Problem sheet 1 (Thermodynamics and Probability):

1. Filament: Problem 1.7 from Kardar's "Statistical Physics of Particles" (MK1)

2. Hard core gas: Problem 1.8 from MK1

3. Superconducting transition: Problem 1.9 from MK1

4. Constant heat capacities (credit: Prof. Deepak Dhar):

   Consider two bodies with temperature independent heat capacities C_1 and C_2, and initial temperatures $T_1 > T_2$.

   (a) If the two bodies are brought into contact so that heat is exchanged only between them, what is the final temperature T_F, and what is the change in entropy.

   (b) What is the final temperature if a Carnot engine is used to transfer heat between the two bodies? What is the amount of work done by the engine in this case?

5. Directed Random Walk (adapted from MK1 2.2)

   A walker walks in 3 dimensions, taking independent steps of size ℓ at an angle θ w.r.t the positive z axis (θ is between 0 to π). The probability density for taking a step between θ and $\theta + d\theta$ is $p(\theta) = (2/\pi)\sin^2(\theta/2)$, while the angle ϕ being uniformly distributed. The walker starts at the origin and takes a large number of steps N.

   a) Calculate the expectation values of x, y, z, x^2, y^2, z^2, and the covariances $\langle xy\rangle$, $\langle yz\rangle$, and $\langle zx\rangle$.

   b) Use central limit theorem to estimate $p(x, y, z)$.

6. Information and Languages

   Both English and Turkish essentially have 26 letters in the alphabet (count the letters with/without accents as the same letter), corresponding to probabilities p_ν with $\nu = (a, b, c, ..z)$. These probabilities are reasonably well-known (see e.g., `https://en.wikipedia.org/wiki/Letter_frequency#Relative_frequencies_of_letters_in_other_languages`).

   a) Compare the information conveyed by these frequencies.

   b) If I have a document where some of the letters are missing, for which language would it be easier to recover that information, based on this data alone? Quantify this.

   c) Now suppose I have more information, i.e., knowledge of bigram frequency (see e.g., `https://en.wikipedia.org/w/index.php?title=Bigram&oldid=698032068`), how does your answer to (b) change?

   Hint: You may want to write a program to do this. Make appropriate approximations.

7. Entropy and Estimation

One can get an unbiased estimate of an unknown probability distribution $p(x)$, for the random variable x taking the values x_i with $i = (1, 2, 3, \ldots, N)$, and obeying k constraints, say $\langle f_k(x) \rangle = c_k$, by maximizing the function

$$\sum_{i=1}^{N} -p(x_i) \ln p(x_i) - \alpha \left(\sum_{i=1}^{N} p(x_i) - 1 \right) - \sum_{k} \beta_k \left(\sum_{i=1}^{N} p(x_i) f_k(x_i) - c_k \right) , \tag{1}$$

w.r.t. all unknowns, i.e., $p(x_i)$, α, and β_k.

a) Derive $p(x)$.

b) A coin was flipped 2 times and produced 2 heads. What is the probability that a fair coin would have led to the above result?

Suppose after the 2 flips you are told that there is a probability κ that the coin is fake and has heads on both faces! Clearly the probability that the coin is fair is $1 - \kappa$, as having tails on both faces is ruled out.

c) What is the best estimate of κ?

d) What is your best estimate for probability of getting a tail on the third flip, if we indeed allow the possibility that the coin may be fake?