The Heston Model

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1 Preliminaries

We start off with the dynamics of the Heston model under the physical measure $\mathbb{P}$:

$$\begin{cases} \mathrm{d}S_t & = \mu S_t \mathrm{d}t + \sqrt{v_t} S_t \mathrm{d}W^{\mathbb{P}}_{1,t}\,, \\ \mathrm{d}v_t & = \kappa(\theta - v_t)\mathrm{d}t + \sigma\sqrt{v_t}\mathrm{d}W^{\mathbb{P}}_{2,t}\,, \end{cases} \tag{1}$$

where

- $r \in \mathbb{R}$ is the risk-free rate;
- $\kappa > 0$ is the volatility mean-reversion rate;
- $\theta > 0$ is the long-term volatility;
- $\sigma > 0$ is the volatility of variance (or vol-of-vol).

The two Brownian motions are correlated with constant correlation coefficient $|\rho| \leq 1$:

$$\mathrm{d}[W^{\mathbb{P}}_1, W^{\mathbb{P}}_2]_t = \rho \mathrm{d}t\,, \tag{2}$$

In order to price European options, we shall transform to the risk-neutral measure (i.e. take the numéraire to be the money process $B(t) = e^{rt}$). Combining the two Brownian motion processes into the 2-dimensional vector $W^{\mathbb{P}}_t = (W^{\mathbb{P}}_{1,t}, W^{\mathbb{P}}_{2,t})^T$ and defining

$$\eta(t, S_t, v_t) = \left(\frac{\mu - r}{\sqrt{v_t}}, \frac{\tilde{\eta}(t, S_t, v_t)}{\sigma\sqrt{v_t}}\right)^T, \tag{3}$$

we define a new pair of processes

$$\mathrm{d}W^{\mathbb{Q}}_t \coloneqq \mathrm{d}W^{\mathbb{P}}_t + \eta \mathrm{d}t\,, \tag{4}$$

The first component of $\eta(t, S_t, v_t)$ is the familiar *market price of risk* as in the Black-Scholes model (although the denominator is no longer a constant). The second is a still, as yet, unspecified *market price of volatility risk* $\tilde{\eta}(t, S_t, v_t)$ which arises from the fact that we have two Brownian motions but only one risky asset. Under this transformation the Heston SDE becomes

$$\begin{cases} \mathrm{d}S_t &= S_t(r\mathrm{d}t + \sqrt{v_t}\mathrm{d}W_{1,t}^{\mathbb{Q}})\,, \\ \mathrm{d}v_t &= (\kappa(\theta - v_t) - \tilde{\eta}(t, S_t, v_t))\mathrm{d}t + \sigma\sqrt{v_t}\mathrm{d}W_{2,t}^{\mathbb{Q}}\,. \end{cases} \tag{5}$$

The general class of such transformation is not of Heston type. The Heston model further requires the assumption that the market price of volatility risk is linear in the volatility:

$$\eta_2(t, S_t, v_t) = \eta v_t\,, \tag{6}$$

for some constant η. Under this prescription, the volatility process retains a CIR structure:

$$\begin{cases} \mathrm{d}S_t &= S_t(r\mathrm{d}t + \sqrt{v_t}\mathrm{d}W_{1,t}^{\mathbb{Q}})\,, \\ \mathrm{d}v_t &= \kappa^*(\theta^* - v_t)\mathrm{d}t + \sigma\sqrt{v_t}\mathrm{d}W_{2,t}^{\mathbb{Q}}\,, \end{cases} \tag{7}$$

where $\kappa^* = \kappa + \eta$ and $\theta^* = \kappa\theta/(\kappa + \eta)$. The quadratic covariation between these risk-neutral Brownian motions is still evidently

$$\mathrm{d}[W_1^{\mathbb{Q}}, W_2^{\mathbb{Q}}]_t = \rho\mathrm{d}t\,. \tag{8}$$

Note that the parameters of the CIR process coincide under the physical and risk-neutral measures when $\lambda = 0$. Conversely, the value of λ is already embedded in κ^* and θ^* so does not need to be estimated separately from the risk-neutral estimates of κ and θ. Consequently, we shall drop the asterisks on those parameters and denote them simply as κ and θ.

If one is being a bit more careful, one should check that the Radon-Nikodym derivative $\mathcal{Z}$ associated to the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ is indeed a martingale such that Girsanov's theorem holds and $W_t^{\mathbb{Q}}$ are Brownian motions under $\mathbb{Q}$. If the CIR variance process satisfies the *Feller condition*, $2\kappa\theta > \sigma^2$, then v_t is guaranteed to be a.s. positive[1] (as it should be to model a variance) and $\mathcal{Z}$ is a martingale since both the market price of risk and market price of volatility risk satisfy the Novikov condition. Consequently, $W_t^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$. In cases where the Feller condition is violated, one might need to check it specifically.

1.1 The European Call Price

The price of a European call price, at time $t < T$, is priced entirely analogously to the Black-Scholes model and given by the discounted pay-off under the risk-neutral measure. The full description of this pricing method is given in a separate document and so we shall only give a schematic overview of the process here.

The bank process $B(t)$ and risk-neutral measure defines a numéraire pair $(B(T)/B(t), \mathbb{Q})$ and so the process $C(t)/B(t)$ is a martingale under $\mathbb{Q}$. This leads directly to the risk-neutral pricing formula

$$C_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ \,\middle|\, \mathcal{F}_t \right]\,, \tag{9}$$

where $(S_T - K)^+ \equiv C_T$ is the pay-off of the option at maturity. Introducing the indicator function $\mathbf{1}_{S_T > K}$ to denote the event that the terminal stock price exceeds the strike price, we may expand this as

$$C_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[S_T\mathbf{1}_{S_T>K} \,|\, \mathcal{F}_t\right] - Ke^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{S_T>K} \,|\, \mathcal{F}_t\right]\,. \tag{10}$$

[1] Note that, in practise, market data rarely satisfies the Feller condition (which is only a *sufficient* condition for positivity) and so implementations often require a method to deal with negative variances as they arise. Two simple ways to do so are

- the *full truncation scheme* where v_t is floored at zero: $v_t \to v_t^+ = \max(0, v_t)$;
- the *reflection scheme* where negative values of v_t are reflected to $-v_t$: $v_t \to |v_t|$

The second term is evidently the probability of ending in the money under the risk-neutral measure $\mathbb{Q}$. The first term may be evaluated by a change of numéraire to the stock numéraire pair $(S_T/S_t, \tilde{\mathbb{Q}})$ and we obtain

$$C_t = S_t P^{\tilde{\mathbb{Q}}}(S_T > K) - Ke^{-r(T-t)} P^{\mathbb{Q}}(S_T > K)\,, \tag{11}$$

where $P^{\mathbb{Q}}(S_T > K)$ and $P^{\tilde{\mathbb{Q}}}(S_T > K)$ are probabilities of landing ITM taken with respect to the probability measures that are risk-neutral with respect to the bank and stock numéraires respectively. Of course, in the Black-Scholes models, these reduce the standard Gaussian CDF $\Phi(d_\pm)$, resulting in the well-known Black-Scholes formula. However, for the Heston model, it is not possible to obtain a full closed form analytic solution to this. The rest of this document shall be dedicated to pricing these options.

2 The Heston PDE

We construct a portfolio formed from one option $C(t, S_t, v_t)$, Δ units of the underlying stock S_t and ϕ units of another option $\tilde{C}(t, S_t, v_t)$. The value of the portfolio is thus given by

$$\Pi_t = C_t + \Delta S_t + \phi \tilde{C}_t\,. \tag{12}$$

We demand that the portfolio is self-financing, such that there are no cash injections:

$$\mathrm{d}\Pi_t = \mathrm{d}C_t + \Delta \mathrm{d}S_t + \phi \mathrm{d}\tilde{C}_t\,. \tag{13}$$

For brevity, we shall drop the subscript-t wherever there is no ambiguity. Itô's lemma gives

$$\mathrm{d}\Pi = \left(LC + \phi L\tilde{C}\right)\mathrm{d}t + \left(\frac{\partial C}{\partial S} + \phi\frac{\partial \tilde{C}}{\partial S} + \Delta\right)\mathrm{d}S + \left(\frac{\partial C}{\partial v} + \phi\frac{\partial \tilde{C}}{\partial v}\right)\mathrm{d}v\,, \tag{14}$$

where we have defined the differential operator

$$L := \frac{\partial}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2}{\partial S^2} + \rho\sigma v S\frac{\partial^2}{\partial v \partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2}\,. \tag{15}$$

As in the Black-Scholes world, we may hedge the stochastic components by judicious choices of the parameters ϕ and Δ:

$$\phi = -\frac{\partial C/\partial v}{\partial \tilde{C}/\partial v}\,, \qquad \Delta = -\left(\frac{\partial C}{\partial S} + \phi\frac{\partial \tilde{C}}{\partial S}\right)\,. \tag{16}$$

When ϕ and Δ are given by (16), the change in the portfolio is riskless and must thus accrue interest at the risk-free rate and so $\mathrm{d}\Pi = r\Pi\mathrm{d}t$. Equating the two expressions for $\mathrm{d}\Pi$ gives

$$\frac{LC - rC + rS\frac{\partial C}{\partial S}}{\frac{\partial C}{\partial v}} = \frac{L\tilde{C} - r\tilde{C} + rS\frac{\partial \tilde{C}}{\partial S}}{\frac{\partial \tilde{C}}{\partial v}}\,. \tag{17}$$

The left hand side depends on C whereas the right hand side depends on $\tilde{C}$ and so, by separation of variables, we introduce the function $-f(t, S, v)$ and write

$$LC - rC = -rS\frac{\partial C}{\partial S} - f(t, S, v)\frac{\partial C}{\partial v}\,, \tag{18}$$

$$L\tilde{C} - r\tilde{C} = -rS\frac{\partial \tilde{C}}{\partial S} - f(t, S, v)\frac{\partial \tilde{C}}{\partial v}\,. \tag{19}$$

The first term on the right hand side is essentially the sensitivity of the option to the stock price—the Δ (in the sense of the Greeks)—multiplied by the risk-neutral drift of the stock price. Accordingly, it would seem sensible to treat the derivative in the second term on the right hand side as the sensitivity of the option to the volatility and consequently $f(t, S_t, v_t)$ as the risk-neutral drift of the volatility $f(t, S_t, v_t) = \kappa(\theta - v_t) - \eta v_t$.

Under the change of variables

$$x_t \coloneqq \ln S_t\,, \qquad \tau \coloneqq T - t\,, \tag{20}$$

where T is the maturity of the options, the equation for $\tilde{C}$ becomes

$$-\frac{\partial \tilde{C}}{\partial \tau} + \frac{v}{2}\frac{\partial^2 \tilde{C}}{\partial x^2} + \sigma\rho v\frac{\partial^2 \tilde{C}}{\partial x \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 \tilde{C}}{\partial v^2} + \left(r - \frac{v}{2}\right)\frac{\partial \tilde{C}}{\partial x} + \left(\kappa(\theta - v) - \eta v\right)\frac{\partial \tilde{C}}{\partial v} - r\tilde{C} = 0\,. \tag{21}$$

We insert the ansatz (11) into this PDE to determine the equations of motions of the probabilities for the two probabilities $(P^{\tilde{\mathbb{Q}}}(S_t > K), P^{\mathbb{Q}}(S_t > K)) \equiv (P_1, P_2)$. Since the Heston PDE should hold for any contractual features, choosing a call option with $K = 0$ and $S_t = 1$ produces an option whose price is simply $\tilde{C}_t = P_1$. Evaluating various derivatives, the Heston PDE gives the following equation for P_1:

$$-\frac{\partial P_1}{\partial \tau} + \left(r + \frac{v}{2}\right)\frac{\partial P_1}{\partial x} + \left(\sigma v\rho + \kappa(\theta - v) - \eta v\right)\frac{\partial P_1}{\partial v} + \frac{v}{2}\frac{\partial^2 P_1}{\partial x^2} + \sigma v\rho\frac{\partial^2 P_1}{\partial v \partial x} + \frac{\sigma^2 v}{2}\frac{\partial^2 P_1}{\partial v^2} = 0\,. \tag{22}$$

Instead, choosing a contract with $S = 0$ and $K = 1$ yields a negative valuation $\tilde{C}_t = -e^{-r\tau}P_2$. Inserting this ansatz into the Heston PDE gives a very similar equation for P_2:

$$-\frac{\partial P_2}{\partial \tau} + \left(r - \frac{v}{2}\right)\frac{\partial P_2}{\partial x} + \left(\kappa(\theta - v) - \eta v\right)\frac{\partial P_2}{\partial v} + \frac{v}{2}\frac{\partial^2 P_2}{\partial x^2} + \sigma v\rho\frac{\partial^2 P_2}{\partial v \partial x} + \frac{\sigma^2 v}{2}\frac{\partial^2 P_2}{\partial v^2} = 0\,. \tag{23}$$

We may condense the notation by introducing $(u_1, u_2) = (1/2, -1/2)$ and $(b_1, b_2) = (\kappa + \eta - \rho\sigma, \kappa + \eta)$, after which the above equations become

$$-\frac{\partial P_j}{\partial \tau} + (r + u_j v)\frac{\partial P_j}{\partial x} + (\kappa\theta - v b_j)\frac{\partial P_j}{\partial v} + \frac{v}{2}\frac{\partial^2 P_j}{\partial x^2} + \sigma v\rho\frac{\partial^2 P_j}{\partial v \partial x} + \frac{\sigma^2 v}{2}\frac{\partial^2 P_j}{\partial v^2} = 0\,. \tag{24}$$

Recall that P_j is just the cCDF of the terminal stock price under two different measures. For convenience, we shall instead phrase it in terms of the CDF of the terminal log-stock price $x_T \equiv \ln S_T$ which we can write in terms of the characteristic functions $\psi_j(\phi; \tau, x, v)$ via the Gil-Pelaez theorem (see Appendix B for a derivation):

$$\begin{aligned} P_j(S_T > K) &= 1 - P_j(\ln S_T \leq \ln K) && (25)\\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left\{\frac{e^{-i\phi \ln K}\psi_j}{i\phi}\right\}\mathrm{d}\phi\,. && (26) \end{aligned}$$

Note that ϕ is the Fourier dual of $\ln S_T$ and ψ_j is consequently the characteristic function of the log-*terminal* stock price $x_T = \ln S_T$ *given that the current log-stock price and volatility are x_t and v_t*. Under the ansatz

$$\psi_j(\phi; \tau, x, v) = \exp\left(C_j(\tau, \phi) + D_j(\tau, \phi)v + i\phi x\right)\,, \tag{27}$$

one obtains the following equations by matching the coefficients of powers of v:

$$\begin{aligned} \frac{\partial D_j}{\partial \tau} &= i\phi u_j - \frac{\phi^2}{2} + (i\rho\sigma\phi - b_j)D_j + \frac{\sigma^2}{2}D_j^2\,, && (28)\\ \frac{\partial C_j}{\partial \tau} &= i\phi r + \kappa\theta D_j\,. && (29) \end{aligned}$$

The equation for D_j is a Riccati equation, whose general theory we have covered for completeness in Appendix A, with functions

$$q_0(\tau; \phi) = i\phi u_j - \frac{\phi^2}{2}\,, \qquad q_1(\tau; \phi) = i\rho\sigma\phi - b_j\,, \qquad q_2(\tau; \phi) = \frac{\sigma^2}{2}\,. \tag{30}$$

Taking derivatives, the associated ODE is given by

$$m''(\tau;\phi) - (i\rho\sigma\phi - b_j)m'(\tau;\phi) + \frac{\sigma^2}{2}\left(i\phi u_j - \frac{\phi^2}{2}\right)m(\tau;\phi) = 0\,. \tag{31}$$

We solve this by reduction to standard form. We choose a nominal splitting $m(\tau;\phi) = M(\tau;\phi)N(\tau;\phi)$ and choose the two functions M and N to be as convenient as possible. Under this ansatz, we obtain

$$M''N + M'\left(2N' - (i\rho\sigma\phi - b_j)N\right) + M\left(N'' - (i\rho\sigma\phi - b_j)N' + \frac{\sigma^2}{2}\left(i\phi u_j - \frac{\phi^2}{2}\right)N\right) = 0 \tag{32}$$

We then choose the form of $N(\tau;\phi)$ that eliminates the coefficient of $M'(\tau;\phi)$:

$$2N' - (i\rho\sigma\phi - b_j)N = 0 \qquad \Rightarrow \qquad N(\tau;\phi) = e^{\frac{1}{2}(i\rho\sigma\phi - b_j)\tau}\,, \tag{33}$$

where we have chosen to absorb the constant of integration into $M(\tau;\phi)$. Back-substituting, we obtain the following ODE in standard form for M:

$$M'' + M\left(-\frac{1}{4}(i\rho\sigma\phi - b_j)^2 + \frac{\sigma^2}{2}\left(i\phi u_j - \frac{\phi^2}{2}\right)\right) = 0\,. \tag{34}$$

Solving for $M(\tau;\phi)$, we obtain

$$\begin{aligned} M(\tau;\phi) = k_1 \exp&\left(\tau\sqrt{\frac{1}{4}(i\rho\sigma\phi - b_j)^2 - \frac{\sigma^2}{2}\left(i\phi u_j - \frac{\phi^2}{2}\right)}\right) \\ &+ k_2 \exp\left(-\tau\sqrt{\frac{1}{4}(i\rho\sigma\phi - b_j)^2 - \frac{\sigma^2}{2}\left(i\phi u_j - \frac{\phi^2}{2}\right)}\right)\,, \end{aligned} \tag{35}$$

where k_1 and k_2 are constants of integration. Accordingly, we obtain

$$m(\tau;\phi) = k_1 \exp\left[\frac{\tau}{2}\left(d_j + i\rho\sigma\phi - b_j\right)\right] + k_2 \exp\left[\frac{\tau}{2}\left(-d_j + i\rho\sigma\phi - b_j\right)\right]\,, \tag{36}$$

where

$$d_j \coloneqq \sqrt{(i\rho\sigma\phi - b_j)^2 - \sigma^2\left(2i\phi u_j - \phi^2\right)}\,. \tag{37}$$

Working backwards, and using the notation from Appendix A, we compute

$$w(\tau;\phi) = -\frac{m'(\tau;\phi)}{m(\tau;\phi)} = -\frac{\frac{1}{2}(d_j + i\rho\sigma\phi - b_j) + \frac{k_2}{k_1}\cdot\frac{1}{2}(-d_j + i\rho\sigma\phi - b_j)e^{-d_j\tau}}{1 + \frac{k_2}{k_1}e^{-d_j\tau}}\,. \tag{38}$$

Setting this equal to $w(\tau;\phi) = D_j(\tau,\phi)q_2(\tau;\phi) = \frac{\sigma^2}{2}D_j$ yields the solution for D_j. For the boundary conditions, we know that $\psi_j(\phi;\tau,x,v) = \mathbb{E}^{(j)}[\exp(i\phi x)] = \exp\left(C_j(\tau,\phi) + D_j(\tau,\phi)v + i\phi x\right)$, where the expectation is taken with respect to the j^{th} measure. At maturity $\tau = 0$, the terminal normalised log-stock price x_T is a known constant and so comes out of the expectation, yielding

$$\psi_j(\phi;0,x_T,v) = \exp\left(i\phi x_T\right) = \exp\left(C_j(0,\phi) + D_j(0,\phi)v + i\phi x_T\right)\,. \tag{39}$$

We thus obtain the boundary conditions

$$C_j(0,\phi) = D_j(0,\phi) = 0\,. \tag{40}$$

Applying $D_j(0,\phi) = 0$, we obtain

$$\frac{k_2}{k_1} = \frac{b_j - i\rho\sigma\phi + d_j}{b_j - i\rho\sigma\phi - d_j} \coloneqq g_j\,, \tag{41}$$

giving

$$D_j(\tau,\phi) = \frac{b_j - i\rho\sigma\phi + d_j}{\sigma^2} \frac{1-e^{d_j\tau}}{1-g_j e^{d_j\tau}} \,. \tag{42}$$

The equation for C_j can be solved by decomposition into partial fractions:

$$\begin{aligned}
C_j(\tau,\phi) &= c_0 + i\phi r\tau + \frac{\kappa\theta(b_j - i\rho\sigma\phi + d_j)}{\sigma^2} \int \mathrm{d}\tau \frac{1-e^{d_j\tau}}{1-g_j e^{d_j\tau}} && (43)\\
&= c_0 + i\phi r\tau + \frac{\kappa\theta(b_j - i\rho\sigma\phi + d_j)}{\sigma^2} \int \frac{\mathrm{d}y}{d_j y} \frac{1-y}{1-g_j y} && (44)\\
&= c_0 + i\phi r\tau + \frac{\kappa\theta(b_j - i\rho\sigma\phi + d_j)}{\sigma^2 d_j} \int \mathrm{d}y \left(\frac{1}{y} + \frac{g_j - 1}{1-g_j y}\right) && (45)\\
&= c_0 + i\phi r\tau + \frac{\kappa\theta(b_j - i\rho\sigma\phi + d_j)}{\sigma^2 d_j} \left[\ln y - \frac{(g_j-1)}{g_j}\ln(1-g_j y)\right] && (46)\\
&= c_0 + i\phi r\tau + \frac{\kappa\theta(b_j - i\rho\sigma\phi + d_j)}{\sigma^2 d_j} \left[d_j\tau + \frac{1-g_j}{g_j}\ln(1-g_j e^{d_j\tau})\right] && (47)\\
&= c_0 + i\phi r\tau + \frac{\kappa\theta}{\sigma^2}\left[(b_j - i\rho\sigma\phi + d_j)\tau - 2\ln(1-g_j e^{d_j\tau})\right] \,, && (48)
\end{aligned}$$

where the second line uses a change of variables to $y = \exp(d_j\tau)$. Applying (40), we obtain

$$C_j(\tau,\phi) = i\phi r\tau + \frac{\kappa\theta}{\sigma^2}\left[(b_j - i\rho\sigma\phi + d_j)\tau - 2\ln\frac{1-g_j e^{d_j\tau}}{1-g_j}\right] \,. \tag{49}$$

2.1 The Little Heston Trap

Albrecher et al. show that the stability of the numerical integration of P_j can be improved by the following very simple transformation. Let

$$c_j = \frac{1}{g_j} = \frac{b_j - i\rho\sigma\phi - d_j}{b_j - i\rho\sigma\phi + d_j} = \frac{1}{g_j} \,. \tag{50}$$

Then, the two functions (C_j, D_j) can be rewritten in the form

$$\begin{aligned}
C_j(\tau,\phi) &= i\phi r\tau + \frac{\kappa\theta}{\sigma^2}\left[(b_j - i\rho\sigma\phi - d_j)\tau - 2\ln\frac{1-c_j e^{-d_j\tau}}{1-c_j}\right] && (51)\\
D_j(\tau,\phi) &= \frac{b_j - i\rho\sigma\phi - d_j}{\sigma^2}\left(\frac{1-e^{-d_j\tau}}{1-c_j e^{-d_j\tau}}\right) \,. && (52)
\end{aligned}$$

This is the form that we have chosen to implement. With these analytic expressions, we may now construct the characteristic function (27) and use it to compute the relevant probabilities (26) that appear in the option price (11).

Appendices

A The Riccati Equation

The Riccati equation is a first-order non-linear ODE of the form

$$y'(x) = q_0(x) + q_1(x)y(x) + q_2(x)y^2(x)\,. \tag{53}$$

Nevertheless, it can be transformed into a second order linear ODE by suitable transformations, as demonstrated below. Let $w(x) = y(x)q_2(x)$. Then,

$$w'(x) = q_0(x)q_2(x) + \left(q_1(x) + \frac{q_2'(x)}{q_2(x)}\right) w(x) + w^2(x)\,. \tag{54}$$

Reparametrising as $w(x) = -m'(x)/m(x)$, we obtain a second order linear ODE for $m(x)$,

$$m''(x) - \left(q_1(x) + \frac{q_2'(x)}{q_2(x)}\right) m'(x) + q_0(x)q_2(x)m(x) = 0\,, \tag{55}$$

which may be solved via standard techniques. In the main text, we have referred to this transformed equation as the 'associated ODE'.

B The Gil-Pelaez Theorem

Here, we give a simplified derivation of the Gil-Pelaez inversion theorem that relates the CDF and the characteristic function. We shall denote the PDF, CDF and characteristic functions as $f(x), F(x)$ and $\psi(k) = \mathbb{E}[\exp(ikx)]$ respectively. We begin by considering the following convolution:

$$\begin{aligned}
(f \star \mathrm{sgn})(x) &= \int_{-\infty}^{\infty} f(y)\,\mathrm{sgn}(x-y)\mathrm{d}y && (56)\\
&= -\int_{-\infty}^{\infty} f(y)\,\mathrm{sgn}(y-x)\mathrm{d}y && (57)\\
&= -\int_{-\infty}^{x} f(y)\,\mathrm{sgn}(y-x)\mathrm{d}y - \int_{x}^{\infty} f(y)\,\mathrm{sgn}(y-x)\mathrm{d}y && (58)\\
&= F(x) - (1 - F(x)) = 2F(x) - 1\,. && (59)
\end{aligned}$$

Taking the Fourier transform[2] of both sides, using the convolution theorem and noting that $\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} \mathrm{d}x e^{-ikx} f(x) = \mathbb{E}[e^{-ikx}] = \psi(-k)$, we obtain

$$\psi(-k) \cdot \frac{2}{ik} = 2\mathcal{F}\{F(x)\} - \mathcal{F}\{1\} = 2\mathcal{F}\{F(x)\} - 2\pi\delta(k)\,. \tag{62}$$

Rearranging and taking the inverse Fourier transform, we obtain

$$F(x) = \mathcal{F}^{-1}\left\{\pi\delta(k) + \frac{\psi(-k)}{ik}\right\} = \frac{1}{2} + \underbrace{\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{e^{ikx}\psi(-k)}{ik}}_{:=I}\,. \tag{63}$$

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[2] We shall use the following conventions for the Fourier transform:

$$\mathcal{F}\{f(x)\} = \hat{f}(k) = \int_{-\infty}^{\infty} \mathrm{d}x e^{-ikx} f(x)\,, \qquad \mathcal{F}^{-1}\{\hat{f}(k)\} = f(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{ikx} \hat{f}(k)\,. \tag{60}$$

In our conventions, we have

$$\mathcal{F}\{1\} = 2\pi\delta(k)\,, \qquad \mathcal{F}\{\mathrm{sgn}(x)\} = \frac{2}{ik}\,. \tag{61}$$

Focusing only on the integral, we obtain

$$I = \int_{-\infty}^{0} \frac{\mathrm{d}k}{2\pi} \frac{e^{ikx}\psi(-k)}{ik} + \int_{0}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{e^{ikx}\psi(-k)}{ik} \tag{64}$$

$$= \int_{\infty}^{0} \frac{\mathrm{d}k}{2\pi} \frac{e^{-ikx}\psi(k)}{ik} + \int_{0}^{\infty} \frac{\mathrm{d}k}{2\pi} \frac{e^{ikx}\psi(-k)}{ik} \tag{65}$$

$$= -\int_{0}^{\infty} \frac{\mathrm{d}k}{2\pi \cdot ik} \left(e^{-ikx}\psi(k) - e^{ikx}\psi(-k) \right) . \tag{66}$$

From here, we give two equivalent formulations. Firstly, we can rewrite the integral as

$$I = -\frac{1}{2\pi} \int_{0}^{\infty} \frac{\mathrm{d}k}{ik} \left(e^{-ikx}\psi(k) - e^{ikx}\psi(-k) \right) , \tag{67}$$

where $\mathrm{Im}(z) = (z - \bar{z})/(2i)$ and we have used the fact that $\psi(-k) = \overline{\psi(k)}$, to obtain

$$F(x) = \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \mathrm{d}k \frac{\mathrm{Im}\left\{ e^{-ikx}\psi(k) \right\}}{k} . \tag{68}$$

Alternatively, we may proceed from (66) as

$$I = -\int_{0}^{\infty} \frac{\mathrm{d}k}{2\pi} \left(\frac{e^{-ikx}\psi(k)}{ik} + \frac{e^{ikx}\psi(-k)}{-ik} \right) \tag{69}$$

to obtain

$$F(x) = \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \mathrm{d}k \, \mathrm{Re} \left\{ \frac{e^{-ikx}\psi(k)}{ik} \right\} , \tag{70}$$

where $\mathrm{Re}(z) = (z + \bar{z})/2$ is the real component. This is the form that we use in the main text.