

1 Separating and convergence-determining classes

Definition 1.1 (Separating class)

Suppose Ω is a non-empty set, \mathcal{A} is a σ -algebra of subsets of Ω , (S, \mathcal{A}) is the corresponding measurable space, and $\mathcal{M}_1(\Omega, \mathcal{A})$ is the set of all probability measures defined on \mathcal{A} . A **separating class** of subsets of (Ω, \mathcal{A}) is a collection $\mathcal{S} \subset \mathcal{A}$ of subsets of Ω which satisfies the following condition: For every two probability measures $\mu, \nu \in \mathcal{M}_1(\Omega, \mathcal{A})$,

$$\mu(S) = \nu(S), \text{ for every } S \in \mathcal{S} \implies \mu(A) = \nu(A), \text{ for every } A \in \mathcal{A}$$

Definition 1.2 (Convergence-determining class)

Suppose Ω is a topological space, $\mathcal{B}(\Omega)$ is its Borel σ -algebra, $(\Omega, \mathcal{B}(\Omega))$ is the corresponding measurable space, and $\mathcal{M}_1(\Omega, \mathcal{B}(\Omega))$ is the set of all probability measures defined on $\mathcal{B}(\Omega)$. A **convergence-determining class** of subsets of $(\Omega, \mathcal{B}(\Omega))$ is a collection $\mathcal{C} \subset \mathcal{B}(\Omega)$ of Borel subsets of Ω which satisfies the following condition: For any $\mu, \mu_1, \mu_2, \dots \in \mathcal{M}_1(\Omega, \mathcal{B}(\Omega))$,

$$\lim_{n \rightarrow \infty} \mu_n(C) = \mu(C), \text{ for every } C \in \mathcal{C} \implies \mu_n \xrightarrow{w} \mu.$$

Theorem 1.3

Suppose Ω is a non-empty set, \mathcal{A} is a σ -algebra of subsets of Ω , and (Ω, \mathcal{A}) is the corresponding measurable space. If

- $\mathcal{S} \subset \mathcal{A}$ is closed under finite intersections, and
- \mathcal{S} generates \mathcal{A} (i.e. $\sigma(\mathcal{S}) = \mathcal{A}$),

then \mathcal{S} is a separating class of subsets of (Ω, \mathcal{A}) .

PROOF Let μ and ν be two probability measures defined on (Ω, \mathcal{A}) such that $\mu(S) = \nu(S)$ for each $S \in \mathcal{S}$. We need to show that $\mu(A) = \nu(A)$ for each $A \in \mathcal{A}$. To this end, let

$$\mathcal{L} := \{ A \in \mathcal{A} = \sigma(\mathcal{S}) \mid \mu(A) = \nu(A) \}.$$

Note that $\mathcal{S} \subset \mathcal{L}$, by the hypothesis that μ and ν agree on \mathcal{S} , and $\mathcal{L} \neq \emptyset$ since $\Omega \in \mathcal{L}$. By Corollary B.8, it suffices to establish that \mathcal{L} is a λ -system, since then it will follow that

$$\mathcal{A} = \sigma(\mathcal{S}) \subset \mathcal{L} := \{ A \in \mathcal{A} = \sigma(\mathcal{S}) \mid \mu(A) = \nu(A) \} \subset \sigma(\mathcal{S}) = \mathcal{A},$$

i.e., $\mathcal{A} = \sigma(\mathcal{S}) = \mathcal{L}$, or equivalently, μ and ν agree on all of $\mathcal{A} = \sigma(\mathcal{S})$. Now, we have already noted that $\Omega \in \mathcal{L}$. For $A \in \mathcal{L}$, we have

$$\mu(\Omega \setminus A) = 1 - \mu(A) = 1 - \nu(A) = \nu(\Omega \setminus A),$$

hence $\Omega \setminus A \in \mathcal{L}$. Thus, \mathcal{L} is closed under complementations. Lastly, let $A_1, A_2, \dots \in \mathcal{L}$ be pairwise disjoint. Then,

$$\mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \nu(A_n) = \nu\left(\bigsqcup_{n=1}^{\infty} A_n\right),$$

thus $\bigsqcup_{n=1}^{\infty} A_n \in \mathcal{L}$, which proves that \mathcal{L} is closed under countable disjoint unions. \mathcal{L} is therefore indeed a λ -system and the proof of the Theorem is complete. \square

Corollary 1.4 Suppose S is a topological space and $\mathcal{B}(S)$ is its Borel σ -algebra (i.e. the σ -algebra generated by the collection of open subsets of S). Then, the collection of open subsets of S is a separating class of subsets of the measurable space $(S, \mathcal{B}(S))$.

2 Examples of separating and convergence-determining classes of \mathbb{R}^∞

Definition 2.1 (The metric on \mathbb{R}^∞ , Example 1.2, [1])

Let \mathbb{R}^∞ denotes the set of all infinite sequences of real numbers, i.e.

$$\mathbb{R}^\infty := \{ (x_1, x_2, \dots) \mid x_i \in \mathbb{R}, \text{ for each } i \in \mathbb{N} \}.$$

Define $\rho : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow [0, 1]$ as follows:

$$\rho(x, y) := \sum_{n=1}^{\infty} \frac{\min\{1, |x_n - y_n|\}}{2^n}.$$

Remark 2.2 Recall that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2} \cdot \left(\frac{1}{1 - \frac{1}{2}} \right) = 1,$$

which proves indeed that $0 \leq \rho(x, y) \leq 1$, for any $x, y \in \mathbb{R}^\infty$.

Theorem 2.3 (The metric space properties of \mathbb{R}^∞)

- (i) $(\mathbb{R}^\infty, \rho)$ is a metric space. Let \mathbb{R}^∞ denote also this metric space in the remainder of this Theorem.
- (ii) For $x, x^{(1)}, x^{(2)}, x^{(3)}, \dots \in \mathbb{R}^\infty$, we have:

$$\rho(x^{(n)}, x) \longrightarrow 0 \iff \text{for each } i \in \mathbb{N}, \lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0$$

- (iii) For each $n \in \mathbb{N}$, the “natural projection to the initial segment of length n ”

$$\pi_n : \mathbb{R}^\infty \longrightarrow \mathbb{R}^n : x \longmapsto (x_1, x_2, \dots, x_n)$$

is continuous, where \mathbb{R}^n has the usual Euclidean topology.

- (iv) For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, let $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)$ denote the open hypercube in \mathbb{R}^n of side length 2ε centred at $\pi_n(x) \in \mathbb{R}^n$, i.e.

$$C_{\mathbb{R}^n}(\pi_n(x), \varepsilon) := \left\{ y \in \mathbb{R}^n \mid \begin{array}{l} |y_i - x_i| < \varepsilon, \\ i = 1, 2, \dots, n \end{array} \right\}$$

Then, its pre-image in \mathbb{R}^∞ under π_n

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) = \left\{ y \in \mathbb{R}^\infty \mid \begin{array}{l} |y_i - x_i| < \varepsilon, \\ i = 1, 2, \dots, n \end{array} \right\}$$

is an open subset of \mathbb{R}^∞ .

- (v) For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, we have:

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right),$$

where $B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right)$ is the open ball in \mathbb{R}^∞ centred at x of radius $\varepsilon + \frac{1}{2^n}$, i.e.

$$B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right) := \left\{ y \in \mathbb{R}^\infty \mid \rho(y, x) < \varepsilon + \frac{1}{2^n} \right\}$$

(vi) *The collection*

$$\left\{ \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset \mathbb{R}^\infty \mid n \in \mathbb{N}, x \in \mathbb{R}^\infty, \varepsilon > 0 \right\}$$

of all pre-images under π_n of open hypercubes in \mathbb{R}^n , for all $n \in \mathbb{N}$, forms a basis for the topology of \mathbb{R}^∞ .

(vii) \mathbb{R}^∞ is a separable metric space.

(viii) \mathbb{R}^∞ is a complete metric space.

PROOF

(i) Clearly, ρ is non-negative and symmetric. We now show that, for any $x, y \in \mathbb{R}^\infty$, we have $\rho(x, y) = 0$ implies $x = y$. Indeed,

$$\begin{aligned} \rho(x, y) = 0 &\iff \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - y_i|\}}{2^i} = 0 \\ &\iff \min\{1, |x_i - y_i|\} = 0, \text{ for each } i \in \mathbb{N} \\ &\iff |x_i - y_i| = 0, \text{ for each } i \in \mathbb{N} \\ &\iff x = y. \end{aligned}$$

In order to show that ρ is a metric, it remains only to establish the Triangle Inequality. By Lemma A.2, for any $x, y, z \in \mathbb{R}^\infty$, we have

$$\begin{aligned} \rho(x, y) &= \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - y_i|\}}{2^i} \\ &\leq \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - z_i|\} + \min\{1, |z_i - y_i|\}}{2^i} \\ &= \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - z_i|\}}{2^i} + \sum_{i=1}^{\infty} \frac{\min\{1, |z_i - y_i|\}}{2^i} \\ &= \rho(x, z) + \rho(z, y), \end{aligned}$$

where we have used the fact that $0 \leq \rho \leq 1$ to split the infinite sum into two terms in second-to-last equality. This proves that ρ satisfies the Triangle Inequality, and it is thus a metric on \mathbb{R}^∞ .

(ii) $\lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = 0 \implies \lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0$, for each $i \in \mathbb{N}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = 0 &\implies \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} = 0 \\ &\implies \lim_{n \rightarrow \infty} \min\{1, |x_i^{(n)} - x_i|\} = 0, \text{ for each } i \in \mathbb{N} \\ &\implies \lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0, \text{ for each } i \in \mathbb{N} \end{aligned}$$

$$\lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0, \text{ for each } i \in \mathbb{N} \implies \lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = 0$$

This follows from the Weierstrass M -test. Suppose $\lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0$, for each $i \in \mathbb{N}$. Then,

$$\lim_{n \rightarrow \infty} \frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} = 0 =: y_i, \text{ for each } i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$, let $M_i := \frac{1}{2^i}$. Then,

$$\frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} \leq M_i \quad \text{and} \quad \sum_{i=1}^{\infty} M_i < \infty.$$

Hence, by the Weierstrass M -test (Lemma A.3), we have

$$\lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} = \sum_{i=1}^{\infty} y_i = 0.$$

(iii) Immediate by (ii).

(iv) Since $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon) \subset \mathbb{R}^n$ is an open subset of \mathbb{R}^n , its pre-image under the continuous (by (iii)) map $\pi_n : \mathbb{R}^\infty \rightarrow \mathbb{R}^n$ is an open subset of \mathbb{R}^∞ .

(v) For $y \in \mathbb{R}^\infty$, we have

$$\begin{aligned} y \in \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) &\implies |y_i - x_i| < \varepsilon, \text{ for each } i = 1, 2, \dots, n \\ &\implies \rho(x, y) := \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - y_i|\}}{2^i} \leq \sum_{i=1}^n \frac{\varepsilon}{2^i} + \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \varepsilon + \frac{1}{2^n}. \end{aligned}$$

This proves:

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right).$$

(vi) It suffices to show that every open ball in $B_{\mathbb{R}^\infty}(x, r) \subset \mathbb{R}^\infty$, $r > 0$, contains the pre-image of an open hypercube centred at $\pi_n(x) \in \mathbb{R}^n$ under π_n . To this end, for $r > 0$, choose $\varepsilon > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large such that $\varepsilon + \frac{1}{2^n} < r$. Then, for any $x \in \mathbb{R}^\infty$, by (v), we have:

$$x \in \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right) \subset B_{\mathbb{R}^\infty}(x, r),$$

as required.

(vii) It suffices to exhibit a countable subset of \mathbb{R}^∞ that intersects every open ball in \mathbb{R}^∞ . To this end, let

$$D := \bigcup_{n=1}^{\infty} \left\{ x = (x_1, x_2, \dots) \in \mathbb{R}^\infty \mid \begin{array}{l} x_i \in \mathbb{Q}, \text{ for each } i \in \mathbb{N} \\ x_i = 0, \text{ for all } i \geq n \end{array} \right\}.$$

Clearly, D is a countable subset of \mathbb{R}^∞ . Now let $B_{\mathbb{R}^\infty}(x, \varepsilon)$ be an arbitrary open ball in \mathbb{R}^∞ . Choose $\delta > 0$ small enough and $n \in \mathbb{N}$ large enough such that $\delta + \frac{1}{2^n} < \varepsilon$. Then,

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \delta)) \subset B_{\mathbb{R}^\infty}\left(x, \delta + \frac{1}{2^n}\right) \subset B_{\mathbb{R}^\infty}(x, \varepsilon),$$

Now, for each $i = 1, 2, \dots, n$, choose $z_i \in \mathbb{Q} \cap (x_i - \delta, x_i + \delta)$. Let $z = (z_1, z_2, \dots, z_n, 0, 0, \dots) \in \mathbb{R}^\infty$. Then, we have

$$z \in D \cap \left\{ y \in \mathbb{R}^\infty \mid \begin{array}{l} y_i \in (x_i - \delta, x_i + \delta), \\ \text{for each } i = 1, 2, \dots, n \end{array} \right\} = D \cap \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \delta)) \subset D \cap B_{\mathbb{R}^\infty}(x, \varepsilon).$$

This proves the countable subset $D \subset \mathbb{R}^\infty$ has non-empty intersection with every open ball in \mathbb{R}^∞ , i.e. D is dense in \mathbb{R}^∞ . Hence, \mathbb{R}^∞ is separable.

(viii) We need to show that every Cauchy sequence in \mathbb{R}^∞ converges to any element in \mathbb{R}^∞ .

$$\begin{aligned}
 & \left\{ x^{(n)} \right\}_{n \in \mathbb{N}} \subset \mathbb{R}^\infty \text{ is a Cauchy sequence in } \mathbb{R}^\infty \\
 \iff & \text{ for each } \varepsilon > 0, \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that } \rho(x^{(m)}, x^{(n)}) < \varepsilon, \text{ for any } m, n > N_\varepsilon \\
 \implies & \text{ for each } i \in \mathbb{N}, \text{ we have:} \\
 & \text{ for each } \varepsilon > 0, \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that } |x_i^{(m)} - x_i^{(n)}| < \varepsilon, \text{ for any } m, n > N_\varepsilon \\
 \implies & \text{ for each } i \in \mathbb{N}, \left\{ x_i^{(n)} \right\}_{n \in \mathbb{N}} \subset \mathbb{R} \text{ is a Cauchy sequence in } \mathbb{R}; \text{ hence } x_i := \lim_{n \rightarrow \infty} x_i^{(n)} \in \mathbb{R} \text{ exists} \\
 \implies & \lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = 0, \text{ where } x := (x_1, x_2, \dots) \in \mathbb{R}^\infty \quad (\text{by (ii)})
 \end{aligned}$$

This proves that \mathbb{R}^∞ indeed is a complete metric space.

□

Definition 2.4

The **finite-dimensional class** of subsets of \mathbb{R}^∞ is, by definition, the following:

$$\mathcal{B}_f(\mathbb{R}^\infty) := \left\{ \pi_k^{-1}(B) \subset \mathbb{R}^\infty \mid \begin{array}{l} k \in \mathbb{N} \\ B \in \mathcal{B}(\mathbb{R}^k) \end{array} \right\},$$

where $\pi_k : \mathbb{R}^\infty \longrightarrow \mathbb{R}^k : x = (x_1, x_2, \dots) \longmapsto (x_1, \dots, x_k)$ is the projection of \mathbb{R}^∞ onto \mathbb{R}^k .

Theorem 2.5

- (i) $\mathcal{B}_f(\mathbb{R}^\infty) \subset \mathcal{B}(\mathbb{R}^\infty)$.
- (ii) $\mathcal{B}_f(\mathbb{R}^\infty)$ is a separating class of Borel subsets of \mathbb{R}^∞ .
- (iii) $\mathcal{B}_f(\mathbb{R}^\infty)$ is a convergence-determining class of Borel subsets of \mathbb{R}^∞ .

PROOF

- (i) Note that

$$\mathcal{B}_f(\mathbb{R}^\infty) := \left\{ \pi_k^{-1}(B) \subset \mathbb{R}^\infty \mid \begin{array}{l} k \in \mathbb{N} \\ B \in \mathcal{B}(\mathbb{R}^k) \end{array} \right\} = \bigcup_{k=1}^{\infty} \pi_k^{-1}(\mathcal{B}(\mathbb{R}^k)).$$

Thus, (i) is equivalent to the statement that each $\pi_k : \mathbb{R}^\infty \longrightarrow \mathbb{R}^k$ is Borel measurable. But each π_k is continuous, hence Borel measurable (Corollary B.12). This proves (i).

□

A Technical Lemmas

Lemma A.1 Define

$$\phi : [0, \infty) \longrightarrow [0, 1] : t \longmapsto \min\{1, t\}.$$

Then, ϕ satisfies:

$$\phi(s+t) \leq \phi(s) + \phi(t), \text{ for each } s, t \in [0, \infty).$$

PROOF For any $s, t \in [0, \infty)$, either $s + t \geq 1$ or $s + t < 1$. If $s + t \geq 1$, then

$$\phi(s + t) = \min\{1, s + t\} = 1 < 2 = 1 + 1 \leq \min\{1, s\} + \min\{1, t\} = \phi(s) + \phi(t),$$

hence, the required inequality holds. On the other hand, if $s + t < 1$, then we must also have $s < 1$ and $t < 1$ (since $s, t \geq 0$). Hence,

$$\phi(s + t) = \min\{1, s + t\} = s + t = \min\{1, s\} + \min\{1, t\} = \phi(s) + \phi(t),$$

thus, the required inequality also holds. □

Lemma A.2 For any $x, y, z \in \mathbb{R}$, we have:

$$\min\{1, |x - y|\} \leq \min\{1, |x - z|\} + \min\{1, |z - y|\}.$$

PROOF Observe that $|x - y| \leq |x - z| + |z - y|$ implies

$$\min\{1, |x - y|\} \leq |x - z| + |z - y|.$$

The above inequality, together with $\min\{1, |x - y|\} \leq 1$, thus in turn imply:

$$\min\{1, |x - y|\} \leq \min\{1, |x - z| + |z - y|\}.$$

By Lemma A.1, we therefore have:

$$\min\{1, |x - y|\} \leq \min\{1, |x - z| + |z - y|\} \leq \min\{1, |x - z|\} + \min\{1, |z - y|\},$$

which proves the present Lemma. □

Lemma A.3 (The Weierstrass M -test, Theorem A.28, [2])

Suppose that $\lim_{n \rightarrow \infty} x_i^{(n)} = x_i$, for each $i \in \mathbb{N}$, and that $|x_i^{(n)}| \leq M_i$, where $\sum_{i=1}^{\infty} M_i < \infty$. Then,

(i) $\sum_{i=1}^{\infty} x_i$ exists, and $\sum_{i=1}^{\infty} x_i^{(n)}$ exists for each $n \in \mathbb{N}$.

(ii) Furthermore,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

PROOF

(i) $\sum_{i=1}^{\infty} M_i < \infty$ and $|x_i^{(n)}| \leq M_i \implies$ the series $\sum_{i=1}^{\infty} x_i$ and $\sum_{i=1}^{\infty} x_i^{(n)}$, $n \in \mathbb{N}$, converge absolutely.

(ii) Let $\varepsilon > 0$ be given. Choose $K \in \mathbb{N}$ sufficiently large such that $\sum_{j=K+1}^{\infty} M_i < \frac{\varepsilon}{3}$. Next, choose $N \in \mathbb{N}$ sufficiently large such that

$$|x_i^{(n)} - x_i| < \frac{\varepsilon}{3K}, \text{ for any } n > N \text{ and } i = 1, 2, \dots, K.$$

Then, we have, for each $n > N$,

$$\begin{aligned} \left| \sum_{i=1}^{\infty} x_i^{(n)} - \sum_{i=1}^{\infty} x_i \right| &= \left| \sum_{i=1}^K (x_i^{(n)} - x_i) + \sum_{i=K+1}^{\infty} x_i^{(n)} - \sum_{i=K+1}^{\infty} x_i \right| \\ &\leq \sum_{i=1}^K |x_i^{(n)} - x_i| + \sum_{i=K+1}^{\infty} |x_i^{(n)}| + \sum_{i=K+1}^{\infty} |x_i| \\ &\leq K \cdot \frac{\varepsilon}{3K} + \sum_{i=K+1}^{\infty} M_i + \sum_{i=K+1}^{\infty} M_i \leq \frac{\varepsilon}{3} + 2 \cdot \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

□

B σ -algebras and λ -systems

Definition B.1

Suppose Ω is a non-empty set. A σ -algebra of subsets of Ω is a collection \mathcal{A} of subsets of Ω which satisfies the following conditions:

- $\Omega \in \mathcal{A}$.
- $\Omega \setminus A \in \mathcal{A}$, for every $A \in \mathcal{A}$.
- $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, whenever $A_1, A_2, \dots \in \mathcal{A}$

Definition B.2

Suppose Ω is a non-empty set. A λ -system of subsets of Ω is a collection \mathcal{L} of subsets of Ω which satisfies the following conditions:

- $\Omega \in \mathcal{L}$.
- $\Omega \setminus A \in \mathcal{L}$, for every $A \in \mathcal{L}$.
- $\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{L}$, whenever $A_1, A_2, \dots \in \mathcal{L}$ and $A_i \cap A_j = \emptyset$, for any $i, j \in \mathbb{N}$ with $i \neq j$.

Remark B.3 Clearly, every σ -algebra is also a λ -system.

Theorem B.4

Suppose Ω is a non-empty set and \mathcal{L} is a λ -system of subsets of Ω .

- (i) \mathcal{L} is closed under proper set-theoretic differences, i.e. $A, B \in \mathcal{L}$ and $A \subset B$ together imply $B \setminus A \in \mathcal{L}$.
- (ii) If \mathcal{L} is closed under finite intersections, then \mathcal{L} is a σ -algebra of subsets of Ω .

PROOF For each $X \subset \Omega$, write $\Omega \setminus X$ as X^c .

- (i) Suppose $A, B \in \mathcal{L}$ with $A \subset B$. Then, $B^c \cap A = \emptyset$. Hence, $B \setminus A = B \cap A^c = (B^c \cup A)^c = (B^c \sqcup A)^c \in \mathcal{L}$, since \mathcal{L} is closed under complementations and finite disjoint unions.
- (ii) Since \mathcal{L} is a λ -system, we immediately have $\Omega \in \mathcal{L}$, and hence $\Omega \setminus A \in \mathcal{L}$, for every $A \in \mathcal{L}$. It remains to show that \mathcal{L} closed under countable unions, i.e. for $A_1, A_2, \dots \in \mathcal{L}$, we need to show $\bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$. To this end, define:

$$\begin{aligned} B_1 &:= A_1 \\ B_2 &:= A_2 \cap A_1^c \\ B_3 &:= A_3 \cap A_1^c \cap A_2^c \\ &\vdots \\ B_n &:= A_n \cap A_1^c \cap A_2^c \cap \dots \cap A_{n-1}^c \end{aligned}$$

Being a λ -system, \mathcal{L} is closed under complementations. By hypothesis, \mathcal{L} is furthermore closed under finite intersections. We thus see that $B_n \in \mathcal{L}$, for each $n \in \mathbb{N}$. Note also that the B_n 's are pairwise disjoint, and

$$\bigcup_{i=1}^n A_i = \bigsqcup_{i=1}^n B_i, \quad \text{for each } n \in \mathbb{N}.$$

Hence,

$$\bigcup_{i=1}^{\infty} A_i = \bigsqcup_{i=1}^{\infty} B_i \in \mathcal{L},$$

since \mathcal{L} is closed under countable pairwise disjoint unions (\mathcal{L} being a λ -system). This proves that \mathcal{L} is a σ -algebra of subsets of Ω . □

Theorem B.5 *Let Ω be a non-empty set.*

- (i) *The intersection of a non-empty collection of σ -algebras of subsets of Ω is itself a σ -algebra of subsets of Ω .*
- (ii) *The intersection of a non-empty collection of λ -systems of subsets of Ω is itself a λ -system of subsets of Ω .*

PROOF

- (i) Suppose Γ is an (arbitrary) non-empty set, and, for each $\gamma \in \Gamma$, \mathcal{A}_γ is a σ -algebra of subsets of Ω . We need to prove that $\mathcal{A} := \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma$ is itself a σ -algebra of subsets of Ω .

$$\underline{\Omega \in \mathcal{A} := \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma}$$

Since, for each $\gamma \in \Gamma$, \mathcal{A}_γ is a σ -algebra of subsets of Ω , we have $\Omega \in \mathcal{A}_\gamma$. Thus, $\Omega \in \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma$.

$$\underline{A \in \mathcal{A} \implies \Omega \setminus A \in \mathcal{A}}$$

$$A \in \mathcal{A} := \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma \iff A \in \mathcal{A}_\gamma, \forall \gamma \in \Gamma \implies \Omega \setminus A \in \mathcal{A}_\gamma, \forall \gamma \in \Gamma \implies \Omega \setminus A \in \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma =: \mathcal{A}$$

$$\underline{A_1, A_2, \dots \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}}$$

$$\begin{aligned} A_1, A_2, \dots \in \mathcal{A} := \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma &\implies A_1, A_2, \dots \in \mathcal{A}_\gamma, \forall \gamma \in \Gamma \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_\gamma, \forall \gamma \in \Gamma \\ &\implies \bigcup_{i=1}^{\infty} A_i \in \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma =: \mathcal{A} \end{aligned}$$

- (ii) Suppose Γ is an (arbitrary) non-empty set, and, for each $\gamma \in \Gamma$, \mathcal{L}_γ is a λ -system of subsets of Ω . We need to prove that $\mathcal{L} := \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma$ is itself a λ -system of subsets of Ω .

$$\underline{\Omega \in \mathcal{L} := \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma}$$

Since, for each $\gamma \in \Gamma$, \mathcal{L}_γ is a λ -system of subsets of Ω , we have $\Omega \in \mathcal{L}_\gamma$. Thus, $\Omega \in \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma$.

$$\underline{A \in \mathcal{L} \implies \Omega \setminus A \in \mathcal{L}}$$

$$A \in \mathcal{L} := \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma \iff A \in \mathcal{L}_\gamma, \forall \gamma \in \Gamma \implies \Omega \setminus A \in \mathcal{L}_\gamma, \forall \gamma \in \Gamma \implies \Omega \setminus A \in \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma =: \mathcal{L}$$

$$\underline{A_1, A_2, \dots \in \mathcal{L} \text{ and } A_i \cap A_j \text{ whenever } i \neq j \implies \bigsqcup_{i=1}^{\infty} A_i \in \mathcal{L}}$$

$$\begin{aligned} A_1, A_2, \dots \in \mathcal{L} &:= \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma, \text{ and } A_i \cap A_j \text{ whenever } i \neq j \\ \implies A_1, A_2, \dots &\in \mathcal{L}_\gamma, \forall \gamma \in \Gamma, \text{ and } A_i \cap A_j \text{ whenever } i \neq j \\ \implies \bigsqcup_{i=1}^{\infty} A_i &\in \mathcal{L}_\gamma, \forall \gamma \in \Gamma \\ \implies \bigsqcup_{i=1}^{\infty} A_i &\in \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma =: \mathcal{L} \end{aligned}$$

□

Theorem B.6 Suppose Ω is a non-empty set, \mathcal{S} is non-empty collection of subsets of Ω . Denote the power set of Ω by $\mathcal{P}(\Omega)$. Define

$$\begin{aligned} \sigma(\mathcal{S}) &:= \bigcap_{\mathcal{A} \in \Sigma(\mathcal{S})} \mathcal{A}, \text{ where } \Sigma(\mathcal{S}) := \left\{ \mathcal{A} \subset \mathcal{P}(\Omega) \mid \begin{array}{l} \mathcal{A} \text{ is a } \sigma\text{-algebra of subsets of } \Omega, \\ \text{and } \mathcal{S} \subset \mathcal{A} \end{array} \right\}, \text{ and} \\ \lambda(\mathcal{S}) &:= \bigcap_{\mathcal{L} \in \Lambda(\mathcal{S})} \mathcal{L}, \text{ where } \Lambda(\mathcal{S}) := \left\{ \mathcal{L} \subset \mathcal{P}(\Omega) \mid \begin{array}{l} \mathcal{L} \text{ is a } \lambda\text{-system of subsets of } \Omega, \\ \text{and } \mathcal{S} \subset \mathcal{L} \end{array} \right\}. \end{aligned}$$

Then, $\sigma(\mathcal{S})$ is the unique smallest σ -algebra of subsets of Ω that contains $\mathcal{S} \subset \mathcal{P}(\Omega)$, and $\lambda(\mathcal{S})$ is the unique smallest λ -system of subsets of Ω that contains $\mathcal{S} \subset \mathcal{P}(\Omega)$. More precisely, we have

- $\mathcal{S} \subset \sigma(\mathcal{S})$, $\mathcal{S} \subset \lambda(\mathcal{S})$, and
- $\sigma(\mathcal{S})$ is a σ -algebra of subsets of Ω , and $\lambda(\mathcal{S})$ is a λ -system of subsets of Ω , and
- if $\mathcal{A} \subset \mathcal{P}(\Omega)$ is a σ -algebra and $\mathcal{S} \subset \mathcal{A}$, then $\sigma(\mathcal{S}) \subset \mathcal{A}$.
- if $\mathcal{L} \subset \mathcal{P}(\Omega)$ is a λ -system and $\mathcal{S} \subset \mathcal{L}$, then $\lambda(\mathcal{S}) \subset \mathcal{L}$.

PROOF First, note that $\Sigma(\mathcal{S}) \neq \emptyset$ since $\mathcal{P}(\Omega) \in \Sigma(\mathcal{S})$. Similarly, $\Lambda(\mathcal{S}) \neq \emptyset$ since $\mathcal{P}(\Omega) \in \Lambda(\mathcal{S})$. It is immediate that $\mathcal{S} \subset \sigma(\mathcal{S})$, and $\sigma(\mathcal{S})$ is contained in every σ -algebra which contains \mathcal{S} . Similarly, $\mathcal{S} \subset \lambda(\mathcal{S})$, and $\lambda(\mathcal{S})$ is contained in every λ -system which contains \mathcal{S} . Since $\sigma(\mathcal{S})$ is, by definition, an intersection of σ -algebras, it itself is a σ -algebra of subsets of Ω by Theorem B.5. Similarly, since $\lambda(\mathcal{S})$ is, by definition, an intersection of λ -systems, it itself is a λ -system of subsets of Ω by Theorem B.5. □

Theorem B.7 Suppose Ω is a non-empty set and \mathcal{S} is a non-empty collection of subsets of Ω . Then,

$$\mathcal{S} \text{ is closed under finite intersections} \implies \lambda(\mathcal{S}) \text{ is a } \sigma\text{-algebra of subsets of } \Omega,$$

where $\lambda(\mathcal{S})$ is λ -system of subsets of Ω generated by \mathcal{S} .

PROOF By Theorem B.4(ii), it suffices to show that $\lambda(\mathcal{S})$ is closed under finite intersections. We establish the proof in the following series of claims:

Claim 1: For each $A \in \lambda(\mathcal{S})$,

$$\mathcal{L}(A) := \{ B \subset \Omega \mid A \cap B \in \lambda(\mathcal{S}) \}$$

is a λ -system of subsets of Ω .

Proof of Claim 1: Clearly, $\Omega \in \mathcal{L}(A)$, since $A \cap \Omega = A \in \lambda(\mathcal{S})$. Next, we prove that $\mathcal{L}(A)$ is closed under complementations. Let $B \in \mathcal{L}(A)$. Then, $A \cap B \in \lambda(\mathcal{S})$. Note that $A = (A \cap B) \sqcup (A \cap B^c)$, hence $A \cap B^c = A \setminus (A \cap B) \in \lambda(\mathcal{S})$, since $A, A \cap B \in \lambda(\mathcal{S})$ and $\lambda(\mathcal{S})$ is closed under proper set-theoretic differences by Theorem B.4(i). This proves that $\mathcal{L}(A)$ is indeed closed under complementations. We now prove that $\mathcal{L}(A)$ is closed under countable disjoint unions. Let $B_1, B_2, \dots \in \mathcal{L}(A)$ be pairwise disjoint. Then, $A \cap B_1, A \cap B_2, \dots \in \lambda(\mathcal{S})$ are pairwise disjoint. Hence,

$$A \cap \left(\bigsqcup_{i=1}^{\infty} B_i \right) = \bigsqcup_{i=1}^{\infty} (A \cap B_i) \in \lambda(\mathcal{S}),$$

since $\lambda(\mathcal{S})$ is closed under countable disjoint unions. This proves that $\mathcal{L}(A)$ is a λ -system and thus completes the proof of the Claim 1.

Claim 2: $\mathcal{S} \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$. Consequently, $\lambda(\mathcal{S}) \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$.

Proof of Claim 2: Suppose $A \in \mathcal{S}$. Then, $A \cap B \in \mathcal{S}$ for each $B \in \mathcal{S}$, **by the hypothesis that \mathcal{S} is closed under finite intersections**. Thus, $A \cap B \in \lambda(\mathcal{S})$, since $\mathcal{S} \subset \lambda(\mathcal{S})$. Hence, $B \in \mathcal{L}(A)$, for any $A, B \in \mathcal{S}$. This proves that $\mathcal{S} \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$. By Claim 1, $\mathcal{L}(A)$ is a λ -system. Hence, $\mathcal{L}(A) \supset \lambda(\mathcal{S})$, the smallest λ -system containing \mathcal{S} . This proves Claim 2.

Claim 3: $A \cap B \in \lambda(\mathcal{S})$, for each $A \in \mathcal{S}$ and $B \in \lambda(\mathcal{S})$.

Proof of Claim 3: Let $A \in \mathcal{S}$ and $B \in \lambda(\mathcal{S})$. By Claim 2, we have $\lambda(\mathcal{S}) \subset \mathcal{L}(A)$. Thus we have $B \in \mathcal{L}(A)$, which is equivalent to $A \cap B \in \lambda(\mathcal{S})$. This proves Claim 3.

Claim 4: $\mathcal{S} \subset \mathcal{L}(B)$, for each $B \in \lambda(\mathcal{S})$. Consequently, $\lambda(\mathcal{S}) \subset \mathcal{L}(B)$, for each $B \in \lambda(\mathcal{S})$.

Proof of Claim 4: Suppose $B \in \lambda(\mathcal{S})$. Then, $A \cap B \in \lambda(\mathcal{S})$ for each $A \in \mathcal{S}$, by Claim 3. This proves that $\mathcal{S} \subset \mathcal{L}(B)$. By Claim 1, $\mathcal{L}(B)$ is a λ -system. Hence, $\mathcal{L}(B) \supset \lambda(\mathcal{S})$, the smallest λ -system containing \mathcal{S} . This proves Claim 4.

Claim 5: $A \cap B \in \lambda(\mathcal{S})$, for each $A, B \in \lambda(\mathcal{S})$.

Proof of Claim 5: Let $A, B \in \lambda(\mathcal{S})$. By Claim 4, we have $\lambda(\mathcal{S}) \subset \mathcal{L}(B)$. Thus we have $A \in \mathcal{L}(B)$, which is equivalent to $A \cap B \in \lambda(\mathcal{S})$. This proves Claim 5.

Claim 5 states precisely that $\lambda(\mathcal{S})$ is closed under finite intersections, and completes the proof. □

Corollary B.8 Suppose Ω is a non-empty set and \mathcal{S} is a non-empty collection of subsets of Ω . If \mathcal{S} is closed under finite intersections, then

- (i) $\sigma(\mathcal{S}) \subset \lambda(\mathcal{S})$, and

(ii) $\sigma(\mathcal{S}) \subset \mathcal{L}$, for any λ -system \mathcal{L} of subsets of Ω such that $\mathcal{S} \subset \mathcal{L}$,

where $\sigma(\mathcal{S})$ and $\lambda(\mathcal{S})$ are, respectively, the σ -algebra and λ -system of subsets of Ω generated by \mathcal{S} .

PROOF

(i) By Theorem B.6, $\lambda(\mathcal{S})$ is the smallest λ -system containing \mathcal{S} . Since \mathcal{S} is, by hypothesis, closed under finite intersections, $\lambda(\mathcal{S})$ is furthermore a σ -algebra, by Theorem B.7. Thus, by Theorem B.6 again, we have $\sigma(\mathcal{S}) \subset \lambda(\mathcal{S})$.

(ii) This is now immediate since

$$\sigma(\mathcal{S}) \subset \lambda(\mathcal{S}) \subset \mathcal{L},$$

where the first inclusion follows by (i), and the second inclusion follows by Theorem B.6.

□

Lemma B.9 (The pre-image of a σ -algebra is itself a σ -algebra.)

Suppose Ω is a non-empty set, (X, \mathcal{X}) is a measurable space, and $f : \Omega \rightarrow X$ is a map from Ω into X . Then,

$$f^{-1}(\mathcal{X}) := \{ f^{-1}(V) \subset \Omega \mid V \in \mathcal{X} \}$$

is a σ -algebra of subsets of Ω .

PROOF

$$\underline{\Omega \in f^{-1}(\mathcal{X})} \quad f(\Omega) \subset X \implies \Omega = f^{-1}(X) \in f^{-1}(\mathcal{X}).$$

$f^{-1}(\mathcal{X})$ is closed under complementations Let $V \in \mathcal{X}$. Then, $X \setminus V \in \mathcal{X}$, and

$$\Omega \setminus f^{-1}(V) = \{ \omega \in \Omega \mid f(\omega) \notin V \} = \{ \omega \in \Omega \mid f(\omega) \in X \setminus V \} = f^{-1}(X \setminus V) \in f^{-1}(\mathcal{X}),$$

which shows that $f^{-1}(\mathcal{X})$ is indeed closed under complementations.

$f^{-1}(\mathcal{X})$ is closed countable unions Let $V_1, V_2, \dots \in \mathcal{X}$. Then, $\bigcup_{i=1}^{\infty} V_i \in \mathcal{X}$, and

$$\bigcup_{i=1}^{\infty} f^{-1}(V_i) = \left\{ \omega \in \Omega \mid \begin{array}{l} f(\omega) \in V_i \\ \text{for some } i \in \mathbb{N} \end{array} \right\} = f^{-1}\left(\bigcup_{i=1}^{\infty} V_i\right) \in f^{-1}(\mathcal{X}),$$

which proves that $f^{-1}(\mathcal{X})$ is indeed closed under countable unions.

This concludes the proof that that $f^{-1}(\mathcal{X})$ is a σ -algebra of subsets of Ω .

□

Lemma B.10

Suppose (Ω, \mathcal{A}) is a measurable space, X is a non-empty set, and $f : \Omega \rightarrow X$ is a map from Ω into X . Then,

$$\mathcal{F} := \{ V \subset X \mid f^{-1}(V) \in \mathcal{A} \}$$

is a σ -algebra of subsets of X .

PROOF

$$\underline{X \in \mathcal{F}} \quad f^{-1}(X) = \Omega \in \mathcal{A} \implies X \in \mathcal{F}.$$

\mathcal{F} is closed under complementations $V \in \mathcal{F} \implies f^{-1}(V) \in \mathcal{A} \implies f^{-1}(X \setminus V) = \Omega \setminus f^{-1}(V) \in \mathcal{A} \implies X \setminus V \in \mathcal{F}$, which proves that \mathcal{F} is indeed closed under complementations.

\mathcal{F} is closed under countable unions

$$\begin{aligned} V_1, V_2, \dots \in \mathcal{F} &\implies f^{-1}(V_1), f^{-1}(V_2), \dots \in \mathcal{A} \\ &\implies f^{-1}\left(\bigcup_{i=1}^{\infty} V_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(V_i) \in \mathcal{A} \\ &\implies \bigcup_{i=1}^{\infty} V_i \in \mathcal{F}, \end{aligned}$$

which proves that \mathcal{F} is indeed closed under countable unions. □

Theorem B.11

Suppose (Ω, \mathcal{A}) and (X, \mathcal{X}) are measurable spaces, and $f : \Omega \longrightarrow X$ is a map from Ω into X . Then, f is $(\mathcal{A}, \mathcal{X})$ -measurable if there exists $\mathcal{S} \subset \mathcal{X}$ satisfying the following conditions:

- \mathcal{S} generates \mathcal{X} , i.e. $\sigma(\mathcal{S}) = \mathcal{X}$, and
- $f^{-1}(\mathcal{S}) \subset \mathcal{A}$.

PROOF By Lemma B.10,

$$\mathcal{F} := \{ V \subset X \mid f^{-1}(V) \in \mathcal{A} \}$$

is a σ -algebra of subsets of X . By hypothesis, $\mathcal{S} \subset \mathcal{F}$; hence, $\mathcal{X} = \sigma(\mathcal{S}) \subset \mathcal{F}$. Thus, $f^{-1}(\mathcal{X}) \subset \mathcal{A}$; equivalently, f is $(\mathcal{A}, \mathcal{X})$ -measurable. □

Corollary B.12 (Continuous maps are Borel measurable.)

Suppose X_1, X_2 are topological spaces, and $\mathcal{B}_1, \mathcal{B}_2$ are their respective Borel σ -algebras. Then, every continuous map $f : X_1 \longrightarrow X_2$ is $(\mathcal{B}_1, \mathcal{B}_2)$ -measurable.

References

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