

1 Chapter 1

Exercise 1.1(a)

Let X be the sum of the two number obtained.

Let X_1 be the number obtained on Die 1.

Let X_2 be the number obtained on Die 2.

Thus, $X = X_1 + X_2$, and

$$E_x = \{X = x\} = \{X_1 + X_2 = x\} = \{X_1 = x_1, X_2 = x - x_1 \mid 1 \leq x_1, x - x_1 \leq 6\}$$

Now,

$$1 \leq x - x_1 \leq 6 \iff -1 \geq x_1 - x \geq -6 \iff x - 1 \geq x_1 \geq x - 6 \iff x - 6 \leq x_1 \leq x - 1$$

Hence,

$$E_x = \{X = x\} = \{X_1 + X_2 = x\} = \{X_1 = x_1, X_2 = x - x_1 \mid \max\{1, x - 6\} \leq x_1 \leq \min\{6, x - 1\}\}$$

$$\begin{aligned} P(E_x) &= \sum_{x_1=\max\{1, x-6\}}^{\min\{6, x-1\}} P(X_1 = x_1, X_2 = x - x_1) = \sum_{x_1=\max\{1, x-6\}}^{\min\{6, x-1\}} \frac{1}{6^2} \\ &= \frac{1}{6^2} (\min\{6, x - 1\} - \max\{1, x - 6\} + 1) \end{aligned}$$

Next, note that

$$\min\{6, x - 1\} = \begin{cases} x - 1, & \text{if } x = 2, 3, \dots, 6 \\ 6, & \text{if } x = 7, 8, \dots, 12 \end{cases} \quad \text{and} \quad \max\{1, x - 6\} = \begin{cases} 1, & \text{if } x = 2, 3, \dots, 6 \\ x - 6, & \text{if } x = 7, 8, \dots, 12 \end{cases}$$

Hence,

$$\begin{aligned} P(E_x) &= \frac{1}{6^2} (\min\{6, x - 1\} - \max\{1, x - 6\} + 1) = \frac{1}{36} \begin{cases} (x - 1) - 1 + 1, & \text{if } x = 2, 3, \dots, 6 \\ 6 - (x - 6) + 1, & \text{if } x = 7, 8, \dots, 12 \end{cases} \\ &= \frac{1}{36} \begin{cases} x - 1, & \text{if } x = 2, 3, \dots, 6 \\ 13 - x, & \text{if } x = 7, 8, \dots, 12 \end{cases} \end{aligned}$$

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Exercise 1.18

Recapitulation of the rules of craps: Let x be the number obtained on the first roll. If $x \in \{7, 11\}$, then the player wins. If $x \in \{2, 3, 12\}$, then the player loses. If $x \in \{4, 5, 6, 8, 9, 10\}$, then the player keeps rolling, until either 7 is rolled or x is rolled. If x is rolled first (before 7 is rolled), then the player wins. If 7 is rolled first (before x is rolled), then the player loses.

Let W be the $\{0, 1\}$ -valued random variable such that $W = 1$ if the player wins, and $W = 0$ if the player loses. We thus seek to compute $P(W = 1)$.

Let X be (the random variable of) the sum of the two numbers obtained on the first roll. Note that $\text{Range}(X) = \{2, 3, 4, \dots, 12\}$. Then,

$$\begin{aligned} P(W = 1) &= \sum_{x=2}^{12} P(W = 1 \mid X = x) \cdot P(X = x) \\ &= P(W = 1 \mid X = 7) P(X = 7) + P(W = 1 \mid X = 11) P(X = 11) + \sum_{x \in \{4, 5, 6, 8, 9, 10\}} P(W = 1 \mid X = x) \cdot P(X = x) \end{aligned}$$

Exercises and Solutions in Biostatistical Theory

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Now, note that $P(W = 1|X = 7) = P(W = 1|X = 11) = 1$, $P(X = 7) = \frac{6}{36} = \frac{1}{6}$, and $P(X = 11) = \frac{2}{36} = \frac{1}{18}$.

From Exercise 1.1(a), we have:

$$\begin{aligned} P(X = x) &= \frac{1}{6^2} (\min\{6, x-1\} - \max\{1, x-6\} + 1) = \frac{1}{36} \begin{cases} (x-1) - 1 + 1, & \text{if } x = 2, 3, \dots, 6 \\ 6 - (x-6) + 1, & \text{if } x = 7, 8, \dots, 12 \end{cases} \\ &= \frac{1}{36} \begin{cases} x-1, & \text{if } x = 2, 3, \dots, 6 \\ 13-x, & \text{if } x = 7, 8, \dots, 12 \end{cases} \end{aligned}$$

Next, let Y_n be the random variable of the sum of the two numbers obtained on the $(n+1)$ st roll. Then,

$$\begin{aligned} P(W = 1|X = x) &= \sum_{n=1}^{\infty} [1 - P(Y_n = 7) - P(Y_n = x)]^{n-1} \cdot P(X = x) \\ &= P(X = x) \cdot \sum_{n=1}^{\infty} [1 - P(Y_n = 7) - P(Y_n = x)]^{n-1} \\ &= P(X = x) \cdot \frac{1}{1 - [1 - P(Y = 7) - P(Y = x)]} \\ &= \frac{P(X = x)}{P(Y = 7) + P(Y = x)} \\ &= \frac{P(X = x)}{\frac{1}{6} + P(Y = x)} \end{aligned}$$

Hence,

$$\begin{aligned} P(W = 1) &= \sum_{x=2}^{12} P(W = 1|X = x) \cdot P(X = x) \\ &= P(W = 1|X = 7) P(X = 7) + P(W = 1|X = 11) P(X = 11) + \sum_{x \in \{4,5,6,8,9,10\}} P(W = 1|X = x) \cdot P(X = x) \\ &= \frac{6}{36} + \frac{2}{36} + \sum_{x \in \{4,5,6,8,9,10\}} \frac{P(X = x)^2}{\frac{1}{6} + P(Y = x)} \\ &= \frac{6}{36} + \frac{2}{36} + \frac{(\frac{4-1}{36})^2}{\frac{1}{6} + \frac{4-1}{36}} + \frac{(\frac{5-1}{36})^2}{\frac{1}{6} + \frac{5-1}{36}} + \frac{(\frac{6-1}{36})^2}{\frac{1}{6} + \frac{6-1}{36}} + \frac{(\frac{13-8}{36})^2}{\frac{1}{6} + \frac{13-8}{36}} + \frac{(\frac{13-9}{36})^2}{\frac{1}{6} + \frac{13-9}{36}} + \frac{(\frac{13-10}{36})^2}{\frac{1}{6} + \frac{13-10}{36}} \\ &= \frac{6}{36} + \frac{2}{36} + \frac{(1/36)^2}{1/36} \left(\frac{3^2}{6+3} + \frac{4^2}{6+4} + \frac{5^2}{6+5} + \frac{5^2}{6+5} + \frac{4^2}{6+4} + \frac{3^2}{6+3} \right) \\ &= \frac{6}{36} + \frac{2}{36} + \frac{2}{36} \left(\frac{3^2}{6+3} + \frac{4^2}{6+4} + \frac{5^2}{6+5} \right) = \frac{1}{36} \left[6 + 2 + 2 \left(\frac{9}{9} + \frac{16}{10} + \frac{25}{11} \right) \right] \\ &= \frac{1}{36} \left[8 + 2 \left(\frac{536}{110} \right) \right] = \frac{1}{36} \left[\frac{1952}{110} \right] = \frac{1}{2^2 \cdot 3^2} \left[\frac{2^5 \cdot 61}{2 \cdot 5 \cdot 11} \right] \\ &= \frac{2^2 \cdot 61}{3^2 \cdot 5 \cdot 11} \approx 0.4929293 \end{aligned}$$

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References