Lecture 1

Reflections on the Notion of Space I

The purpose of these first lectures is to understand the notion of a manifold in different contexts (topological/differential/analytic manifolds). We begin in this first lecture by looking at the case of topological manifolds.

1.1 Reminders on topological manifolds

Definition 1.1.1. *1. A **topological manifold** is a topological space X that has an open cover $\{U_i\}_{i\in I}$, such that for each $i \in I$, there exists a homeomorphism from U_i to an open set of $\mathbb{R}^{n_i}$ (for some $n_i \geq 0$ depedent on i).*

2. The category of topological manifolds is the full subcategory of topological spaces whose objects are topological manifolds. It is denoted by **TopMfd**.

Let X be a topological manifold and $\{U_i\}_{i\in I}$ an open cover as in definition 1.1(1). We write, for i and j in I, $U_{i,j} = U_n \cap U_j$. There's a diagram of topological spaces

$$\coprod_{(i,j)\in I^2} U_{i,j} \rightrightarrows \coprod_{i\in I} U_i$$

The first morphism sends $U_{i,j}$ to U_i via the natural inclusion $U_{i,j} subset U_i$, and the second morphism sends $U_{i,j}$ to U_j via the natural inclusion $U_{i,j} \subset U_j$. There's also another morphism

$$\coprod_{i \in I} U_i \to X$$

which are inclusions $U_i \subset X$. This coequalizes the two morphisms above. So we get a morphism of topological spaces

$$Colim \left(\coprod_{(i,j) \in I^2} U_{i,j} \rightrightarrows \coprod_{i \in I} U_i \right) \to X$$

The important fact is the following

Lemma 1.1.2. *The morphism*

$$Colim \left(\coprod_{(i,j) \in I^2} U_{i,j} \rightrightarrows \coprod_{i \in I} U_i \right) \to X$$

is an isomorphism.

Proof. The lemma says for a topological space Y, to give a morphism $f : X \to Y$ is the same as choosing, for each given $i \in I$, a morphism $f_i : U_i \to Y$ such that $(f_i)|_{U_{i,j}} = (f_j)|_{U_{i,j}}$ for all $(i, j) \in I^2$.

(Exercise: provide the details.) $\square$

One can interpret the above lemma as follows: all topological manifolds are obtains as the colimit of a diagram of opens in $\mathbb{R}^n$ (for some n). We draw from this the following principle:

Principle. *The category $TopMfd$ of topological manifolds is deduced from the category of opens in $\mathbb{R}^n$ (and continiuous maps.).*

this is the principle that we're going to clarify in the following.

1.2 Manifolds and sheaves

Let C be the full subcategory of $TopMfd$ whose objects are opens of $\mathbb{R}^n$. We denote by $Pr(C)$ the category of presheaves of sets over C (also denoted $\hat{C}$). We consider the Yoneda embedding restricted to C:

$$h_{(-)} : VarTop \to Pr(C) \tag{1.1}$$
$$X \mapsto h_X \tag{1.2}$$

where the presheaf h_X is defined by

$$h_X(Y) := Hom_{TopMfd}(Y, X)$$

Lemma 1.2.1. *The functor $h_{(-)}$ above is fully faithful.*

Proof. The functor is faithful: ...

The functor is full: ...

TODO: finish □

The lemma 2.1 is a good point of depature, for $TopMfd$ is identified as (is equivalent to) a full subcategory of $Pr(C)$. We seek a characterization of this subcategory.

We start with C a Grothendieck site by declaring a family of morphisms $\{U_i \to U\}_{i\in I}$ in C to be a covering family if each morphism $U_i \to U$ is an open immersion, and the total morphism $\sqcup_{i\in I} U_i \to U$ is surjective. This defines a pre-topology on C (exercise: verify). The associated topology is denoted τ.

Lemma 1.2.2. *For all $X \in TopMfd$, the presheaf $h_X \in Pr(C)$ is a sheaf for the topology τ.*

Proof. ... □

Lemma 2.2 implies we have a full and faithful functor

$$h_{(-)} : TopMfld \to Sh(C, \tau)$$

A sheaf isomorphic to h_X is said to be representable by X. More generally we identify the category $TopMfd$ with its image in $Sh(C, \tau)$

To chacaterize this image we make a definition

Definition 1.2.3. *1. A morphism $f : F \to G$ in $Sh(C, \tau)$ is a **local homeomphism** if for all $X \in C$, and all morphisms $h_X \to G$, the sheaf*

$F \times_G h_X$ is representable by some $Y \in TopMfd$, and the induced morphism $Y \to X$ by the projection $h_y \simeq F \times_G h_X \to h_X$ is a local homeomorphism of topological spaces [1].

2. *A morphism in $Sh(C, \tau)$ is an **open immersion** if it's a monomorphism and a local homeomorphism.*

It's easy to verify that open immersions in $Sh(C, \tau)$ are stable under compositions (Exercise: verify). One can also verify that local heomorphisms are stable under compositions, but this needs corrolary 2.5 below (Exercise: verify). We see also that a morphism of topological manifolds is a local homeomorphism if and only if $h_X \to h_Y$ is a local homeomorphism in the sense of the above definition (Exercise: verify).

We then have the following proposition

Proposition 1.2.4. *A sheaf $F \in Sh(C, \tau)$ is representable by a topological manifold (i.e. $F \simeq h_X$ for some $X \in TopMfd$), if there exists a family $\{U_i\}_{i \in I}$ of objects in C, and a morphism of sheaves*

$$p : \bigsqcup_{i \in I} h_{U_i} \to F$$

satisfying the following two conditions

1. *The morphism p is an epimorphism of sheaves.*

2. *For all $i \in I$, the morphism $U_i \to F$ is an open immersion (in the sense of definition 2.3)*

Proof. □

Corollary 1.2.5. *Let $X \in TopMfd$, and $F \to h_X$ a morphism of sheaves. If there exists an open cover of X such that for all $i \in I$, the sheaf $f \times_{h_X} h_{U_i}$ is representable by a topological manifold, then F is representable by a topological manifold.*

Proof. For all $i \in I$, choose ${V_{i,j}}_{j \in J} \cdots$ □

[1] Recall: a continuous map of topological spaces is a local homeomorphism if for each $x \in X$, there exists U an open neighborhood of x in X and V and open neighborhood of $f(x)$ in Y, such that f induces a homeomorphism from $U \to V$

1.3 Quotient manifolds

Let G be a (discrete) group acting on a topological manifold $X \in TopMfd$. By functoriality, the group G acts on the sheaf h_X. Recall a group action on X is ... . Recall also a group action G on X is properly discontinuous if all points $x \in X$ has an open neighborhood $U \subset X$ such that for all $g \in G$, we have

$$g(U) \cap U \neq \emptyset \implies (g = e)$$

In the following, we will take care not to confuse the sheaf quotient h_X/G and the sheaf $h_{X/G}$ represented by the quotient topological space.

Proposition 1.3.1. *1. If the action G on X is free, the the quotient morphism*

$$h_X \to h_X/G$$

is a local homeomorphism

2. If the action G on X is properly discontinuous, then the quotient $h_X/G \in Sh(C, \tau)$ is a topological manifold.

Proof. ... □

1.4 Shortcomings of manifolds

The proposition 3.1 is a good reason for cosntructing examples of topological manifolds by properly discontinuous actions. However, when G acts on a manifold X but the action is not propertly discontinuous, the topological space quotient X/G is in general very pathological. The sheaf quotient h_X/G. The sheaf quotient h_X/G has good properties (e.g. point (1) of proposition 3.1) similar to representability of a topological manifold.

An example is the following: we take the action of the discrete group $\mathbb{Q}$ (under addition on the topological space $\mathbb{R}$ via the morphism

$$\mathbb{R} \times \mathbb{Q} \to \mathbb{R}$$

given by $(x, t) \mapsto x + t$. We note this action is free, but not properly discontinuous. IN addition, the morphism $\mathbb{R} \to \mathbb{R}/\mathbb{Q}$ is not a local homeomorphism nor is it locally injective. Finally, the topological space quotient $\mathbb{R}/\mathbb{Q}$ has a gross topology. We see that the quotient $\mathbb{R}/\mathbb{Q}$ is not a reasonable object from the point of view of geometry. On the otherhand, the sheaf quotient $h_{\mathbb{R}}/\mathbb{Q}$ is more interesting, for the morphisms $h_{\mathbb{R}} \to h_{\mathbb{R}}/\mathbb{Q}$ is a local homeomorphism. The sheaf $h_{\mathbb{R}}/\mathbb{Q}$ is a primary example of a geometric space

Definition 1.4.1. *A sheaf $F \in Sh(C, \tau)$ is a geometric space if there exists a family of objects $\{U_i\}_{i \in I}$ of C, and a morphism of sheaves*

$$p : \bigsqcup_{i \in I} h_{U_i} \to F$$

satisfies the following two conditions

1. *The morphism p is an epimorphism of sheaves*
2. *For all $i \in I$, the morphism $U_i \to F$ is a local homeomorphism.*