

Photos as Projective Spaces

In the previous post we deduced the formula which relates the 3D coordinates of a point (x_1, x_2, x_3) with the 2D coordinates (ξ_1, ξ_2) of the image of that point on the photo (formula (2) from the previous post):

$$\xi_1 = \frac{x_1}{x_3}, \quad \xi_2 = \frac{x_2}{x_3} \quad (1)$$

By construction, this formula is true for points which lie in front of the camera, i.e. $x_3 > 0$.

For our purposes, however, we would like to have a nice mathematical model of the photo, a relation between the 3D points and, possibly, some imaginary extension of the picture, such that it includes (almost) all 3D points and we should not worry if, after a small movement of the camera, a certain point is still on the photo, has appeared or disappeared from it. For $x_3 > 0$ the model should give (1).

Since (1) is defined for $x_3 > 0$ hemi-space only and, taking into account its geometric derivation, we can interpret it as follows: the camera identifies all points on a ray starting from the origin. In other words, all points which in the 3D space lie on a ray starting from the origin (the optical center of the camera) are projected on a single point on the photo. Hence, a natural generalization would be to interpret a photo as the set of rays starting from the origin.

Topologically, the space of rays starting from the origin is isomorphic to sphere. Indeed, if we place the center of the sphere in the origin, then a point on the sphere uniquely identifies a ray and, vice versa, a ray uniquely identifies a point on the sphere.

Although justified physically (opposite rays really correspond to different directions), this assumption leads to serious modeling problems. Indeed, all points which lie behind the camera, i.e. $x_3 < 0$, should be projected on the opposite side of the sensor and hence, we will not have a single formula for all points. Besides, any small rotation which involves the axis x_3 will move some points from one side of the sensor to the other.

What if we simply extend the domain of the (1) to the whole 3D-space, i.e. assume that the formulae are true also for $x_3 < 0$?

With this assumption, the formulae (1) imply that all points which lie on a line passing through the origin are projected on a single point on the photo. Although this is counter-intuitive from the physical perspective it makes the mathematical model very simple. The projection of the whole world on the photo (except the plane $x_3 = 0$ which we will consider in the upcoming post) is given via the simple formula (1).

The problem with such a model is that it identifies two opposite directions. On the other hand, we assume that two antipodal points never appear on the same photo.

The space of lines passing through the origin is called real projective space and denoted \mathbb{RP}^2 . We will focus on the study of real projective spaces of different dimensions and their applications in the upcoming posts.

For now, let us notice, that \mathbb{RP}^2 is a sphere where we identify two antipodal points. Such a space cannot be even embedded in \mathbb{R}^3 , i.e. we cannot neither draw it, nor even print it on a 3D-printer.