

A higher structure identity principle

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Tsementzis

paigenorth.github.io/hottest.pdf
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Outline

- 1 Motivation
- 2 Lower structure identity principles in univalent foundations
- 3 First-order logic with dependent sorts for lower structures
- 4 FOLDS categories

Different notions of equality

Synthetic vs. analytic equalities

In MLTT, we always have a (*synthetic*) equality type between $a, b : T$

$$a =_T b.$$

Depending on the type T , we might have a type of “*analytic* equalities”

$$a \cong b.$$

A “univalence principle” for this T and this \cong states that

$$(a =_T b) \rightarrow (a \cong b)$$

is an equivalence.

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The univalence *axiom* in type theory states that

$$S =_{\mathcal{U}} T \rightarrow S \simeq T$$

is an equivalence.

Identicals and indiscernibilites

Identity of indiscernibles

Leibniz: two things are equal when they are *indiscernible* (have the same properties).

$$(a = b) \leftarrow (\forall P. P(a) \leftrightarrow P(b))$$

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$$(a =_T b) \leftrightarrow \left(\prod_{P:T \rightarrow \mathcal{U}} P(a) \simeq P(b) \right)$$

- This holds in MLTT.
- Given a ‘univalence principle’ $(a =_T b) \simeq (a \cong b)$, we would find a *structure identity principle* (in the sense of Aczel):

$$(a \cong b) \rightarrow \left(\prod_{P:T \rightarrow \mathcal{U}} P(a) \simeq P(b) \right).$$

Goal

Our goal

To define a large class of (higher) *structures* and a notion of *equivalence* between them validating a univalence principle. This then automatically validates a structure identity principle.

Using ideas from:

- *First Order Logic with Dependent Sorts*, Makkai, 1995.
- *Univalent categories and the Rezk completion*, Ahrens, Kapulkin, Shulman, 2015.

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Propositions

Theorem (univalence for propositions)

Given two mere propositions P and Q ,

$$(P =_{\text{Prop}} Q) \simeq (P \leftrightarrow Q).$$

Corollary (structure identity principle for propositions)

Given two mere propositions P and Q ,

$$(P \leftrightarrow Q) \rightarrow \left(\prod_{S:\text{Prop} \rightarrow \mathcal{U}} S(P) \simeq S(Q) \right).$$

Magmas

Magmas

A *magma* is a set M and a binary operation $M \times M \rightarrow M$.

There are two notions of ‘sameness’ for elements m, n of a magma:

1. Equality: $m =_M n$

2. Indiscernibility:

$$\prod_{x,y:M} (mx = nx) \times (xm = xn) \times ((xy = m) \leftrightarrow (xy = n))$$

This produces two notions of equivalence of magmas:

1. $M \cong_e N$ if there are morphisms $f : M \hookrightarrow N : g$ respecting the operation such that gfm is *equal* to m for all $m : M$ and likewise for fgn
2. $M \cong_i N$ if there are morphisms $f : M \hookrightarrow N : g$ respecting the operation such that gfm is *indiscernible* from m for all $m : M$ and likewise for fgn

Preorders and topological spaces

Preorders

A *preorder* is a set P and a reflexive, transitive relation $\leq: P \times P \rightarrow \text{Prop}$. Two elements p, q of a preorder P are *indiscernible* if

$$\prod_{x:P} (p \leq x \leftrightarrow q \leq x) \times (x \leq p \leftrightarrow x \leq q) \times (p \leq p \leftrightarrow q \leq q)$$

or, equivalently, if $p \leq q \times q \leq p$.

Topological spaces

A *topological space* is a set T and a collection $O: (T \rightarrow \text{Prop}) \rightarrow \text{Prop}$ of subsets closed under union and finite intersection.

Two elements s, t of a topological space T are *indiscernible* if $U(s) \leftrightarrow U(t)$ for every open set U of T .

Motivation

Equivalences between (higher) categorical structures are up to indiscernibility.

A lower structure identity principle in UF

Theorem (univalence for magmas with \cong_e)

Given two magmas M, N ,

$$(M =_{\text{Mag}} N) \simeq (M \cong_e N).$$

- This is a special case of the general result for all ‘standard’ structures on sets (Thm 9.8.2 of the HoTT Book).
- The same holds for preorders with \cong_e and for topological spaces with \cong_e .

Another lower structure identity principle in UF?

Univalence with \cong_i

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A: No: in particular, the projection $U : \mathbf{Mag} \rightarrow \mathbf{Set}$ would then take an equivalence $M \cong_i N$ to an equivalence $UM \cong_i UN$ between the underlying sets, making it an equivalence $M \cong_e N$.

For example, let $\mathbf{1}$ be the poset whose underlying set has one element, and let $\mathbf{2}$ be the poset whose underlying set has two elements a and b for which $a \leq b$ and $b \leq a$.



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Univalence with \cong_i

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A: Yes: if we identify equality and indiscernibility.

For example, let **1** be the poset whose underlying set has one element, and let **2** be the poset whose underlying set has two elements a and b for which $a \leq b$ and $b \leq a$.



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First-order logic with dependent sorts

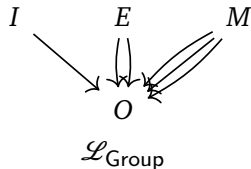
Inverse category

An *inverse category* is a strict category \mathcal{I} and a function $\rho : \mathcal{I} \rightarrow \mathbf{Nat}^{\text{op}}$ whose fibers are discrete.

The *height* of an inverse category (\mathcal{I}, ρ) is the maximum value of ρ .

Signatures

Signatures are inverse categories of finite height.



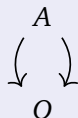
Structures

An \mathcal{L} -structure is roughly a functor from \mathcal{L} into \mathcal{U} .

$\mathcal{L}_{\text{Proset}}$ -structures

An $\mathcal{L}_{\text{Proset}}$ -structure S is

1. A type SO ,
2. A type $SA(x,y)$ for every $x,y : O$ (meaning $x \leq y$)



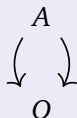
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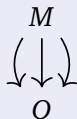
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$\mathcal{L}_{\text{Magma}}$ -structures

An $\mathcal{L}_{\text{Magma}}$ -structure S is

1. A type SO ,
2. A type $SM(x,y,z)$ for every $x,y,z : O$ (meaning z is the product of x and y)



We can impose axioms on these structures.

Indiscernibilities

Indiscernibilities between O -elements of $\mathcal{L}_{\text{Proset}}$ -structures

An indiscernibility between two terms $p, q : SO$ consists of

- $\prod_{x:SO} SA(p, x) \cong SA(q, x)$
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Indiscernibilities between O -elements of $\mathcal{L}_{\text{Magma}}$ -structures

An indiscernibility between two terms $m, n : SO$ consists of

- $\prod_{x,y:SO} SM(m, x, y) \cong SM(n, x, y)$
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Indiscernibilities at the top-level

Indiscernibilities between A-elements of $\mathcal{L}_{\text{Proset}}$ -structures

An indiscernibility between two terms $a, b : SA(p, q)$ consists of

- -

so all terms of $a, b : SA(p, q)$ are (trivially) indiscernible.

Definition (univalent structure)

A structure M of a signature \mathcal{L} is *univalent* if the type of indiscernibilities between any two terms of any one sort is equivalent to the type of equalities between them.

In particular, this means that all of the top-level sorts are propositions, and all of the next-level sorts are sets.

Univalent structures

Proposition

A $\mathcal{L}_{\text{Proset}}$ -structure S is univalent when each $p \leq q$ is a proposition and $(p = q) \rightarrow (p \leq q) \times (q \leq p)$ is an equivalence - in other words, when A is a poset.

Proposition

A $\mathcal{L}_{\text{Magma}}$ -structure S is univalent when each $SM(m, n, p)$ is a proposition and $(m = n) \rightarrow \prod_{x, y: M} (mx = nx) \times (xm = xn) \times ((xy = m) \leftrightarrow (xy = n))$ is an equivalence.

Proposition

A topological space T is univalent when $(x = y) \rightarrow \prod_{U \text{ open in } T} (x \in U \leftrightarrow y \in U)$ is an equivalence - in other words, T is a T_0 space.

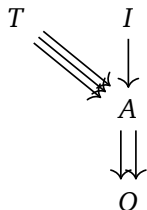
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\mathcal{L}_{cat} -structures

We can define the data of a category \mathcal{C} to be

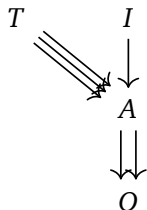
- A type $\mathcal{C}O : \mathcal{U}$
- A family $\mathcal{C}A : \mathcal{C}O \times \mathcal{C}O \rightarrow \mathcal{U}$
- A family $\mathcal{C}I : \prod_{(x:\mathcal{C}O)} \mathcal{C}A(x,x) \rightarrow \mathcal{U}$
- A family $\mathcal{C}T : \prod_{(x,y,z:\mathcal{C}O)} \mathcal{C}A(x,y) \rightarrow \mathcal{C}A(y,z) \rightarrow \mathcal{C}A(x,z) \rightarrow \mathcal{U}$



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We want to add axioms such as

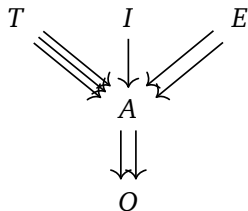
$$\forall (x,y,z : O). \forall (f : A(x,y)). \forall (g : A(y,z)). \forall (h, h' : A(x,z)). \\ T(x,y,z,f,g,h) \rightarrow T(x,y,z,f,g,h') \rightarrow (h = h')$$

(composites are unique), so we add an equality ‘predicate’.

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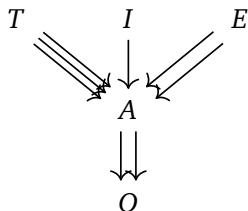
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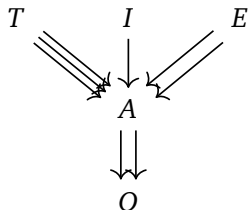
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$$\forall (x,y,z : O). \forall (f : A(x,y)). \forall (g : A(y,z)). \forall (h, h' : A(x,z)). \\ T(x,y,z,f,g,h) \rightarrow T(x,y,z,f,g,h') \rightarrow E(h,h')$$

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Univalent \mathcal{L}_{cat} -structures

- Every two elements of $\mathcal{C}I_x(f)$, $\mathcal{C}E_{x,y}(f,g)$, or $\mathcal{C}T_{x,y,z}(f,g,h)$ are indiscernible
 - so each of these types should be a proposition.
- The axioms making E a congruence for T and I make $\mathcal{C}E(f,g)$ the type of indiscernibilities between $f,g : \mathcal{C}A(x,y)$
 - so we should have $(f = g) = \mathcal{C}E(f,g)$, making each $\mathcal{C}A(x,y)$ a set.
- The indiscernibilities between $a,b : \mathcal{C}O$ consist of
 1. $\phi_{x\bullet} : \mathcal{C}A(x,a) \simeq \mathcal{C}A(x,b)$ for each $x : \mathcal{C}O$
 2. $\phi_{\bullet z} : \mathcal{C}A(a,z) \simeq \mathcal{C}A(b,z)$ for each $z : \mathcal{C}O$
 3. $\phi_{\bullet\bullet} : \mathcal{C}A(a,a) \simeq \mathcal{C}A(b,b)$
 4. The following for all appropriate w,x,y,z,f,g,h :

$$T_{x,y,a}(f,g,h) \leftrightarrow T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$$

$$I_{a,a}(f) \leftrightarrow I_{b,b}(\phi_{\bullet\bullet}(f))$$

$$T_{x,a,z}(f,g,h) \leftrightarrow T_{x,b,z}(\phi_{x\bullet}(f), \phi_{\bullet z}(g), h)$$

$$E_{x,a}(f,g) \leftrightarrow E_{x,b}(\phi_{x\bullet}(f), \phi_{x\bullet}(g))$$

$$T_{a,z,w}(f,g,h) \leftrightarrow T_{b,z,w}(\phi_{\bullet z}(f), g, \phi_{\bullet w}(h))$$

$$E_{a,x}(f,g) \leftrightarrow E_{b,x}(\phi_{\bullet x}(f), \phi_{\bullet x}(g))$$

$$T_{x,a,a}(f,g,h) \leftrightarrow T_{x,b,b}(\phi_{x\bullet}(f), \phi_{\bullet\bullet}(g), \phi_{x\bullet}(h))$$

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Univalent \mathcal{L}_{cat} -structures continued

Proposition

The type of indiscernibilities between $a, b : \mathcal{C}O$ is equivalent to $a \cong b$.

(The isomorphisms $\phi_{x\bullet} : \mathcal{C}A(x, a) \cong \mathcal{C}A(x, b)$ are natural by $\mathcal{C}T_{x,y,a}(f, g, h) \leftrightarrow \mathcal{C}T_{x,y,b}(f, \phi_{y\bullet}(g), \phi_{x\bullet}(h))$ (saying $\phi_{y\bullet}(g) \circ f = \phi_{x\bullet}(g \circ f)$). The rest of the data is redundant.)
Thus, in a univalent \mathcal{L}_{cat} -structure, $(a = b) \simeq a \cong b$.

Theorem

Univalent \mathcal{L}_{cat} -structures are equivalent to the univalent categories of Ahrens-Kapulkin-Shulman.

Categorical equivalences

Theorem (univalence for univalent categories) (AKS 2015)

Given univalent categories \mathcal{C}, \mathcal{D} ,

$$(\mathcal{C} = \mathcal{D}) \simeq (\mathcal{C} \simeq \mathcal{D})$$

A categorical equivalence arises as a very surjective morphism.

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of $\mathcal{L}_{\text{cat}+\mathbf{E}}$ -structures consists of surjections

- $FO : \mathcal{C}O \rightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x,y) \rightarrow \mathcal{D}A(Fx,Fy)$ for every $x,y : \mathcal{C}O$
- $FT : \mathcal{C}T(f,g,h) \rightarrow \mathcal{D}T(Ff,Fg,Fh)$ for all $f : \mathcal{C}A(x,y), g : \mathcal{C}A(y,z), h : \mathcal{C}A(x,z)$
- $FE : \mathcal{C}E(f,g) \rightarrow \mathcal{D}E(Ff,Fg)$ for all $f,g : \mathcal{C}A(x,y)$
- $FI : \mathcal{C}I(f) \rightarrow \mathcal{D}I(Ff)$ for all $f : \mathcal{C}A(x,x)$

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- $FA : \mathcal{C}A(x,y) \rightarrow \mathcal{D}A(Fx,Fy)$ for every $x,y : \mathcal{C}O$
- $FT : \mathcal{C}T(f,g,h) \leftrightarrow \mathcal{D}T(Ff,Fg,Fh)$ for all $f : \mathcal{C}A(x,y), g : \mathcal{C}A(y,z), h : \mathcal{C}A(x,z)$
- $FE : (f = g) \leftrightarrow (Ff = Fg)$ for all $f,g : \mathcal{C}A(x,y)$
- $FI : \mathcal{C}I(f) \leftrightarrow \mathcal{D}I(Ff)$ for all $f : \mathcal{C}A(x,x)$

Categorical equivalences

Theorem (univalence for univalent categories) (AKS 2015)

Given univalent categories \mathcal{C}, \mathcal{D} ,

$$(\mathcal{C} = \mathcal{D}) \simeq (\mathcal{C} \simeq \mathcal{D})$$

A categorical equivalence arises as a very surjective morphism.

A *very surjective morphism* or *equivalence* $F : \mathcal{C} \simeq \mathcal{D}$ of **univalent** $\mathcal{L}_{\text{cat}+\mathbf{E}}$ -structures consists of surjections

- $FO : \mathcal{C}O \rightarrow \mathcal{D}O$
- $FA : \mathcal{C}A(x,y) \xrightarrow{\cong} \mathcal{D}A(Fx,Fy)$ for every $x,y : \mathcal{C}O$
- $FT : \mathcal{C}T(f,g,h) \leftrightarrow \mathcal{D}T(Ff,Fg,Fh)$ for all $f : \mathcal{C}A(x,y), g : \mathcal{C}A(y,z), h : \mathcal{C}A(x,z)$
- $FE : (f = g) \leftrightarrow (Ff = Fg)$ for all $f,g : \mathcal{C}A(x,y)$
- $FI : \mathcal{C}I(f) \leftrightarrow \mathcal{D}I(Ff)$ for all $f : \mathcal{C}A(x,x)$

Equivalences in general

Definition (equivalence)

An *equivalence* $M \simeq N$ between two \mathcal{L} -structures is a very split-surjective morphism $M \rightarrow N$.

Theorem

Given two univalent \mathcal{L} -structures M and N ,

$$(M = N) \simeq (M \simeq N).$$

Theorem

For a signature $L : \text{Sig}(n)$, the type of univalent L -structures is of h -level $n + 1$.

Example: magmas

Equivalences of univalent magmas

An equivalence of magmas N, P consists of surjections

- $FO : NO \rightarrow PO$
- $FM : NM(x, y, z) \rightarrow PM(Fx, Fy, Fz)$

Example: magmas

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Summary

For every signature \mathcal{L} , we have

- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem,
- and thus a (higher) structure identity principle.

Summary

For every signature \mathcal{L} , we have

- a notion of indiscernibility within each sort,
- a notion of univalent structures,
- a notion of equivalence,
- a univalence theorem,
- and thus a (higher) structure identity principle.

The paper includes examples of

- \dagger -categories,
- presheaves,
- profunctors,
- semi-displayed categories,
- bicategories,
- ...

Further work

- Drop the splitness condition for certain structures
- Formulate an analogue to the Rezk completion

Thank you!