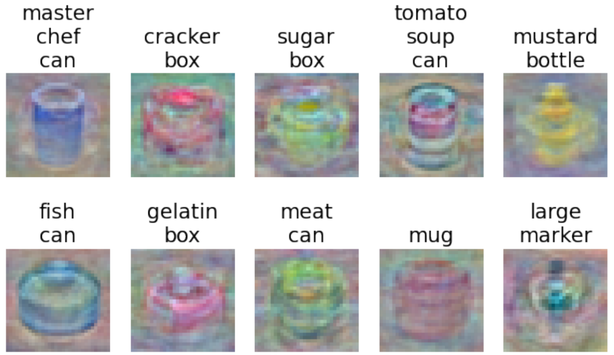
Recap from Previous Lecture

From linear classifiers to fully-connected networks

$$f(x) = W_2 \max(0, W_1 x + b_1) + b_2$$



Linear classifier: One template per class



Neural networks: Many reusable templates

Recap from Previous Lecture

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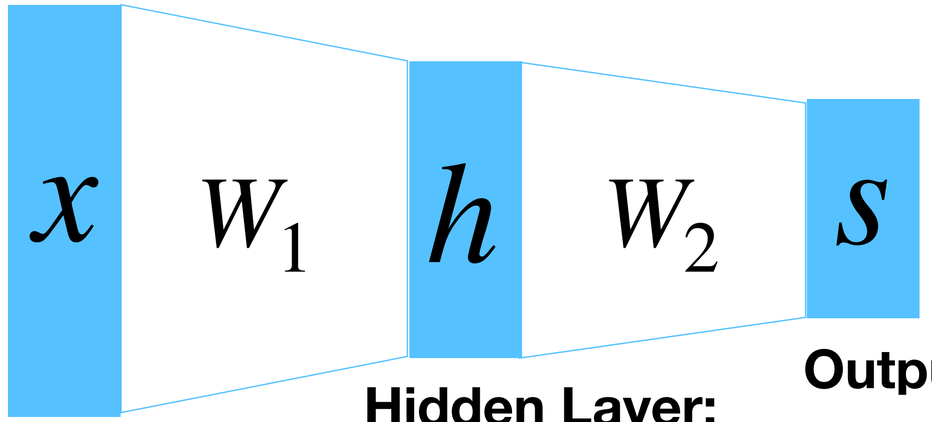
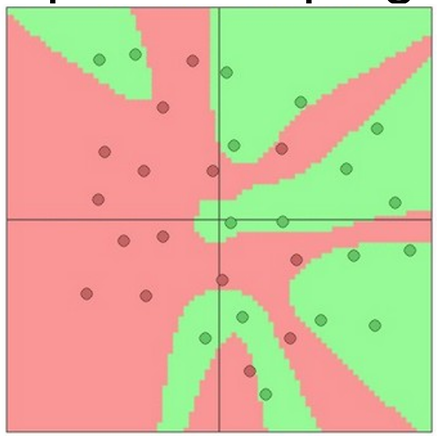


Input: 3072

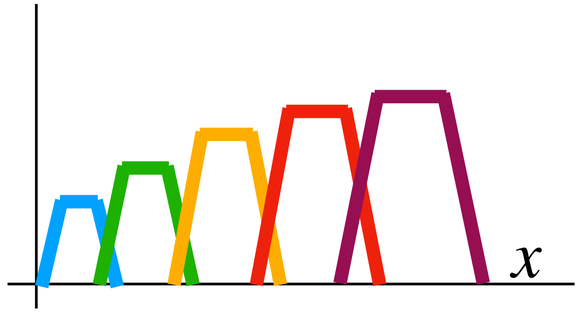
Hidden Layer: 100

Output: 10

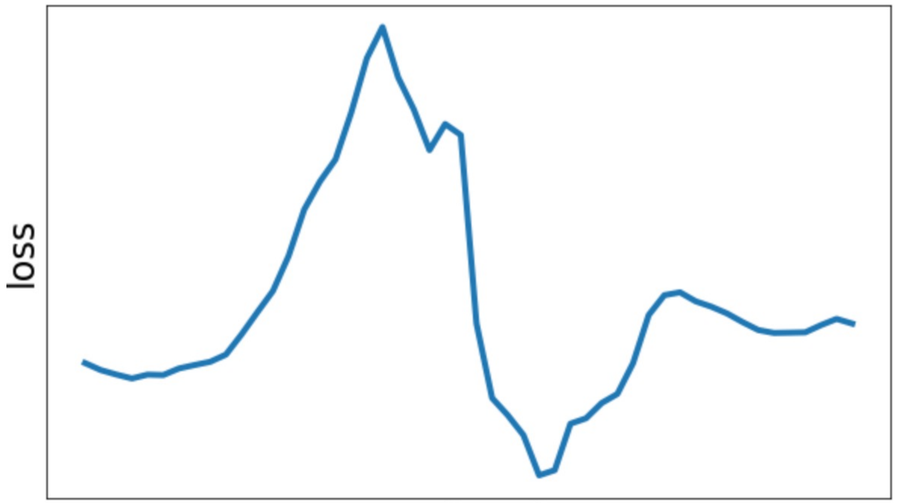
Space Warping



Universal approximation



Nonconvex



Problem: How to compute gradients?

$$s = W_2 \max(0, W_1 x + b_!) + b_2$$ Nonlinear score function

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$ Per-element data loss

$$R(W) = \sum_k W_k^2$$ L2 regularization

$$L(W_1, W_2, b_1, b_2) = \frac{1}{N} \sum_{i=1}^{N} L_i + \lambda R(W_1) + \lambda R(W_2)$$ Total loss

If we can compute $\frac{\delta L}{\delta W_1}, \frac{\delta L}{\delta W_2}, \frac{\delta L}{\delta b_1}, \frac{\delta L}{\delta b_2}$ then we can optimize with SGD

(Bad) Idea: Derive $\nabla_W L$ on paper

$$s = f(x; W) = Wx$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$= \sum_{j \neq y_i} \max(0, W_{j,:}\, x - W_{y_i,:}\, x + 1)$$

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + \lambda \sum_k W_k^2$$

$$= \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, W_{j,:}\, x - W_{y_i,:}\, x + 1) + \lambda \sum_k W_k^2$$

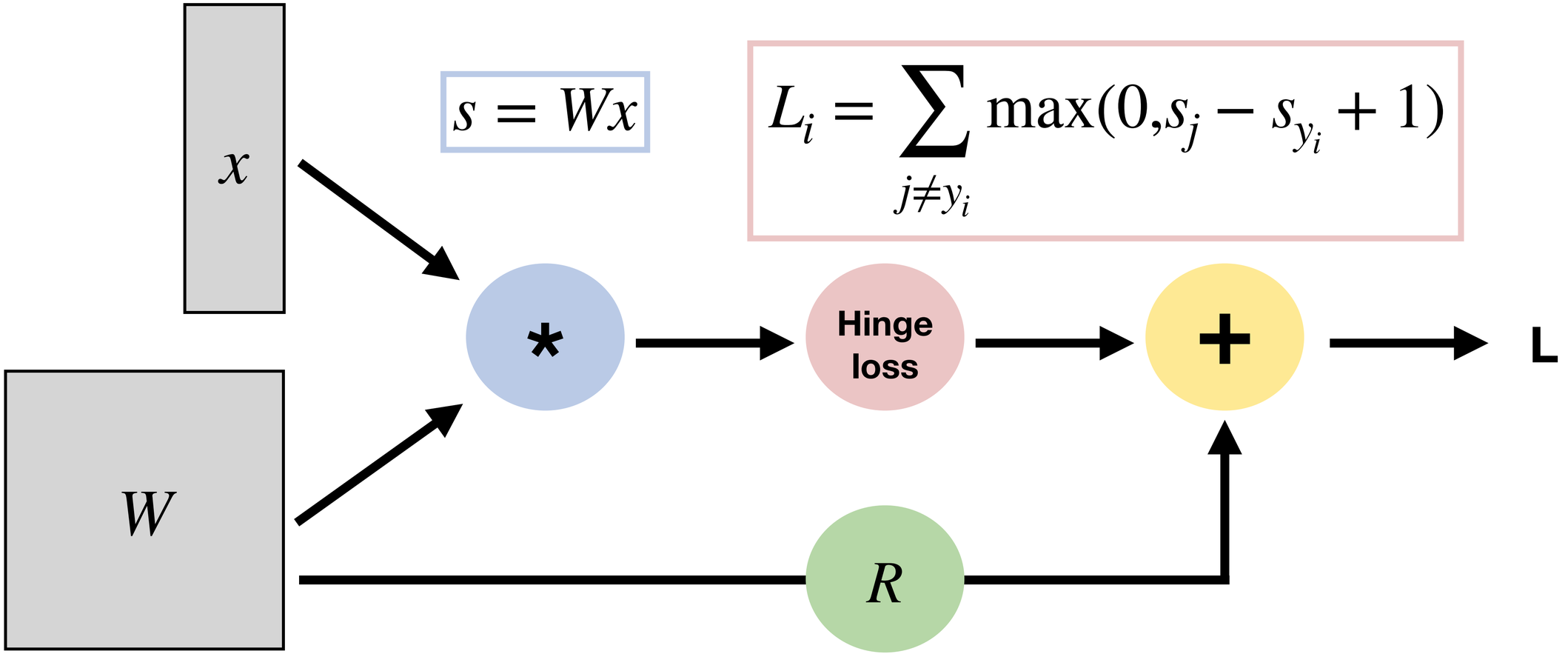
$$\nabla_W L = \nabla_W \left(\frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, W_{j,:}\, x - W_{y_i,:}\, x + 1) + \lambda \sum_k W_k^2 \right)$$

Problem: Very tedious with lots of matrix calculus

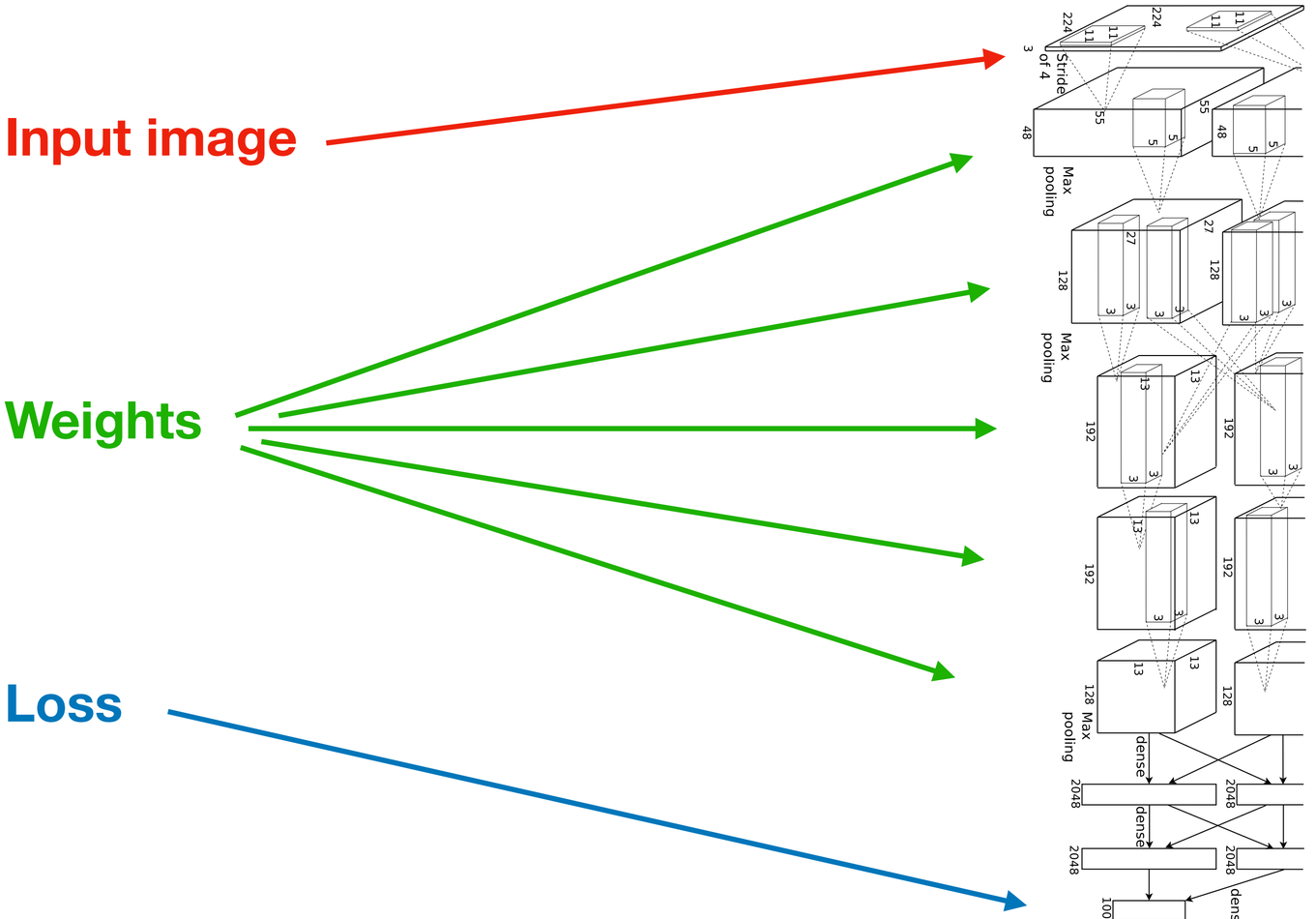
Problem: What if we want to change the loss? E.g. use softmax instead of SVM? Need to re-derive from scratch. Not modular!

Problem: Not feasible for very complex models!

Better Idea: Computational Graphs

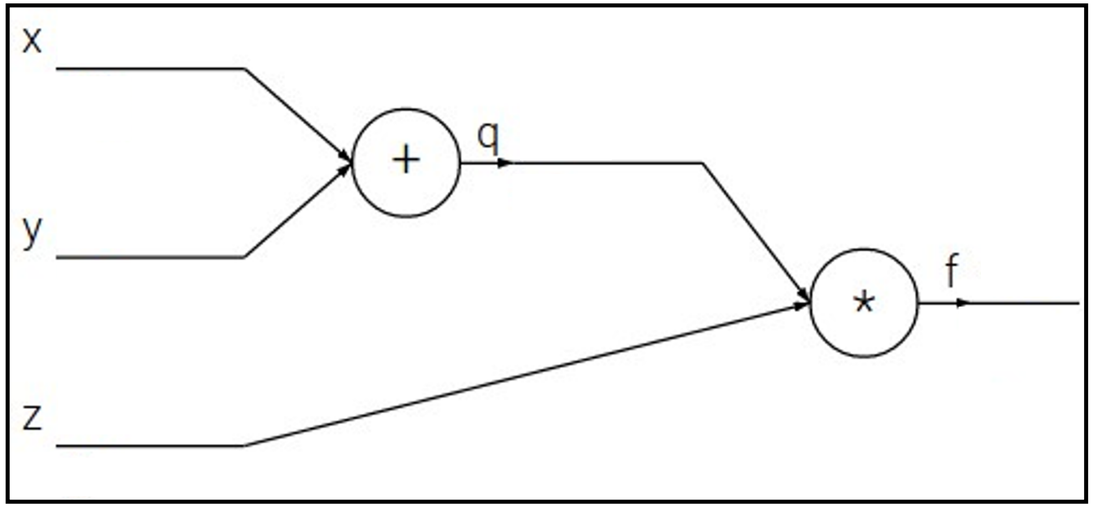


Deep Network (AlexNet)



Backpropagation: Simple Example

$$f(x, y, z) = (x + y) \cdot z$$



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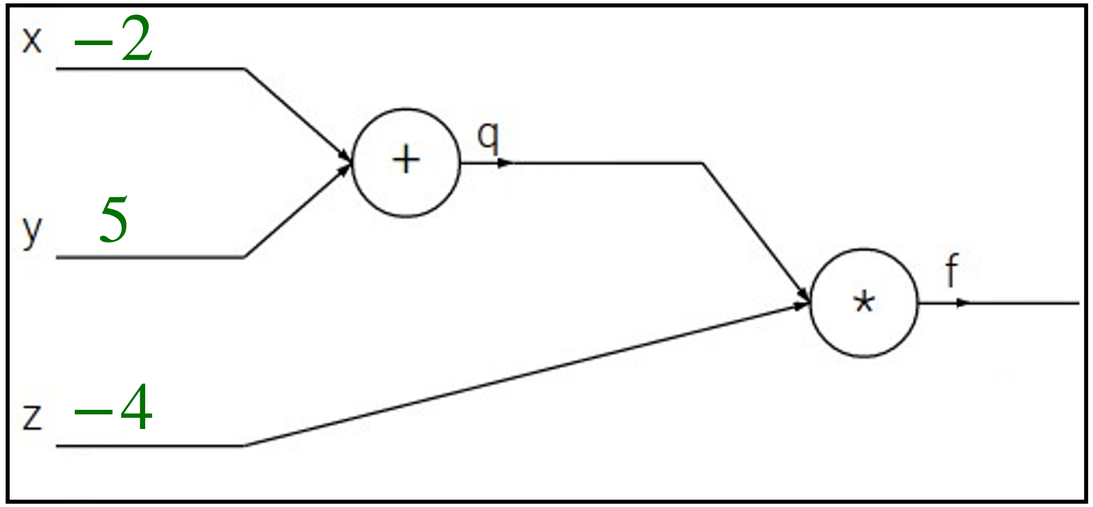

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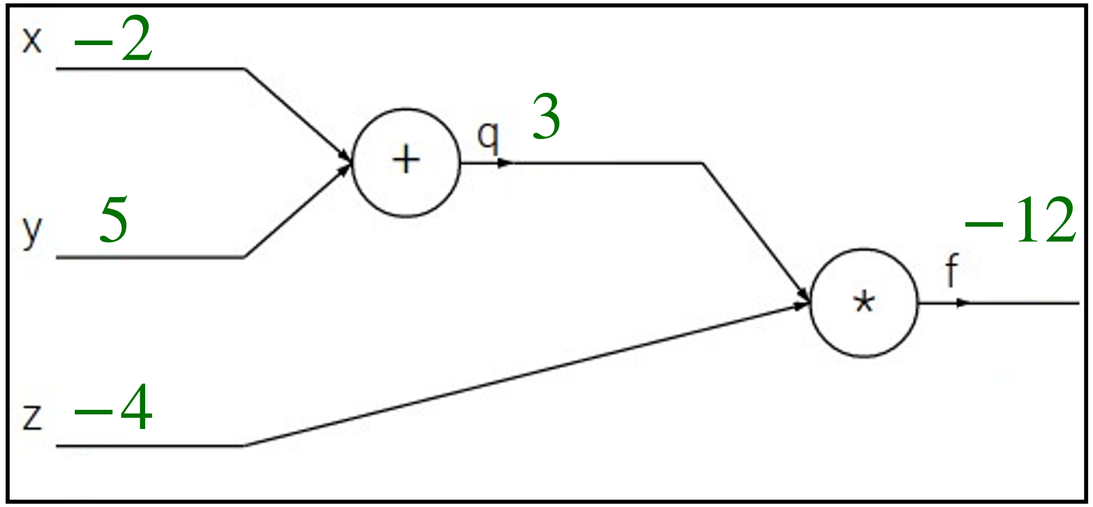

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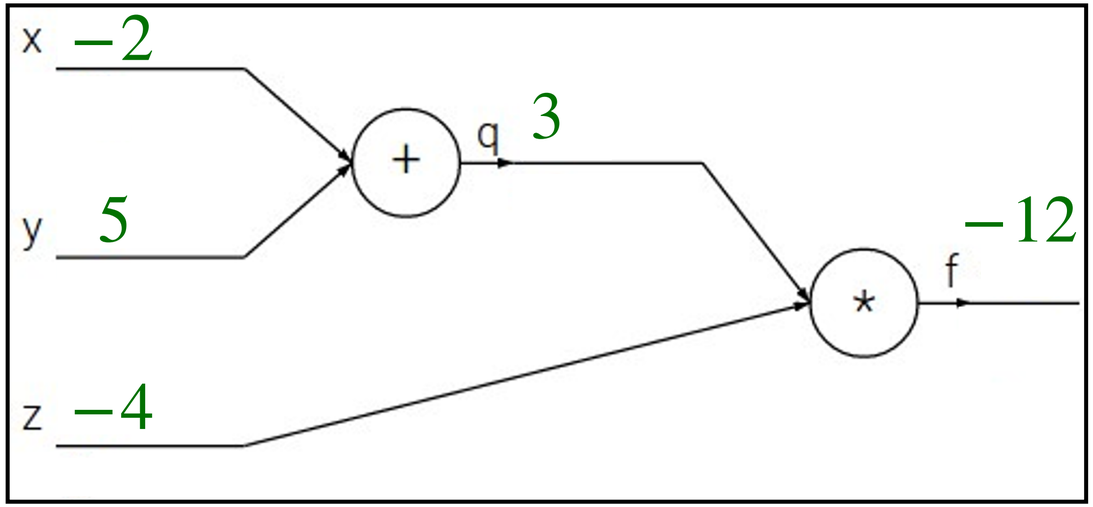

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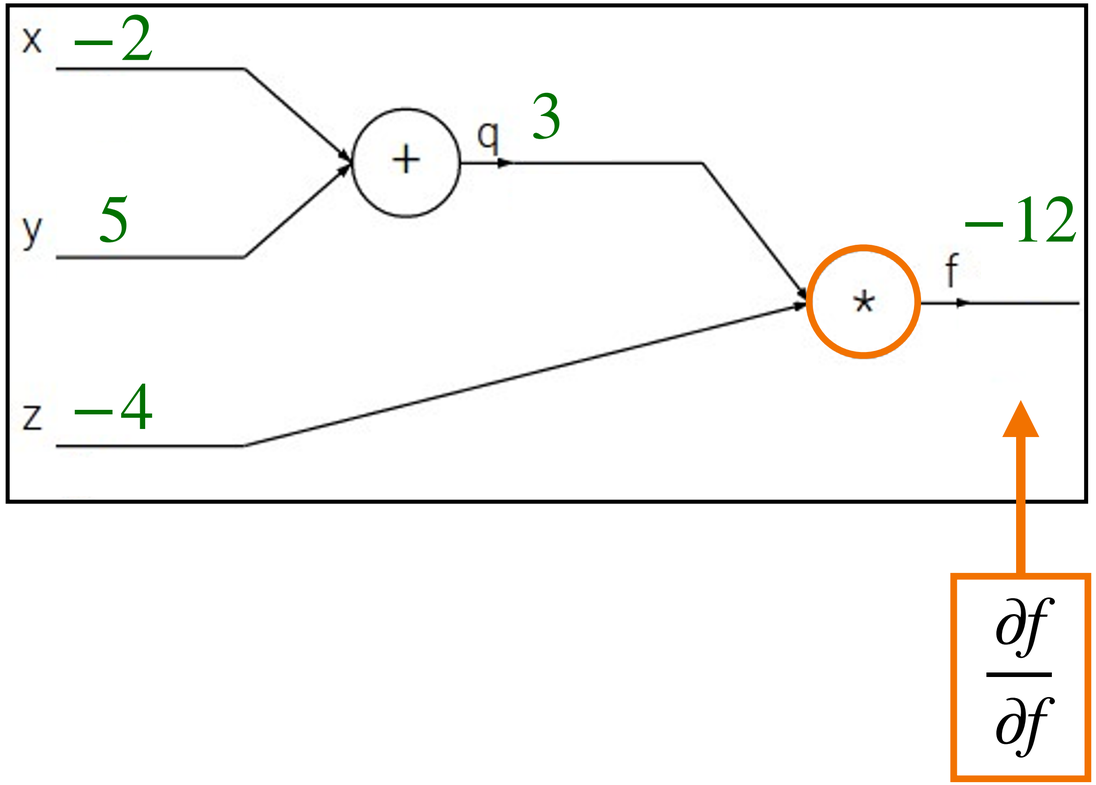

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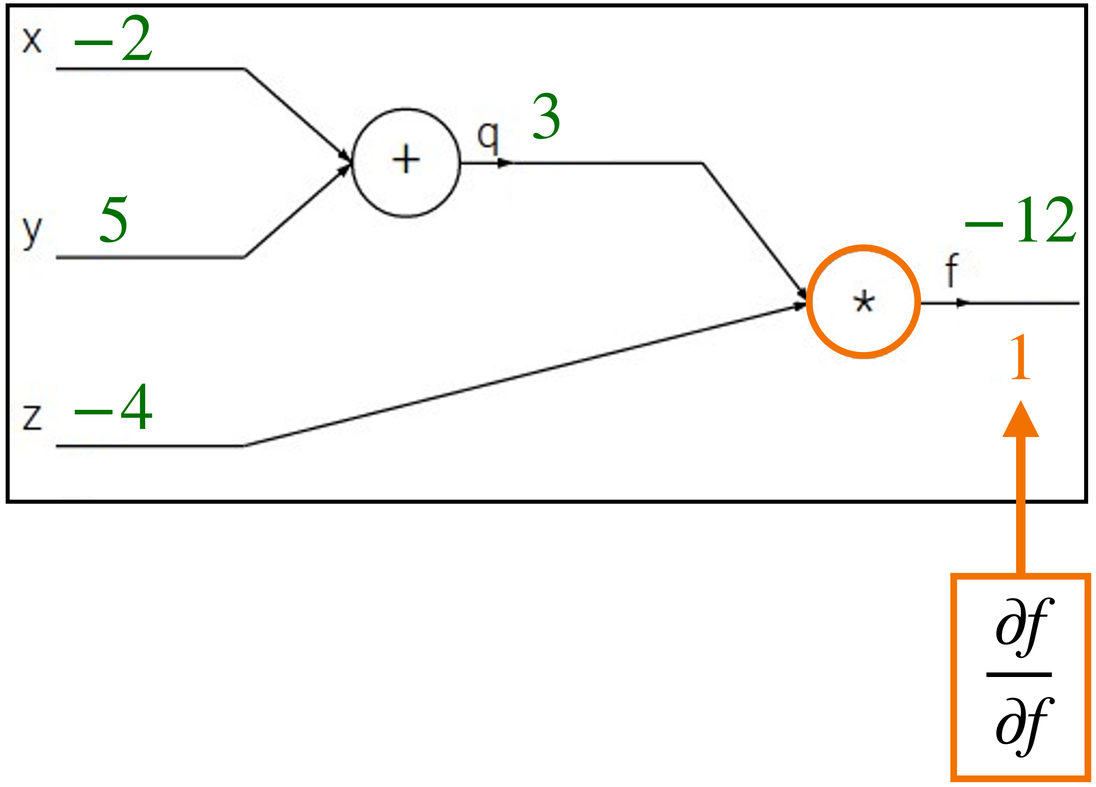

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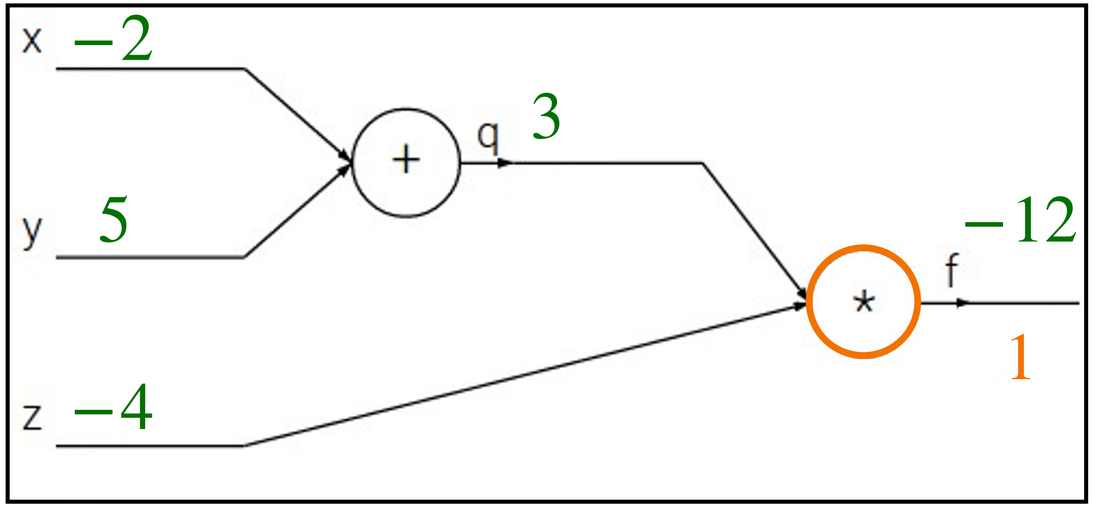

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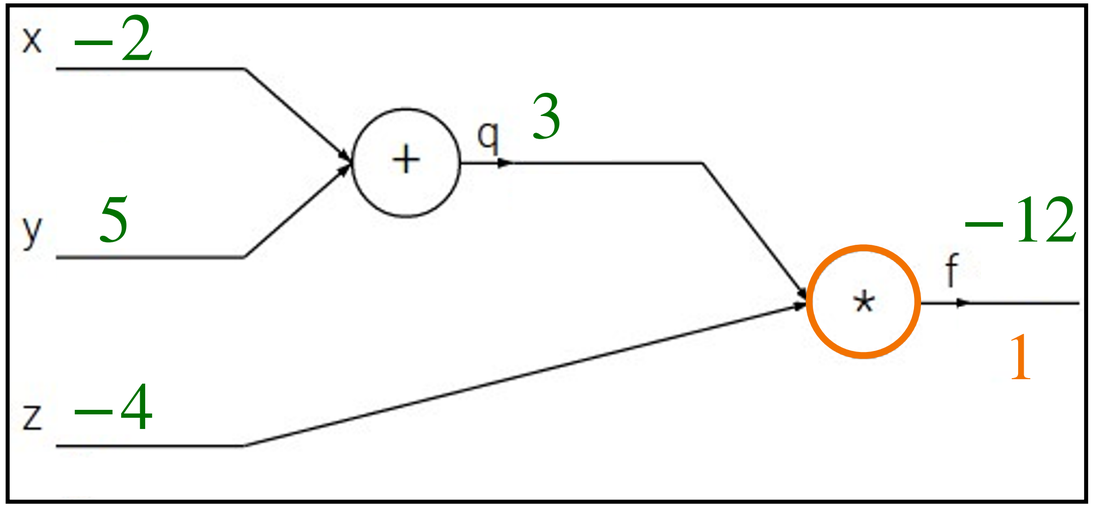

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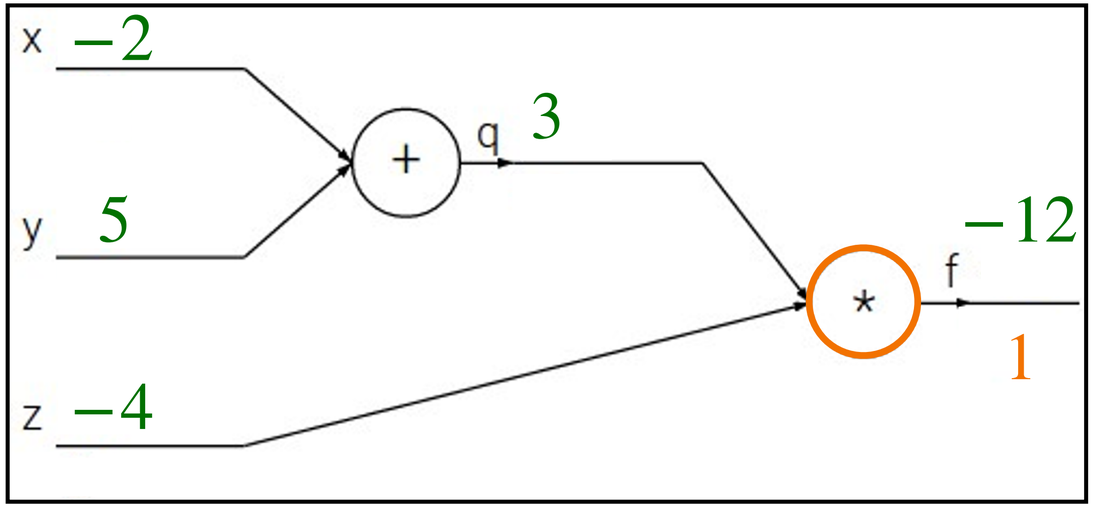

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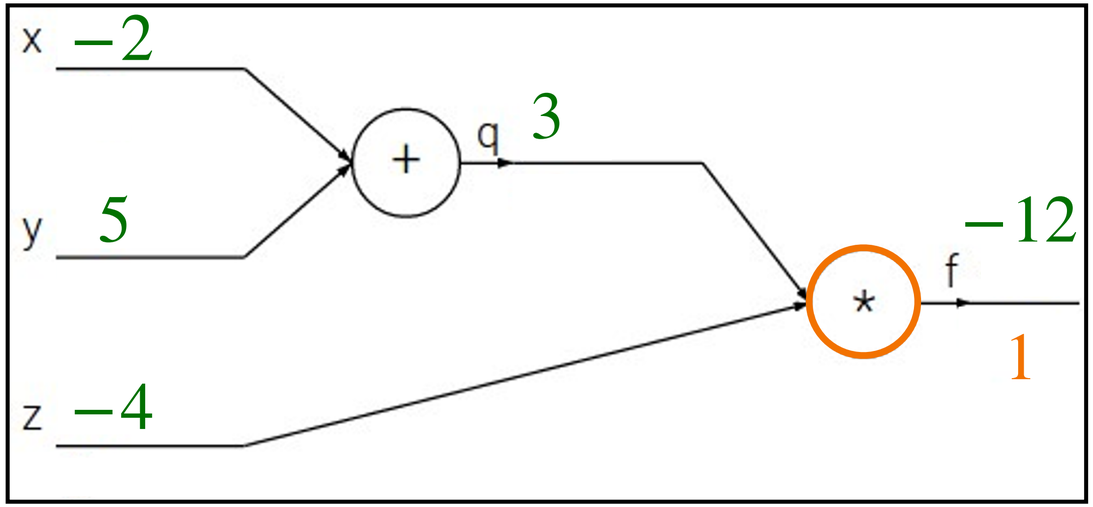

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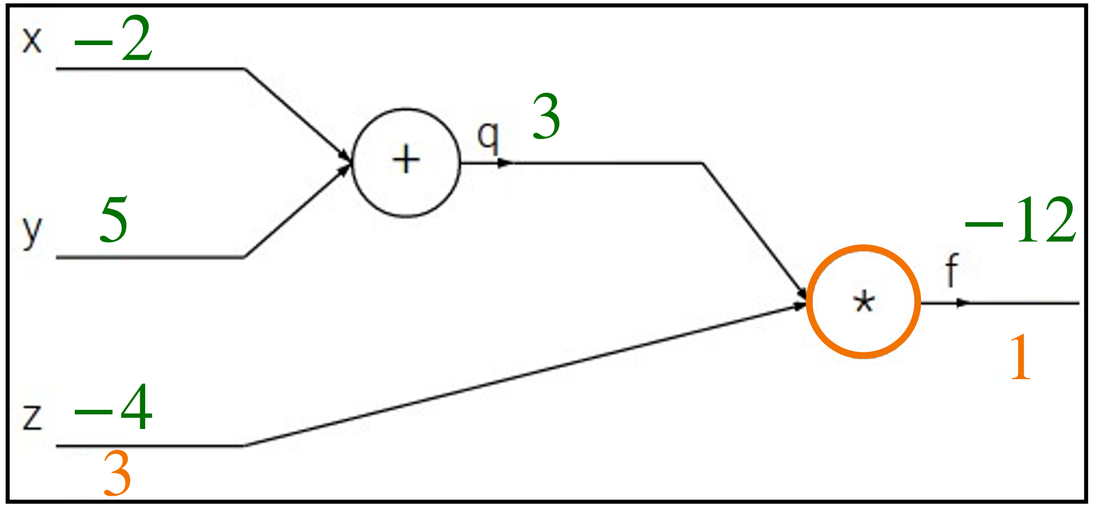

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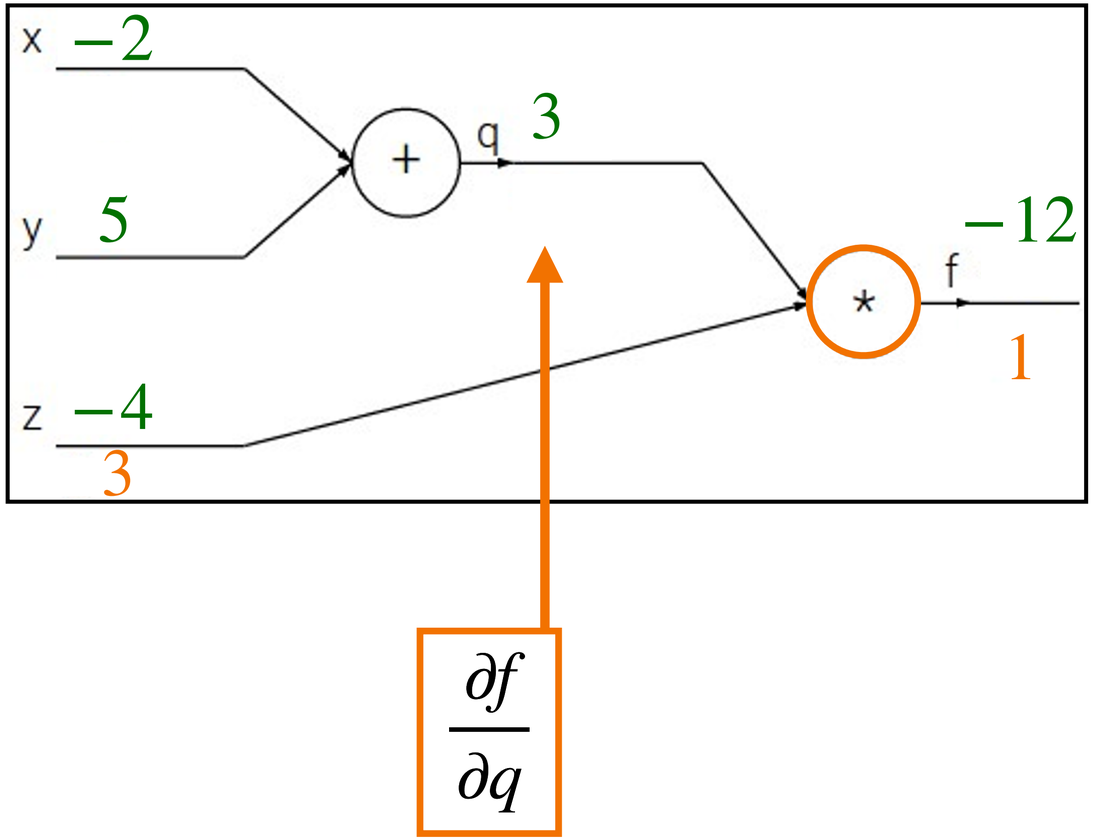

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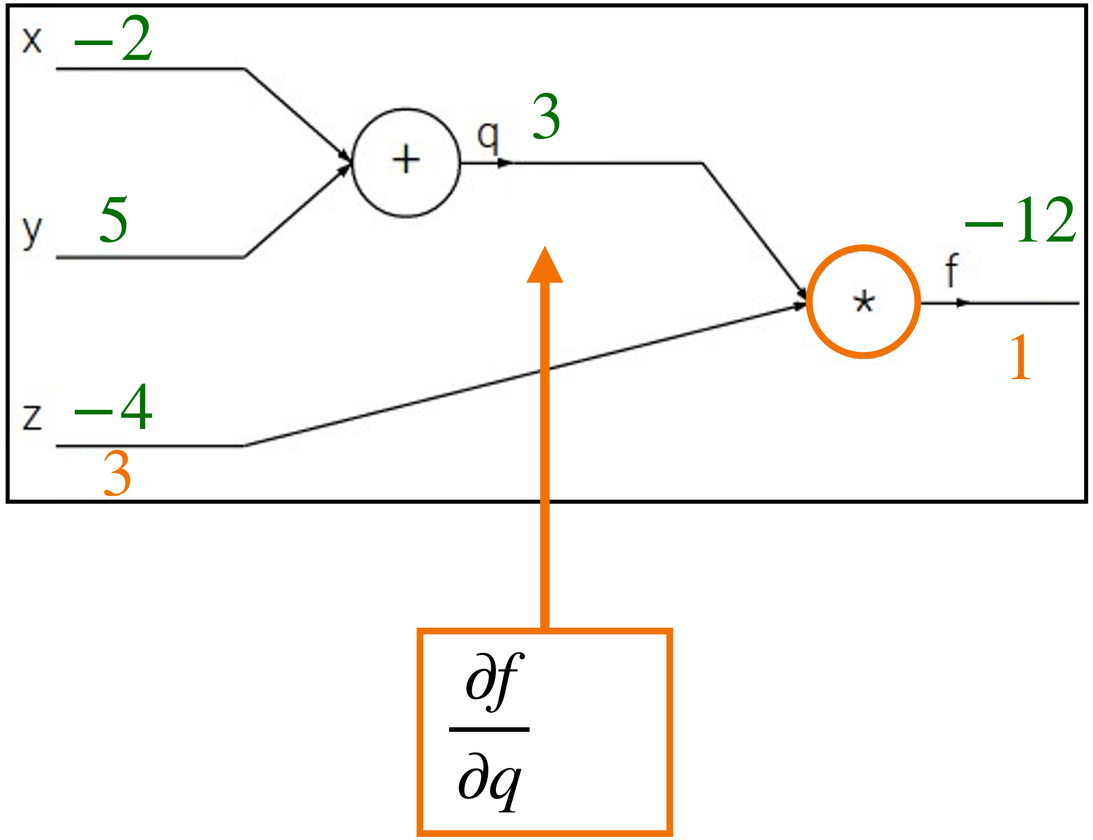

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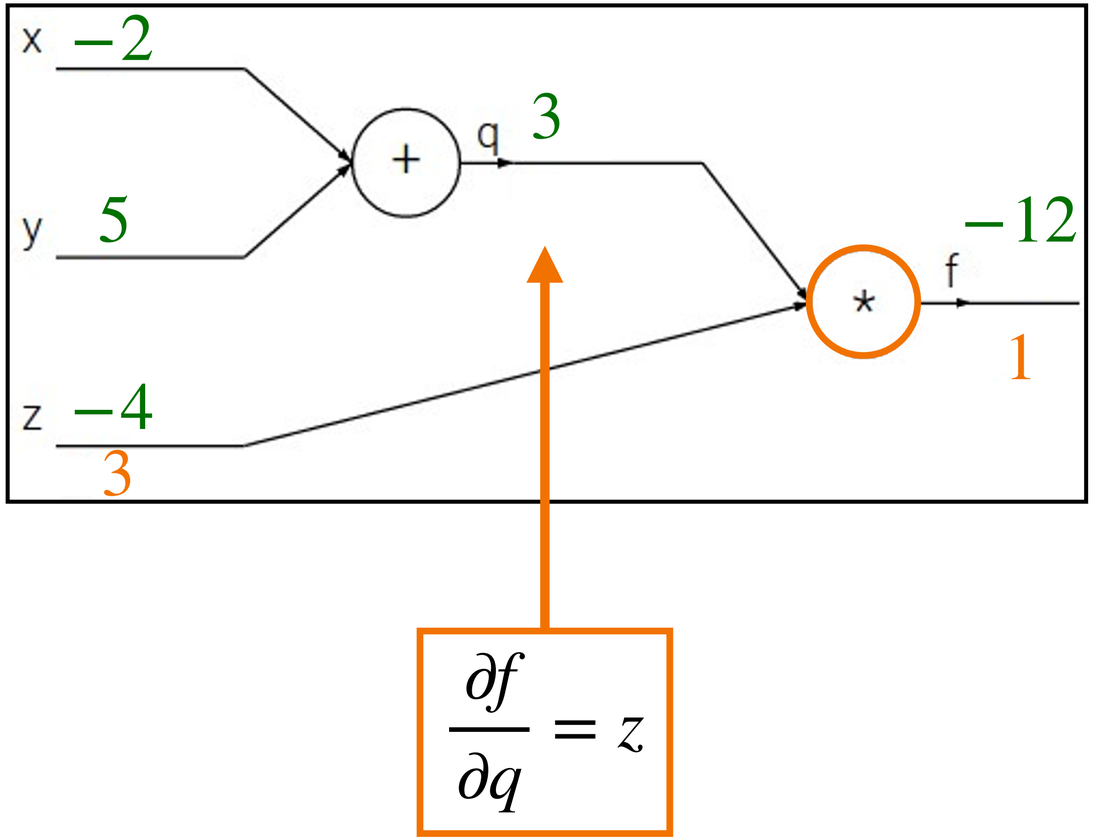

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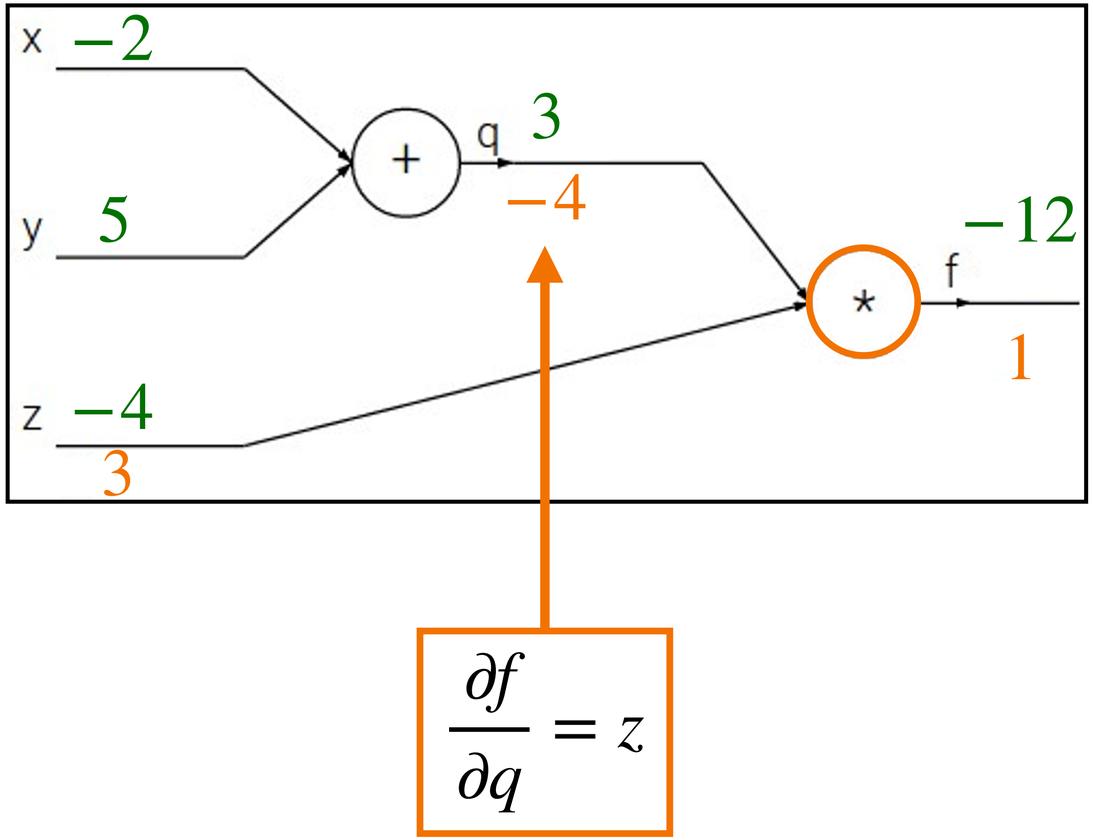

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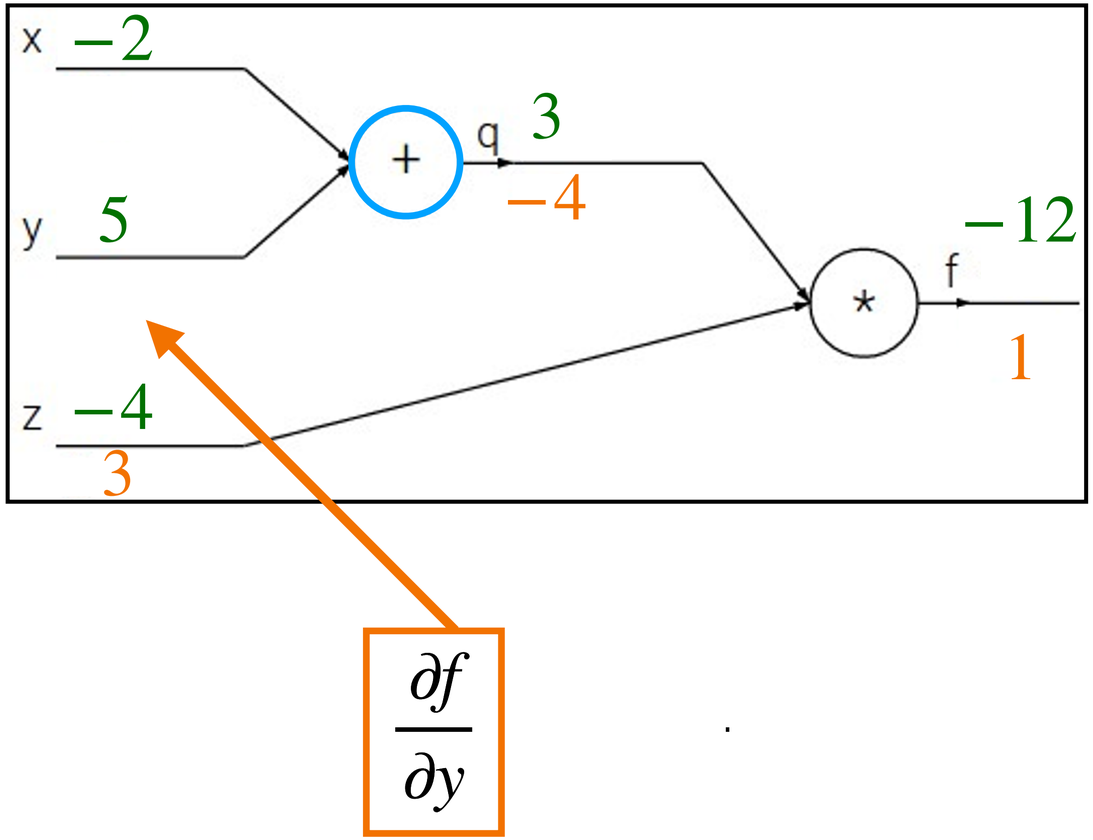

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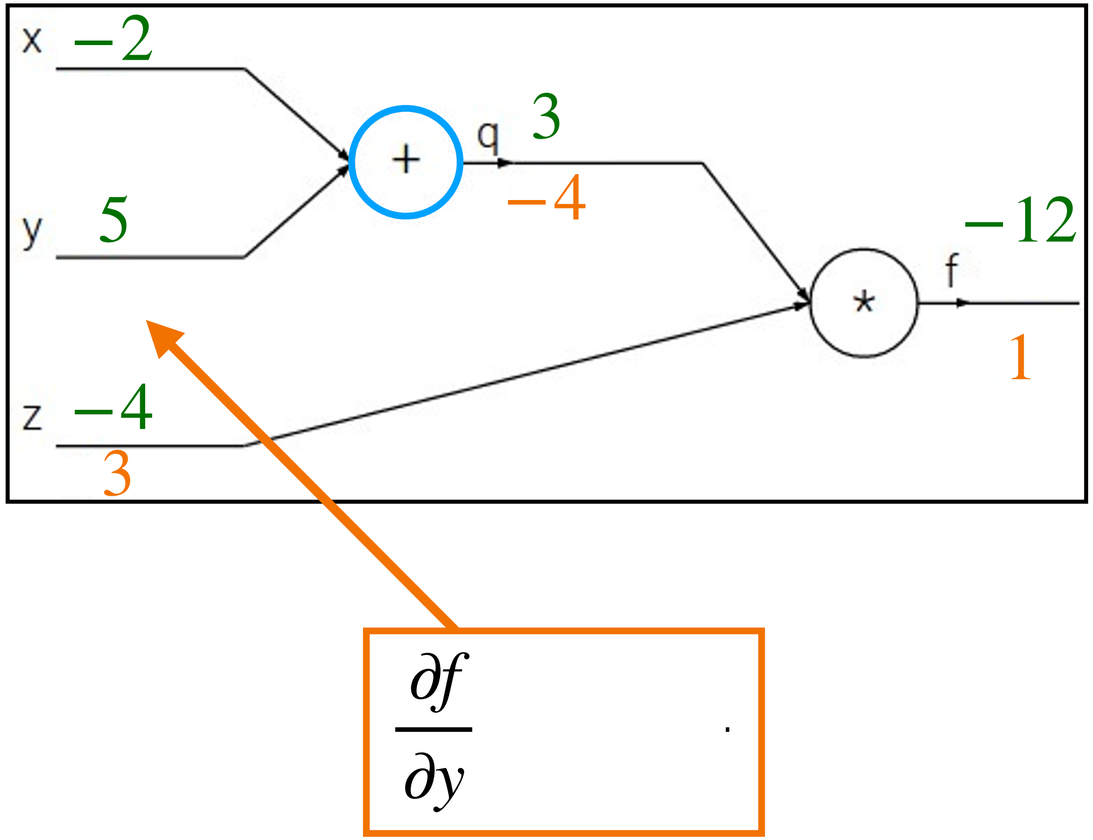

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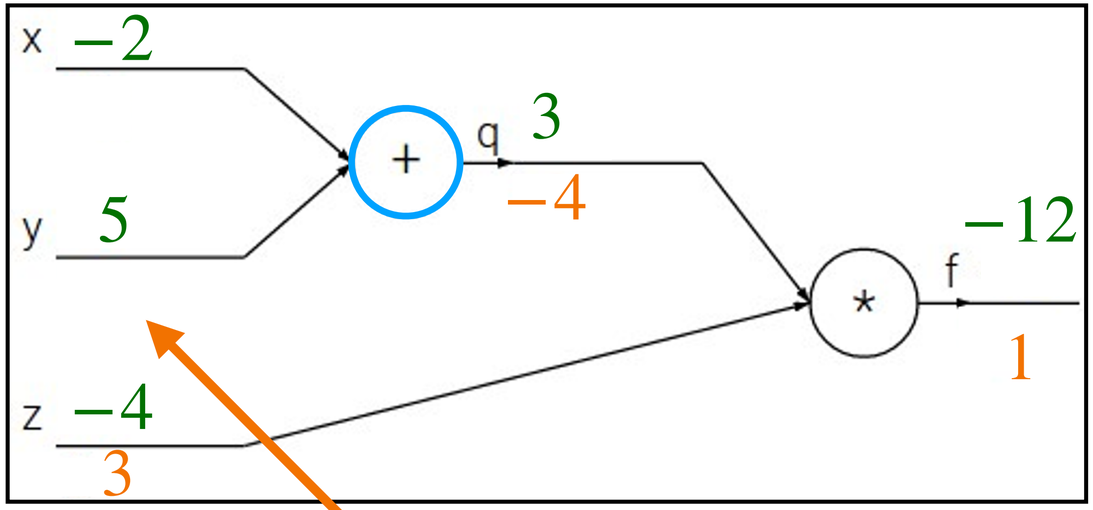

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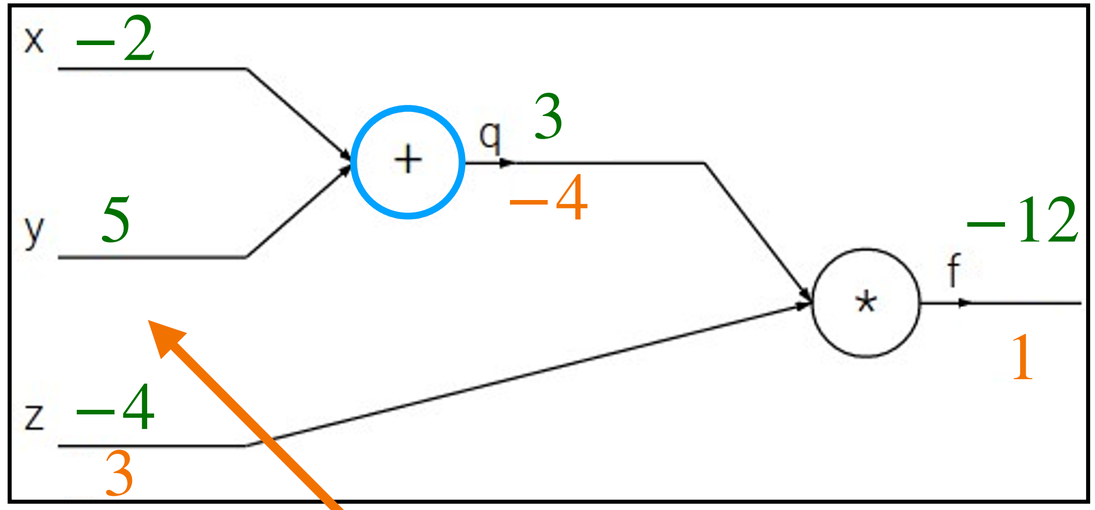

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Downstream Gradient | Local Gradient | Upstream Gradient

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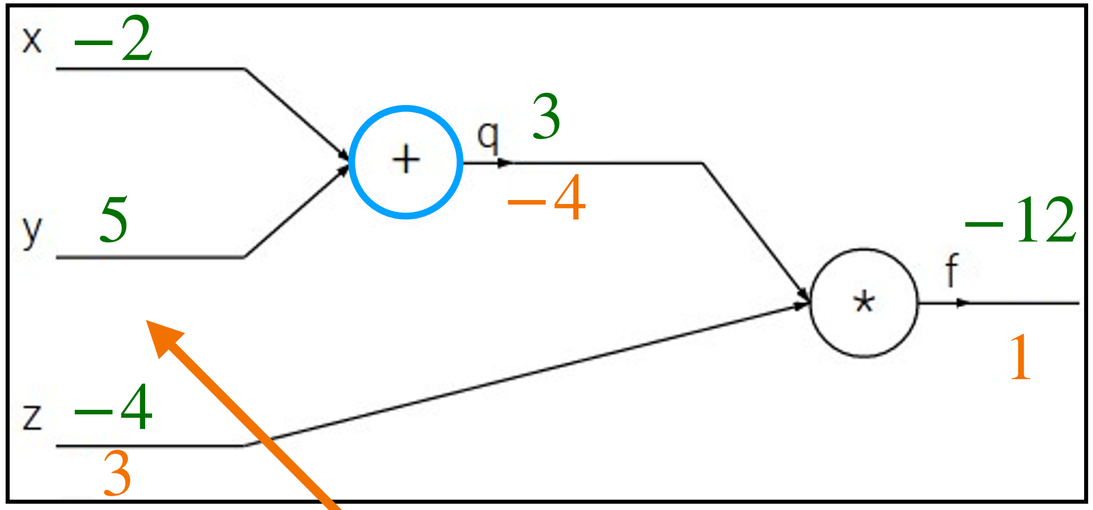

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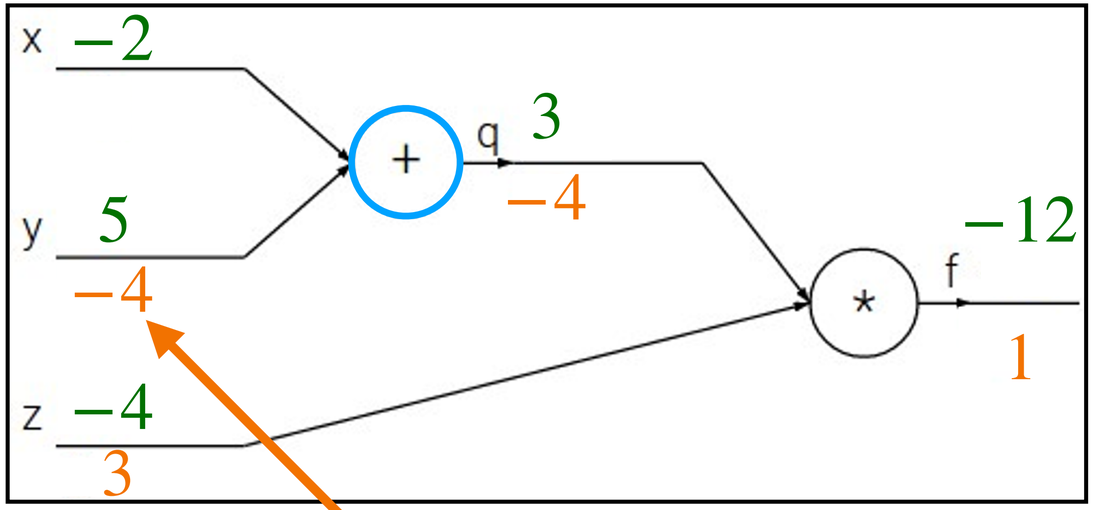

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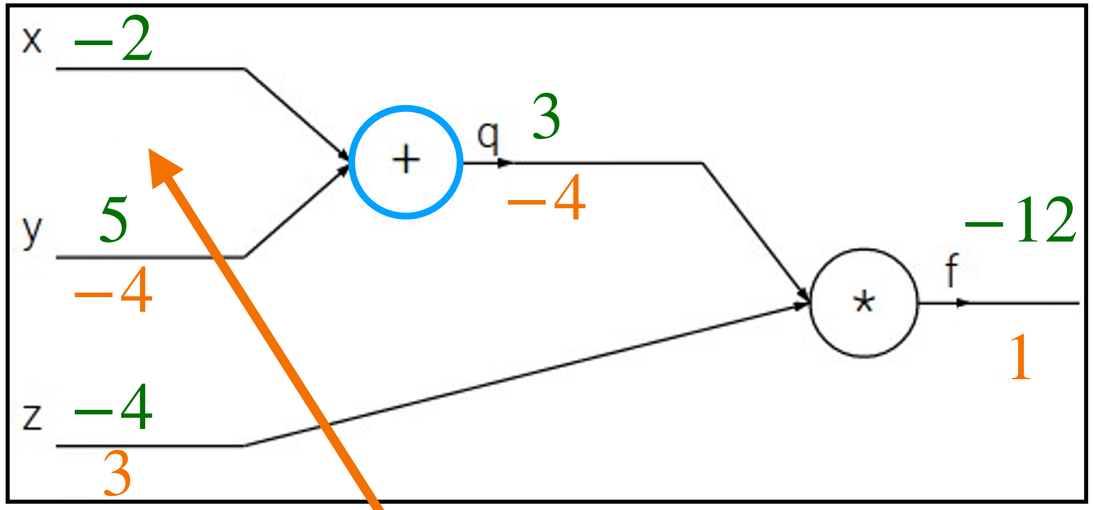
$$q = x + y \qquad f = q \cdot z$$

2. Backward pass: Compute derivatives

Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



$$\frac{\partial f}{\partial x} = \frac{\partial q}{\partial x}\frac{\partial f}{\partial q}$$

$$\frac{\partial q}{\partial x} = 1$$

Downstream Gradient | Local Gradient | Upstream Gradient

Backpropagation: Simple Example

$$f(x, y, z) = (x + y) \cdot z$$

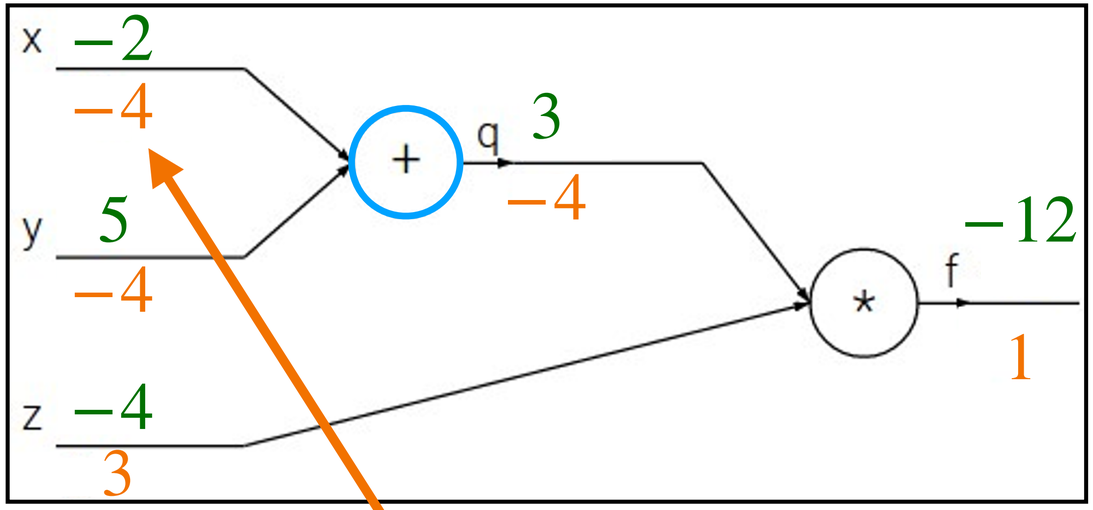
e.g. $x = -2, y = 5, z = -4$

1. Forward pass: Compute outputs

$$q = x + y \qquad f = q \cdot z$$

2. Backward pass: Compute derivatives

Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



$$\frac{\partial f}{\partial x} = \frac{\partial q}{\partial x}\frac{\partial f}{\partial q}$$

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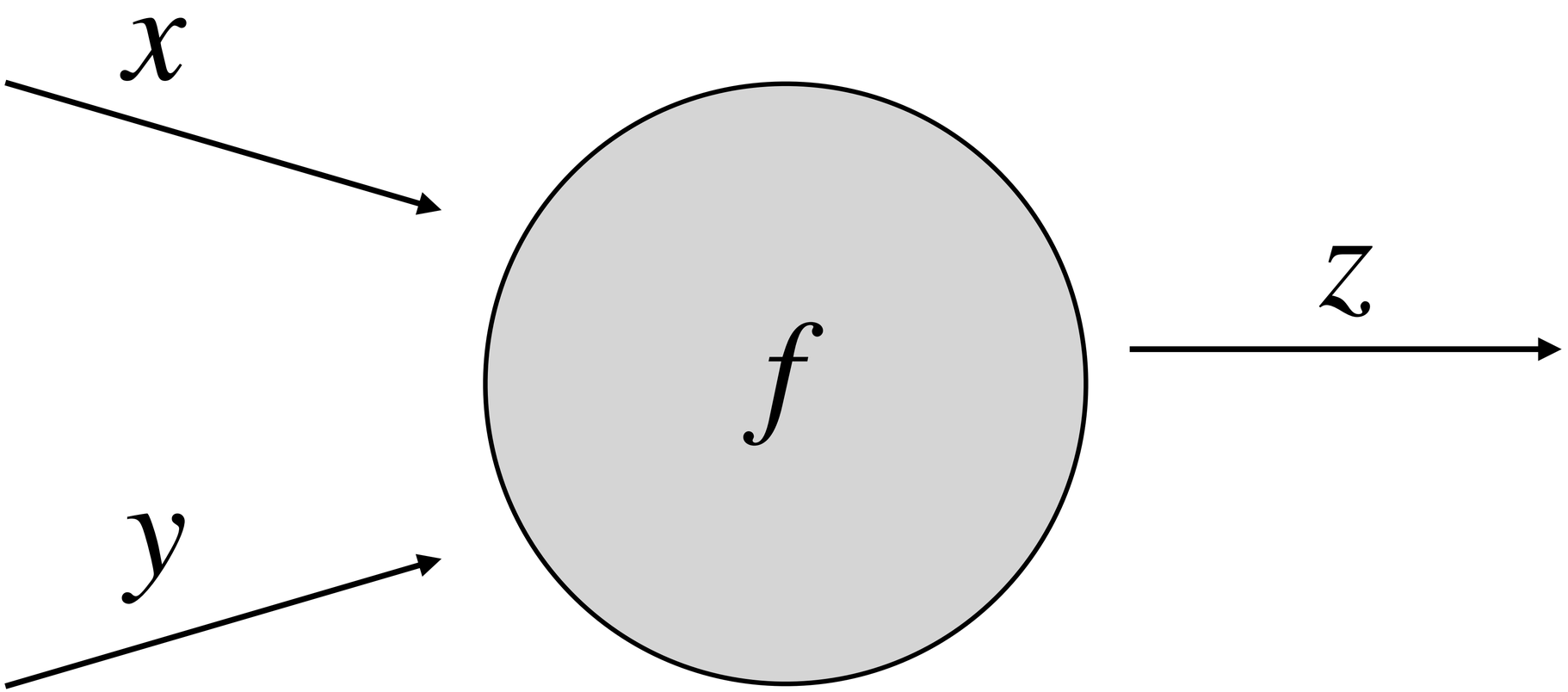
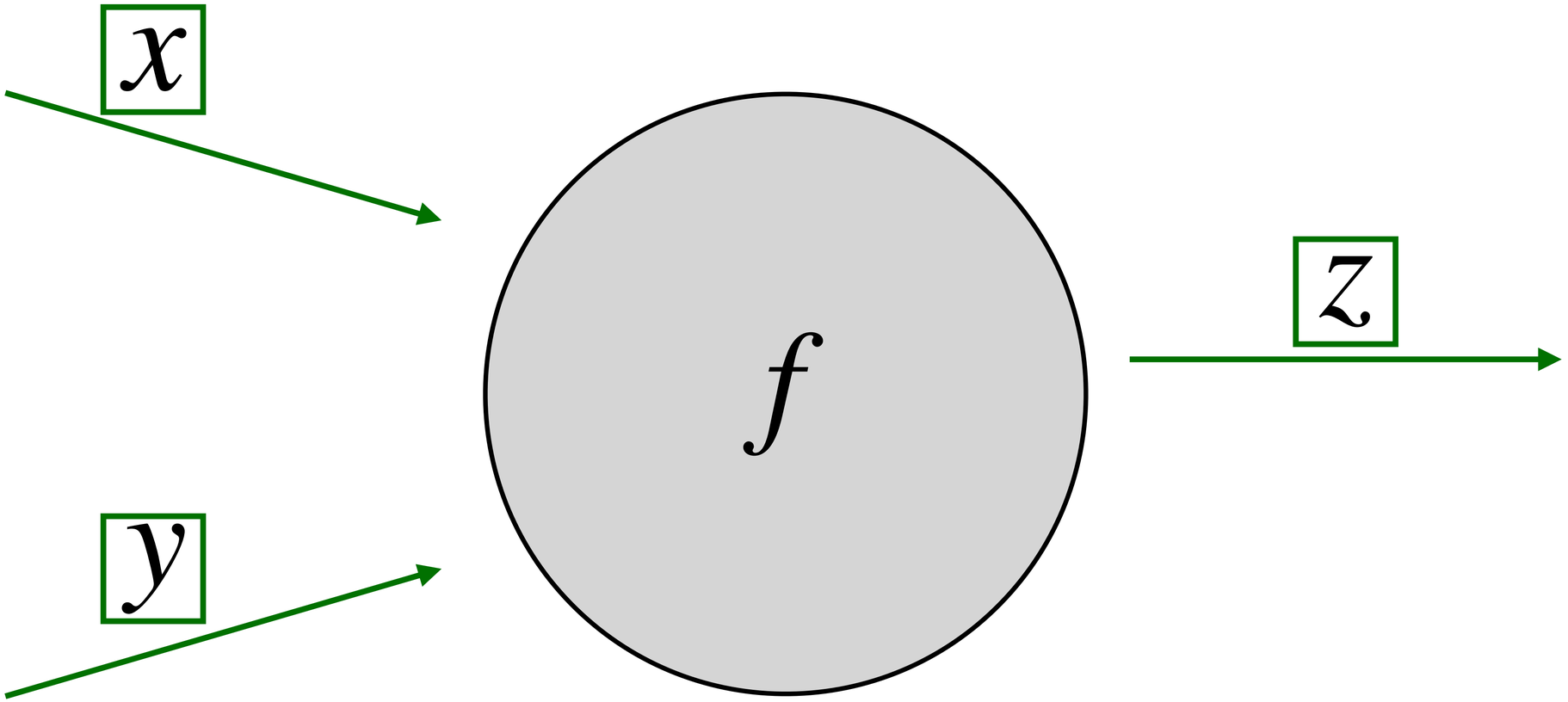
Downstream Gradient | Local Gradient | Upstream Gradient

Generalizing Backpropagation



Generalizing Backpropagation



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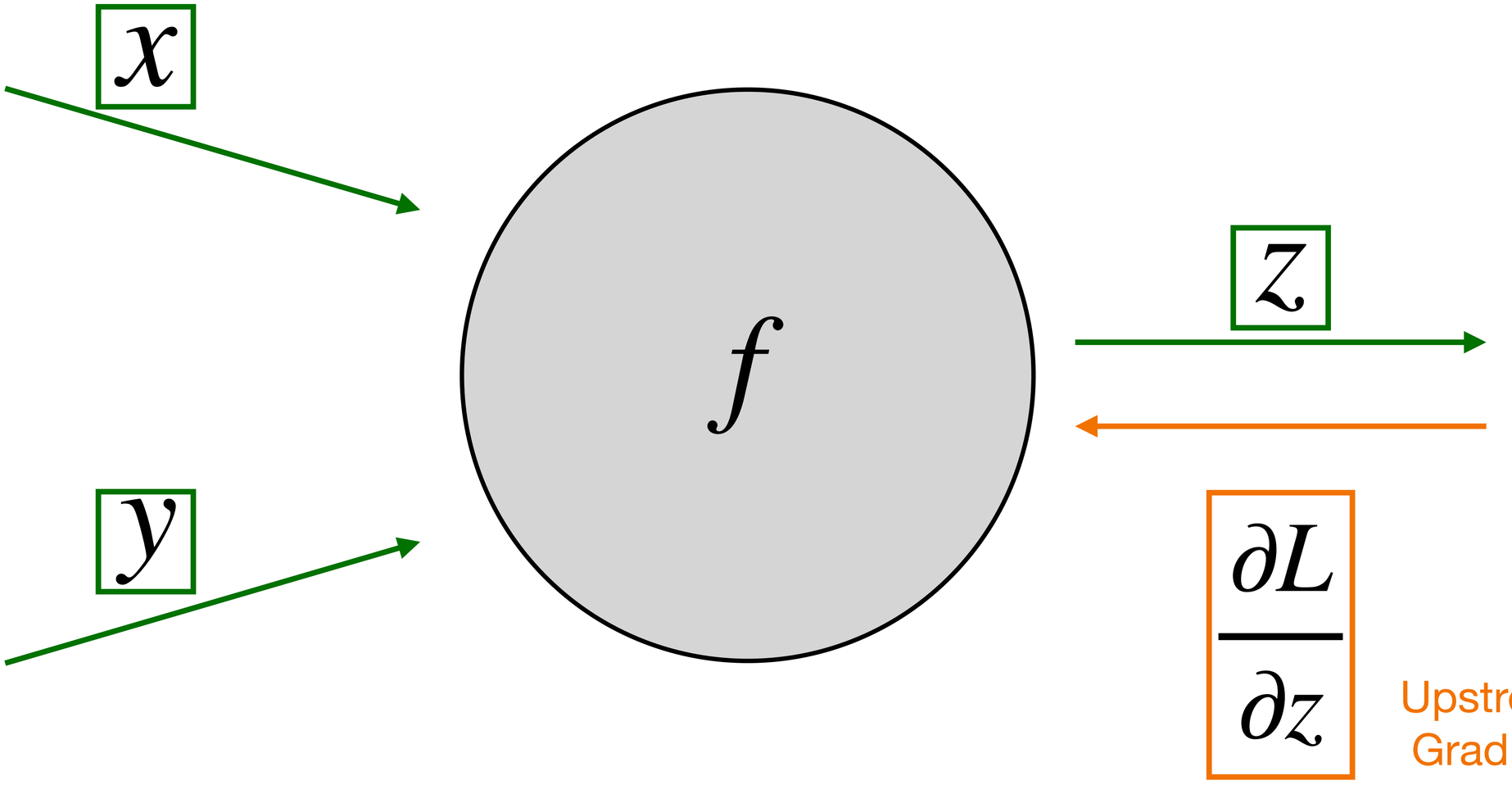

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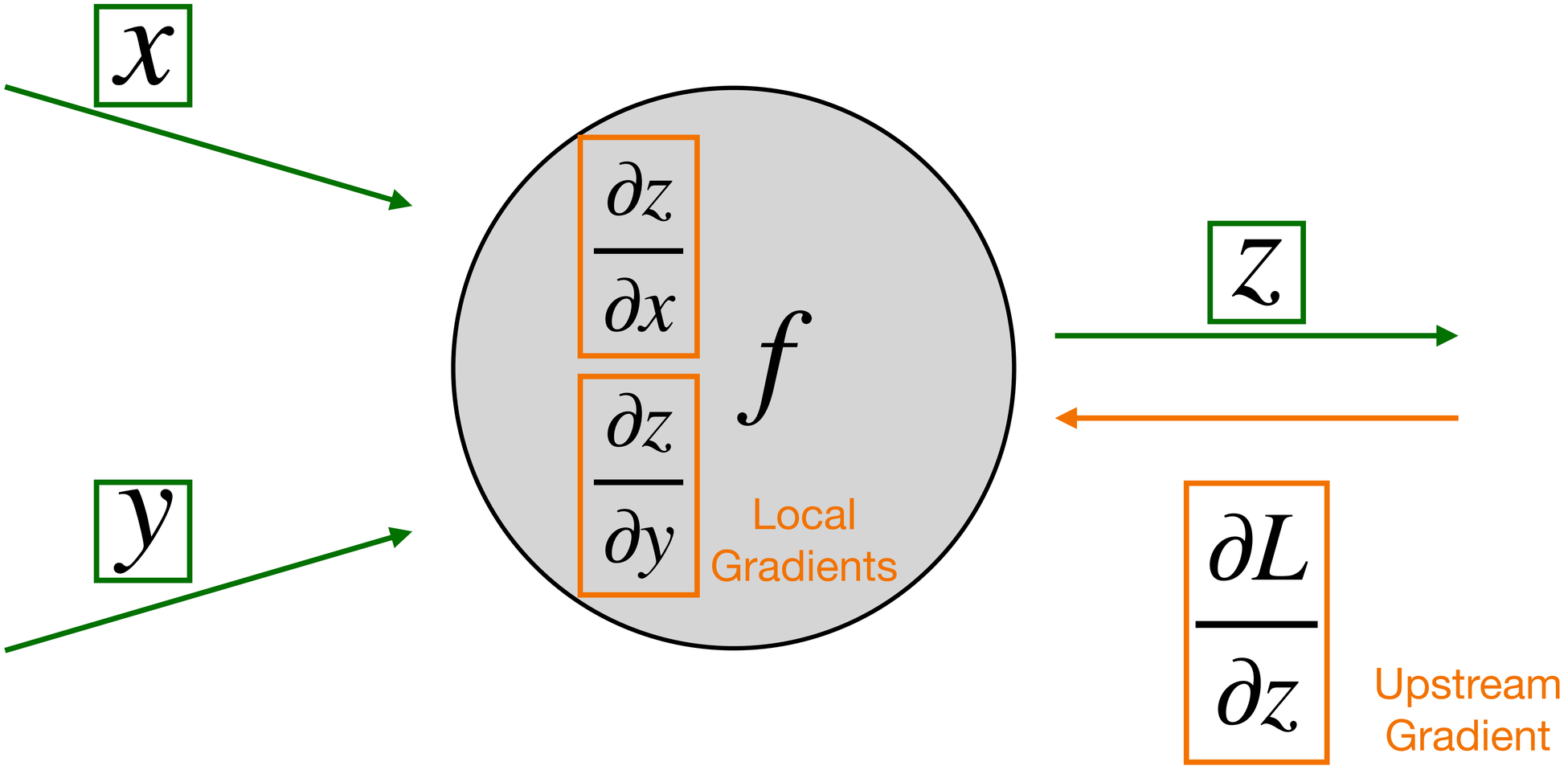
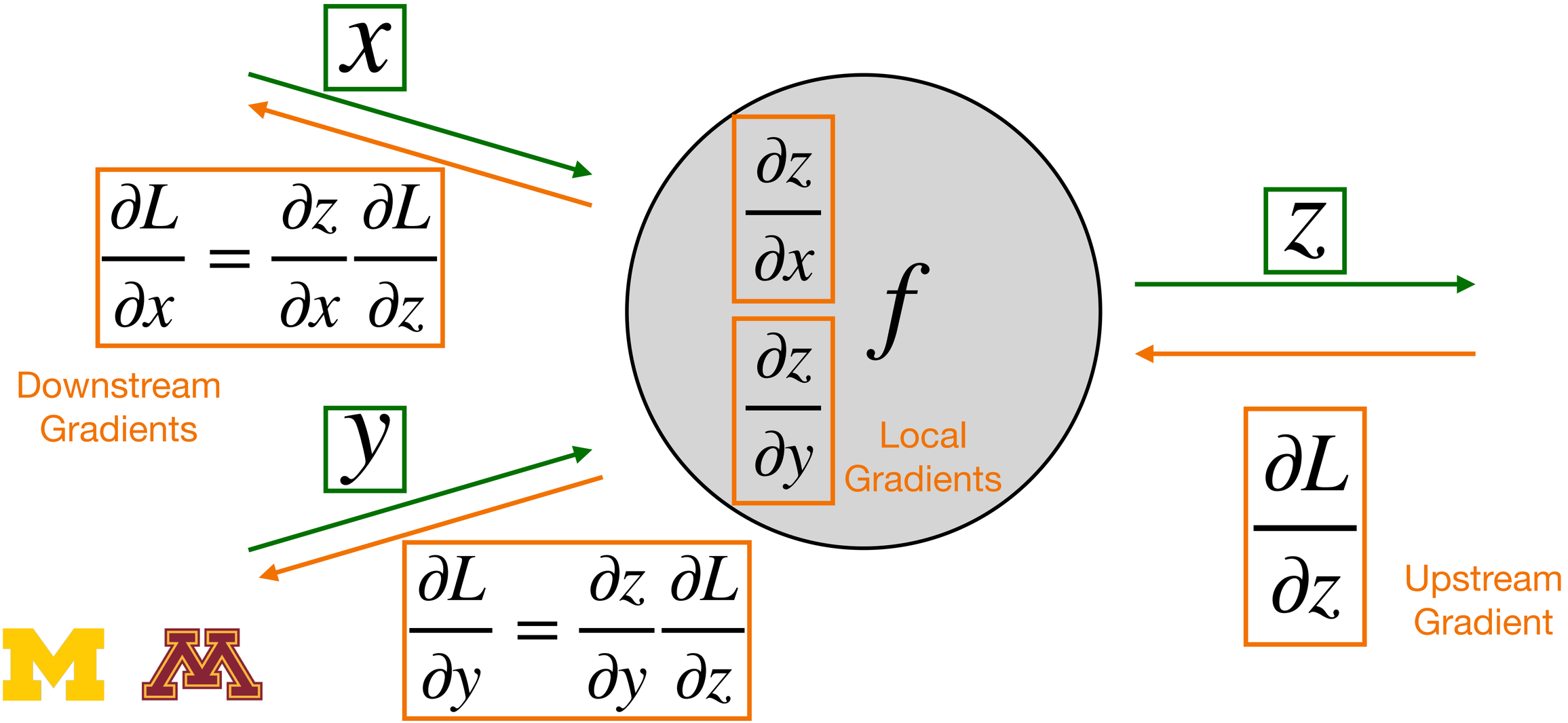


Generalizing Backpropagation



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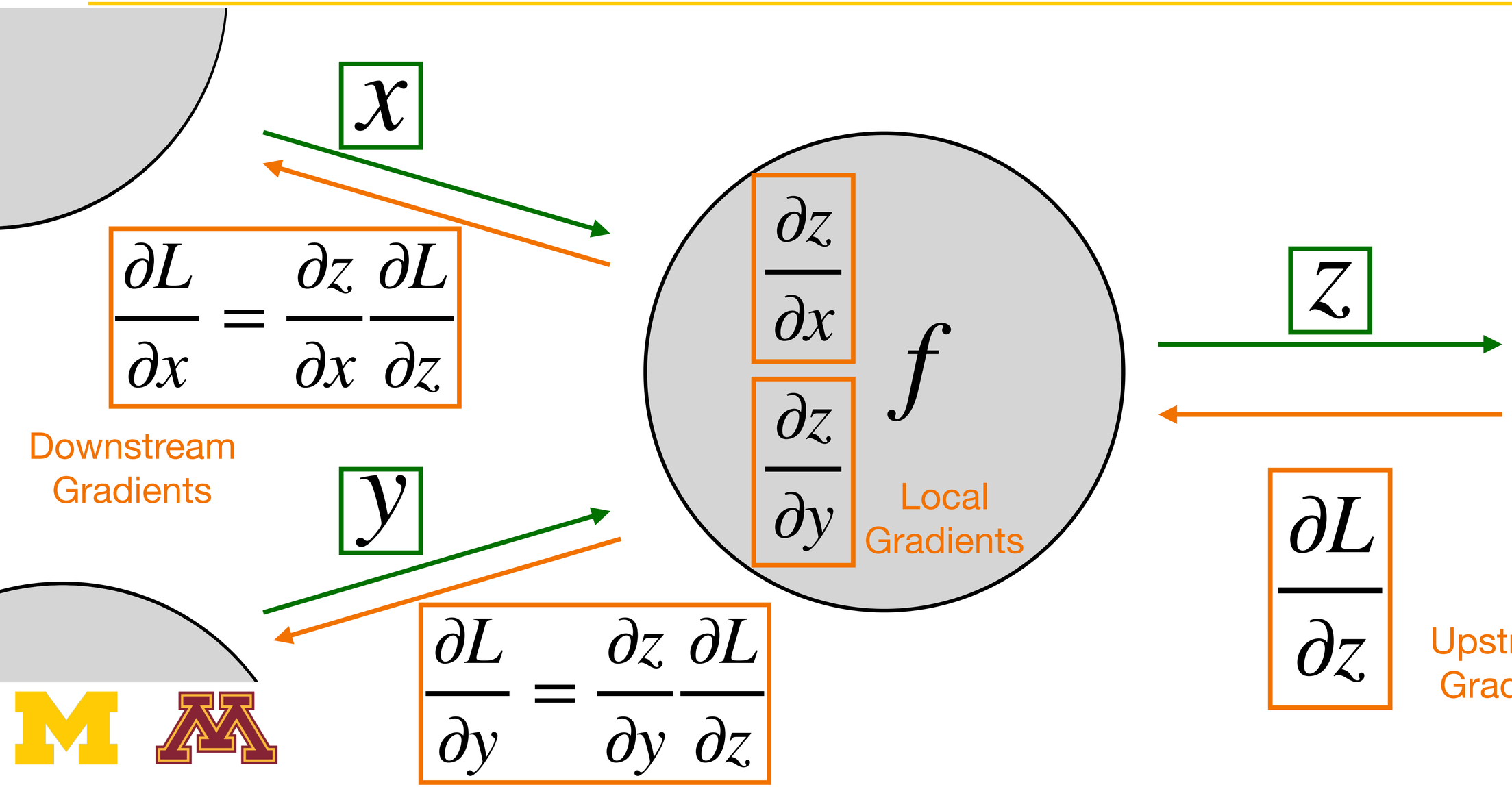

Another example

$$f(x, w) = \frac{1}{1 + e^{-(w_0 x_0 + w_1 x_1 + w2)}}$$

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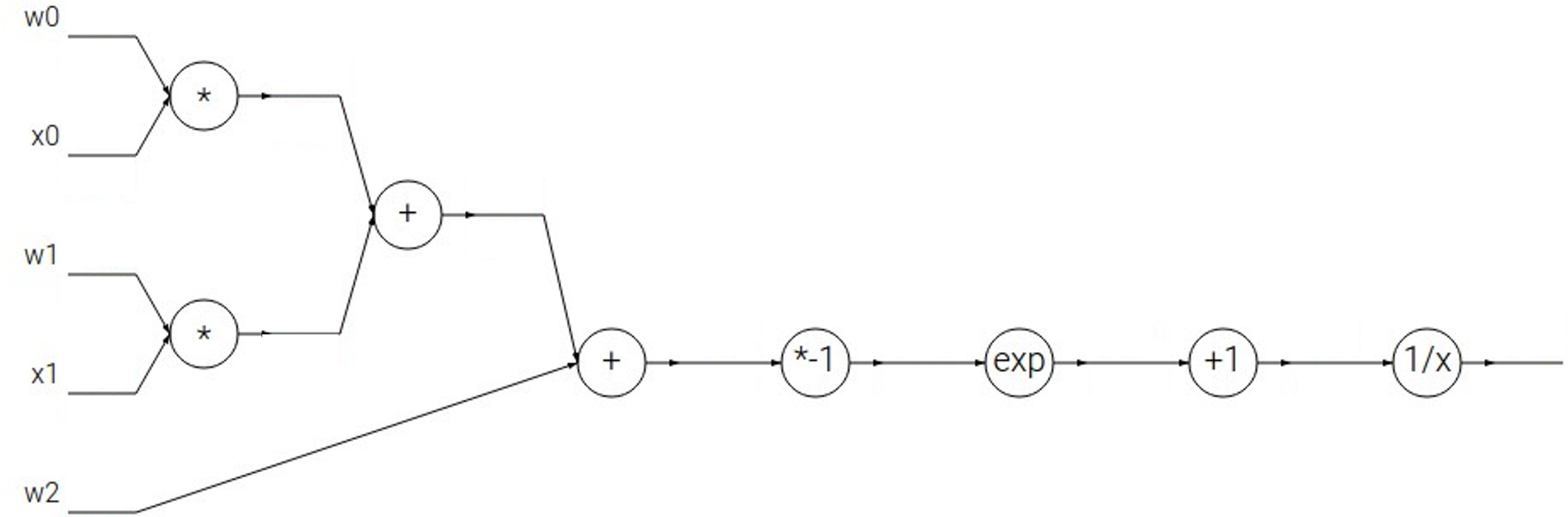

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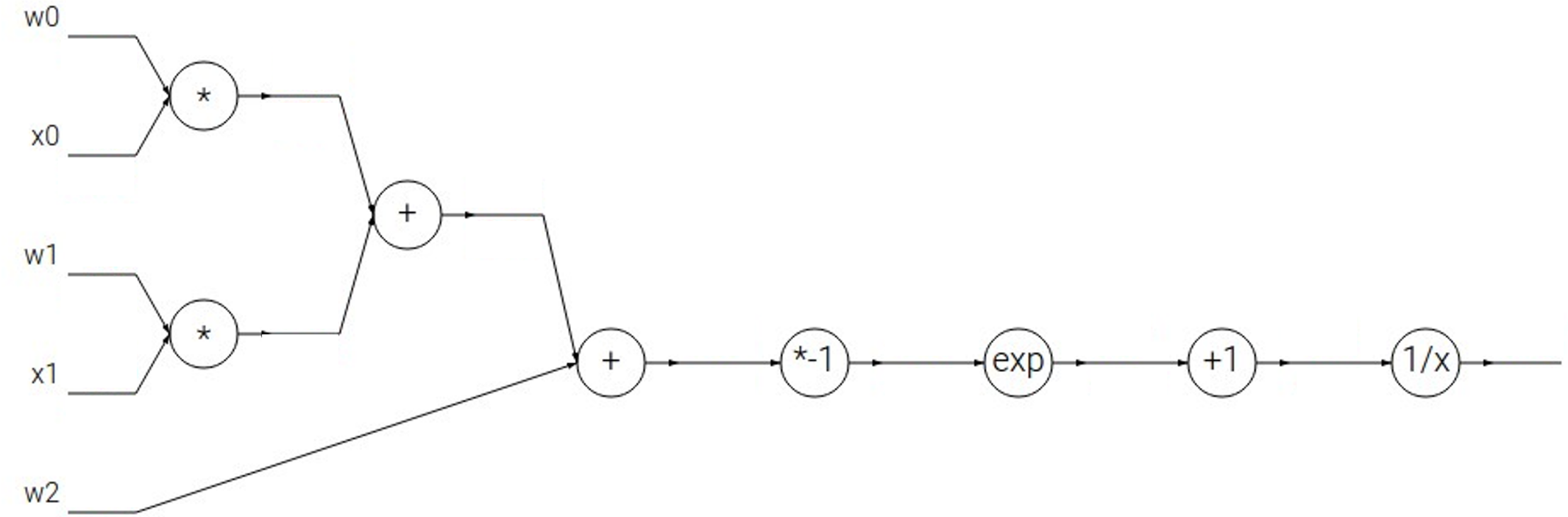

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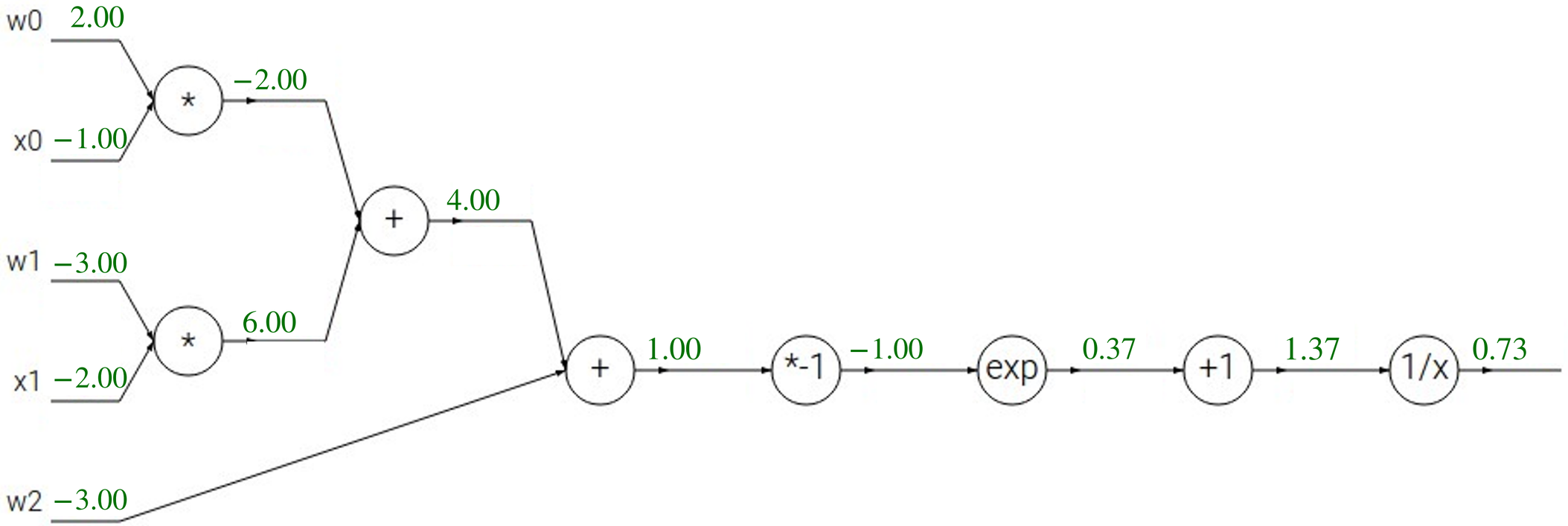
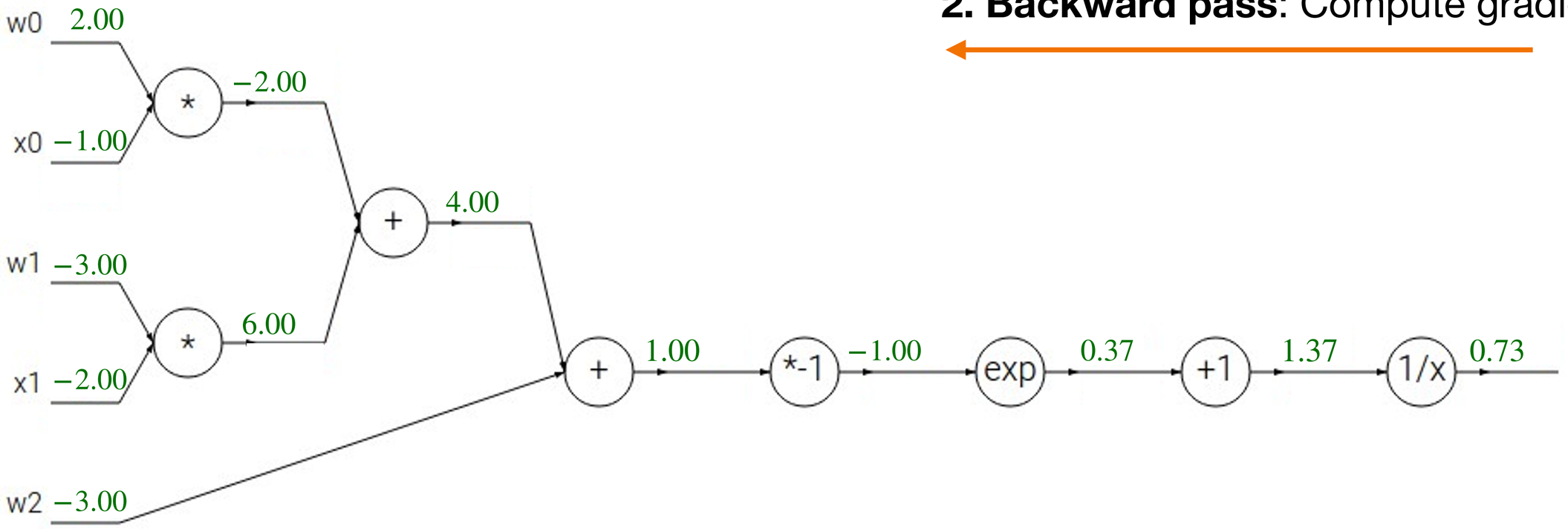


Another example

$$f(x, w) = \frac{1}{1 + e^{-(w_0 x_0 + w_1 x_1 + w2)}}$$

1. Forward pass: Compute outputs

2. Backward pass: Compute gradients



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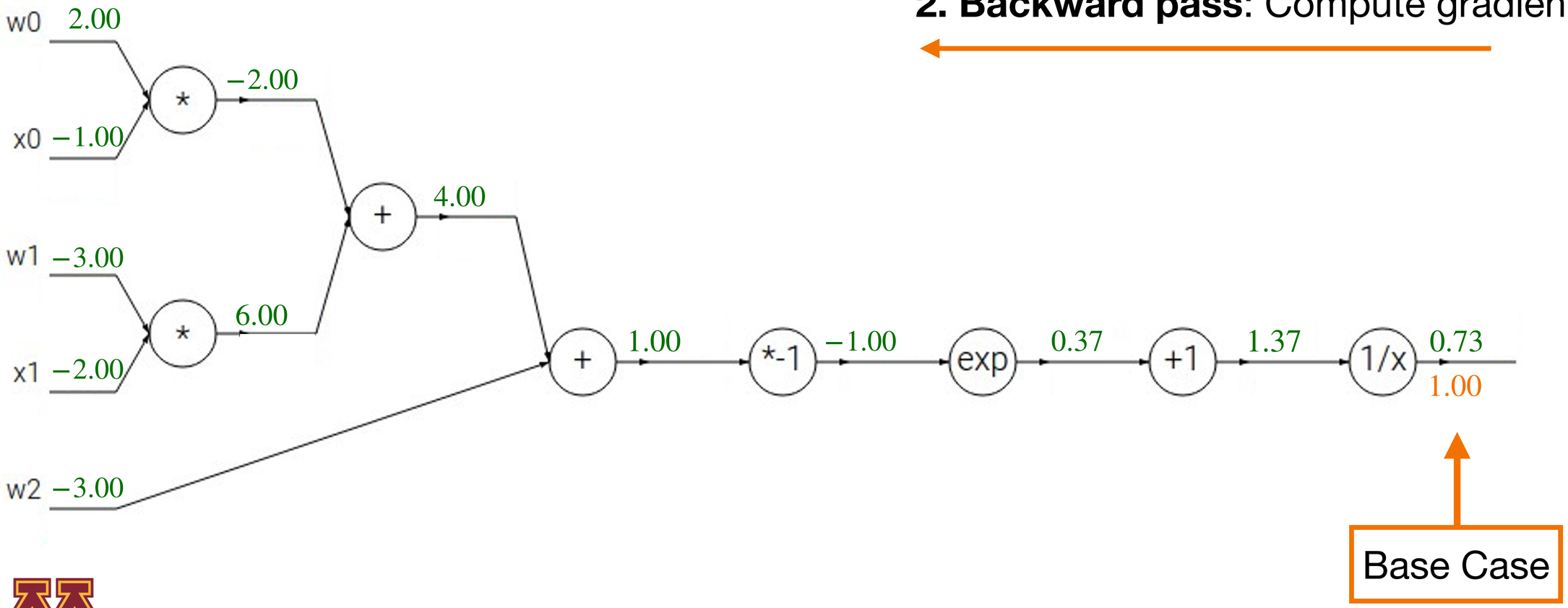

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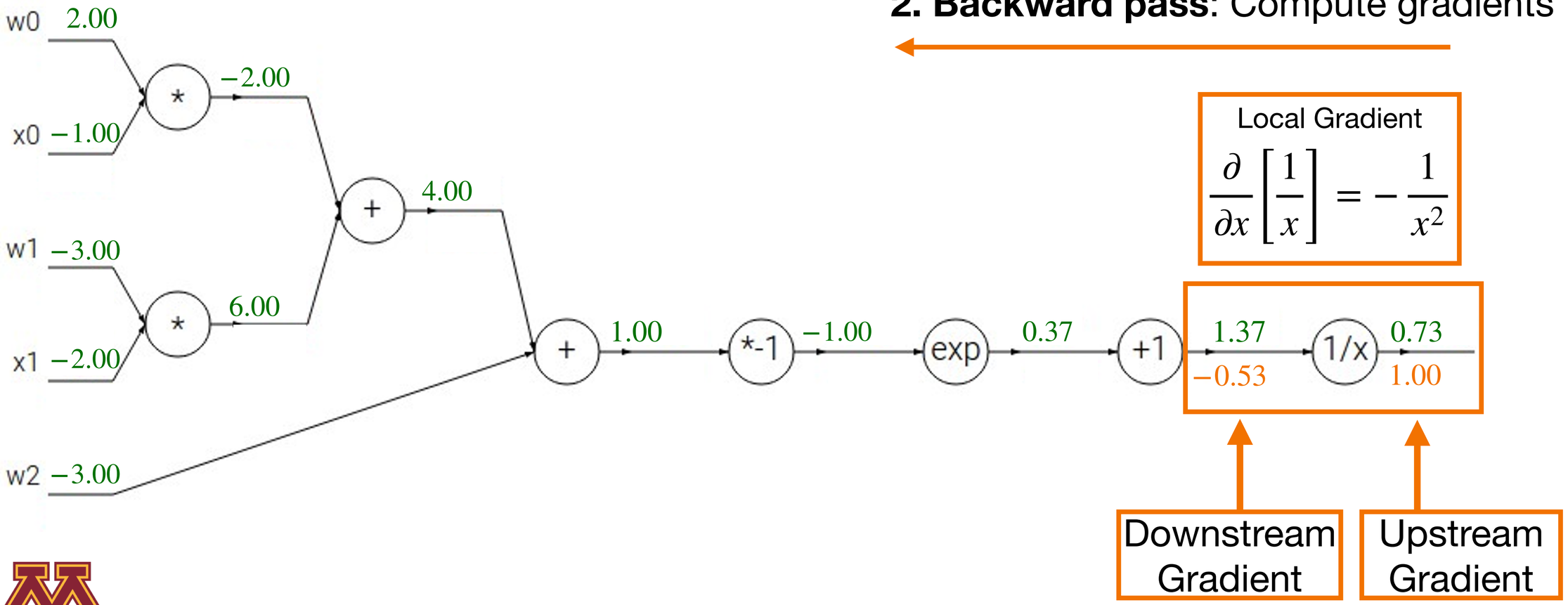

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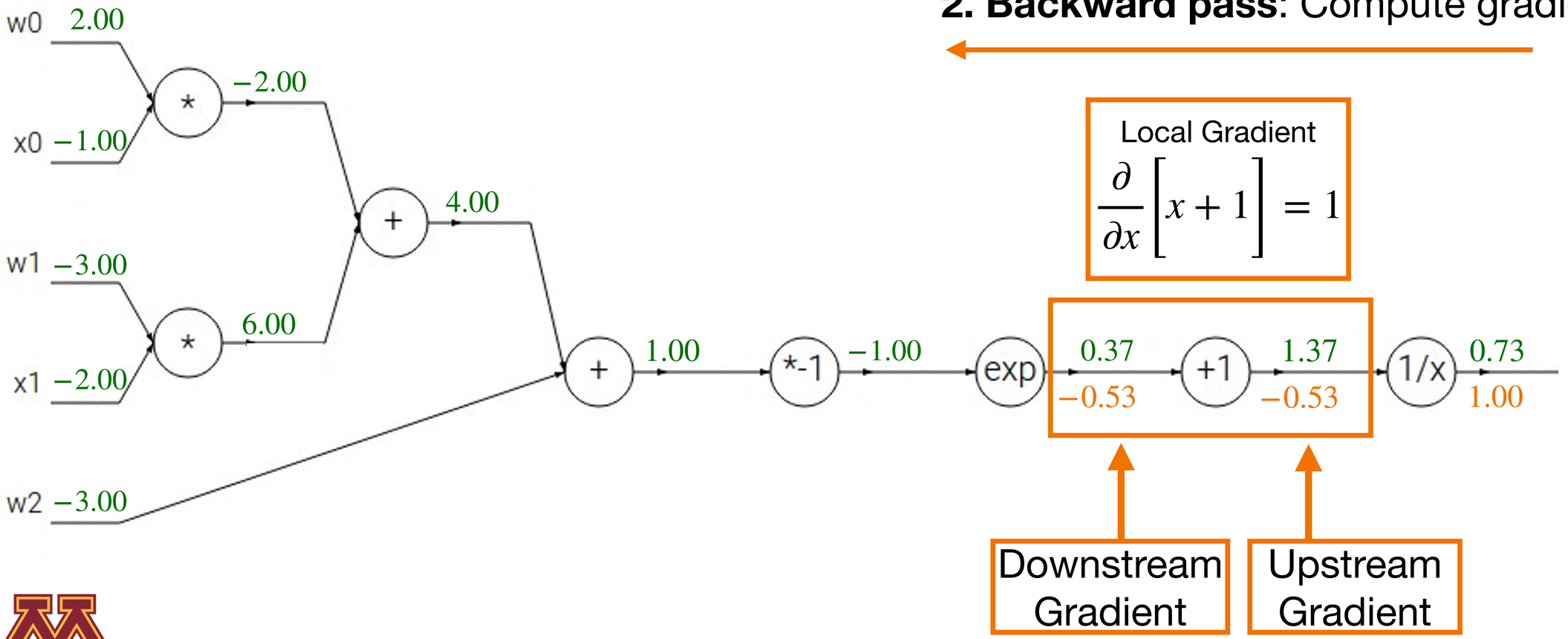

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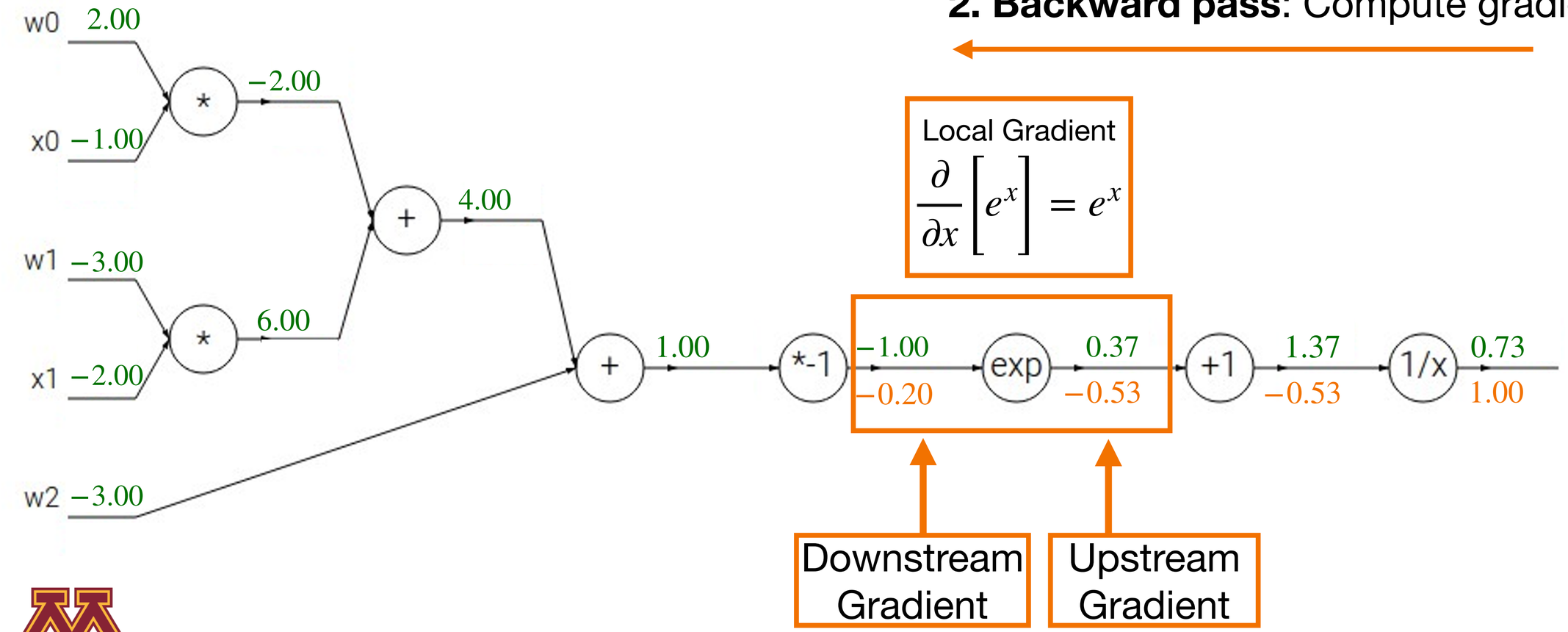

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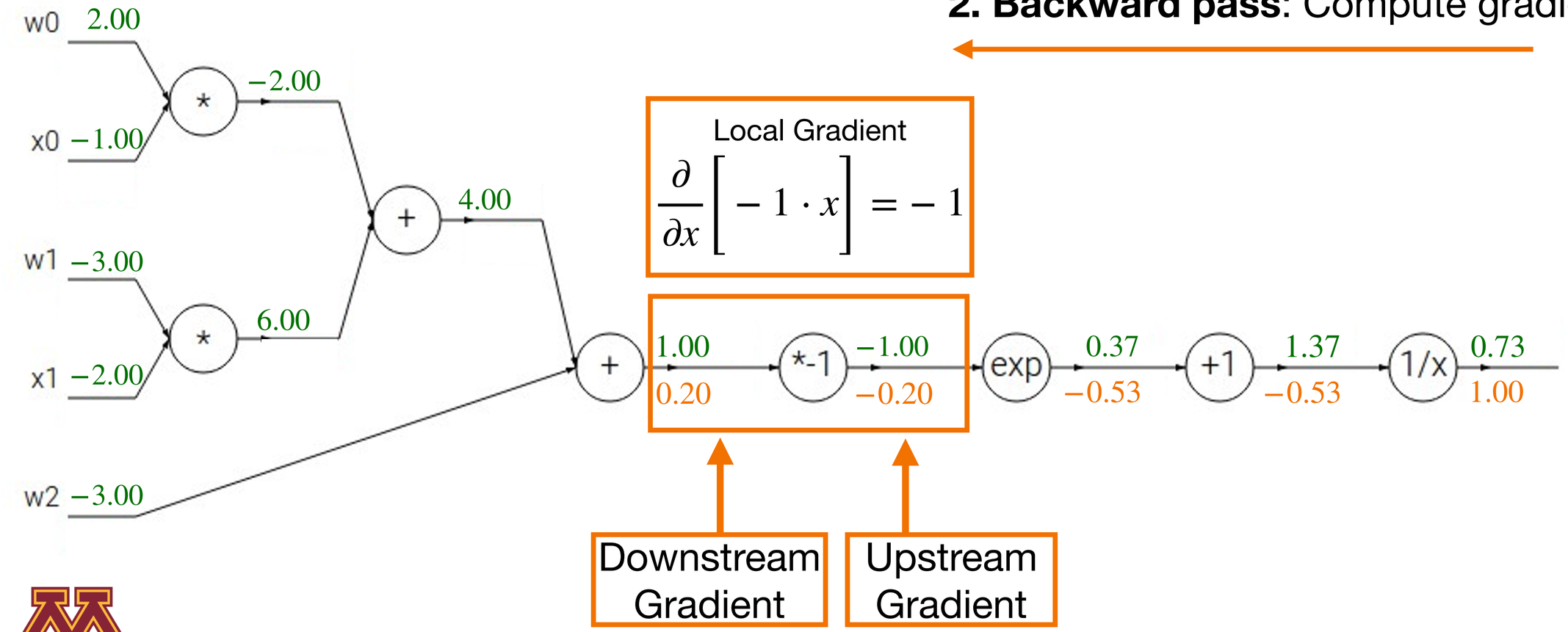

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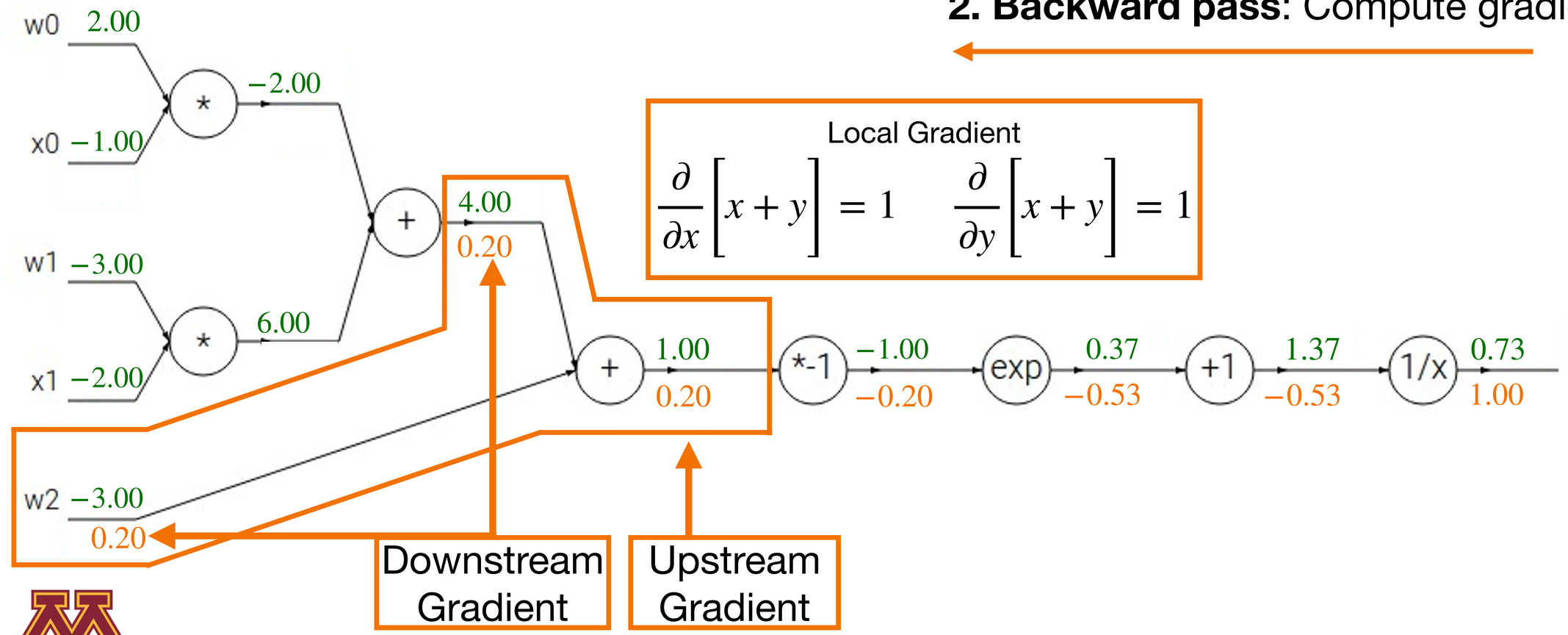

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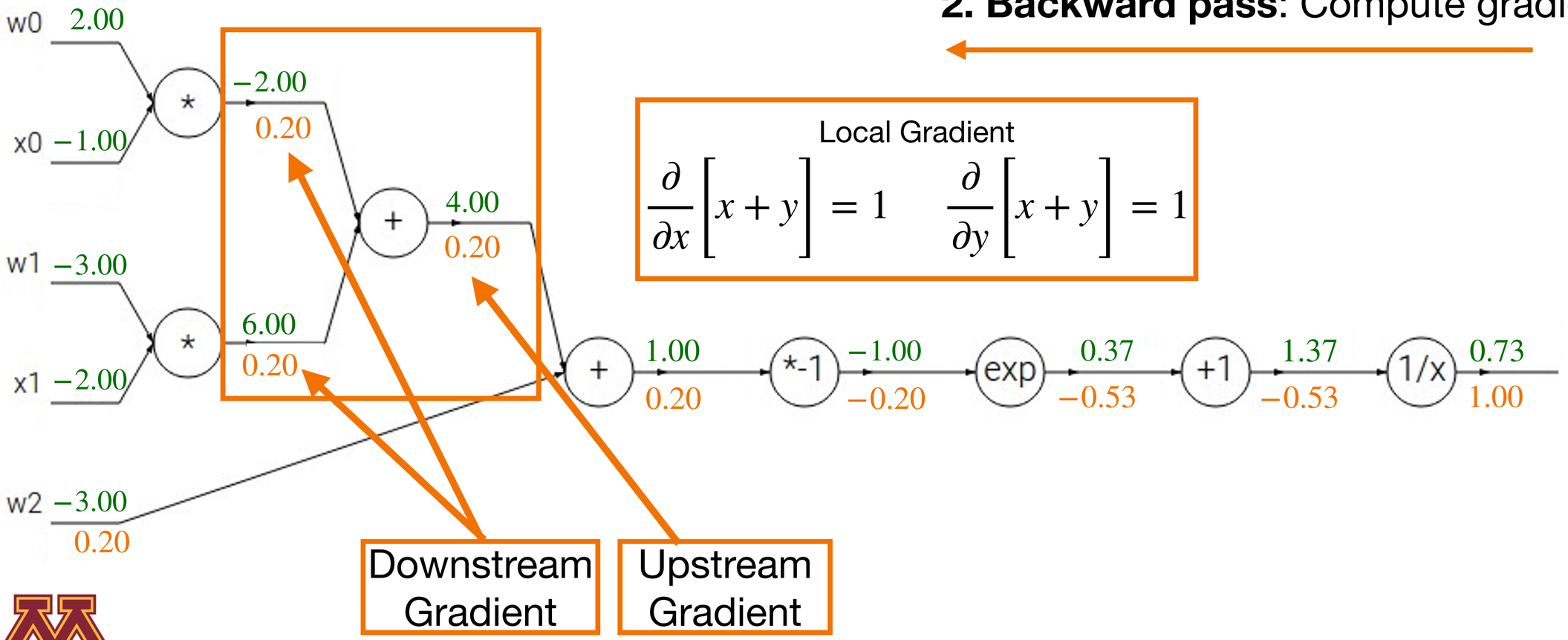

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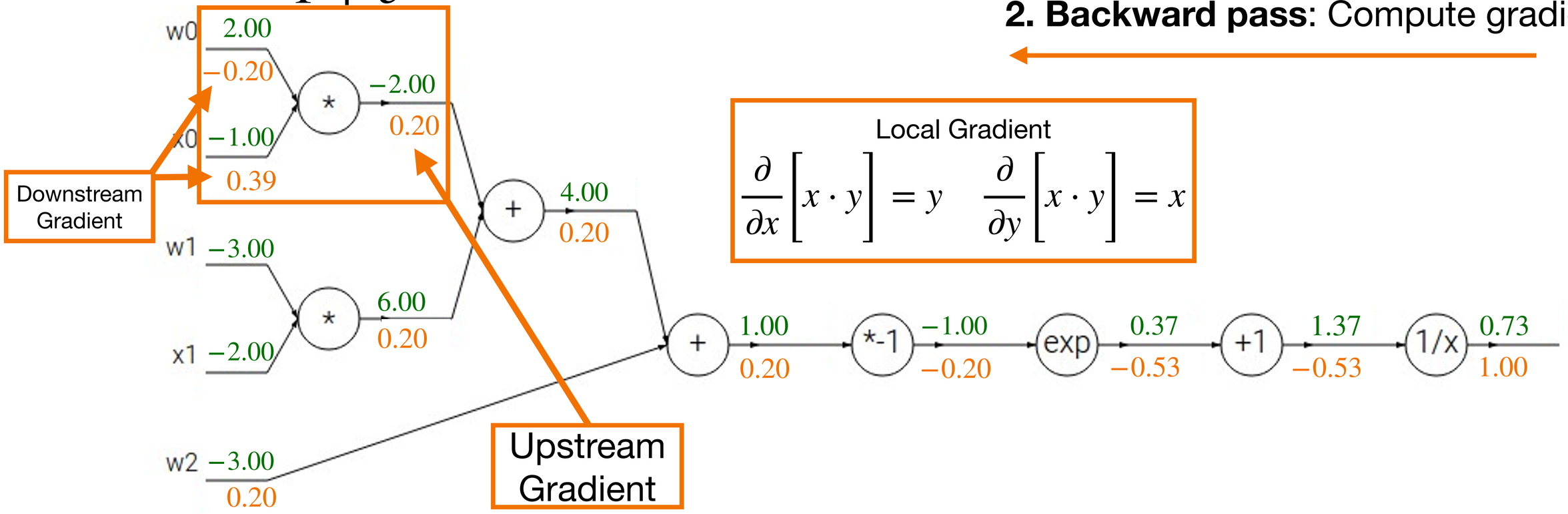

Local Gradient

$$\frac{\partial}{\partial x}\left[x \cdot y\right] = y \qquad \frac{\partial}{\partial y}\left[x \cdot y\right] = x$$

Another example

$$f(x, w) = \frac{1}{1 + e^{-(w_0 x_0 + w_1 x_1 + w2)}}$$

1. Forward pass: Compute outputs

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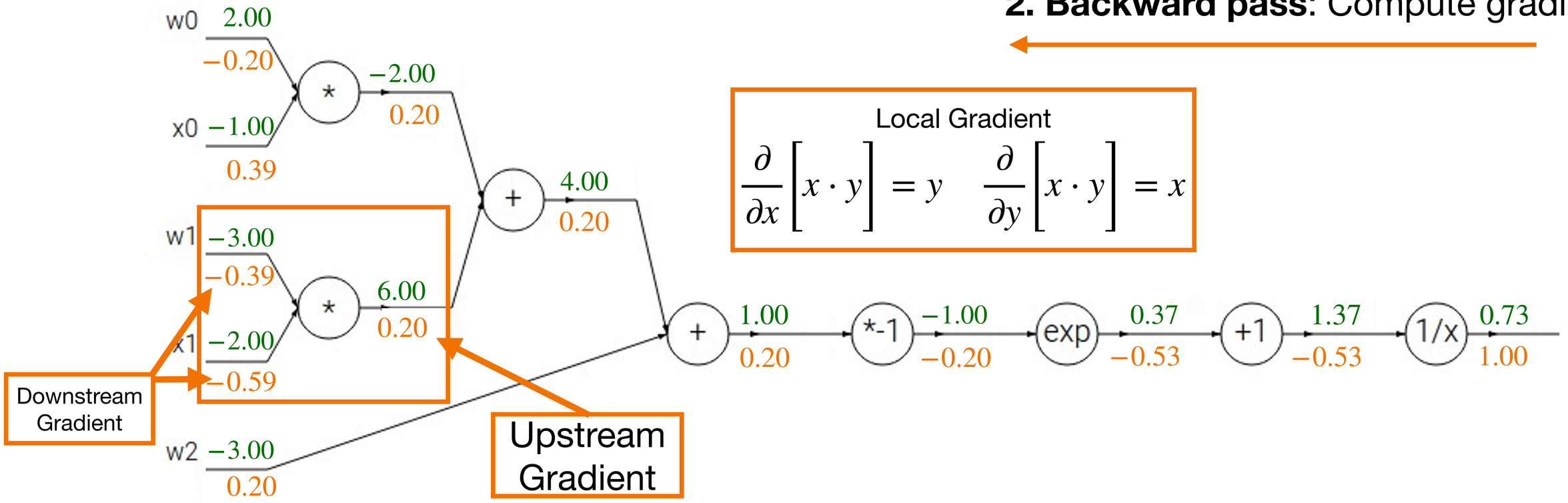

Another example

$$f(x,w) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w2)}}$$

$$= \sigma(w_0x_0 + w_1x_1 + w2)$$

1. Forward pass: Compute outputs

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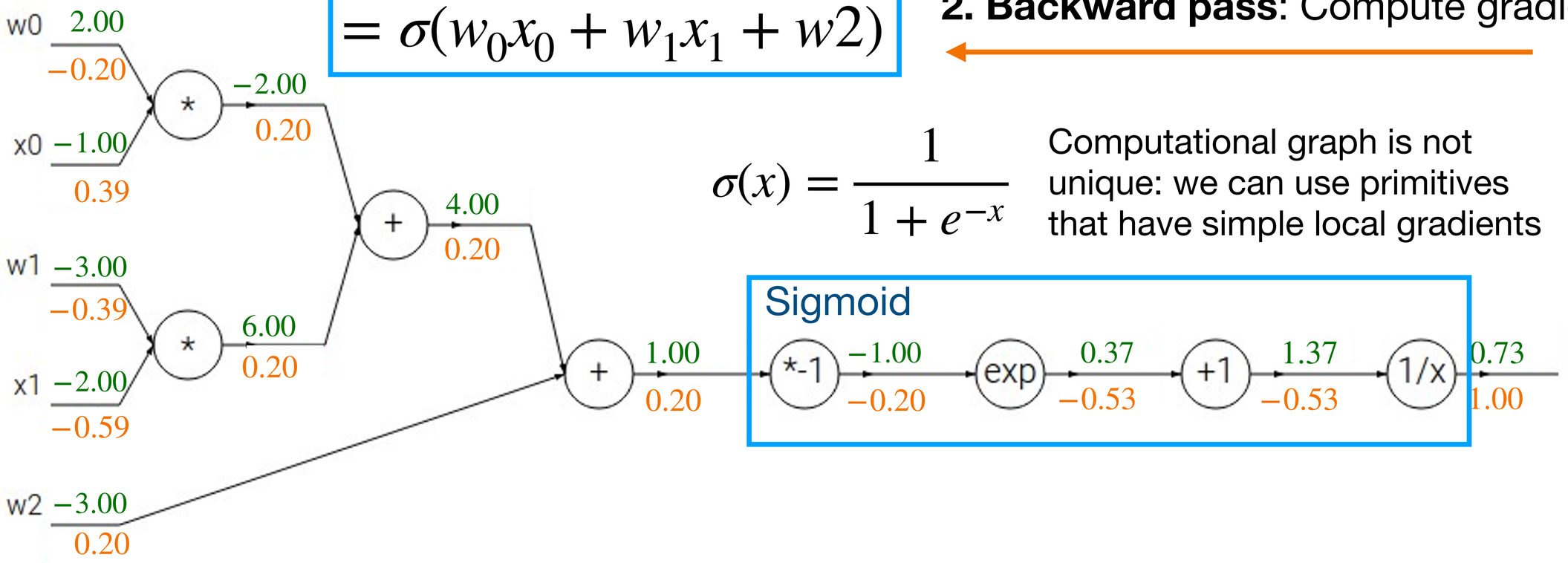

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

Computational graph is not unique: we can use primitives that have simple local gradients

Another example

$$f(x,w) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w2)}}$$

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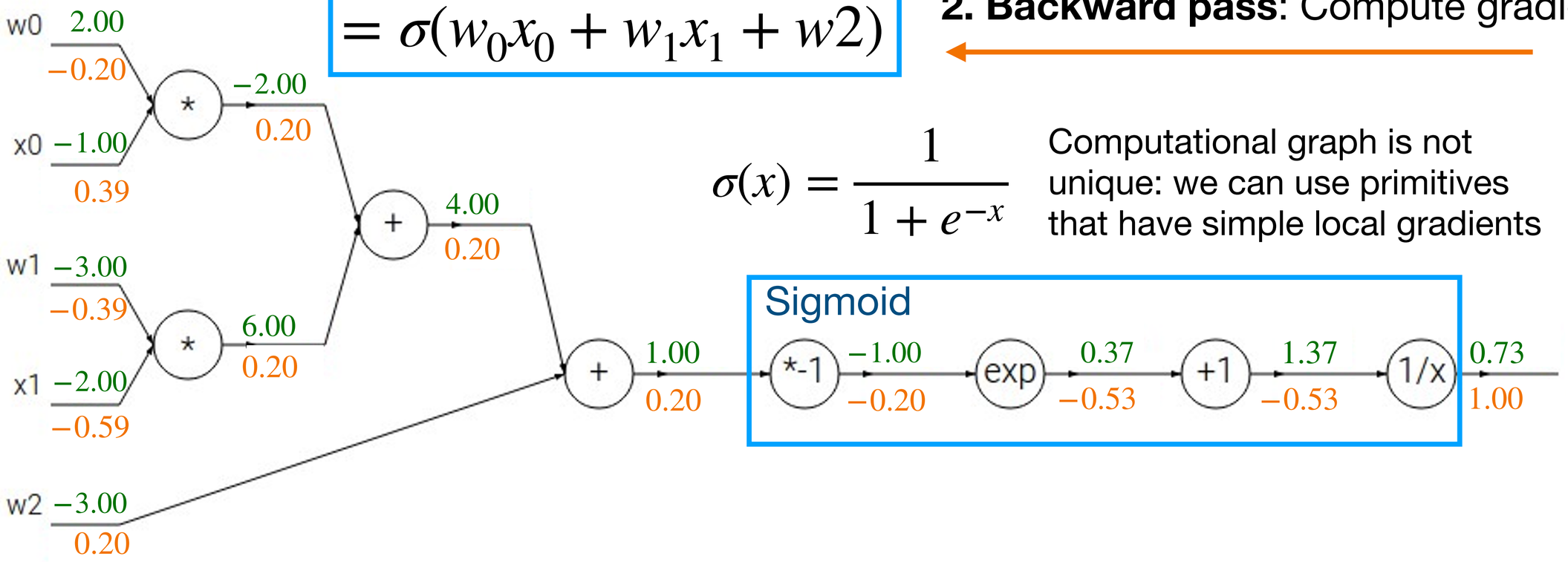

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

Computational graph is not unique: we can use primitives that have simple local gradients

Sigmoid local gradient:

$$\frac{\partial}{\partial x}\left[\sigma(x)\right] = \frac{e^{-x}}{(1+e^{-x})^2} = \left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)\left(\frac{1}{1+e^{-x}}\right) = (1-\sigma(x))\sigma(x)$$

Another example

$$f(x,w) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w2)}}$$

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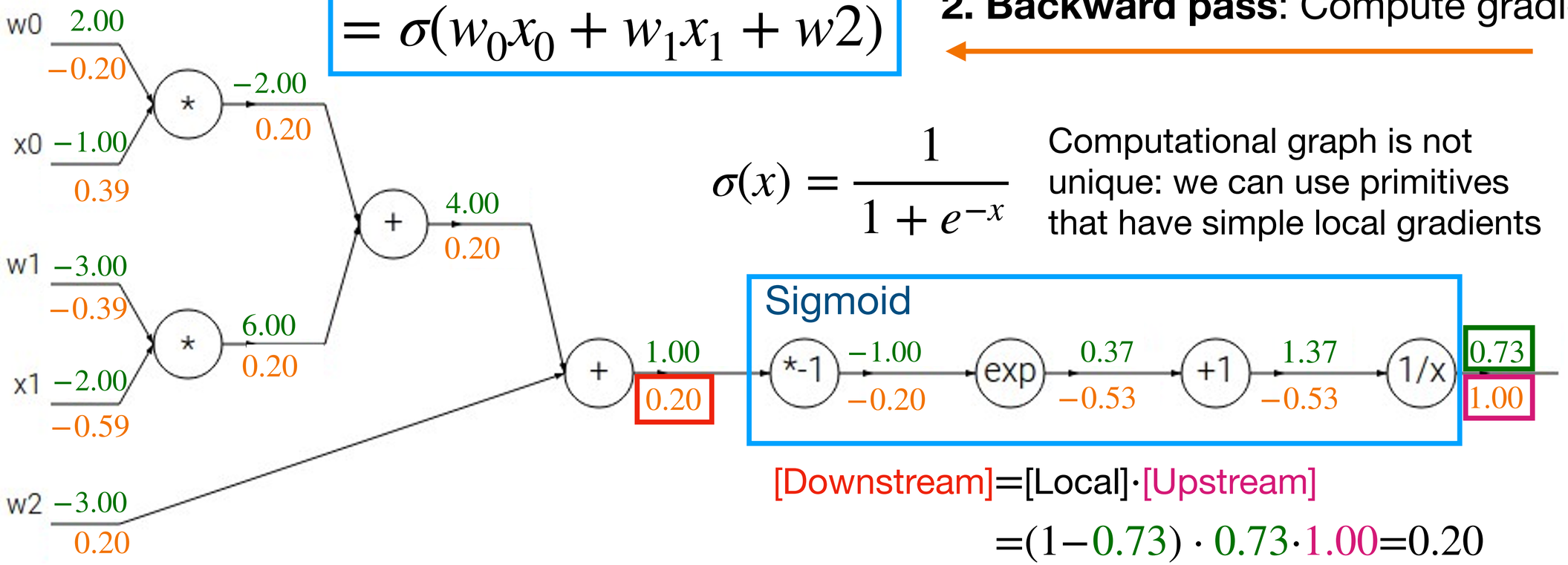

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Computational graph is not unique: we can use primitives that have simple local gradients

[Downstream]=[Local]·[Upstream]

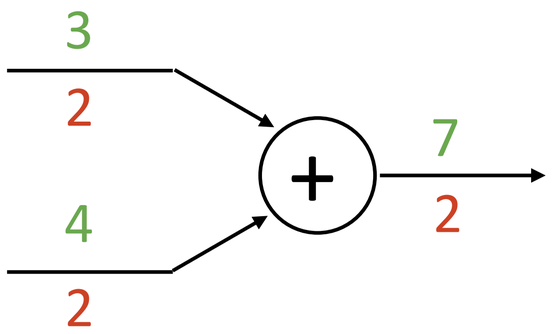
$$=(1-0.73)\cdot 0.73\cdot 1.00=0.20$$

Sigmoid local gradient:

$$\frac{\partial}{\partial x}\left[\sigma(x)\right] = \frac{e^{-x}}{(1+e^{-x})^2} = \left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)\left(\frac{1}{1+e^{-x}}\right) = (1-\sigma(x))\sigma(x)$$

Patterns in Gradient Flow

add gate: gradient distributor



Patterns in Gradient Flow

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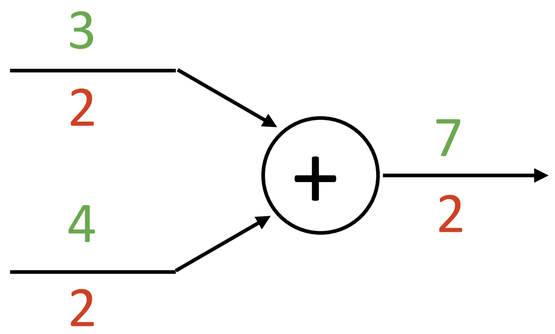
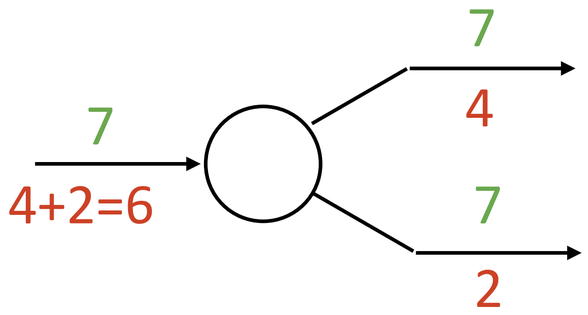


copy gate: gradient adder



Patterns in Gradient Flow

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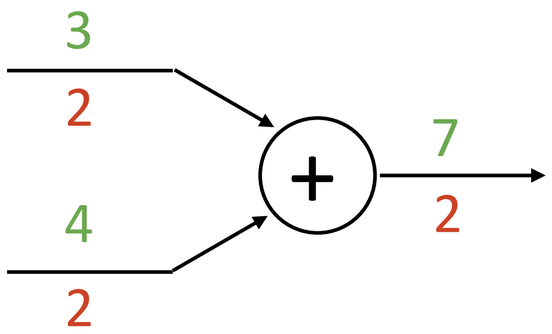
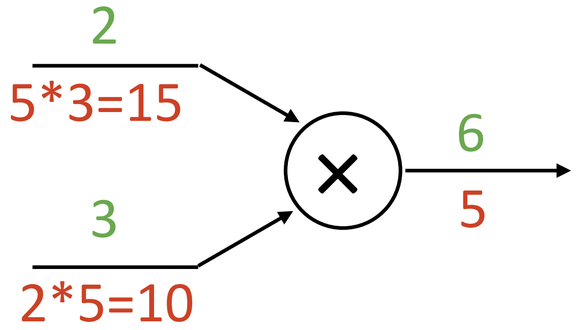


mul gate: “swap multiplier”



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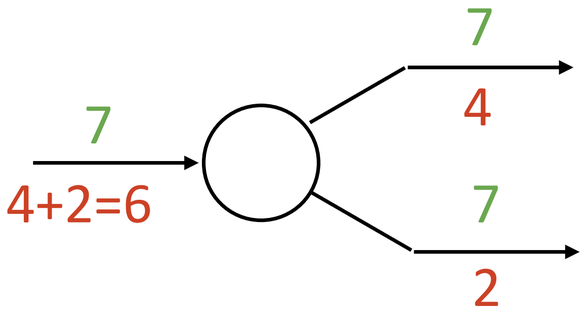

Patterns in Gradient Flow

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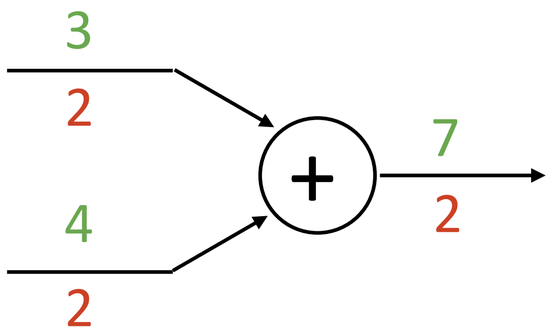

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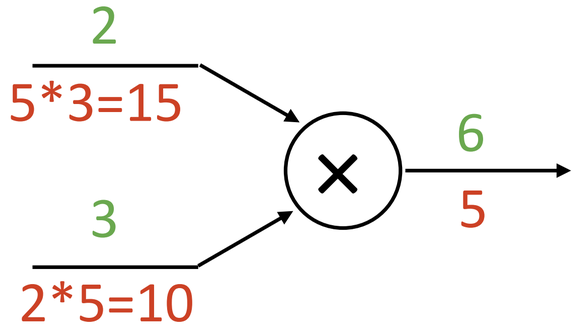

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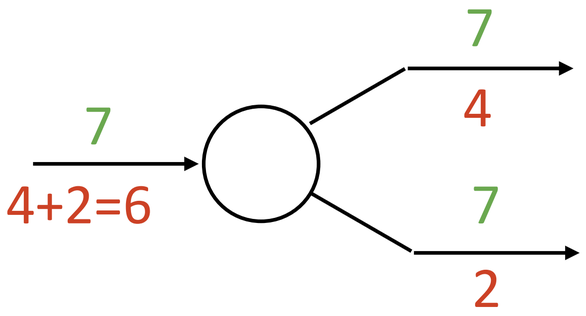
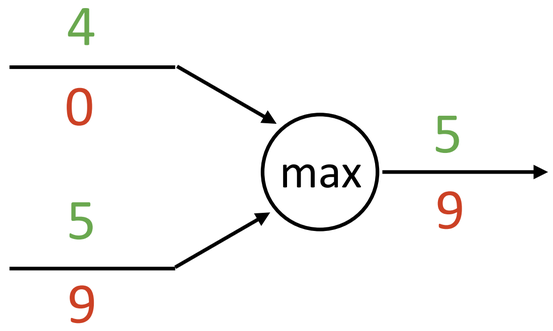
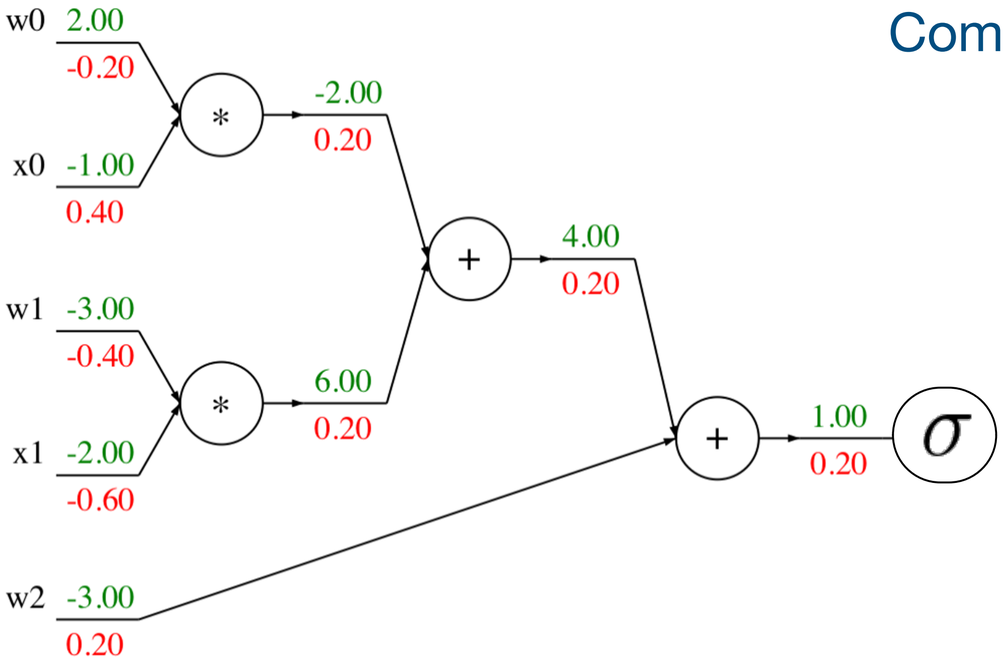


max gate: gradient router



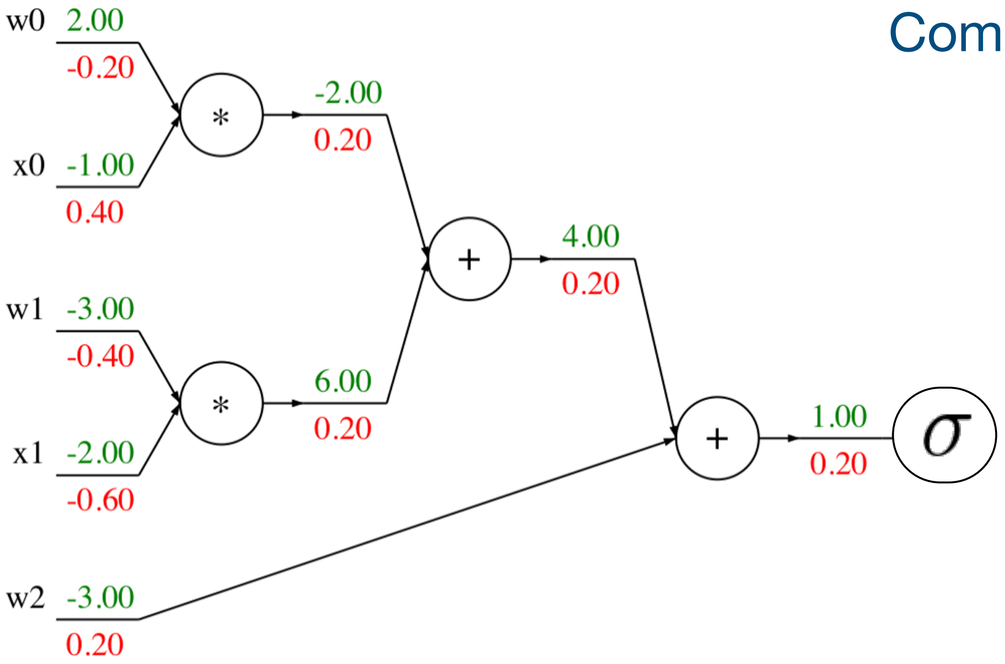
Backprop Implementation: “Flat” gradient code



Forward pass:
Compute outputs

```
def f(w0, x0, w1, x1, w2):
  s0 = w0 * x0
  s1 = w1 * x1
  s2 = s0 + s1
  s3 = s2 + w2
  L = sigmoid(s3)
```

Backprop Implementation: “Flat” gradient code



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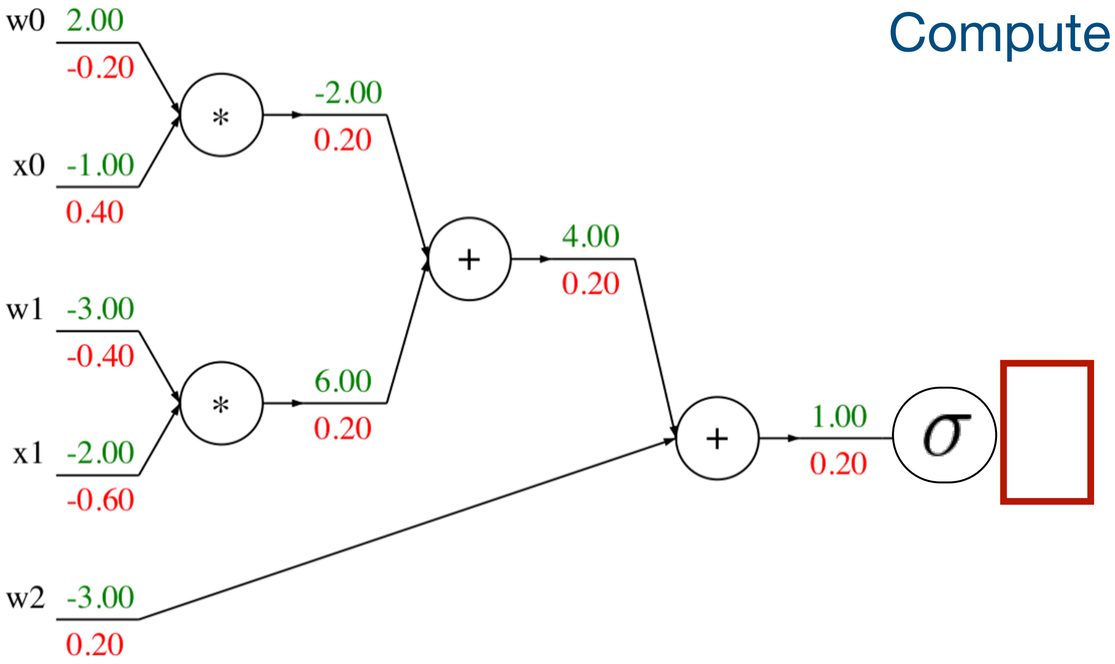
  s2 = s0 + s1
  s3 = s2 + w2
  L = sigmoid(s3)
```

Backward pass:
Compute gradients

```
grad_L = 1.0
grad_s3 = grad_L * (1 - L) * L
grad_w2 = grad_s3
grad_s2 = grad_s3
grad_s0 = grad_s2
grad_s1 = grad_s2
grad_w1 = grad_s1 * x1
grad_x1 = grad_s1 * w1
grad_w0 = grad_s0 * x0
grad_x0 = grad_s0 * w0
```

Backprop Implementation: “Flat” gradient code



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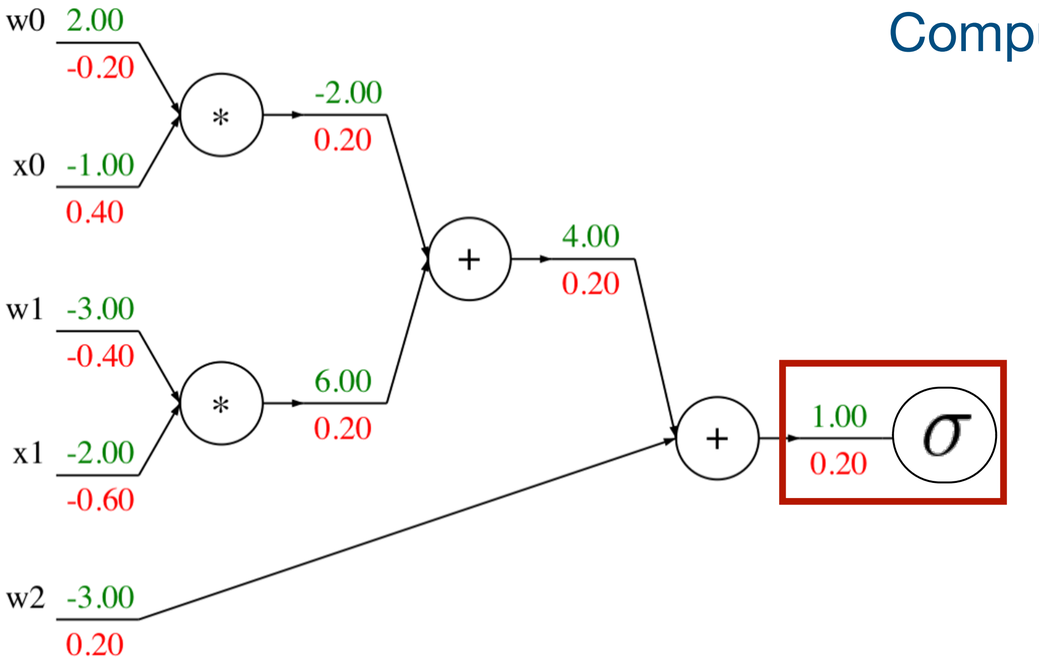
  s2 = s0 + s1
  s3 = s2 + w2
  L = sigmoid(s3)
```

Base case

```
grad_L = 1.0
grad_s3 = grad_L * (1 - L) * L
grad_w2 = grad_s3
grad_s2 = grad_s3
grad_s0 = grad_s2
grad_s1 = grad_s2
grad_w1 = grad_s1 * x1
grad_x1 = grad_s1 * w1
grad_w0 = grad_s0 * x0
grad_x0 = grad_s0 * w0
```

Backward pass:
Compute gradients

Backprop Implementation: “Flat” gradient code



Forward pass:
Compute outputs

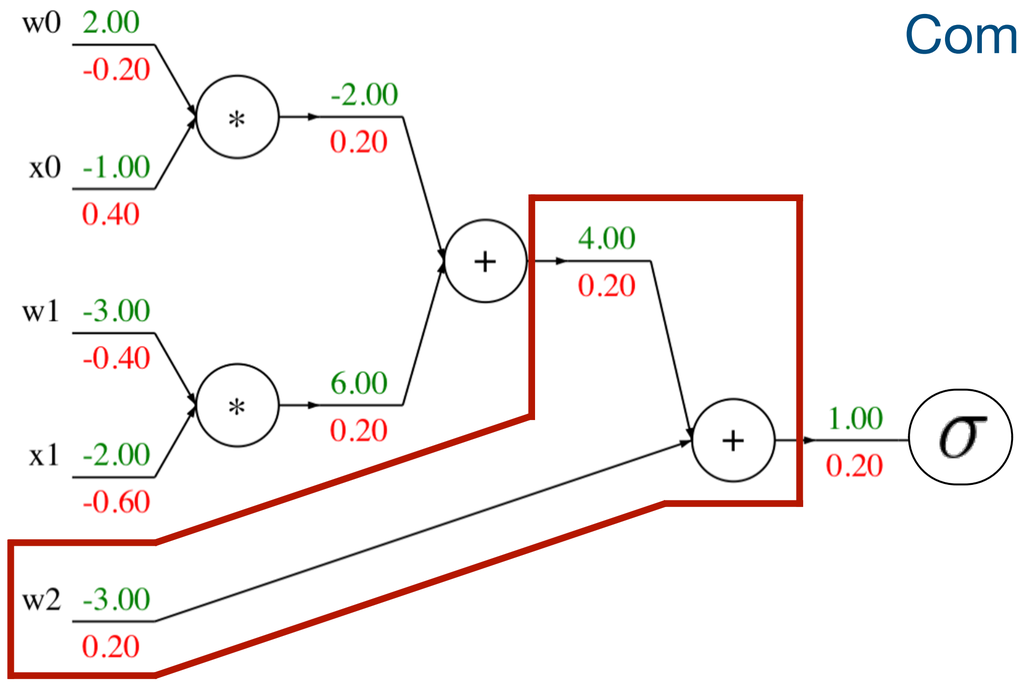
```
def f(w0, x0, w1, x1, w2):
  s0 = w0 * x0
  s1 = w1 * x1
  s2 = s0 + s1
  s3 = s2 + w2
  L = sigmoid(s3)
```

Sigmoid

```
grad_L = 1.0
grad_s3 = grad_L * (1 - L) * L
grad_w2 = grad_s3
grad_s2 = grad_s3
grad_s0 = grad_s2
grad_s1 = grad_s2
grad_w1 = grad_s1 * x1
grad_x1 = grad_s1 * w1
grad_w0 = grad_s0 * x0
grad_x0 = grad_s0 * w0
```

Backward pass:
Compute gradients

Backprop Implementation: “Flat” gradient code



Forward pass:
Compute outputs

```
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  s1 = w1 * x1
  s2 = s0 + s1
  s3 = s2 + w2
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```

Add

```
grad_L = 1.0
grad_s3 = grad_L * (1 - L) * L
grad_w2 = grad_s3
grad_s2 = grad_s3
grad_s0 = grad_s2
grad_s1 = grad_s2
grad_w1 = grad_s1 * x1
grad_x1 = grad_s1 * w1
grad_w0 = grad_s0 * x0
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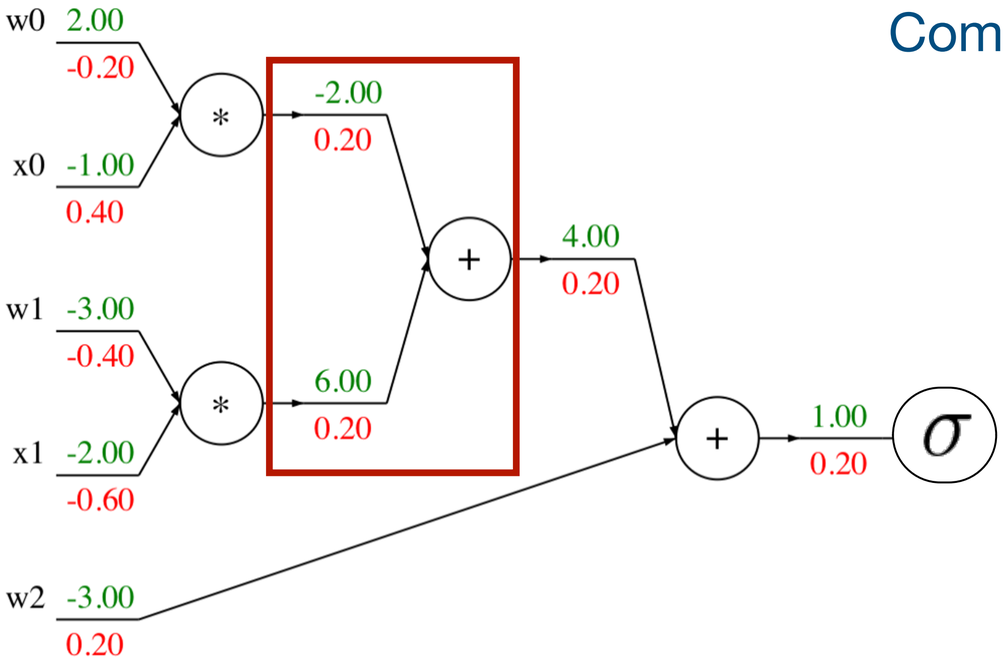
```

Backward pass:
Compute gradients

Backprop Implementation: “Flat” gradient code



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Compute outputs

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grad_s1 = grad_s2
grad_w1 = grad_s1 * x1
grad_x1 = grad_s1 * w1
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```

Backward pass:
Compute gradients

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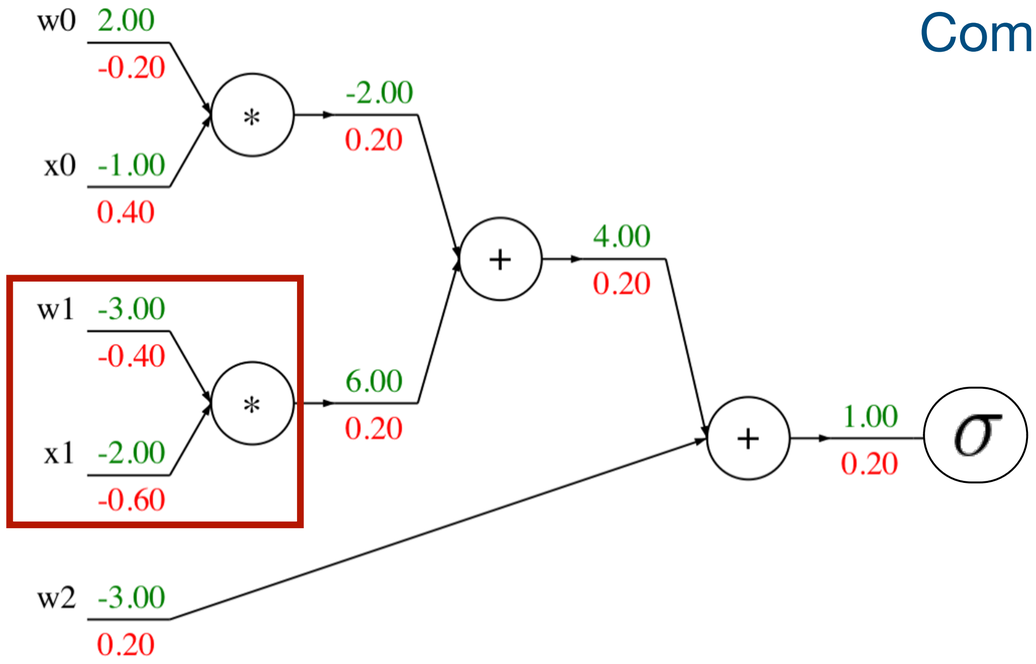

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Compute outputs

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```
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grad_s2 = grad_s3
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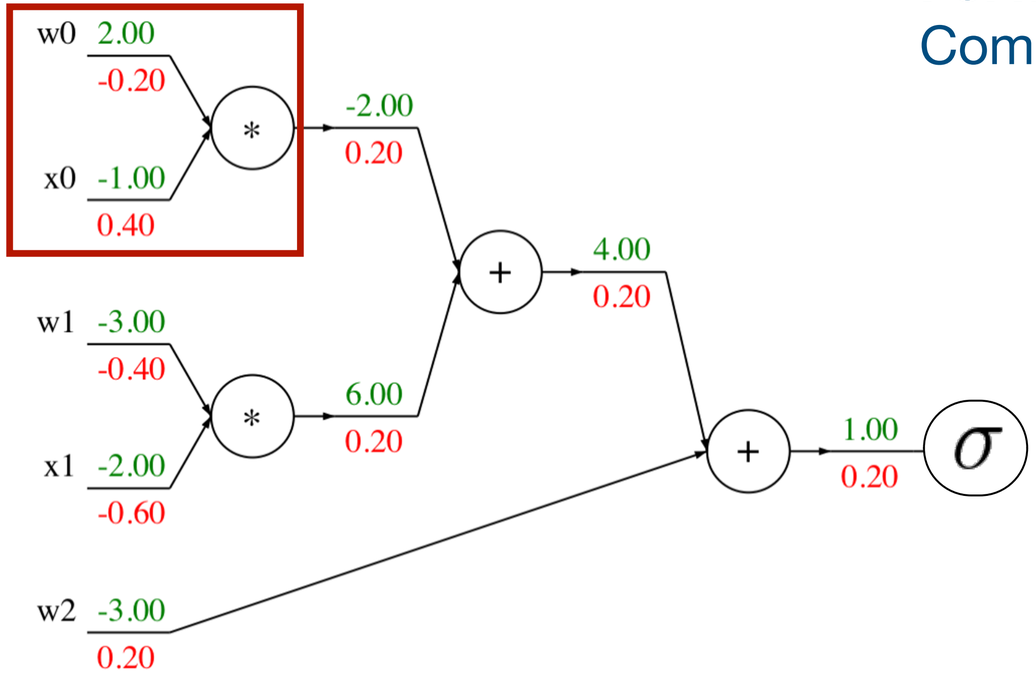
grad_s1 = grad_s2
grad_w1 = grad_s1 * x1
grad_x1 = grad_s1 * w1
grad_w0 = grad_s0 * x0
grad_x0 = grad_s0 * w0
```

Multiply

Backward pass:
Compute gradients

Backprop Implementation: “Flat” gradient code



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```
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grad_x0 = grad_s0 * w0
```

Multiply

Backward pass:
Compute gradients

“Flat” Backprop: Do this for Project 1 & 2

Forward pass:
Compute outputs

```
#############################################################################
# TODO:                                                                     #
# Implement a vectorized version of the structured SVM loss, storing the    #
# result in loss.                                                           #
#############################################################################
# Replace "pass" statement with your code
num_classes = W.shape[1]
num_train = X.shape[0]
score = # ...
correct_class_score = # ...
margin = # ...
data_loss = # ...
reg_loss = # ...
loss += data_loss + reg_loss
#############################################################################
#                             END OF YOUR CODE                              #
#############################################################################
```

Backward pass:
Compute gradients

```
#############################################################################
# TODO:                                                                     #
# Implement a vectorized version of the gradient for the structured SVM     #
# loss, storing the result in dW.                                           #
#                                                                           #
# Hint: Instead of computing the gradient from scratch, it may be easier    #
# to reuse some of the intermediate values that you used to compute the     #
# loss.                                                                     #
#############################################################################
# Replace "pass" statement with your code
dmargins = # ...
dscores = # ...
dW = # ...
#############################################################################
#                             END OF YOUR CODE                              #
#############################################################################
```

“Flat” Backprop: Do this for Project 1 & 2

Forward pass:
Compute outputs

```
#############################################################################
# TODO:                                                                     #
# Implement a vectorized version of the structured SVM loss, storing the    #
# result in loss.                                                           #
#############################################################################
# Replace "pass" statement with your code
num_classes = W.shape[1]
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loss += data_loss + reg_loss
#############################################################################
#                             END OF YOUR CODE                              #
#############################################################################
```

Backward pass:
Compute gradients

```
#############################################################################
# TODO:                                                                     #
# Implement a vectorized version of the gradient for the structured SVM     #
# loss, storing the result in dW.                                           #
#                                                                           #
# Hint: Instead of computing the gradient from scratch, it may be easier    #
# to reuse some of the intermediate values that you used to compute the     #
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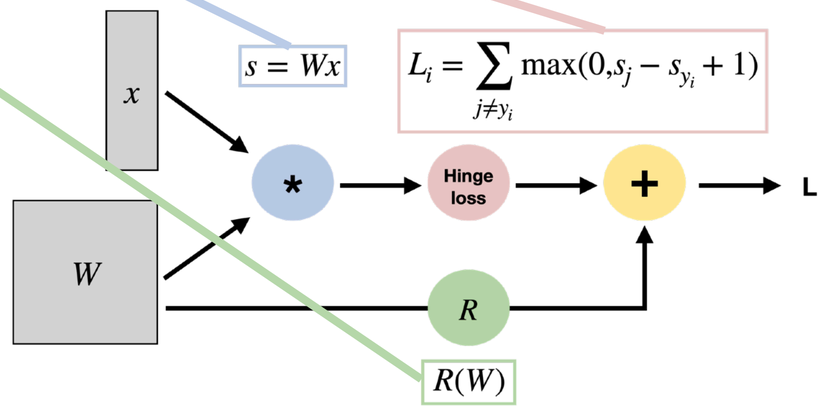
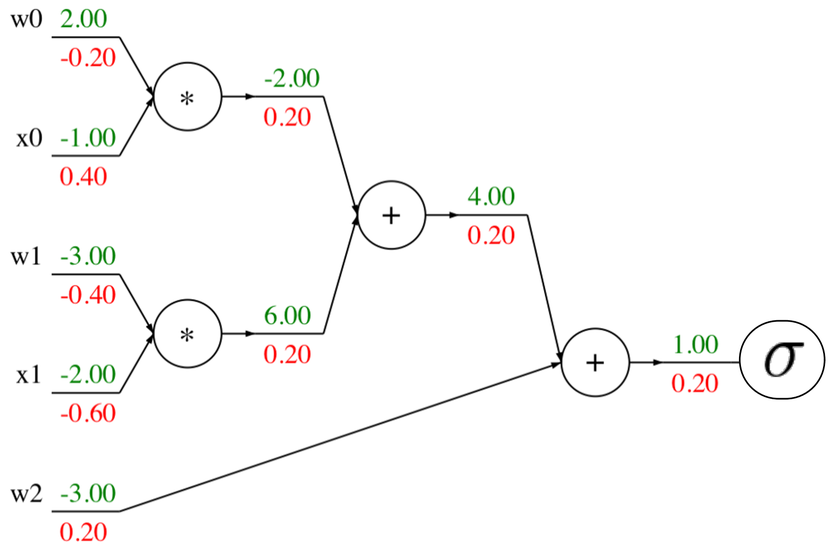
```



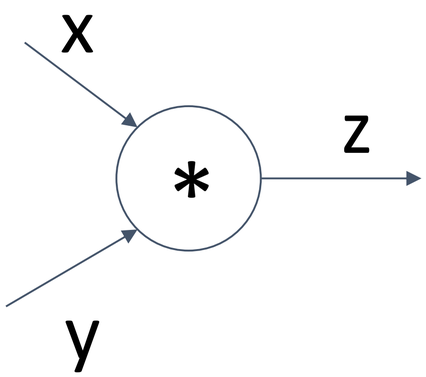
Backprop Implementation: Modular API



Graph (or Net) object *(rough pseudo code)*

```
class ComputationalGraph(object):
    #...
    def forward(inputs):
        # 1. [pass inputs to input gates...]
        # 2. forward the computational graph:
        for gate in self.graph.nodes_topologically_sorted():
            gate.forward()
        return loss # the final gate in the graph outputs the loss
    def backward():
        for gate in reversed(self.graph.nodes_topologically_sorted()):
            gate.backward() # little piece of backprop (chain rule applied)
        return inputs_gradients
```

Example: PyTorch Autograd Functions



(x,y,z are scalars)

```
class Multiply(torch.autograd.Function):
  @staticmethod
  def forward(ctx, x, y):
    ctx.save_for_backward(x, y)
    z = x * y
    return z
  @staticmethod
  def backward(ctx, grad_z):
    x, y = ctx.saved_tensors
    grad_x = y * grad_z   # dz/dx * dL/dz
    grad_y = x * grad_z   # dz/dy * dL/dz
    return grad_x, grad_y
```

Need to stash some values for use in backward

Upstream gradient

Multiply upstream and local gradients

So far: backprop with scalars

What about vector-valued functions?

Recap: Vector Derivatives

$$x \in \mathbb{R}, y \in \mathbb{R}$$

Regular derivative:

$$\frac{\partial y}{\partial x} \in \mathbb{R}$$

If x changes by a small amount, how much will y change?

Recap: Vector Derivatives

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If x changes by a small amount, how much will y change?

$x \in \mathbb{R}^N, y \in \mathbb{R}$

Derivative is **Gradient**:

$$\frac{\partial y}{\partial x} \in \mathbb{R}^N,$$

$$\left(\frac{\partial y}{\partial x}\right)_i = \frac{\partial y}{\partial x_i}$$

For each element of x, if it changes by a small amount then how much will y change?

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$$\left(\frac{\partial y}{\partial x}\right)_i = \frac{\partial y}{\partial x_i}$$

For each element of x, if it changes by a small amount then how much will y change?

$x \in \mathbb{R}^N, y \in \mathbb{R}^M$

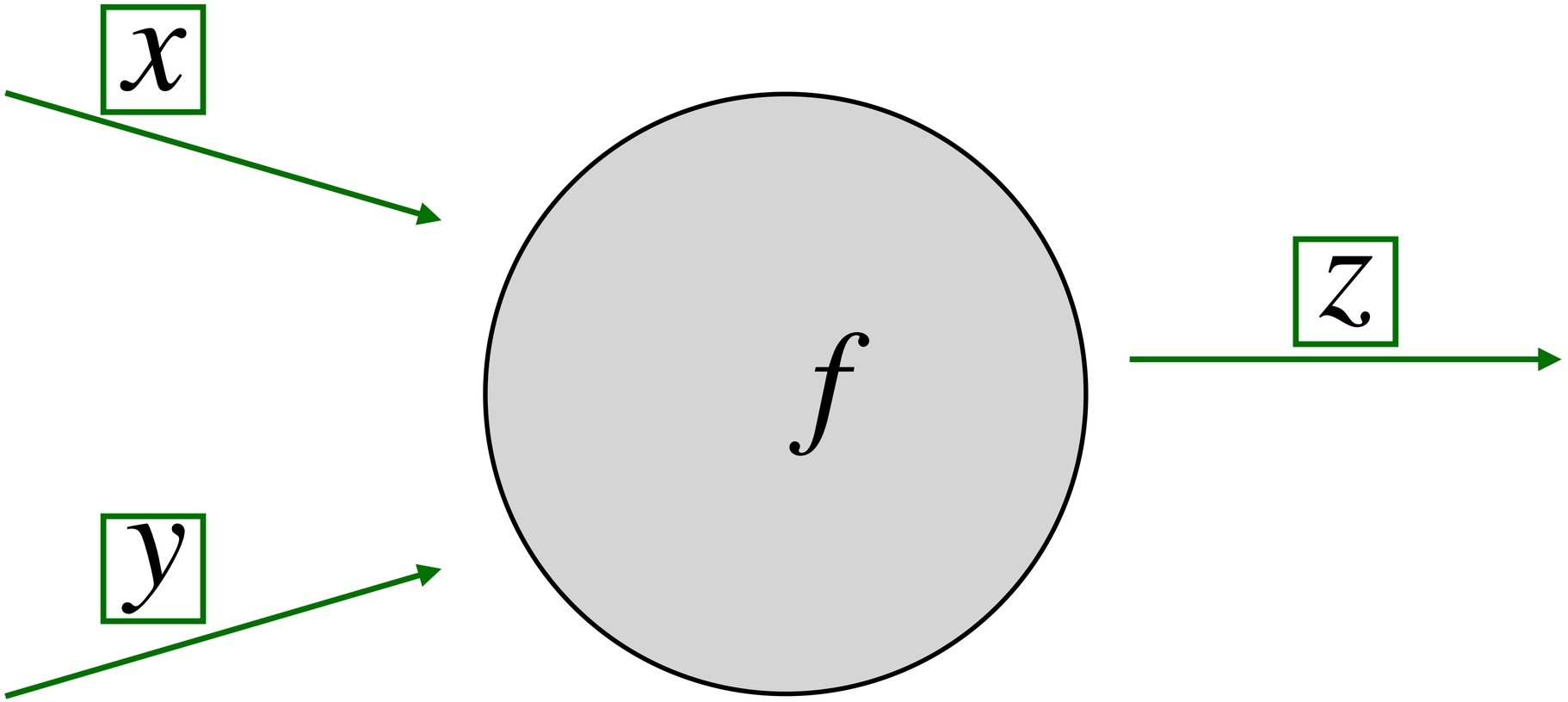
Derivative is **Jacobian**:

$$\frac{\partial y}{\partial x} \in \mathbb{R}^{N \times M}$$

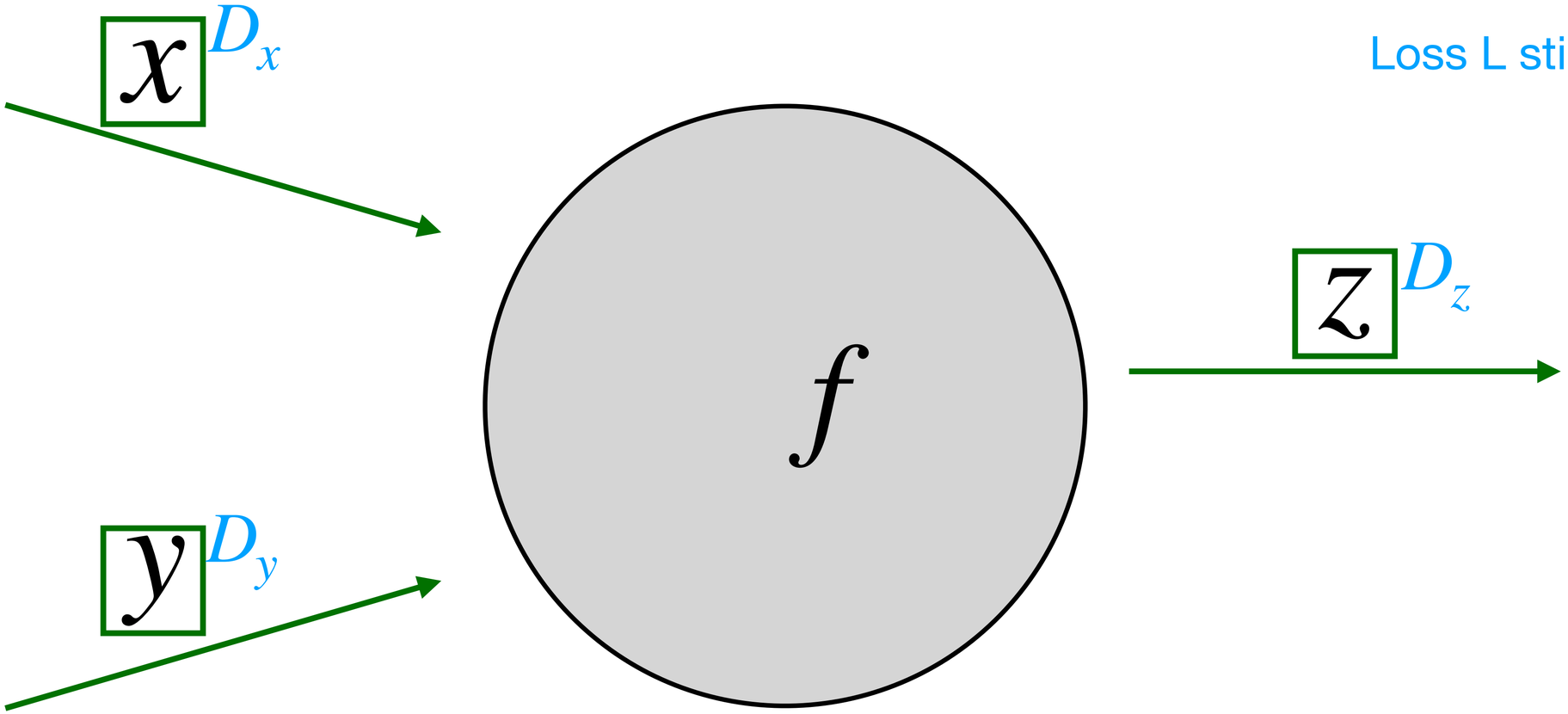
$$\left(\frac{\partial y}{\partial x}\right)_{i,j} = \frac{\partial y_j}{\partial x_i}$$

For each element of x, if it changes by a small amount then how much will each element of y change?

Backprop with Vectors

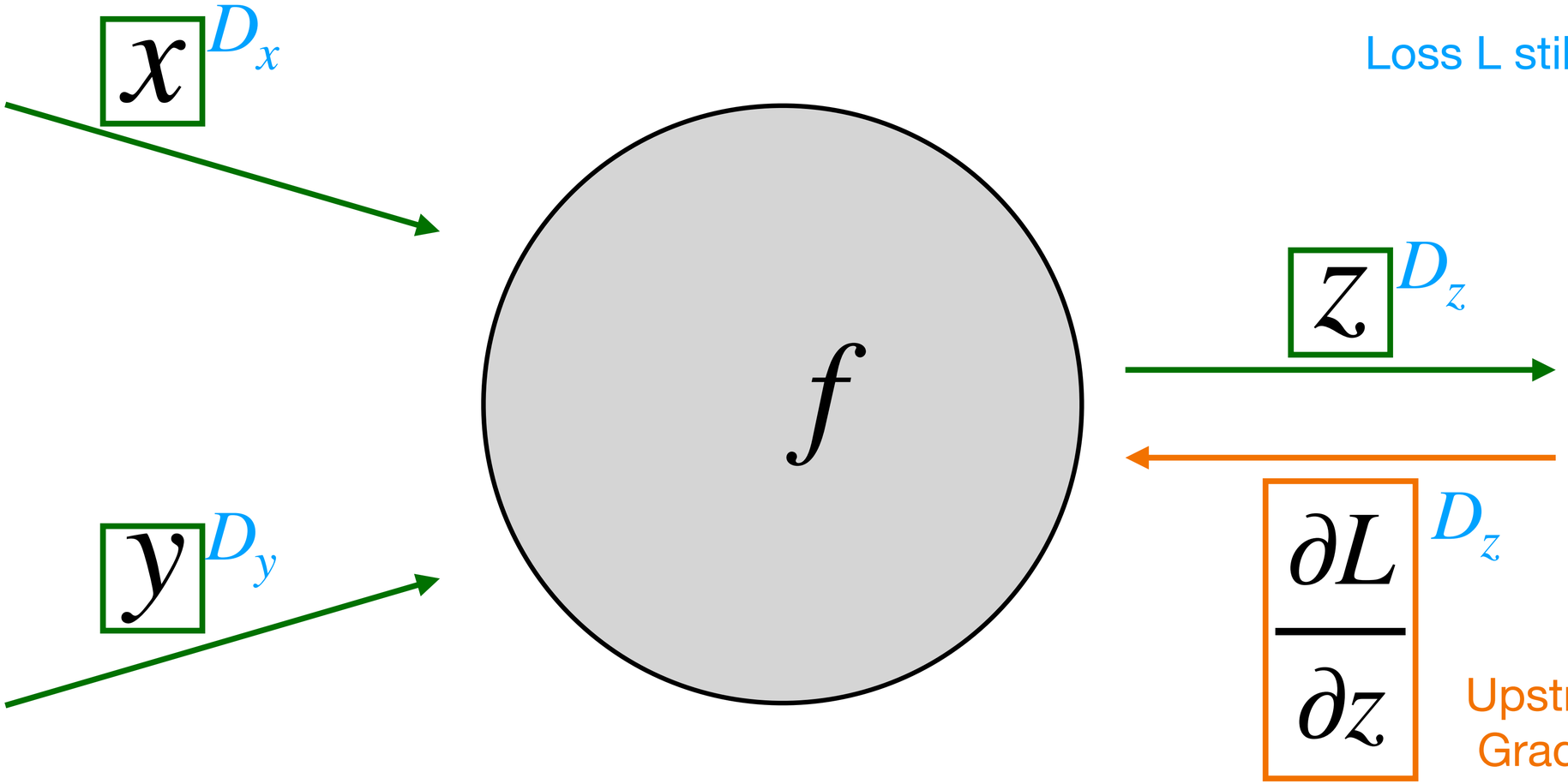


Backprop with Vectors



Loss L still a scalar!

Backprop with Vectors

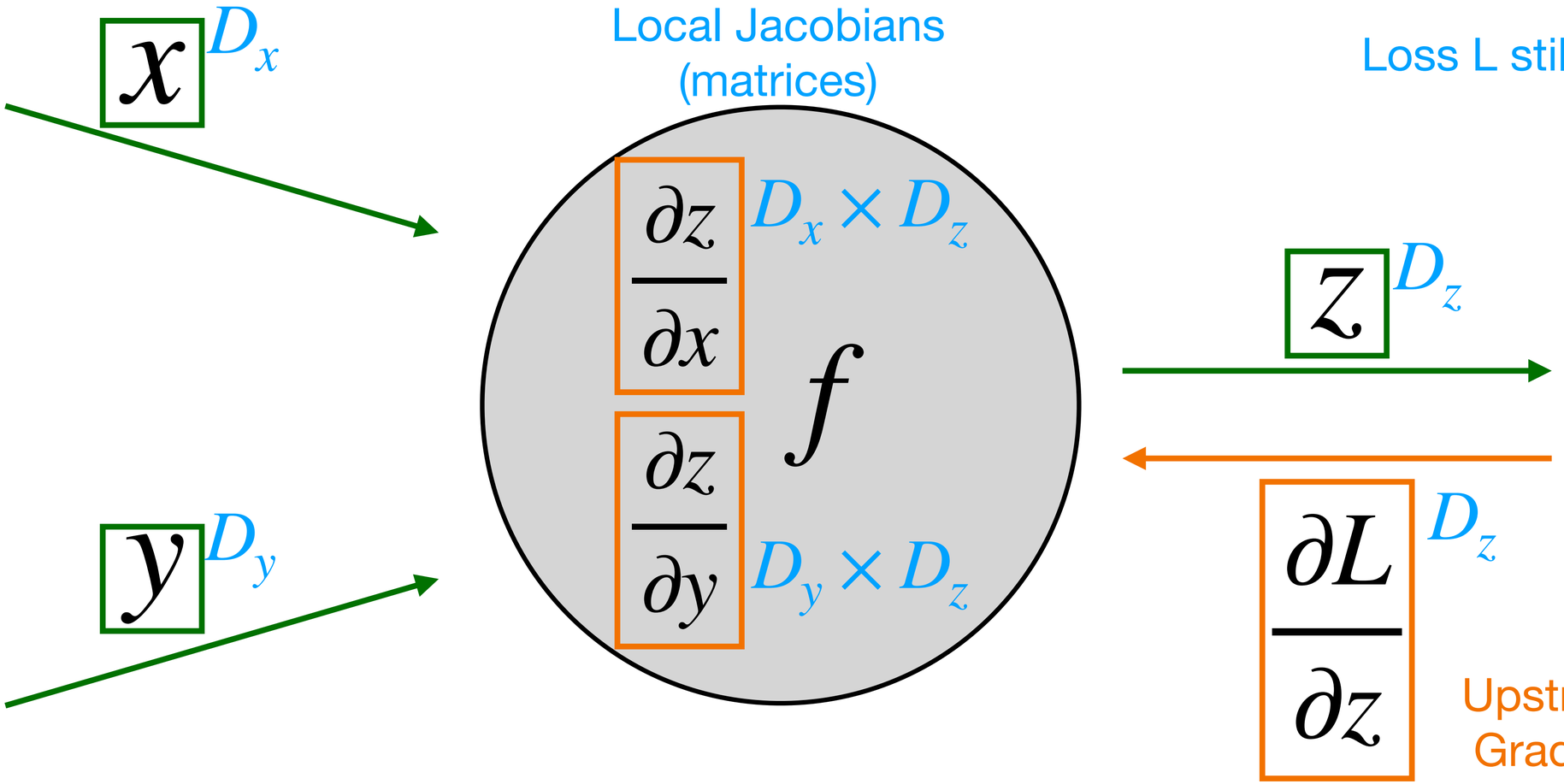


Loss L still a scalar!

Upstream Gradient

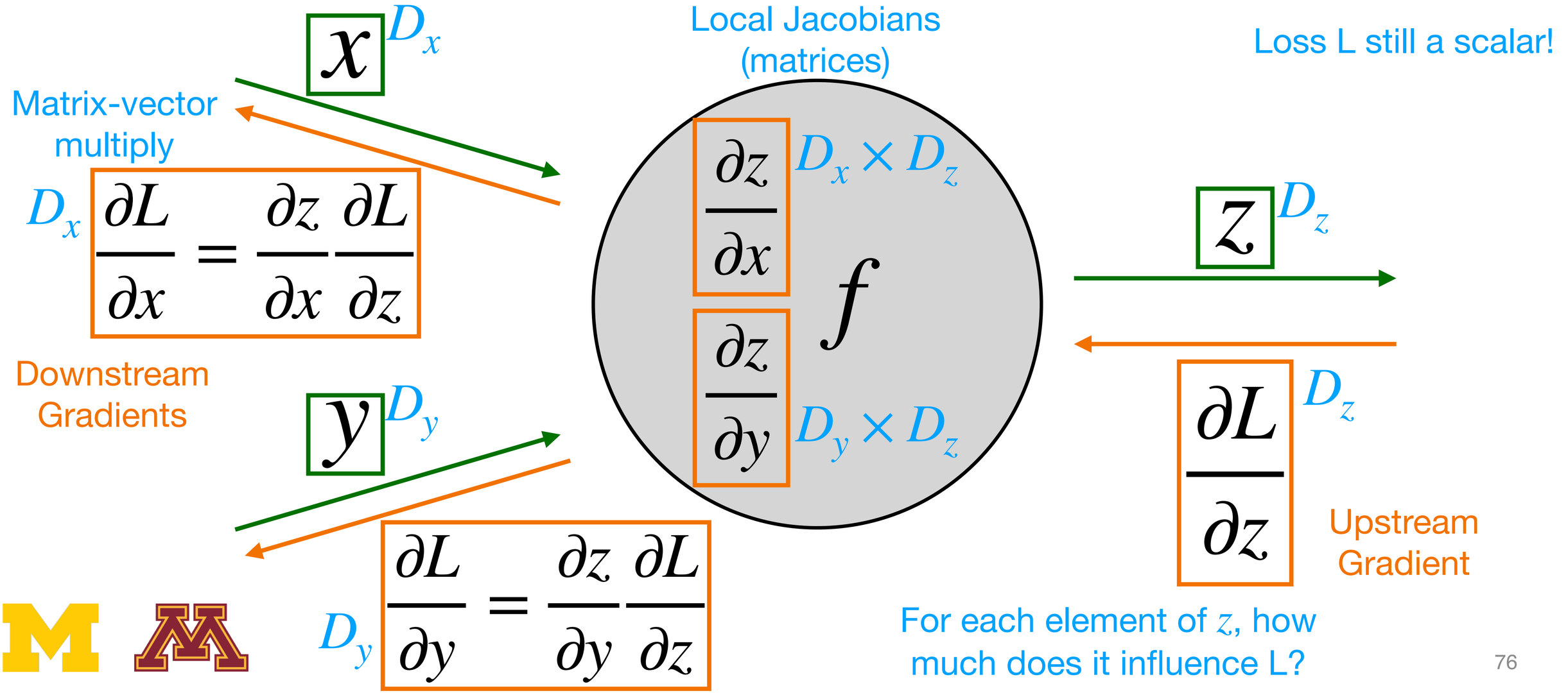
For each element of z, how much does it influence L?

Backprop with Vectors



For each element of z, how much does it influence L?

Backprop with Vectors



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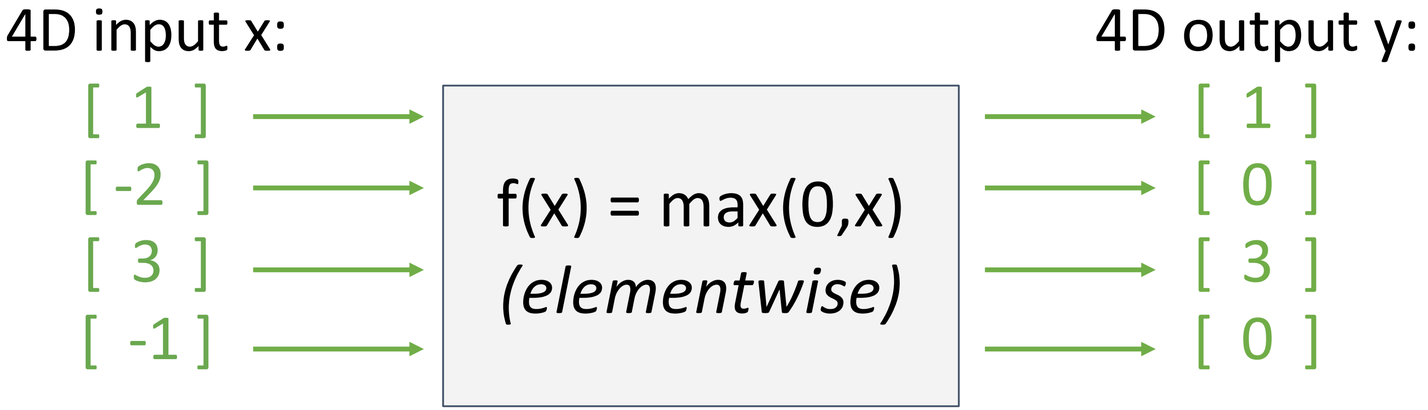
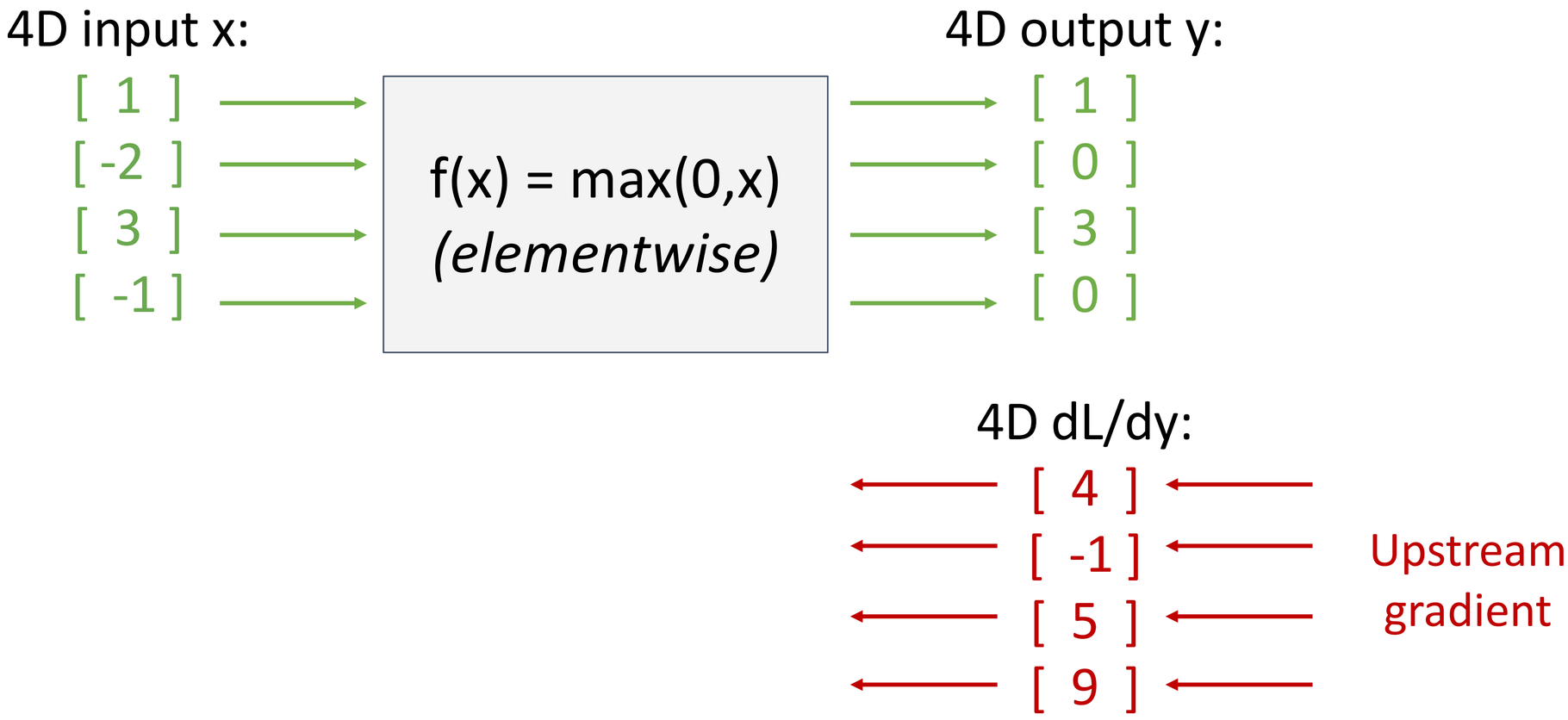


Backprop with Vectors



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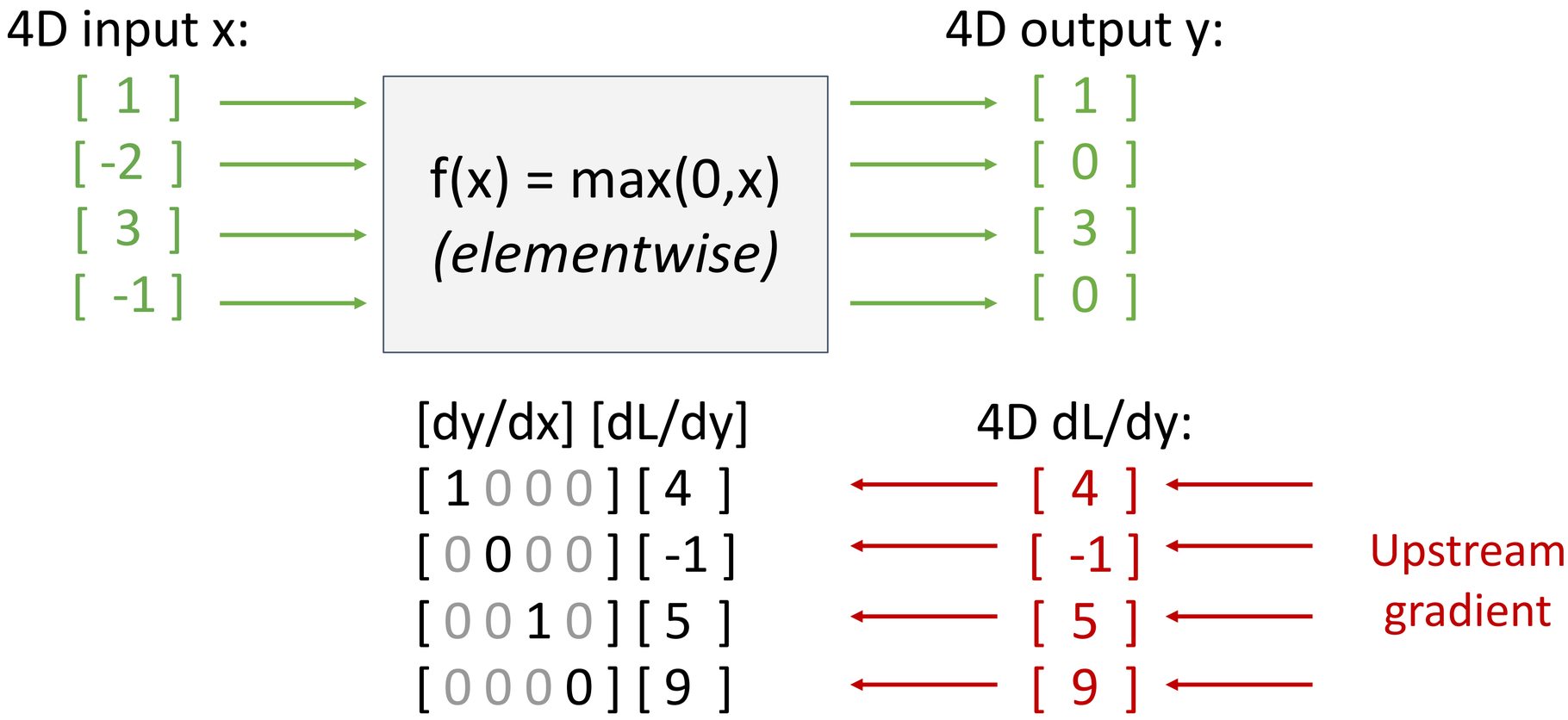

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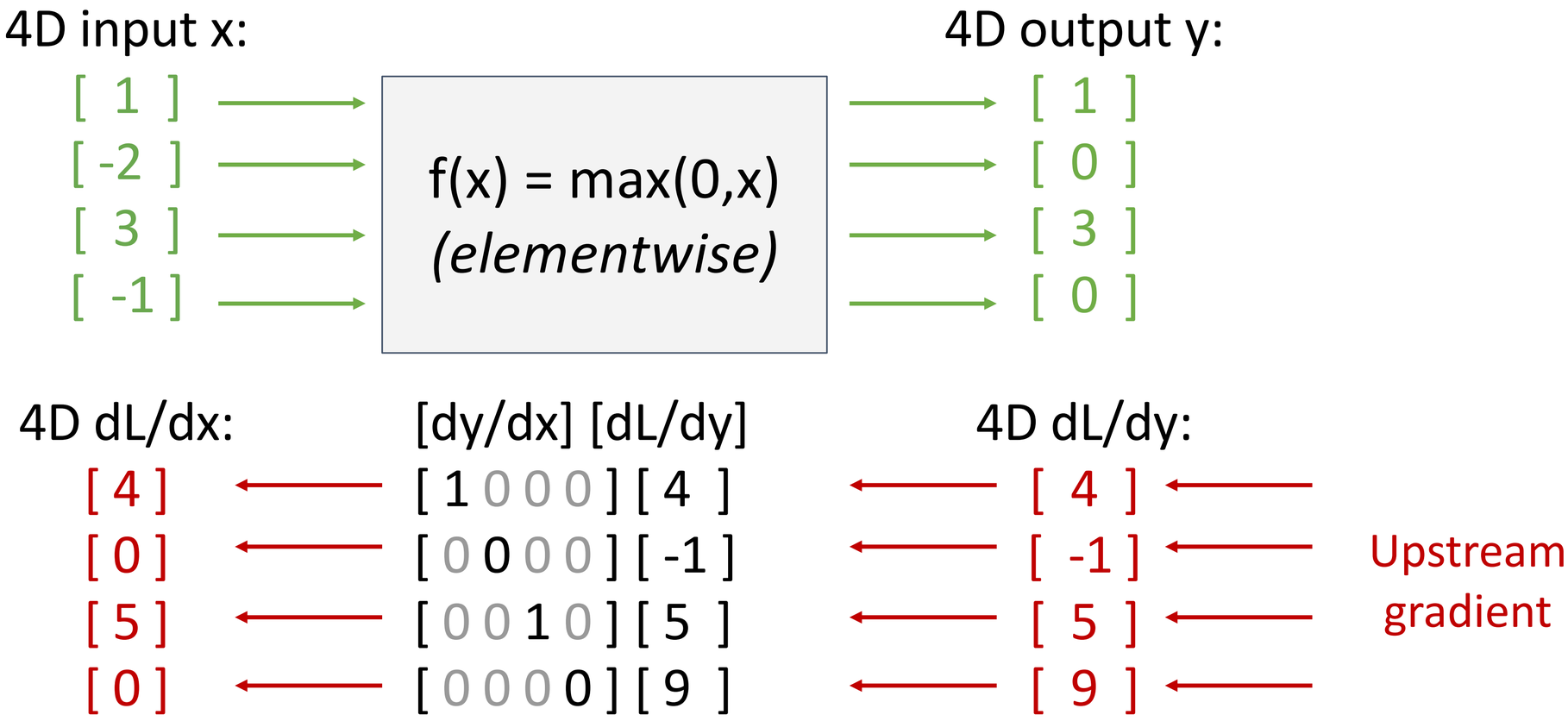

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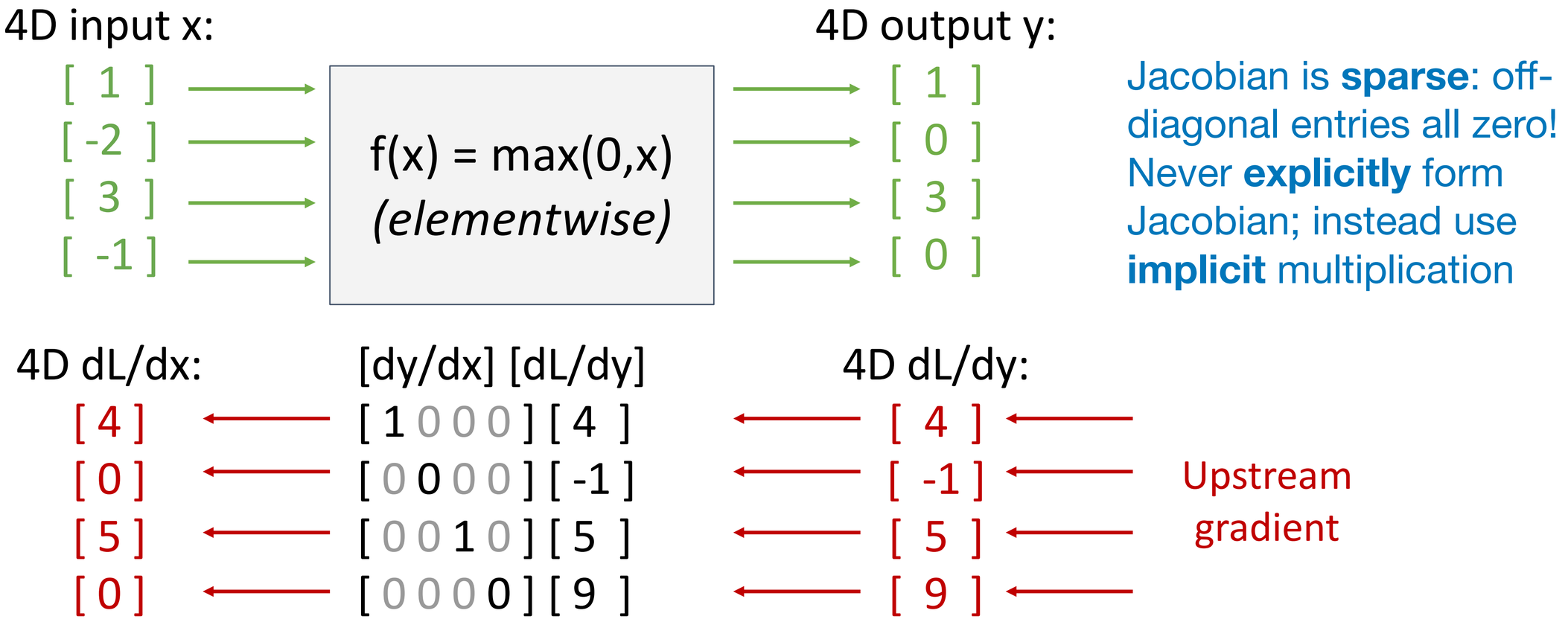

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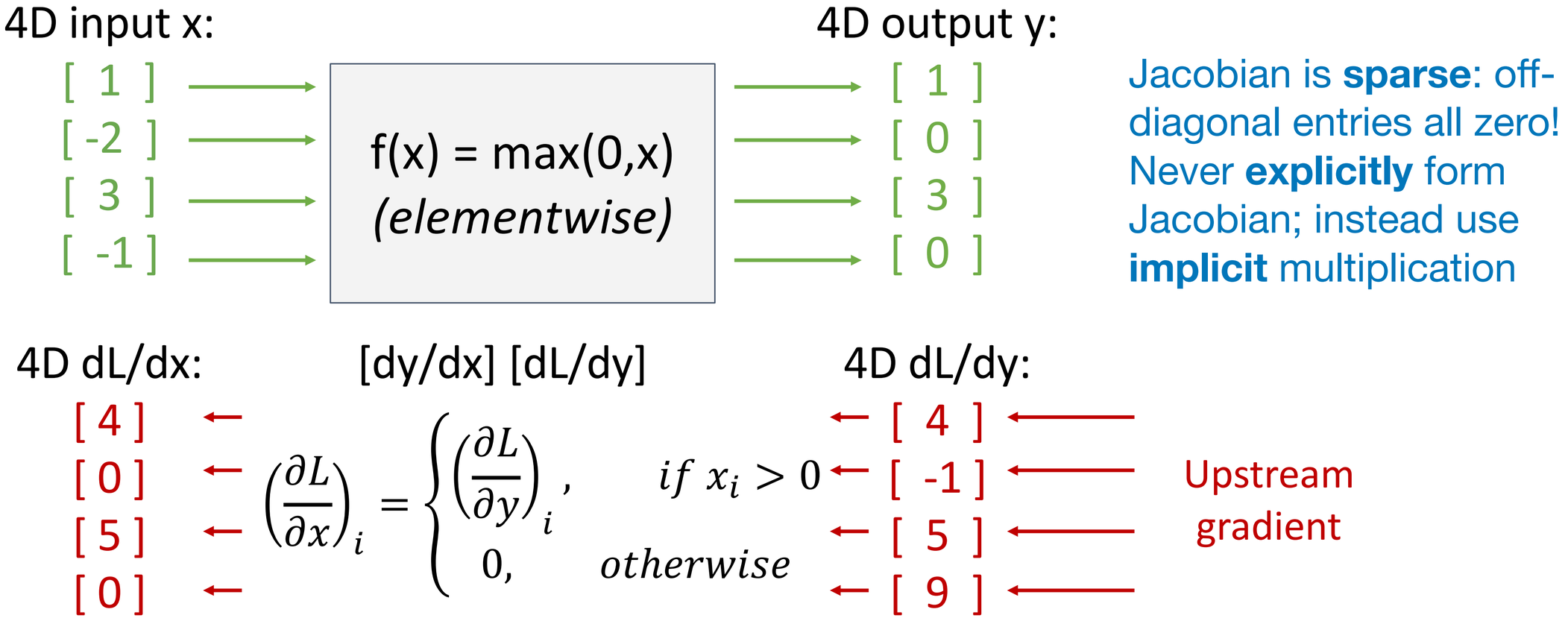
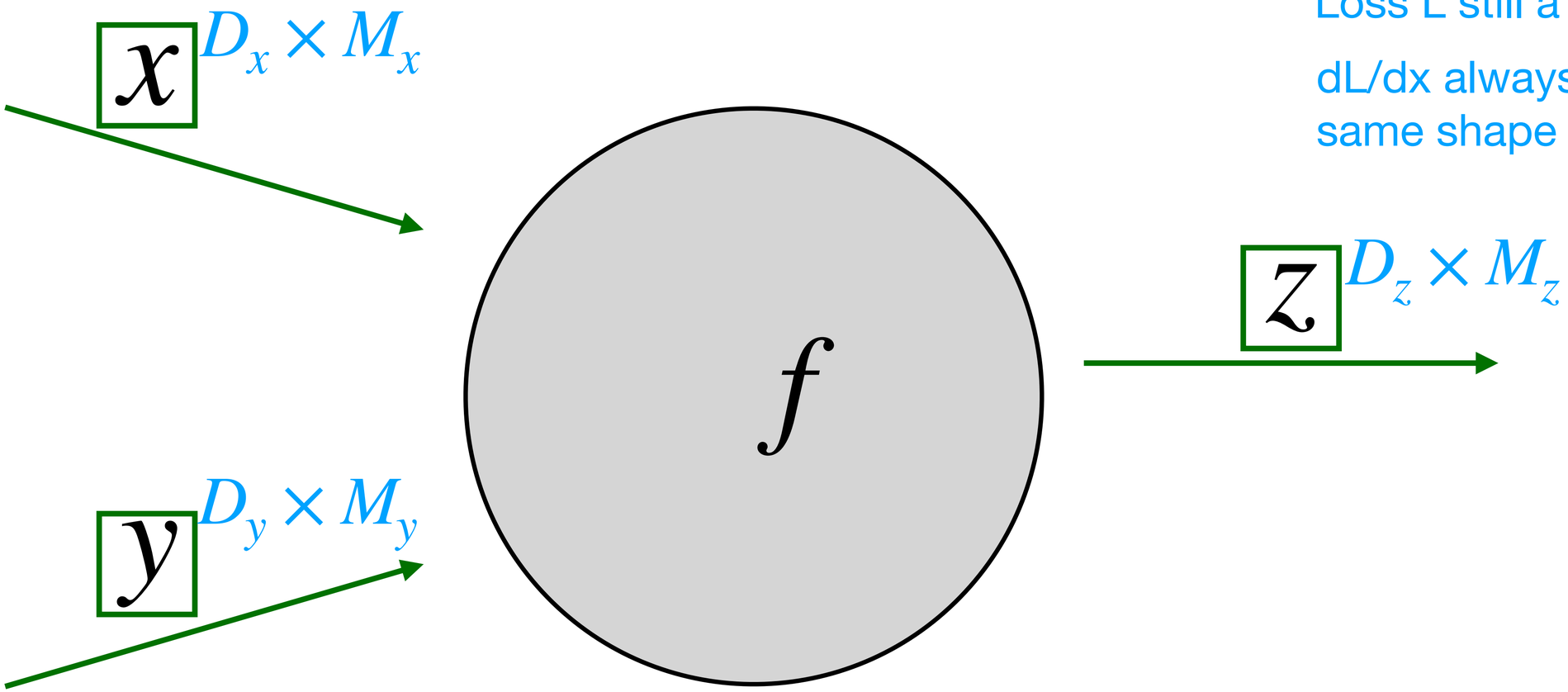


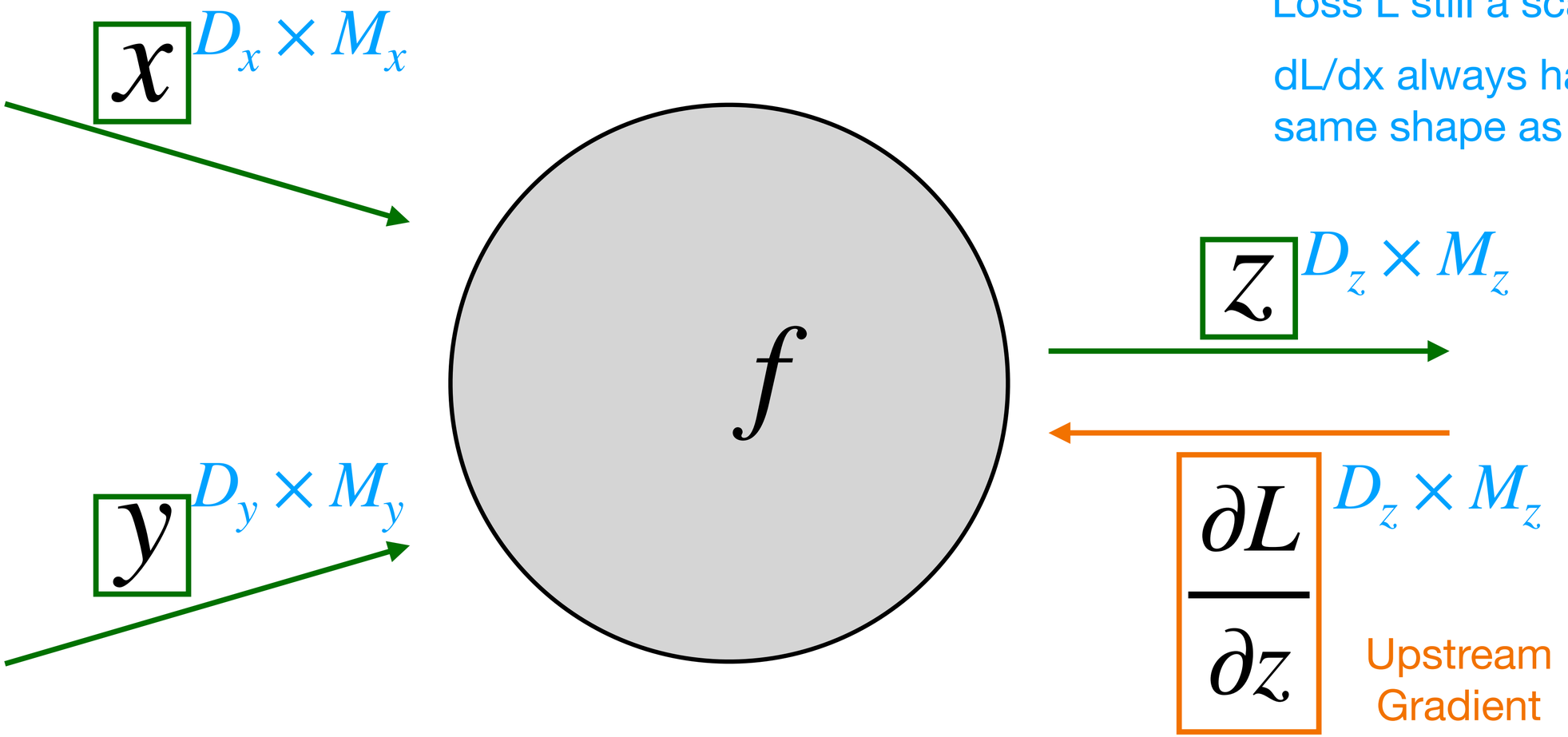
Backprop with Matrices (or Tensors)



Loss L still a scalar!

dL/dx always has the same shape as x!

Backprop with Matrices (or Tensors)



Loss L still a scalar!

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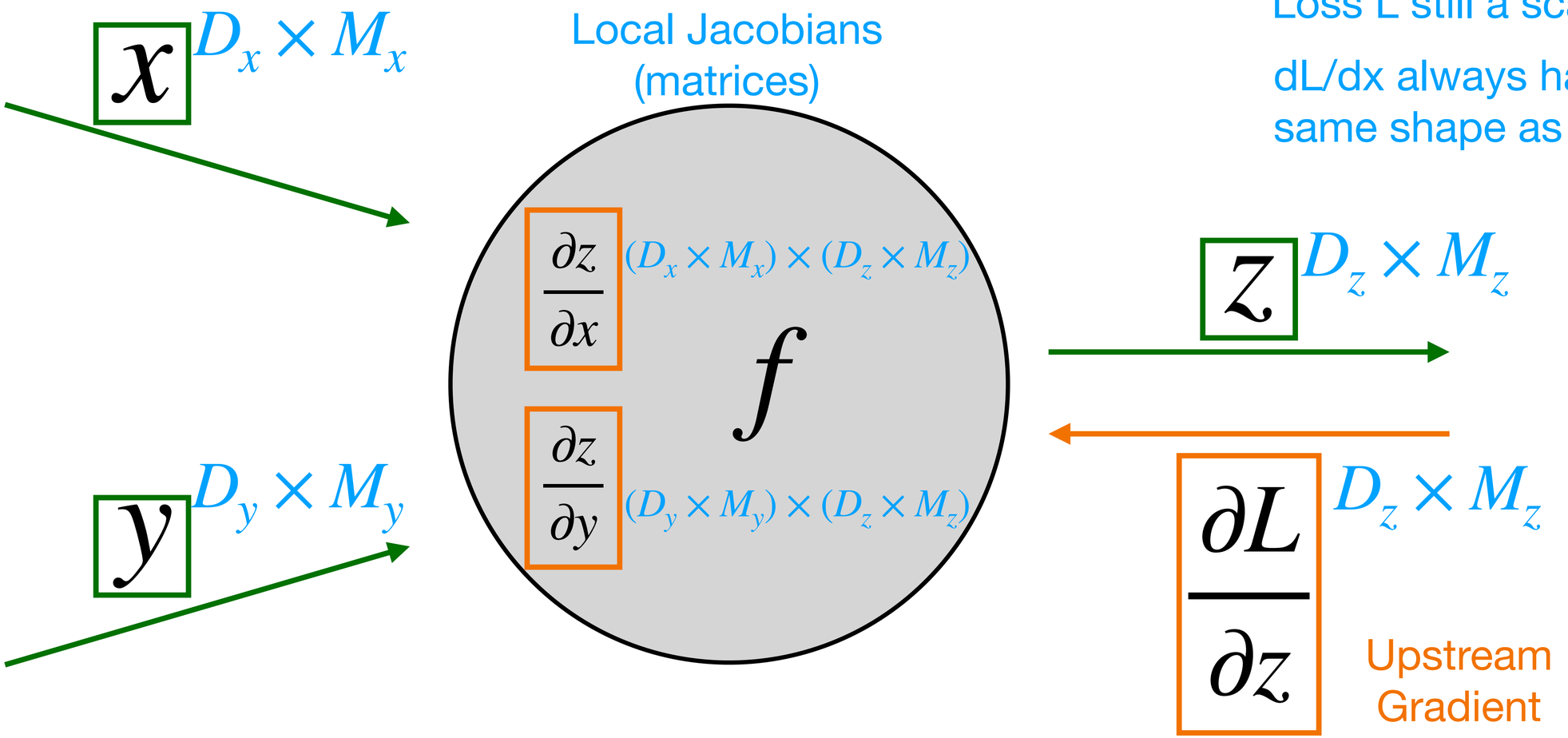
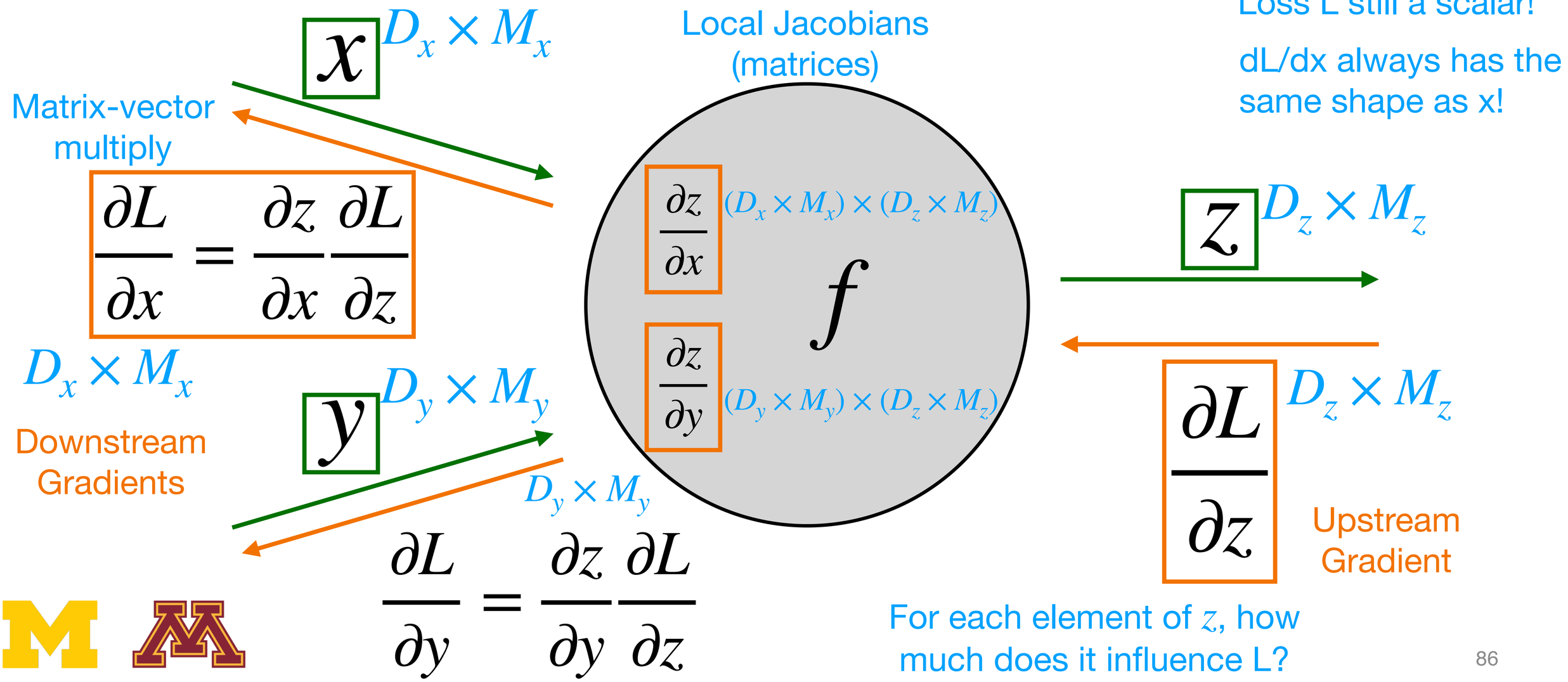
For each element of z, how much does it influence L?

Backprop with Matrices (or Tensors)



Backprop with Matrices (or Tensors)



Example: Matrix Multiplication

x: [N×D]

[2 1 -3]
[-3 4 2]

w: [D×M]

[3 2 1 -1]
[2 1 3 2]
[3 2 1 -2]

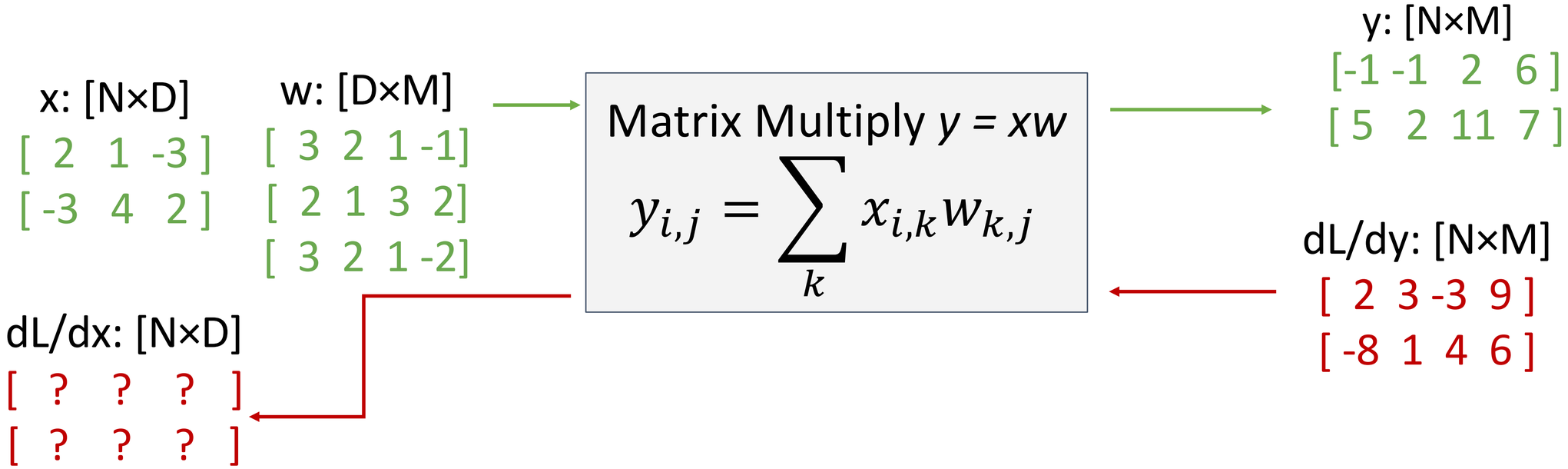
Matrix Multiply $y = xw$

$$y_{i,j} = \sum_k x_{i,k} w_{k,j}$$

y: [N×M]

[-1 -1 2 6]
[5 2 11 7]

Example: Matrix Multiplication



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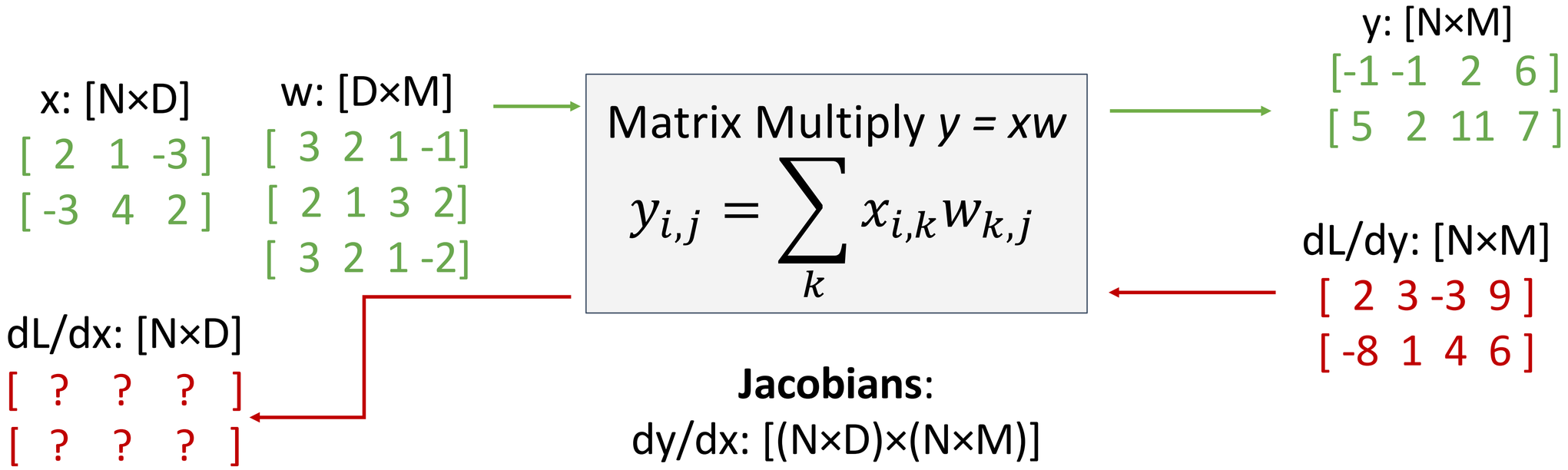

Jacobians:

dy/dx: [(N×D)×(N×M)]

dy/dw: [(D×M)×(N×M)]

For a neural net we may have
N=64, D=M=4096
Each Jacobian takes 256 GB of memory! Must work with them implicitly!

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[2 1 3 2]
[3 2 1 -2]

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$$y_{i,j} = \sum_k x_{i,k} w_{k,j}$$

y: [N×M]
[-1 -1 2 6]
[5 2 11 7]

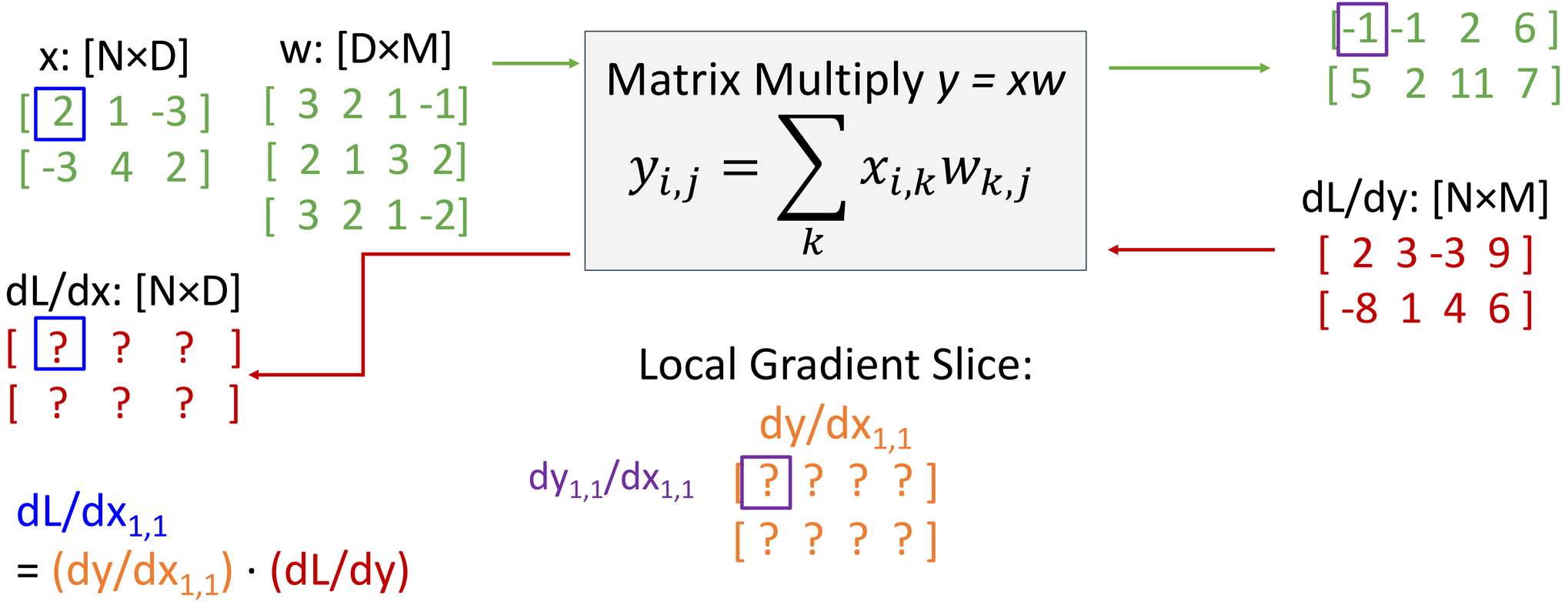
dL/dy: [N×M]
[2 3 -3 9]
[-8 1 4 6]

dL/dx: [N×D]
[? ? ?]
[? ? ?]

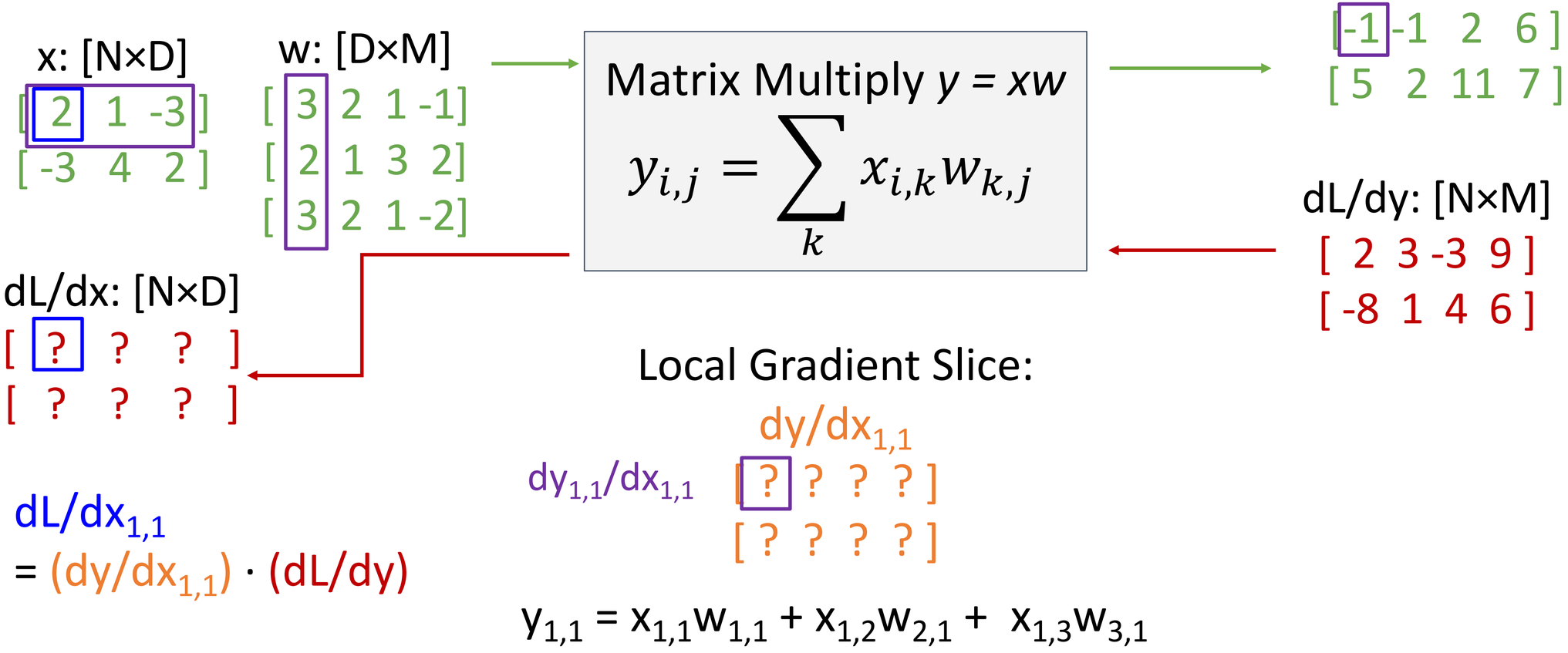
Local Gradient Slice:
$dy/dx_{1,1}$
[? ? ? ?]
[? ? ? ?]

$dL/dx_{1,1}$
$= (dy/dx_{1,1}) \cdot (dL/dy)$

Example: Matrix Multiplication



Example: Matrix Multiplication



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[5 2 11 7]

dL/dy: [N×M]

[2 3 -3 9]
[-8 1 4 6]

dL/dx: [N×D]

[? ? ?]
[? ? ?]

$dL/dx_{1,1}$
$= (dy/dx_{1,1}) \cdot (dL/dy)$

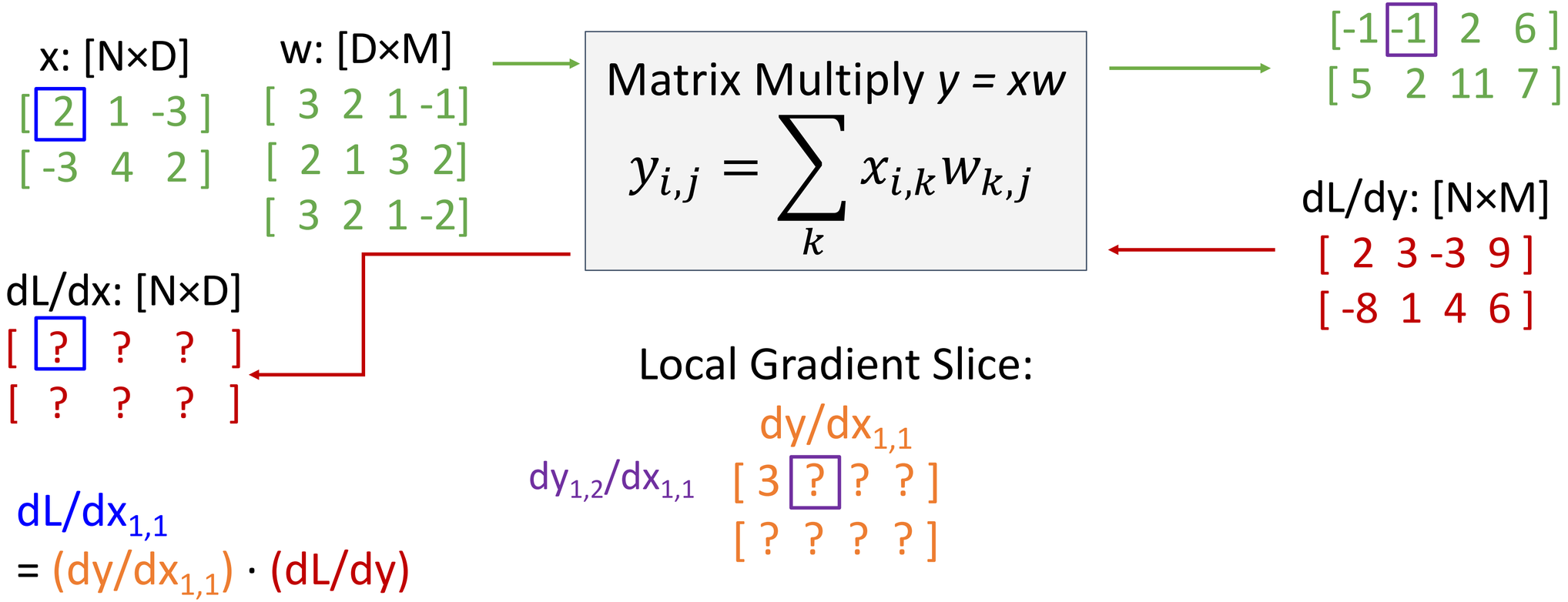
Local Gradient Slice:

$dy/dx_{1,1}$

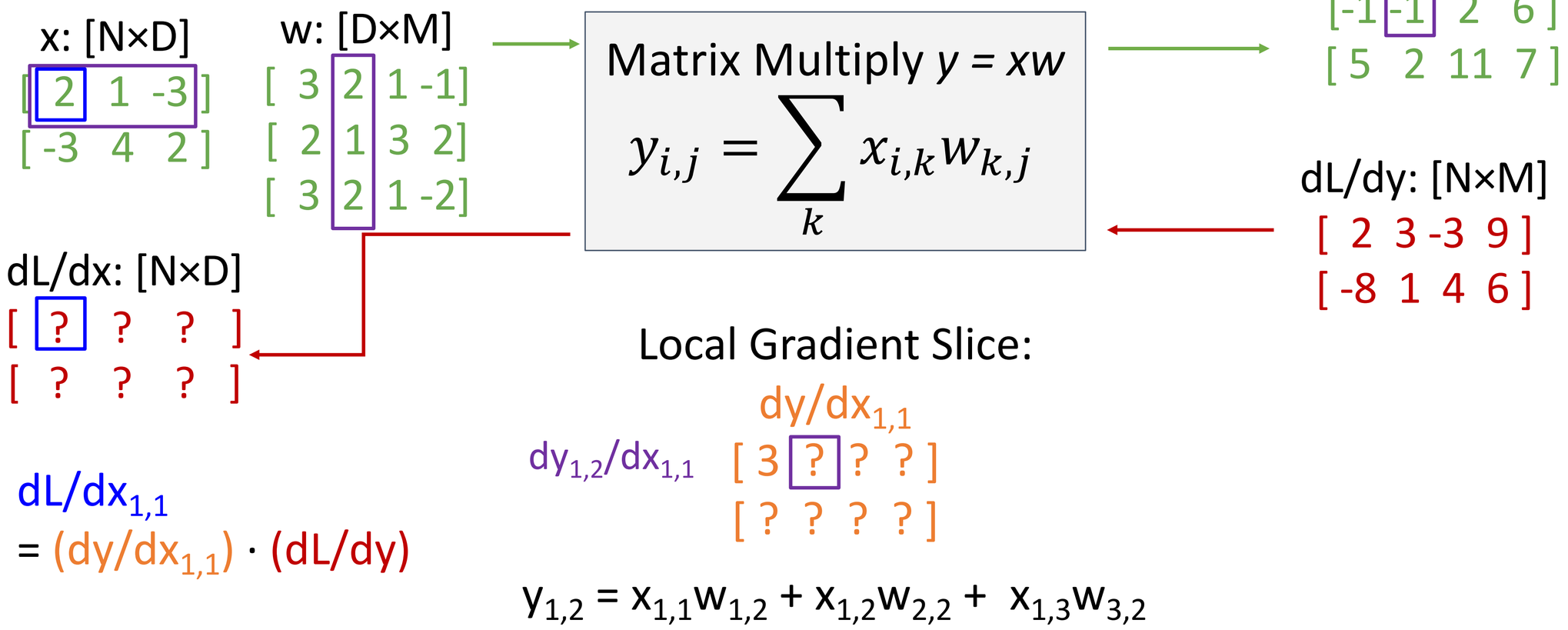
$dy_{1,1}/dx_{1,1}$ [3 ? ? ?]
[? ? ? ?]

$y_{1,1} = x_{1,1}w_{1,1} + x_{1,2}w_{2,1} + x_{1,3}w_{3,1}$
$\Rightarrow dy_{1,1}/dx_{1,1} = w_{1,1}$

Example: Matrix Multiplication



Example: Matrix Multiplication



Example: Matrix Multiplication

x: [N×D]

[2 1 -3]
[-3 4 2]

w: [D×M]

[3 2 1 -1]
[2 1 3 2]
[3 2 1 -2]

Matrix Multiply $y = xw$

$$y_{i,j} = \sum_k x_{i,k} w_{k,j}$$

[-1 -1 2 6]
[5 2 11 7]

dL/dy: [N×M]

[2 3 -3 9]
[-8 1 4 6]

dL/dx: [N×D]

[? ? ?]
[? ? ?]

$dL/dx_{1,1}$
$= (dy/dx_{1,1}) \cdot (dL/dy)$

Local Gradient Slice:

$dy/dx_{1,1}$

$dy_{1,2}/dx_{1,1}$ [3 2 ? ?]
[? ? ? ?]

$y_{1,2} = x_{1,1}w_{1,2} + x_{1,2}w_{2,2} + x_{1,3}w_{3,2}$
$\Rightarrow dy_{1,2}/dx_{1,1} = w_{1,2}$

Example: Matrix Multiplication

x: [N×D]
[2 1 -3]
[-3 4 2]

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[3 2 1 -1]
[2 1 3 2]
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$dy_{1,2}/dx_{1,1}$ [3 2 1 -1]
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dL/dx: [N×D]

[? ? ?]

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Local Gradient Slice:

$dy/dx_{1,1}$

$dy_{1,2}/dx_{1,1}$ [3 2 1 -1]

[? ? ? ?]

$dL/dx_{1,1}$

$= (dy/dx_{1,1}) \cdot (dL/dy)$

$y_{2,1} = x_{2,1}w_{1,1} + x_{2,2}w_{2,1} + x_{2,3}w_{3,1}$

Example: Matrix Multiplication

x: [N×D]

[2 1 -3]
[-3 4 2]

w: [D×M]

[3 2 1 -1]
[2 1 3 2]
[3 2 1 -2]

Matrix Multiply $y = xw$

$$y_{i,j} = \sum_k x_{i,k} w_{k,j}$$

[-1 -1 2 6]
[5 2 11 7]

dL/dy: [N×M]

[2 3 -3 9]
[-8 1 4 6]

dL/dx: [N×D]

[? ? ?]
[? ? ?]

Local Gradient Slice:

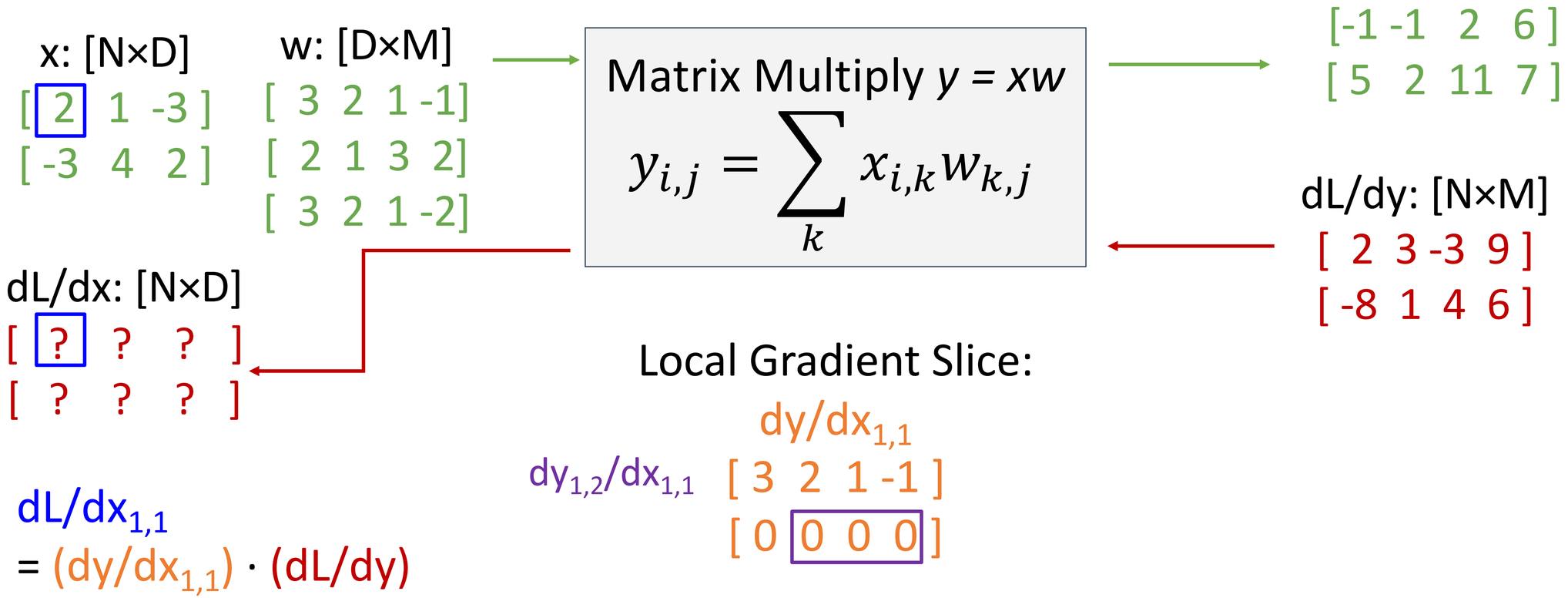
$dy/dx_{1,1}$

$dy_{1,2}/dx_{1,1}$ [3 2 1 -1]
[0 ? ? ?]

$dL/dx_{1,1}$
$= (dy/dx_{1,1}) \cdot (dL/dy)$

$y_{2,1} = x_{2,1}w_{1,1} + x_{2,2}w_{2,1} + x_{2,3}w_{3,1}$
$\Rightarrow dy_{2,1}/dx_{1,1} = 0$

Example: Matrix Multiplication



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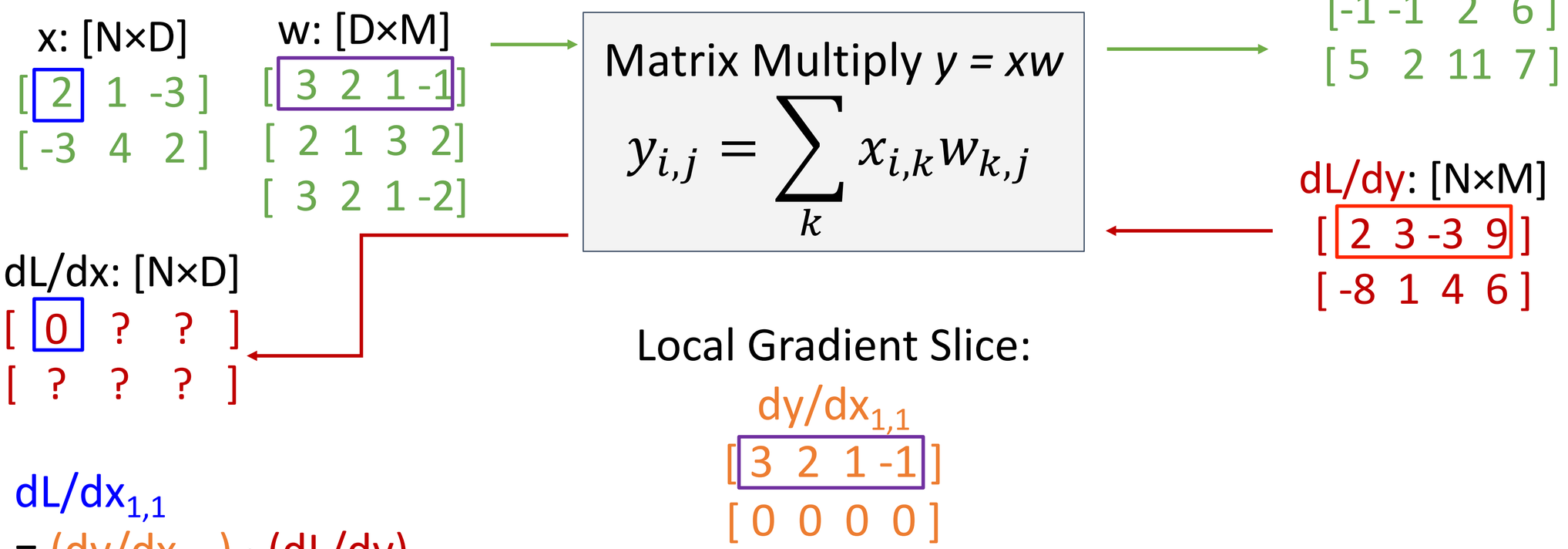
dL/dx: [N×D]
[? ? ?]
[? ? ?]

Local Gradient Slice:

$dy/dx_{1,1}$
[3 2 1 -1]
[0 0 0 0]

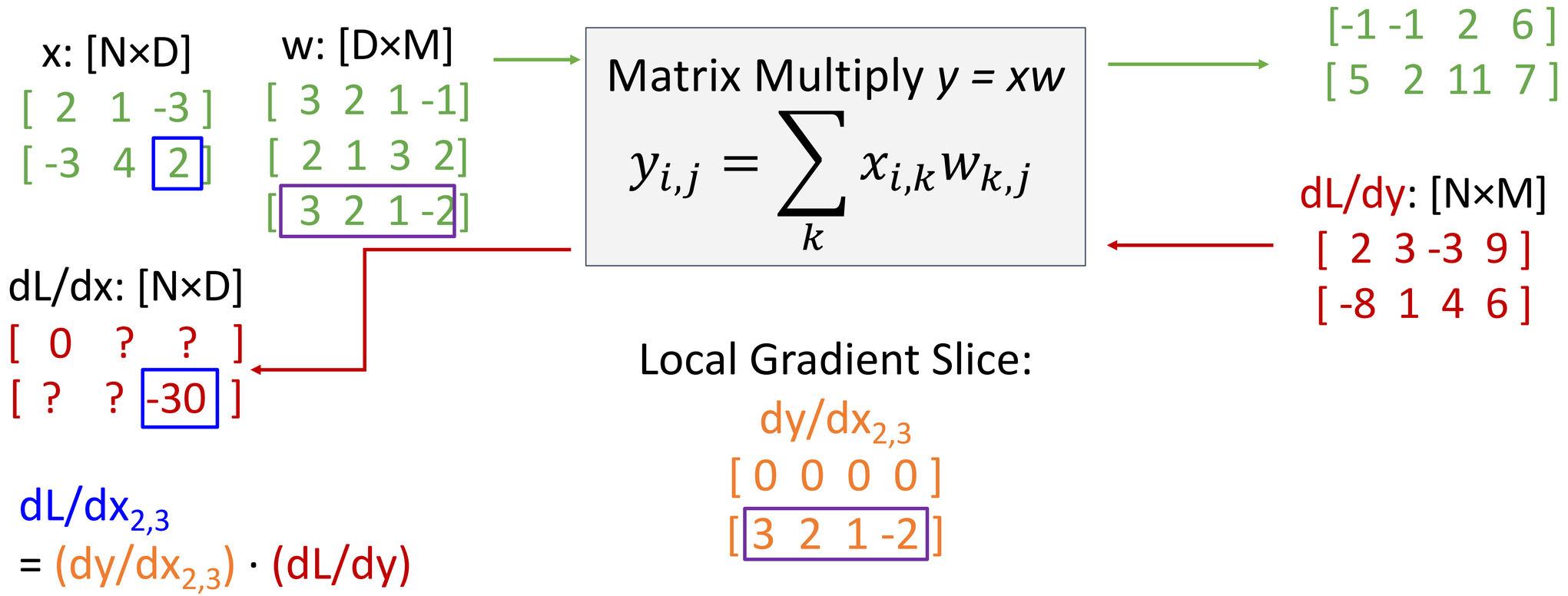
$dL/dx_{1,1}$
$= (dy/dx_{1,1}) \cdot (dL/dy)$

Example: Matrix Multiplication

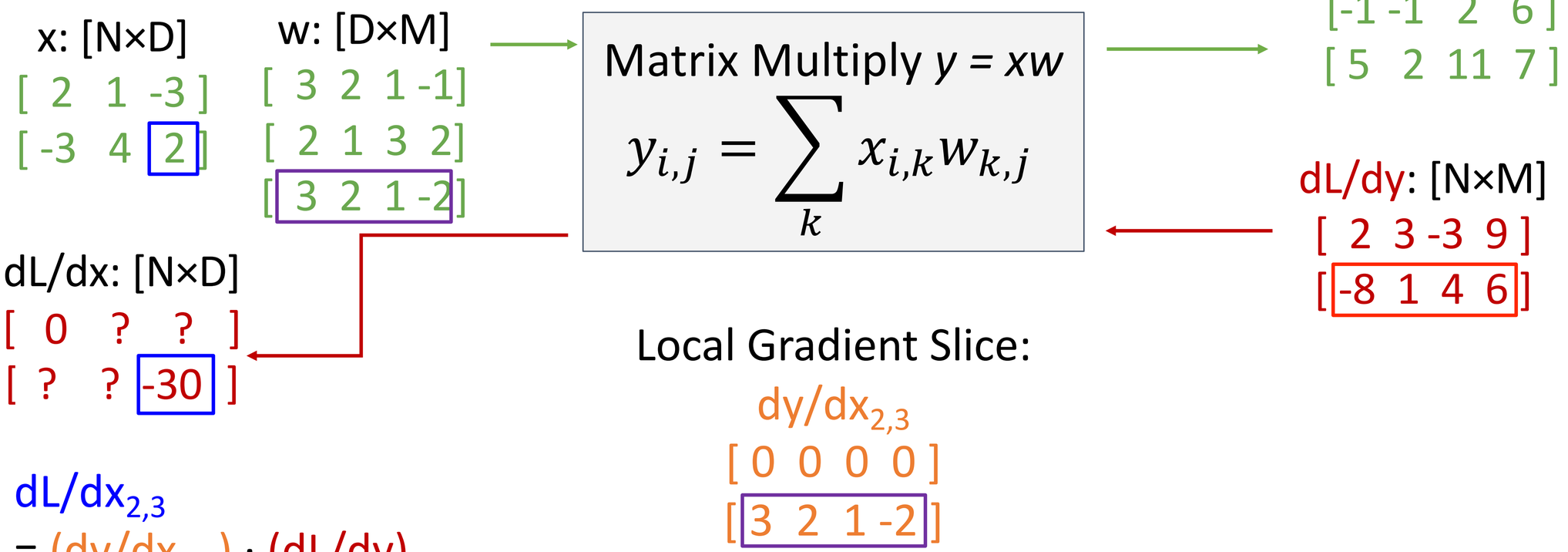


$dL/dx_{1,1}$
$= (dy/dx_{1,1}) \cdot (dL/dy)$
$= (w_{1,:}) \cdot (dL/dy_{1,:})$
$= 3*2 + 2*3 + 1*(-3) + (-1)*9 = 0$

Example: Matrix Multiplication

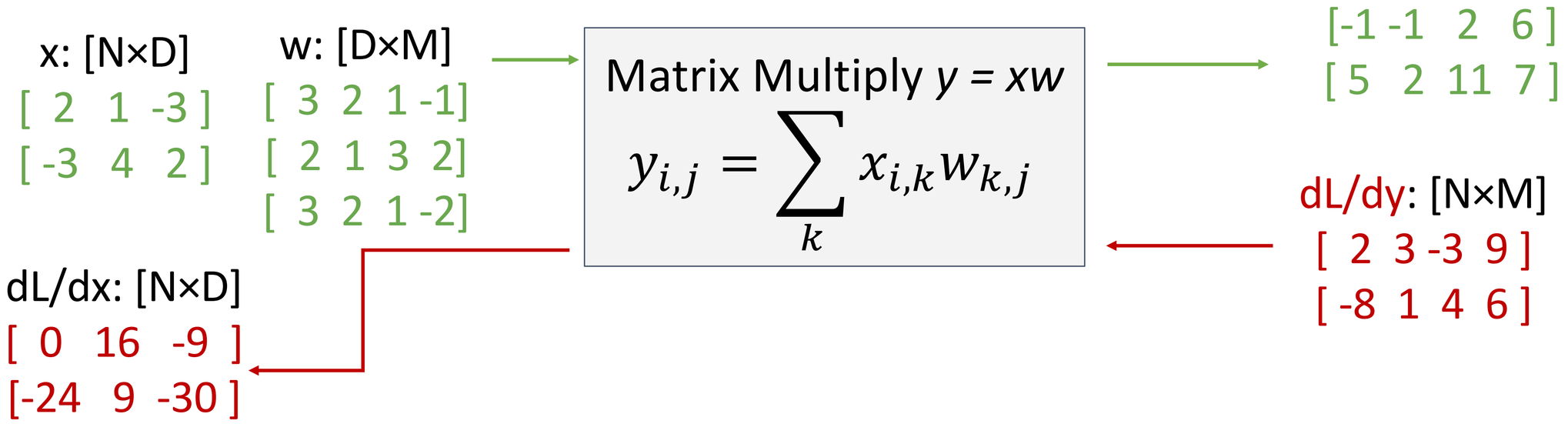


Example: Matrix Multiplication



$dL/dx_{2,3}$
$= (dy/dx_{2,3}) \cdot (dL/dy)$
$= (w_{3,:}) \cdot (dL/dy_{2,:})$
$= 3*(-8) + 2*1 + 1*4 + (-2)*6 = -30$

Example: Matrix Multiplication



$dL/dx_{i,j}$
$= (dy/dx_{i,j}) \cdot (dL/dy)$
$= (w_{j,:}) \cdot (dL/dy_{i,:})$

Example: Matrix Multiplication

x: [N×D]
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[-3 4 2]

w: [D×M]
[3 2 1 -1]
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[-1 -1 2 6]
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dL/dy: [N×M]
[2 3 -3 9]
[-8 1 4 6]

dL/dx: [N×D]
[0 16 -9]
[-24 9 -30]

$dL/dx = (dL/dy)\, w^T$

[N x D] [N x M] [M x D]

$dL/dx_{i,j}$
$= (dy/dx_{i,j}) \cdot (dL/dy)$
$= (w_{j,:}) \cdot (dL/dy_{i,:})$

Easy way to remember:
It's the only way the shapes work out!

Example: Matrix Multiplication

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w: [D×M]
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[0 16 -9]
[-24 9 -30]

$dL/dx = (dL/dy)\, w^T$

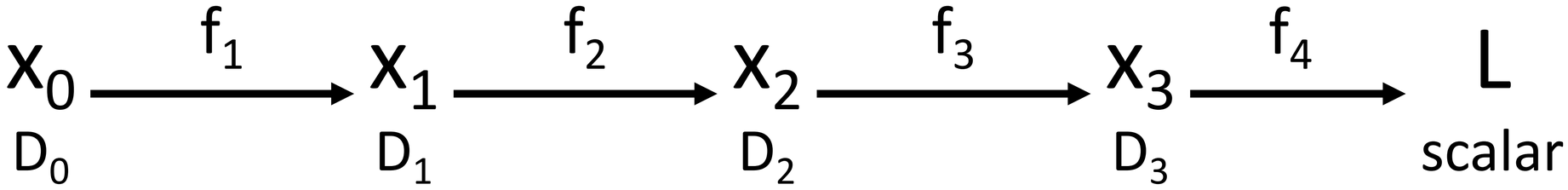
[N x D] [N x M] [M x D]

$dL/dw = x^T\, (dL/dy)$

[D x M] [D x N] [N x M]

Easy way to remember:
It's the only way the shapes work out!

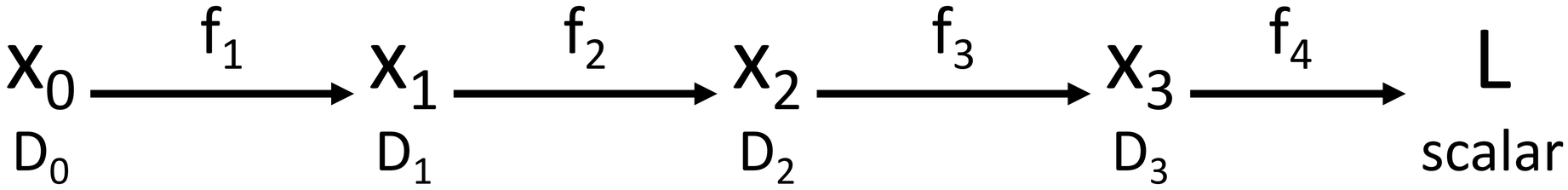
Backpropagation: Another View



Chain rule

$$\frac{\partial L}{\partial x_0} = \left(\frac{\partial x_1}{\partial x_0}\right)\left(\frac{\partial x_2}{\partial x_1}\right)\left(\frac{\partial x_3}{\partial x_2}\right)\left(\frac{\partial L}{\partial x_3}\right)$$

Backpropagation: Another View



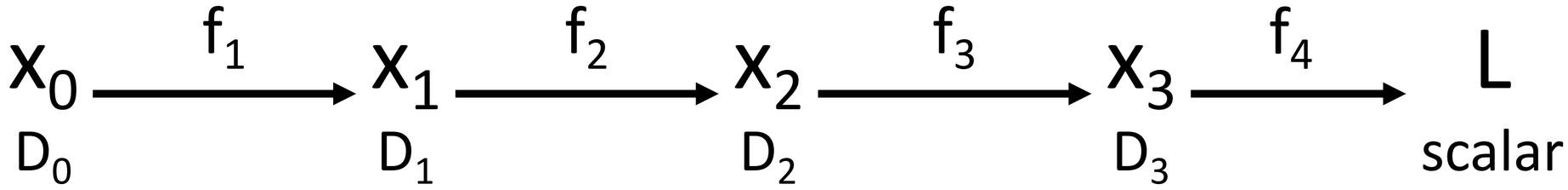
Matrix multiplication is **associative**: we can compute products in any order

Chain rule

$$\frac{\partial L}{\partial x_0} = \left(\frac{\partial x_1}{\partial x_0}\right)\left(\frac{\partial x_2}{\partial x_1}\right)\left(\frac{\partial x_3}{\partial x_2}\right)\left(\frac{\partial L}{\partial x_3}\right)$$

$[D_0 \times D_1]$ $[D_1 \times D_2]$ $[D_2 \times D_3]$ $[D_3]$

Reverse-Mode Automatic Differentiation



Matrix multiplication is **associative**: we can compute products in any order

Computing products right-to-left avoids matrix-matrix products; only needs matrix-vector

Chain rule

$$\frac{\partial L}{\partial x_0} = \left(\frac{\partial x_1}{\partial x_0}\right)\left(\frac{\partial x_2}{\partial x_1}\right)\left(\frac{\partial x_3}{\partial x_2}\right)\left(\frac{\partial L}{\partial x_3}\right)$$

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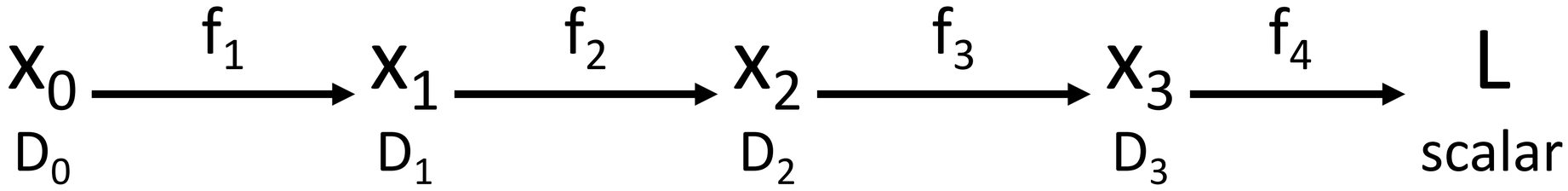
Reverse-Mode Automatic Differentiation



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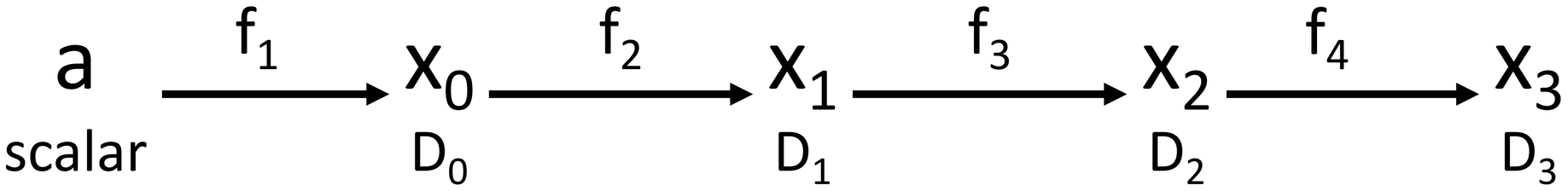
$[D_0 \times D_1]$ $[D_1 \times D_2]$ $[D_2 \times D_3]$ $[D_3]$

Compute grad of scalar output w/respect to all vector inputs

What if we want grads of scalar input w/respect to vector outputs?

Forward-Mode Automatic Differentiation

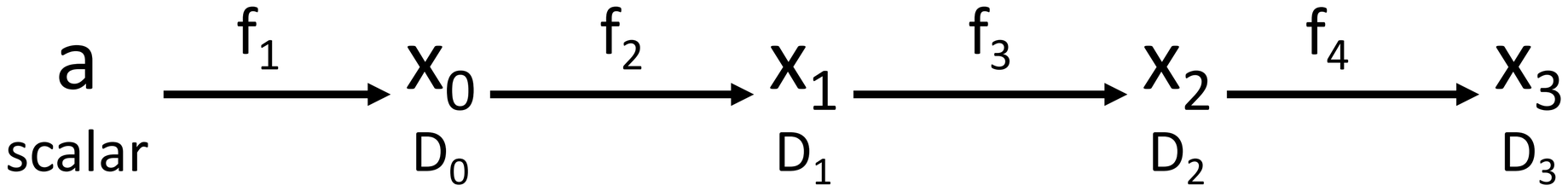


Chain rule

$$\frac{\partial x_3}{\partial a} = \left(\frac{\partial x_0}{\partial a}\right)\left(\frac{\partial x_1}{\partial x_0}\right)\left(\frac{\partial x_2}{\partial x_1}\right)\left(\frac{\partial x_3}{\partial x_2}\right)$$

$[D_0]$ $[D_0 \times D_1]$ $[D_1 \times D_2]$ $[D_2 \times D_3]$

Forward-Mode Automatic Differentiation



Computing products left-to-right avoids matrix-matrix products; only needs matrix-vector

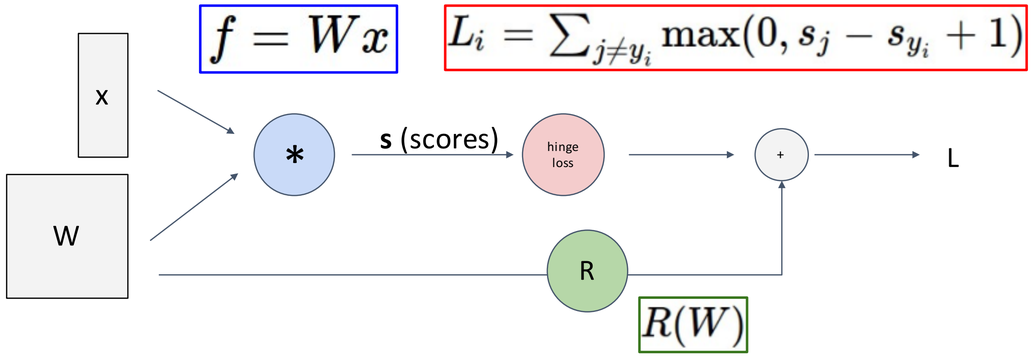
Chain rule

$$\frac{\partial x_3}{\partial a} = \left(\frac{\partial x_0}{\partial a}\right)\left(\frac{\partial x_1}{\partial x_0}\right)\left(\frac{\partial x_2}{\partial x_1}\right)\left(\frac{\partial x_3}{\partial x_2}\right)$$

$[D_0]$ $[D_0 \times D_1]$ $[D_1 \times D_2]$ $[D_2 \times D_3]$

Summary

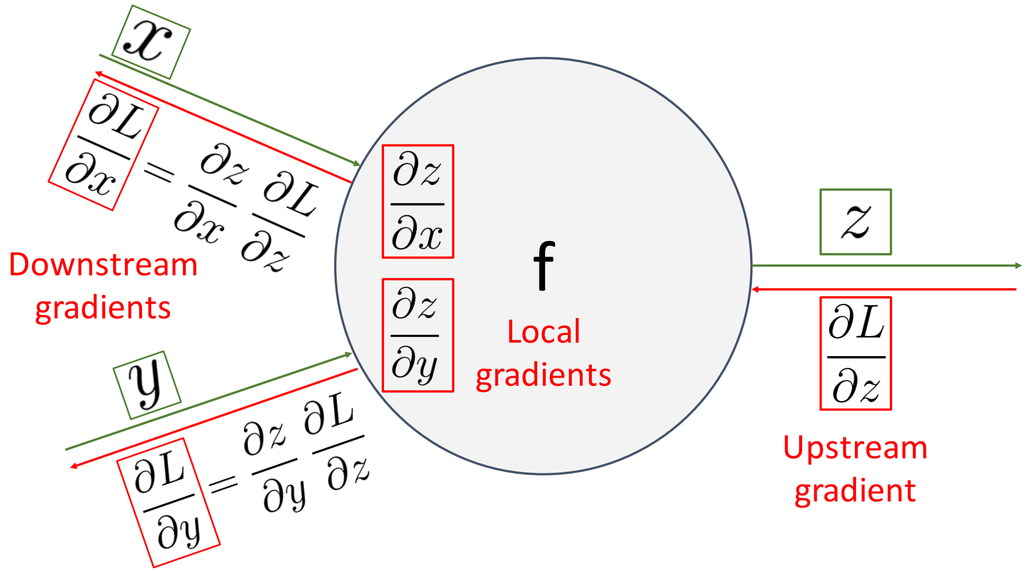
Represent complex expressions as **computational graphs**



Forward pass computes outputs

Backward pass computes gradients

During the backward pass, each node in the graph receives **upstream gradients** and multiplies them by **local gradients** to compute **downstream gradients**



Summary

Backprop can be implemented with “flat” code where the backward pass looks like forward pass reversed (Use this for A2!)

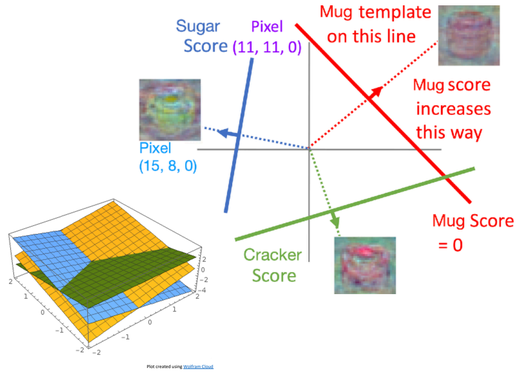
```
def f(w0, x0, w1, x1, w2):
  s0 = w0 * x0
  s1 = w1 * x1
  s2 = s0 + s1
  s3 = s2 + w2
  L = sigmoid(s3)

  grad_L = 1.0
  grad_s3 = grad_L * (1 - L) * L
  grad_w2 = grad_s3
  grad_s2 = grad_s3
  grad_s0 = grad_s2
  grad_s1 = grad_s2
  grad_w1 = grad_s1 * x1
  grad_x1 = grad_s1 * w1
  grad_w0 = grad_s0 * x0
  grad_x0 = grad_s0 * w0
```

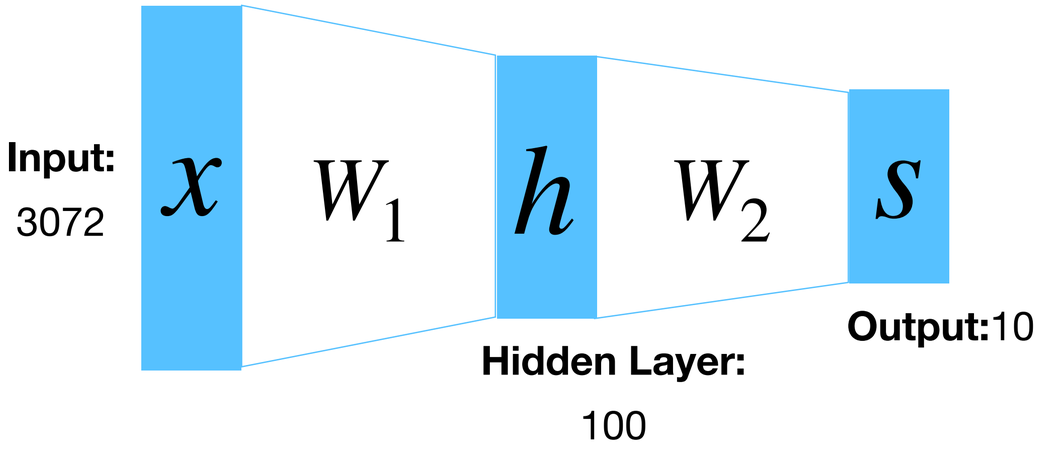
Backprop can be implemented with a modular API, as a set of paired forward/backward functions

```
class Multiply(torch.autograd.Function):
  @staticmethod
  def forward(ctx, x, y):
    ctx.save_for_backward(x, y)
    z = x * y
    return z
  @staticmethod
  def backward(ctx, grad_z):
    x, y = ctx.saved_tensors
    grad_x = y * grad_z   # dz/dx * dL/dz
    grad_y = x * grad_z   # dz/dy * dL/dz
    return grad_x, grad_y
```

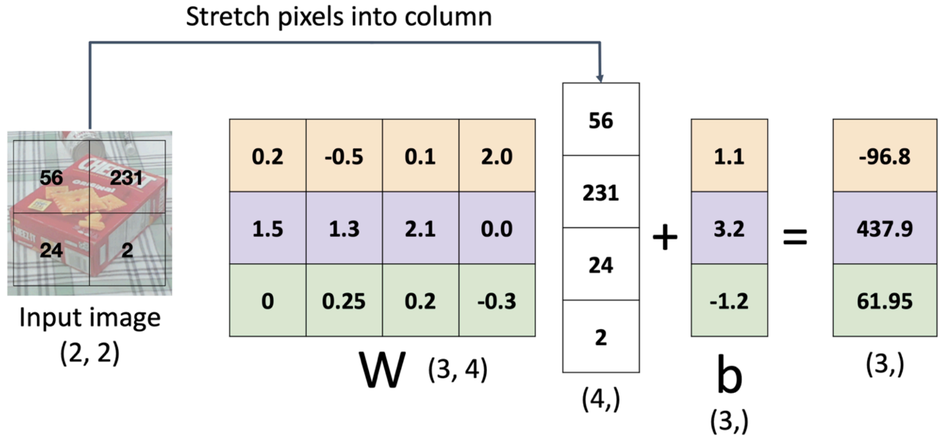
Summary



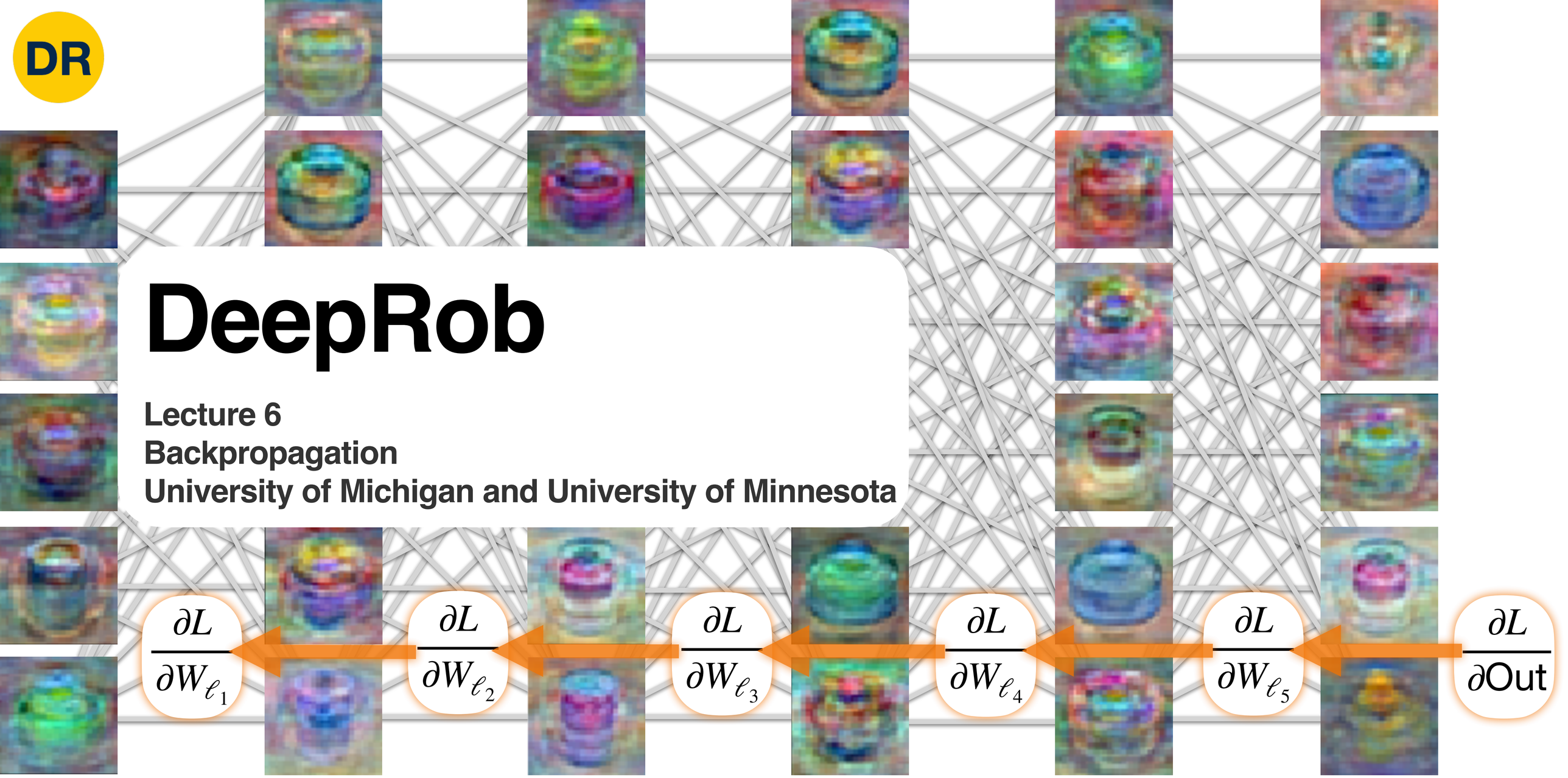
$$f(x) = W_2 \max(0, W_1 x + b_1) + b_2$$



Problem: So far our classifiers don't respect the spatial structure of images!



Next time: Convolutional Neural Networks


DeepRob
Lecture 6
Backpropagation
University of Michigan and University of Minnesota
$\frac{\partial L}{\partial W_{\ell_1}}$
$\frac{\partial L}{\partial W_{\ell_2}}$
$\frac{\partial L}{\partial W_{\ell_3}}$
$\frac{\partial L}{\partial W_{\ell_4}}$
$\frac{\partial L}{\partial W_{\ell_5}}$
$\frac{\partial L}{\partial \text{Out}}$