

TYPE THEORY HW 2

All problems are in the simply typed lambda calculus

Exercise 1. Define addition on the natural numbers in a different way from that done in class.

Exercise 2. Define an exponential function (that takes two natural numbers m, n to m^n) on the natural numbers.

Exercise 3. Consider the type \mathbb{B} defined in the last homework. Show that this acts like $\mathbb{N}/2$; that is:

- (1) Define a function $\text{mod2} : \mathbb{N} \Rightarrow \mathbb{B}$ that sends every even number to 0 and every odd number to 1
- (2) Check for a few specific natural numbers n that $\text{mod2}(\text{mult}(2, n)) = 0$. Think about what would be needed in the type theory to prove that

$$\text{mod2}(\text{mult}(2, n)) = 0.$$

for all $n : \mathbb{N}$.

- (3) Define functions $f : \mathbb{N} \times \mathbb{B} \Rightarrow \mathbb{N}$ and $g : \mathbb{N} \Rightarrow \mathbb{N} \times \mathbb{B}$ that are metatheoretically inverse to each other (that is, for given any specific $(n, b) : \mathbb{N} \times \mathbb{B}$, you could show that $gf(n, b) = (n, b)$ and given any specific $n : \mathbb{N}$, you could show that $fg(n) = n$).

Exercise 4.

- (1) Consider the following rule that is part of the definition of \mathbb{N} (in the empty context).

$$\frac{\vdash z : T \quad x : T \vdash t(x) : T}{n : \mathbb{N} \vdash j_{z,t}(n) : T}$$

Turn this rule into a function in the type theory.

- (2) Generalize this to take into account any context.

$$\frac{\Gamma \vdash z : T \quad \Gamma, x : T \vdash t(x) : T}{\Gamma, n : \mathbb{N} \vdash j_{z,t}(n) : T}$$

Exercise 5. In set-based mathematics, a *pointed magma structure* on a set M is a point $e \in M$ and a binary operation that takes $m, n \in M$ to some $m \cdot n \in M$. The operation is not required to be associative and the point is not required to be a unit. Note that every group and every monoid has an underlying magma.

- (1) Using the types that we have already defined in the simply typed lambda calculus, for any type T , construct the type of magma structures on T , emulating the set-based definition above. Call this type $\mathbf{Magma}(T)$.
- (2) Construct an interesting magma structure on \mathbb{N} .
- (3) Construct an interesting magma structure on $T \Rightarrow T$ for any type T .
- (4) Construct, for any types S and T , a magma structure on $S \times T$ from a magma structure on S and a magma structure T .