

## A Cumulative distribution functions

**Definition A.1** Let  $X : (\Omega, \mathcal{A}, \mu) \rightarrow \mathbb{R}$  be a  $\mathbb{R}$ -valued random variable. The **cumulative distribution function** of  $X$  is, by definition, the function  $F_X : \mathbb{R} \rightarrow [0, 1]$  defined as follows:

$$F_X(x) := P(X \leq x) = \mu(\{\omega \in \Omega \mid X(\omega) \leq x\}), \quad \text{for each } x \in \mathbb{R}.$$

**Definition A.2** A function  $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is said to be

- **non-decreasing** if  $f(x) \leq f(y)$ , for any  $x, y \in D$  with  $x \leq y$ .
- **non-increasing** if  $f(x) \geq f(y)$ , for any  $x, y \in D$  with  $x \leq y$ .
- **monotone** if  $f$  is either non-decreasing or non-increasing.

**Theorem A.3** Let  $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a monotone function. Then,

$$\lim_{x \rightarrow a^-} f(x) \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x)$$

exist for every  $a \in \text{interior}(D)$ .

**Definition A.4** Let  $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ . A point  $a \in \text{interior}(D)$  is a **jump discontinuity** of  $f$  if both

$$\lim_{x \rightarrow a^-} f(x) \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x)$$

exist but they are unequal.

**Corollary A.5** A monotone  $\mathbb{R}$ -valued function defined on an interval of  $\mathbb{R}$  can have only jump discontinuities.

**Theorem A.6** A function  $F : \mathbb{R} \rightarrow [0, 1]$  is a cumulative distribution function of some  $\mathbb{R}$ -valued random variable if and only if each of following four conditions holds:

- $F$  is non-decreasing.
- $F$  is right-continuous.
- $\lim_{x \rightarrow -\infty} F(x) = 0$ .
- $\lim_{x \rightarrow +\infty} F(x) = 1$ .

**PROOF** If  $F : \mathbb{R} \rightarrow [0, 1]$  is a cumulative distribution function of some  $\mathbb{R}$ -valued random variable  $X : (\Omega, \mathcal{A}, \mu) \rightarrow \mathbb{R}$ , then the four conditions follow immediately from the property of the probability measure  $\mu$ . Conversely, suppose the four conditions hold. Let  $\Omega := [0, 1]$  and  $\mathcal{B}(\Omega)$  the Borel subsets of  $\Omega$ . Let  $\mu$  be the Lebesgue measure on  $(\Omega, \mathcal{B}(\Omega))$ , i.e.  $\mu$  is determined by:

$$\mu([0, \omega]) := \omega, \quad \text{for each } \omega \in \Omega = [0, 1].$$

Define the random variable  $X : (\Omega, \mathcal{B}(\Omega), \mu) \rightarrow \mathbb{R}$  by:

$$X(\omega) := \inf \{x \in \mathbb{R} \mid \omega \leq F(x)\}, \quad \text{for each } \omega \in \Omega = [0, 1].$$

Note that  $X$  is simply the quantile function of  $F$ .

**Claim:** Suppose  $G : \mathbb{R} \rightarrow [0, 1]$  is non-decreasing and right-continuous. Then, for any  $\omega \in [0, 1]$  and  $x \in \mathbb{R}$ ,

$$\inf \{ \xi \in \mathbb{R} \mid \omega \leq G(\xi) \} \leq x \iff \omega \leq G(x).$$

**Proof of Claim:** Suppose  $\omega \leq G(x)$ . Then,  $x \in \{ \xi \in \mathbb{R} \mid \omega \leq G(\xi) \}$ . Hence,  $\inf \{ \xi \in \mathbb{R} \mid \omega \leq G(\xi) \} \leq x$ . Conversely, suppose  $\inf \{ \xi \in \mathbb{R} \mid \omega \leq G(\xi) \} \leq x$ . Since  $G$  is non-decreasing and right-continuous, we have:

$$\begin{aligned} \inf \{ \xi \in \mathbb{R} \mid \omega \leq G(\xi) \} \leq x &\implies \text{for any } \varepsilon > 0, \exists \xi \in \mathbb{R}, \text{ satisfying } \omega \leq G(\xi), \text{ such that } \xi \leq x + \varepsilon \\ &\implies \text{for any } \varepsilon > 0, \exists \xi \in \mathbb{R}, \text{ satisfying } \omega \leq G(\xi) \text{ and } G(\xi) \leq G(x + \varepsilon) \\ &\implies \text{for any } \varepsilon > 0, \omega \leq G(x + \varepsilon) \\ &\implies \omega \leq \lim_{\varepsilon \rightarrow 0^+} G(x + \varepsilon) = G(x). \end{aligned}$$

This completes the proof of the Claim.

Noting that, by hypothesis,  $F$  is non-decreasing right-continuous, and invoking the Claim above, we see that

$$\begin{aligned} P(X \leq x) &= P(\{ \omega \in \Omega \mid X(\omega) \leq x \}) = P(\{ \omega \in \Omega \mid \inf \{ \xi \in \mathbb{R} \mid \omega \leq F(\xi) \} \leq x \}) \\ &= P(\{ \omega \in \Omega \mid \omega \leq F(x) \}) = \mu(\{ \omega \in [0, 1] \mid \omega \leq F(x) \}) = \mu([0, F(x)]) \\ &= F(x) \end{aligned}$$

This shows that if  $F$  satisfies the four conditions, then  $F$  is the cumulative distribution function of the random variable  $X$  constructed above. The proof of the Theorem is now complete.  $\square$

## Theorem A.7 (Darboux-Froda)

*The set of discontinuities of a monotone  $\mathbb{R}$ -valued function defined on an interval of  $\mathbb{R}$  is at most countable.*

## B The $O_P$ and $o_P$ notations; convergence in distribution implies boundedness in probability

[1]

### Definition B.1 (The Big- $O_P$ notation)

Let  $\{X_n : (\Omega_n, \mathcal{A}_n, \mu_n) \rightarrow \mathbb{R}^k\}_{n \in \mathbb{N}}$  be a sequence of  $\mathbb{R}^k$ -valued random variables. Let  $\{a_n\}_{n \in \mathbb{N}}$  be sequence of positive numbers. The notation  $X_n = O_P(a_n)$  means:

*For every  $\varepsilon > 0$ , there exist  $C_\varepsilon > 0$  and  $n_\varepsilon \in \mathbb{N}$  such that  $P(|X_n| \leq C_\varepsilon \cdot a_n) > 1 - \varepsilon$ , for every  $n \geq n_\varepsilon$ .*

### Proposition B.2 The following are equivalent:

- (a)  $X_n = O_P(a_n)$ .
- (b) For every  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that  $P(|X_n| \leq C_\varepsilon \cdot a_n) > 1 - \varepsilon$ , for each  $n \in \mathbb{N}$ .
- (c) For every  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that  $\limsup_{n \rightarrow \infty} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon$ .
- (d) For every  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that  $\sup_{n \in \mathbb{N}} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon$ .
- (e)  $\lim_{C \rightarrow \infty} \limsup_{n \rightarrow \infty} P(|X_n| > C \cdot a_n) = 0$ .
- (f)  $\lim_{C \rightarrow \infty} \sup_{n \in \mathbb{N}} P(|X_n| > C \cdot a_n) = 0$ .

# Large Sample Theory for Finite Population Sampling

Study Notes

April 26, 2015

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PROOF

(a)  $\implies$  (b)

Let  $\varepsilon > 0$  be given. By (a), there exist  $B_\varepsilon > 0$  and  $n_\varepsilon \in \mathbb{N}$  such that  $P(|X_n| \leq B_\varepsilon \cdot a_n) > 1 - \varepsilon$ , for each  $n \geq n_\varepsilon$ .

**Claim:** Let  $Y$  be an  $\mathbb{R}^k$ -valued random variable. Then, for each  $\varepsilon > 0$ , there exists  $A_\varepsilon > 0$  such that  $P(|Y| \leq A_\varepsilon) > 1 - \varepsilon$ .

Proof of Claim: Suppose the Claim were false. Then, there exists some  $\varepsilon > 0$  such that  $P(|Y| \leq A) \leq 1 - \varepsilon$ , for every  $A > 0$ ; equivalently,  $P(|Y| > A) > \varepsilon$ , for every  $A > 0$ . This implies  $\lim_{A \rightarrow \infty} P(|Y| > A) = \limsup_{A \rightarrow \infty} P(|Y| > A) \geq \varepsilon > 0$ . But this is a contradiction since  $\lim_{A \rightarrow \infty} P(|Y| > A) = 0$ , for every  $\mathbb{R}^k$ -valued random variable  $Y$ . This proves the Claim.

By the Claim, for each  $i = 1, 2, \dots, n_\varepsilon - 1$ , there exists  $B_\varepsilon^{(i)} > 0$  such that  $P(|X_i| \leq B_\varepsilon^{(i)} \cdot a_i) > 1 - \varepsilon$ . Now, let  $C_\varepsilon := \max \{B_\varepsilon^{(1)}, B_\varepsilon^{(1)}, \dots, B_\varepsilon^{(n_\varepsilon-1)}, B_\varepsilon\}$ . Then,  $P(|X_n| \leq C_\varepsilon \cdot a_n) > 1 - \varepsilon$ , for every  $n \in \mathbb{N}$ . This proves the implication (a)  $\implies$  (b).

(b)  $\implies$  (a) Trivial: Suppose (b) holds. Then (a) immediately follows with  $n_\varepsilon = 1$ .

(a)  $\iff$  (c) Let  $\varepsilon > 0$  be given.

- (a)  $\iff$  There exist  $C_\varepsilon > 0$  and  $n_\varepsilon \in \mathbb{N}$  such that  $P(|X_n| \leq C_\varepsilon \cdot a_n) > 1 - \varepsilon$ , for every  $n \geq n_\varepsilon$ .
- $\iff$  There exist  $C_\varepsilon > 0$  and  $n_\varepsilon \in \mathbb{N}$  such that  $P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon$ , for every  $n \geq n_\varepsilon$ .
- $\iff$  There exist  $C_\varepsilon > 0$  such that  $\limsup_{n \rightarrow \infty} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon \iff$  (c)

(b)  $\iff$  (d) Let  $\varepsilon > 0$  be given.

- (b)  $\iff$  There exists  $C_\varepsilon > 0$  such that  $P(|X_n| \leq C_\varepsilon \cdot a_n) > 1 - \varepsilon$ , for every  $n \in \mathbb{N}$ .
- $\iff$  There exists  $C_\varepsilon > 0$  such that  $P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon$ , for every  $n \in \mathbb{N}$ .
- $\iff$  There exist  $C_\varepsilon > 0$  such that  $\sup_{n \in \mathbb{N}} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon \iff$  (d)

(d)  $\iff$  (f) Let  $\varepsilon > 0$  be given. We first establish that (f)  $\implies$  (d).

- (f)  $\iff$  There exists  $C_\varepsilon > 0$  such that  $\sup_{n \in \mathbb{N}} P(|X_n| > C \cdot a_n) \leq \varepsilon$ , for each  $C \geq C_\varepsilon$ .
- $\implies$  There exists  $C_\varepsilon > 0$  such that  $\sup_{n \in \mathbb{N}} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon \iff$  (d)

Conversely, suppose (d) holds and  $C \geq C_\varepsilon$ . Then,  $|X_n| > C \cdot a_n \implies |X_n| > C_\varepsilon \cdot a_n$ . Hence,  $\{|X_n| > C \cdot a_n\} \subseteq \{|X_n| > C_\varepsilon \cdot a_n\}$ , which in turn implies  $P(|X_n| > C \cdot a_n) \leq P(|X_n| > C_\varepsilon \cdot a_n)$ . Thus, we have

$$\sup_{n \in \mathbb{N}} P(|X_n| > C \cdot a_n) \leq \sup_{n \in \mathbb{N}} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon,$$

i.e. (f) holds.

(c)  $\iff$  (e) Let  $\varepsilon > 0$  be given. We first establish that (e)  $\implies$  (c).

- (e)  $\iff$  There exists  $C_\varepsilon > 0$  such that  $\limsup_{n \rightarrow \infty} P(|X_n| > C \cdot a_n) \leq \varepsilon$ , for each  $C \geq C_\varepsilon$ .
- $\implies$  There exists  $C_\varepsilon > 0$  such that  $\limsup_{n \rightarrow \infty} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon \iff$  (c)

Conversely, suppose (c) holds and  $C \geq C_\varepsilon$ . Then,  $|X_n| > C \cdot a_n \implies |X_n| > C_\varepsilon \cdot a_n$ . Hence,  $\{|X_n| > C \cdot a_n\} \subseteq \{|X_n| > C_\varepsilon \cdot a_n\}$ , which in turn implies  $P(|X_n| > C \cdot a_n) \leq P(|X_n| > C_\varepsilon \cdot a_n)$ . Thus, we have

$$\limsup_{n \rightarrow \infty} P(|X_n| > C \cdot a_n) \leq \limsup_{n \rightarrow \infty} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon,$$

i.e. (e) holds.

This completes the proof of the Proposition. □

### Definition B.3 (Bounded in probability)

A sequence  $\{X_n : (\Omega_n, \mathcal{A}_n, \mu_n) \longrightarrow \mathbb{R}^k\}_{n \in \mathbb{N}}$  of  $\mathbb{R}^k$ -valued random variables is said to be **bounded in probability** if  $X_n = O_P(1)$ .

### Theorem B.4

If a sequence  $\{X_n : (\Omega_n, \mathcal{A}_n, \mu_n) \longrightarrow \mathbb{R}^k\}_{n \in \mathbb{N}}$  of  $\mathbb{R}^k$ -valued random variables converges in distribution to some random variable  $X : (\Omega, \mathcal{A}, \mu) \longrightarrow \mathbb{R}^k$ , then the sequence  $\{X_n\}$  is bounded in probability.

PROOF    Let □

## References

- [1] DEVILLE, J.-C., AND SÄRNDAL, C.-E. Calibration estimators in survey sampling. *Journal of the American Statistical Association* 87, 418 (1992), 376–382.