Mathematical Methods I (80114)

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Fall 2021

Compiled: 2023-03-17 12:46:34

https://github.com/outofink/notes

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MATHEMATICAL METHODS I

1

VECTORS AND EUCLIDEAN GEOMETRY

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1.1 Set Theory

$$A \times B = \{(x, y) \mid x \in A, y \in B\}$$
$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

Circle:

$$\left\{(x, y) \in \mathbb{R}^2 \mid (x - x_0)^2 + (y - y_0)^2 = r^2\right\}$$

Graph:

$$G_f = \left\{(x, y) \in \mathbb{R}^2 \mid y = f(x)\right\}$$

1.2 Vectors in $\mathbb{R}^n$

1.2.1 What is a vector?

A *vector* is an arrow with a *direction* and *magnitude*.

It can express:

- Displacement
- Angular velocity

Two vectors with the same magnitude and direction are said to be *equal*.

1.2.2 Definitions and Symbols

What is $\mathbb{R}^n$?

Plane: $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$

Space: $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

$$\mathbb{R}^n = \{(x_0, x_1, x_2, \dots) \mid \underset{1 \leq i \leq n}{x_i} \in \mathbb{R}\}$$

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid \underset{1 \leq i \leq n}{x_i \in \mathbb{R}} \right\}$$

Points: A, B, C

Parallel lines: $AB \parallel CD$

Perpendicular lines: $AB \perp CD$

Distance between points (on a plane):

$$AB = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

Distance between points (in $\mathbb{R}^3$ space):

$$AB = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

Distance (in $\mathbb{R}^n$):

$$AB = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

1.2.3 Expressing vectors mathematically

We place the *tail* of the vector and put it at the *origin* $(0, 0)$ and the *head* at point $P(x, y)$.

If $\underline{u} = \vec{(x, y)}$ that means that u is a vector that starts at $(0, 0)$ and ends x to the right and y up.

Symbol. Vector: $\underline{u}$, $\vec{AB}$

Zero vector: $\underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Magnitude: $\|\underline{u}\|$

Position vector at (x, y, z): $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Unit vector (magnitude of 1): $\underline{\hat{e}}$

On $\mathbb{R}^2$: $\underline{\hat{\imath}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \underline{\hat{\jmath}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1.2.4 Adding vectors

$$\underline{u} + \underline{v} = \underline{v} + \underline{u}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

1.2.5 Multiplying a vector with a scalar

$$\lambda \cdot \underline{a} = \lambda \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda \cdot a_1 \\ \lambda \cdot a_2 \\ \vdots \\ \lambda \cdot a_n \end{pmatrix}$$

$$-\underline{a} = \begin{pmatrix} -a_1 \\ \vdots \\ -a_n \end{pmatrix} = (-1) \cdot \underline{a}$$

1.2.6 Finding getting a unit vector from a vector (normal)

$$\underline{\hat{a}} = \frac{\underline{a}}{||\underline{a}||}$$

Sometimes we symbolize the length of $\underline{r}$ as r.

$$\underline{r} = r \cdot \underline{\hat{r}}$$

1.2.7 Subtracting vectors

$$\underline{u} + (-\underline{v}) = \underline{u} - \underline{v}$$

1.2.8 Expressing vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = x\ \underline{\hat{\imath}} + y\ \underline{\hat{\jmath}} + z\ \underline{\hat{k}}$$

1.2.9 Finding a line that passes through two position vectors $\underline{a}, \underline{b}$

Parametric Form

$$\underline{v} = \underline{b} - \underline{a}$$

$$\begin{aligned} l &= \{\underline{a} + t\underline{v} \mid t \in \mathbb{R}\} \\ &= \{(1-t)\underline{a} + t\underline{b} \mid t \in \mathbb{R}\} \\ &= \{(1-t)\underline{a} + t\underline{b} \mid 0 \leq t \leq 1\} \end{aligned}$$

If $t \in (0,1)$, the point is between $\underline{a}$ and $\underline{b}$ (AB).

Example 1.2.1.

$$A = (1,0,6) \quad B = (2,1,3)$$

$$\left\{ (1-t)\begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

$$\left\{ \begin{pmatrix} 1+t \\ t \\ 6-9t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

$$AB = \underline{r} = \begin{pmatrix} 1+t \\ t \\ 6-9t \end{pmatrix}_{0 \leq t \leq 1}$$

1.2.10 Scalar multiplication (between two vectors)

$\underline{a}, \underline{b} \in \mathbb{R}^n$:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n$$

$$\underline{a} \cdot \underline{b} = \sum_{i=1}^{n} a_i b_i$$

Example 1.2.2.

$$\begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} = 2 \cdot 1 + 1 \cdot (-5) + 7 \cdot 2 = 11$$

Characteristics:

1. $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$
2. $(\lambda \underline{a}) \cdot \underline{b} = \lambda(\underline{a} \cdot \underline{b})$
3. $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$
4. $\underline{a} \cdot \underline{a} = \|a\|^2$

Geometric meaning:

$\underline{a}, \underline{b} \in \mathbb{R}^n$:

$$\begin{aligned} \|\underline{a} - \underline{b}\|^2 &= (\underline{a} - \underline{b}) \cdot (\underline{a} - \underline{b}) \\ &= \|a\|^2 + \|b\|^2 - 2(\underline{a} \cdot \underline{b}) \end{aligned}$$

Reminder (Law of Cosines).

$$c^2 = a^2 + b^2 + 2ab\cos\theta$$

Conclusion.

$$\underline{a} \cdot \underline{b} = \|a\| \cdot \|b\| \cos\theta$$
$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{\|a\| \cdot \|b\|}$$

$\underline{a}, \underline{b} \neq \underline{0}$:

$$\begin{aligned} \underline{a} \cdot \underline{b} = 0 &\iff \underline{a} \perp \underline{b} \\ \underline{a} \cdot \underline{b} > 0 &\iff \theta \text{ is acute } \left(0 \leq \theta < \frac{\pi}{2}\right) \\ \underline{a} \cdot \underline{b} < 0 &\iff \theta \text{ is obtuse } \left(\frac{\pi}{2} < \theta \leq \pi\right) \end{aligned}$$

1.3 Projection

Let $\hat{\underline{e}}$ be a unit vector and $\underline{a}$ be some vector, such that $\underline{a} \cdot \underline{e} = \|a\| \cos\theta$ (*projection* of $\underline{a}$ on $\hat{\underline{e}}$).

1.4 Planes in $\mathbb{R}^3$

Consider a plane $\Pi \subseteq \mathbb{R}^3$, the *normal vector* to Π, $\underline{n}$ (perpendicular to the plane), and the *unit vector* in the direction of the normal, $\hat{\underline{n}}$.

$$\forall\, \underline{r} \in \Pi : \underline{r} \cdot \hat{\underline{n}} = \mathbf{0}$$

Example 1.4.1. Given plane $3x + 2y - z = 10$, find the normal vector and the distance to the origin.

Normal vector: $\underline{n} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

Distance: $\frac{10}{\|\underline{n}\|} = \frac{10}{\sqrt{3^2+2^2+(-1)^2}} = \frac{10}{\sqrt{14}}$

In general, given the plane $ax + by + cz = d$, the distance from the origin is $\frac{d}{\sqrt{a^2+b^2+c^2}}$.

1.5 Determinant

A *determinant* of a square matrix of the size 2×2 and 3×3:

2×2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

It represents the *signed* area of a parallelogram made from the vectors $\binom{a}{c}$, $\binom{b}{d}$.

3×3:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + g \begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

It represents the *signed* area of a *parallelepiped* (3D object made up of 6 parallelograms).

1.6 Vector Multiplication

Symbol. $\wedge$ or $\times$

A vector ($\in \mathbb{R}^3$) $\wedge$ a vector ($\in \mathbb{R}^3$) = a vector ($\in \mathbb{R}^3$).

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ \\ -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\ \\ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$\underline{a} \wedge \underline{b} = \underline{\hat{\imath}} \begin{vmatrix} a_2 & b_2 \\ a_3 & c_3 \end{vmatrix} - \underline{\hat{\jmath}} \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + \underline{\hat{k}} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} \underline{\hat{\imath}} & a_1 & b_1 \\ \underline{\hat{\jmath}} & a_2 & b_2 \\ \underline{\hat{k}} & a_3 & b_3 \end{vmatrix}$$

1.6.1 Characteristics

1. $\underline{a} \wedge \underline{b}$ is perpendicular to both $\underline{a}$ and $\underline{b}$
2. $\|\underline{a} \wedge \underline{b}\| = \|\underline{a}\| \cdot \|\underline{b}\| \sin\theta$

1.6.2 Triple scalar product $\in \mathbb{R}^3$

Given $\underline{a}, \underline{b}, \underline{c} \in \mathbb{R}^3$, their *triple scalar product* is:

$$[\underline{a}, \underline{b}, \underline{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \underline{a} \cdot (\underline{b} \wedge \underline{c})$$

MATHEMATICAL METHODS I

2

COORDINATE SYSTEMS AND SIMPLE GEOMETRIC SHAPES

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2.1 Introduction

Example 2.1.1 (Circle).

$$x^2 + y^2 = 1 \tag{Equation}$$

$$\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_{0 \le t \le 2\pi} \tag{Parametric}$$

Two operations on shapes:

1. Transformation by a vector $\binom{a}{b}$
2. Stretching by c width-wise, d height-wise

Given parameterization $\binom{x(t)}{y(t)}$ or an equation $T(x, y)$:

Transformation by a vector $\binom{a}{b}$:

$$\begin{pmatrix} x(t) + a \\ y(t) + b \end{pmatrix} \quad T(x - a, y - b)$$

Stretching by c width-wise, d height-wise:

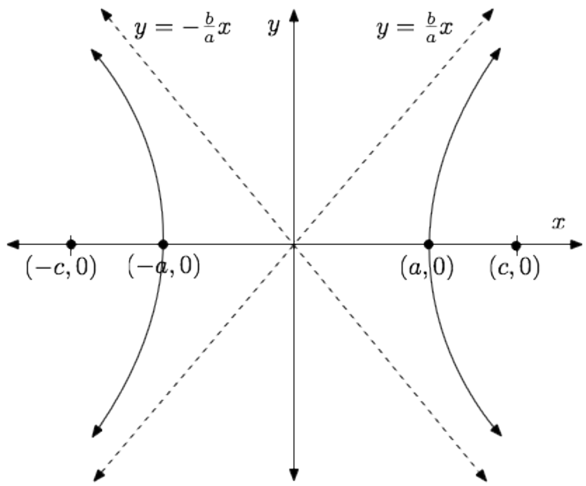
$$\begin{pmatrix} c \cdot x(t) \\ d \cdot y(t) \end{pmatrix} \quad T\left(\frac{x}{c}, \frac{y}{d}\right)$$

2.2 Hyperbola

We start from the following equation:

$$xy = c \implies y = \frac{c}{x}$$

Let's rotate this function $45°$ counter-clockwise:



To define our new function, we now define u, v on the plane in how they relate to our initial (x, y).

We can do this by using projections of x and y onto our new u, v axes:

$$u = (x, y) \cdot \underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}_{\text{Unit vector of } u} = \frac{1}{\sqrt{2}}(x + y)$$

$$v = (x, y) \cdot \underbrace{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)}_{\text{Unit vector of } v} = \frac{1}{\sqrt{2}}(-x + y)$$

Conclusion.

$$uv = \alpha$$
$$y^2 - x^2 = \frac{\alpha}{2}$$
$$x^2 - y^2 = c \iff uv = -\frac{c}{2}$$

After stretching by a in the direction of the x-axis and by b in the direction of the y-axis:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = c$$

With the new asymptotes being:

$$y = \pm x \implies y = \pm x \cdot \frac{b}{a}$$

2.3 Polar Coordinates

Besides for the *Cartesian* coordinate system (which is defined by how far a point is from the x and y-axis), we have the *polar* coordinate system, which is defined by the distance between the point and the origin (radius r) and the angle between the x-axis and the point (angle, θ).

Converting between representations:

$$x = r\cos\theta \quad y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \underbrace{(+\pi)}_{\text{If } x < 0} + 2\pi k$$

A circle shifted by a to horizontally with a radius of a:

$$(x - a)^2 + y^2 = a^2$$
$$r = 2a\cos\theta$$

2.4 Cylindrical Coordinates

The *cylindrical coordinate system* is essentially the polar system just with a third dimension z, where the radius is the distance from the z-axis (*rho*, ρ), the angle (θ), and height (z), such that:

$$x = \rho\cos\theta \quad y = \rho\sin\theta \quad z = z$$

2.5 Solid of Revolution

A *solid of revolution* is made by taking a line (such as a parabola) and rotating around in the three dimensions to make a three-dimensional solid (in the case of starting with a parabola, we create a *paraboloid*).

Suppose a shape that is defined by $z = f(x)$. To rotate around the z-axis:

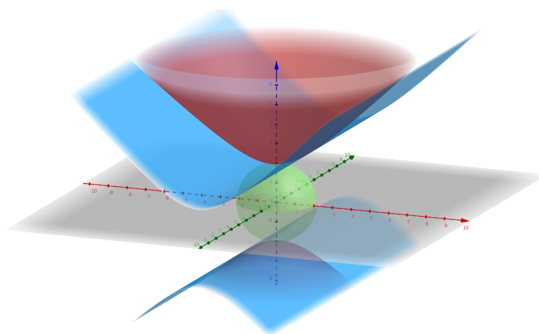
$$\rho = \sqrt{x^2 + y^2}$$
$$z = f\left(\sqrt{x^2 + y^2}\right)$$

Parameterially:

$$z = f(x) = \begin{pmatrix} t \\ \sqrt{1-t^2} \\ f(t) \end{pmatrix} \quad z = f\left(\sqrt{x^2+y^2}\right) = \begin{pmatrix} t\cos\theta \\ t\sin\theta \\ f(t) \end{pmatrix}$$

Example 2.5.1.

$$z^2 - y^2 - y^2 = 4 \implies z^2 - (\overbrace{x^2 + y^2}^{\rho^2}) = 4$$



Red: $z^2 - y^2 - x^2 = 4$
Blue: $z^2 - x^2 = 4$
Green: $z^2 + y^2 + x^2 = 4$

We have three points on a plane:

$$A = (1,2,3) \quad B = (1,1,1) \quad C = (6,-4,0)$$

To express the plane parametrically, we take two vectors from our points (such as $\vec{AB}, \vec{BC}$)

$$\vec{AB} = B - A = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}$$

$$\vec{BC} = C - B = \begin{pmatrix} 5 \\ -5 \\ -1 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$$

2.6 Parameterization of Hyperbolae

$$x^2 = y^2 = c$$
$$\begin{cases} x + y = t \\ x - y = \frac{c}{t} \end{cases}$$
$$x = \frac{1}{2}\left(t + \frac{c}{t}\right)$$
$$y = \frac{1}{2}\left(t - \frac{c}{t}\right)$$

$x^2 - y^2 = -1 :$ $\begin{pmatrix} \frac{1}{2}\left(t - \frac{1}{t}\right) \\ \frac{1}{2}\left(t + \frac{1}{t}\right) \end{pmatrix}$	$(x+1)^2 - \left(\frac{y+1}{\sqrt{\frac{1}{2}}}\right) = -1 :$ $\begin{pmatrix} \frac{1}{2}\left(t - \frac{1}{t}\right) - 1 \\ \frac{1}{2\sqrt{2}}\left(t + \frac{1}{t}\right) - 1 \end{pmatrix}$

$y = x^2$ to rotate around the x-axis:

$$y \to \pm\sqrt{y^2 + z^2}$$
$$x^2 = \sqrt{y^2 + z^2}$$
$$x^4 = y^2 + z^2$$

Parametrically:

$$\begin{pmatrix} t \\ t^2 \cos\theta \\ t^2 \sin\theta \end{pmatrix}$$

2.7 Spherical Coordinate System

The *spherical coordinate system* consists of the radius r (the distance of the point from the origin), θ (the angle above the x-axis towards the y-axis), and ϕ (the angle above the z-axis along towards the point), such that:

$$0 \leq \theta \leq 2\pi \quad 0 \leq \phi \leq \pi$$

In Cartesian coordinates:

$$
\begin{aligned}
x &= r \sin\phi \cos\theta \\
y &= r \sin\phi \sin\theta \\
z &= r \cos\phi
\end{aligned}
$$

$$
\begin{gathered}
r = \sqrt{x^2 + y^2 + z^2} \\
\phi = \cos^{-1}\left(\frac{z}{r}\right) \\
\theta = \tan^{-1}\left(\frac{y}{x}\right) \underbrace{+\pi}_{y<0} = \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right)
\end{gathered}
$$

3

COMPLEX NUMBERS

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3.1 Introduction

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

Inception: To help solve 3rd degree polynomials.

Definition (Fundamental Theorem of Algebra)**.** For all polynomials:

$$a_n x_n + a_{a-1} x^{x-1} + \ldots + a_1 x + a_0$$

that has *complex coefficients* has *complex roots*.

3.2 Operations

$$(a + bi) + (c + di) = (a + d) + (b + d)i \tag{3.1}$$
$$(a + bi) - (c + di) = (a - d) + (b - d)i \tag{3.2}$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \tag{3.3}$$
$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{e + fi}{c^2 + d^2} = \frac{e}{c^2 + d^2} + \frac{f}{c^2 + d^2} i \tag{3.4}$$

3.3 Complex Plane

Every complex number $z = a + bi$ can be represented on the complex plane at the point (a, b).

It can also be represented in the polar form:

$$r = |z| \quad \theta = \arg(z)$$
$$z = r\cos\theta + r\sin\theta i \quad (\operatorname{cis}\theta)$$

3.3.1 Characteristics

1. Properties of four algebraic operations of the real numbers also apply to the complex ones (i.e. associative, distributive, etc.)
2. *Complex conjugate*:

$$\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$$
$$\overline{z_1 \cdot z_2} = \overline{z}_1 \cdot \overline{z}_2$$
$$\overline{\frac{z_1}{z_2}} = \frac{\overline{z}_1}{\overline{z}_2}$$
$$\frac{1}{z} = \frac{\overline{z}}{|z|^2},\ z \cdot \overline{z} = |z|^2$$

3.3.2 Analysis

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$
$$z_1 z_2 = r_1 r_2[(\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)}) + i(\underbrace{\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1}_{\sin(\theta_1+\theta_2)})]$$

Conclusion.

$$|z_1 z_2| = r_1 r_2$$
$$\arg(z_1 z_2) = \theta_1 + \theta_2$$

Definition (De Moivre's Formula)**.**

$$(r\operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta)$$

Definition (nth-root of a complex number)**.**

$$z^n = r\operatorname{cis}\theta$$
$$z = \sqrt[n]{r}\operatorname{cis}\left(\frac{\theta + 2\pi k}{n}\right), \quad k = 0, 1, 2, \ldots, n-1$$

Definition (Euler's Formula)**.**

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$
$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

e to a complex number:

$$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i\sin b)$$

Our goal is to:

1. Express $\cos(nx)$ in terms of $\sin x, \cos x$.
2. Express $\sin^n(x)$ as a sum of $\sin x, \cos x$, without multiplying them.

Example 3.3.1.

$$\begin{gathered}\cos(5x) = \mathrm{Re}(e^{i5x}) = \mathrm{Re}((e^{ix})^5) \\ = \mathrm{Re}((\cos x + i\sin x)^5) \\ (a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + \dots\end{gathered}$$

To simplify our calculation since we are only looking for the real part of our solution, we can ignore any place where $\sin$ is raised to an odd power (since $i^2 = -1$).

$$= \cos^5 x - 10\cos^x \sin^2 x + 5\cos x \sin^4 x$$

Now for an example in the opposite direction:

$$\begin{gathered}\sin^5 x = \left(\frac{1}{2i}\right)^4 (e^{ix} - e^{-ix})^4 \\ \frac{1}{16}(e^{i4x} - 4e^{i2x} + 6 - 4e^{-i2x} + e^{-i4x}) \\ = \frac{1}{16}(2\cos(4x) - 8\cos(2x) + 6)\end{gathered}$$

Example 3.3.2.

$$\begin{gathered}a\cos(\omega t) + b\sin(\omega t) \\ \mathrm{Re}(\underbrace{(a+bi)}_{re^{i\theta}}\underbrace{(\cos(\omega t) - i\sin(\omega t))}_{e^{-i\omega t}}) \\ = \mathrm{Re}\left(re^{i(\theta-\omega t)}\right) = r\cos(\theta - \omega t) = r\cos(\omega t - \theta) \\ = \sqrt{a^2+b^2}\cos\left(\omega t - \tan^{-1}\left(\frac{a}{b}\right)(+\pi)\right)\end{gathered}$$

3.4 What is $\ln(a+bi)$?

$\ln z$ is the solution to the equation $e^{\omega} = z \to e^u \cdot e^{iv} = a + bi = re^{i\theta}$.

$$\begin{gathered}u = \ln r = \ln|z| \\ v = \theta + 2\pi k\end{gathered}$$

Conclusion:

$$\ln(z) = \ln|z| + i(\arg(z) + 2\pi k)$$

Example 3.4.1.

$$\ln\left(-\sqrt{3}+i\right)$$

$$-\sqrt{3}+i = 2\operatorname{cis}\left(\underbrace{\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)}_{-\frac{\pi}{6}} + \pi\right)$$

$$= 2\operatorname{cis}\left(\frac{5\pi}{6}\right)$$

$$\ln\left(-\sqrt{3}+i\right) = \ln 2 + i\left(\frac{5\pi}{6} + 2\pi k\right)$$

3.5 Solving Complex Equations

$$z^4 + z^3 + z^2 + z + 1 = 0 \setminus : z^2$$

$$z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$$

$$t = z + \frac{1}{z} \quad t^2 = z^2 + 2 + \frac{1}{z^2}$$

$$t^2 + t - 1 = 0$$

$$t_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$$

$$\Downarrow$$

$$2z^2 - (-1+\sqrt{5}) + 2 = 0$$

z can be found given that we know how to find the square root of complex numbers.

$$z_{1,2} = \frac{-1+\sqrt{5} \pm \sqrt{(-1+\sqrt{5})^2 - 16}}{4}$$

$$z^4 + z^3 + z^2 + z + 1 = 0$$

$$(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$$

$$z^5 = 1 = \boxed{1 \cdot e^{i \cdot 0}}$$

3.6 Fundamental Theorem of Algebra

All polynomials can be factored into a product of linear elements in the complex world.

$$\begin{aligned} x^4 - 1 &= (x^2-1)(x^2+1) \\ &= (x-1)(x+1)(x-i)(x+i) \end{aligned} \tag{1}$$

$$
\begin{gathered}
x^3 - 3x^2 + 2 = (x-1)(x^2 - 2x - 2) \\
x_{1,2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3} \\
\Downarrow \\
= (x-1)(x-(1+\sqrt{3}))(x-(1-(1-\sqrt{3})))
\end{gathered} \tag{2}
$$

$$
\begin{gathered}
x^2 + 6 + 9 = (x+3)^2 \\
-3 \text{ is the root (double root)}
\end{gathered} \tag{3}
$$

$$
\begin{gathered}
x^4 + 2x^2 + 1 = (x^2+1)^2 \quad \text{(division over the real numbers)} \\
(x-i)^2(x+i)^2 \quad \text{(division over the complex numbers)} \\
\pm i \text{ are each double roots}
\end{gathered} \tag{4}
$$

Claim.

$$p(x)a_n x^n + a_{n-1}x^{x-1} + \ldots + a_1 x + a_0$$

Polynomials with real coefficients if z is a root of $p(x)$ then $\overline{z}$ is also a root of $p(x)$.

Example 3.6.1.

$$
\begin{gathered}
x^3 + 3x^2 + 4x + 2 \\
\pm 1, \pm 2 = (x+1)(x^2 + 2x + 2) \\
x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 8}}{2} = \boxed{-1 \pm i}
\end{gathered}
$$

Proof. Given $a_n z^n + \ldots + a_0 = 0$, we have to prove:

$$a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \ldots = 0$$

$$
\begin{gathered}
\text{Reminder:} \\
\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2 \quad \overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2 \\
\Downarrow \\
(\overline{z^n}) = (\overline{z})^n
\end{gathered}
$$

Given: $a_n z^n + a_{a-1} z^{n-1} + \ldots = 0$:

$$\overline{a_n z^n + a_{a-1} z^{n-1} + \ldots = 0} = \overline{0}$$
$$\Downarrow$$
$$\overline{a_n z^n} + \ldots + \overline{a}_n = 0$$
$$\overline{a_n}\,\overline{z^n} + \ldots + \overline{a}_n = 0$$
$$a_n(\overline{z})^n + \ldots + a_n = 0$$
$$\implies \overline{z} \text{ is a root of the polynomial.}$$

Note on $\ln(z)$: There are infinite solutions because $\omega = \ln|z| + i(\arg z + 2\pi k)$ where $k \in \mathbb{N}$.

■

Example 3.6.2. $x^4 + x^2 + 1$ can be factored over the real numbers and over the complex numbers.

$$t = x^2 \to t^2 + t + 1$$
$$t_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$
$$z_1 = \pm\sqrt{t} = \pm\sqrt{e^{i+\frac{2\pi}{3}}} = \pm e^{i\frac{\pi}{3}}$$
$$z_2 = \pm\sqrt{e^{\frac{4\pi}{3}i}} = \pm e^{i\frac{2\pi}{3}}$$

$$(x - e^{i\frac{\pi}{3}})(x + e^{i\frac{\pi}{3}})(x - e^{i\frac{2\pi}{3}})(x + e^{i\frac{2\pi}{3}})$$

$$e^{i\frac{\pi}{3}} = \operatorname{cis}\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i \implies$$
$$\pm e^{i\frac{2\pi}{3}} = \pm\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= (x^2 + x + 1)(x^2 + x + 1)$$

$$(x - z)(x - \overline{z})$$
$$= x^2 - (\underbrace{z + \overline{z}}_{\text{real}})x + \underbrace{z \cdot \overline{z}}_{\text{real}}$$

All real polynomials can be factored into real linear or double roots.

Example 3.6.3.

$$(\underbrace{2+i}_{\theta_1})(\underbrace{3+i}_{\theta_2}) = 5 + 2i + 3i + i^2 = \underbrace{5+5i}_{\theta_1+\theta_2}$$

$$\theta_1 = \tan^{-1}\left(\frac{1}{2}\right) \quad \theta_2 = \tan^{-1}\left(\frac{1}{3}\right)$$

$$\theta_1 + \theta_2 = \frac{\pi}{4}$$

MATHEMATICAL METHODS I

4

SINGLE VARIABLE FUNCTIONS

Moshe Krumbein - Fall 2021

4.1 Functions

Definition. For all inputs x into a function $f(x)$, return a single output y.

Graph: collection of all the points in the form $(x, f(x))$.

Image: collection of all possible y values.

Note. A *graph* of a function is not necessarily the *image* of a function.

4.1.1 Hyperbolic Functions

$$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right) \text{ (even)}$$
$$\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right) \text{ (odd)}$$
$$\sinh^2(x) - \cosh^2(x) = 1$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

The hyperbolic functions maintain many of the same identities of the trigonometric functions.

The area between the *ray* to $(\cosh(t), \sinh(t))$ and the hyperbolic function $x^2 - y^2 = 1$ is exactly equal to $\frac{t}{2}$.

$$\sinh : \mathbb{R} \to \mathbb{R}$$
$$\cosh : \mathbb{R} \to [1, \infty)$$
$$\sinh^{-1} : \mathbb{R} \to \mathbb{R}$$
$$\cosh^{-1} : [1, \infty) \to [0, \infty)$$

Formula for $\cosh^{-1} x$:

$$\begin{gathered} u = \cosh^{-1} x \\ \Updownarrow \\ \cosh(x) = x \\ \Updownarrow \\ e^u + e^{-u} = 2x \setminus : e^u \\ e^{2u} + 1 - 2xe^u = 0 \\ e^u = x \pm \sqrt{x^2 - 1} \\ u = \ln\left(x \pm \sqrt{x^2 - 1}\right) \end{gathered}$$

We found 2 values for u who are opposite from one either, so we define u to be the greater one:

$$\boxed{\cosh^{-1} x = \ln x + \sqrt{x^2 - 1}}$$

4.2 Exponential Functions

$$f(x) = x^n, \quad n \in \mathbb{R}$$

4.2.1 With Natural Exponents

If n is odd then $f(x) = x^n$ is odd.

If n is even then $f(x) = x^n$ is even.

4.2.2 With Reciprocal Natural Exponents

$$f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$$

This function is the inverse of x^n.

If n is odd then $f(x) = x^n$ is odd.

If n is even then $f(x) = x^n$ is even.

4.2.3 With Rational Exponents

$$f(x) = x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

If n is odd: $x^{\frac{m}{n}}$ is defined for all x.

If m is even the function is even and if m is odd then the function is odd.

If $\frac{m}{n} > 1$ the function is *convex* (up) and if $\frac{m}{n} < 1$ the function is *concave* (down).

If n is even (therefore m is odd) $x^{\frac{m}{n}}$ is defined $x \in [0, \infty)$.

If $\frac{m}{n} > 1$ the function is *convex* (up) and if $\frac{m}{n} < 1$ the function is *concave* (down).

4.2.4 With Negative Rational Exponents

$$f(x) = x^{-\frac{m}{n}}$$

The right side of the function always looks like $\frac{1}{x}$

Just the right side of the function: n is even.

Odd function that looks like $\frac{1}{x}$: n, m are odd.

Even function that looks like $\frac{1}{x}$: n is odd and m is even.

4.2.5 With Real Exponents

$$x^{\alpha} : \alpha \in \mathbb{R} \backslash \mathbb{Q}$$
$$x^{\alpha} := e^{\alpha \ln x} \quad (x > 0)$$

4.3 Composing Functions

$$g \circ f(x) = g(f(x))$$

Note.

$$\sin x = \frac{1}{i} \sinh x$$

4.3.1 Graphing functions with Square Roots

$$f(x) = \sqrt{(1-x)(2-x)(3-x)}$$

First we graph what's under the square root, then we see where the function is defined with a square root and how it affects the slope of the function (if the power is greater than or less than 1).

4.4 Continuous Functions

f is *continuous* at point a if $x \to a : f(x) \to f(a)$.

f is a *continuous function* if for all a in the *domain* of f fulfills: $x \to a : f(x) \to f(a)$.

4.4.1 Continuity of Functions on a Closed Interval

Theorem (Intermediate value theorem)**.** If the function f is continuous on $[a, b]$ and c is between $f(b)$ and $f(a)$ then there exists $x \in [a, b]$ such that $f(x) = c$.

Theorem (Intermediate value theorem)**.** If f is continuous on $[a, b]$ then f has a *minimum* and *maximum* on $[a, b]$.

MATHEMATICAL METHODS I

5

DERIVATIVES AND INTEGRALS

Moshe Krumbein - Fall 2021

5.1 Introduction

The *integral* of a function is the area under the curve of a graph and the *derivative* is the slope of the function.

Definition (Fundamental Theorem of Calculus)**.** *Integration* and *differentiation* are inverse functions of each other (under certain conditions).

A *definite integral* is also known as the *antiderivative.*

5.2 Derivative

Definition (Derivative)**.**

$$\frac{\Delta s}{\Delta t} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t} \quad \Delta t \to 0$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Symbol (Derivative).

$$\frac{ds}{dt}, \dot{s}(t), s'(t)$$

5.2.1 Geometric Significance of Derivatives

A function is *differential* at the point x_0 when there are no “sharp edges” or jumps at x_0.

5.2.2 Linear Approximation

If x is close to x_0, then:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

5.3 Integral

The area under the graph is the approximation: (*Riemann Sum*)

$$\sum_{i=1}^{n} f(t_i)(\overbrace{t_i - t_{i-1}}^{\Delta t_i})$$

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} \to \int_0^1 f(x)dx$$

If there does exist an approximation for this sum (that is not dependent on specific division), then we say that f is *integratable* on $[a, b]$:

$$\int_a^b f(t)dt$$

5.4 Operations Between Derivatives

Note. Differentiability $\implies$ continuity (continuity $\not\Rightarrow$ differentiability)

$$(f \pm g)' = f' \pm g' \tag{5.1}$$

$$(fg)' = f'g + fg' \tag{5.2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \tag{5.3}$$

$$(g \circ f)' = f' \cdot g'(f) \tag{5.4}$$

5.5 Derivative of an Inverse Function

Given function f differentiable at the point a:

$$f^{-1}(x) \cdot f(x) = x$$

Differentiate in terms of x:

$$(f^{-1}(f(x)))' \cdot f'(x) = 1$$

$$(f^{-1}(f(x)))' = \frac{1}{f'(x)}$$

5.6 Differentiating Implicit Functions

Example 5.6.1. $x^2 + y^2 - 1 = 0$

We differentiate both sides in terms of x and attempt to isolate $\frac{dy}{dx}$.

$$x^y - \ln y = 2x \implies x^{f(x)} - \ln(f(x)) = 2x$$
$$x^{f(x)}\left(\ln(x) \cdot f'(x) + \frac{1}{x} f(x)\right) - \frac{f'(x)}{f(x)} = 2$$

5.7 Linear Approximation

Reminder. If f is differentiable at the point a, then near a exists:

$$f(x) \approx f(a) + f'(a)(x - a)$$

Example 5.7.1.

$$\tan^{-1}(1.05)$$

$$\tan^{-1}(1) = \frac{\pi}{4} \to \quad \begin{aligned} a &= 1 \\ x &= 1.05 \\ f(x) &= \tan^{-1} x \\ f'(x) &= \frac{1}{1+x^2} \end{aligned}$$

$$f(1.05) \approx \overbrace{f(1)}^{\frac{\pi}{4}} + \overbrace{f'(1)}^{\frac{1}{2}}(0.05)$$
$$\tan^{-1}(1.05) \approx \frac{\pi}{4} + 0.025$$

5.8 Differential

Definition. The *differential* is the *linear approximation* of the change of a function.

If y is a function of x:

$$\Delta y \approx \frac{dy}{dx} \Delta x \implies dy = \frac{dy}{dx} dx$$

Example 5.8.1.

$$y = x^2 \quad z = \tan^{-1}(y)$$

Calculate the relation between dx, dy, dz around the points:

$$x = 1,\ y = 1,\ z = \frac{\pi}{4}$$

$$\frac{dy}{dx} = 2x \implies \frac{dy}{dx} = 2$$
$$dy = 2dx \quad dz = \frac{1}{2}dy$$

5.9 Lagrange's Mean Value Theorem

Definition. If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $a < c < b$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

6

TAYLOR SERIES

Moshe Krumbein - Fall 2021

6.1 Introduction

Goal: to find an approximation for a function that is better than a linear approximation.

Symbol (Derivatives).

$$f', f'', f''', f^{(4)}, \ldots, f^{(n)}$$
$$\dot{f}, \ddot{f} \text{ (derivative in terms of time)}$$
$$\frac{df}{dx}, \frac{d^2 f}{dx^2}, \ldots, \frac{d^n f}{dx^n}$$

6.2 High-order derivatives of products

$$f(x) = u(x) \cdot v(x)$$
$$f' = u'v + uv'$$
$$\vdots$$
$$f^{(3)} = u^{(3)}v + 3u''v' + 3u'v'' + uv^{(3)}$$

Definition (Leibniz product rule)**.**

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

Note. This is similar to *Newton's generalized binomial theorem.*

Example 6.2.1.

$$f(x) = x^2 \sin x$$
$$f^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (x^2)^{(n-k)} (\sin x)^{(k)}$$

If $n - k \geq 3$ then $(x^2)^{(n-k)} = 0$. So what's left is:

$$f^{(n)} = \underbrace{\binom{n}{n}}_{1} x^2 \sin^{(n)} x + \underbrace{\binom{n}{n-1}}_{n} 2x \sin^{(n-1)} x + \underbrace{\binom{n}{n-2}}_{\frac{n(n-1)}{2}} 2 \sin^{(n-2)} x$$

$$\boxed{f^{(n)}(x) = x^2 \sin^{(n)} x + 2nx \sin^{(n-1)} x + n(n-1) \sin^{(n-2)} x}$$

6.3 Polynomials

$$\begin{aligned}
p(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n & p(0) &= 0! \cdot a_0 \\
p'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} & p'(0) &= 1! \cdot a_1 \\
p''(x) &= 2a_2 + \cdots + n(n-1) a_n x^{n-2} & p''(0) &= 2! \cdot a_2 \\
p'''(x) &= 3 \cdot 2a_3 + \cdots + n(n-1)(n-2) a_n x^{n-3} & p'''(0) &= 3! \cdot a_3 \\
&\vdots & &\vdots \\
p^{(n)}(x) &= n! \cdot a_n & p^{(n)}(0) &= n! \cdot a_n
\end{aligned}$$

We can now rewrite p in terms of its derivatives at 0:

$$p(x) = p(0) + p'(0)x + \frac{p''(0)}{2!} x^2 + \cdots + \frac{p^{(n)}(0)}{n!} x^n$$

If we wanted to evaluate p at a point that isn't 0 we can do it by utilizing transformations:

$$q(x) = p(x - a)$$

$$\begin{aligned}
q(0) &= p(a) \\
q'(x) &= p'(a) \\
&\vdots \\
q^{(n)}(0) &= p^{(n)}(a)
\end{aligned}$$

$$q(x) = q(0) + q'(0)x + \frac{q''(0)}{2!} x^2 + \cdots + \frac{q^{(n)}(0)}{n!} x^n$$
$$p(x) = p(a) + p'(a)(x - a) + \frac{p''(a)}{2!} (x - a)^2 + \cdots + \frac{p^{(n)}(a)}{n!} (x - a)^n$$

(Taylor series of $p(x)$ at $x = a$)

Example 6.3.1.

Write the Taylor series of $p(x)$ at $x = -2$:

$$\begin{aligned} p(x) &= 3x^2 - 5x + 17 & p(-2) &= 39 \\ p'(x) &= 6x - 5 & p'(-2) &= -17 \\ p''(x) &= 6 & p''(-2) &= 6 \end{aligned}$$

$$\Downarrow$$

$$p(x) = 39 + (-17)(x+2) + \frac{6}{2!}(x+2)^2$$

6.4 Taylor Polynomial of order n at $x = a$ of $f(x)$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

$$\boxed{p_n^f = \sum_{k=0}^{n} \frac{f^{(n)}(a)}{k!}(x-a)^k}$$

Note. If $a = 0$, then we call this polynomial a *Maclaurin polynomial*.

6.5 Three Questions on Taylor Polynomials

1. For any x, does the Taylor polynomial converge?
2. If it does converge is it converging to the value of the function it's approximating?
3. If it does converge towards the value it's trying to approximate, what is its *error* after n iterations?

The answers to questions 1 and 2 depend on the function.

For example, the function $f(x) = e^{-\frac{1}{x^2}}$ as a Taylor polynomial is $p_n^f = 0$, which answers the first question as yes and the second as no.

If the answer of the first and second question is yes, then we can say:

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k}$$

Note. A *series* is an infinite sum.

6.6 What is n such that we get an error which is smaller than a specific value?

General Case

Reminder (Lagrange's Intermediate Value Theorem).

$$f'(c) = \frac{f(x) - f(a)}{x - a} \implies f(x) = f(a) + f'(c)(x - a)$$
$$\exists c : a < c < x$$

Definition (Lagrange Remainder)**.** If f is continuous on $[a, x]$ (or $[x, a]$) and differential $n + 1$ times on the aforementioned interval, then there exists a c on the interval such that:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + \underbrace{\frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}}_{R_n(x)\text{—Lagrange Remainder}}$$

Example 6.6.1. Calculate $e^{0.1}$ with an accuracy of 10^{-5}.

$$|R_n(0.1)| < 10^{-5}$$
$$R_n(0.1) = \frac{f^{(n+1)}(c)}{(n + 1)!}(0.1 - 0)^{n+1}$$
$$0 < c < 0.1 \implies e^0 < e^c < e^{0.1}$$
$$R_n(0.1) = \frac{e^c}{(n + 1)!}(0.1)^{n+1} < \frac{e^{0.1}}{(n + 1)!}(0.1)^{n+1} < \frac{3}{(n + 1)!}(0.1)^{n+1} < 10^{-5}$$

$n = 4$ definitely works, and $n = 3$ does not. Therefore we can say:

$$e^{0.1} \approx 1 + 0.1 + \frac{0.1^2}{2!} + \frac{0.1^3}{3!} + \frac{0.1^4}{4!}$$

With an error that is less than 10^{-5}.

Example 6.6.2. Calculate $\ln 1.2$ with an accuracy of 10^{-5}.

$$\ln(1.2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(0.2)^n$$
$$0.2 - \frac{0.2^2}{2} + \frac{0.2^3}{3} - \frac{0.2^4}{4} + \dots$$

Definition (Liebniz's Law). If we have a series with alternating signs:

$$\sum_n (-1)^n a_n = \underbrace{\underbrace{\underbrace{a_1 - a_2}_{S_1} + a_3}_{S_2} - a_4}_{S_3} + \dots$$

Where:

1. The series a_n is decreasing
2. The series a_n is approaches 0

Then:

1. The sum of the entire series will always be between S_n and S_{n+1}
2. This series is called a *Leibniz Series*

$$\boxed{|R_n(x)| < a_{n+1}}$$

Back to our example:

Since our series here is a Leibniz series, then we know that:

$$|R_n(0.2)| < \frac{0.2^{n+1}}{n+1} < 10^{-5}$$

We can see from this that $n = 7$ is a suitable answer.

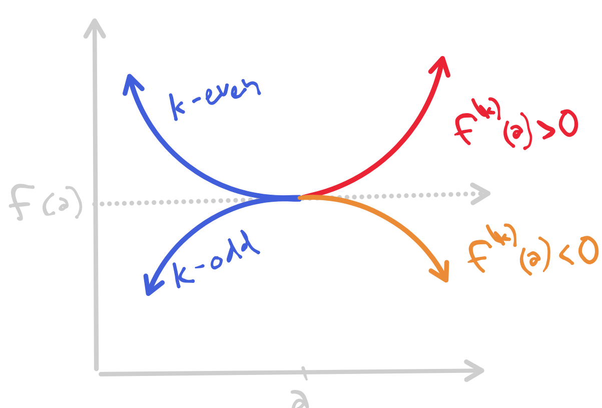
$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

6.7 Leading and Sub-Leading Order Term Approximation

Consider the Taylor Series of $f(x)$ where $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \ldots$$

Let's assume that all terms before $n = k$ are zero. How will f look around a?



$f^{(k)}(a) > 0$ and k is even: *local minimum*

$f^{(k)}(a) > 0$ and k is odd: *inflection point*

$f^{(k)}(a) < 0$ and k is even: *local maximum*

$f^{(k)}(a) < 0$ and k is odd: *inflection point*

In short: If $f'(a) = 0$, what do we know about a?

We check which is the smallest k (lowest derivative) such that $f^{(k)}(a) \neq 0$.

- If k is even then a is a *local extremum*.
- If k is odd then a is an *inflection point*.

Note. Even if $f'(x) \neq 0$, if $f''(x) = 0$, we need to check the next time $f^{(k)}(x) \neq 0$. If k is odd, then a is an *inflection point*.

6.8 Asymptotes

The line $y = ax + b$ is an *asymptote* at ∞ if:

$$\lim_{x \to \infty} (f(x) - (ax + b)) = 0$$

Another way of calculating a and b (besides for polynomial division):

$$a = \lim_{x \to \infty} \frac{f(x)}{x} \qquad b = \lim_{x \to \infty} (f(x) - ax)$$

Definition (Big O notation)**.** We say that $f(x) = O(g(x))$ if for some $A \in \mathbb{R} : f(x) \leq A(g(x))$.

$$f(x) = \frac{1}{2}x^2 + 2x - 1 = O(x^2) \quad x \to \infty$$

We say that $O(x^2)$ the "most dominant term" or the "size" of the function $f(x)$.

As far as we're concerned, if the leading term of a Taylor series is $\frac{f^{(k)}(a)}{k!}(x-a)^k$, we say that around the point a:

$$f \approx O((x-a)^k)$$

and that f zeros at a of the k-th order.

Definition (Little o notation)**.**

$$f \approx o(g(x)) \quad f \ll g(x)$$

Example 6.8.1 (Big O and little o notation).

$$x^6 + 1 = O(x^6) \qquad x^6 + 1 = o(x^7)$$

6.9 Additional Theorems

Definition (L'Hôpital's rule). L'Hôpital's rule helps us calculate limits of the form: "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$".

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

Example 6.9.1.

$$\lim_{x\to 0} \frac{1-\cos 10x}{x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{10\sin 10x}{2x} \overset{L'H}{=} \lim_{x\to 0} \frac{100\cos 10x}{2} = 50$$

Definition (Weierstrass extreme value theorem). If f is continuous on $[a, b]$, then f has a *minimum* and *maximum* on $[a, b]$.

Example 6.9.2 (function as a Taylor Polynomial).

$$\ln\left(1-5x^2\right)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$
$$\ln\left(1-5x^2\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-5x^2)^n}{n} = \sum_{n=1}^{\infty} -\frac{5^n}{n} x^{2n}$$

$$-1 < -5x^2 \leq 1 \implies -\frac{1}{5} \geq x^2 > \frac{1}{5} \implies -\sqrt{\frac{1}{5}} < x < \sqrt{\frac{1}{5}}$$

MATHEMATICAL METHODS I

7

INDEFINITE INTEGRALS

MOSHE KRUMBEIN - FALL 2021

7.1 Introduction

Definition. We say that $F(x)$ is a *primitive function (anti-derivative)* of $f(x)$ if:

$$F'(x) = f(x)$$

We also symbolize:

$$\boxed{\int f(x)\,dx = F(x) + c}$$

Theorem. If $F(x)$ is an *anti-derivative* of f, then for all c, $F(x) + c$ is also a *anti-derivative* of f.

Theorem. If F and G are *anti-derivatives* of f for a given interval, then for that same interval $F - G = c$.

Example 7.1.1.

$$f(x) = \frac{1}{x}$$

$$F(x) = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

$F(x)$ is an anti-derivative of $f(x)$

$$G(x) = \begin{cases} \ln x + 1 & x > 0 \\ \ln(-x) + 3 & x < 0 \end{cases}$$

$G(x)$ is an anti-derivative of $f(x)$

We see here that although $G - F$ isn't always the same constant, it is always a constant.

$$\int \frac{1}{x} = \ln|x| + c$$

7.2 Rules of Integration

Integral of a sum is the same of the integral:

$$\int f + g = \int f + \int g \tag{7.1}$$

Integral of composed functions:

$$\int f(g(x))g'(x) = F(g(x)) + c \tag{7.2}$$

Integration by parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

or more simply: (7.3)

$$\int uv' = uv - \int u'v$$

Example 7.2.1.

$$\int f(ax+b)\,dx = \frac{1}{a}\int a \cdot f(ax+b)\,dx = \frac{1}{a}\int f(t)\,dt$$

$$\frac{1}{a}F(t) + c = \frac{F(ax+b)}{a} + c$$

7.3 Examples of Integration by Substitution

Example 7.3.1.

$$\int \frac{dx}{1+(ax)^2} = \frac{\tan^{-1}(ax)}{a} + c$$

Example 7.3.2.

$$\int \frac{dx}{a^2+x^2} = \int \frac{1}{a^2} \cdot \frac{dx}{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a^2} \cdot \frac{\tan^{-1}\left(\frac{x}{a}\right)}{\frac{1}{a}} + c = \frac{\tan^{-1}\left(\frac{x}{a}\right)}{a} + c$$

Example 7.3.3.

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{1}{\sqrt{a^2}} \cdot \frac{dx}{\sqrt{1+\left(\frac{x}{a}\right)^2}} = \frac{1}{a} \cdot \frac{\sin^{-1}\left(\frac{x}{a}\right)}{\frac{1}{a}} + c = \sin^{-1}\left(\frac{x}{a}\right) + c$$

Example 7.3.4.
$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln|\cos x| + c$$

Example 7.3.5.
$$\int 5x \cdot e^{x^2} \, dx = \frac{5}{2} \int e^t \, dt = \frac{5}{2} e^t + c = \frac{5}{2} e^{x^2} + c$$

In General:
$$\int f'(x) e^{f(x)} \, dx = e^{f(x)} + c \tag{7.4}$$
$$\int f'(x) \cos(f(x)) \, dx = \sin(f(x)) + c \tag{7.5}$$
$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + c \tag{7.6}$$

Example 7.3.6.
$$\int \frac{2x+1}{x^2+x+17} \, dx = \ln \underbrace{(x^2+x+17)}_{\text{always positive}} + c$$

Completing the square:

Example 7.3.7.
$$\int \frac{8}{x^2+10x} \, dx = \int \frac{8}{(x-5)^2-25} \, dx = -\frac{8}{5} \tanh^{-1}\left(\frac{x+5}{5}\right) + c$$

Example 7.3.8.
$$\int \frac{8}{4x^2+4x+7} \, dx = \int \frac{8}{(2x+1)^2+6} \, dx = \frac{8 \tan^{-1}\left(\frac{2x+1}{\sqrt{6}}\right)}{2\sqrt{6}} + c$$

Example 7.3.9.
$$\int \frac{e^{\tan^{-1}(x)}}{1+x^2} \, dx = \int e^t \, dt = e^t + c = e^{\tan^{-1}(x)} + c$$

Example 7.3.10.
$$\int \frac{dx}{\sqrt{1-3x-2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{17}{16} - \left(x+\frac{3}{4}\right)^2}} = \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x+\frac{3}{4}}{\sqrt{\frac{17}{16}}}\right) + c$$

7.4 Examples for Integration by Parts

Example 7.4.1.

$$\int xe^{3x}\,dx = \frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}\,dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + c$$

Example 7.4.2.

$$\int \ln x\,dx = x\ln x - \int \frac{1}{x}\cdot x\,dx = x\ln x - x + c$$

Example 7.4.3.

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1} x - \frac{1}{2}\ln\bigl(1+x^2\bigr) + c$$

Continuing from last week:

Example 7.4.1.

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\cdot\ln x - \frac{1}{1+n}\int x^n\,dx = \frac{x^{n+1}}{n+1}\cdot\ln x - \frac{x^{n+1}}{(1+n)^2} + c$$

Example 7.4.2.

$$\underbrace{\int x^n e^x\,dx}_{I_n} = x^n e^x - \underbrace{\int x^{n-1}e^x\,dx}_{I_{n-1}}$$

$$I_n = x^n e^x - n\cdot I_{n-1} \quad I_0 = e^x + c$$

$$\frac{I_n}{n!} = \frac{x^n e^x}{n!} - \frac{I_{n-1}}{(n-1)!}$$

Example 7.4.3.

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

And we can continue similar to how we continued in the previous example using I_n and I_{n-1}. Alternatively:

$$\int (\ln x)^n\,dx = \int t^n e^t\,dt \qquad \ln x = t$$

Which is identical to our previous example.

Example 7.4.4.

$$I_n = \int (\sin x)^n \, dx$$
$$I_n = -(\sin x)^{n-1} \cos x + (n+1) \int (\sin x)^{n-2}(1 - \sin^2 x) \, dx$$
$$I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n}(\cos x)(\sin x)^{n-2}$$
$$I_0 = x + c \qquad I_1 = -\cos x + c$$

Alternatively, if n is odd:

$$\int \sin^{11} x \, dx = \int \sin x \sin^{10} x$$
$$(t = \cos x, dt = -\sin x \, dx)$$
$$\int (1 - t^2)^5 \, dt$$

If n is even:

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$$
$$\int \sin^6 x \, dx = \int (\sin^2 x)^3 \, dx = \int \left(\frac{1 - \cos 2x}{x}\right)^3 dx$$
$$= \frac{1}{8} \int (1 - 3\cos 2x + \underbrace{3\cos^2 2x}_{\frac{1+\cos 4x}{2}} - \underbrace{\cos^3 2x}_{\text{Like our odd example}}) \, dx$$

Example 7.4.5.

$$\int \frac{r(t)}{\sqrt{1 - t^2}} \, dt$$

Where $r(t)$ is a *rational function*. With substitution:

$$\int r(\sin \theta) \, d\theta$$

Which is often much easier to solve.

7.5 Trigonometric Substitutions

7.5.1 Universal Substitution

With integrals of rational polynomials containing $\sin x$ or $\cos x$, i.e.:

$$\int \frac{1 + \cos \theta}{2 + \sin \theta} \, d\theta \qquad \int \frac{\cos^3 \theta}{1 + 2\sin \theta} \, d\theta$$

We can use the following *universal substitution*:

$$
\begin{gathered}
t=\tan\left(\frac{\theta}{2}\right)\rightarrow dt=\frac{1}{2}(1+t^2)\,d\theta \\
d\theta=\frac{2}{1+t^2}\,dt \quad \tan\theta=\frac{2t}{1-t^2} \\
\sin\theta=\frac{2t}{1+t^2} \quad \cos\theta=\frac{1-t^2}{1+t^2}
\end{gathered}
$$

7.5.2 Special Case Substitutions

1. $f(\pi-\theta)=-f(\theta)$

   We can substitute $t=\sin x$.

2. $f(-\theta)=-f(\theta)$

   We can substitute $t=\cos x$.

3. $f(\pi+\theta)=f(\theta)$

   We can substitute $t=\tan x$.

7.6 Integral of Rational Functions

1. Sometimes was divide the polynomials to simplify our integral such that the degree of the polynomial in the numerator is lower than the degree of the polynomial in the denominator.

2. $$\int\frac{f'}{f}=\ln|f|+c$$

3. $$\int\frac{dx}{ax+b}=\frac{\ln|ax+b|}{a}+c$$

   $$\int\frac{dx}{a^2\pm x}=\begin{cases}\frac{\tan^{-1}\left(\frac{x}{a}\right)}{a}+c\\ \frac{\tanh^{-1}\left(\frac{x}{a}\right)}{a}+c\end{cases}$$

   This works for all fraction that you can complete the square.

4. Partial Fractions
   $$\frac{1}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}$$

Important Note: If we have a denominator of the form $(ax+b)^n$, we split into partial fractions in the following way:

$$\frac{p_n(x)}{(ax+b)^n} = \frac{c_1}{ax+b} + \frac{c_2}{(ax+b)^2} + \cdots + \frac{c_n}{(ax+b)^n}$$

Example 7.6.1.

$$\int \frac{x+5}{x^3-x^2-x+1}\,dx$$
$$\int \frac{x+5}{(x-1)^2(x+1)} = \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}\right) dx$$
$$= A\ln|x-1| - \frac{B}{x-1} + C\ln|x+1| + c$$

7.7 Integrating Rational Functions Using Complex Numbers

Reminder (Complex Numbers).

$$\ln(z) = |z| + \arg(z)i$$
$$z + \overline{z} = 2\operatorname{Re}(z)$$
$$z - \overline{z} = 2i\operatorname{Im}(z)$$
$$\tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}(x)$$

Example 7.7.1.

$$\int \frac{x+1}{x^4+x^2}\,dx$$
$$x^4+x^2 = x^2(x^2+1) = x^2(x-i)(x+i)$$
$$\frac{x+1}{x^4+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+i} + \frac{D}{x-i}$$
$$A=1,\ B=1,\ C=\frac{-1-i}{2},\ D=\frac{-1+i}{2}$$
$$\frac{x+1}{x^4} = \int \left(\frac{1}{x} + \frac{1}{x^2} + \frac{-1-i}{2(x+i)} + \frac{-1+i}{2(x-i)}\right) dx$$
$$= \ln|x| - \frac{1}{x} + \underbrace{\left(\frac{-1-i}{2}\right)\ln(x+i)}_{z} + \underbrace{\left(\frac{-1+i}{2}\right)\ln(x-i)}_{\overline{z}} + c$$
$$\ln(x+i) = \ln\sqrt{x^2+1} + \tan^{-1}\left(\frac{1}{x}\right) i$$
$$z+\overline{z} = 2\operatorname{Re}\left(\frac{1}{2}(-1-i)\left(\ln\sqrt{x^2+1} + \tan^{-1}\left(\frac{1}{x}\right) i\right)\right)$$
$$= -\ln\sqrt{x^2+1} + \tan^{-1}\left(\frac{1}{x}\right) = -\frac{1}{2}\ln\left(x^2+1\right) - \tan^{-1}x + \frac{\pi}{2}$$
$$\int \frac{x+1}{x^4+x^2}\,dx = \ln|x| - \frac{1}{x} - \frac{1}{2}\ln\left(x^2+1\right) - \tan^{-1}x + c$$

Example 7.7.2.

$$\int \frac{x+1}{x^4+2x^2+1}\,dx$$

Using real numbers:

$$\frac{1}{2}\int \frac{2x}{(x^2+1)^2}\,dx + \int \frac{1}{(x^2+1)^2}\,dx$$
$$x = \tan t, \quad dx = \frac{1}{\cos^2 t}\,dt$$
$$= -\frac{\frac{1}{2}}{(x^2+1)^2} + \int \frac{\frac{1}{\cos^2 t}}{\left(\frac{1}{\cos^2 t}\right)^2}\,dt$$
$$\int \cos^2 t\,dx = \int \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\underbrace{t}_{\tan^{-1}x} - \frac{1}{4}\underbrace{\sin 2t}_{?} + c$$
$$\ldots$$

In essence, using real numbers here is fairly difficult.

However, using imaginary numbers:

$$\int \frac{x+1}{(x+i)^2(x-i)^2}\,dx = \int \left(\frac{\frac{i}{4}}{x+i} + \frac{\frac{1}{4}(-1+i)}{(x+i)^2} + \frac{-\frac{i}{4}}{x-i} - \frac{\frac{1}{4}(1+i)}{(x-i)^2} \right) dx$$

$$= \underbrace{\frac{i}{4}\ln(x+i)}_{z} + \underbrace{\left(-\frac{i}{4}\right)\ln(x-i)}_{\overline{z}} \underbrace{-\frac{\frac{1}{4}(-1+i)}{x+i}}_{+\overline{\omega}} + \underbrace{\frac{\frac{1}{4}(1+i)}{x-i}}_{\omega} + c$$

$$z+\overline{z} = 2\,\mathrm{Re}(z) = \mathrm{Re}\left(\frac{i}{2} \left(\frac{1}{2}\ln\left(x^2+1\right) + i \underbrace{\left(-\tan^{-1}x + \frac{\pi}{2}\right)}_{\tan^{-1}\left(\frac{1}{x}\right)} \right)\right)$$

$$= \frac{1}{2}\left(\tan^{-1}x + \frac{\pi}{2}\right)$$

$$\omega+\overline{\omega} = 2\,\mathrm{Re}(\omega) = \mathrm{Re}\left(\frac{\frac{1}{2}(1+i)}{x-i}\right) = \frac{1}{2}\,\mathrm{Re}\left(\frac{(1+i)(x+i)}{(x-i)(x+i)}\right) = \frac{1}{2}\frac{x-1}{x^2+1}$$

$$\int \frac{x+1}{x^4+2x^2+1}\,dx = \frac{1}{2}\tan^{-1}x + \frac{x-1}{2(x^2+1)} + c$$

MATHEMATICAL METHODS I

8

DEFINITE INTEGRALS

MOSHE KRUMBEIN - FALL 2021

8.1 Introduction

We define $\int_a^b f(x)\,dx$ to be the area underneath the graph of $f(x)$, above the x-axis on the interval $[a, b]$.

8.2 The Fundamental Theorem of Calculus

Definition (First Fundamental Theorem of Calculus)**.** Under certain conditions:

$$\frac{d}{dx}\underbrace{\left(\int_a^x f(t)\,dt\right)}_{F(x)} = f(x)$$

Proof.

$$F'(x) = \lim_{h\to 0}\frac{f(x+h) - F(x)}{h}$$
$$= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h} = \frac{h\cdot f(x) + \text{negligible}}{h} = f(x)$$

■

Example 8.2.1.

$$\frac{d}{dx}\left(\int_1^x \frac{\sin t}{t}\,dt\right) = \frac{\sin x}{x}$$

Example 8.2.2.

$$\frac{d}{dx}\left(\int_1^{x^2} \frac{\sin t}{t}\,dt\right) = \frac{\sin(x^2)}{x^2}\cdot 2x$$

Example 8.2.3.

$$\frac{d}{dx}\left(\int_{5x}^{e^x} e^{-t^2}\,dt\right) = \frac{d}{dx}\left(\int_0^{e^x} e^{-t^2}\,dt - \int_0^{5x} e^{-t^2}\,dt\right) = e^{-(e^x)^2}\cdot e^x - e^{-(5x)^2}\cdot 5$$

Definition (Upgraded Theorem)**.**

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) = f(h)\cdot h' - f(g)\cdot g'$$

Definition (Second Fundamental Theorem of Calculus)**.** Also known as the *Newton-Leibniz axiom.*

If f is continuous and F is an antiderivative of f, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof. According to the first theorem:

$$\int_a^x f(t)\,dt = F(x) + c$$

If we set $x = a$:

$$0 = \int_a^a f(t)\,dt = F(a) + c \implies C = -F(a)$$

■

Essentially, there are a few things to keep in mind:

- When doing substitution, it is important to also adjust the limits of the integral accordingly (this only works in the substitution is *one-to-one*.)
- Often we can solve definite integrals simply by seeing if we substitute the limits of the integral are equal to each other, which means the integral is equal to zero, without having to work out the antiderivative at all.

8.3 Improper Integrals

There two types of improper integrals:

1. Divergent improper integrals, e.g.:

$$\int_3^\infty x\,dx = \infty \qquad \int_0^1 \frac{1}{x}\,dx = \infty$$

2. Convergent improper integrals, e.g.:

$$\int_3^\infty \frac{1}{x^2}\,dx = \frac{1}{3}$$

Note.

$$\int_{-1}^1 \frac{\sin x}{x}\,dx$$

This is a *proper integral* because the function is *bounded* on the interval even though there is an undefined point at $x = 0$.

Theorem.

$$\int_1^\infty \frac{1}{x^a}\,dx$$

If $a > 1$, the integral *converges*, and if $a \leq 1$, the integral *diverges*.

However:

$$\int_0^1 \frac{1}{x^a}\,dx$$

If $a < 1$, the integral *converges*, and if $a \geq 1$, the integral *diverges*.

$$\int_c^d \frac{1}{(x - x_0)^a}\,dx\,, \quad c \leq x_0 \leq d$$

If $a < 1$, the integral *converges*, and if $a \geq 1$, the integral *diverges*.

MATHEMATICAL METHODS I

9

VECTOR FUNCTIONS ($\mathbb{R} \to \mathbb{R}^n$)

Moshe Krumbein - Fall 2021

9.1 Introduction

Definition (Vector Function).

$$f : \mathbb{R} \to \mathbb{R}^n \implies f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

When drawing these functions, instead of drawing the *graph*, we draw the *image* of the function.

Example 9.1.1.

$$f : \mathbb{R} \to \mathbb{R}^2 \implies f(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

The *image* of this function is the unit circle.

9.2 Derivative of a Vector Function

Definition (Derivative of a Vector Function).

$$\frac{d}{dt} \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} = \begin{pmatrix} f_1'(t) \\ \vdots \\ f_n'(t) \end{pmatrix} = \dot{f}(t) = \underline{r}(t)$$

Definition (Linear Approximation).

$$f(t + \Delta t) = f(t) + \dot{f}(t)\Delta t$$

9.2.1 Characteristics

$$\frac{d}{dt}(\underline{r} \cdot \underline{w}) = \underline{\dot{r}} \cdot \underline{w} + \underline{r} \cdot \underline{\dot{w}} \tag{9.1}$$

$$\frac{d}{dt}(\underline{r} \wedge \underline{w}) = \underline{\dot{r}} \wedge \underline{w} + \underline{r} \wedge \underline{\dot{w}} \tag{9.2}$$

$$\frac{d}{dt}[\underline{r}, \underline{w}, \underline{u}] = [\underline{\dot{r}}, \underline{w}, \underline{u}] + [\underline{r}, \underline{\dot{w}}, \underline{u}] + [\underline{r}, \underline{w}, \underline{\dot{u}}] \tag{9.3}$$

Example 9.2.1.

$$\frac{d}{dt}(\underline{r} \cdot \underline{r}) = 2\underline{r} \cdot \underline{\dot{r}}$$

9.3 Integral of a Vector Function

Definition (Integral of a Vector Function)**.**

$$\int \underline{f}(t)\, dt = \begin{pmatrix} \int f_1 \\ \vdots \\ \int f_n \end{pmatrix}$$

9.4 Complex Functions

$$f : \mathbb{R} \to \mathbb{C}$$

We can look at this function as if it was $f : \mathbb{R} \to \mathbb{R}^2$

Example 9.4.1.

$$\begin{aligned} f(t) &= e^{i\omega t} = \cos(\omega t) + i \sin(\omega t) \\ \dot{f}(t) &= -\omega \sin(\omega t) + i\omega \cos(\omega t) \\ &= i\omega(\cos(\omega t) + i \sin(\omega t)) = i\omega e^{i\omega t} \end{aligned}$$

9.5 Length of a Curve

Given curve $\underline{r}(t)$, its velocity vector is $\underline{\dot{r}}(t)$.

Its scalar velocity is defined as $\|\underline{\dot{r}}(t)\|$, or simply $\dot{r}(t)$.

The length of the curve from $\underline{r}(a)$ to $\underline{r}(b)$ is defined as:

$$\int_a^b \|\underline{\dot{r}}(t)\|\, dt$$

Example 9.5.1.

$$\underline{r}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_{0 \le t \le 2\pi}$$

The length of the curve is:

$$\int_0^{2\pi} \|\dot{\underline{r}}(t)\| \, dt = \int_0^{2\pi} \underbrace{\left\| \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\|}_{1} dt = \int_0^{2\pi} dt = 2\pi$$

Example 9.5.2.

$$\underline{r}(t) = \begin{pmatrix} t\cos t \\ t\sin t \\ t \end{pmatrix}$$

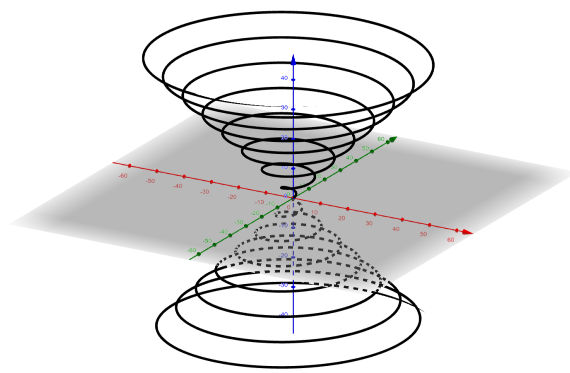


Figure 9.1: Graph of $\underline{r}(t)$ in Example 9.5.2

$$\dot{\underline{r}}(t) = \begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \\ 1 \end{pmatrix}$$

$$\|\dot{\underline{r}}(t)\| = \sqrt{t^2 + 2}$$

The length of the curve:

$$\int_a^b \sqrt{t^2+2} \, dt$$

MATHEMATICAL METHODS I

10

MULTIVARIABLE FUNCTIONS $\mathbb{R}^m \to \mathbb{R}$

Moshe Krumbein - Fall 2021

10.1 Introduction

We are mostly working with $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x, y) = z$$

Example 10.1.1.

$$f : z = x^2 + y^2$$
$$g : z = \sqrt{1 - x^2 - y^2}$$
$$h : z = \sqrt{x^2 + y^2}$$

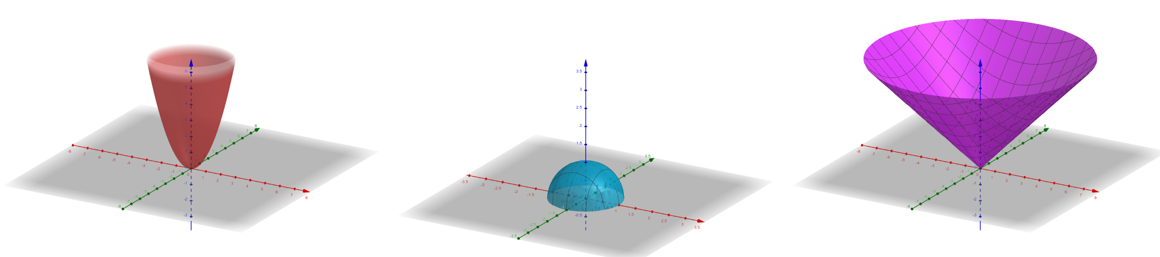


Figure 10.1: Graphs of the functions f, g, h respectively

10.2 Domain of Multivariable Functions

The *domain* is part of the plane $\mathbb{R}^2$.

Example 10.2.1.

$$\begin{aligned} z = x^2 + y^2 &\quad \text{Domain: } \mathbb{R}^2 \\ z = \sqrt{1 - x^2 - y^2} &\quad \text{Domain: } x^2 + y^2 \le 1 \\ z = \sqrt{x^2 + y^2} &\quad \text{Domain: } \mathbb{R}^2 \\ z = \sin^{-1}(x + y) &\quad \text{Domain: } |x + y| \le 1 \\ z = e^x + 2y &\quad \text{Domain: } \mathbb{R}^2 \\ z = \ln\big(\sin\big(x^2 + y^2\big)\big) &\quad \text{Domain: } 2\pi k < x^2 + y^2 < (2k + 1)\pi \end{aligned}$$

10.3 Continuity of Multivariable Functions

Definition (Continuity of Multivariable Functions). f is continuous at the point (x_0, y_0) if for all (x, y) that is "in the neighborhood" of (x_0, y_0) exists $f(x, y)$ that is "in the neighborhood" of $f(x_0, y_0)$.

Example 10.3.1.

$$f(x,y) = \begin{cases} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

Is continuous on $\mathbb{R}^2$.

10.4 Cross Sections of Multivariable Functions with Planes

Regarding planes of the form $x = x_0$ or $y = y_0$, we simply replace the variable with the constant in the equation.

Example 10.4.1.

$$z = \sqrt{1 - x^2 - y^2} \quad x_0 = \frac{1}{2}$$
$$z = \sqrt{\frac{3}{4} - y^2}$$

If $z = z_0$:

Example 10.4.2.

$$z = x^2 + y^2 \quad z_0 = 4$$
$$x^2 + y^2 = 4$$

This also is called a function's *contour line* of height z_0.

10.5 Partial Derivatives

Definition (Partial Derivatives)**.** The *partial derivative* of a multivariable function is the derivative with respect to only one of its variables (with the others held constant).

Symbol. The partial derivative of f with respect to y:

$$\frac{\partial f}{\partial y}, f_y, f_2$$

The partial derivative of f with respect to x:

$$\frac{\partial f}{\partial x}, f_x, f_1$$

Example 10.5.1.

$$f(x,y) = x^3 y + e^{2y+x}$$

Find $f_x(x,y)$ and $f_y(x,y)$.

$$\begin{aligned} f_x(x,y) &= 3x^2 y + e^{2y+x} \\ f_y(x,y) &= x^3 + 2e^{2y+x} \end{aligned}$$

Example 10.5.2 (Question 1). If f is continuous at the point (x_0, y_0), does that mean it's always partially differential at that point?

Conclusion. No, a counterexample being a cone, which is continuous everywhere but not differentiable at its bottom point $(0,0)$ (see h in Example 10.1.1).

Example 10.5.3 (Question 2). Does a function f having both a partial derivative with respect to x and y (f_x, f_y) at a given point imply continuity at that same point?

Conclusion. No, counterexample:

$$f(x,y) = \begin{cases} 1 & x = 0 \text{ or } y = 0 \\ 0 & \text{otherwise} \end{cases}$$

10.6 Directional Derivatives

Definition (Directional Derivatives)**.** A *directional derivative* is a partial derivative, but instead of finding the derivative with respect to either the x or y-axes, we can find the derivative and any direction.

In other words, given a point (x_0, y_0) and a direction $\binom{a}{b}$, we want to differentiate $f(x,y)$ in in that direction, at the point (x_0, y_0).

In order to find the direction derivative, we can express our function f in terms of t:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\partial f_{\underline{v}} \equiv \frac{dz}{dt}(0)$$

Note. The length of the directional vector has to be equal to 1:

$$\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\| = 1$$

Example 10.6.1. Find the directional derivative of the function $z = x^2 + y$ at the point $(2, 3)$ in the direction $\begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$.

$$f\left(2 + \frac{\sqrt{2}}{2}t, 3 - \frac{1}{2}t\right) = \left(2 + \frac{\sqrt{3}}{2}t\right)^2 + 3 - \frac{1}{2}t$$

Example 10.6.2.

$$f(x, y) = \begin{cases} \sqrt{x^2 + y^2} & y > 0 \\ -\sqrt{x^2 + y^2} & y < 0 \\ x & y = 0 \end{cases}$$

$$\underline{\hat{e}} = \begin{pmatrix} a \\ b \end{pmatrix}, \quad f(0 + ta, 0 + tb) = t\sqrt{a^2 + b^2}$$
$$\partial f_{\underline{\hat{e}}}(0, 0) = \sqrt{a^2 + b^2} = 1$$

Example 10.6.3. Differentiate $f(x, y) = x^2y$ at the point (x_0, y_0) in the direction $\underline{\hat{e}} = (a, b)$.

$$g(t) = f(x_0 + at, y_0 + bt) = (x_0 + at)^2(y_0 + bt)$$
$$g'(t) = \frac{dg}{dt} = 2(x_0 + at)a(y_0 + bt) + (x_0 + at)^2 b$$
$$g'(0) = 2ax_0y_0 + bx_0^2 = \partial f_{\underline{\hat{e}}}(x_0, y_0)$$

Example 10.6.4 (Question 3). For a function f, if there exists a directional derivative in all directions at a given point, is f continuous at that point?

Conclusion. No, a counterexample being a function that resembles a piece of paper torn in half though still connected at a single point in the middle.

10.7 Differentiability

Definition (Differentiability)**.** We say that $f(x, y)$ differentiable at (x_0, y_0) if exists a linear approximation at that point:

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + A\Delta x + B\Delta y$$

In other words, $f(x, y)$ is differentiable at point (x_0, y_0) if there exists a tangent plane to that point.

What are A and B in our equation?

$$f(x_0 + \Delta x, y_0) \approx f(x_0, y_0) + A\Delta x$$

We know that:

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = A$$

And likewise:

$$\frac{\partial f}{\partial y}(x_0, y_0) = B$$

Therefore the equation to find the linear approximation is:

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

Where the equation for the tangent plane is:

$$z = z_0 + f_x \cdot (x - x_0) + f_y \cdot (y - y_0)$$

Suppose that f is differential at the point (x_0, y_0). Find $\partial f_{\underline{\binom{\hat{a}}{b}}}(x_0, y_0)$.

$$g(t) = f(x_0 + at, y_0 + bt) \approx f(x_0, y_0) + A \cdot at + B \cdot bt$$

$$\begin{aligned} g'(0) = \lim_{\Delta t \to 0} \frac{g(0 + t) - g(0)}{\Delta t} &= \lim_{\Delta t \to 0} \frac{\overbrace{f(x_0 + a\Delta t, y_0 + b\Delta t) - f(x_0, y_0)}^{A \cdot a\Delta t + B \cdot b\Delta t}}{\Delta t} \\ &= A \cdot a + B \cdot b = \boxed{f_x \cdot a + f_y \cdot b} \end{aligned}$$

Note. This only works for differentiable functions!

Succinctly put:

$$\partial f_{\underline{\hat{e}}}(x_0, y_0) = \begin{pmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

How do we know if a function is differentiable at a given point?

If there are partial, continuous derivatives in the “neighborhood” of that point.

10.8 Chain Rule

Reminder (Single Variable Functions). If $y = f(x)$ and there's a function $g(y)$, then:

$$\frac{dg}{dx} = \frac{dg}{dy} \cdot \frac{dy}{dx}$$

For multivariable functions, there are two chain rules:

1. If $f(x, y)$ is *differentiable* and there is a curve on the plane $\underline{r}(t) = (x(t), y(t))$:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

   Or in other words:

$$f'(t) = f_x \cdot x'(t) + f_y \cdot y'(t)$$

   Example 10.8.1.

$$f(x, y) = xy^2 + e^x \quad \underline{r}(t) = (t^2, t^3)$$

   Using the chain rule:

$$\begin{aligned} f'(t) &= (y^2 + e^x) \cdot 2t + (2xy) \cdot 3t^2 \\ &= ((t^3)^2 + e^{t^2}) \cdot 2t + (2t^2 t^3) \cdot 3t^2 \end{aligned}$$

   Directly:

$$\begin{aligned} f(x, y) = f(t^2, t^3) &= t^8 + e^{t^2} \\ \frac{df}{dt} &= 8t^7 + 2te^{t^2} \end{aligned}$$

2. If $f(x, y)$ is *differentiable* and there are two *differentiable* functions $x = x(u, v)$ and $y = y(u, v)$, then:

$$\left.\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} \end{aligned}\right| \quad \begin{aligned} f_u &= f_x \cdot x_u + f_y \cdot y_u \\ f_v &= f_v \cdot x_v + f_y \cdot y_v \end{aligned}$$

In other words, the first chain rule can be written as:

$$f'(t) = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

And the second chain rule can be written as:

$$(f_u, f_v) = \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

A nice use of the chain rule is for *implicit functions*:

Given the function $f(x, y) = 0$, and we consider the contour line 0 (which is the curve). We consider the following curve:

$$x = x \quad y = y(x)$$

And how we differentiate with respect to x (which is our t in this case):

$$f_x \cdot \frac{dx}{dx} + f_y \cdot \frac{dy}{dx} = 0 \implies \boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

Example 10.8.2 (Implicit Functions).

$$xe^y + \ln(2x + 3y) = 0$$
$$\frac{dy}{dx} = -\frac{e^y + \frac{2}{2x+3y}}{xe^y + \frac{3}{2x+3y}}$$

10.9 Gradient

Reminder. If f is *differentiable* at (x_0, y_0), then the *directional derivative* at (x_0, y_0) in the direction $\underline{\hat{\binom{a}{b}}}$ is:

$$\partial f_{\underline{\hat{\binom{a}{b}}}}(x_0, y_0) = f_x \cdot a + f_y \cdot b = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

Definition (Gradient)**.** The vector of partial derivatives of f is called the *gradient* of f and is symbolized as $\underline{\boldsymbol{\nabla} f}$.

Example 10.9.1.

$$f(x, y) = x^2 y$$
$$\underline{\boldsymbol{\nabla} f} = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}$$

And at the point $(1, 3)$ the *gradient* is $\binom{6}{1}$.

Assuming f is differential:

$$\partial f_{\underline{\hat{e}}}(x_0, y_0) = \underline{\boldsymbol{\nabla} f}(x_0, y_0) \cdot \underline{\hat{e}} = \|\underline{\boldsymbol{\nabla} f}\| \cdot \underbrace{\|\underline{\hat{e}}\|}_{1} \cos\alpha = \|\underline{\boldsymbol{\nabla} f}\| \cos\alpha$$

Where α is the angle between the $\underline{\boldsymbol{\nabla} f}$ and $\underline{\hat{e}}$.

Conclusion. We can draw the following conclusions based on α:

1. The maximum directional derivative is when $\underline{\hat{e}}$ is in the same direction as $\underline{\boldsymbol{\nabla} f}$ $(= \|\underline{\boldsymbol{\nabla} f}\|)$.

2. The minimum directional derivative is when $\underline{\hat{e}}$ is in the opposite direction as $\underline{\boldsymbol{\nabla} f}$ $(= -\|\underline{\boldsymbol{\nabla} f}\|)$.
3. The directional derivative is zero when $\underline{\hat{e}}$ is perpendicular to $\underline{\boldsymbol{\nabla} f}$.

Additionally, we can also conclude that the gradient is always perpendicular to the contour line.

This can help us better understand the graph of f by first drawing out the contour line and then overlay the field of gradient vectors.

Note. If $z = f(x, y)$, then the *normal* of the tangent plane is $\begin{pmatrix} f_x \\ f_y \\ -1 \end{pmatrix}$.

Definition (Stationary/Critical Point)**.** A point (x_0, y_0) where the *gradient* is zero or doesn't exist is called a *stationary* or *critical point* of f.

A *critical point* is:

a. A *local minimum*
b. A *local maximum*
c. A *saddle point*

Example 10.9.2.

$$f(x, y) = x^3 + y^3 - 3x - 3y$$
$$f_x = 3x^2 - 3 = 0 \quad f_y = 3y^2 - 3 = 0$$

There are four *critical points*: $(\pm 1, \pm 1)$.

$$(1, -1)$$

For $(1, -1)$, we'll take a close point $(1 + h, -1 + k)$:

$$f(1 + h, -1 + k) = (1 + h)^3 + (-1 + k)^3 - 3(1 + h) - 3(-1 + k) = \boxed{3h^2 - 3k^2 + h^3 + k^3}$$
$$f(1, -1) = 0$$

We see that at the point $(1, -1)$ it isn't the minimum or maximum in its neighborhood, and therefore it is a *saddle point.*

10.10 High Order Partial Derivatives

Example 10.10.1.

$$f(x, y) = e^{x^2 y}$$
$$f_x = 2xye^{x^2 y}$$
$$f_y = x^2 e^{x^2 y}$$

Symbol.

$$\begin{array}{l|l} \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial \frac{\partial f}{\partial x}}{\partial x} = f_{xx} & \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial \frac{\partial f}{\partial y}}{\partial y} = f_{yy} \\ \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial \frac{\partial f}{\partial x}}{\partial y} = f_{xy} & \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial \frac{\partial f}{\partial y}}{\partial x} = f_{yx} \end{array}$$

We find of the derivative of each one with respect to x and y:

$$\begin{aligned} f_{xx} &= 2ye^{x^2y} + 4x^2y^2e^{x^2y} \quad & f_{xy} &= 2xe^{x^2y} + 2x^3ye^{x^2y} \\ f_{yx} &= 2xe^{x^2y} + 2x^3ye^{x^2y} \quad & f_{yy} &= x^4e^{x^2y} \end{aligned}$$

We notice here that $f_{xy} = f_{yx}$.

Theorem. If f_{xy} and f_{yx} are defined surrounding (x_0, y_0), then they are equal.

Theorem (Taylor's Theorem for Multivariate Functions)**.** We are trying to find an approximation to $f(x_0 + h, y_0 + k)$.

We define $u(t) = f(x_0 + th, y_0 + tk)$, which is a single variable function, which we can find a Taylor polynomial for.

Reminder. Maclaurin polynomial of $u(t)$:

$$u(0) + u'(0)t + \frac{u''(0)}{2!}t^2 + \dots$$

$$\begin{aligned} u'(t) &= f_x \cdot x'(t) + f_y \cdot y'(t) = hf_x + kf_y \\ u''(t) = \frac{df_x}{dt} \cdot h + \frac{df_y}{dt} \cdot k &= (f_{xx} \cdot h + f_{xy} \cdot k)h + (f_{yy} \cdot h + f_{yx} \cdot k)k \\ &= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \end{aligned}$$

Therefore, an approximation of order 2 (where $t = 1$):

$$f(x_0 + h, y_0 + k) \approx f(x_0 + y_0) + (f_xh + f_yk) + \frac{1}{2}(f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2)$$

10.11 Investigating Stationary (Critical) Points

Suppose (x_0, y_0) is a *stationary* point. (In other words: $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$).

We consider an arbitrary point surrounding (x_0, y_0): $(x_0 + h, y_0 + k)$. According the second order approximation we just derived:

$$f(x_0 + h, y_0 + k) \approx f(x_0 + y_0) + \underbrace{(f_xh + f_yk)}_{=0} + \frac{k^2}{2}\underbrace{\left(f_{xx}\left(\frac{h}{k}\right)^2 + 2f_{xy}\left(\frac{h}{k}\right) + f_{yy}\right)}_{\text{Let's investigate!}}$$

That second order term is always positive when:

1. $f_{xx} > 0$
2. $(2f_{xy})^2 - 4f_{xx}f_{yy} < 0$

That second order term is always negative when:

1. $f_{xx} < 0$
2. $(2f_{xy})^2 - 4f_{xx}f_{yy} < 0$

It is sometimes positive and sometimes negative when:

1. $(2f_{xy})^2 - 4f_{xx}f_{yy} > 0$

Summary

If $f_{xx}f_{yy} - f_{xy}^2 > 0$, then (x_0, y_0) is a local critical point.

In such a case, if $f_{xx} > 0$ it is a *minimum* point and if $f_{xx} < 0$ then it is a *maximum* point.

Alternatively, $f_{xx}f_{yy} - f_{xy}^2 < 0$, then (x_0, y_0) is a *saddle* point.

Note. If $f_{xx} = 0$, we don't know.

Another way to look at it:

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \to \begin{cases} > 0 & \textit{extreme} \text{ point} \\ < 0 & \textit{saddle} \text{ point} \end{cases}$$

Example 10.11.1.

$$f(x, y) = x^3 - x - y^2$$
$$f_x = 3x^2 - 1 = 0 \qquad f_y = -2y = 0$$
$$x = \pm\frac{1}{\sqrt{3}} \quad y = 0$$
$$\left(\frac{1}{\sqrt{3}}, 0\right) - \text{saddle point} \qquad \left(-\frac{1}{\sqrt{3}}, 0\right) - \text{maximum}$$

10.12 Critical Points under Constraints

Theorem (Weierstrass Extreme Value Theorem)**.** If $f(x, y)$ is continuous, on a closed and bounded domain, f has *minimum* and *maximum* points.

How do we find them? They are either:

a. Local *extrema*
b. They are are on bounds of the domain

Example 10.12.1.

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$
$$D = \{(x,y) \mid -6 \leq x, y \leq 6\} = [-6,6] \times [-6,6]$$

First, let's find critical points:

$$f_x = 4x^3 - 4x + 4y = 0$$
$$f_y = 4y^3 - 4y + 4x = 0$$
$$x = 0, \pm\sqrt{2} \quad y = -x$$
$$(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$$

Now, to investigate the bounds, we can break them up into four parts:

$$x = 6, -6 \leq y \leq 6 : f'(6,y) = 4(y^3 - y + 6) = 4(y+2)(y^2 - 3y + 3)$$
$$(6,-2), (6,6), (6,-6)$$

And continue to the other sides of the bounds to create one list of all potential critical points.

Finally, we plug in all of the points into our original function and see which are the *minimum* and *maximum* points in the bounded region.

10.13 Differentials

Reminder. If $z = f(x,y)$ and *differentiable*, then $z \approx z_0 + f_x\Delta x + f_y\Delta y$, or in other words:

$$\Delta z \approx f_x\Delta x + f_y\Delta y$$

Then the *differential* of z is:

$$\boxed{dz = f_x\,dx + f_y\,dy}$$

10.14 Lagrange Multipliers

The goal is to find *extrema* of a function $f(x_1, \ldots, x_n)$ under constants $g_1(x_1, \ldots, x_n) = g_2(x_1, \ldots, x_n) = \cdots = g_m(x_1 \ldots, x_n)$.

Example 10.14.1. We want to find the *extrema* of $f(x,y)$ under constraint of $g(x,y) = 0$.

Lagrange states the an *extremum* will be at a point under the constraint of $g(x,y) = 0$ is tangent to the contour line of f. In other words:

$$\boldsymbol{\nabla} f = \lambda \cdot \boldsymbol{\nabla} g$$

Where λ is the *Lagrange multiplier.*

Example 10.14.2. Find the *extremum* of $f(x, y, z) = xyz$ on the domain:

$$D = \left\{(x, y, z) \mid {}^{x+y+z=c}_{z,y,z \geq 0}\right\}$$

First, D is a closed and bounded region, which means it contains minimum and maximum points.

Our constraints:

$$\underbrace{x + y + z - c}_{g(x,y,z)} = 0$$

We will now find points such that $\boldsymbol{\nabla} f = \lambda \cdot \boldsymbol{\nabla} g$. In other words:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = 0 \end{cases} \left| \begin{array}{c} yz = \lambda \cdot 1 \\ xz = \lambda \cdot 1 \\ xy = \lambda \cdot 1 \\ x + y + z - c = 0 \end{array} \right.$$

$$(c, 0, 0), (0, c, 0), (0, 0, c), \left(\frac{c}{3}, \frac{c}{3}, \frac{c}{3}\right)$$

For the first three points (the boundary of f) we see that $f = 0$ (which is our minimum) and for our fourth point we see that $f(\frac{c}{3}, \frac{c}{3}, \frac{c}{3}) = \frac{c^3}{27}$, which is our maximum.

In a case such that we have more than one constraint ($h = g = 0$), we use:

$$\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g + \mu \boldsymbol{\nabla} h$$

Example 10.14.3. Find the minimum of $f(x, y, z) = x^2 + 2y^2 + 3z^2$ under constraint of $x + y + z = 1$.

Our constraint is closed but not bounded, and therefore we cannot be sure if there will be a maximum or minimum within our constraint. Indeed in our case, there is no maximum.

Here we will consider the region $-100 \leq x, y, z \leq 100$. It is clear that the minimum will be within this region.

From here we can find our *extrema* using *Lagrange multipliers* and we will see that the one *extremum* we find is indeed the global minimum.

11 MULTIVARIABLE FUNCTIONS $\mathbb{R}^m \to \mathbb{R}^n$

Moshe Krumbein - Fall 2021

11.1 Introduction

We will be examining functions of the form:

$$f(x_1, \ldots, x_m) = \begin{pmatrix} f_1(x_1, \ldots, x_m) \\ \vdots \\ f_n(x_1 \cdots, x_m) \end{pmatrix}$$

Example 11.1.1.

$$f(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \quad \Bigg| \quad f(r, \theta, \phi) = \begin{pmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{pmatrix}$$

Definition (Derivative)**.** For the partial derivative:

$$\frac{\partial f}{\partial x_i} = \begin{pmatrix} \frac{\partial f_1}{\partial x_i} \\ \vdots \\ \frac{\partial f_n}{\partial x_i} \end{pmatrix}$$

In general:

$$d\underline{f} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}\, dx_1 + \frac{\partial f_1}{\partial x_2}\, dx_2 + \cdots + \frac{\partial f_1}{\partial x_m}\, dx_m \\ \vdots \\ \frac{\partial f_n}{\partial x_1}\, dx_1 + \frac{\partial f_n}{\partial x_2}\, dx_2 + \cdots + \frac{\partial f_n}{\partial x_m}\, dx_m \end{pmatrix} = \frac{\partial \underline{f}}{\partial x_1}\, dx_1 + \cdots + \frac{\partial \underline{f}}{\partial x_m}\, dx_m$$

$$D\underline{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix} \qquad D\underline{f} \cdot \begin{pmatrix} dx_1 \\ \vdots \\ dx_m \end{pmatrix} = d\underline{f}$$

Example 11.1.2.

$$\underline{f}(r\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \quad \frac{\partial \underline{f}}{\partial r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \frac{\partial \underline{f}}{\partial \theta} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}$$

$$d\underline{f} = \frac{\partial \underline{f}}{\partial r}\,dr + \frac{\partial \underline{f}}{\partial \theta}\,d\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} dr + \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix} d\theta$$

$$D\underline{f} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Example 11.1.3. We will consider the following cases

1. $m = 1$

$$f : \mathbb{R} \to \mathbb{R}^n$$

   Describes a path in $\mathbb{R}^n$ (Chapter 9).

2. $n = 1$

$$f : \mathbb{R}^m \to \mathbb{R}$$

   What we did in Chapter 10.

3. $m = n$ (2 or 3)

   There are two different ways to examine these functions:

   (a) Substitution: $f(r, \theta) = \binom{r\cos\theta}{r\sin\theta}$

   (b) Using *vector fields*: each point in $\mathbb{R}^n$ can be represented as an n-dimensional vector.

4. $m = 2, n = 3$

   Parameterization of a *surface* in $\mathbb{R}^3$

   Example 11.1.4.

$f : \mathbb{R}^2 \to R :$	$f : \mathbb{R}^2 \to \mathbb{R}^3 :$
$f(x, y) = x^2 + y^2$	$f(x, y) = \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix}$ $f(\rho, \theta) = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ \rho^2 \end{pmatrix}$

11.2 Surfaces in $\mathbb{R}^3$

Or in other words, the *image* of functions $f : \mathbb{R}^2 \to \mathbb{R}^3$.

Example 11.2.1 (Paraboloid).

$$f(\rho, \theta) = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ \rho^2 \end{pmatrix} \quad f_\rho = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2\rho \end{pmatrix} \quad f_\theta = \begin{pmatrix} -\rho\sin\theta \\ \rho\cos\theta \\ 0 \end{pmatrix}$$

$$D\underline{f} = \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \\ 2\rho & 0 \end{pmatrix}$$

$$d\underline{f} = f_\rho\, d\rho + f_\theta\, d\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2\rho \end{pmatrix} d\rho + \begin{pmatrix} -\rho\sin\theta \\ \rho\cos\theta \\ 0 \end{pmatrix} d\theta$$

$d\underline{f}$ is how much $\underline{f}$ changes when there is a small change in ρ or θ. We want to find the *area* of $d\underline{f}$:

$$\|hf_\rho \times kf_\theta\| = hk\|f_\rho \times f_\theta\|$$

Basically we see for small unit on the plane u, v that the area is surface is approximately the area on the plane uv times $\|f_u \times f_v\|$.

Definition. A point such that $f_u \parallel f_v$ is called a *singular point.*

11.3 Tangent plane on the surface $f(u, v)$

It's easy to see that f_u, f_v span the plane and therefore the *normal* of the plane is:

$$N = f_u \times f_v \quad \hat{N} = \frac{f_u \times f_v}{\|f_u \times f_v\|}$$

Note. There are many parameterizations for each plane, for example:

$$f(u, v) = \begin{pmatrix} u \\ v \\ u^2 + v^2 \end{pmatrix} \qquad g(u, v) = \begin{pmatrix} u\cos v \\ u\sin v \\ u^2 \end{pmatrix}$$

11.4 Change of Coordinates and Jacobian

Reminder. $f : \mathbb{R}^m \to \mathbb{R}^n$:

- Paths ($m = 1$)
- $f : \mathbb{R}^2 \to \mathbb{R}$ (all of Chapter 10)

- Surfaces ($m = 2, n = 3$)
- $m = n$
  Either changing coordinates or *vector fields*

Suppose $f(x, y)$ is differentiable and:

$$f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

To find the area of a small section from x, y, we see that we can approximate with the area of the parallelogram of the vectors u, v, which we know can calculated by the *determinant*:

$$\begin{vmatrix} hu_x & ku_y \\ hv_x & kv_y \end{vmatrix}$$

Note. This matrix is the derivative Df of f.

Definition (Jacobian)**.** This determinant is called the *Jacobian* symbolized by $\dfrac{\partial(u,v)}{\partial(x,y)}$.

We see that the area on the plane u, v is approximately the area on the plane x, y times $\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right|$.

Example 11.4.1.

$$f(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$$J = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Note. This is the same for working in three dimensions.

Example 11.4.2.

$$f(r,\theta,\phi) = \begin{pmatrix} r\cos\theta\sin\phi \\ r\sin\theta\sin\phi \\ r\cos\phi \end{pmatrix}$$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| = \left|\begin{pmatrix} | & | & | \\ f_r & f_\theta & f_\phi \\ | & | & | \end{pmatrix}\right| = r^2\sin\phi$$

11.5 Operations on Vector and Scalar Fields

Suppose we have a *scalar field* $\phi : \mathbb{R}^n \to \mathbb{R}$ and a *vector field* $f : \mathbb{R}^n \to \mathbb{R}^n$.

1. Gradient:

$$\boldsymbol{\nabla}\phi = \begin{pmatrix} \phi_x \\ \phi_y \\ \phi_z \end{pmatrix}$$

2. Divergence:

$$\operatorname{div} f = \boldsymbol{\nabla} \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \cdots + \frac{\partial f_n}{\partial x_n}$$

3. Curl:

   If f is a *vector field* in $\mathbb{R}^3$ then:

$$\operatorname{curl} f = \boldsymbol{\nabla} \times f = \begin{vmatrix} i & \frac{\partial}{\partial x} & f_x \\ j & \frac{\partial}{\partial y} & f_y \\ k & \frac{\partial}{\partial z} & f_z \end{vmatrix}$$

Example 11.5.1.

$$\phi(x, y, z) = x^2 e^{2x+3z}$$

$$\operatorname{grad} \phi = \boldsymbol{\nabla}\phi = \begin{pmatrix} 2xy + 2e^{2x+3z} \\ x^2 \\ 3e^{2x+3z} \end{pmatrix}$$

Example 11.5.2.

$$f(x, y, z) = \begin{pmatrix} x^2 y \\ yz \\ z \end{pmatrix}$$

$$\operatorname{div} f = \boldsymbol{\nabla} \cdot f = 2xy + z + 1$$

$$\operatorname{curl} f = \begin{vmatrix} i & \frac{\partial}{\partial x} & x^2 y \\ j & \frac{\partial}{\partial y} & yz \\ k & \frac{\partial}{\partial z} & z \end{vmatrix} = \begin{pmatrix} -y \\ 0 \\ -x^2 \end{pmatrix}$$

11.5.1 Characteristics of div, grad, and curl

1. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times A) = 0$

   $\operatorname{div}(\operatorname{curl} A) = 0$

11.6 Integrating a Scalar Field Along a Curve

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a *vector field* and $\underline{r} : [a, b] \to \mathbb{R}^2$ be the *curve* C.

$$\int\limits_c f(x, y) \underbrace{ds}_{\text{small section of } C} \qquad ds = \|\underline{r}'(t)\|\, dt$$

$$\int\limits_c f\, ds = \int_a^b f(\underline{r}(t))\|\underline{r}'(t)\|\, dt$$

Example 11.6.1.

$$f(x, y) = y, \quad C = \text{ the upper half of the unit circle}$$

$$r(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_{0 \le t \le \pi} \qquad r'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}_{0 \le t \le \pi} \qquad \|r'(t)\| = 1$$

$$\int\limits_c y\, ds = \int_0^\pi \sin t \cdot 1\, dt = (-\cos t)\Big|_0^\pi = 2$$

11.7 Integrating a Vector Field Along a Curve

Given a *vector field* (power field) $\underline{f}$ and *curve* C. The work done by F on C is:

$$\int \underline{f} \cdot d\underline{S}$$

Example 11.7.1.

$$f(x, y) = \begin{pmatrix} x^2 y \\ y - 2x \end{pmatrix}$$

With the *curve* C on the section from $(1, 1)$ to $(3, 7)$.

$$r(t) = \begin{pmatrix} 1 + 2t \\ 1 + 6t \end{pmatrix}$$

$$\int\limits_c \underline{f} \cdot d\underline{S} = \int_a^b f(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^1 \begin{pmatrix} (1 + 2t)^2(1 + 6t) \\ 1 + 6t - 2(t + 2t) \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 6 \end{pmatrix} dt = \int_0^1 2(t + 2t)^2(1 + 6t) + 6(1 + 6t - 2(1 + 2t))\, dt$$

Another form:

$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} \quad r(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\int\limits_c \underline{f} \cdot d\underline{s} = \int\limits_c \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} dt = \int\limits_c f_1 \underbrace{x'(t)\, dt}_{dx} + f_2 \underbrace{y'(t)\, dt}_{dy}$$

Theorem. If ϕ is a *scalar vector*, then:

$$\int_c \boldsymbol{\nabla}\phi \cdot d\underline{S} = \phi(B) - \phi(A)$$

Regardless of the path.

11.8 Examples

Example 11.8.1.

$$f(u,v) = \begin{pmatrix} uv \\ u+v \\ u^2+v^2 \end{pmatrix}$$

For the point $(u,v) = (1,2)$

1. Normal:

$$N = f_u \times f_v$$
$$N = \begin{pmatrix} v \\ 1 \\ 2u \end{pmatrix} \times \begin{pmatrix} u \\ 1 \\ 2v \end{pmatrix} = \begin{pmatrix} 2v-2u \\ 2u^2-2v^2 \\ v-u \end{pmatrix}$$

   Substituting in our point:

$$N = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 1 \end{pmatrix}$$
$$\hat{N} = \frac{1}{\sqrt{41}} \begin{pmatrix} 2 \\ -6 \\ 1 \end{pmatrix}$$

2. Tangent plane:

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$
$$2(x-2) - 6(y-3) + 1(z-5) = 0$$

3. Linear approximation:

$$df = f_u\,du + f_v\,dv = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}(u-1) + \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}(v-2)$$
$$f(u,v) \approx f_0 + df = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}(u-1) + \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}(v-2)$$

Example 11.8.2. Find the *Jacobean* of:

$$\begin{pmatrix} uv \\ \frac{u}{v} \end{pmatrix}$$

$$J = \begin{vmatrix} f_{1u} & f_{1v} \\ f_{2u} & f_{2v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -2\frac{u}{v}$$

Example 11.8.3. $\int\limits_c xy\, ds$, on the upper half a circle with the radius of 2 centered at the origin.

$$r(t) = \begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix}_{0 \le t \le \pi}$$

$$\int_0^\pi 2\cos t \cdot 2\sin t \cdot \underbrace{\left\| \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} \right\|}_{2} dt = 8 \int_0^\pi \sin t \cos t\, dt = 0$$

Example 11.8.4. $\int\limits_c x^2\, dy + y\, dx$, on the fourth quadrant on a circle of radius 3 centered at the origin, going clockwise.

$$r(t) = \begin{pmatrix} 3\sin t \\ 3\cos t \end{pmatrix}_{\frac{\pi}{2} \le t \le \pi} \left(\text{or } r(t) = \begin{pmatrix} 3\cos(-t) \\ 3\sin(-t) \end{pmatrix}_{0 \le t \le \frac{\pi}{2}} \right)$$

$$\int_{\frac{\pi}{2}}^{\pi} \big(\underbrace{(3\sin t)^2}_{x^2} \underbrace{(-3\sin t)}_{y'(t)} + \underbrace{(3\cos t)}_{y} \underbrace{(3\cos t)}_{x'(t)}\big)\, dt$$

Example 11.8.5. $\int\limits_c \begin{pmatrix} \cos z \\ z \\ y - x\sin z \end{pmatrix} d\underline{r}$ on the curve $(2\cos t, t^2, \cos(2t))$, $0 \le t \le \pi$.

$$\int_0^\pi \begin{pmatrix} \cos(\cos 2t) \\ \cos 2t \\ t^2 - 2\cos\sin(\cos 2t) \end{pmatrix} \begin{pmatrix} -2\sin t \\ 2t \\ -2\sin 2t \end{pmatrix} dt$$

This integral is difficult to do, so instead we will look for a *potential* function $\phi(x, y, z)$ such that:

$$\phi_x = \cos z \implies \phi = \int \cos z\, dx = x\cos z + c(y, z)$$

$$\phi_y = z \implies \phi = \int z\, dy = zx + c(x, z)$$

$$\phi_z = y - x\sin z \implies \boxed{\phi = \int (y - x\sin z)\, dz = yz + x\cos z + c(x, y)}$$

$$\int_c \nabla \underbrace{(yz + x\cos z)}_{\phi}\, dr = \phi(B) - \phi(A) = \phi(-2, \pi^2, 1) - \phi(2, 0, 1)$$

11.9 Conservative Vector Field

Definition (Conservative Vector Field). A *conservative vector field* is a field F such that for all closed curves C on the *domain* D:

$$\oint_C f\, d\underline{r} = 0$$

$\iff$ for an open curve C, $\int_c f\, dr$ is only dependent the initial and final points of C.

$\iff$ f is a *gradient* of a scalar function ϕ.

Example 11.9.1.

$$\underline{f} = -\frac{r}{r^3} \quad r(x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad r = |\underline{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\underline{f} = -\frac{1}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We can see if $\underline{\phi} = \frac{1}{r}$, then $\nabla\phi = f$.

Reminder. If ϕ is a *scalar function*, then $\operatorname{curl}(\nabla\phi) = \underline{0}$. $(\nabla \times \nabla\phi)$

Conclusion. If f is a *conservative field*, then $\operatorname{curl} \underline{f} = 0$.

However, if $\operatorname{curl} \underline{f} = 0$, we can't necessarily be sure f is a *conservative field.*

Example 11.9.2.

$$\underline{f}(x, y, z) = \left(\frac{y}{x^2 + y^2}, -\frac{x}{x^2 + y^2}, 0\right)$$

We see that $\operatorname{curl} f = \underline{0}$.

Let C be the unit circle:

$$r(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}_{0 \le t \le 2\pi} \qquad r'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}_{0 \le t \le 2\pi}$$

$$\oint_C f \cdot dr = \int_0^{2\pi} \begin{pmatrix} \sin t \\ -\cos t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt = -1 \int_0^{2\pi} dt = -2\pi \neq 0$$

Which means despite the fact that $\operatorname{curl} f = \underline{0}$, f is not a *conservative field.*

If $\operatorname{curl} \underline{f} = \underline{0}$ on a *simply connected* domain D, then $\underline{f}$ is *conservative.*

Definition (Simply Connected Space)**.** If we imagine our curve as a physical string, if it is able to be pulled through without getting "stuck" on a "hole" in the domain, then the domain is *simply connected.*

MATHEMATICAL METHODS I

12

INTEGRATION OF MULTIVARIABLE FUNCTIONS

MOSHE KRUMBEIN - FALL 2021

12.1 Double Integrals

Given the function $z = f(x, y)$ and the domain D on the plane. We want to find:

$$\iint\limits_D f(x, y) = \iint f(x, y)\, dy\, dx$$

Which is essentially the volume of the domain between D and f.

Definition (Fubini's theorem). If $D = [a, b] \times [c, d]$:

$$\iint\limits_D f\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Example 12.1.1.

$$f(x, y) = x^2 + 3y^2 + xy \quad D = [1, 2] \times [0, 1]$$

$$\iint\limits_D f\, dA = \int_0^1 \left(\int_1^2 (x^2 + 3y^2 + xy)\, dx \right) dy = \int_0^1 \left(\frac{x^3}{3} + 3xy^2 + \frac{x^2}{2} y \right)\Bigg|_{x=1}^{2} dy$$

$$\int_0^1 \left(\frac{7}{3} + 3y^2 + \frac{3}{2} y \right) dy = \frac{49}{12}$$

Note. If $f(x, y) = g(x)h(y)$ and $D = [a, b] \times [c, d]$:

$$\iint\limits_D f(x, y) = \int_a^b \left(\int_c^d g(x)h(y)\, dy \right) dx \int_a^b g(x) \int_c^b h(y)\, dy\, dx = \int_a^b g(x)\, dx \cdot \int_c^d h(y)\, dy$$

12.2 Double Integrals of Non-rectangular Domains

Simple Vertical Domain:

$$a \leq x \leq b \quad g(x) \leq y \leq h(x)$$
$$\iint\limits_D f = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

Simple Vertical Domain:

$$c \leq y \leq d \quad g(y) \leq x \leq h(y)$$
$$\iint\limits_D f = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$

12.3 Uses of Double Integrals

12.3.1 Volume of Solids of Revolution

In essence, to find the area of a solid of revolution, we consider the volume to be the sum of thin disks with the radius on $f(x)$:

$$\lim \sum_{k=1}^{\infty} \underbrace{\pi(f(x))^2}_{\text{area of the circle}} \Delta x_k = \pi \int_a^b (f(x))^2\,dx$$

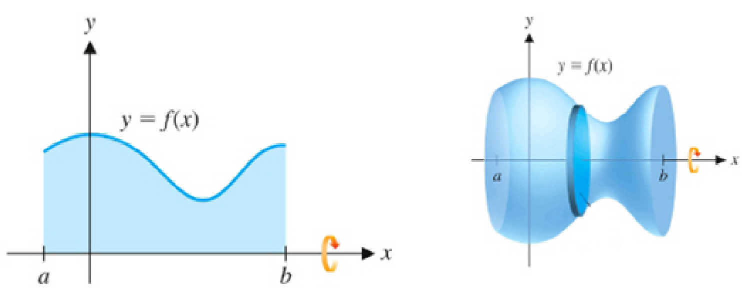


Figure 12.1: Volume of Solids of Revolution

12.3.2 Center of Gravity

$$\text{Mass: } \iint\limits_D f(x,y)\,dx\,dy$$

$$\text{Center of Mass:}$$

$$\overline{x} = \frac{\iint x \cdot f(x,y)\,dx\,dy}{\text{mass}}$$

$$\overline{y} = \frac{\iint y \cdot f(x,y)\,dx\,dy}{\text{mass}}$$

12.3.3 Useful Formulas

$$\int_a^b 1\,dx = \underbrace{b-a}_{\text{length}}$$

$$\iint\limits_D 1\,dx\,dy = \text{ Area of } D$$

Example 12.3.1.

$$\int_{-6}^{6}\int_{0}^{\sqrt{26-x^2}} 1\,dx\,dy$$

This may be solved in the conventional way, though it'll take some time. However we can just consider the area of the domain which is the upper half of a circle with a radius of 3, giving us:

$$\frac{1}{2}\pi 6^2 = 18\pi$$

12.4 Triple Integrals

$$D \subseteq \mathbb{R}^3 \qquad \iiint\limits_D f(x,y,z)\,dv$$

What does it represent?

If $f(x,y,z)$ is the density, then the integral is the mass.

If $f(x,y,z) \equiv 1$, then the integral is the volume of the domain.

Example 12.4.1.

$$f(x,y,z) = x + yz \quad D = [1,3] \times [0,1] \times [-2,6]$$
$$\int_1^3 \int_{-2}^6 \int_0^1 (x+yz)\,dy\,dz\,dx = \int_1^3 \int_{-2}^6 \underbrace{\left[xy + \frac{y^2 a}{2}\right]\Big|_{y=0}^1}_{x+\frac{z}{2}} dz\,dx$$
$$\vdots$$

Example 12.4.2.

$$\iiint\limits_D xyz\,dv \quad D: x=0, y=0, z=0, x+2y+z=1$$
$$0 \le z \le 2-x-2y \quad 0 \le y \le \frac{1}{2} - \frac{1}{2}x \quad 0 \le x \le 1$$
$$\int_0^1 \int_0^{\frac{1}{2}-\frac{1}{2}x} \int_0^{2-x-2y} xyz\,dz\,dy\,dx$$
$$\vdots$$

12.5 Change of Coordinates in Integrals

Similar to finding the area of a function after changing coordinates using the *Jacobean*, when we change coordinates on a double (or triple) integral:

$$dx\,dy = |J|\,du\,dv$$

Example 12.5.1.

$$f(x,y) = x+1 \quad D: y < x, x < \sqrt{1-y^2}$$
$$\iint\limits_D (x+1)\,dx\,dy$$

Let's convert this to the polar coordinate system:

$$0 \le r \le 1 \quad 0 \le \theta \le \frac{\pi}{4}$$
$$\int_0^1 \int_0^{\frac{\pi}{4}} (1 + r\cos\theta) r\,dr\,d\theta$$

MATHEMATICAL METHODS I

13

INTEGRATION ON SURFACES

Moshe Krumbein - Fall 2021

13.1 Introduction

Reminder. A *surface* on $\mathbb{R}^3$ is the *image* of a function:

$$\underline{r} : D \to \mathbb{R}^3 \qquad r(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

The symbolize the surface as Σ ($\Sigma = \operatorname{Im} \underline{r}$).

Example 13.1.1. A sphere with a radius of 3:

$$\underline{r}(u,v) = \begin{pmatrix} 3\sin v \cos u \\ 3 \sin v \sin u \\ 3\cos v \end{pmatrix} \qquad D : 0 \le u \le 2\pi, 0 \le v \le \pi$$

A small rectangle whose area is $\delta_u \cdot \delta_v$ becomes similar to a parallelogram on Σ:

$$\underbrace{\|\underline{r}_u \times \underline{r}_v\|}_{\|\underline{N}\|} \delta_u \delta_v$$

Symbol.

$$\begin{aligned} dS &= \overbrace{\|\underline{r}_u \times \underline{r}_v\|}^{\text{scalar}} \, du \, dv \\ d\underline{S} &= \overbrace{(\underline{r}_u \times \underline{r}_v)}^{\text{vector}} \, du \, dv \\ \underline{N} &= \underline{r}_u \times \underline{r}_v \\ \underline{\hat{N}} &= \frac{\underline{r}_u \times \underline{r}_v}{\|\underline{r}_u \times \underline{r}_v\|} \end{aligned}$$

Example 13.1.2 (Half-Torus).

$$r(u,v) = \begin{pmatrix} (2+\cos u)\cos v \\ (2+\cos u)\sin v \\ \sin u \end{pmatrix} \quad D: 0 \le u \le \pi, 0 \le v \le 2\pi$$

$$\underline{r}_u = \begin{pmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{pmatrix} \quad \underline{r}_v = \begin{pmatrix} -(2+\cos u)\sin v \\ (2+\cos u)\cos v \\ 0 \end{pmatrix}$$

$$\underline{N} = \underline{r}_u \times \underline{r}_v = (2+\cos u)\cdot \underline{r}_u \times \begin{pmatrix} -\sin v \\ \cos v \\ 0 \end{pmatrix} = (2+\cos u)\begin{pmatrix} -\cos u\cos v \\ -\cos u \sin v \\ -\sin u \end{pmatrix}$$

$$\begin{aligned} \|\underline{N}\| = \|\underline{r}_u \times \underline{r}_v\| &= (2+\cos u)\sqrt{(-\cos u\cos v)^2 + (-\cos u \sin v)^2 + (-\sin u)^2} \\ &= 2+\cos u \end{aligned}$$

$$\hat{\underline{N}} = \frac{\underline{N}}{\|\underline{N}\|} = -\begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}$$

$$dS = \|\underline{r}_u \times \underline{r}_v\|\, du\, dv = (2+\cos u)\, du\, dv$$

$$d\underline{S} = (\underline{r}_u \times \underline{r}_v)\, du\, dv = -(2+\cos u)\begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix} du\, dv$$

13.2 Integration on Surfaces

Reminder. We learned how to do integration on a curve:

$$\underline{r}: \mathbb{R} \to \mathbb{R}^3$$

and integration on a *scalar field* ϕ:

$$\int\limits_C \phi\, ds = \int_a^b \phi(r(t))\|r'(t)\|\, dt$$

(which represents either *mass* or *area*)

and integration on a *vector field* $\underline{f}$:

$$\int\limits_C f \cdot dS = \int_a^b f(\underline{r}(t)) \cdot r'(t)\, dt$$

(which represents *work*)

Integration on a *scalar field* ϕ:

$$\iint\limits_{\Sigma} \phi \, dS = \iint\limits_{D} \phi(r(u,v)) \| \underline{r}_u \times \underline{r}_v \| \, du \, dv$$

(which represents *mass*)

Note. If the *density* $\phi = 1$, then the integral represents the *surface area.*

Integration on a *vector field* $\underline{f}$:

$$\iint\limits_{\Sigma} f \cdot d\underline{S} = \iint\limits_{D} f(\underline{r}(u,v)) \cdot (\underline{r}_u \times \underline{r}_v) \, du \, dv$$

(which represents *flux*)

Example 13.2.1. Find the *surface area* of our half-torus:

We will take $\phi = 1$ and calculate:

$$\iint\limits_{\Sigma} 1 \cdot dS = \int_0^{2\pi} \int_0^{\pi} (2 + \cos u) \, du \, dv = 2\pi [2u + \sin u] \Big|_0^{\pi} = 4\pi^2$$

Example 13.2.2. Integrate:

$$\iint\limits_{\Sigma} \underline{r} \cdot d\underline{S} = \int_0^{2\pi} \int_0^{\pi} \underbrace{\begin{pmatrix} (2+\cos u)\cos v \\ (2+\cos u)\sin v \\ \sin u \end{pmatrix}}_{\underline{r}} \cdot \underbrace{-(2+\cos u) \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix}}_{d\underline{S}} du \, dv$$

$$= \int_0^{2\pi} \int_0^{\pi} (2+\cos u)((2+\cos u) \cdot (-\cos u \cos^2 v - \cos u \sin^2 v) - \sin^2 u) \, du \, dv$$

$$= \int_0^{2\pi} \int_0^{\pi} (2+\cos u)((2+\cos u)(-\cos u) - \sin^2 u) \, du \, dv$$

$$= -\int_0^{2\pi} \int_0^{\pi} (2+\cos u)(2\cos u + 1) \, du \, dv = -2\pi \int_0^{\pi} (2\cos^2 u + 5\cos u + 2) \, du$$

$$= -6\pi^2$$

Note. Here, the *normal* is facing “inwards,” into the half-torus, and which mean the volume of liquid that *enters* the half-torus is $6\pi^2$.

Note. If the *vector field* $\underline{f}$ is a *constant unit vector*, then:

$$\iint\limits_{\Sigma} \underline{f} \cdot d\underline{S}$$

represents the area of the *projection* of Σ on the plane perpendicular to $\underline{f}$.

Example 13.2.3. Calculate the *projection* of a half-torus onto the xy plane:

$$\iint_{\Sigma} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot d\underline{S} = \int_0^{2\pi} \int_0^{\pi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot -(2+\cos u) \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{pmatrix} du\, dv$$

$$= \int_0^{2\pi} \int_0^{\pi} (2+\cos u)(-\sin u)\, du\, dv = 2\pi \int_0^{\pi} (-2\sin u - \sin u \cos u)\, du = |-8\pi|$$

$$= 8\pi$$