Complex Variables and Applications (80314)

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https://github.com/outofink/notes

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COMPLEX VARIABLES AND APPLICATIONS

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INTRODUCTION

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1.1 Introduction

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\} \quad i \equiv \sqrt{-1}$$
$$\arg(z) = \{\theta + 2\pi k \mid k \in \mathbb{Z}\} \quad \operatorname{Arg}(z) = \theta, \quad -\pi < \theta \leq \pi$$

$$\operatorname{Arg}(z) = \begin{cases} \arctan \frac{y}{x} & x > 0 \\ \frac{\pi}{2} & x = 0, y > 0 \\ -\frac{\pi}{2} & x = 0, y < 0 \\ \pi + \arctan \frac{y}{x} & x < 0, y > 0 \\ -\pi + \arctan \frac{y}{x} & x < 0, y > 0 \end{cases}$$

1.2 Operations

$$(a + bi) + (c + di) = (a + d) + (b + d)i \tag{1.1}$$
$$(a + bi) - (c + di) = (a - d) + (b - d)i \tag{1.2}$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \tag{1.3}$$
$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{e + fi}{c^2 + d^2} = \frac{e}{c^2 + d^2} + \frac{f}{c^2 + d^2}i \tag{1.4}$$

1.3 Complex Plane

Every complex number $z = a + bi$ can be represented on the complex plane at the point (a, b).

It can also be represented in the polar form:

$$r = |z| \quad \theta = \arg(z)$$
$$z = r\cos\theta + r\sin\theta i \quad (\operatorname{cis}\theta)$$

1.3.1 Characteristics

1. Properties of four algebraic operations of the real numbers also apply to the complex ones (i.e. associative, distributive, etc.)
2. *Complex conjugate*:

$$\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$$
$$\overline{z_1 \cdot z_2} = \overline{z}_1 \cdot \overline{z}_2$$
$$\overline{\frac{z_1}{z_2}} = \frac{\overline{z}_1}{\overline{z}_2}$$
$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}, \ z \cdot \overline{z} = |z|^2$$

1.3.2 Analysis

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$
$$z_1 z_2 = r_1 r_2[(\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)}) + i(\underbrace{\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1}_{\sin(\theta_1+\theta_2)})]$$

Conclusion.

$$|z_1 z_2| = r_1 r_2$$
$$\arg(z_1 z_2) = \theta_1 + \theta_2$$

Definition (De Moivre's Formula)**.**

$$(r\operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta)$$

Definition (nth-root of a complex number)**.**

$$z^n = r\operatorname{cis}\theta$$
$$z = \sqrt[n]{r}\operatorname{cis}\left(\frac{\theta + 2\pi k}{n}\right), \quad k = 0, 1, 2, \ldots, n-1$$

Definition (Euler's Formula)**.**

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$
$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

e to a complex number:

$$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b)$$

Our goal is to:

1. Express $\cos(nx)$ in terms of $\sin x, \cos x$.
2. Express $\sin^n(x)$ as a sum of $\sin x, \cos x$, without multiplying them.

Example 1.3.1.

$$\begin{aligned} \cos(5x) &= \mathrm{Re}(e^{i5x}) = \mathrm{Re}((e^{ix})^5) \\ &= \mathrm{Re}((\cos x + i \sin x)^5) \\ (a+b)^5 &= a^5 + 5a^4 b + 10a^3 b^2 + 10a^2 b^3 + \dots \end{aligned}$$

To simplify our calculation since we are only looking for the real part of our solution, we can ignore any place where $\sin$ is raised to an odd power (since $i^2 = -1$).

$$= \cos^5 x - 10 \cos^x \sin^2 x + 5 \cos x \sin^4 x$$

Now for an example in the opposite direction:

$$\begin{aligned} \sin^5 x &= \left(\frac{1}{2i}\right)^4 (e^{ix} - e^{-ix})^4 \\ &\frac{1}{16}(e^{i4x} - 4e^{i2x} + 6 - 4e^{-i2x} + e^{-i4x}) \\ &= \frac{1}{16}(2\cos(4x) - 8\cos(2x) + 6) \end{aligned}$$

Example 1.3.2.

$$\begin{aligned} &a\cos(\omega t) + b\sin(\omega t) \\ &\mathrm{Re}(\underbrace{(a+bi)}_{re^{i\theta}}\underbrace{(\cos(\omega t) - i\sin(\omega t))}_{e^{-i\omega t}}) \\ &= \mathrm{Re}\left(re^{i(\theta-\omega t)}\right) = r\cos(\theta - \omega t) = r\cos(\omega t - \theta) \\ &= \sqrt{a^2+b^2}\cos\left(\omega t - \tan^{-1}\left(\frac{a}{b}\right)(+\pi)\right) \end{aligned}$$

1.4 Continuity

Example 1.4.1.

$$\begin{gathered} f_0(z) = \mathrm{Arg}(z) = u(x,y) + iv(x,y) \\ f_0 : \mathbb{C} \setminus \{0\} \to \mathbb{C} \\ (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} \quad v(x,y) = 0 \end{gathered}$$

This function isn't continuous when $y = 0, x < 0$ because the limit from above equals π but the limit from below is $-\pi$.

$$f_k(z) = \text{Arg}(z) + 2\pi k$$

Note. We can define $\theta(z)$ as a continuous function on D_1 such that:

$$f(D_1) = [0, 2\pi)$$

such that we define the domain to be:

$$D_1 = \mathbb{C} \setminus \{(t + i0), t \geq 0\} \quad f(D_1) = (0, 2\pi)$$

In general, we can define $f : U \to \mathbb{C}$ to be continuous such that $\forall z \in U$:

$$z = |z|e^{if(z)}$$

if U does not contain *closed paths* that surround 0.

Example 1.4.2.

$$U = \mathbb{C} \setminus \{(it), t \geq 0\})$$

Given the example of $f : U \to \mathbb{C}$ such that $\forall z \in U$:

$$z = |z|e^{if(z)}$$

We can define:

$$f(z) = \begin{cases} \arctan \frac{y}{x} & x > 0 \\ \frac{\pi}{2} & x = 0, y > 0 \\ \pi + \arctan \frac{y}{x} & x < 0 \end{cases}$$

Example 1.4.3.

$$U = \mathbb{C} \setminus \{(t + it^2), t \geq 0\}$$

Where $f : U \to \mathbb{C}$ is continuous such that $\forall z \in U$:

$$\begin{cases} z = |z|e^{if(z)} \\ f(2 + 2i) = \frac{\pi}{4} \end{cases}$$

We define γ to be the path that is not defined in U. In other words: $\gamma : y = x^2, x \geq 0$.

We see that the discontinuity will be between when $y < x^2$ and $y > x^2$.

$$\begin{cases} \arctan \frac{y}{x} - 2\pi & x > 0, y > x^2 \\ \arctan \frac{y}{x} & x > 0, y < x^2 \\ -\frac{\pi}{2} & x = 0, y < 0 \\ -\frac{3\pi}{2} & x = 0, y > 0 \\ \arctan \frac{y}{x} - \pi & x < 0 \end{cases}$$

1.5 Linear Translation

$$f(z) = az + b \quad a, b \in \mathbb{C}$$

Reminder.

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$$
$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot e^{i(\theta_1 + \theta_2)}$$

An uninteresting case would be when $a = 0$ because then $f(z) = b$ so we'll asssume $a \neq 0$.

f is not a linear function since $f(z_1 + z_2) \neq f(z_1) + f(z_2)$.

Claim. f is injective and surjective, and therefore bijective, and in that sense is linear.

Proof. Let $w \in \mathbb{C}$. We find for all $z \in \mathbb{C}$:

$$f(z) = w \iff az + b = w \iff z = \frac{1}{a}(w - b) \iff z = \frac{1}{a}w - \frac{b}{a} = f^{-1}(w)$$

■

Example 1.5.1.

$$\gamma = \{(t + i\cos t) \mid t \in [-\pi, \pi]\}$$
$$f(z) = 2iz - 3$$

We first take γ, scale it by 2, rotate it a quarter turn counter-clockwise, and then translate it left by 3.

Important: remember that the scaling is done in both the imaginary and real directions!

Claim. $f : \mathbb{C} \to \mathbb{C}$ is continuous.

Proof. Let $z_0 \in \mathbb{C}$ and $\varepsilon > 0$ and we see that there exists $\delta > 0$ such that:

$$|f(z) - f(z_0)| < \varepsilon \Leftarrow |z - z_0| < \delta$$

■

Claim. f translates straight lines to straight lines and circles to circles.