Risk-Neutral Pricing in the Black-Scholes Model

Ray Otsuki

August 10, 2020

Contents

1 Risk Neutral Measures

For this chapter, we shall adorn measure-specific objects with a superscript of the measure that they are taken with respect to. Thus, $P^{\mathbb{P}}(A)$ denotes the probability of event A occurring *under the measure* $\mathbb{P}$, $\mathbb{E}^{\mathbb{P}}[f(X)]$ denotes the expectation under $f(X)$ *under the measure* $\mathbb{P}$ and $W^{\mathbb{P}}$ denotes a Brownian motion *under the measure* $\mathbb{P}$.

1.1 Girsanov's Theorem

Definition 1. *Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are said to be* equivalent *if they agree on the null set of events: $P^{\mathbb{P}}(A) = 0$ if, and only if, $P^{\mathbb{Q}}(A) = 0$ for all $A \in \mathcal{F}$.*

Let $\mathcal{Z}$ be a r.v. such that $\mathbb{E}^{\mathbb{P}}[\mathcal{Z}] = 1$ and $\mathcal{Z} > 0$. Let

$$P^{\mathbb{Q}}(A) := \mathbb{E}^{\mathbb{P}}[\mathcal{Z}\mathbf{1}_A] = \int_A \mathcal{Z}\mathrm{d}\mathbb{P}\,, \qquad A \in \mathcal{F}\,. \tag{1}$$

Remark 1. *The requirement $\mathbb{E}^{\mathbb{P}}[\mathcal{Z}] = 1$ is to ensure that $P^{\mathbb{Q}}(\Omega) = 1$.*

Since $P^{\mathbb{Q}}(A) \equiv \int_A \mathrm{d}\mathbb{Q}$, this can be written as

$$\mathrm{d}\mathbb{Q} = \mathcal{Z}\mathrm{d}\mathbb{P} \qquad \text{or} \qquad \mathcal{Z} = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\,, \tag{2}$$

and $\mathcal{Z}$ is called the *Radon-Nikodym derivative* (or *density*) of $\mathbb{Q}$ with respect to $\mathbb{P}$.

Theorem 1. *(Radon-Nikodym) Two measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if, and only if, there exists a random variable $\mathcal{Z}$ such that $\mathbb{E}[\mathcal{Z}] = 1$, $\mathcal{Z} > 0$ and $\mathbb{Q}$ is given by* (1)*.*

Proof. The proof is beyond the scope of these notes but the theorem above will be used extensively in pricing options. □

Let that $\mathcal{Z}$ is a martingale. Define a new measure for some fixed $T > 0$ by

$$\mathrm{d}\mathbb{Q} = \mathcal{Z}(T)\mathrm{d}\mathbb{P}\,. \tag{3}$$

The expectations with respect to the two measures are related as follows:

$$\mathbb{E}^{\mathbb{Q}}[X] = \int X \mathrm{d}\mathbb{Q} = \int \mathcal{Z}(T) X \mathrm{d}\mathbb{P} = \mathbb{E}^{\mathbb{P}}[\mathcal{Z}(T)X] \,. \tag{4}$$

Similarly, given a σ-algebra $\mathcal{F}$, we define the conditional expectation $\mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}]$ as the unique $\mathcal{F}$-measurable r.v. such that

$$\int_A \mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}] \mathrm{d}\mathbb{Q} = \int_A X \mathrm{d}\mathbb{Q} \qquad \forall\, A \in \mathcal{F} \,. \tag{5}$$

The remainder of this section will be devoted to proving Girsanov's theorem.

Theorem 2. *(Girsanov) Let $b(t) = (b_1(t), b_2(t), \ldots, b_d(t))$ be a d-dimensional adapted process, $W^{\mathbb{P}}(t)$ be a d-dimensional Brownian motion with respect to measure $\mathbb{P}$ and let*

$$W^{\mathbb{Q}}(t) = W^{\mathbb{P}}(t) + \int_0^t b(s)\, ds \,. \tag{6}$$

Let $\mathcal{Z}$ be the exponential process induced by $b(t)$:

$$\mathcal{Z}(t) = exp\left(-\frac{1}{2}\int_0^t |b(s)|^2 ds - \int_0^t b(s) \cdot dW^{\mathbb{P}}(s)\right) \,. \tag{7}$$

Define a new measure $d\mathbb{Q} = \mathcal{Z}(T) d\mathbb{P}$. If $\mathcal{Z}$ is a martingale with respect to $\mathbb{P}$, then $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$ up to time T.

Remark 2. *Note that*

$$b(s) \cdot dW^{\mathbb{P}}(s) = \sum_{i=1}^{d} b_i(s) dW_i^{\mathbb{P}}(s) \,, \qquad \textit{and} \qquad |b(s)|^2 = \sum_{i=1}^{d} b_i(s)^2 \,, \tag{8}$$

and that the differential form of the relation between $W^{\mathbb{Q}}$ and $W^{\mathbb{P}}$ is given by

$$dW_i^{\mathbb{Q}}(t) = dW_i^{\mathbb{P}}(t) + b_i(t) \,. \tag{9}$$

Remark 3. *The processes $W_i^{\mathbb{Q}}$ are* not *Brownian motions under $\mathbb{P}$ and the processes $W_i^{\mathbb{P}}$ are* not *Brownian motions under $\mathbb{Q}$; they are only Brownian motions under their respective measures.*

Before giving the proof, we first give two lemmas.

Lemma 1. *Let $0 \le s \le t \le T$ and let $\mathbb{P}$ and $\mathbb{Q}$ be two measures related by a martingale density $\mathcal{Z}(T)$. If X is a $\mathcal{F}_t$-measurable random variable, then*

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}_s] = \frac{1}{\mathcal{Z}(s)} \mathbb{E}^{\mathbb{P}}[\mathcal{Z}(t) X|\mathcal{F}_s] \,. \tag{10}$$

Proof. Let $A \in \mathcal{F}_s$ (and hence $A \in \mathcal{F}_t$ for $t \ge s$). Then,

$$\int_A \mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_s] \mathrm{d}\mathbb{Q} = \int_A \mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_s] \mathcal{Z}(T) \mathrm{d}\mathbb{P} \tag{11}$$

$$= \int_A \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_s]\mathcal{Z}(T) \,\middle|\, \mathcal{F}_s\right] \mathrm{d}\mathbb{P} \tag{12}$$

$$= \int_A \mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_s] \mathbb{E}^{\mathbb{P}}[\mathcal{Z}(T) \,|\, \mathcal{F}_s] \mathrm{d}\mathbb{P} \tag{13}$$

$$= \int_A \mathbb{E}^{\mathbb{Q}}[X \,|\, \mathcal{F}_s] \mathcal{Z}(s) \mathrm{d}\mathbb{P} \,, \tag{14}$$

where the second line treats $\mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{F}_s]\mathcal{Z}(T)$ as a single r.v. and replaces it with a conditional expectation and the fourth line follows from the martingale assumption of $\mathcal{Z}(t)$. On the other hand, using the fact that X is $\mathcal{F}_t$-measurable and that $\mathcal{Z}(T)$ is a martingale, we have

$$\int_A \mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{F}_s]\mathrm{d}\mathbb{Q} = \int_A X\mathrm{d}\mathbb{Q} = \int_A X\mathcal{Z}(T)\mathrm{d}\mathbb{P} \tag{15}$$

$$= \int_A \mathbb{E}^{\mathbb{P}}\left[X\mathcal{Z}(T) \mid \mathcal{F}_t\right]\mathrm{d}\mathbb{P} \tag{16}$$

$$= \int_A X\mathbb{E}^{\mathbb{P}}\left[\mathcal{Z}(T) \mid \mathcal{F}_t\right]\mathrm{d}\mathbb{P} \tag{17}$$

$$= \int_A X\mathcal{Z}(t)\mathrm{d}\mathbb{P} \tag{18}$$

$$= \int_A \mathbb{E}^{\mathbb{P}}\left[X\mathcal{Z}(t) \mid \mathcal{F}_s\right]\mathrm{d}\mathbb{P}\,. \tag{19}$$

The integrands in both expressions are $\mathcal{F}_s$-measurable and thus must be equal. $\square$

Lemma 2. *An adapted process M is a martingale under $\mathbb{Q}$ if, and only if, $M\mathcal{Z}$ is a martingale under $\mathbb{P}$.*

Proof. Suppose, first, that $M\mathcal{Z}$ is a martingale with respect to $\mathbb{P}$. Then, using the previous lemma, we have

$$\mathbb{E}^{\mathbb{Q}}[M(t) \mid \mathcal{F}_s] = \frac{1}{\mathcal{Z}(s)}\mathbb{E}^{\mathbb{P}}[\mathcal{Z}(t)M(t) \mid \mathcal{F}_s] = \frac{1}{\mathcal{Z}(s)}\mathcal{Z}(s)M(s) = M(s)\,, \tag{20}$$

and so M is a martingale under $\mathbb{Q}$. Instead, assume that M is a martingale under $\mathbb{Q}$. Then,

$$\mathbb{E}^{\mathbb{P}}[M(t)\mathcal{Z}(t) \mid \mathcal{F}_s] = \mathcal{Z}(s)\mathbb{E}^{\mathbb{Q}}[M(t) \mid \mathcal{F}_s] = \mathcal{Z}(s)M(s) \tag{21}$$

and so $\mathcal{Z}(s)M(s)$ is a martingale under $\mathbb{P}$. $\square$

We now turn to the proof of Girsanov's theorem.

Proof. Clearly, $W^{\mathbb{Q}}$ is continuous and the quadratic variation between two components of the d-dimensional process is given by

$$\mathrm{d}[W_i^{\mathbb{Q}}, W_j^{\mathbb{Q}}] = \mathrm{d}[W_i^{\mathbb{P}}, W_j^{\mathbb{P}}](t) = \mathbf{1}_{i=j}\mathrm{d}t\,, \tag{22}$$

since the two processes differ only by a deterministic function which has vanishing quadratic variation. We first compute $\mathrm{d}\mathcal{Z}(t)$ from Itô's formula:

$$\mathrm{d}\mathcal{Z}(t) = -\frac{1}{2}|b(t)|^2\mathcal{Z}(t)\mathrm{d}t - \mathcal{Z}(t)b(t)\cdot\mathrm{d}W^{\mathbb{P}}(t) + \frac{1}{2}\mathcal{Z}(t)|b(t)|^2\mathrm{d}t = -\mathcal{Z}(t)b(t)\cdot\mathrm{d}W^{\mathbb{P}}(t)\,. \tag{23}$$

Although Itô integrals with respect to Brownian motions are only guaranteed to be local martingales, we shall assume that the square-integrable condition $\mathbb{E}^{\mathbb{P}}\left[\int_0^T |\mathcal{Z}(s)b(s)|^2\mathrm{d}s\right] < \infty$ holds such that $\mathcal{Z}(t)$ is a full martingale (see remark below). Then,

$$\mathrm{d}(\mathcal{Z}W_i^{\mathbb{Q}}) = \mathcal{Z}\mathrm{d}W_i^{\mathbb{Q}} + W_i^{\mathbb{Q}}\mathrm{d}\mathcal{Z} + \mathrm{d}[\mathcal{Z}, W_i^{\mathbb{Q}}] \tag{24}$$

$$= \mathcal{Z}\mathrm{d}W_i^{\mathbb{P}} + \mathcal{Z}b_i\mathrm{d}t - W_i^{\mathbb{Q}}\mathcal{Z}b\cdot\mathrm{d}W^{\mathbb{P}} - b_i\mathcal{Z}\mathrm{d}t = \mathcal{Z}\mathrm{d}W_i^{\mathbb{P}} - W_i^{\mathbb{Q}}\mathcal{Z}b\cdot\mathrm{d}W^{\mathbb{P}}\,. \tag{25}$$

Integrating both sides, the two terms on the right hand side are Itô integrals with respect to Brownian motions, and are thus local martingales. Square-integrability follows from the Cauchy-Schwarz inequality, making $\mathcal{Z}W_i^{\mathbb{Q}}$ a martingale under $\mathbb{P}$. The previous lemma then gives that $W_i^{\mathbb{Q}}$ is a martingale under $\mathbb{Q}$. Together with (22), Lévy's characterisation ensures that each $W_i^{\mathbb{Q}}(t)$ is a Brownian motion under $\mathbb{Q}$. $\square$

Remark 4. *The process $\mathcal{Z}$ is actually guaranteed to be a supermartingale $\mathbb{E}^{\mathbb{P}}[\mathcal{Z}] \leq 1$. However, for the present context, we require that $\mathcal{Z}$ is moreover a martingale. The* Novikov condition *or* Kazamaki condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |b(s)|^2 ds\right)\right] < \infty\,, \qquad or \qquad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t b(s)\cdot dW^{\mathbb{P}}(s)\right)\right] < \infty \tag{26}$$

give sufficient *conditions for $\mathcal{Z}$ to be a martingale. However, in many practical situations, it turns out that these conditions do not hold and so the martingale property of $\mathcal{Z}$ must be checked manually.*

1.2 Risk-Neutral Pricing

Consider a stock price following a generalised geometric Brownian motion:

$$\mathrm{d}S(t) = \alpha(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W^{\mathbb{P}}(t)\,, \tag{27}$$

where $\alpha(t)$ and $\sigma(t)$ are mean returns and volatilities respectively. They need not be constants, unlike in a conventional geometric Brownian motion, but must be adapted processes. We shall, however, assume that $\sigma(t) > 0$. The measure $\mathbb{P}$ here is called the *physical measure* and is the one dictated by the observed values in the stock market.

Suppose an investor places money in a risk-free account with variable interest rate $R(t)$ (again an adapted process). We define the *discount process D* by

$$D(t) = \exp\left(-\int_0^t R(s)\mathrm{d}s\right) \qquad \Leftrightarrow \qquad \mathrm{d}D(t) = -D(t)R(t)\mathrm{d}t\,. \tag{28}$$

In this notation, the discounted stock price is given by DS.

Definition 2. *A* risk-neutral measure *is a measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ and under which the discounted stock price process $D(t)S(t)$ is a martingale.*

Consider the dynamics of the discounted process (note that D is deterministic and so has vanishing quadratic covariance with S):

$$\mathrm{d}\left(D(t)S(t)\right) = D(t)\mathrm{d}S(t) + S(t)\mathrm{d}D(t) = (\alpha - R)DS\mathrm{d}t + DS\sigma\mathrm{d}W^{\mathbb{P}}(t)\,. \tag{29}$$

Define the *market price of risk* $\theta(t)$ by

$$\theta(t) := \frac{\alpha(t) - R(t)}{\sigma(t)} \tag{30}$$

and let

$$\mathrm{d}W^{\mathbb{Q}}(t) = \theta(t)\mathrm{d}t + \mathrm{d}W^{\mathbb{P}}(t)\,. \tag{31}$$

One may verify that

$$\mathrm{d}\left(D(t)S(t)\right) = \sigma(t)D(t)S(t)\mathrm{d}W^{\mathbb{Q}}(t)\,. \tag{32}$$

By Girsanov's theorem, $W^{\mathbb{Q}}(t)$ must be a Brownian motion with respect to the measure $\mathrm{d}\mathbb{Q} = \mathcal{Z}(T)\mathrm{d}\mathbb{P}$, where the Radon-Nikodym derivative (in this case) is the exponential process induced by the market price of risk[1]:

$$\mathcal{Z}(t) = \exp\left(-\int_0^t \theta(s)\mathrm{d}W^{\mathbb{P}}(s) - \frac{1}{2}\int_0^t \theta(s)^2\mathrm{d}s\right)\,. \tag{33}$$

[1] As per Remark 4, one should really check that $\mathcal{Z}$ is a martingale. This is evidently dependent on the functional form of all of the processes and so leave this computation for the application to the Black-Scholes model later.

Then, (32) tells us that the discounted stock price DS is a martingale under $\mathbb{Q}$, making $\mathbb{Q}$ a risk-neutral measure. Note that, under $\mathbb{Q}$, the dynamics of the stock price are given by

$$\mathrm{d}S(t) = R(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W^{\mathbb{Q}}(t)\,; \tag{34}$$

the mean return function $\alpha(t)$ has been replaced with the riskless rate $R(t)$ and so this measure has made the process risk-neutral.

In fact this idea of risk-neutrality holds more generally. Suppose that the value of a derivative on the stock is $C(t)$. We construct a replicating portfolio $\Pi(t)$ (whose value is equal to the value of the derivative $C(t)$ for all t) from $\Delta(t)$ units of stock and the remainder of the value held in a money market account. If we demand that the portfolio is self-financing, we have

$$\mathrm{d}\Pi(t) = \Delta(t)\mathrm{d}S(t) + R(t)\left(\Pi(t) - \Delta(t)S(t)\right)\mathrm{d}t\,. \tag{35}$$

Consider, now, the discounted portfolio value. After some algebra, and using the risk-neutral form of the stock price (34), we obtain

$$\mathrm{d}\left(D(t)\Pi(t)\right) = \Pi(t)\mathrm{d}D(t) + D(t)\mathrm{d}\Pi(t) = \sigma(t)\Delta(t)D(t)S(t)\mathrm{d}W^{\mathbb{Q}} \tag{36}$$

and so the discounted replicating portfolio of a derivative on the stock is also a martingale under the same measure $\mathbb{Q}$ that made the discounted stock a martingale.

Remark 5. *In the above, we have assumed the existence of a replicating portfolio for the security $C(t)$. This is justified by the* martingale representation theorem *which states that any martingale can be expressed as an Itô integral with respect to a Brownian motion (recall we already know that Itô integrals are martingales; the martingale representation theorem provides a partial converse).*

1.3 Risk-Neutral Pricing Formula

A particularly important application of the risk-neutral measure is the following theorem:

Theorem 3. *Let $C(T)$ be a $\mathcal{F}_T$-measurable random variable, representing the payoff of a derivative security with maturity T, and let $\mathbb{Q}$ be the risk-neutral measure. The arbitrage-free price at time t is given by the discounted measure under the risk-neutral measure:*

$$C(t) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T R(s)\,ds\right)C(T)\,\middle|\,\mathcal{F}_t\right]. \tag{37}$$

Proof. We shall price the derivative through a replicating portfolio $\Pi(t)$ which we showed was a martingale under the risk-neutral measure.

$$C(t) = \frac{1}{D(t)}D(t)\Pi(t) = \frac{1}{D(t)}\mathbb{E}^{\mathbb{Q}}\left[D(T)\Pi(T)\,|\,\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{D(T)}{D(t)}C(T)\,\middle|\,\mathcal{F}_t\right]. \tag{38}$$

□

1.4 Numéraire Pairs and Change of Numéraire Formula

The risk-neutral pricing method above can be extended by introducing the notion of *numéraires.*

Definition 3. *A* numéraire *is a process $N(t)$ that is a.s. strictly positive for all $t \in [0,T]$.*

Remark 6. *An almost surely strictly positive process is a process that can, in principle, hit zero but for which the probability of occurrence of such events is zero. The restriction ensures that the division of a process by a numéraire remains well-defined. In this context, we talk of pricing an object 'in terms of' (or 'in units of') the numéraire.*

Definition 4. *A* numéraire pair *$(N(t),\mathbb{Q})$ consists of a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that the process $\Pi(t)/N(t)$ is a local martingale under $\mathbb{Q}$ for any portfolio processes $\Pi(t)$.*

The utility of being able to do such a transformation hinges on the following:

Proposition 1. *Let $(N(t), \mathbb{Q})$ be a numéraire pair and let $\Pi(t) = \sum_{i=1}^{n} w_i(t) S_i(t)$ be the value of a self-financing portfolio holding weights $w_i(t)$ in basic assets $S_i(t)$. Then, the process $\Pi(t)/N(t)$ is also self-financing.*

Proof. Recall that a self-financing portfolio satisfies

$$\mathrm{d}\Pi(t) = \sum_{i=1}^{n} w_i(t) \mathrm{d}S_i(t) \,. \tag{39}$$

From Itô's lemma[2] we have

$$\begin{aligned}
\mathrm{d}\left(\frac{\Pi(t)}{N(t)}\right) &= \frac{1}{N(t)}\mathrm{d}\Pi(t) - \frac{\Pi(t)}{N(t)^2}\mathrm{d}N(t) - \frac{1}{N(t)^2}\mathrm{d}[\Pi, N](t) + \frac{\Pi(t)}{N(t)^3}\mathrm{d}[N, N](t) && (41) \\
&= \frac{\sum_{i=1}^{n} w_i(t)\mathrm{d}S_i(t)}{N(t)} - \frac{\sum_{i=1}^{n} w_i(t)S_i(t)}{N(t)^2}\mathrm{d}N(t) - \frac{\sum_{i=1}^{n} w_i(t)}{N(t)^2}\mathrm{d}[S_i, N](t) \\
&\quad + \frac{\sum_{i=1}^{n} w_i(t)S_i(t)}{N(t)^3}\mathrm{d}[N, N](t) && (42) \\
&= \sum_{i=1}^{n} w_i(t)\left(\frac{\mathrm{d}S_i(t)}{N(t)} - \frac{S_i(t)\mathrm{d}N(t)}{N(t)^2} - \frac{\mathrm{d}[S_i, N](t)}{N(t)^2} + \frac{S_i(t)\mathrm{d}[N, N](t)}{N(t)^3}\right) \,. && (43)
\end{aligned}$$

One may verify (again using Itô's lemma) that each summand on the right is equal to $\mathrm{d}(S_i(t)/N(t))$ and so we obtain

$$\mathrm{d}\left(\frac{\Pi(t)}{N(t)}\right) = \sum_{i=1}^{n} w_i(t) \mathrm{d}\left(\frac{S_i(t)}{N(t)}\right) \tag{44}$$

□

The above result means that we do not need to worry that a choice of numéraire might affect the self-financing property of a portfolio. We now close with the main result of this section:

Lemma 3. *Let $(N, \mathbb{Q})$ be a numéraire pair and let $\tilde{N}$ be a second numéraire. If*

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\tilde{N}(T)}{N(T)}\right] = 1 \,, \tag{45}$$

then $(\tilde{N}, \tilde{\mathbb{Q}})$ is also a numéraire pair, where $\tilde{\mathbb{Q}}$ is defined by the Radon-Nikodym derivative

$$\mathcal{Z}(T) = \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{\tilde{N}(T)}{N(T)} \,. \tag{46}$$

Proof. By definition of a numériare pair, $\tilde{N}(T)/N(T)$ is a $\mathbb{Q}$-martingale. Recall that a process $Y(t)$ is a local $\tilde{\mathbb{Q}}$-martingale if, and only if, $Y(t)\mathcal{Z}$ is a local $\mathbb{Q}$-martingale. Parametrising $Y(t) = X(t)/\tilde{N}(t)$ for any portfolio process $X(t)$, then $Y(t)\mathcal{Z} = X(t)/N(t)$ which is indeed a $\mathbb{Q}$-martingale. Thus, any process of the form $X(t)/\tilde{N}(t)$ is a local $\tilde{\mathbb{Q}}$-martingale and so $(\tilde{N}, \tilde{\mathbb{Q}})$ is a numéraire pair. □

[2] Recall, if $X(t) = (X_1(t), \ldots, X_n(t))$ be a n-dimensional continuous stochastic semimartingales and $f(X(t))$ be a function of it. Then,

$$\mathrm{d}f(X(t)) = \sum_{i=1}^{n} \partial_i f(X(t))\mathrm{d}X_i(t) + \frac{1}{2}\sum_{i,j=1}^{n} \partial_i\partial_j f(X(t))\mathrm{d}[X_i, X_j](t) \,. \tag{40}$$

Remark 7. *More generally, since* $(N, \mathbb{Q})$ *is a numéraire pair, the process* $\tilde{N}(t)/N(t)$ *is a local martingale and so*

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\tilde{N}(t)}{N(t)}\,\middle|\,\mathcal{F}_s\right] = \frac{\tilde{N}(s)}{N(s)} \qquad \Rightarrow \qquad \mathbb{E}^{\mathbb{Q}}\left[\frac{\tilde{N}(t)/\tilde{N}(s)}{N(t)/N(s)}\,\middle|\,\mathcal{F}_s\right] = 1 \tag{47}$$

for $0 \leq s \leq t \leq T$. *Taking the expectation of both sides, and using the law of total expectation, we see that*

$$\mathcal{Z} = \frac{\tilde{N}(t)/\tilde{N}(s)}{N(t)/N(s)} \tag{48}$$

has expectation 1 and can thus be interpreted as a density for a new numéraire pair. In this setup, our starting numéraire pair is $(N(t)/N(s), \mathbb{Q})$. *Parametrise any portfolio process* $Y(t)$ *as*

$$Y(t) = \frac{X(t)}{\tilde{N}(t)/\tilde{N}(s)}\,. \tag{49}$$

Since $Y(t)\mathcal{Z} = \frac{X(t)}{N(t)/N(s)}$ *is a local martingale by definition of the numéraire pair, this means that* $Y(t)$ *is a local martingale under the measure defined by* $d\tilde{\mathbb{Q}} = \mathcal{Z}d\mathbb{Q}$ *and so* $(\tilde{N}(t)/\tilde{N}(s), \tilde{\mathbb{Q}})$ *is also a numéraire pair.*

The result of this is that once we have obtained a numéraire pair $(N, \mathbb{Q})$, we can essentially switch to the most convenient numéraire pair $(\tilde{N}, \tilde{\mathbb{Q}})$ at will by the prescription above. The crucial point to note is that the risk-neutral pricing formula precisely gives us a numéraire pair $(B, \mathbb{Q})$, where $\mathbb{Q}$ in the present context is precisely the risk-neutral measure, that we may use as the starting point of a change of numéraire. In more detail, we rephrase the risk-neutral pricing formula in terms of the bank account $B(t)$ rather than the discount process and this will allow us to exploit the results above. We first rewrite (37) as

$$\frac{C(t)}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{C(T)}{B(T)}\,\middle|\,\mathcal{F}_t\right] \qquad \forall\, t \leq T\,, \tag{50}$$

where

$$B(t) = e^{\int_0^t R(s)\mathrm{d}s} = \frac{1}{D(t)} \tag{51}$$

is the risk-free bank account. This demonstrates that the process $C(t)/B(t)$ is a martingale under $\mathbb{Q}$ and so $(B(t), \mathbb{Q})$ is a numéraire pair. By the previous lemma, if there exists an alternate numéraire $\tilde{N}(t) \neq B(t)$ such that

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\tilde{N}(t)}{B(t)}\right] = 1\,, \tag{52}$$

then $(\tilde{N}, \tilde{\mathbb{Q}})$, with $\tilde{\mathbb{Q}}$ defined by $\mathcal{Z} = \tilde{N}(t)/B(t)$, is also a numéraire pair and it follows that

$$\frac{C(t)}{\tilde{N}(t)} = \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\frac{C(T)}{\tilde{N}(T)}\,\middle|\,\mathcal{F}(t)\right]\,. \tag{53}$$

2 Pricing Options in the Black-Scholes world

In the Black-Scholes worlds, we assume that $\sigma(t) = \sigma, \alpha(t) = \mu$ and $R = r$ are constants and that the dynamics of the stock price are given by

$$\mathrm{d}S_t = S_t(\mu\mathrm{d}t + \sigma\mathrm{d}W^{\mathbb{P}}(t))\,, \tag{54}$$

where $\mathbb{P}$ is the physical measure and $W^{\mathbb{P}}$ is a martingale under $\mathbb{P}$. As before, we define the market price of risk (here a constant)

$$\theta = \frac{\mu - r}{\sigma}\,. \tag{55}$$

We define the new process (which *a priori* is not know to be a Brownian motion)

$$W^{\mathbb{Q}} = W^{\mathbb{P}} + \int_0^t \theta \mathrm{d}s = W^{\mathbb{P}} + \theta t\,, \tag{56}$$

and also the exponential process generated by $b(t) = \theta$:

$$\mathcal{Z}(T) = \exp\left(-\frac{1}{2}\theta T^2 - \theta W_T\right)\,. \tag{57}$$

Since $b(t) = \theta$ satisfies the Novikov condition, $\mathcal{Z}$ is a martingale and so Girsanov's theorem holds and $W^{\mathbb{Q}}$ is a Brownian motion under the (risk-neutral) measure $\mathrm{d}\mathbb{Q} = \mathcal{Z}(T)\mathrm{d}\mathbb{P}$. The dynamics of the stock price under $\mathbb{Q}$ are given by

$$\mathrm{d}S(t) = S(t)\left(r\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{Q}}(t)\right)\,. \tag{58}$$

2.1 European Options and The Black-Scholes Formula

The risk-neutral pricing formula implies that the arbitrage-free price of a European call option is given by the discounted price under the risk-neutral measure $\mathbb{Q}$:

$$C_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r\mathrm{d}s}(S_T - K)^+ \,|\, \mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)\mathbf{1}_{S_T>K} \,|\, \mathcal{F}_t\right]\,. \tag{59}$$

The indicator function can be imposed in the limits in the integral, taken with respect to the stock price:

$$\begin{aligned} C_t &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T \mathbf{1}_{S_T>K} \,|\, \mathcal{F}_t\right] - Ke^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{S_T>K} \,|\, \mathcal{F}_t\right] && (60)\\ &= c_1 + c_2 && (61)\end{aligned}$$

The second term is easier to evaluate; it is simply the probability that the terminal price is greater than the strike K under the risk-neutral measure:

$$c_2 = -Ke^{-r(T-t)}\int_\Omega \mathbf{1}_{S_T>K}\mathrm{d}\mathbb{Q} = -Ke^{-r(T-t)}\int_K^\infty \mathrm{d}\mathbb{Q} = -Ke^{-r(T-t)}P^{\mathbb{Q}}(S_T > K)\,. \tag{62}$$

To evaluate this probability, note that the dynamics of the terminal stock price under $\mathbb{Q}$ is given by

$$\ln S_T = \ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma W^{\mathbb{Q}}_{T-t}\,, \tag{63}$$

where the process $W^{\mathbb{Q}}_t$ is a Brownian motion with respect to $\mathbb{Q}$. Since $W^{\mathbb{Q}}_{T-t} \sim N(0, T-t)$, we have

$$\ln S_T \sim N\left(\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)\,, \tag{64}$$

and use this to define the standard Gaussian variate

$$Z = \frac{\ln|S_T/S_t| - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \sim N(0,1)\,. \tag{65}$$

Then, we have

$$P^{\mathbb{Q}}(S_T > K) = 1 - P^{\mathbb{Q}}(S_T \leq K) = 1 - P^{\mathbb{Q}}(\ln S_T \leq \ln K) \tag{66}$$

$$= 1 - P^{\mathbb{Q}}\left(Z \leq -\frac{\ln|S_t/K| + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \tag{67}$$

$$= 1 - \Phi\left(-\frac{\ln|S_t/K| + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right). \tag{68}$$

The first term c_1 is more difficult but can be simplified by a change of numéraire. Recall that, by using the risk-neutral measure $\mathbb{Q}$, we are implicitly using the bank account $B(t) = \exp(rt)$ as the numéraire[3]. Here, we shall change numéraires to use the stock price instead $\tilde{N}(t) = S(t)$ and denote the associated measure as $\tilde{\mathbb{Q}}$. The Radon-Nikodym derivative describing this change of measure is

$$\mathcal{Z} = \frac{S(T)/S(t)}{B(T)/B(t)} = e^{-r(T-t)} e^{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma W^{\mathbb{Q}}_{T-t}} = e^{-\frac{\sigma^2}{2}(T-t) + \sigma W^{\mathbb{Q}}_{T-t}}. \tag{69}$$

Then,

$$c_1 = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(T)/B(t)} S_T \mathbf{1}_{S_T > K} \,|\, \mathcal{F}_t\right] \tag{70}$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\mathcal{Z} S_t \mathbf{1}_{S_T > K} \,|\, \mathcal{F}_t\right] \tag{71}$$

$$= S_t \mathbb{E}^{\mathbb{Q}}\left[\mathcal{Z} \mathbf{1}_{S_T > K} \,|\, \mathcal{F}_t\right] \tag{72}$$

$$= S_t \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\mathbf{1}_{S_T > K} \,|\, \mathcal{F}_t\right] = \int_K^\infty \mathrm{d}\tilde{\mathbb{Q}}, \tag{73}$$

which is the probability of the terminal stock price exceeding the strike under this new measure $\tilde{\mathbb{Q}}$. To evaluate this, we compare (69) to the exponential process to determine that $b(t) = -\sigma$. This evidently satisfies the Novikov condition and so $\mathcal{Z}$ is a martingale and Grisanov's theorem holds. It follows that the process

$$\mathrm{d}W^{\tilde{\mathbb{Q}}}(t) = \mathrm{d}W^{\mathbb{Q}}(t) - \sigma \mathrm{d}t \tag{74}$$

is a Brownian motion under $\mathrm{d}\tilde{\mathbb{Q}} = \mathcal{Z}\mathrm{d}\mathbb{Q}$. The dynamics of the stock price under this measure is given by

$$\mathrm{d}S_t = S_t\left((r + \sigma^2)\mathrm{d}t + \mathrm{d}W^{\tilde{\mathbb{Q}}}(t)\right), \tag{75}$$

i.e. the stock price is still log-normally distributed under this new measure $\tilde{\mathbb{Q}}$ (albeit with a different drift). We proceed as before. Noting that

$$\ln S_T \sim N\left(\ln S_t + \left(r + \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right), \tag{76}$$

we define the standard Gaussian variate

$$\tilde{Z} = \frac{\ln|S_T/S_t| - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \sim N(0,1). \tag{77}$$

Then,

$$P^{\tilde{\mathbb{Q}}}(S_T > Q) = 1 - P^{\tilde{\mathbb{Q}}}(S_T \leq K) = 1 - P^{\tilde{\mathbb{Q}}}(\ln S_T \leq \ln K) \tag{78}$$

$$= 1 - P^{\tilde{\mathbb{Q}}}\left(\tilde{Z} \leq -\frac{\ln|K/S_t| + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \tag{79}$$

$$= 1 - \Phi\left(-\frac{\ln|K/S_t| + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right). \tag{80}$$

[3] Or rather $B(T)/B(t) = e^{r(T-t)}$ as numéraire.

Finally, we use the fact that $1-\Phi(-x)=\Phi(x)$ in both c_1 and c_2 to obtain the Black-Scholes formula

$$C_t(t,S_t) = S_t\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)\,, \tag{81}$$

where

$$d_\pm = \frac{\ln|S_t/K| + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\,. \tag{82}$$

For completeness, we shall demonstrate how c_1 can also be computed by directly. Note that

$$c_1 = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[e^{\ln S_T}\mathbf{1}_{S_T>K}\,|\,\mathcal{F}_t]\,, \tag{83}$$

where $\mathbb{E}^{\mathbb{Q}}[e^{\ln S_T}]$ looks like the moment-generating function of a Gaussian variate $\ln S_T$ (note that the indicator function only affects the limits of integration). To this end, we make use of the following lemma:

Lemma 4. *Let $X \sim N(\mu,\sigma^2)$. Then, the truncated moment-generating function for $X \in [A,B]$ is given by*

$$\mathbb{E}[e^{tX}\mathbf{1}_{A\le X\le B}] = e^{t\mu+\frac{t^2\sigma^2}{2}}\left[\Phi\left(\frac{B-(\mu+t\sigma)}{\sigma}\right) - \Phi\left(\frac{A-(\mu+t\sigma)}{\sigma}\right)\right]. \tag{84}$$

Proof. The proof is entirely analogous to the standard calculation of the m.g.f. and will thus be omitted. ☐

For our purposes, we have $A=\ln K$, $B=\infty$ and $t=1$ giving

$$\begin{aligned} c_1 &= e^{-r(T-t)}e^{\ln S_t + \left(r-\frac{\sigma^2}{2}\right)(T-t)+\frac{\sigma^2(T-t)}{2}} \\ &\quad\times\left(\Phi(\infty) - \Phi\left(\frac{\ln K - \left(\ln S_t + \left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma^2(T-t)\right)}{\sigma\sqrt{T-t}}\right)\right) && (85)\\ &= S_t\left(1-\Phi\left(-\frac{\ln|S_t/K| + \left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right). && (86)\end{aligned}$$

We see that there are two probabilities in the expression for the call option. The first turns out to be the delta of the option; using $d_- = d_+ - \sigma\sqrt{T-t}$, we obtain

$$\begin{aligned} \Delta_t = \frac{\partial C_t}{\partial S_t} &= \Phi(d_+) + S_t\frac{\partial\Phi(d_+)}{\partial S_t} - Ke^{-r(T-t)}\frac{\partial\Phi(d_-)}{\partial S_t} && (87)\\ &= \Phi(d_+) + \frac{\phi(d_+)}{\sigma\sqrt{T-t}} - Ke^{-r(T-t)}\frac{\phi(d_-)}{S_t\sigma\sqrt{T-t}} && (88)\\ &= \Phi(d_+) + \frac{1}{\sigma\sqrt{T-t}}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}} - \frac{K}{S_t}e^{-r(T-t)}\frac{1}{\sqrt{2\pi}}e^{-\frac{(d_+-\sigma\sqrt{T-t})^2}{2}}\right) && (89)\\ &= \Phi(d_+) + \frac{1}{\sigma\sqrt{T-t}}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}}\left(1-\frac{K}{S_t}e^{-r(T-t)}e^{d_+\sigma\sqrt{T-t}-\frac{\sigma^2(T-t)}{2}}\right) && (90)\\ &= \Phi(d_+)\,. && (91)\end{aligned}$$

The second probability $\Phi(d_-)$ was the probability of $S_T > K$ under the risk-neutral measure and is thus, equivalently, the risk-neutral probability of exercise of the option.