# Estimation and Control Library

Pan

***Abstract*— A library for optimal estimation and control, as well as classical/modern control theories. Algorithms are applied on simulated robots using Ignition Gazebo.**

## I. INTRODUCTION

A library for optimal estimation and control, as well as classical/modern control theories. Algorithms are applied on simulated robots using ROS [1] and Ignition Gazebo [2].

## II. CARTPOLE DYNAMICS

This section derives the equation of motion of the cartpole system in the following two ways: (1) standard vector analysis; (2) using Lagrangian.

### A. *Standard vector analysis*

The position of the cart is given as

$$\mathbf{r}_1 = x\mathbf{E}_x \tag{1}$$

Computing the 1st and 2nd order rate of change of $\mathbf{r}_1$ in reference frame $\mathcal{F}_i$, we obtain the velocity and acceleration as

$$\mathbf{v}_1 =^{\mathcal{F}_i} \frac{d}{dt}(\mathbf{r}_1) = \dot{x}\mathbf{E}_x \tag{2}$$

$$\mathbf{a}_1 =^{\mathcal{F}_i} \frac{d}{dt}(\mathbf{v}_1) = \ddot{x}\mathbf{E}_x \tag{3}$$

The position of the pole is given as

$$\mathbf{r}_2 = x\mathbf{E}_x + l\mathbf{e}_r \tag{4}$$

Similarly, the velocity and acceleration of the pole can be computed as

$$\mathbf{v}_2 =^{\mathcal{F}_i} \frac{d}{dt}(\mathbf{r}_2) = \dot{x}\mathbf{E}_x + l\dot{\theta}\mathbf{e}_\theta \times \mathbf{e}_r = \dot{x}\mathbf{E}_x - l\dot{\theta}\mathbf{e}_z \tag{5}$$

$$\mathbf{a}_2 =^{\mathcal{F}_i} \frac{d}{dt}(\mathbf{v}_2) = \ddot{x}\mathbf{E}_x - l\ddot{\theta}\mathbf{e}_z - l\dot{\theta}^2\mathbf{e}_r \tag{6}$$

Rewriten $\mathbf{a}_2$ in $\mathbf{E}_x$ and $\mathbf{E}_z$ as

$$\mathbf{a}_2 = (\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta)\mathbf{E}_x - (l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta)\mathbf{E}_z \tag{7}$$

Applying Netwon 2nd law on the $\mathbf{E}_x$-direciton of the cart and pole, we have

$$F - T_x = m_1\ddot{x} \tag{8}$$

$$T_x = m_2(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta) \tag{9}$$

Adding the above two euqations we get the first equation,

$$F = (m_1 + m_2)\ddot{x} + m_2(l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta) \tag{10}$$

[1] The Pennsylvania State University, University Park, PA 16802, USA. pan.liu@psu.edu

The cartpole system is consisted of two systems, i.e. the cart and the pole. The internal force can be canceled out by analyzing the two systems as a whole. A simplier way is to apply the balance of angular momemtun relative to the cart.

$$\frac{d}{dt}\mathbf{H}_c = \mathbf{M}_c - (\mathbf{r}_2 - \mathbf{r}_1) \times m_2\mathbf{a}_1 \tag{11}$$

where $\mathbf{H}_c$ is the angular momemtun of the cartpole system relative to the cart.

$$\mathbf{H}_c = (\mathbf{r}_2 - \mathbf{r}_1) \times m_2(\mathbf{v}_2 - \mathbf{v}_1) = m_2 l^2\dot{\theta}\mathbf{E}_y \tag{12}$$

And $\mathbf{M}_c$ is the moment relative to the cart.

$$\mathbf{M}_c = (\mathbf{r}_2 - \mathbf{r}_1) \times (-m_2 g)\mathbf{E}_z = m_2 gl\sin\theta\mathbf{E}_y \tag{13}$$

$$(\mathbf{r}_2 - \mathbf{r}_1) \times m_2\mathbf{a}_1 = l\mathbf{e}_r \times m_2\ddot{x}\mathbf{E}_x = m_2 l\ddot{x}\cos\theta\mathbf{E}_y \tag{14}$$

Thus, we get the second equation

$$\cos\theta\ddot{x} + l\ddot{\theta} = g\sin\theta \tag{15}$$

In summary, the equation of motion is

$$(m_1 + m_2)\ddot{x} + m_2(l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta) = F \tag{16}$$

$$\cos\theta\ddot{x} + l\ddot{\theta} = g\sin\theta \tag{17}$$

### B. *Linearization*

Assuming $\mathbf{q} = [x, \theta]^T$, the equations of motion of the cartpole system can be written in the standard form as

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{B}\mathbf{u} \tag{18}$$

where $\mathbf{q} = [x, \theta]^T$ is a $n$-vector called the *generalized coordinates vector*, $\mathbf{H}(\mathbf{q})$ is a $n \times n$ nonsingular symmetric positive-definite matrix called the *mass matrix*, $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is a $n \times n$ matrix called the *centrifugal/Coriolis/friction matrix*, $\mathbf{G}(\mathbf{q})$ is a $n$-vector sometimes called the *conservative forces vector*. **This equation of motion is valid for systems that follow classical Newton-Euler mechanics or Lagrangian mechanics** with a kinetic energy that is quadratic in the derivative of the generalized coordinates and a potential energy that may depend on the generalized coordinates, but not on its derivative. Such systems include robot arms, mobile robots, airplanes, helicopters,underwatervehicles,hovercraft,etc.

$$\mathbf{H}(\mathbf{q}) = \begin{bmatrix} m_1 + m_2 & m_2 l\cos\theta \\ \cos\theta & l \end{bmatrix}, \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 & -m_2 l\dot{\theta}\sin\theta \\ 0 & 0 \end{bmatrix},$$
$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} 0 \\ -g\sin\theta \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{19}$$

Let $x_1 = \mathbf{q}$, $x_2 = \dot{\mathbf{q}}$, and $x = [x_1, x_2]^T$. Then we have the **standard form for linearization**, i.e. $\dot{x} = f(x, u)$

$$\dot{x} = \begin{bmatrix} x_2 \\ \mathbf{H}(\mathbf{q})^{-1}(-\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} - \mathbf{G}(\mathbf{q}) + \mathbf{Bu}) \end{bmatrix} \tag{20}$$

Linearize around the equlibrium point $x^* = 0$,

$$\dot{x} = Ax + Bu \tag{21}$$

where

$$A = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{H}(\mathbf{q})^{-1}\frac{\partial}{\partial \mathbf{q}}\mathbf{G}(\mathbf{q}) & -\mathbf{H}(\mathbf{q})^{-1}\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) \end{bmatrix}, \tag{22}$$

$$B = \begin{bmatrix} 0 \\ \mathbf{H}^{-1}\mathbf{B} \end{bmatrix} \tag{23}$$

Note that the term involving $\frac{\partial}{\partial \mathbf{q}}\mathbf{H}(\mathbf{q})^{-1}$ disappears because $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} - \mathbf{G}(\mathbf{q}) + \mathbf{Bu}$ must be zero at the fixed point. Many of the $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}$ derivatives drop out, too, because $\dot{\mathbf{q}}^* = 0$. The $-\mathbf{H}(\mathbf{q})^{-1}$ can be calculated by Matlab symbolic inversion,

$$\mathbf{H}(\mathbf{q})^{-1} = \begin{bmatrix} \frac{1}{m_1+m_2-m_2\cos^2\theta} & \frac{-m_2\cos\theta}{m_1+m_2-m_2\cos^2\theta} \\ \frac{-\cos\theta}{m_1 l+m_2 l-m_2 l\cos^2\theta} & \frac{m_1+m_2}{m_1+m_2-m_2\cos^2\theta} \end{bmatrix} \tag{24}$$

and

$$\frac{\partial}{\partial \mathbf{q}}\mathbf{G}(\mathbf{q}) = \begin{bmatrix} 0 & 0 \\ 0 & -g\cos\theta \end{bmatrix} \tag{25}$$

Substitute $x^* = 0$ and rearrange $x$ to be $x = [x, \dot{x}, \theta, \dot{\theta}]$, we have

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m_2}{m_1}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{m_1+m_2}{m_1 l}g & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1/m \\ 0 \\ -1/(m_1 l) \end{bmatrix} \tag{26}$$

## References


[1] M. Quigley, K. Conley, B. Gerkey, J. Faust, T. Foote, J. Leibs, R. Wheeler, and A. Y. Ng, "Ros: an open-source robot operating system," in *ICRA workshop on open source software*, vol. 3, no. 3.2. Kobe, Japan, 2009, p. 5.

[2] I. Gazebo, "https://ignitionrobotics.org."