Electricity and Magnetism (83313)

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https://github.com/outofink/notes

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ELECTRICITY AND MAGNETISM

1

INTRODUCTION

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Definition (Operator)**.** An *operator* is an mathematical object which takes a *function* and returns a *function*.

For example, the *derivative*: $\frac{df}{dx}$.

Note. This is not a fraction!

The most common *operator* is electromagnatism is the *nabla* (∇):

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

1.1 Gradient

$$\vec{f}(x, y, z) = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k} = \mathrm{grad}(\phi)$$

What can we use the *gradient* for?

1.1.1 Directional derivative

The *derivative* of the function in the certain direction.

$$\frac{\partial\phi}{\partial\hat{n}} = \nabla\phi \cdot \hat{n}$$

Where $\hat{n}$ is the unit vector in the desired direction.

Conservative Fields

$$\phi(\vec{r}_2) - \phi(\vec{r}_1) = \int_\gamma \nabla\phi \cdot d\vec{r}$$

1.2 Divergence

$$\vec{f}(x,y,z) = f_x(x,y,z)\hat{\imath} + f_y(x,y,z)\hat{\jmath} + f_z(x,y,z)\hat{k}$$
$$\operatorname{div}(\vec{f}) = \boldsymbol{\nabla} \cdot \vec{f} = \frac{df_x}{dx} + \frac{df_y}{dy} + \frac{df_z}{dz}$$

It's much easier to work in cases that are symmetrical (cylinders and infinite planes).

$$\vec{f}(x,y,z) = xy\hat{x} + yz\hat{y} + xz\hat{z} \quad \boldsymbol{\nabla} \cdot \vec{f} = y + z + x$$

Integration on Surfaces

Suppose we have a vector field $\vec{f}(x,y,z)$ and a surface defined by $s(x,y,z) = c$ (a sphere).

$$\iint_S \vec{f}(x,y,z)\, d\vec{s} \quad d\vec{s} = ds \cdot \hat{n}$$

Note. We define the direction to be *outward*.

$$ds = R^2 \sin(\theta)\, d\theta\, d\phi\, \hat{r} \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi$$
$$\vec{f}(\vec{r}) = \vec{r}$$
$$\int \vec{f}(\vec{r})\, d\vec{r} = \oiint \hat{r} R^2 \sin(\theta)\, d\phi\, d\theta \cdot \hat{r} = R^2 \oiint \sin(\theta)\, d\phi\, d\theta = 4\pi R^2$$

1.2.1 Gauss's Law

$$\oiint_{S(V)} \vec{f} \cdot d\vec{s} = \iiint_V (\boldsymbol{\nabla} \cdot \vec{f}) \cdot dV$$

Where $S(V)$ is the area which encloses the V.

1.3 Curl

$$f(x,y,z) = f_x\hat{x} + f_y\hat{y} + f_z\hat{z}$$
$$\operatorname{curl}(f) = \boldsymbol{\nabla} \times f = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\hat{x} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\hat{y} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\hat{z}$$

1.3.1 Stokes' Law

$$\oint_{l(A)} \vec{f} \cdot d\vec{l} = \int_A (\boldsymbol{\nabla} \times \vec{f})\, d\vec{s} \quad d\vec{s} = ds \cdot \hat{n}$$

Note. We define the direction using the *right hand rule*.

1.4 What is electricity?

$$e = 1.602 \cdot 10^{-19} \text{ C}$$

There exists *conservation of charge.*

1.4.1 Coulomb's Law

Definition (Coulomb's Law)**.** *Coulomb's Law* defines the vector (magnitude and direction) of the force between two charges in space given charges q_1, q_2 at distances r_1, r_2 respectively.

$$\vec{r}_{1,2} = \vec{r}_1 - \vec{r}_2 \quad \hat{r}_{1,2} = \frac{\vec{r}_1 - \vec{r}_2}{\|\vec{r}_1 - \vec{r}_2\|}$$
$$\vec{F}_{1,2} = \frac{kq_1q_2}{\|\vec{r}_{1,2}\|^2}\hat{r}_{1,2}$$

Where $k = 8.99 \cdot 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}$. Sometimes we define k in terms of ε_0 (dielectric constant in a vacuum):

$$k = \frac{1}{4\pi\varepsilon_0} \quad \varepsilon_0 = 8.8542 \cdot 10^{12} \frac{\text{C}^2}{\text{N} \cdot \text{m}^2}$$

More simply, we say:

$$\vec{F} = \frac{kq_1q_2}{r^2}\hat{r}$$

1.5 Superposition Principle

Definition (Superposition Principle)**.** Given a system of charges, the force on a single particle is the *vector sum* of all of the forces in the system on that particle.

In order to more simple use the superposition principle, we can use:

$$\frac{\hat{r}_{1,2}}{\|\vec{r}_{1,2}\|^2} = \frac{\vec{r}_{1,2}}{\|\vec{r}_{1,2}\|^3}$$

$$\vec{F}_i = \sum_{i \neq j} \frac{kq_iq_j}{\|\vec{r}_i - \vec{r}_j\|}(\vec{r}_i - \vec{r}_j)$$

1.5.1 Work and Energy

$$\vec{F}_{2,1} = k\frac{q_1q_2}{|r_2' - r_1|^2}\hat{r}_{2,1}'$$
$$W = -\int_{\infty}^{r_2} \vec{F}_{2,1} \cdot dr_2'$$

Note. The work being done is *conservative*, which means it is not dependent on the path taken.

Doing a change a variables:

$$\begin{gathered}\overline{r}' = \overline{r}_2' - \overline{r}_1 \\ |\overline{r}_2' - \overline{r}_1| = |r'| \\ dr' = dr_2'\end{gathered}$$

Back to finding the work:

$$W = -\int_{\infty}^{\overline{r}_2 - \overline{r}_1} k\frac{q_1 q_2}{|r'|^2}\,dr' = k\frac{q_1 q_2}{|r'|}\bigg|_{\infty}^{\overline{r}_2 - \overline{r}_1} = \frac{kq_1 q_2}{|\overline{r}_2 - \overline{r}_1|}$$

The about of work required to bring q_2 from infinity to distance r from r_1 is equal to the amount of work q_1 needs to push q_2 from r to infinity.

If we wanted to bring in a third particle q_3:

$$\begin{gathered}U_{1,2,3} = \frac{kq_1 q_2}{|\overline{r}_2 - \overline{r}_1|} + \frac{kq_1 q_3}{|\overline{r}_3 - \overline{r}_1|} + \frac{kq_2 q_3}{|\overline{r}_3 - \overline{r}_2|} \\ U = \sum_i \sum_{j<i} \frac{kq_i q_j}{|\overline{r}_i - \overline{r}_j|} = \frac{1}{2}\sum_i \sum_{i \neq j} \frac{kq_i q_j}{|\overline{r}_i - \overline{r}_j|}\end{gathered}$$

1.6 Electric Fields

Given a fixed charge q_0 and a free charge q_1:

$$\begin{gathered}\overline{F}_{0,1} = k\frac{q_0 q_1}{|\vec{r}_0 - \vec{r}_1|^2}\hat{r}_{0,1} \\ \vec{E}(\vec{r}_0) = \frac{\vec{F}(\vec{r}_0)}{q_0} = k\frac{q_1}{|\vec{r}_0 - \vec{r}_1|^2}\hat{r}_{0,1}\left[\frac{\mathrm{N}}{\mathrm{C}}\right]\end{gathered}$$

Where $\vec{E}$ represents the electric field and $\vec{r}_0$ is the location at which we are measuring the field.

For an arbitrary number of points in the field:

$$\vec{E}(\vec{r}_0) = \sum_i \frac{kq_i}{|\vec{r}_0 - \vec{r}_i|^2}\hat{r}_{0,i}$$

And assuming there's one large charge Q:

$$\vec{E}(\vec{r}) = k\frac{Q}{r^2}\hat{r}$$

1.6.1 Characteristics

- $\oint \vec{E} \cdot d\vec{\ell} = 0$
- $\nabla \times \vec{E} = 0$

1.7 Flux

Definition (Flux).

$$\phi = EA$$

Where ϕ is *flux*, E is the electric field, and A is the area.

If the field and the area are perpendicular, then we can say:

$$\phi = \vec{E} \cdot \vec{A} = |E||A| \cos \theta$$
$$\phi = \int \vec{E} \cdot d\vec{s}$$

1.8 Charge Densities

Definition (Volume Charge Density).

$$\rho \left[\frac{\mathrm{C}}{\mathrm{m}^3} \right] \implies q = \int_V \rho \, dV$$

Definition (Surface Charge Density).

$$\sigma \left[\frac{\mathrm{C}}{\mathrm{m}^2} \right] \implies q = \int_S \sigma \, dS$$

Definition (Linear Charge Density).

$$\lambda \left[\frac{\mathrm{C}}{\mathrm{m}} \right] \implies q = \int_\ell \lambda \, d\ell$$

1.9 Electric Field

Per *Coulomb's Law*, we can build a "force field" by a single charge q_1:

$$\vec{F}(\vec{r}_0) = \frac{kq_0q_1}{|\vec{r}_0 - \vec{r}_1|^2} \hat{r}_{0,1}$$

Where q_0 is a *test charge* at any point in space. From this, we can build an *electric field*:

$$\vec{E}(\vec{r}_0) = \frac{\vec{F}(\vec{r}_0)}{q_0} = \frac{kq_1}{|\vec{r}_0 - \vec{r}_1|^2}\hat{r}_{0,1}$$

Or in general:

$$\vec{E}(\vec{r}_0) = \frac{\vec{F}(\vec{r}_0)}{q_0} = \sum_i \frac{kq_i}{|\vec{r}_0 - \vec{r}_i|^2}\hat{r}_{0,i}$$

Example 1.9.1. Suppose we have a charged two-dimensional disk on the xy-plane with radius r and *charge density* σ (constant). What in the *electric field* on the z-axis?

$$\vec{E}(z) = \int \frac{k\sigma\, ds'}{|\vec{r} - \vec{r'}|^2}\hat{r}_{0,'} \quad \hat{r}_{0,'} = \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|} \implies$$

$$\vec{E}(z) = \int k\frac{\sigma(\vec{r} - \vec{r'})\, ds'}{|\vec{r} - \vec{r'}|^3}$$

$$\vec{r}(0,0,z) \quad \vec{r'}(r'\cos(\phi'), r'\sin(\phi'), 0)$$

$$0 \leq r' \leq R \quad 0 > \phi' \leq 2\pi$$

$$\vec{r} - \vec{r'} = (-r'\cos(\phi'), -r'\sin(\phi'), z)$$

$$|\vec{r} - \vec{r'}|^3 = (r'^2 + z^2)^{\frac{3}{2}}$$

$$ds' = r'\, dr'\, d\phi'$$

$$\vec{E}(z) = \int_0^{2\pi}\int_0^R \frac{k\sigma}{(r'^2 + z^2)^{\frac{3}{2}}} \cdot (-r'\cos\phi', -r'\sin\phi', z) \cdot r'\, dr'\, d\phi'$$

$$= 2\pi k\sigma z \int_0^R \frac{r'\, dr'}{(r'^2 + z^2)^{\frac{3}{2}}}$$

$$\vec{E}(z) = 2\pi k\sigma z\hat{z} \cdot \left.\frac{-1}{\sqrt{r'^2 + z^2}}\right|_0^R = \left(2\pi k\sigma - \frac{2\pi k\sigma z}{\sqrt{R^2 + z^2}}\right)\hat{z}$$

$$\boxed{\vec{E}(z \to 0) = 2\pi k\sigma\hat{z}}$$

Which is the *electric field* for an infinite plane.

$$z \gg R$$

$$\vec{E}(z) \approx 2\pi k\sigma\left(1\left(-\frac{R^2}{2z^2}\right)\right)\hat{z} = \frac{\pi R^2 k\sigma}{z^2}\hat{z} = \frac{kq}{z^2}\hat{z}$$

Which is reminiscent of *Coulomb's law*, which makes sense intuitively, for something with a finite radius from very far away appears and behaves point-like.

1.10 Gauss's Law

Where Φ is the *electric flux* through a closed surface S:

$$\Phi = \oint \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0} = 4\pi k \cdot q$$

This expresses that in order to find the *flux* on a closed surface, we only need to know how much charge in within the surface, regardless of the shape of the surface and the charges outside the surface.

$$\boxed{\oint \vec{E}\, d\vec{S} = \frac{q}{\varepsilon_0}} \quad \text{(Integral Form)}$$

Gauss's law is useful because it allows us to choose whatever surface we would like, namely a sphere, and to take advantage of its symmetries, in order to simplify our calculations.

1.10.1 Charged sphere without thickness of radius a

$$\vec{E}(r) = \begin{cases} 0 & r < a \\ \frac{kq}{r^2}\hat{r} & a > a \quad q = 4\pi a^2 \sigma \end{cases}$$

1.10.2 Differential form of Gauss's Law

$$\oiint \vec{E}\, d\vec{S} = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \iiint \rho \, dV$$

We can used *Gauss's theorem* (*divergence theorem*), which states:

$$\oiint_{S(\sigma)} \vec{E}\, ds = \iiint_{\sigma} \boldsymbol{\nabla} \cdot \vec{E}\, dV$$

which provides:

$$\int \boldsymbol{\nabla} \cdot \vec{E}\, dV = \int \frac{\rho}{\varepsilon_0}\, dV$$
$$\implies \boxed{\boldsymbol{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$$