Mathematical Methods II (80157)

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https://github.com/outofink/notes

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MATHEMATICAL METHODS II

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VECTOR ANALYSIS

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1.1 Review

$$\int_a^b f' = [f]_a^b$$

Given an n^{th} dimensional function ($n = 2, 3$):

- Scalar Field:

$$f : \underbrace{D}_{\subset \mathbb{R}^n} \to \mathbb{R}$$

- Vector Field:

$$\underline{f} : D \to \mathbb{R}^n$$

$$(D_{\underline{x}}\underline{f})(d\underline{x}) = (d\underline{f}(\underline{x}))$$

$$f : \mathbb{R}^2 \to \mathbb{R} \quad f(x, y)$$
$$\underline{D} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$
$$\underline{\boldsymbol{\nabla}} D = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

$$f : \mathbb{R}^3 \to \mathbb{R} \quad f(x, y, z)$$
$$(\underline{D}f)^T = \underline{\boldsymbol{\nabla}} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

1.1.1 Three Dimensional Vector Field

$$\underline{f} : \mathbb{R}^3 \to \mathbb{R}^3$$
$$\underline{D}f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial y} \\ \frac{\partial f_1}{\partial z} & \frac{\partial f_2}{\partial z} & \frac{\partial f_3}{\partial z} \end{pmatrix}$$
$$\operatorname{div} \underline{f} = \boldsymbol{\nabla} \cdot \underline{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$
$$\operatorname{curl} \underline{f} = \boldsymbol{\nabla} \times \underline{f} = \begin{pmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix}$$

$$\int_a^b f(x)\,dx$$
$$\sum_{r=1}^{N} f(x_r)(x_r - x_{r-1})$$

$$\iint_D f\,dA$$
$$\sum_{\text{parts}} \underline{f} \cdot dr$$
$$\int_a^b \underline{f}(\underline{r}(t)) \cdot \underline{r}(t)\,dt$$

$$\iint_\Sigma = \iint_D \underline{f}(x(u,v), y(u,v), z(u,v)) \cdot \left(\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}\right) du\,dv$$

$$\iiint_R f\,dV$$
$$dS = \|\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}\|\,du\,dv$$
$$\underline{\hat{N}} = \frac{\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}}{\|\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}\|}$$

Example 1.1.1.

$$\underline{f}(x,y) = \begin{pmatrix} y \\ x \end{pmatrix} \quad \underline{r}(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix} \quad 0 \le t \le 1$$

$$\int_c \underline{f} \cdot d\underline{r} = \int_0^1 \begin{pmatrix} t^3 \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} dt = \int_0^1 5t^3\, dt = \left[t^3\right]_0^1 = 1$$

$$\int_c y\, dx + x\, dy = \int_c f_1\, dx + f_2\, dy = \int_c d(xy) = [xy]_A^B = 1$$

1.2 Meaning of grad, div, curl

1.2.1 grad

$$\boldsymbol{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

1.2.2 curl

Essentially the "angular velocity".

$$\underline{v} = \underline{\omega} \times \underline{r}$$

1.3 Integral Theorems

1.3.1 One Dimensional

$$[\phi]_a^b = \int_a^b \phi'$$

1.3.2 Two Dimensional

$$[\phi]_A^B = \int_c \boldsymbol{\nabla}\phi \cdot dr$$

Green's Theorem

$$\oint_{\partial D} dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

1.3.3 Three Dimensional

$$[\phi]_A^B = \int_c \boldsymbol{\nabla}\phi \cdot dr$$

Stokes' Theorem

$$\oint_{\partial\Sigma} \underline{f}\, ds = \iint_{\Sigma} (\boldsymbol{\nabla} \times \underline{f})\, dS$$

Gauss's Theorem

$$\oiint_{\partial R} \underline{f}\, d\underline{S} = \iiint_{R} (\boldsymbol{\nabla} \cdot \underline{f})\, dV$$

1.3.4 First Type

One Dimensional

$$[f]_a^b = \int_a^b f'$$

n-Dimensional

$$[\phi]_A^B = \int_c \boldsymbol{\nabla}\phi \cdot dr$$

Suppose $\underline{r} = \underline{r}(t)$:

$$\int_c \boldsymbol{\nabla}\phi \cdot dr = \int_a^b \underbrace{(\boldsymbol{\nabla}\phi)(\underline{r}(t))}_{\boldsymbol{\nabla} phi} \cdot \underbrace{\dot{\underline{r}}(t)\, dt}_{d\underline{r}}$$

$$= \int_a^b \frac{d}{dt}(\phi(\underline{r}(t)))\, dt = [\phi(\underline{r}(t))]_a^b = \phi((\underline{r}(b)) - \phi(\underline{r}(a)))$$

$$= \phi(B) - \phi(A) = [\phi]_A^B$$

1.3.5 Second Type

Green's Theorem

Where the edge is ∂D and it is a closed path and with a anticlockwise orientation:

$$\boxed{\oint_{\partial D} P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Proof. It's enough to prove:

$$\oint_{\partial D} P\, dx \iint_D -\frac{\partial P}{\partial y}\, dA$$

$$\oint_{\partial D} Q\, dy = \iint_D \frac{\partial Q}{\partial x}\, dA$$

When D is a *simple region*:

$$D = \{a \leq x \leq b, c(x) \leq y \leq d(x)\}$$

$$\gamma_1 : \underline{r}(t) = \begin{pmatrix} t \\ c(t) \end{pmatrix}_{a \leq t \leq b}$$
$$\gamma_2 : \underline{r}(t) = \begin{pmatrix} t \\ d(t) \end{pmatrix}_{a \leq t \leq b}$$
$$\partial D = \gamma_1 \cup (-\gamma_2)$$

$$\oint_{\partial D} P\,dx = \int_{\gamma_1} P \underbrace{dx}_{\binom{P}{0} d\underline{r}} - \int_{\gamma_2} P\,dx$$
$$= \int_a^b P(t, c(t))\,dt - \int_a^b P(t, d(t))\,dt$$
$$\vdots$$

Where D is not a *simple region* it can be split into *simple regions*. ■

Essentially, we see that Stokes', Green's and Gauss's theorems are all just implementations of the first fundamental theorem of calculus.

$$\oint_{\partial D} \begin{pmatrix} P \\ Q \end{pmatrix} \cdot d\underline{r}$$

is the *circulation* of $\binom{P}{Q}$ on ∂D.

$$\frac{\oint_{\partial D} \binom{P}{Q} \cdot d\underline{r}}{\text{area over } D} = \frac{\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA}{\text{area over } D} = \text{average of } \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \text{ over } D$$

In other words, this is the 2D rotor (curl) over D.

$$\frac{1}{2} \oint_{\partial D} x\,dy - y\,dx = \text{area of } D$$

1.4 Scalar and Vector Potential

Suppose that $\underline{f}$ is an n-th dimensional vector field ($n = 2, 3$).

$$\underline{f} : D \to \mathbb{R}^n$$

Scalar potential is a function V on D, such that:

$$\boldsymbol{\nabla} V = -\underline{f}$$

If $\underline{f}$ is a *conservative* field ($\iff$):

- $\int_\gamma \underline{f} \cdot d\underline{r}$ is not dependent on γ
- $\oint_\gamma \underline{f} \cdot d\underline{r} = 0$

$$\boldsymbol{\nabla} \times \underline{f} = \underline{0} \Leftarrow \underline{f} \text{ is a } \textit{conservative field } (n = 3) \iff \oint_\gamma \underline{f} \cdot d\underline{r} = 0$$

Note. If we know that D is *simply connected* then we know that:

$$\boldsymbol{\nabla} \times \underline{f} = \underline{0} \iff \underline{f} \text{ is a } \textit{conservative field}$$

1.5 Finding Scalar Potential from a Vector Field $\underline{\mathbf{f}}$

$$\int_\gamma -\boldsymbol{\nabla} V \cdot d\underline{r} = V(A) - V(B)$$

$$\underline{f} = V(\underline{r}_0) - V(\underline{r})$$

$$\implies \boxed{V(\underline{r}) = \underbrace{V(\underline{r}_0)}_{\text{constant}} - \int_\gamma \underline{f} \cdot d\underline{r}}$$

Note. If V is a scalar potential of $\underline{f}$, then $V + c$ is also a scalar potential of $\underline{f}$.

If $\underline{f}$ is a vector field, $D = \mathbb{R}^3$, $\boldsymbol{\nabla} \times \underline{f} = \underline{0} \implies$ there exists a scalar potential for $\underline{f}$:

$$V = -\int_0^x f_1(t, 0, 0)\, dt - \int_0^y f_2(x, t, 0)\, dt - \int_0^z f_3(x, y, t)\, dt$$

Note. Finding a scalar potential for f is equivalent to writing the differential as an *exact differential.*

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \qquad f_1\, dx + f_2\, dy + f_3\, dz$$

$$dV = \frac{\partial V}{\partial x}\, dx + \frac{\partial V}{\partial y}\, dy + \frac{\partial V}{\partial z}\, dz$$