1 The population total, population mean, and population variance of a population characteristic

Let $n, N \in \mathbb{N}$, with $n \leq N$. Let $\mathcal{U} = \{ 1, 2, \ldots, N \}$, which represents the finite population, or universe, of N elements.

Definition 1.1 *A* population characteristic *is an $\mathbb{R}$-valued function $y : \mathcal{U} \longrightarrow \mathbb{R}$ defined on the population $\mathcal{U}$. We denote the value of y evaluated at $i \in \mathcal{U}$ by y_i. The population total, denoted by t, of y is defined:*

$$t \;:=\; \sum_{i=1}^{N} y_i \in \mathbb{R}\,.$$

The population mean*, denoted by $\overline{y}$, of y is defined by:*

$$\overline{y} \;:=\; \dfrac{1}{N}\sum_{i=1}^{N} y_i \in \mathbb{R}\,.$$

The population variance*, denoted by S^2, of y is defined by:*

$$S^2 \;:=\; \dfrac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \overline{y}\right)^2 \;=\; \dfrac{1}{N-1}\left\{\left(\sum_{i=1}^{N} y_i^2\right) - N\cdot\overline{y}^{\,2}\right\} \in \mathbb{R}\,.$$

In survey sampling, we seek to estimate population total t and population mean $\overline{y}$ of a population characteristic $y : \mathcal{U} \longrightarrow \mathbb{R}$ by making observations of values of y on only a (usually proper) subset of $\mathcal{U}$, and extrapolate from these observations. The subset on which observations of values of y are made is called a *sample*.

2 Simple Random Sampling (SRS)

Definition 2.1 *Let $\mathcal{U}$ be a nonempty finite set, $N := \#(\mathcal{U}) \in \mathbb{N}$, and let $n \in \{ 1, 2 \ldots, N \}$ be given. We define the probability space $\Omega_{\mathrm{SRS}}(\mathcal{U}, n)$ as follows: Let $\Omega(\mathcal{U}, n)$ be the set of all subsets of $\mathcal{U}$ with n elements, i.e.*

$$\Omega(\mathcal{U}, n) \;:=\; \{\, \omega \subset \mathcal{U} \mid \#(\omega) = n \,\}\,.$$

Note that $\#(\Omega(\mathcal{U}, n)) = \left(\begin{array}{c} N \\ n \end{array}\right)$. Let $\mathcal{P}(\Omega(\mathcal{U}, n))$ be the power set of $\Omega(\mathcal{U}, n)$. Define $\mu : \Omega \longrightarrow \mathbb{R}$ to be the "uniform" probability measure on the (finite) σ-algebra $\mathcal{P}(\Omega(\mathcal{U}, n))$ determined by:

$$\mu(\omega) \;=\; \dfrac{1}{\left(\begin{array}{c} N \\ n \end{array}\right)} \;=\; \dfrac{n!(N-n)!}{N!}\,, \quad \text{for each } \omega \in \Omega(\mathcal{U}, n).$$

Then, $\Omega_{\mathrm{SRS}}(\mathcal{U}, n)$ is defined to be the probability space $(\, \Omega(\mathcal{U}, n)\,,\, \mathcal{P}(\Omega(\mathcal{U}, n))\,,\, \mu\,)$.

Definition 2.2 *The* simple-random-sampling sample total $\widehat{t}_{\mathrm{SRS}}$ *of the population characteristic y is, by definition, the random variable $\widehat{t}_{\mathrm{SRS}} : \Omega_{\mathrm{SRS}}(\mathcal{U}, n) \longrightarrow \mathbb{R}$ defined by*

$$\widehat{t}_{\mathrm{SRS}}(\omega) \;:=\; \dfrac{N}{n}\sum_{i\in\omega} y_i\,, \quad \text{for each } \omega \in \Omega.$$

The simple-random-sampling sample mean $\widehat{\overline{y}}_{\mathrm{SRS}}$ *of the population characteristic y is, by definition, the random variable $\widehat{\overline{y}}_{\mathrm{SRS}} : \Omega_{\mathrm{SRS}}(\mathcal{U}, n) \longrightarrow \mathbb{R}$ defined by*

$$\widehat{\overline{y}}_{\mathrm{SRS}}(\omega) \;:=\; \dfrac{1}{n}\sum_{i\in\omega} y_i\,, \quad \text{for each } \omega \in \Omega.$$

The simple-random-sampling sample variance $\widehat{s^2}_{\mathrm{SRS}}$ *of the population characteristic* y *is, by definition, the random variable* $\widehat{s^2}_{\mathrm{SRS}} : \Omega_{\mathrm{SRS}}(\mathcal{U}, n) \longrightarrow \mathbb{R}$ *defined by*

$$\widehat{s^2}_{\mathrm{SRS}}(\omega) \;:=\; \frac{1}{n-1}\sum_{i\in\omega}\left(y_i - \widehat{\overline{y}}_{\mathrm{SRS}}(\omega)\right)^2, \quad \text{for each } \omega \in \Omega.$$

Proposition 2.3

1. $\widehat{\overline{y}}_{\mathrm{SRS}}$ *is an unbiased estimator of the population mean* $\overline{y}$*, and* $\mathrm{Var}\left[\,\widehat{\overline{y}}_{\mathrm{SRS}}\,\right] = \left(1 - \dfrac{n}{N}\right)\dfrac{S^2}{n}$.
2. $\widehat{t}_{\mathrm{SRS}}$ *is an unbiased estimator of the population total* t*, and* $\mathrm{Var}\left[\,\widehat{t}_{\mathrm{SRS}}\,\right] = N^2\left(1 - \dfrac{n}{N}\right)\dfrac{S^2}{n}$.
3. $\widehat{s^2}_{\mathrm{SRS}}$ *is an unbiased estimator of the population variance* S^2.
4. $\widehat{\mathrm{Var}}\left[\,\widehat{\overline{y}}_{\mathrm{SRS}}\,\right] := \left(1 - \dfrac{n}{N}\right)\dfrac{\widehat{s^2}_{\mathrm{SRS}}}{n}$ *is an unbiased estimator of* $\mathrm{Var}\left[\,\widehat{\overline{y}}_{\mathrm{SRS}}\,\right]$.
5. $\widehat{\mathrm{Var}}\left[\,\widehat{t}_{\mathrm{SRS}}\,\right] := N^2\left(1 - \dfrac{n}{N}\right)\dfrac{\widehat{s^2}_{\mathrm{SRS}}}{n}$ *is an unbiased estimator of* $\mathrm{Var}\left[\,\widehat{t}_{\mathrm{SRS}}\,\right]$.

A quote from Lohr [2], p.37: *Hájek [1] proves a central limit theorem for simple random sampling without replacement. In practical terms, Hájek's theorem says that if certain technical conditions hold, and if* n*,* N*, and* $N - n$ *are all "sufficiently large," then the sampling distribution of*

$$\frac{\widehat{\overline{y}}_{\mathrm{SRS}} - \overline{y}}{\sqrt{\left(1 - \dfrac{n}{N}\right)\dfrac{S^2}{n}}}$$

is "approximately" normal (Gaussian) with mean 0 *and variance* 1.

Corollary 2.4 (to Hájek's theorem) *For a simple random sampling procedure, an approximate* $(1-\alpha)$*-confidence interval,* $0 < \alpha < 1$*, for the population mean* $\overline{y}$ *is given by:*

$$\widehat{\overline{y}}_{\mathrm{SRS}} \,\pm\, z_{\alpha/2}\cdot\sqrt{\left(1 - \frac{n}{N}\right)\frac{S^2}{n}}$$

For sufficiently large samples, the above approximate confidence interval can itself be estimated from observations by:

$$\widehat{\overline{y}}_{\mathrm{SRS}} \pm \mathrm{SE}\left[\,\widehat{\overline{y}}_{\mathrm{SRS}}\,\right] \;=\; \widehat{\overline{y}}_{\mathrm{SRS}} \pm \sqrt{\left(1 - \frac{n}{N}\right)\frac{\widehat{s^2}_{\mathrm{SRS}}}{n}}$$

where

$$\mathrm{SE}\left[\,\widehat{\overline{y}}_{\mathrm{SRS}}\,\right] \;:=\; \sqrt{\widehat{\mathrm{Var}}\left[\,\widehat{\overline{y}}_{\mathrm{SRS}}\,\right]} \;=\; \sqrt{\left(1 - \frac{n}{N}\right)\frac{\widehat{s^2}_{\mathrm{SRS}}}{n}}$$

In order to prove Proposition 2.3, we introduce some auxiliary random variables:

Definition 2.5 *Let* $n, N \in \mathbb{N}$*, with* $n < N$*,* $\mathcal{U} := \{\, 1, 2, \ldots, N \,\}$*, and* $\Omega := \{\, \omega \subset \mathcal{U} \mid \#(\omega) = n \,\}$*. For each* $i \in \mathcal{U} = \{\, 1, 2, \ldots, N \,\}$*, we define the random variable* $Z_i : \Omega \longrightarrow \{0, 1\}$ *as follows:*

$$Z_i(\omega) \;=\; \begin{cases} 1, & \text{if } i \in \omega\,, \\ 0, & \text{if } i \notin \omega \end{cases}.$$

Immediate observations:

- $\widehat{t}_{\mathrm{SRS}} = \dfrac{N}{n}\sum_{i=1}^{N} Z_i\, y_i,$, as random variables on (Ω, P), i.e.

$$\widehat{t}_{\mathrm{SRS}}(\omega) \;=\; \frac{N}{n}\sum_{i=1}^{N} Z_i(\omega)\, y_i, \quad \text{for each } \;\omega\in\Omega.$$

- $\widehat{\overline{y}}_{\mathrm{SRS}} = \dfrac{1}{n}\sum_{i=1}^{N} Z_i\, y_i$, as random variables on (Ω, P), i.e.

$$\widehat{\overline{y}}_{\mathrm{SRS}}(\omega) \;=\; \frac{1}{n}\sum_{i=1}^{N} Z_i(\omega)\, y_i, \quad \text{for each } \;\omega\in\Omega.$$

- $E[\,Z_i\,] = \dfrac{n}{N}$. Indeed,

$$E[\,Z_i\,] = 1\cdot P(Z_i=1) + 0\cdot P(Z_i=0) = P(Z_i=1) = \frac{\text{number of samples containing } i}{\text{number of all possible samples}} = \frac{\left(\begin{array}{c} N-1 \\ n-1 \end{array}\right)}{\left(\begin{array}{c} N \\ n \end{array}\right)} = \frac{n}{N}$$

- $Z_i^2 = Z_i$, since $\mathrm{range}(Z_i) = \{\,0, 1\,\}$. Consequently,

$$E[\,Z_i^2\,] \;=\; E[\,Z_i\,] \;=\; \frac{n}{N}\,.$$

- $\mathrm{Var}[\,Z_i\,] = \dfrac{n}{N}\left(1-\dfrac{n}{N}\right)$. Indeed,

$$\begin{aligned}
\mathrm{Var}[\,Z_i\,] \;&:=\; E\left[\,(Z_i - E[\,Z_i\,])^2\,\right] \;=\; E\left[Z_i^2\right] - (E[\,Z_i\,])^2 \\
&=\; E[\,Z_i\,] - \left(\frac{n}{N}\right)^2 \;=\; \frac{n}{N} - \left(\frac{n}{N}\right)^2 \\
&=\; \frac{n}{N}\left(1-\frac{n}{N}\right).
\end{aligned}$$

- For $i \neq j$, we have $E[\,Z_i\cdot Z_j\,] = \left(\dfrac{n-1}{N-1}\right)\cdot\left(\dfrac{n}{N}\right)$. Indeed,

$$\begin{aligned}
E[\,Z_i\cdot Z_j\,] \;&=\; 1\cdot P(Z_i=1 \;\text{ and }\; Z_j=1) + 0\cdot P(Z_i=0 \;\text{ or }\; Z_j=0) \\
&=\; P(Z_i=1 \;\text{ and }\; Z_j=1) \;=\; P(Z_j=1|Z_i=1)\cdot P(Z_i=1) \\
&=\; \left(\frac{n-1}{N-1}\right)\cdot\left(\frac{n}{N}\right)
\end{aligned}$$

- For $i \neq j$, we have $\mathrm{Cov}\,(Z_i, Z_j) = -\dfrac{1}{N-1}\left(1-\dfrac{n}{N}\right)\left(\dfrac{n}{N}\right) \leq 0$. Indeed,

$$\begin{aligned}
\mathrm{Cov}\,(Z_i, Z_j) \;&:=\; E\,[\,(Z_i - E[\,Z_i\,])\cdot(Z_j - E[\,Z_j\,])\,] \;=\; E\,[\,Z_i\,Z_j\,] - E[\,Z_i\,]\cdot E[\,Z_j\,] \\
&=\; \left(\frac{n-1}{N-1}\right)\cdot\left(\frac{n}{N}\right) - \left(\frac{n}{N}\right)^2 \;=\; \frac{n}{N}\left(\frac{nN - N - nN + n}{N(N-1)}\right) \\
&=\; -\frac{1}{N-1}\left(1-\frac{n}{N}\right)\left(\frac{n}{N}\right)
\end{aligned}$$

Proof of Proposition 2.3

1.

$$E\left[\,\widehat{\overline{y}}_{\mathrm{SRS}}\,\right] \;=\; E\left[\,\frac{1}{n}\sum_{i=1}^{N} Z_i\,y_i\,\right] \;=\; \frac{1}{n}\sum_{i=1}^{N} E[\,Z_i\,]\cdot y_i \;=\; \frac{1}{n}\sum_{i=1}^{N}\left(\frac{n}{N}\right)\cdot y_i \;=\; \frac{1}{N}\sum_{i=1}^{N} y_i \;=:\; \overline{y}.$$

$$\begin{aligned}
\mathrm{Var}\left[\,\widehat{\overline{y}}_{\mathrm{SRS}}\,\right] &= \mathrm{Var}\left[\,\frac{1}{n}\sum_{i=1}^{N} Z_i\,y_i\,\right] \;=\; \frac{1}{n^2}\,\mathrm{Var}\left[\,\sum_{i=1}^{N} Z_i\,y_i\,\right] \;=\; \frac{1}{n^2}\,\mathrm{Cov}\left[\,\sum_{i=1}^{N} Z_i\,y_i\,,\;\sum_{j=1}^{N} Z_j\,y_j\,\right] \\
&= \frac{1}{n^2}\left\{\,\sum_{i=1}^{N} y_i^2\,\mathrm{Var}(Z_i) + \sum_{i=1}^{N}\sum_{i\neq j=1}^{N} y_i y_j\,\mathrm{Cov}(Z_i,Z_j)\,\right\} \\
&= \frac{1}{n^2}\left\{\,\sum_{i=1}^{N} y_i^2\,\frac{n}{N}\left(1-\frac{n}{N}\right) - \sum_{i=1}^{N}\sum_{i\neq j=1}^{N} y_i y_j\,\frac{1}{N-1}\left(1-\frac{n}{N}\right)\left(\frac{n}{N}\right)\,\right\} \\
&= \frac{1}{n^2}\,\frac{n}{N}\left(1-\frac{n}{N}\right)\left\{\,\sum_{i=1}^{N} y_i^2 - \frac{1}{N-1}\sum_{i=1}^{N}\sum_{i\neq j=1}^{N} y_i y_j\,\right\} \\
&= \frac{1}{n}\left(1-\frac{n}{N}\right)\frac{1}{N(N-1)}\left\{\,(N-1)\sum_{i=1}^{N} y_i^2 - \sum_{i=1}^{N}\sum_{i\neq j=1}^{N} y_i y_j\,\right\} \\
&= \frac{1}{n}\left(1-\frac{n}{N}\right)\frac{1}{N(N-1)}\left\{\,(N-1)\sum_{i=1}^{N} y_i^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} y_i y_j + \sum_{i=1}^{N} y_i^2\,\right\} \\
&= \frac{1}{n}\left(1-\frac{n}{N}\right)\frac{1}{N(N-1)}\left\{\,N\sum_{i=1}^{N} y_i^2 - \left(\sum_{i=1}^{N} y_i\right)\left(\sum_{j=1}^{N} y_j\right)\,\right\} \\
&= \frac{1}{n}\left(1-\frac{n}{N}\right)\frac{1}{N-1}\left\{\,\sum_{i=1}^{N} y_i^2 - N\left(\frac{1}{N}\sum_{i=1}^{N} y_i\right)^2\,\right\} \\
&= \frac{1}{n}\left(1-\frac{n}{N}\right)\frac{1}{N-1}\left\{\,\sum_{i=1}^{N} y_i^2 - N\cdot\overline{y}^2\,\right\} \\
&= \left(1-\frac{n}{N}\right)\frac{S^2}{n}
\end{aligned}$$

2.

$$E\left[\,\widehat{t}_{\mathrm{SRS}}\,\right] \;=\; E\left[\,N\cdot\widehat{\overline{y}}_{\mathrm{SRS}}\,\right] \;=\; N\cdot E\left[\,\widehat{\overline{y}}_{\mathrm{SRS}}\,\right] \;=\; N\cdot\overline{y} \;=\; N\cdot\left(\frac{1}{N}\sum_{i=1}^{N} y_i\right) \;=\; \sum_{i=1}^{N} y_i \;=:\; t.$$

$$\mathrm{Var}\left[\,\widehat{t}_{\mathrm{SRS}}\,\right] \;=\; \mathrm{Var}\left[\,N\cdot\widehat{\overline{y}}_{\mathrm{SRS}}\,\right] \;=\; N^2\cdot\mathrm{Var}\left[\,\widehat{\overline{y}}_{\mathrm{SRS}}\,\right] \;=\; N^2\left(1-\frac{n}{N}\right)\frac{S^2}{n}$$

3.

$$
\begin{aligned}
E\left[\,\widehat{s^2}_{\mathrm{SRS}}\,\right] &= E\left[\frac{1}{n-1}\sum_{i\in\omega}\left(y_i-\widehat{\overline{y}}_{\mathrm{SRS}}\right)^2\right] \;=\; \frac{1}{n-1}\,E\left[\sum_{i\in\omega}\left((y_i-\overline{y})-(\widehat{\overline{y}}_{\mathrm{SRS}}-\overline{y})\right)^2\right] \\
&= \frac{1}{n-1}\,E\left[\left(\sum_{i\in\omega}(y_i-\overline{y})^2\right)-n\left(\widehat{\overline{y}}_{\mathrm{SRS}}-\overline{y}\right)^2\right] \\
&= \frac{1}{n-1}\left\{E\left[\sum_{i=1}^{N}Z_i(y_i-\overline{y})^2\right]-n\,\mathrm{Var}\left[\,\widehat{\overline{y}}_{\mathrm{SRS}}\,\right]\right\} \\
&= \frac{1}{n-1}\left\{\sum_{i=1}^{N}E[\,Z_i\,]\,(y_i-\overline{y})^2-n\left(1-\frac{n}{N}\right)\frac{S^2}{n}\right\} \\
&= \frac{1}{n-1}\left\{\sum_{i=1}^{N}\frac{n}{N}(y_i-\overline{y})^2-\left(1-\frac{n}{N}\right)S^2\right\} \\
&= \frac{1}{n-1}\left\{\frac{n(N-1)}{N}\,\frac{1}{N-1}\sum_{i=1}^{N}(y_i-\overline{y})^2-\left(1-\frac{n}{N}\right)S^2\right\} \\
&= \frac{1}{n-1}\left\{\frac{n(N-1)}{N}-\left(1-\frac{n}{N}\right)\right\}S^2 \\
&= \frac{1}{n-1}\left\{\frac{nN-n-N+n}{N}\right\}S^2 \;=\; S^2
\end{aligned}
$$

4. Immediate from preceding statements.

5. Immediate from preceding statements. □

3 Stratified Simple Random Sampling

Let $\mathcal{U}=\{\,1,2,\ldots,N\,\}$ be the population, as before. Let

$$\mathcal{U} \;=\; \bigsqcup_{h=1}^{H}\mathcal{U}_h$$

be a partition of $\mathcal{U}$. Such a partition is called a *stratification* of the population $\mathcal{U}$. Each of $\mathcal{U}_1,\mathcal{U}_2,\ldots,\mathcal{U}_H$ is called a *stratum*. Let $N_h := \#(\mathcal{U}_h)$, for $h=1,2,\ldots,H$. Note that $N_1+N_2+\cdots+N_H=N$.

In *stratified simple random sampling*, an SRS is taken within each stratum $\mathcal{U}_h$, $h=1,2,\ldots,H$. Let n_h, $h=1,2,\ldots,H$, be the number elements in the simple random sample taken in the stratum $\mathcal{U}_h$. In other words, a stratified simple random sample ω of the stratified population $\mathcal{U}=\bigsqcup_{h=1}^{H}\mathcal{U}_h$ has the form:

$$\omega \;=\; \bigsqcup_{h=1}^{H}\omega_h\,, \quad \text{where}\;\; \omega_h\in\Omega_{\mathrm{SRS}}(\mathcal{U}_h,n_h)\,, \quad \text{for each}\;\; h=1,2,\ldots,h\,.$$

Note that $n_1+n_2+\cdots+n_H =: n = \#(\omega)$.

We now give unbiased estimators, and their variances, of the population total and population mean of a population characteristic under stratified simple random sampling. Let $y:\mathcal{U}\longrightarrow\mathbb{R}$ be a population characteristic. Define:

$$\widehat{t}_{\mathrm{Str}} \;:=\; \sum_{h=1}^{H}N_h\cdot\widehat{\overline{y}}_{h,\mathrm{SRS}}$$

$$\widehat{\overline{y}}_{\mathrm{Str}} \;:=\; \frac{1}{N}\cdot\widehat{t}_{\mathrm{Str}} \;=\; \sum_{h=1}^{H}\frac{N_h}{N}\cdot\widehat{\overline{y}}_{h,\mathrm{SRS}}$$

Here,

$$\widehat{\overline{y}}_{h,\mathrm{SRS}} \;:\; \Omega_{\mathrm{SRS}}(\mathcal{U}_h, n_h) \longrightarrow \mathbb{R} \;:\; \omega_h \;\longmapsto\; \frac{1}{n_h}\sum_{i\in\omega_h} y_i$$

is the SRS estimator of

$$\overline{y}_h \;:=\; \overline{y|_{\mathcal{U}_h}} \;=\; \frac{1}{N_h}\sum_{i\in\mathcal{U}_h} y_i \in \mathbb{R},$$

the "stratum mean" of the "stratum characteristic" $y|_{\mathcal{U}_h} : \mathcal{U}_h \longrightarrow \mathbb{R}$, the restriction of the population characteristic $y : \mathcal{U} \longrightarrow \mathbb{R}$ to the stratum $\mathcal{U}_h$. Then,

$$E\left[\,\widehat{t}_{\mathrm{Str}}\,\right] \;=\; t \;:=\; \sum_{i=1}^{N} y_i, \quad \text{and} \quad E\left[\,\widehat{\overline{y}}_{\mathrm{Str}}\,\right] \;=\; \overline{y} \;:=\; \frac{1}{N}\sum_{i=1}^{N} y_i.$$

In other words, $\widehat{t}_{\mathrm{Str}}$ and $\widehat{\overline{y}}_{\mathrm{Str}}$ are unbiased estimators of the population total t and population mean $\overline{y}$ of the population characteristic $y : \mathcal{U} \longrightarrow \mathbb{R}$, respectively. Indeed,

$$\begin{aligned}
E\left[\,\widehat{t}_{\mathrm{Str}}\,\right] \;&=\; E\left[\sum_{h=1}^{H} N_h \cdot \widehat{\overline{y}}_{h,\mathrm{SRS}}\right] \;=\; \sum_{h=1}^{H} N_h\, E\left[\,\widehat{\overline{y}}_{h,\mathrm{SRS}}\,\right] \;=\; \sum_{h=1}^{H} N_h\,\overline{y}_h \\
&=\; \sum_{h=1}^{H} N_h\left(\frac{1}{N_h}\sum_{i\in\mathcal{U}_h} y_i\right) \;=\; \sum_{h=1}^{H}\left(\sum_{i\in\mathcal{U}_h} y_i\right) \;=\; \sum_{i=1}^{N} y_i \;=:\; t\,.
\end{aligned}$$

And,

$$E\left[\,\widehat{\overline{y}}_{\mathrm{Str}}\,\right] \;=\; E\left[\frac{1}{N}\cdot\widehat{t}_{\mathrm{Str}}\right] \;=\; \frac{1}{N}\,E\left[\,\widehat{t}_{\mathrm{Str}}\,\right] \;=\; \frac{1}{N}\sum_{i=1}^{N} y_i \;=:\; \overline{y}\,.$$

Furthermore,

$$\mathrm{Var}\left[\,\widehat{t}_{\mathrm{Str}}\,\right] \;=\; \mathrm{Var}\left[\sum_{h=1}^{H} N_h \cdot \widehat{\overline{y}}_{h,\mathrm{SRS}}\right] \;=\; \sum_{h=1}^{H} N_h^2 \cdot \mathrm{Var}\left[\,\widehat{\overline{y}}_{h,\mathrm{SRS}}\,\right] \;=\; \sum_{h=1}^{H} N_h^2\left(1 - \frac{n_h}{N_h}\right)\frac{S_h^2}{n_h}.$$

$$\mathrm{Var}\left[\,\widehat{\overline{y}}_{\mathrm{Str}}\,\right] \;=\; \mathrm{Var}\left[\frac{1}{N}\cdot\widehat{t}_{\mathrm{Str}}\right] \;=\; \frac{1}{N^2}\cdot\mathrm{Var}\left[\,\widehat{t}_{\mathrm{Str}}\,\right] \;=\; \sum_{h=1}^{H}\left(\frac{N_h}{N}\right)^2\left(1 - \frac{n_h}{N_h}\right)\frac{S_h^2}{n_h}.$$

Comparing variances of SRS and stratified simple random sampling with proportional allocation via ANOVA (analysis of variance):

By definition, in stratified simple random sampling with proportional allocation, the stratum sample size n_h, for each $h = 1, 2, \ldots, H$, is chosen such that $n_h/N_h = n/N$. Consequently,

$$\begin{aligned}
\mathrm{Var}\left[\,\widehat{t}_{\mathrm{PropStr}}\,\right] \;&=\; \sum_{h=1}^{H} N_h^2\left(1 - \frac{n_h}{N_h}\right)\frac{S_h^2}{n_h} \;=\; \frac{N}{n}\left(1 - \frac{n}{N}\right)\sum_{h=1}^{H} N_h\,S_h^2 \\
&=\; \frac{N}{n}\left(1 - \frac{n}{N}\right)\left\{\,\sum_{h=1}^{H}(N_h - 1)\,S_h^2 + \sum_{h=1}^{H} S_h^2\,\right\} \\
&=\; \frac{N}{n}\left(1 - \frac{n}{N}\right)\left\{\,\mathrm{SSW} + \sum_{h=1}^{H} S_h^2\,\right\},
\end{aligned}$$

where

$$\mathrm{SSW} \;:=\; \sum_{h=1}^{H}\sum_{i\in\mathcal{U}_h}\left(y_i - \overline{y}_{\mathcal{U}_h}\right)^2 \;=\; \sum_{h=1}^{H}(N_h - 1)\,S_h^2.$$

is called the *inter-strata squared deviation* (or *within-strata squared deviation*), and

$$S_h^2 \;:=\; \frac{1}{N_h - 1}\sum_{i\in\mathcal{U}_h}\left(y_i - \overline{y}_{\mathcal{U}_h}\right)^2$$

is the stratum variance of the population characteristic $y:\mathcal{U}\longrightarrow\mathbb{R}$ over the stratum $\mathcal{U}_h$. The following relation between $\mathrm{Var}\big[\,\widehat{t}_{\mathrm{SRS}}\,\big]$ and $\mathrm{Var}\big[\,\widehat{t}_{\mathrm{PropStr}}\,\big]$ always holds (see [2], p.106):

$$\mathrm{Var}\big[\,\widehat{t}_{\mathrm{SRS}}\,\big] \;=\; \mathrm{Var}\big[\,\widehat{t}_{\mathrm{PropStr}}\,\big] + \left(1-\frac{n}{N}\right)\frac{N}{n}\,\frac{N}{N-1}\left\{\,\mathrm{SSB} - \sum_{h=1}^{H}\left(1-\frac{N_h}{N}\right)S_h^2\,\right\},$$

where

$$\mathrm{SSB} \;:=\; \sum_{h=1}^{H} N_h\left(\overline{y}_{\mathcal{U}_h} - \overline{y}_{\mathcal{U}}\right)^2 \;=\; \sum_{h=1}^{H}\sum_{i\in\mathcal{U}_h}\left(\overline{y}_{\mathcal{U}_h} - \overline{y}_{\mathcal{U}}\right)^2$$

is the *inter-strata squared deviation* (or *between-strata squared deviation*). It is also an easily established fact that the sum of the inter-strata squared deviation SSB and the intra-strata squared deviation SSW is always the total population squared deviation SSTO:

$$\mathrm{SSTO} \;:=\; \sum_{i=1}^{N}\left(y_i - \overline{y}_{\mathcal{U}}\right)^2 \;=\; \sum_{h=1}^{H}\sum_{i\in\mathcal{U}_h}\left(y_i - \overline{y}_{\mathcal{U}}\right)^2 \,.$$

Most importantly, we see from above that, for stratified simple random sampling with proportional allocation, the following implication holds:

$$\sum_{h=1}^{H}\left(1-\frac{N_h}{N}\right)S_h^2 \,\leq\, \mathrm{SSB} \quad\Longrightarrow\quad \mathrm{Var}\big[\,\widehat{t}_{\mathrm{PropStr}}\,\big] \leq \mathrm{Var}\big[\,\widehat{t}_{\mathrm{SRS}}\,\big]\,.$$

In heuristic terms, in proportional-allocation stratification for which each stratum is relatively homogeneous and the strata are relatively dissimilar to each other (intra-strata variation being smaller than inter-strata variation), then the unbiased estimator for the population total from the proportional-allocation stratified simple random sampling is more precise than that from SRS.

4 Two-stage Cluster Sampling

The universe $\mathcal{U} = \bigsqcup_{i=1}^{N}\mathcal{C}_i$ of observation units is partitioned into N *clusters* (or *primary sampling units*, psu's) $\mathcal{C}_i$. In two-stage cluster sampling, the *secondary sampling units* (or ssu's) are the observation units. Let M_i be the number of ssu's in the ith psu; in other words, $M_i := \#(\mathcal{C}_i)$.

First Stage: Select a simple random sample (SRS) $\omega_1 = \{\,\mathcal{C}_{i_1}, \mathcal{C}_{i_2}, \ldots, \mathcal{C}_{i_n}\,\}$ of n psu's from the collection of N psu's.

Second Stage: From each psu $\mathcal{C}\in\omega_1$ selected in the First Stage, select a simple random sample (SRS) $\omega_{\mathcal{C}}$ of m_i secondary sampling units (ssu's) from the collection of M_i ssu's in $\mathcal{C}$.

The sample is then $\omega := \bigsqcup_{\mathcal{C}\in\omega_1}\omega_{\mathcal{C}}$. In other words, the sample ω consists of all the secondary sampling units (or observation units) selected (during the Second Stage) from all the primary sampling units selected in the First Stage.

The Horvitz-Thompson estimator $\widehat{t}_{\mathrm{HT}}$, as defined below, is an unbiased estimator for the total of an $\mathbb{R}$-valued population characteristic $y:\mathcal{U}\longrightarrow\mathbb{R}$.

$$\widehat{t}_{\mathrm{HT}} \;\;:=\;\; \sum_{k\in\omega}\left(\frac{N}{n}\,\frac{M_{y_k}}{m_{y_k}}\right)y_k \;\;=\;\; \sum_{k\in\omega}\left(\frac{1}{\pi_k}\right)y_k \;\;=\;\; \sum_{\mathcal{C}\in\omega_1}\sum_{k\in\omega_{\mathcal{C}}}\left(\frac{N}{n}\,\frac{M_{y_k}}{m_{y_k}}\right)y_k,$$

where $M_{y_k} := M_i := \#(\mathcal{C}_i)$ and $m_{y_k} := m_i := \#(\omega_{\mathcal{C}_i})$ such that $\mathcal{C}_i$ is the unique psu containing the ssu $k \in \mathcal{U} = \bigsqcup_i^N \mathcal{C}_i$.

The variance of the Horvitz-Thompson estimator $\widehat{t}_{\mathrm{HT}}$ is given by:

$$\mathrm{Var}\big(\,\widehat{t}_{\mathrm{HT}}\,\big) \;=\; N^2\left(1-\frac{n}{N}\right)\frac{S_t^2}{n} \;+\; \sum_{i=1}^{N}\frac{N}{n}\cdot M_i^2\left(1-\frac{m_i}{M_i}\right)\frac{S_i^2}{m_i},$$

where

$$S_t^2 \;:=\; \frac{1}{N-1}\sum_{i=1}^{N}\left(t_i-\frac{t}{N}\right)^2, \quad S_i^2 \;:=\; \frac{1}{M_i-1}\sum_{j=1}^{M_i}\left(y_j-\frac{t_i}{M_i}\right)^2, \quad t \;:=\; \sum_{k\in\mathcal{U}} y_k\,, \quad \text{and} \quad t_i \;:=\; \sum_{k\in\mathcal{C}_i} y_k$$

IMPORTANT OBSERVATION: The first summand in the expression of $\mathrm{Var}\big(\,\widehat{t}_{\mathrm{HT}}\,\big)$ is due to variability in the First-Stage sampling, whereas the second summand is due to variability in the Second-Stage sampling.

5 One-stage Cluster Sampling

One-stage cluster sampling is a special form of two-stage cluster sampling in which all second-stage samples are censuses. In other words, following the notation introduced for two-stage cluster sampling, in one-stage cluster sampling, we have $\omega_{\mathcal{C}} = \mathcal{C}$, for each first-stage-selected $\mathcal{C} \in \omega_1$. This also implies $m_i = M_i$ for each $i = 1, 2, \ldots, N$.

Then, the Horvitz-Thompson estimator $\widehat{t}_{\mathrm{HT}}$ and its variance reduces to:

$$\begin{aligned}
\widehat{t}_{\mathrm{HT}} \;&:=\; \sum_{\mathcal{C}\in\omega_1}\sum_{k\in\mathcal{C}}\left(\frac{N}{n}\frac{M_{y_k}}{m_{y_k}}\right)y_k \;=\; \frac{N}{n}\cdot\sum_{\mathcal{C}\in\omega_1}\sum_{k\in\mathcal{C}} y_k \;=\; \frac{N}{n}\cdot\sum_{\mathcal{C}\in\omega_1} t_{\mathcal{C}}\,, \quad \text{where } t_{\mathcal{C}} := \sum_{k\in\mathcal{C}} y_k \\
\mathrm{Var}\big(\,\widehat{t}_{\mathrm{HT}}\,\big) \;&=\; N^2\left(1-\frac{n}{N}\right)\frac{S_t^2}{n} \;+\; \sum_{i=1}^{N}\frac{N}{n}\cdot M_i^2\left(1-\frac{m_i}{M_i}\right)\frac{S_i^2}{m_i} \\
&=\; N^2\left(1-\frac{n}{N}\right)\frac{S_t^2}{n} \;+\; \sum_{i=1}^{N}\frac{N}{n}\cdot M_i^2\,(1-1)\,\frac{S_i^2}{m_i} \;=\; N^2\left(1-\frac{n}{N}\right)\frac{S_t^2}{n}
\end{aligned}$$

6 Stratified Simple Random Sampling as a special case of Two-stage Cluster Sampling

Stratified simple random sampling is a special case of two-stage cluster sampling in which the first-stage sampling is a census. In other words, if $\mathcal{U} = \bigsqcup_{i=1}^{N} \mathcal{C}_i$, then $\omega_1 = \{\,\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_N\,\}$. In particular, $n = N$.

Then, the Horvitz-Thompson estimator $\widehat{t}_{\mathrm{HT}}$ and its variance reduces to:

$$\begin{aligned}
\widehat{t}_{\mathrm{HT}} \;&:=\; \sum_{\mathcal{C}\in\omega_1}\sum_{k\in\omega_{\mathcal{C}}}\left(\frac{N}{n}\frac{M_{y_k}}{m_{y_k}}\right)y_k \;=\; \sum_{i=1}^{N} M_i\left(\frac{1}{m_i}\sum_{k\in\omega_{\mathcal{C}_i}} y_k\right) \;=\; \sum_{i=1}^{N} M_i\,\overline{y}_{\omega_{\mathcal{C}_i}} \\
\mathrm{Var}\big(\,\widehat{t}_{\mathrm{HT}}\,\big) \;&=\; N^2\left(1-\frac{n}{N}\right)\frac{S_t^2}{n} \;+\; \sum_{i=1}^{N}\frac{N}{n}\cdot M_i^2\left(1-\frac{m_i}{M_i}\right)\frac{S_i^2}{m_i} \\
&=\; N^2\,(1-1)\,\frac{S_t^2}{n} \;+\; \sum_{i=1}^{N} 1\cdot M_i^2\left(1-\frac{m_i}{M_i}\right)\frac{S_i^2}{m_i} \;=\; \sum_{i=1}^{N} M_i^2\left(1-\frac{m_i}{M_i}\right)\frac{S_i^2}{m_i}
\end{aligned}$$

The above formula agree exactly with those derived earlier for stratified simple random sampling (apart from obvious notational changes).

7 General linear estimators for (multivariate) population totals

Let $U = \{1, 2, \ldots, N\}$ be a finite population. Let $\mathbf{y} : U \longrightarrow \mathbb{R}^m$ be an $\mathbb{R}^m$-valued function defined on U (commonly called a "population parameter"). We will use the common notation $\mathbf{y}_k$ for $\mathbf{y}(k)$. We wish to estimate $\mathbf{T}_{\mathbf{y}} := \sum_{k\in U} \mathbf{y}_k \in \mathbb{R}^m$ via survey sampling. Let $p : \mathcal{S} \longrightarrow (0, 1]$ be our chosen sampling design, where $\mathcal{S} \subseteq \mathcal{P}(U)$ is the set of all possible samples in the design, and $\mathcal{P}(U)$ is the power set of U.

Definition 7.1

A random variable $\widehat{\mathbf{T}}_{\mathbf{y}} : \mathcal{S} \longrightarrow \mathbb{R}^m$ is said to be linear *in the population parameter $\mathbf{y} : U \longrightarrow \mathbb{R}$ if it has the following form:*

$$\begin{array}{rccl} \widehat{\mathbf{T}}_{\mathbf{y}} : & \mathcal{S} & \longrightarrow & \mathbb{R}^m \\ & s & \longmapsto & \sum_{k\in s} w_k(s)\,\mathbf{y}_k \;=\; \sum_{k\in U} I_k(s)\,w_k(s)\,\mathbf{y}_k, \end{array}$$

where, for each $k \in U$, $w_k : \mathcal{S} \longrightarrow \mathbb{R}$ is itself an $\mathbb{R}$-valued random variable, and $I_k : \mathcal{S} \longrightarrow \{0, 1\}$ is the indicator random variable defined by:

$$I_k(s) \;=\; \begin{cases} 1, & \text{if } k \in s, \\ 0, & \text{otherwise} \end{cases}$$

We call the w_k's the weights *of $\widehat{\mathbf{T}}_{\mathbf{y}}$, and we use the notation $\widehat{\mathbf{T}}_{\mathbf{y};w}$ to indicate that the random variable depends on the weights w_k.*

Nomenclature In the context of finite-population probability sampling, under a design $p : \mathcal{S} \longrightarrow (0, 1]$, an "estimator" is precisely just a random variable defined on the space $\mathcal{S}$ of all admissible samples in the design.

Proposition 7.2

Let $\widehat{\mathbf{T}}_{\mathbf{y};w} : \mathcal{S} \longrightarrow \mathbb{R}^m$, with $\widehat{\mathbf{T}}_{\mathbf{y};w}(s) = \sum_{k\in U} I_k(s)\,w_k(s)\,\mathbf{y}_k = \sum_{k\in s} w_k(s)\,\mathbf{y}_k$, be a random variable linear in the population parameter $\mathbf{y} : U \longrightarrow \mathbb{R}$. Then,

$$E\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\Big] \;=\; \mathbf{T}_{\mathbf{y}},\ \text{for arbitrary } \mathbf{y} \quad\Longleftrightarrow\quad E[\,I_k\,w_k\,] \;=\; 1,\ \text{for each } k \in U.$$

PROOF Note:

$$E\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\Big] \;=\; E\left[\sum_{k\in s} w_k\,\mathbf{y}_k\right] \;=\; E\left[\sum_{k\in U} I_k\,w_k\,\mathbf{y}_k\right] \;=\; \sum_{k\in U} E[\,I_k\,w_k\,]\;\mathbf{y}_k$$

Hence, since $\mathbf{y} : U \longrightarrow \mathbb{R}$ is arbitrary,

$$E\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\Big] \;=\; \mathbf{T}_{\mathbf{y}} \;:=\; \sum_{k\in U} \mathbf{y}_k \quad\Longleftrightarrow\quad \sum_{k\in U} (E[\,I_k\,w_k\,] - 1)\cdot\mathbf{y}_k \;=\; \mathbf{0} \quad\Longleftrightarrow\quad E[\,I_k\,w_k\,] \;=\; 1,\ \text{for each } k \in U.$$

The proof of the Proposition is now complete. □

Corollary 7.3

Let $U = \{1, 2, \ldots, N\}$ be a finite population. For any fixed but arbitrary population parameter $\mathbf{y} : U \longrightarrow \mathbb{R}^m$ and for any sampling design $p : \mathcal{S} \longrightarrow (0, 1]$ such that each of its first-order inclusion probabilities is strictly positive, the Horvitz-Thompson estimator $\widehat{\mathbf{T}}^{\mathrm{HT}}_{\mathbf{y}}$ is well-defined and it is the unique unbiased estimator for $\mathbf{T}_{\mathbf{y}}$, which is linear in $\mathbf{y}$ and whose weights are constant in s.

PROOF Recall that the Horvitz-Thompson estimator is defined as:

$$\widehat{\mathbf{T}}^{\mathrm{HT}}_{\mathbf{y}}(s) \;:=\; \sum_{k\in s} \frac{1}{\pi_k}\,\mathbf{y}_k \;:=\; \sum_{k\in U} I_k(s)\,\frac{1}{\pi_k}\,\mathbf{y}_k,$$

where $\pi_k := E[\,I_k\,] = \sum_{k\in U} p(s)\,I_k(s) = \sum_{s\ni k} p(s)$ is the inclusion probability of $k \in U$ under the sampling design $p : \mathcal{S} \longrightarrow (0,1]$. Clearly, $\widehat{\mathbf{T}}^{\mathrm{HT}}_{\mathbf{y}}$ is linear in $\mathbf{y}$ with weights constant in s. Next, note that:

$$E\Big[\,\widehat{\mathbf{T}}^{\mathrm{HT}}_{\mathbf{y}}\,\Big] \;=\; E\left[\sum_{k\in s}\frac{1}{\pi_k}\,\mathbf{y}_k\right] \;=\; E\left[\sum_{k\in U} I_k\,\frac{\mathbf{y}_k}{\pi_k}\right] \;=\; \sum_{k\in U} E[\,I_k\,]\,\frac{\mathbf{y}_k}{\pi_k} \;=\; \sum_{k\in U}\pi_k\,\frac{\mathbf{y}_k}{\pi_k} \;=\; \sum_{k\in U}\mathbf{y}_k \;=\; \mathbf{T}_y$$

Hence, $\widehat{\mathbf{T}}^{\mathrm{HT}}_{\mathbf{y}}$ is an unbiased estimator for $\mathbf{T}_{\mathbf{y}}$. Conversely, let

$$\widehat{\mathbf{T}}_{y;w}(s) \;=\; \sum_{k\in s} w_k\,\mathbf{y}_k$$

be any unbiased estimator for $\mathbf{T}_{\mathbf{y}}$ which linear in $\mathbf{y}$ with weights w_k constant in s. Thus,

$$\sum_{k\in U}\mathbf{y}_k \;=\; \mathbf{T}_{\mathbf{y}} \;=\; E\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\Big] \;=\; E\left[\sum_{k\in s} w_k\,\mathbf{y}_k\right] \;=\; E\left[\sum_{k\in U} I_k\,w_k\,\mathbf{y}_k\right] \;=\; \sum_{k\in U} E[\,I_k\,]\,w_k\,\mathbf{y}_k \;=\; \sum_{k\in U}\pi_k\,w_k\,\mathbf{y}_k.$$

Since $\mathbf{y}$ is arbitrary, the above equation immediately implies that

$$\pi_k w_k - 1 = 0,$$

or equivalently, $w_k = \dfrac{1}{\pi_k}$; in other words, $\widehat{\mathbf{T}}_{\mathbf{y};w}$ is in fact equal to the Horvitz-Thompson estimator. The proof of the Corollary is now complete. □

Lemma 7.4

Let $(\Omega,\mathcal{A},p)$ be a probability space, $X, Y : \Omega \longrightarrow \mathbb{R}$ be two $\mathbb{R}$-valued random variables defined on Ω, and $\mathbf{u},\mathbf{v}\in\mathbb{R}^m$ be two fixed vectors in $\mathbb{R}^m$. Then,

$$\mathrm{Cov}(\,X\cdot\mathbf{u}\,,\,Y\cdot\mathbf{v}\,) \;=\; \mathrm{Cov}(X,Y)\cdot\mathbf{u}\cdot\mathbf{v}^T \;\in\; \mathbb{R}^{m\times m}$$

PROOF Note:

$$\begin{aligned}
\mathrm{Cov}(\,X\cdot\mathbf{u}\;,\;Y\cdot\mathbf{v}\,) &:= E\Big[\,(X\,\mathbf{u}-\mu_X\mathbf{u})\cdot(Y\,\mathbf{v}-\mu_Y\mathbf{v})^T\,\Big] \;=\; E\big[\,(X-\mu_X)\,\mathbf{u}\cdot(Y-\mu_Y)\,\mathbf{v}^T\,\big] \\
&= E\big[\,(X-\mu_X)\,(Y-\mu_Y)\cdot\mathbf{u}\cdot\mathbf{v}^T\,\big] \;=\; E[\,(X-\mu_X)\,(Y-\mu_Y)\,]\cdot\mathbf{u}\cdot\mathbf{v}^T \\
&= \mathrm{Cov}(X,Y)\cdot\mathbf{u}\cdot\mathbf{v}^T,
\end{aligned}$$

as required. □

Proposition 7.5

Let $\widehat{\mathbf{T}}_{\mathbf{y};w} : \mathcal{S}\longrightarrow\mathbb{R}$, with $\widehat{\mathbf{T}}_{\mathbf{y};w}(s) = \sum_{k\in s} w_k(s)\,\mathbf{y}_k = \sum_{k\in U} I_k(s)\,w_k(s)\,\mathbf{y}_k$, be a random variable linear in the population parameter $\mathbf{y} : U \longrightarrow \mathbb{R}$. Then, the covariance matrix of $\widehat{\mathbf{T}}_{\mathbf{y};w}$ is given by:

$$\mathrm{Var}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\Big] \;=\; \sum_{i\in U}\sum_{k\in U}\mathrm{Cov}[\,I_i\,w_i\,,\,I_k\,w_k\,]\;\mathbf{y}_i\cdot\mathbf{y}_k^T \;\in\mathbb{R}^{m\times m}$$

Furthermore, if the first-order and second-order inclusion probabilities of the sampling design $p : \mathcal{S}\longrightarrow(0,1]$ are all strictly positive, i.e. $\pi_k = \pi_{kk} := \sum_{s\ni k} p(s) > 0$, for each $k\in U$, and $\pi_{ik} := \sum_{s\ni i,k} p(s) > 0$, for any distinct $i,k\in U$, then an unbiased estimator for $\mathrm{Var}\Big[\,\widehat{\mathbf{T}}_{y;w}\,\Big]$ is given by:

$$\widehat{\mathrm{Var}}\Big[\,\widehat{\mathbf{T}}_{y;w}\,\Big](s) \;:=\; \sum_{i,k\in s}\frac{\mathrm{Cov}(I_iw_i,I_kw_k)}{\pi_{ik}}\,\mathbf{y}_i\cdot\mathbf{y}_k^T \;=\; \sum_{k\in s}\frac{\mathrm{Var}(I_kw_k)}{\pi_k}\,\mathbf{y}_k\cdot\mathbf{y}_k^T + \sum_{\substack{i,k\in s\\ i\neq k}}\frac{\mathrm{Cov}(I_iw_i,I_kw_k)}{\pi_{ik}}\,\mathbf{y}_i\cdot\mathbf{y}_k^T,\;\text{ for each } s\in\mathcal{S}.$$

PROOF First, note that Lemma 7.4 implies:

$$\mathrm{Var}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\Big] \;=\; \mathrm{Cov}\left[\,\sum_{i\in U} I_i\,w_i\,\mathbf{y}_i\;,\;\sum_{k\in U} I_k\,w_k\,\mathbf{y}_k\,\right] \;=\; \sum_{i\in U}\sum_{k\in U}\mathrm{Cov}[\,I_i\,w_i\,,\,I_k\,w_k\,]\cdot\mathbf{y}_i\cdot\mathbf{y}_k^T \;\in\; \mathbb{R}^{m\times m}$$

Next,

$$\begin{aligned}
E\Big(\,\widehat{\mathrm{Var}}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\Big]\,\Big) &= \sum_{s\in\mathcal{S}} p(s)\cdot\widehat{\mathrm{Var}}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\Big](s) \;=\; \sum_{s\in\mathcal{S}} p(s)\cdot\left(\sum_{i,k\in s}\frac{\mathrm{Cov}(I_iw_i,I_kw_k)}{\pi_{ik}}\cdot\mathbf{y}_i\cdot\mathbf{y}_k^T\right)\\
&= \sum_{s\in\mathcal{S}} p(s)\cdot\left(\sum_{i,k\in U} I_i(s)I_k(s)\cdot\frac{\mathrm{Cov}(I_iw_i,I_kw_k)}{\pi_{ik}}\cdot\mathbf{y}_i\cdot\mathbf{y}_k^T\right)\\
&= \sum_{i,k\in U}\left(\sum_{s\in\mathcal{S}} p(s)I_i(s)I_k(s)\right)\cdot\frac{\mathrm{Cov}(I_iw_i,I_kw_k)}{\pi_{ik}}\cdot\mathbf{y}_i\cdot\mathbf{y}_k^T\\
&= \sum_{i,k\in U}\left(\sum_{s\ni i,k} p(s)\right)\cdot\frac{\mathrm{Cov}(I_iw_i,I_kw_k)}{\pi_{ik}}\cdot\mathbf{y}_i\cdot\mathbf{y}_k^T\\
&= \sum_{i,k\in U}\pi_{ik}\cdot\frac{\mathrm{Cov}(I_iw_i,I_kw_k)}{\pi_{ik}}\cdot\mathbf{y}_i\cdot\mathbf{y}_k^T \;=\; \sum_{i,k\in U}\mathrm{Cov}(I_iw_i,I_kw_k)\cdot\mathbf{y}_i\cdot\mathbf{y}_k^T\\
&= \mathrm{Var}\Big[\,\widehat{\mathbf{T}}_{y;w}\,\Big]
\end{aligned}$$

Lastly, recall that $\pi_{kk}=\pi_k$ and $\mathrm{Cov}(I_kw_k,I_kw_k)=\mathrm{Var}[\,I_kw_k\,]$, and the validity of the following identity is thus trivial:

$$\sum_{i,k\in s}\frac{\mathrm{Cov}(I_iw_i,I_kw_k)}{\pi_{ik}}\cdot\mathbf{y}_i\cdot\mathbf{y}_k^T \;=\; \sum_{k\in s}\frac{\mathrm{Var}(I_kw_k)}{\pi_k}\cdot\mathbf{y}_k\cdot\mathbf{y}_k^T + \sum_{\substack{i,k\in s\\ i\neq k}}\frac{\mathrm{Cov}(I_iw_i,I_kw_k)}{\pi_{ik}}\cdot\mathbf{y}_i\cdot\mathbf{y}_k^T$$

The proof of the Proposition is complete. □

8 Calibrated linear estimators for (multivariate) population totals

Definition 8.1

Let $\widehat{\mathbf{T}}_{\mathbf{y};w}:\mathcal{S}\longrightarrow\mathbb{R}^m$ be an $\mathbb{R}^m$-valued random variable which is linear in the $\mathbb{R}^m$-valued population parameter $\mathbf{y}:U\longrightarrow\mathbb{R}^m$, i.e.

$$\begin{array}{cccccc}
\widehat{\mathbf{T}}_{\mathbf{y};w}: & \mathcal{S} & \longrightarrow & \mathbb{R}^m & & \\
& s & \longmapsto & \sum_{k\in s} w_k(s)\cdot\mathbf{y}_k & = & \sum_{k\in U} I_k(s)\,w_k(s)\cdot\mathbf{y}_k,
\end{array}$$

where, for each $k\in U$, $w_k:\mathcal{S}\longrightarrow\mathbb{R}$ is itself an $\mathbb{R}$-valued random variable, and $I_k:\mathcal{S}\longrightarrow\{0,1\}$ is the indicator random variable defined by:

$$I_k(s) \;=\; \begin{cases}1, & \text{if } k\in s,\\ 0, & \text{otherwise}\end{cases}$$

Let $x:U\longrightarrow\mathbb{R}$ be an $\mathbb{R}$-valued population parameter and $T_x:=\sum_{k\in U}x_k$.

Then, $\widehat{\mathbf{T}}_{\mathbf{y};w}$ is said to be calibrated with respect to x *if*

$$\sum_{k\in s} w_k(s)\,x_k \;=\; T_x,\quad\text{for each } s\in\mathcal{S}.$$

Example 8.2
If the sampling design has fixed sample size and each of its first-order inclusion probabilities is strictly positive, then Horvitz-Thompson estimator is calibrated with respect to the first-order inclusion probabilities.

To see this, let $U = \{1, 2, \ldots, N\}$ *be a finite population,* $\mathbf{y} : U \longrightarrow \mathbb{R}^m$ *a population parameter, and* $p : \mathcal{S} \subset \mathcal{P}(U) \longrightarrow (0, 1]$ *a sampling design such that* $\pi_k := \sum_{s \ni k} p(s) > 0$*, for each* $k \in U$*. The Horvitz-Thompson estimator* $\widehat{\mathbf{T}}^{\mathrm{HT}}_{\mathbf{y}} : \mathcal{S} \longrightarrow \mathbb{R}$ *is then well-defined and is given by:*

$$\widehat{\mathbf{T}}^{\mathrm{HT}}_{\mathbf{y}}(s) \;:=\; \sum_{k \in s} \frac{\mathbf{y}_k}{\pi_k}$$

Let $x : U \longrightarrow \mathbb{R}$ *be defined by*

$$x_k \;=\; \pi_k, \;\; \text{for each } k \in U,$$

i.e. x_k *is simply the inclusion probability of* $k \in U$ *under the sampling design* $p : \mathcal{S} \longrightarrow (0, 1]$.

Now, suppose that the sampling design has a fixed sample size n*, and we shall show that* $\widehat{\mathbf{T}}^{\mathrm{HT}}_{\mathbf{y}}$ *is consequently calibrated with respect to* $x : U \longrightarrow \mathbb{R}$*. Indeed, recall that the weights of the Horvitz-Thompson estimator are simply* $w_k(s) = 1/\pi_k$*, for each* $k \in U$ *and each* $s \in \mathcal{S}$*. Hence,*

$$\sum_{k \in s} w_k(s)\, x_k \;=\; \sum_{k \in s} \frac{1}{\pi_k}\, \pi_k \;=\; \sum_{k \in s} 1 \;=\; \left(\begin{array}{c}\text{sample}\\ \text{size of } s\end{array}\right) \;=\; n,$$

since the sampling design has fixed size n*. On the other hand,*

$$T_x \;=\; \sum_{k \in U} x_k \;=\; \sum_{k \in U} \pi_k \;=\; \sum_{k \in U} E[\, I_k \,] \;=\; E\left[\, \sum_{k \in U} I_k \,\right] \;=\; E\left[\begin{array}{c}\text{sample}\\ \text{size}\end{array}\right] \;=\; n,$$

again since the sample size is fixed and equals n*. Therefore, we have, for any* $s \in \mathcal{S}$,

$$\sum_{k \in s} w_k(s)\, x_k \;=\; n \;=\; T_x$$

Therefore, the Horvitz-Thompson estimator, under the assumption of fixed sample size, is indeed calibrated with respect to the inclusion probabilities $x : U \longrightarrow \mathbb{R}$, $x_k = \pi_k := \sum_{s \ni k} p(s)$*, for each* $k \in U$. □

Proposition 8.3
Let $\widehat{\mathbf{T}}_{\mathbf{y};w,x} : \mathcal{S} \longrightarrow \mathbb{R}^m$ *be an* $\mathbb{R}^m$*-valued random variable which is linear in the* $\mathbb{R}^m$*-valued population parameter* $\mathbf{y} : U \longrightarrow \mathbb{R}^m$ *and calibrated with respect to the population parameter* $x : U \longrightarrow \mathbb{R}$*, with* $x_k \neq 0$ *for each* $k \in U$.
Then, the mean squared error matrix of $\widehat{\mathbf{T}}_{\mathbf{y};w,x}$ *as an estimator of* $\mathbf{T}_{\mathbf{y}}$ *is given by:*

$$\mathrm{MSE}\Big[\, \widehat{\mathbf{T}}_{\mathbf{y};w,x} \,\Big] \;=\; -\frac{1}{2} \sum_{\substack{i,k \in U \\ i \neq k}} a_{ik} \left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k}\right) \cdot \left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k}\right)^T x_i\, x_k \;\in\; \mathbb{R}^{m \times m}, \;\; \text{where} \;\; a_{ik} \;:=\; E[\,(I_i\, w_i - 1)\,(I_k\, w_k - 1)\,]\,.$$

Proof

$$\begin{aligned}
\mathrm{MSE}\Big[\, \widehat{\mathbf{T}}_{\mathbf{y};w,x} \,\Big] \;&=\; E\left[\, \left(\widehat{\mathbf{T}}_{\mathbf{y};w,x} - \mathbf{T}_{\mathbf{y}}\right) \cdot \left(\widehat{\mathbf{T}}_{\mathbf{y};w,x} - \mathbf{T}_{\mathbf{y}}\right)^T \,\right] \;=\; E\left[\, \left(\sum_{i \in U} (I_i w_i - 1)\, \mathbf{y}_i\right) \cdot \left(\sum_{k \in U} (I_k w_k - 1)\, \mathbf{y}_k\right)^T \,\right] \\
&=\; \sum_{i \in U} \sum_{k \in U} E[\,(I_i w_i - 1)\,(I_k w_k - 1)\,] \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T \;=\; \sum_{k \in U} a_{kk} \cdot \mathbf{y}_k \cdot \mathbf{y}_k^T + \sum_{\substack{i,k \in U \\ i \neq k}} a_{ik} \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T \\
&=\; \sum_{k \in U} a_{kk} \left(\frac{\mathbf{y}_k \cdot \mathbf{y}_k^T}{x_k^2}\right) x_k^2 + \sum_{\substack{i,k \in U \\ i \neq k}} a_{ik} \left(\frac{\mathbf{y}_i}{x_i}\right) \cdot \left(\frac{\mathbf{y}_k}{x_k}\right)^T x_i\, x_k
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& -\frac{1}{2}\sum_{\substack{i,k\in U\\ i\neq k}} a_{ik}\left(\frac{\mathbf{y}_i}{x_i}-\frac{\mathbf{y}_k}{x_k}\right)\cdot\left(\frac{\mathbf{y}_i}{x_i}-\frac{\mathbf{y}_k}{x_k}\right)^T x_i\,x_k \\
= & -\frac{1}{2}\sum_{\substack{i,k\in U\\ i\neq k}} a_{ik}\left[\left(\frac{\mathbf{y}_i}{x_i}\right)\left(\frac{\mathbf{y}_i}{x_i}\right)^T-\left(\frac{\mathbf{y}_i}{x_i}\right)\left(\frac{\mathbf{y}_k}{x_k}\right)^T-\left(\frac{\mathbf{y}_k}{x_k}\right)\left(\frac{\mathbf{y}_i}{x_i}\right)^T+\left(\frac{\mathbf{y}_k}{x_k}\right)\left(\frac{\mathbf{y}_k}{x_k}\right)^T\right] x_i\,x_k \\
= & -\frac{1}{2}\sum_{\substack{i,k\in U\\ i\neq k}} a_{ik}\left[\left(\frac{\mathbf{y}_i}{x_i}\right)\left(\frac{\mathbf{y}_i}{x_i}\right)^T+\left(\frac{\mathbf{y}_k}{x_k}\right)\left(\frac{\mathbf{y}_k}{x_k}\right)^T\right] x_i\,x_k+\sum_{\substack{i,k\in U\\ i\neq k}} a_{ik}\left(\frac{\mathbf{y}_i}{x_i}\right)\left(\frac{\mathbf{y}_k}{x_k}\right)^T x_i\,x_k
\end{aligned}
$$

Thus, the proof of the present Proposition will be complete once we show:

$$
\underbrace{\sum_{k\in U} a_{kk}\left(\frac{\mathbf{y}_k}{x_k}\right)\left(\frac{\mathbf{y}_k}{x_k}\right)^T x_k^2}_{\frac{1}{2}\sum_{\substack{i,k\in U\\ i=k}} a_{ik}\left[\left(\frac{\mathbf{y}_i}{x_i}\right)\left(\frac{\mathbf{y}_i}{x_i}\right)^T+\left(\frac{\mathbf{y}_k}{x_k}\right)\left(\frac{\mathbf{y}_k}{x_k}\right)^T\right] x_i\,x_k} \quad = \quad -\frac{1}{2}\sum_{\substack{i,k\in U\\ i\neq k}} a_{ik}\left[\left(\frac{\mathbf{y}_i}{x_i}\right)\left(\frac{\mathbf{y}_i}{x_i}\right)^T+\left(\frac{\mathbf{y}_k}{x_k}\right)\left(\frac{\mathbf{y}_k}{x_k}\right)^T\right] x_i\,x_k,
$$

which is equivalent to:

$$
\sum_{i\in U}\sum_{k\in U} a_{ik}\left[\left(\frac{\mathbf{y}_i}{x_i}\right)\left(\frac{\mathbf{y}_i}{x_i}\right)^T+\left(\frac{\mathbf{y}_k}{x_k}\right)\left(\frac{\mathbf{y}_k}{x_k}\right)^T\right] x_i\,x_k \;=\; 0. \tag{8.1}
$$

Observe that

$$
\begin{aligned}
\mathrm{LHS}(8.1) &= \sum_{i\in U}\sum_{k\in U} a_{ik}\left(\frac{\mathbf{y}_i}{x_i}\right)\left(\frac{\mathbf{y}_i}{x_i}\right)^T x_i\,x_k+\sum_{i\in U}\sum_{k\in U} a_{ik}\left(\frac{\mathbf{y}_k}{x_k}\right)\left(\frac{\mathbf{y}_k}{x_k}\right)^T x_i\,x_k \\
&= 2\sum_{i\in U}\sum_{k\in U} a_{ik}\left(\frac{\mathbf{y}_i}{x_i}\right)\left(\frac{\mathbf{y}_i}{x_i}\right)^T x_i\,x_k \;=\; 2\sum_{i\in U} x_i\left(\frac{\mathbf{y}_i}{x_i}\right)\left(\frac{\mathbf{y}_i}{x_i}\right)^T\left(\sum_{k\in U} a_{ik}x_k\right).
\end{aligned}
$$

Hence, (8.1) follows once we show

$$
\sum_{k\in U} a_{ik}x_k = 0, \quad \text{for each } i\in U. \tag{8.2}
$$

Lastly, we now claim that (8.2) follows from the hypothesis that $\widehat{T}_{y;w;x}$ is calibrated with respect to x. Indeed,

$$
\begin{aligned}
\sum_{k\in U} a_{ik}x_k &= \sum_{k\in U} E[\,(I_iw_i-1)(I_kw_k-1)\,]\,x_k \;=\; \sum_{k\in U}\left[\sum_{s\in\mathcal{S}} p(s)(I_i(s)w_i(s)-1)(I_k(s)w_k(s)-1)\right]x_k \\
&= \sum_{s\in\mathcal{S}} p(s)\cdot(I_i(s)w_i(s)-1)\cdot\left[\sum_{k\in U}(I_k(s)w_k(s)-1)\cdot x_k\right] \\
&= \sum_{s\in\mathcal{S}} p(s)\cdot(I_i(s)w_i(s)-1)\cdot\left[\underbrace{\left(\sum_{k\in s} w_k(s)\,x_k\right)-T_x}_{0}\right] \\
&= 0
\end{aligned}
$$

The proof of the present Proposition is now complete. □

Proposition 8.4 (The Yates-Grundy-Sen Variance Estimator for calibrated linear population total estimators)

Let $p : \mathcal{S} \longrightarrow (0,1]$ be a sampling design each of whose first-order and second-order inclusion probabilities is strictly positively. Let $\widehat{\mathbf{T}}_{\mathbf{y};w,x} : \mathcal{S} \longrightarrow \mathbb{R}^m$ be a random variable which is linear in the population parameter $\mathbf{y} : U \longrightarrow \mathbb{R}^m$ and calibrated with respect to the population parameter $x : U \longrightarrow \mathbb{R}$, with $x_k \neq 0$ for each $k \in U$. Suppose that $\widehat{\mathbf{T}}_{\mathbf{y};w,x}$ is an unbiased estimator for $\mathbf{T}_{\mathbf{y}} := \sum_{k\in U} \mathbf{y}_k$, for arbitrary $\mathbf{y}$. Then, the following is an unbiased estimator of the variance $\mathrm{Var}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w,x}\,\Big]$ of $\widehat{\mathbf{T}}_{\mathbf{y};w,x}$: For each $s \in \mathcal{S}$ admissible in the sampling design $p : \mathcal{S} \longrightarrow (0,1]$,

$$\widehat{\mathrm{Var}}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w,x}\,\Big](s) \;:=\; -\frac{1}{2}\sum_{\substack{i,k\in s\\ i\neq k}} \left(w_i(s)w_k(s) - \frac{1}{\pi_{ik}} \right) \left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k} \right) \cdot \left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k} \right)^T x_i\, x_k$$

Terminology: $\widehat{\mathrm{Var}}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w,x}\,\Big]$ is called the Yates-Grundy-Sen Variance Estimator.

PROOF Since $\widehat{\mathbf{T}}_{\mathbf{y};w,x}$ is an unbiased estimator for $\mathbf{T}_{\mathbf{y}}$ by hypothesis, we have $\mathrm{Var}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w,x}\,\Big] = \mathrm{MSE}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w,x}\,\Big]$. By Proposition 8.3, we thus have:

$$\mathrm{Var}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w,x}\,\Big] \;=\; -\frac{1}{2}\sum_{\substack{i,k\in U\\ i\neq k}} a_{ik} \left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k} \right)^2 x_i\, x_k, \quad \text{where } a_{ik} \;:=\; E[\,(I_i\, w_i - 1)\,(I_k\, w_k - 1)\,]\,.$$

On the other hand,

$$E\Big(\,\widehat{\mathrm{Var}}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w,x}\,\Big]\,\Big) \;=\; -\frac{1}{2}\sum_{\substack{i,k\in U\\ i\neq k}} E\left[\,I_i I_k \left(w_i w_k - \frac{1}{\pi_{ik}} \right)\right] \left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k} \right) \cdot \left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k} \right)^T x_i\, x_k$$

Thus, it remains only to show:

$$a_{ik} \;=\; E\left[\,I_i I_k \left(w_i w_k - \frac{1}{\pi_{ik}} \right)\right].$$

Now,

$$E\left[\,I_i I_k \left(w_i w_k - \frac{1}{\pi_{ik}} \right)\right] \;=\; E[\,I_i I_k w_i w_k\,] - \frac{1}{\pi_{ik}} E[\,I_i I_k\,] \;=\; E[\,I_i I_k w_i w_k\,] - \frac{1}{\pi_{ik}}\,\pi_{ik} \;=\; E[\,I_i I_k w_i w_k\,] - 1,$$

and

$$\begin{aligned} a_{ik} \;&=\; E[\,(I_i\, w_i - 1)\,(I_k\, w_k - 1)\,] \;=\; E[\,I_i\, I_k\, w_i\, w_k\,] - E[\,I_i\, w_i\,] - E[\,I_k\, w_k\,] + 1 \\ &=\; E[\,I_i\, I_k\, w_i\, w_k\,] - 1 - 1 + 1 \;=\; E[\,I_i\, I_k\, w_i\, w_k\,] - 1 \\ &=\; E\left[\,I_i I_k \left(w_i w_k - \frac{1}{\pi_{ik}} \right)\right], \end{aligned}$$

where third last equality follows from Proposition 7.2 and the unbiasedness hypothesis on $\widehat{\mathbf{T}}_{\mathbf{y};w,x}$ as an estimator for $\mathbf{T}_{\mathbf{y}}$. The proof of the present Proposition is now complete. □

9 Unbiased variance estimators for the Horvitz-Thompson Estimator

Let $U = \{1, 2, \ldots, N\}$ be a finite population. Let $\mathbf{y} = (y_1, y_2, \ldots, y_m) : U \longrightarrow \mathbb{R}^m$ be an $\mathbb{R}^m$-valued function defined on U (commonly called a "population parameter"). We will use the common notation $\mathbf{y}_k$ for $\mathbf{y}(k)$. We wish to estimate $\mathbf{T}_{\mathbf{y}} := \sum_{k\in U} \mathbf{y}_k \in \mathbb{R}^m$ via survey sampling. Let $p : \mathcal{S} \longrightarrow (0,1]$ be our chosen sampling design, where $\mathcal{S} \subseteq \mathcal{P}(U)$ is the set of all possible samples in the design, and $\mathcal{P}(U)$ is the power set of U.

Proposition 9.1
Suppose the first-order and second-order inclusion probabilities of $p : \mathcal{S} \longrightarrow (0,1]$ are all strictly positive, i.e.

$$\pi_k := \sum_{s \ni k} p(s) = \sum_{k \in U} I_k(s) p(s) > 0 \quad \text{and} \quad \pi_{ik} := \sum_{s \ni i,k} p(s) = \sum_{i,k \in U} I_i(s) I_k(s) p(s) > 0,$$

for any $i, k \in U$. Then, the Horvitz-Thompson estimator for $\mathbf{T}_{\mathbf{y}}$ is:

$$\widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}}(s) \;\; := \;\; \sum_{k \in s} \frac{1}{\pi_k} \, \mathbf{y}_k,$$

and the covariance matrix of the Horvitz-Thompson estimator can be given by:

$$\mathrm{Var}\Big[\, \widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}} \,\Big] \;\; = \;\; \sum_{i,k \in U} (\pi_{ik} - \pi_i \pi_k) \cdot \left(\frac{\mathbf{y}_i}{\pi_i} \right) \cdot \left(\frac{\mathbf{y}_k}{\pi_k} \right)^T$$

An unbiased estimator for the covariance matrix of the Horvitz-Thompson estimator is given by:

$$\widehat{\mathrm{Var}}\Big[\, \widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}} \,\Big](s) \;\; = \;\; \sum_{i,k \in s} \left(\frac{\pi_{ik} - \pi_i \pi_k}{\pi_{ik}} \right) \cdot \left(\frac{\mathbf{y}_i}{\pi_i} \right) \cdot \left(\frac{\mathbf{y}_k}{\pi_k} \right)^T, \;\; \text{for each } s \in \mathcal{S}.$$

Furthermore, if the sampling design has fixed sample size, then an alternative expression of the covariance matrix of the Horvitz-Thompson estimator is:

$$\mathrm{Var}\Big[\, \widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}} \,\Big] \;\; = \;\; -\frac{1}{2} \sum_{i,k \in U} (\pi_{ik} - \pi_i \pi_k) \cdot \left(\frac{\mathbf{y}_i}{\pi_i} - \frac{\mathbf{y}_k}{\pi_k} \right) \cdot \left(\frac{\mathbf{y}_i}{\pi_i} - \frac{\mathbf{y}_k}{\pi_k} \right)^T$$

and the corresponding Yates-Grundy-Sen variance estimator is:

$$\widehat{\mathrm{Var}}^{\mathrm{YGS}}\Big[\, \widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}} \,\Big](s) \;\; := \;\; -\frac{1}{2} \sum_{i,k \in s} \left(\frac{\pi_{ik} - \pi_i \pi_k}{\pi_{ik}} \right) \cdot \left(\frac{\mathbf{y}_i}{\pi_i} - \frac{\mathbf{y}_k}{\pi_k} \right) \cdot \left(\frac{\mathbf{y}_i}{\pi_i} - \frac{\mathbf{y}_k}{\pi_k} \right)^T$$

PROOF By Proposition 7.5, for any random variable (a.k.a. estimator) $\widehat{\mathbf{T}}_{\mathbf{y};w}$ linear in the population parameter $\mathbf{y} : \mathcal{S} \longrightarrow \mathbb{R}^m$ with weights $w_k : \mathcal{S} \longrightarrow \mathbb{R}$, $k \in U$, the following

$$\widehat{\mathrm{Var}}\Big[\, \widehat{\mathbf{T}}_{y;w} \,\Big](s) \;\; := \;\; \sum_{i,k \in s} \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}} \, \mathbf{y}_i \cdot \mathbf{y}_k^T \tag{9.3}$$

always gives an unbiased estimator for the covariance matrix of $\widehat{\mathbf{T}}_{y;w}$. For the Horvitz-Thompson estimator, the weights are $w_k = 1/\pi_k$, for each $k \in U$, and the weights are independent of the sample $s \in \mathcal{S}$. Thus, for the Horvitz-Thompson estimator, the right-hand side of equation (9.3) becomes:

$$\begin{aligned}
\sum_{i,k \in s} \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}} \, \mathbf{y}_i \cdot \mathbf{y}_k^T
&= \sum_{i,k \in s} \frac{\mathrm{Cov}(I_i, I_k)}{\pi_{ik}} \left(\frac{\mathbf{y}_i}{\pi_i} \right) \cdot \left(\frac{\mathbf{y}_k}{\pi_k} \right)^T \\
&= \sum_{i,k \in s} \frac{E(I_i I_k) - E(I_i) E(I_k)}{\pi_{ik}} \left(\frac{\mathbf{y}_i}{\pi_i} \right) \cdot \left(\frac{\mathbf{y}_k}{\pi_k} \right)^T \\
&= \sum_{i,k \in s} \frac{\pi_{ik} - \pi_i \pi_k}{\pi_{ik}} \left(\frac{\mathbf{y}_i}{\pi_i} \right) \cdot \left(\frac{\mathbf{y}_k}{\pi_k} \right)^T,
\end{aligned}$$

which coincides with the right-hand side of the equation of the conclusion of the present Proposition. Thus this present Proposition is but a special case of Proposition 7.5, specialized to the Horvitz-Thompson estimator, and the proof is now complete. □

10 Estimation of Domain Totals

11 Conditional inference in finite-population sampling

In this section, we give a justification for making inference conditional on the observed sample size for sampling designs with random sample size.

Observation ("mixture" of experiments) [see [3], p.15.]

Consider a population $\mathcal{U}$ of 1000 units. We wish to estimate the total T_y of a certain population characteristic $\mathbf{y} = (y_1, y_2, \ldots, y_{1000})$. Suppose we use the following two-step sampling scheme:

- Step 1: We first flip a fair coin.
  Define the random variable X by letting $X = 1$ if the coin lands heads, and $X = 0$ if it lands tails.
- Step 2: If $X = 1$, we select an SRS from $\mathcal{U}$ of size 100. If $X = 0$, we take a census on all of $\mathcal{U}$.

Let $\mathcal{S} \subset \mathcal{P}(\mathcal{U})$ denote the probability space of all possible samples induced by the (two-step) sampling design above. Note that $\mathcal{S} = \mathcal{S}_0 \sqcup \mathcal{S}_1$, where $\mathcal{S}_0 = \{\,\mathcal{U}\,\}$ and $\mathcal{S}_1$ is the set of all subsets of $\mathcal{U}$ of size 100. The sampling design is determined by the following probability distribution on $\mathcal{S}$:

$$P(\mathcal{U}) = \frac{1}{2} \quad \text{and} \quad P(\,s\,) = \frac{1}{2\begin{pmatrix}1000\\100\end{pmatrix}}, \quad \text{for each } s \in \mathcal{S}_1.$$

Let $\widehat{T}_y : \mathcal{S} \longrightarrow \mathbb{R}$ denote our chosen estimator for T_y. Then the (unconditional) probability distribution of $\widehat{T}_y$ can be "decomposed" as follows:

$$\begin{aligned}
P\left(\widehat{T}_y = t \,\middle|\, \mathbf{y}\right) &= P\left(\widehat{T}_y = t,\, X = 0 \,\middle|\, \mathbf{y}\right) + P\left(\widehat{T}_y = t,\, X = 1 \,\middle|\, \mathbf{y}\right) \\
&= P\left(\widehat{T}_y = t \,\middle|\, X = 0,\, \mathbf{y}\right) \cdot P(\,X = 0 \mid \mathbf{y}\,) + P\left(\widehat{T}_y = t \,\middle|\, X = 1,\, \mathbf{y}\right) \cdot P(\,X = 1 \mid \mathbf{y}\,) \\
&= P\left(\widehat{T}_y = t \,\middle|\, X = 0,\, \mathbf{y}\right) \cdot P(\,X = 0\,) + P\left(\widehat{T}_y = t \,\middle|\, X = 1,\, \mathbf{y}\right) \cdot P(\,X = 1\,),
\end{aligned}$$

where the last equality follows because the distribution of X is independent of $\mathbf{y}$. Suppose the observation we make consists of $\left(\widehat{T}_y\,,\,X\right)$. The unconditional probability distribution of $\widehat{T}_y$, given by $P\left(\widehat{T}_y = t \,\middle|\, \mathbf{y}\right)$ above, describes of course the randomness of the estimator $\widehat{T}_y$ as induced by both the randomness of the sample $s \in \mathcal{S} = \mathcal{S}_0 \sqcup \mathcal{S}_1$ as well as that of X (the outcome of the coin flip in Step 1). Now, suppose we have indeed carried out the sampling procedure and have obtained an observation of $\left(\widehat{T}_y\,,\,X\right)$. Suppose it happened that $X = 1$. Hence, we know that the estimate $\widehat{T}_y(s)$ we actually obtained was generated from an SRS of size 100 (rather than a census). Note also that the probability distribution of X is independent of $\mathbf{y}$ and the observation of X gives no information about $\mathbf{y}$. One school of thought therefore argues that downstream inferences about $\mathbf{y}$ should be carried out using the conditional probability $P\left(\widehat{T}_y = t \,\middle|\, X = 1\,,\,\mathbf{y}\right)$, rather than the unconditional probability $P\left(\widehat{T}_y = t \,\middle|\, \mathbf{y}\right)$. In other words, in the present example, as far as making inferences about $\mathbf{y}$ is concerned, only the randomness in Step 2 is relevant, and the randomness in Step 1 (i.e. the randomness of X, the outcome of the coin flip) is irrelevant to any inference about $\mathbf{y}$. Consequently randomness of X "should" be removed in any inference procedure for $\mathbf{y}$, and this is achieved by conditioning on the observed value of X. $\square$

Conditioning on obtained sample size for sample designs with random sample size

Suppose $\mathcal{U}$ is a finite population. We wish to estimate the total $T_y = \sum_{i\in\mathcal{U}} y_i$ of a population characteristic $\mathbf{y} : \mathcal{U} \longrightarrow \mathbb{R}$, using a sample design $p : \mathcal{S} \longrightarrow [\,0\,,1\,]$ and a estimator $\widehat{T} : \mathcal{S} \longrightarrow \mathbb{R}$. We make the assumption that the sampling design p is independent of $\mathbf{y}$. Let $N : \mathcal{S} \longrightarrow \mathbb{N} \cup \{\,0\,\}$ be the random variable of sample size, i.e. $N(s) =$ number of elements in s, for each possible sample $s \in \mathcal{S}$. Then,

$$\begin{aligned}
P\left(\widehat{T}=t \,\middle|\, \mathbf{y}\right) &= \sum_{n} P\left(\widehat{T}=t,\, N=n \,\middle|\, \mathbf{y}\right) \\
&= \sum_{n} P\left(\widehat{T}=t \,\middle|\, N=n,\, \mathbf{y}\right)\cdot P\left(\, N=n \,|\, \mathbf{y}\,\right) \\
&= \sum_{n} P\left(\widehat{T}=t \,\middle|\, N=n,\, \mathbf{y}\right)\cdot P\left(\, N=n\,\right),
\end{aligned}$$

where the last equality follows from the assumed independence of the probability distribution $p : \mathcal{S} \longrightarrow [\,0\,,\,1\,]$ (hence that of N) from $\mathbf{y}$. The key observation to make now is that: Although the actual sampling procedure operationally may or may not have been a two-step procedure, the independence of p from $\mathbf{y}$ makes it probabilistically equivalent to a two-step procedure, as shown by the above decomposition of $P\left(\widehat{T}=t \,\middle|\, \mathbf{y}\right)$ — Step (1): randomly select a sample size $N=n$ according to the distribution $P(N=n)$, and then Step (2): randomly select a sample s of size n chosen in Step (1) according to the distribution $P(\,s\,|\,N=n\,)$. By the statistical reasoning explained in the preceding observation, it follows that post-sampling inference about $\mathbf{y}$ should be made based on the conditional distribution $P\left(\widehat{T}=t \,\middle|\, N=n\,,\,\mathbf{y}\right)$, rather than the unconditional distribution $P\left(\widehat{T}=t \,\middle|\, \mathbf{y}\right)$. This is because the sampling scheme is probabilistically equivalent to a two-step procedure, with the probability distribution of the first step (choosing a sample size) independent of the parameters of interest (T_y), and thus only the probability distribution of the second step (choosing a sample of the size chosen in first step) should be used to make inference about T_y. □

Caution

In more formal parlance, the random variable $N : \mathcal{S} \longrightarrow \mathbb{N}\cup\{\,0\,\}$ is ancillary to the parameter $\mathbf{y}$. Thus, conditioning on sample size, for finite-population sampling schemes with random sample size, *partially* conforms to the **Conditionality Principle**, which states that statistical inference about a parameter should be made conditioned on observed values of statistics ancillary to that parameter. The conformance is only partial due to the (obvious) fact that it is the sample s itself which is ancillary to the parameter of interest $\mathbf{y}$, not just its sample size $N(s)$. Thus, full conformance to the Conditionality Principle would require inference about $\mathbf{y}$ be made conditioned on the observed sample s itself (rather than its size $N(s)$). However, if we did condition on the obtained sample s itself, the domain of the estimator $\widehat{T}$ would be restricted to the singleton $\{\,s\,\}$, and $\widehat{T}$ could then attain only one value under conditioning on s, and no randomization-based (i.e. design-based) inference — apart from the observed value of $\widehat{T}(s)$ — could be made any longer.

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