1 Equivalence of $(C[0,1]\,,\|\cdot\|_\infty)$-valued random variables and $\mathbb{R}$-valued stochastic processes indexed by $[0,1]$ with continuous sample paths

Proposition 1.1 (The "one-dimensional subsets" of $C[0,1]$ generate its Borel σ-algebra)
Let $(C[0,1]\,,\|\cdot\|_\infty)$ be the metric space of continuous $\mathbb{R}$-valued functions defined on the closed unit interval equipped with the supremum norm. For each $t \in [0,1]$, let $\mathrm{ev}_t : C[0,1] \longrightarrow \mathbb{R} : x \longmapsto x(t)$. Define:

$$\mathcal{S} \;:=\; \left\{ \mathrm{ev}_t^{-1}(H) \,\subset\, C[0,1] \;\middle|\; \begin{array}{c} t\in[0,1] \\ H \in \mathcal{O} \end{array} \right\} \;\subset\; \mathcal{P}(\,C[0,1]\,)\,.$$

Then, $\mathcal{S}$ generates the Borel σ-algebra $\mathcal{B} := \mathcal{B}(C[0,1],\|\cdot\|_\infty)$ of the metric space $(C[0,1],\|\cdot\|_\infty)$; in other words,

$$\sigma(\,\mathcal{S}\,) \;=\; \mathcal{B}.$$

Proof First, note that $\sigma(\,\mathcal{S}\,) \subset \mathcal{B}$. Indeed, recall that, for each $t \in [0,1]$, $\mathrm{ev}_t : C[0,1] \longrightarrow \mathbb{R}$ is continuous, hence $(\mathcal{B},\mathcal{O})$-measurable, by Corollary B.4. In particular, $\mathrm{ev}_t^{-1}(H) \in \mathcal{B}$, for each $t \in [0,1]$ and $H \in \mathcal{O}$. Thus, $\mathcal{S} \subset \mathcal{B}$; hence, $\sigma(\,\mathcal{S}\,) \subset \mathcal{B}$.

It remains to establish the reverse inclusion. To this end, first observe that, for each $x \in C[0,1]$ and each $\varepsilon > 0$, we have

$$\overline{B(x,\varepsilon)} \;=\; \bigcap_{r\,\in\,\mathbb{Q}\,\cap\,[0,1]} \left\{ y \in C[0,1] \;\middle|\; |\,y(r) - x(r)\,| \leq \varepsilon \right\} \;=\; \bigcap_{r\,\in\,\mathbb{Q}\,\cap\,[0,1]} \mathrm{ev}_r^{-1}\Big(\,[x(r)-\varepsilon, x(r)+\varepsilon]\,\Big),$$

which shows that $\sigma(\,\mathcal{S}\,)$ contains all the closed balls in $C[0,1]$. On the other hand, recall that, in any metric space, every open ball can be expressed as a countable union of closed balls; indeed, for any y in the given metric space, and any $\delta > 0$, we have:

$$B(y,\delta) \;=\; \bigcup_{n\in\mathbb{N}} \overline{B\left(y, \delta - \frac{1}{n}\right)}.$$

We thus see that $\sigma(\,\mathcal{S}\,)$ contains all the open balls in $C[0,1]$. By the separability of $C[0,1]$ and Theorem C.1, we see that every open subset of $C[0,1]$ can be expressed as a countable union of open balls. Hence, $\sigma(\,\mathcal{S}\,)$ in fact contains all the open subsets of $C[0,1]$, which immediately yields $\mathcal{B} \subset \sigma(\,\mathcal{S}\,)$. This proves $\sigma(\,\mathcal{S}\,) = \mathcal{B}$. □

Theorem 1.2
Suppose:

- *$(\Omega,\mathcal{A})$ is a measurable space, and $\mathcal{O}$ is the Borel σ-algebra of $\mathbb{R}$ (equipped with usual Euclidean metric).*
- *$(C[0,1]\,,\|\cdot\|_\infty)$ is the metric space of continuous $\mathbb{R}$-valued functions defined on the compact unit interval equipped with the supremum norm, and $\mathcal{B} := \mathcal{B}(C[0,1]\,,\|\cdot\|_\infty)$ is its Borel σ-algebra.*
- *$X : \Omega \longrightarrow C[0,1]$ is a function with domain Ω and codomain $C[0,1]$, but otherwise arbitrary.*
- *For each $t \in [0,1]$, let $\mathrm{ev}_t : C[0,1] \longrightarrow \mathbb{R} : x \longmapsto x(t)$.*
- *For each $t \in [0,1]$, let $X_t := \mathrm{ev}_t \circ X$. In other words, $X_t : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto \mathrm{ev}_t(X(\omega)) = X(\omega)(t)$.*

Then, X is $(\mathcal{A},\mathcal{B})$-measurable if and only if, for each $t \in [0,1]$, X_t is $(\mathcal{A},\mathcal{O})$-measurable.

Proof

$(\Longrightarrow)$

It is trivial to see that, for each $t \in [0,1]$, $\mathrm{ev}_t : (\,C[0,1]\,,\|\cdot\|_\infty\,) \longrightarrow (\,\mathbb{R}\,,|\cdot|\,) : x \longmapsto x(t)$ is continuous. Recall that continuous maps are necessarily Borel measurable; see Corollary B.4. Hence, $\mathrm{ev}_t : (\,C[0,1]\,,\|\cdot\|_\infty\,) \longrightarrow (\,\mathbb{R}\,,|\cdot|\,)$ is $(\mathcal{B}\,,\mathcal{O})$-measurable, for each $t \in [0,1]$. Now, suppose $X : \Omega \longrightarrow C[0,1]$ is $(\mathcal{A},\mathcal{B})$-measurable. Then, for each $t \in [0,1]$, the composition $X_t := \mathrm{ev}_t \circ X$ is $(\mathcal{A}\,,\mathcal{O})$-measurable, as required.

$(\Longleftarrow)$

Suppose that, for each $t \in [0,1]$, $X_t := \mathrm{ev}_t \circ X$ is $(\mathcal{A}\,,\mathcal{O})$-measurable. We seek to establish that $X : (\Omega,\mathcal{A}) \longrightarrow (C[0,1],\mathcal{B})$ is $(\mathcal{A},\mathcal{B})$-measurable. To this end, let

$$\mathcal{S} \;:=\; \left\{\, \mathrm{ev}_t^{-1}(H) \,\subset\, C[0,1] \;\middle|\; \begin{array}{c} t \in [0,1] \\ H \in \mathcal{O} \end{array} \right\} \;\subset\; \mathcal{P}(\,C[0,1]\,)\,.$$

Then, note that the $(\mathcal{A}\,,\mathcal{B})$-measurable of X follows immediately from Theorem B.3, Proposition 1.1, and the following

Claim: $X^{-1}(\,\mathcal{S}\,) \,\subset\, \mathcal{A}$.

Proof of Claim: Every set in $\mathcal{S}$ has the form $\mathrm{ev}_t^{-1}(H)$, for some $t \in [0,1]$ and some $H \in \mathcal{O}$. Note that

$$X^{-1}\big(\,\mathrm{ev}_t^{-1}(H)\,\big) \;=\; (\mathrm{ev}_t \circ X)^{-1}\Big(\,H\,\Big) \;=\; X_t^{-1}\Big(\,H\,\Big) \;\in\; \mathcal{A},$$

where the last containment follows immediately from the $(\mathcal{A}\,,\mathcal{O})$-measurability hypothesis on X_t, for each $t \in [0,1]$. This shows that $X^{-1}(\,\mathcal{S}\,) \subset \mathcal{A}$ and completes the proof of the Claim.

The proof of the Theorem is now complete. $\square$

Theorem 1.3
Suppose:

- $(\Omega,\mathcal{A})$ *is a measurable space, and* $\mathcal{O}$ *is the Borel* σ*-algebra of* $\mathbb{R}$ *(equipped with usual Euclidean metric).*
- $(C[0,1]\,,\|\cdot\|_\infty)$ *is the metric space of continuous* $\mathbb{R}$*-valued functions defined on the closed unit interval equipped with the supremum norm, and* $\mathcal{B} := \mathcal{B}(C[0,1]\,,\|\cdot\|_\infty)$ *is its Borel* σ*-algebra.*
- $X : \Omega \longrightarrow C[0,1]$ *is a function with domain* Ω *and codomain* $C[0,1]$*, but otherwise arbitrary.*
- *For each* $t \in [0,1]$*, let* $\mathrm{ev}_t : C[0,1] \longrightarrow \mathbb{R} : x \longmapsto x(t)$*.*
- *For each* $t \in [0,1]$*, let* $X_t := \mathrm{ev}_t \circ X$*. In other words,* $X_t : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto \mathrm{ev}_t(X(\omega)) = X(\omega)(t)$*.*

Then, the following are equivalent:

(i) X *is a* $(C[0,1]\,,\|\cdot\|_\infty)$*-valued random variable (in other words,* X *is* $(\mathcal{A}\,,\mathcal{B})$*-measurable).*

(ii) *For each* $t \in [0,1]$*,* X_t *is an* $\mathbb{R}$*-valued random variable (in other words, each* X_t *is* $(\mathcal{A}\,,\mathcal{O})$*-measurable).*

(iii) $\{\,X_t : \Omega \longrightarrow \mathbb{R}\,\}_{t\in[0,1]}$ *is a stochastic process indexed by the closed unit interval defined on the probability space* $(\Omega,\mathcal{A},\mu)$ *with state space* $\mathbb{R}$ *and continuous sample paths.*

Proof The equivalence of (i) and (ii) is immediate by the preceding Theorem. The equivalence of (ii) and (iii) is immediate by the definition of stochastic processes. $\square$

2 Scaling limits of finite-dimensional distributions of linearly interpolated random walks are multivariate Gaussian

Proposition 2.1

- *Let $\xi_1, \xi_2, \ldots : \Omega \longrightarrow \mathbb{R}$ be a sequence of independent and identically distributed $\mathbb{R}$-valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mu)$, with expectation value zero and common finite variance $\sigma^2 > 0$.*
- *Define the random variables:*

$$\begin{cases} S_0 \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto 0, & \text{and} \\ S_n \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto \displaystyle\sum_{i=1}^{n} \xi_i(\omega), & \text{for each } n \in \mathbb{N}. \end{cases}$$

- *For each $n \in \mathbb{N}$, define $X^{(n)} : \Omega \longrightarrow C[0,1]$ as follows:*

$$X^{(n)}(\omega)(t) \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) \;+\; n\left(t - \frac{i-1}{n}\right)\xi_i(\omega) \right\}, \quad \text{for each } \omega \in \Omega,\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],\; i = 1,2,3,\ldots,n.$$

- *For each $n \in \mathbb{N}$ and each $t \in [0,1]$, define $X_t^{(n)} : \Omega \longrightarrow \mathbb{R}$ as follows:*

$$X_t^{(n)}(\omega) \;:=\; X^{(n)}(\omega)(t), \quad \text{for each } \omega \in \Omega.$$

Then, the following statements are true:

(i) *For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,*

$$X^{(n)}(\omega)\left(\frac{i}{n}\right) \;=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot S_i(\omega), \quad \text{for } i = 0,1,2,\ldots,n.$$

(ii) *For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,*

$$X^{(n)}(\omega)(t) \;\; \textit{is the linear interpolation from} \;\; \frac{1}{\sigma \cdot \sqrt{n}} S_{i-1}(\omega) \;\; \textit{to} \;\; \frac{1}{\sigma \cdot \sqrt{n}} S_i(\omega) \;\; \textit{over} \;\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],$$

where $i = 1,2,\ldots,n$.

(iii) *For any $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$,*

$$\left(X_{t_1}^{(n)} - X_{t_0}^{(n)}, \;\ldots\;, X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \right) \;\overset{d}{\longrightarrow}\; N\left(\mu = \mathbf{0}\,, \Sigma = \operatorname{diag}(\, t_1 - t_0, \;\ldots\;, t_k - t_{k-1}\,) \right), \;\text{ as } n \longrightarrow \infty.$$

(iv) *For any $0 \leq t_1, t_2, \cdots, t_k \leq 1$,*

$$\left(X_{t_1}^{(n)}, X_{t_2}^{(n)}, \;\ldots\;, X_{t_k}^{(n)} \right) \;\overset{d}{\longrightarrow}\; N\left(\mu = \mathbf{0}\,, \Sigma = \Big[\min\{t_i, t_j\} \Big]_{1 \leq i,j \leq k} \right), \;\text{ as } n \longrightarrow \infty.$$

PROOF

(i) Obvious.

(ii) Obvious.

(iii) First, note that, for each $\omega \in \Omega$, $n \in \mathbb{N}$, and $t \in [0,1]$, we have

$$X^{(n)}_t(\omega) \;=\; \frac{1}{\sigma\cdot\sqrt{n}}\left\{\, S_{\lfloor nt \rfloor}(\omega) \;+\; \Big(nt - \lfloor nt \rfloor\Big)\cdot \xi_{\lfloor nt \rfloor + 1}(\omega) \right\},$$

where $\lfloor\,\cdot\,\rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$, defined by

$$\lfloor\, x \,\rfloor \;:=\; \max\left\{\, k \in \mathbb{Z} \;\middle|\; k \leq x \,\right\}, \quad \text{for each } x \in \mathbb{R},$$

is the round-down function. We next state three Claims, whose proofs will be given below. We note that the desired conclusion follows readily from Claim 3 and the Cramér-Wold Theorem (Theorem 1.9(iii), p.56, [5]); hence the present proof is complete once we establish the three Claims below.

Claim 1: If $\{\, a_n \,\}_{n\in\mathbb{N}}$ is a sequence of non-negative integers and $\{\, b_n \,\}_{n\in\mathbb{N}} \subset \mathbb{N}$ a sequence of positive integers satisfying:

$$a_n \;<\; b_n, \text{ for sufficiently large } n \in \mathbb{N}, \qquad \text{and} \qquad \lim_{n\to\infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$

then

$$\frac{1}{\sigma\cdot\sqrt{n}} \cdot \sum_{i\,=\,1+a_n}^{b_n} \xi_i \;\overset{d}{\longrightarrow}\; \sqrt{c}\cdot Z, \quad \text{where } Z \,\sim\, N(0,1).$$

Claim 2: For each fixed $t \in [0,1]$,

$$W(t)_n \;:=\; \frac{1}{\sigma\cdot\sqrt{n}} \cdot \Big(nt - \lfloor nt \rfloor\Big)\cdot \xi_{\lfloor nt \rfloor + 1} \;\overset{d}{\longrightarrow}\; 0.$$

Claim 3: For $0 \,\leq\, t_0 \,<\, t_1 \,<\, t_2 \,<\, \cdots \,<\, t_k \,\leq\, 1$, and arbitrary $c_1, c_2, \ldots, c_k \in \mathbb{R}$,

$$\sum_{i\,=\,1}^{k} c_i \left(X^{(n)}_{t_i} - X^{(n)}_{t_{i-1}} \right) \;\overset{d}{\longrightarrow}\; N\!\left(0\,,\, \sum_{i\,=\,1}^{k} c_i^2\,(t_i - t_{i-1})\right), \quad \text{as } n \longrightarrow \infty.$$

<u>Proof of Claim 1:</u> Note that, for sufficiently large $n \in \mathbb{N}$, we may write

$$\frac{1}{\sigma\cdot\sqrt{n}} \cdot \sum_{i\,=\,1+a_n}^{b_n} \xi_i \;=\; \frac{\sqrt{b_n - a_n}}{\sqrt{n}} \cdot \left(\frac{1}{\sigma\cdot\sqrt{b_n - a_n}} \cdot \sum_{i\,=\,1+a_n}^{b_n} \xi_i \right).$$

Since, by hypothesis, that

$$\lim_{n\to\infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$

Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [4]), once we establish the following:

$$\frac{1}{\sigma\cdot\sqrt{b_n - a_n}} \cdot \sum_{i\,=\,1+a_n}^{b_n} \xi_i \;\overset{d}{\longrightarrow}\; N(0,1), \quad \text{as } n \longrightarrow \infty.$$

We establish the above convergence by invoking the Lindeberg Central Limit Theorem (Theorem 1.15, §1.5.5, p.67, [5]). In the present context, the Lindeberg Condition is the following:

$$\lim_{n\to\infty} \frac{1}{B_n^2}\cdot E\left[\,\sum_{i\,=\,1+a_n}^{b_n} \xi_i^2\cdot I_{\left\{\,|\,\xi_i\,|\,\geq\,\varepsilon\,S_n\right\}}\,\right] \;=\; 0,\quad\text{for each } \varepsilon>0,$$

where

$$B_n^2 \;:=\; \mathrm{Var}\left[\,\sum_{i\,=\,1+a_n}^{b_n}\xi_i\,\right] \;=\; (b_n-a_n)\,\sigma^2 \;>\; 0.$$

The last equality used the hypothesis that $\xi_1,\ \xi_2,\ \ldots$ are independent and identically distributed with common finite variance $0<\sigma^2<\infty$. Hence, for each $\varepsilon>0$,

$$\begin{aligned}
\frac{1}{B_n^2}\cdot E\left[\,\sum_{i\,=\,1+a_n}^{b_n} \xi_i^2\cdot I_{\left\{\,|\,\xi_i\,|\,\geq\,\varepsilon\,B_n\right\}}\,\right] &= \frac{1}{(b_n-a_n)\,\sigma^2}\cdot(b_n-a_n)\cdot E\left[\,\xi_1^2\cdot I_{\left\{\,|\,\xi_1\,|\,\geq\,\varepsilon\sigma\,\sqrt{b_n-a_n}\right\}}\,\right] \\
&= \frac{1}{\sigma^2}\cdot E\left[\,\xi_1^2\cdot I_{\left\{\,|\,\xi_1\,|/\varepsilon\sigma\,\geq\,\sqrt{b_n-a_n}\right\}}\,\right] \;\longrightarrow\; 0,\quad\text{as } n\longrightarrow\infty,
\end{aligned}$$

since $\displaystyle\lim_{n\to\infty}\sqrt{b_n-a_n}=\infty$ and $\sigma^2=E\left[\,\xi_1^2\,\right]$ is finite. This verifies that the Lindeberg Condition indeed holds in the present context, and completes the proof of Claim 1.

<u>Proof of Claim 2:</u> First, note that $E[\,W(t)_n\,]=0$. We now argue that $W(t)_n\overset{p}{\longrightarrow}0$. To this end, let $\varepsilon>0$ be given. Then,

$$\begin{aligned}
\varepsilon^2\cdot P(\,|\,W(t)_n\,|\,\geq\,\varepsilon\,) &\leq E\left[\,W(t)_n^2\cdot I_{\{\,|\,W(t)_n\,|\,\geq\,\varepsilon\,\}}\,\right] \\
&\leq E\left[\,W(t)_n^2\,\right] \;=\; \mathrm{Var}(\,W(t)_n\,) \;=\; \mathrm{Var}\left[\,\frac{1}{\sigma\cdot\sqrt{n}}\cdot\Big(nt-\lfloor nt\rfloor\Big)\cdot\xi_{\lfloor nt\rfloor+1}\,\right] \\
&= \frac{1}{n\cdot\sigma^2}\cdot\Big(nt-\lfloor\,nt\,\rfloor\Big)^2\cdot\mathrm{Var}\big(\,\xi_{\lfloor nt\rfloor+1}\,\big) \;=\; \frac{1}{n\cdot\sigma^2}\cdot\Big(nt-\lfloor\,nt\,\rfloor\Big)^2\cdot\sigma^2 \\
&\leq \frac{1}{n},
\end{aligned}$$

which implies

$$\lim_{n\to\infty} P(\,|\,W(t)_n\,|\,\geq\,\varepsilon\,) \;=\; 0,\;\;\text{for each } \varepsilon>0,$$

i.e. $W(t)_n\overset{p}{\longrightarrow}0$, as $n\longrightarrow\infty$ (Definition 2, Chapter 1, [4]), which is equivalent to $W(t)_n\overset{d}{\longrightarrow}0$, as $n\longrightarrow\infty$ (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [4]). This proves Claim 2.

<u>Proof of Claim 3:</u> Let $0\,\leq\,t_0\,<\,t_1\,<\,t_2\,<\,\cdots\,<\,t_k\,\leq\,1$, and $c_1,c_2,\ldots,c_k\in\mathbb{R}$ be arbitrary. Observe that:

$$\begin{aligned}
&\sum_{i\,=\,1}^{k} c_i\left(X_{t_i}^{(n)}-X_{t_{i-1}}^{(n)}\right) \\
&= \sum_{i\,=\,1}^{k}\frac{c_i}{\sigma\cdot\sqrt{n}}\left\{\,S_{\lfloor nt_i\rfloor}\,-\,S_{\lfloor nt_{i-1}\rfloor}\right\} + \sum_{i\,=\,1}^{k}\frac{c_i}{\sigma\cdot\sqrt{n}}\left\{\Big(nt_i-\lfloor nt_i\rfloor\Big)\cdot\xi_{\lfloor nt_i\rfloor+1}\,-\,\Big(nt_{i-1}-\lfloor nt_{i-1}\rfloor\Big)\cdot\xi_{\lfloor nt_{i-1}\rfloor+1}\right\} \\
&= \sum_{i\,=\,1}^{k}\frac{c_i}{\sigma\cdot\sqrt{n}}\left\{\,\sum_{j\,=\,1+\lfloor nt_{i-1}\rfloor}^{\lfloor nt_i\rfloor}\xi_j\,\right\} + \sum_{i\,=\,1}^{k} c_i\left\{\,W(t_i)_n\,-\,W(t_{i-1})_n\,\right\} \\
&= \sum_{i\,=\,1}^{k} c_i\,Y_i^{(n)} \;+\; \sum_{i\,=\,1}^{k} c_i\left\{\,W(t_i)_n\,-\,W(t_{i-1})_n\,\right\}
\end{aligned}$$

By Claim 2 and Slutsky's Theorem (Corollary, p.40, [4]),

$$\sum_{i=1}^{k} c_i \left\{ W(t_i)_n \;-\; W(t_{i-1})_n \right\} \;\overset{d}{\longrightarrow}\; 0, \;\text{ as }\; n \longrightarrow \infty. \tag{2.1}$$

Next, note that since $\xi_1, \xi_2, \xi_3, \ldots$ are independent, we see that, for each fixed $n \in \mathbb{N}$,

$$Y_i^{(n)} \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1}\rfloor}^{\lfloor nt_i \rfloor} \xi_j, \quad i \;=\; 1, 2, 3, \ldots, k,$$

are independent. Now, since $0 \le t_{i-1} < t_i \le 1$, it follows that $\lfloor nt_{i-1}\rfloor < \lfloor nt_i \rfloor$ for sufficiently large $n \in \mathbb{N}$. In addition,

$$\begin{aligned}
\frac{\lfloor nt_i\rfloor - \lfloor nt_{i-1}\rfloor}{n} \;&=\; \frac{\lfloor nt_i\rfloor}{n} - \frac{\lfloor nt_{i-1}\rfloor}{n} \;=\; \left(\frac{nt_i}{n} + \frac{\lfloor nt_i\rfloor - nt_i}{n}\right) - \left(\frac{nt_{i-1}}{n} + \frac{\lfloor nt_{i-1}\rfloor - nt_{i-1}}{n}\right) \\
&=\; t_i \;-\; t_{i-1} \;+\; \frac{\lfloor nt_i\rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1}\rfloor - nt_{i-1}}{n},
\end{aligned}$$

which implies

$$\left| \frac{\lfloor nt_i\rfloor - \lfloor nt_{i-1}\rfloor}{n} \;-\; (t_i - t_{i-1}) \right| \;=\; \left| \frac{\lfloor nt_i\rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1}\rfloor - nt_{i-1}}{n} \right| \;\le\; \frac{2}{n} \;\longrightarrow\; 0, \quad \text{as }\; n \longrightarrow \infty.$$

Thus,

$$\lim_{n\to\infty} \frac{\lfloor nt_i\rfloor - \lfloor nt_{i-1}\rfloor}{n} \;=\; t_i \;-\; t_{i-1} \;>\; 0.$$

Thus, by Claim 1, we see that, for each $i \;=\; 1, 2, \ldots, k$,

$$Y_i^{(n)} \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1}\rfloor}^{\lfloor nt_i \rfloor} \xi_j \;\overset{d}{\longrightarrow}\; \sqrt{t_i - t_{i-1}} \cdot N(0,1) \;=\; N\Big(0\,,\, t_i - t_{i-1}\Big), \quad \text{as }\; n \longrightarrow \infty. \tag{2.2}$$

By (2.1), (2.2), Proposition A.1, and Slutsky's Theorem (Corollary, p.40, [4]), we now see that

$$\sum_{i=1}^{k} c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \;=\; \sum_{i=1}^{k} c_i \, Y_i^{(n)} \;+\; \sum_{i=1}^{k} c_i \left\{ W(t_i)_n \;-\; W(t_{i-1})_n \right\} \;\overset{d}{\longrightarrow}\; N\left(0\,,\, \sum_{i=1}^{k} c_i^2 (t_i - t_{i-1})\right).$$

This completes the proof of Claim 3.

(iv) Let $t_0 := 0$, hence, $X_{t_0}^{(n)} \equiv 0$ for each $n \in \mathbb{N}$. We thus have, for each $n \in \mathbb{N}$,

$$\begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} \end{pmatrix} \;=\; \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}}_{T} \cdot \begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} - X_{t_1}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \end{pmatrix}.$$

By (iii), we know that

$$\begin{pmatrix} X^{(n)}_{t_1} \\ X^{(n)}_{t_2} - X^{(n)}_{t_1} \\ \vdots \\ X^{(n)}_{t_k} - X^{(n)}_{t_{k-1}} \end{pmatrix} \overset{d}{\longrightarrow} \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix} \sim N\Big(\,\mu = \mathbf{0}\,,\ \Sigma = \operatorname{diag}(t_1, t_2 - t_1, \ldots, t_k - t_{k-1})\,\Big), \ \text{as} \ n \longrightarrow \infty.$$

Since the map $\mathbb{R}^k \longrightarrow \mathbb{R}^k : x \longmapsto T \cdot x$ is continuous, we see immediately by Slutsky's Theorem (Theorem 6(a), p.39, [4]) that

$$\begin{pmatrix} X^{(n)}_{t_1} \\ X^{(n)}_{t_2} \\ \vdots \\ X^{(n)}_{t_k} \end{pmatrix} \overset{d}{\longrightarrow} T \cdot \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix}, \quad \text{as} \ n \longrightarrow \infty.$$

Since the map $\mathbb{R}^k \longrightarrow \mathbb{R}^k : x \longmapsto T \cdot x$ is an invertible linear automorphism on $\mathbb{R}^k$, we see that

$$L \ = \ \begin{pmatrix} L_{t_1} \\ L_{t_2} \\ \vdots \\ L_{t_k} \end{pmatrix} \ := \ T \cdot \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix}$$

is still an $\mathbb{R}^k$-valued Gaussian random variable, and it clearly has expectation value $\mathbf{0} \in \mathbb{R}^k$, since each of Z_{t_1}, $Z_{t_2-t_1}$, $\ldots$, $Z_{t_k-t_{k-1}}$ has expectation value $0 \in \mathbb{R}$. It remains only to compute the covariance matrix of the $\mathbb{R}^k$-valued Gaussian random variable L. To this end, consider $1 \leq i \leq j \leq k$, i.e. $t_i \leq t_j$. Then, using the alternative notation $Z_{t_1 - t_0} := Z_{t_1}$, we have

$$\begin{aligned}
\operatorname{Cov}\big(L_{t_i}\,,\, L_{t_j}\big) &= \operatorname{Cov}\big(Z_{t_1} + Z_{t_2-t_1} + \cdots + Z_{t_i-t_{i-1}}\,,\, Z_{t_1} + Z_{t_2-t_1} + \cdots + Z_{t_j-t_{j-1}}\big) \\
&= \operatorname{Cov}\left(\sum_{a=1}^{i} Z_{t_a - t_{a-1}}\,,\, \sum_{b=1}^{j} Z_{t_b - t_{b-1}}\right) = \sum_{a=1}^{i}\sum_{b=1}^{j} \operatorname{Cov}\big(Z_{t_a - t_{a-1}}\,,\, Z_{t_b - t_{b-1}}\big) \\
&= \sum_{a=1}^{i} \operatorname{Cov}\big(Z_{t_a - t_{a-1}}\,,\, Z_{t_a - t_{a-1}}\big) = \sum_{a=1}^{i} \operatorname{Var}\big(Z_{t_a - t_{a-1}}\big) = \sum_{a=1}^{i} (\,t_a - t_{a-1}\,) \\
&= (\,t_1 - t_0\,) + (\,t_2 - t_1\,) + \cdots + (\,t_{i-1} - t_{i-2}\,) + (\,t_i - t_{i-1}\,) \\
&= t_i \ = \ \min\{\,t_i\,,t_j\,\},
\end{aligned}$$

as required.

$\square$

A Technical Lemmas

Note that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ does NOT in general imply $X_n + Y_n \xrightarrow{d} X + Y$. But the implication does hold if X_n and Y_n are independent for each $n \in \mathbb{N}$, and both X and Y are Gaussian random variables, as the following Proposition shows.

Proposition A.1 *Let $k \in \mathbb{N}$ be fixed. Suppose:*

- *For each $n \in \mathbb{N}$,*
$$Y_1^{(n)},\, Y_2^{(n)},\, \ldots,\, Y_k^{(n)} \,:\, \Omega^{(n)} \longrightarrow \mathbb{R}$$
are independent $\mathbb{R}$-valued random variables defined on the probability space $\Omega^{(n)}$.
- *For each $i = 1, 2, \ldots, k$,*
$$Y_i^{(n)} \xrightarrow{d} N\big(\,\mu_i,\, \sigma_i^2\,\big), \quad \text{as } n \longrightarrow \infty.$$

Then, for any $c_1,\, c_2,\, \ldots,\, c_k \in \mathbb{R}$,
$$\sum_{i=1}^{k} c_i\, Y_i^{(n)} \xrightarrow{d} N\left(\sum_{i=1}^{k} c_i\,\mu_i\,,\ \sum_{i=1}^{k} c_i^2\,\sigma_i^2\right), \quad \text{as } n \longrightarrow \infty.$$

Proof Let $Y^{(n)} := \sum_{i=1}^{k} c_i\, Y_i^{(n)}$. Let φ_X denote the characteristic function of a $\mathbb{R}$-valued random variable X. Then,

$$\begin{aligned}
\varphi_{Y^{(n)}}(t) &= \varphi_{\sum_i^k c_i Y_i^{(n)}}(t) \\
&= \prod_{i=1}^{k} \varphi_{c_i Y_i^{(n)}}(t), \quad \text{since } Y_1^{(n)},\, \ldots,\, Y_k^{(n)} \text{ are independent} \\
&= \prod_{i=1}^{k} \varphi_{Y_i^{(n)}}(\,c_i t\,) \\
&\longrightarrow \prod_{i=1}^{k} \exp\left\{\sqrt{-1}\,\mu_i\,(c_i\,t) \,-\, \frac{1}{2}\,\sigma_i^2\,(\,c_i t\,)^2\right\} \\
&= \exp\left\{\sqrt{-1}\left(\sum_{i=1}^{k} c_i\mu_i\right) t \,-\, \frac{1}{2}\left(\sum_{i=1}^{k} c_i^2\sigma_i^2\right) t^2\right\}, \quad \text{as } n \longrightarrow \infty,
\end{aligned}$$

where the second and third equalities follow from the properties of characteristic functions of random variables (see p.21, [4]), while the expression of the limit follows from the fact that the characteristic function φ_Z of a random variable Z with distribution $N\big(\,\mu,\,\sigma^2\,\big)$ is

$$\varphi_Z \;=\; \exp\left\{\sqrt{-1}\,\mu t \,-\, \frac{1}{2}\,\sigma^2\,t^2\right\}.$$

The Proposition now follows immediately from the Lévy-Cramér Continuity Theorem (Theorem 1.9(ii), p.56, [5]). $\square$

B Continuous maps are Borel measurable

Lemma B.1 (The pre-image of a σ-algebra is itself a σ-algebra.)
Suppose Ω is a non-empty set, $(X, \mathcal{X})$ is a measurable space, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then,
$$f^{-1}(\mathcal{X}) \;:=\; \left\{\, f^{-1}(V) \subset \Omega \;\middle|\; V \in \mathcal{X} \,\right\}$$
is a σ-algebra of subsets of Ω.

Proof

$\underline{\Omega \in f^{-1}(\mathcal{X})}$ $f(\Omega) \subset X \implies \Omega = f^{-1}(X) \in f^{-1}(\mathcal{X})$.

$\underline{f^{-1}(\mathcal{X}) \text{ is closed under complementations}}$ Let $V \in \mathcal{X}$. Then, $X \setminus V \in \mathcal{X}$, and

$$\Omega \setminus f^{-1}(V) \;=\; \{\, \omega \in \Omega \mid f(\omega) \notin V \,\} \;=\; \{\, \omega \in \Omega \mid f(\omega) \in X \setminus V \,\} \;=\; f^{-1}(X \setminus V) \;\in\; f^{-1}(\mathcal{X}),$$

which shows that $f^{-1}(\mathcal{X})$ is indeed closed under complementations.

$\underline{f^{-1}(\mathcal{X}) \text{ is closed countable unions}}$ Let $V_1, V_2, \ldots \in \mathcal{X}$. Then, $\bigcup_{i=1}^{\infty} V_i \in \mathcal{X}$, and

$$\bigcup_{i=1}^{\infty} f^{-1}(V_i) \;=\; \left\{ \omega \in \Omega \;\middle|\; \begin{array}{c} f(\omega) \in V_i \\ \text{for some } i \in \mathbb{N} \end{array} \right\} \;=\; f^{-1}\left(\bigcup_{i=1}^{\infty} V_i \right) \;\in\; f^{-1}(\mathcal{X}),$$

which proves that $f^{-1}(\mathcal{X})$ is indeed closed under countable unions.

This concludes the proof that that $f^{-1}(\mathcal{X})$ is a σ-algebra of subsets of Ω. □

Lemma B.2 (The push-forward of a σ-algebra is itself a σ-algebra.)
Suppose $(\Omega, \mathcal{A})$ is a measurable space, X is a non-empty set, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then,

$$\mathcal{F} \;:=\; \{\, V \subset X \mid f^{-1}(V) \in \mathcal{A} \,\}$$

is a σ-algebra of subsets of X.

Proof

$\underline{X \in \mathcal{F}}$ $f^{-1}(X) = \Omega \in \mathcal{A} \implies X \in \mathcal{F}$.

$\underline{\mathcal{F} \text{ is closed under complementations}}$ $V \in \mathcal{F} \implies f^{-1}(V) \in \mathcal{A} \implies f^{-1}(X \setminus V) = \Omega \setminus f^{-1}(V) \in \mathcal{A} \implies X \setminus V \in \mathcal{F}$, which proves that $\mathcal{F}$ is indeed closed under complementations.

$\underline{\mathcal{F} \text{ is closed under countable unions}}$

$$\begin{aligned} V_1, V_2, \ldots \;\in\; \mathcal{F} \;&\implies\; f^{-1}(V_1), f^{-1}(V_2), \ldots \;\in\; \mathcal{A} \\ &\implies\; f^{-1}\left(\bigcup_{i=1}^{\infty} V_i \right) \;=\; \bigcup_{i=1}^{\infty} f^{-1}(V_i) \;\in\; \mathcal{A} \\ &\implies\; \bigcup_{i=1}^{\infty} V_i \;\in\; \mathcal{F}, \end{aligned}$$

which proves that $\mathcal{F}$ is indeed closed under countable unions. □

Theorem B.3
Suppose $(\Omega, \mathcal{A})$ and $(X, \mathcal{X})$ are measurable spaces, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then, f is $(\mathcal{A}, \mathcal{X})$-measurable if there exists $\mathcal{S} \subset \mathcal{X}$ satisfying the following conditions:

- *$\mathcal{S}$ generates $\mathcal{X}$, i.e. $\sigma(\mathcal{S}) = \mathcal{X}$, and*
- *$f^{-1}(\mathcal{S}) \subset \mathcal{A}$.*

PROOF By Lemma B.2,

$$\mathcal{F} \;:=\; \left\{\, V \subset X \;\middle|\; f^{-1}(V) \in \mathcal{A} \,\right\}$$

is a σ-algebra of subsets of X. By hypothesis, $\mathcal{S} \subset \mathcal{F}$; hence, $\mathcal{X} = \sigma(\mathcal{S}) \subset \mathcal{F}$. Thus, $f^{-1}(\mathcal{X}) \subset \mathcal{A}$; equivalently, f is $(\mathcal{A}, \mathcal{X})$-measurable. □

Corollary B.4 (Continuous maps are Borel measurable.)
Suppose X_1, X_2 are topological spaces, and $\mathcal{B}_1$, $\mathcal{B}_2$ are their respective Borel σ-algebras. Then, every continuous map $f : X_1 \longrightarrow X_2$ is $(\mathcal{B}_1, \mathcal{B}_2)$-measurable.

C Topology

Theorem C.1 (Appendix M3, [3])
Suppose S is a metric space. Then, the following conditions are equivalent:

(i) *S is separable.*

(ii) *The topology of S has a countable basis.*

(iii) *Every open cover of each subset of S has a countable subcover.*

D Modulus of continuity

Definition D.1 (Modulus of continuity)
*Let $\mathbb{R}^{[0,1]}$ denote the set of all arbitrary $\mathbb{R}$-valued functions defined on the closed unit interval $[0,1]$. The **modulus of continuity** is, by definition, the following function:*

$$w \;:\; \mathbb{R}^{[0,1]} \times (0,1] \;\longrightarrow\; [0,\infty] \;:\; (f,\delta) \;\longmapsto\; \sup\left\{\, |\,f(s) - f(t)\,| \;\middle|\; \begin{array}{c} (s,t) \in [0,1]\times[0,1] \\ |s-t| \;\leq\; \delta \end{array} \right\}.$$

Proposition D.2

(i) *The restriction of w to $C[0,1] \times (0,1]$ takes values in $[0,\infty)$.*

(ii) *For each $\delta \in (0,1]$ and each $f, g \in C[0,1]$, we have*

$$\Big|\, w(f,\delta) \,-\, w(g,\delta) \,\Big| \;\;\leq\;\; 2\,\|\, f \,-\, g \,\|_\infty \,.$$

(iii) *For each $\delta \in (0,1]$, the map $w(\,\cdot\,,\delta) : (C[0,1], \|\cdot\|_\infty) \longrightarrow \mathbb{R} : f \longmapsto w(f,\delta)$ is continuous.*

PROOF
For each $\delta \in (0,1]$, the set

$$D(\delta) \;:=\; \left\{\, (s,t) \in [0,1]\times[0,1] \;\middle|\; |\,s-t\,| \leq \delta \,\right\}$$

is a subset of the compact set $[0,1] \times [0,1]$.

(i) For each $f \in C[0,1]$, the map

$$[0,1]\times[0,1] \;\longrightarrow\; \mathbb{R} \;:\; (s,t) \;\longmapsto\; |\,f(s) - f(t)\,|$$

is continuous on the compact set $[0,1]\times[0,1]$; hence, it is bounded and attains its supremum on $[0,1]\times[0,1]$; in particular, its supremum on $[0,1]\times[0,1]$ is a (finite non-negative) real number. Hence,

$$\begin{aligned} w(f,\delta) \;&:=\; \sup\left\{\,|\,f(s)-f(t)\,| \;\middle|\; \begin{array}{c}(s,t)\in[0,1]\times[0,1]\\ |s-t|\,\leq\,\delta\end{array}\right\} \;=\; \sup_{(s,t)\,\in\,D(\delta)}\left\{\,\left|\,f(s)-f(t)\,\right|\,\right\} \\ &\leq\; \sup_{(s,t)\,\in\,[0,1]\times[0,1]}\left\{\,\left|\,f(s)-f(t)\,\right|\,\right\} \;<\; \infty. \end{aligned}$$

This proves (i).

(ii) Recall that for any $a,b\in\mathbb{R}$, we have

$$|\,a\,| \;=\; |\,a-b+b\,| \;\leq\; |\,a-b\,| \;+\; |\,b\,|,$$

which in turn implies:

$$|\,a\,| \;-\; |\,b\,| \;\leq\; |\,a-b\,|.$$

Consequently, for each $f,g\in C[0,1]$ and each $s,t\in[0,1]$, we have:

$$\begin{aligned} \left|\,f(s)-f(t)\,\right| \;-\; \left|\,g(s)-g(t)\,\right| \;&\leq\; \left|\,f(s)-f(t)-g(s)+g(t)\,\right| \\ &\leq\; \left|\,f(s)-g(s)\,\right| \;+\; \left|\,f(t)-g(t)\,\right| \;\leq\; 2\,\left\|\,f-g\,\right\|_{\infty}. \end{aligned}$$

On the other hand, note that for any $g\in C[0,1]$ and any $(s,t)\in[0,1]\times[0,1]$ with $|\,s-t\,|\,\leq\,\delta$, i.e. $(s,t)\in D(\delta)$, we have

$$\left|\,g(s)-g(t)\,\right| \;\leq\; \sup_{(\xi,\zeta)\,\in\,D(\delta)}\left\{\,\left|\,g(\xi)-g(\zeta)\,\right|\,\right\} \;=:\; w(g,\delta),$$

hence,

$$-\,w(g,\delta) \;\leq\; -\,\left|\,g(s)-g(t)\,\right|, \quad \text{for any } g\in C[0,1] \text{ and any } (s,t)\in D(\delta).$$

Thus, we see that, for any $f,g\in C[0,1]$ and any $(s,t)\in D(\delta)$, we have

$$\left|\,f(s)-f(t)\,\right| \;-\; w(g,\delta) \;\leq\; \left|\,f(s)-f(t)\,\right| \;-\; \left|\,g(s)-g(t)\,\right| \;\leq\; 2\,\left\|\,f-g\,\right\|_{\infty}.$$

Taking supremum of the left-hand side of the preceding inequality over $(s,t)\in D(\delta)$ now yields:

$$\begin{aligned} w(f,\delta) \;-\; w(g,\delta) \;&=\; \sup_{(\xi,\zeta)\,\in\,D(\delta)}\left\{\,\left|\,f(\xi)-f(\zeta)\,\right|\,\right\} \;-\; w(g,\delta) \\ &=\; \sup_{(\xi,\zeta)\,\in\,D(\delta)}\left\{\,\left|\,f(\xi)-f(\zeta)\,\right| \;-\; w(g,\delta)\,\right\} \\ &\leq\; 2\,\left\|\,f-g\,\right\|_{\infty}. \end{aligned}$$

Interchanging f and g in the preceding inequality now yields:

$$\left|\,w(f,\delta) \;-\; w(g,\delta)\,\right| \;\leq\; 2\,\left\|\,f-g\,\right\|_{\infty}.$$

This completes the proof of (ii).

(iii) This is an immediate consequence of (ii).

□

Corollary D.3

For any $\delta \in (0,1]$, the map $w(\,\cdot\,,\delta) : C[0,1] \longrightarrow \mathbb{R}$ is an $\mathbb{R}$-valued random variable (i.e. an $\mathbb{R}$-valued Borel measurable function).

Proof $w(\,\cdot\,,\delta)$ is continuous by the preceding Theorem, and hence, Borel measurable, by Corollary B.4. □

Proposition D.4 (Theorem 7.4, [3])

Suppose:

- $\delta > 0$ *and* $0 = t_0 < t_1 < \cdots < t_n = 1$ *satisfy:*

$$\min_{1\,\leq\,i\,\leq\,n}\left\{\,t_i - t_{i-1}\,\right\} \;\;\geq\;\; \delta.$$

- $C[0,1]$ *is the Banach space of continuous* $\mathbb{R}$*-valued functions defined on* $[0,1]$ *equipped with the supremum norm.*

Then, the following statements are true:

(i) *For each* $f \in C[0,1]$, *we have:*

$$w(f,\delta) \;\;\leq\;\; 3\cdot\max_{1\,\leq\,i\,\leq\,n}\left\{\,\sup_{s\,\in\,[t_{i-1},t_i]}\left|\,f(s)-f(t_{i-1})\,\right|\,\right\}.$$

(ii) *For each* $\varepsilon > 0$ *and each Borel probability measure* P *on the (separable) Banach space* $(C[0,1],\|\cdot\|_\infty)$, *we have:*

$$P\Big(\Big\{\,f\in C[0,1]\;\Big|\;w(f,\delta)\,\geq\,3\,\varepsilon\,\Big\}\Big) \;\;\leq\;\; \sum_{i\,=\,1}^{n} P\left(\left\{\,f\in C[0,1]\;\left|\;\sup_{s\,\in\,[t_{i-1},t_i]}\left|\,f(s)-f(t_{i-1})\,\right|\,\geq\,\varepsilon\,\right.\right\}\right).$$

Proof

(i) First, note that

$$\left.\begin{array}{rcl} |\,s-t\,| & \leq & \delta \\ \displaystyle\min_{1\,\leq\,i\,\leq\,n}\left\{\,t_i-t_{i-1}\,\right\} & \geq & \delta \end{array}\right\} \;\;\Longrightarrow\;\; \left\{\begin{array}{lll} \text{either} & s,t\,\in\,[t_{i-1},t_i], & \text{for some } i\in\{1,2,\ldots,n\} \\ \text{or} & s,t\,\in\,[t_{i-1},t_i]\cup[t_{i-1}-t_{i-2}], & \text{for some } i\in\{2,\ldots,n\} \end{array}\right.$$

For the case in which both s and t lie in the same subinterval $[t_{i-1},t_i]$, for some $i\in\{1,2,\ldots,n\}$, we have

$$\left|\,f(s)-f(t)\,\right| \;\;\leq\;\; \left|\,f(s)-f(t_{i-1})\,\right| + \left|\,f(t_{i-1})-f(t)\,\right| \;\;\leq\;\; 2\cdot\max_{1\,\leq\,i\,\leq\,n}\left\{\,\sup_{s\,\in\,[t_{i-1},t_i]}\left|\,f(s)-f(t_{i-1})\,\right|\,\right\}.$$

For the case in which s and t lie in adjacent subintervals, say $s\in[t_{i-2},t_{i-1}]$ and $t\in[t_{i-1},t_i]$, for some $i\in\{2,\ldots,n\}$, we have

$$\begin{aligned} \left|\,f(s)-f(t)\,\right| \;&\leq\; \left|\,f(s)-f(t_{i-2})\,\right| + \left|\,f(t_{i-2})-f(t_{i-1})\,\right| + \left|\,f(t_{i-1})-f(t)\,\right| \\ &\leq\; 3\cdot\max_{1\,\leq\,i\,\leq\,n}\left\{\,\sup_{s\,\in\,[t_{i-1},t_i]}\left|\,f(s)-f(t_{i-1})\,\right|\,\right\}. \end{aligned}$$

Thus we see that, for any $f\in C[0,1]$ and any $(s,t)\in[0,1]\times[0,1]$ with $|\,s-t\,|\leq\delta$, we have

$$\left|\,f(s)-f(t)\,\right| \;\;\leq\;\; 3\cdot\max_{1\,\leq\,i\,\leq\,n}\left\{\,\sup_{s\,\in\,[t_{i-1},t_i]}\left|\,f(s)-f(t_{i-1})\,\right|\,\right\},$$

which implies, for each $f \in C[0,1]$,

$$w(f,\delta) \;:=\; \sup\left\{\, |\,f(s)-f(t)\,| \;\middle|\; \begin{array}{c}(s,t)\in[0,1]\times[0,1]\\ |s-t|\,\leq\,\delta\end{array}\right\} \;\leq\; 3\cdot\max_{1\leq i\leq n}\left\{\,\sup_{s\,\in\,[t_{i-1},t_i]}\left|\,f(s)-f(t_{i-1})\,\right|\,\right\}.$$

This proves (i).

(ii) By (i), we see that, for any $\varepsilon,\delta>0$ and any $f\in C[0,1]$, we have:

$$3\,\varepsilon \;\leq\; w(f,\delta) \quad\Longrightarrow\quad \varepsilon \;\leq\; \max_{1\leq i\leq n}\left\{\,\sup_{s\,\in\,[t_{i-1},t_i]}\left|\,f(s)-f(t_{i-1})\,\right|\,\right\}.$$

For any $\varepsilon,\delta>0$ and any $C[0,1]$-valued random variable $X:\big(\,\Omega,\mathcal{A},\mu\,\big)\longrightarrow(C[0,1],\|\cdot\|_\infty)$, we have:

$$\begin{aligned}\left\{\,f\in C[0,1]\;\middle|\;3\,\varepsilon\,\leq\,w(f,\delta)\,\right\} \;&\subset\; \left\{\,f\in C[0,1]\;\middle|\;\varepsilon\,\leq\,\max_{1\leq i\leq n}\left\{\,\sup_{s\,\in\,[t_{i-1},t_i]}\left|\,f(s)-f(t_{i-1})\,\right|\,\right\}\,\right\}\\ &=\;\bigcup_{i=1}^{n}\left\{\,f\in C[0,1]\;\middle|\;\varepsilon\,\leq\,\sup_{s\,\in\,[t_{i-1},t_i]}\left|\,f(s)-f(t_{i-1})\,\right|\,\right\},\end{aligned}$$

and (ii) now follows by sub-additivity of measures.

□

Corollary D.5 (Corollary of Theorem 7.4, [3])

Suppose $\{\,P_n\,\}_{n\in\mathbb{N}}$ is a sequence of Borel probability measures on $(C[0,1],\|\cdot\|_\infty)$.
Then, (i) implies (ii):

(i) *For each $\varepsilon,\eta>0$, there exist $\delta\in(0,1)$ and $n_0\in\mathbb{N}$ such that*

$$\frac{1}{\delta}\cdot P_n\left(\left\{\,f\in C[0,1]\;\middle|\;\varepsilon\;\leq\;\sup_{s\,\in\,[\,t\;,\;\min\{1,t+\delta\}\,]}\left|\,f(s)-f(t)\,\right|\,\right\}\right) \;\leq\; \eta,\quad\text{for each } t\in[0,1] \text{ and each } n\geq n_0.$$

(ii) *For each $\varepsilon,\eta>0$, there exist $\delta\in(0,1)$ and $n_0\in\mathbb{N}$ such that*

$$P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;\varepsilon\,\leq\,w(f,\delta)\,\Big\}\Big) \;\leq\; \eta,\quad\text{for each } n\geq n_0.$$

PROOF Suppose (i) holds and let $\varepsilon,\eta>0$ be given. By (i), there exists $\delta\in(0,1)$ and $n_0\in\mathbb{N}$ such that

$$\frac{1}{\delta}\cdot P_n\left(\left\{\,f\in C[0,1]\;\middle|\;\frac{\varepsilon}{3}\;\leq\;\sup_{s\,\in\,[\,t\;,\;\min\{1,t+\delta\}\,]}\left|\,f(s)-f(t)\,\right|\,\right\}\right) \;\leq\; \eta,\quad\text{for each } t\in[0,1] \text{ and each } n\geq n_0.$$

Now, let $t_0=0$, and $t_i=i\delta$, for $i=1,2,3,\ldots,k:=\lfloor\,1/\delta\,\rfloor$. Then, the preceding inequality and the preceding Theorem together imply:

$$\begin{aligned}P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;\varepsilon\,\leq\,w(f,\delta)\,\Big\}\Big) \;&\leq\; \sum_{i\,=\,1}^{k} P_n\left(\left\{\,f\in C[0,1]\;\middle|\;\frac{\varepsilon}{3}\,\leq\,\sup_{s\,\in\,[t_{i-1},t_i]}\left|\,f(s)-f(t_{i-1})\,\right|\,\right\}\right)\\ &\leq\; k\cdot\delta\cdot\eta \;=\; \lfloor\,1/\delta\,\rfloor\cdot\delta\cdot\eta \;\leq\; 1\cdot\eta \;=\; \eta,\quad\text{for each } n\geq n_0.\end{aligned}$$

This completes the proof of the Corollary. □

E The Arzelà-Ascoli Theorem

Recall that the space $C(X)$ of continuous $\mathbb{R}$-valued functions defined on a compact topological space X equipped with the supremum norm is a complete metric space (see Theorem 9.3, [1]). The Arzelà-Ascoli Theorem characterizes compactness of subsets of $C(X)$.

Definition E.1 (Equicontinuity)
Let X be a topological space and (Y,d) a metric space. Let Y^X denote the set of arbitrary functions from X into Y.

- *A subset $S \subset Y^X$ is said to be **equicontinuous at** $x_0 \in X$ if, for each $\varepsilon > 0$, there exists an open subset $V \subset X$ satisfying:*
$$x_0 \in V\,, \quad \text{and} \quad \sup_{(x,f)\,\in\,V\times S}\left\{\, d\Big(f(x), f(x_0)\Big)\,\right\} \;\leq\; \varepsilon.$$
- *A subset $S \subset Y^X$ is said to be **equicontinuous** if it is equicontinuous at each $x_0 \in X$.*

Definition E.2 (Uniform equicontinuity)
*Let (X,ρ) and (Y,d) two metric spaces. Let Y^X denote the set of arbitrary functions from X into Y. A subset $S \subset Y^X$ is said to be **uniformly equicontinuous** if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that:*
$$\sup\left\{\, d\Big(f(x_1), f(x_2)\Big) \;\middle|\; \begin{array}{c} f\in S,\; x_1,x_2\in X, \\ \rho(x_1,x_2)<\delta \end{array} \right\} \;\leq\; \varepsilon.$$

Proposition E.3
Let (X,ρ) and (Y,d) two metric spaces. Let Y^X denote the set of arbitrary functions from X into Y. Then, the following are true:

(i) *Uniform equicontinuity of a subset $S \subset Y^X$ implies equicontinuity of S.*

(ii) *Suppose furthermore that (X,ρ) is compact. Then, equicontinuity of a subset $S \subset Y^X$ implies uniform equicontinuity of S.*

Proof

(i) Suppose $S \subset Y^X$ is uniformly equicontinuous; we seek to prove that S is also equicontinuous. Let $x_0 \in X$ and $\varepsilon > 0$. By uniform equicontinuity of S, there exists $\delta > 0$ such that:
$$\sup\left\{\, d\Big(f(x_1), f(x_2)\Big) \;\middle|\; \begin{array}{c} f\in S,\; x_1,x_2\in X, \\ \rho(x_1,x_2)<\delta \end{array} \right\} \;\leq\; \varepsilon.$$
Let $V(x_0) := \left\{\, x\in X \;\middle|\; \rho(x,x_0)<\delta \,\right\}$. Then, $V(x_0)$ is an open subset of X, with $x_0 \in V(x_0)$, and
$$\sup_{(x,f)\,\in\,V(x_0)\times S}\left\{\, d\Big(f(x), f(x_0)\Big)\,\right\} \;\leq\; \sup\left\{\, d\Big(f(x_1), f(x_2)\Big) \;\middle|\; \begin{array}{c} f\in S,\; x_1,x_2\in X, \\ \rho(x_1,x_2)<\delta \end{array} \right\} \;\leq\; \varepsilon.$$
This proves the equicontinuity of S.

(ii) Suppose (X,ρ) is compact and $S \subset Y^X$ is equicontinuous. Let $\varepsilon > 0$ be given. By equicontinuity of S, for each $x \in X$, there exists an open ball $B(x,\delta_x) \subset X$ such that
$$\sup_{(\xi,f)\,\in\,B(x,\delta_x)\times S}\left\{\, d\Big(f(\xi), f(x)\Big)\,\right\} \;\leq\; \frac{\varepsilon}{2}.$$

Thus, $X = \bigcup_{x \in X} B\Big(x, \delta_x/2\Big)$ is an open cover of X. By compactness of X, this open cover admits a finite subcover:

$$X = \bigcup_{i=1}^{n} B\Big(x_i, \delta_{x_i}/2\Big).$$

Define $\delta := \min_{1 \leq i \leq n} \Big\{ \delta_{x_i}/2 \Big\} > 0$. Now note the uniform equicontinuity of S will be established once we prove the validity of the following:

Claim: For any $\xi_1, \xi_2 \in X$, and any $f \in S$, we have:

$$\rho(\xi_1, \xi_2) < \delta \quad \Longrightarrow \quad d\Big(f(\xi_1), f(\xi_2)\Big) \leq \varepsilon.$$

Proof of Claim: Suppose $\rho(\xi_1, \xi_2) < \delta$. Note that $\xi_1 \in B\Big(x_i, \delta_{x_i}/2\Big)$, for some $i = 1, 2, \ldots, n$. Next, observe that

$$\rho(x_i, \xi_2) \leq \rho(x_i, \xi_1) + \rho(\xi_1, \xi_2) \leq \frac{\delta_{x_i}}{2} + \delta \leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i}.$$

This shows that both $\xi_1, \xi_2 \in B(x_i, \delta_{x_i})$, which implies

$$d\Big(f(\xi_1), f(x_i)\Big) \leq \frac{\varepsilon}{2}, \quad \text{and} \quad d\Big(f(\xi_2), f(x_i)\Big) \leq \frac{\varepsilon}{2}, \quad \text{for each } f \in S,$$

which in turn implies:

$$d\Big(f(\xi_1), f(\xi_2)\Big) \leq d\Big(f(\xi_1), f(x_i)\Big) + d\Big(f(f(x_i), \xi_2)\Big) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \text{for each } f \in S.$$

This completes the proof of the Claim and the uniform equicontinuity of S. □

Theorem E.4 (Arzelà-Ascoli, Theorem 9.10, [1])
Suppose X is a compact topological space and $C(X)$ is the space of continuous $\mathbb{R}$-valued functions defined on X equipped with the supremum norm. Then, for each $S \subset C(X)$, the following conditions are equivalent:

(i) *S is a compact subset of $C(X)$.*

(ii) *S is closed, bounded, and equicontinuous subset of $C(X)$.*

Proof
(i) $\Longrightarrow$ (ii)
Recall that every compact subset in a metric space is closed and bounded. Thus, it remains only to show that $S \subset C(X)$ is equicontinuous. To this end, let $\varepsilon > 0$ be given. Recall that a metric space is compact if and only if it is complete and totally bounded (Theorem 7.8, [1]). Thus, the compactness hypothesis on S implies S is totally bounded; in particular, there exist $f_1, \ldots, f_n \in S$ such that

$$S \subset \bigcup_{i=1}^{n} B\Big(f_i, \frac{\varepsilon}{3}\Big).$$

Hence, for each $x_0 \in X$, we may define

$$V(x_0) := \bigcap_{i=1}^{n} f_i^{-1}\Big(\Big(f_i(x_0) - \frac{\varepsilon}{3}, f_i(x_0) + \frac{\varepsilon}{3}\Big)\Big).$$

Note that $V(x_0)$ is open and $x_0 \in V(x_0)$. Now, let $f \in S$ and $x \in V(x_0)$ be given. We may choose $i \in \{1,2,\ldots,n\}$ such that $f \in B\!\left(f_i\,,\,\dfrac{\varepsilon}{3}\right)$, i.e. $\|\,f - f_i\,\|_\infty \leq \dfrac{\varepsilon}{3}$. Hence,

$$\left|\,f(x) - f(x_0)\,\right| \;\leq\; \left|\,f(x) - f_i(x)\,\right| + \left|\,f_i(x) - f_i(x_0)\,\right| + \left|\,f_i(x_0) - f(x_0)\,\right| \;\leq\; \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \;=\; \varepsilon.$$

Thus,

$$\sup_{(x,f)\,\in\,V(x_0)\times S}\left\{\,\left|\,f(x) - f(x_0)\,\right|\,\right\} \;\leq\; \varepsilon.$$

This shows equicontinuity of S at $x_0 \in X$. Since $x_0 \in X$ is arbitrary, we may conclude that S is equicontinuous.

<u>(ii) $\Longrightarrow$ (i)</u>

Suppose $S \in C(X)$ is closed, bounded, and equicontinuous. We need to show that S is a compact subset of $C(X)$. Recall that every subset of a metric space is compact if and only if it is sequentially compact (Theorem 7.3, [1]). Thus, it suffices to show that every sequence $\{\,f_n\,\}_{n\in\mathbb{N}} \subset S$ has a convergent subsequence with limit in S. We start by stating and proving the following:

<u>Claim 1:</u>

For each $k \in \mathbb{N}$, there exists a finite subset $F_k \subset X$ and open neighbourhoods $\{\,V_y\,\}_{y\in F_k}$ such that

$$X \;=\; \bigcup_{y\,\in\,F_k} V_y,\,, \quad\text{and}\quad \sup\left\{\,\left|\,f(x) - f(y)\,\right| \;\middle|\; \begin{array}{c} x \in V_y,\; y \in F_k \\ f \in S \end{array}\right\} \;\leq\; \frac{1}{3k}.$$

Proof of Claim 1: By equicontinuity of S, for each $y \in X$, there exists an open neighbourhood $V_y \subset X$ of $y \in X$ such that

$$\sup\left\{\,\left|\,f(x) - f(y)\,\right| \;\middle|\; \begin{array}{c} x \in V_y \\ f \in S \end{array}\right\} \;\leq\; \frac{1}{3k}.$$

Thus, $X = \bigcup_{y\,\in\,X} V_y$ is an open cover of X. Compactness of X now implies that this open cover of X admits a finite subcover, i.e.

$$X \;=\; \bigcup_{y\,\in\,F_k} V_y, \quad \text{for some finite subset } F_k \subset X.$$

Lastly, note that

$$\sup\left\{\,\left|\,f(x) - f(y)\,\right| \;\middle|\; \begin{array}{c} x \in V_y,\; y \in F_k \\ f \in S \end{array}\right\} \;=\; \sup_{y\,\in\,F_k}\left\{\,\sup\left\{\,\left|\,f(x) - f(y)\,\right| \;\middle|\; \begin{array}{c} x \in V_y \\ f \in S \end{array}\right\}\right\} \;\leq\; \frac{1}{3k}.$$

This completes the proof of Claim 1.

Next, let $F := \bigcup_{k\,=\,1}^{\infty} F_k$. Note that F is a countably infinite set. Let $F = \{\,x_1, x_2, \ldots\,\}$ be an enumeration of F. Recall that we wish to prove that every sequence $\{\,f_n\,\}_{n\in\mathbb{N}} \subset S$ contains a convergent subsequence with limit in S. Now, consider the array of real numbers:

$$\begin{array}{cccc} f_1(x_1) & f_2(x_1) & f_3(x_1) & \cdots \\ f_1(x_2) & f_2(x_2) & f_3(x_2) & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

Since $S \subset C(X)$ is bounded (with respect to the $\|\cdot\|_\infty$ norm on $C(X)$), there exists $M > 0$ such that $\sup_{f\,\in\,S} \left\|\,f\,\right\|_\infty \leq M$. In particular, every row in the above array is bounded. By Theorem A.14, p.538, [2], there exists an increasing sequence of positive integers $n(1), n(2), n(3), \ldots$ such that the limit

$$\lim_{i\,\to\,\infty} f_{n(i)}(x_k) \;\text{ exists, for each } k = 1, 2, \ldots.$$

Claim 2: $\{ f_{n(i)} \}_{i\in\mathbb{N}}$ is a Cauchy sequence in $(C(X) , \|\cdot\|_\infty)$.

Proof of Claim 2: For each $k \in \mathbb{N}$, the convergence of $\{ f_{n(i)}(x_k) \}_{i\in\mathbb{N}}$ in $\mathbb{R}$ implies that each $\{ f_{n(i)}(x_k) \}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Since the set F_k is finite, we see that there exists $m_k \in \mathbb{N}$ such that

$$\left| f_{n(i)}(y) - f_{n(j)}(y) \right| < \frac{1}{3k}, \quad \text{for any } i,j \geq m_k, \text{ and each } y \in F_k.$$

Now, for each $x \in X$, there exists $y \in F_k$ such that $x \in V_y$. Hence, for any $i, j \geq m_k$ and any $x \in X$, we have

$$\begin{aligned} \left| f_{n(i)}(x) - f_{n(j)}(x) \right| &\leq \left| f_{n(i)}(x) - f_{n(i)}(y) \right| + \left| f_{n(i)}(y) - f_{n(j)}(y) \right| + \left| f_{n(j)}(y) - f_{n(j)}(x) \right| \\ &\leq \frac{1}{3k} + \frac{1}{3k} + \frac{1}{3k} = \frac{1}{k}. \end{aligned}$$

In other words,

$$\left\| f_{n(i)} - f_{n(j)} \right\|_\infty \leq \frac{1}{k}, \quad \text{for any } i,j \geq m_k.$$

This shows that $\{ f_{n(i)} \}_{i\in\mathbb{N}}$ is indeed a Cauchy sequence in $(C(X) , \|\cdot\|_\infty)$ and completes the proof of Claim 2.

Lastly, by Theorem 9.3, [1], $(C(X) , \|\cdot\|_\infty)$ is a complete metric space. Thus, the Cauchy sequence $\{ f_{n(i)} \}_{i\in\mathbb{N}} \subset C(X)$ converges to some element $f_0 \in C(X)$. Since $S \subset C(X)$ is, by hypothesis, a closed subset of $C(X)$, we see furthermore that $f_0 \in S$. This proves the sequential compactness of S and completes the proof of the Arzelà-Ascoli Theorem. □

Proposition E.5

Suppose X is a compact topological space and $C(X)$ is the space of continuous $\mathbb{R}$-valued functions defined on X equipped with the supremum norm. Let $S \subset C(X)$.

(i) *If S is equicontinuous at $x_0 \in X$, then its closure $\overline{S}$ in $C(X)$ is equicontinuous at x_0.*

(ii) *If a subset $S \subset C(X)$ is equicontinuous, then its closure $\overline{S}$ in $C(X)$ is equicontinuous.*

Proof It is obvious that (ii) is an immediate consequence of (i). Thus, it suffices to establish (i). Since, by hypothesis, $S \subset C(X)$ is equicontinuous at $x_0 \in X$, we have: for each $\varepsilon > 0$, there exists an open subset $V \subset X$ satisfying:

$$x_0 \in V, \quad \text{and} \quad \sup_{(x,f)\,\in\, V\times S} \left\{ \left| f(x) - f(x_0) \right| \right\} \leq \varepsilon.$$

Observe that, in order to show the equicontinuity of $\overline{S}$ at x_0, it suffices to show that the following inequality is also valid:

$$\sup_{(x,g)\,\in\, V\times \overline{S}} \left\{ \left| g(x) - g(x_0) \right| \right\} \leq \varepsilon.$$

To this end, let $g \in \overline{S} \subset C(X)$. Then, there exist a sequence $f_1, f_2, \ldots \in S$ such that

$$\lim_{n\to\infty} \| f_n - g \|_\infty = \lim_{n\to\infty} \sup_{x\in X} \left\{ \left| f_n(x) - g(x) \right| \right\} = 0.$$

Consequently, for any $x \in V$ and $n \in \mathbb{N}$, we have:

$$\begin{aligned} \left| g(x) - g(x_0) \right| &\leq \left| g(x) - f_n(x) \right| + \left| f_n(x) - f_n(x_0) \right| + \left| f_n(x_0) - g(x_0) \right| \\ &\leq \| g - f_n \|_\infty + \varepsilon + \| f_n - g \|_\infty \longrightarrow 0 + \varepsilon + 0 = \varepsilon. \end{aligned}$$

This implies:

$$\sup_{(x,g)\,\in\, V\times \overline{S}} \left\{ \left| g(x) - g(x_0) \right| \right\} \leq \varepsilon,$$

as desired. This completes the proof of the Proposition. □

Theorem E.6 (Theorem 7.2, p.81, [3])
Let $C[0,1]$ denote the space of continuous $\mathbb{R}$-valued functions defined on the closed unit interval $[0,1]$ equipped with the supremum norm. Then, for each subset $S \subset C[0,1]$, the following are equivalent:

(i) *S is a relatively compact subset of $C[0,1]$, i.e. the closure of S is a compact subset of $C[0,1]$.*

(ii) $\displaystyle\sup_{f \in S}\left\{ \|f\|_\infty \right\} < \infty$, *and S is uniformly equicontinuous.*

(iii) $\displaystyle\sup_{f \in S}\left\{ |f(0)| \right\} < \infty$, *and for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sup\left\{ \Big| f(t_1) - f(t_2) \Big| \;\middle|\; \begin{array}{c} f \in S,\ t_1, t_2 \in [0,1], \\ |t_1 - t_2| < \delta \end{array} \right\} \;\leq\; \varepsilon.$$

(iv) $\displaystyle\sup_{f \in S}\left\{ |f(0)| \right\} < \infty$, *and*

$$\lim_{\delta \to 0^+} \sup_{f \in S}\left\{ \sup\left\{ \Big| f(t_1) - f(t_2) \Big| \;\middle|\; \begin{array}{c} t_1,\, t_2 \in [0,1] \\ |t_1 - t_2| < \delta \end{array} \right\} \right\} \;=\; 0.$$

(v) $\displaystyle\sup_{f \in S}\left\{ |f(0)| \right\} < \infty$, *and* $\displaystyle\lim_{\delta \to 0^+} \sup_{f \in S}\left\{ w(f,\delta) \right\} \;=\; 0.$

Proof

(i) $\Longleftrightarrow$ (ii)

$$\begin{array}{rcll}
\text{(i)} & \Longleftrightarrow & \overline{S} \text{ is compact,} & \text{(by definition of relative compactness)} \\
& \Longleftrightarrow & \overline{S} \text{ is bounded and equicontinuous,} & \text{(by the Arzelà-Ascoli Theorem)} \\
& \Longleftrightarrow & S \text{ is bounded and equicontinuous,} & \text{(by Proposition E.5)} \\
& \Longleftrightarrow & S \text{ is bounded and uniformly equicontinuous,} & \text{(by Proposition E.3)} \\
& \Longleftrightarrow & \text{(ii).} &
\end{array}$$

(ii) $\Longleftrightarrow$ (iii)

Noting that the second condition in (iii) is precisely uniform equicontinuity of S, we see immediately that (ii) $\Longrightarrow$ (iii). Conversely, we may conclude that (iii) $\Longrightarrow$ (ii) once we prove that (iii) implies $\displaystyle\sup_{f \in S}\left\{ \|f\|_\infty \right\} < \infty$. To this end, take $\varepsilon = 1$. Then, by the second condition in (iii) (uniform equicontinuity of S), there exists $\delta > 0$ such that

$$\sup\left\{ \Big| f(t_1) - f(t_2) \Big| \;\middle|\; \begin{array}{c} f \in S,\ t_1, t_2 \in [0,1], \\ |t_1 - t_2| < \delta \end{array} \right\} \;\leq\; \varepsilon \;:=\; 1.$$

Next, choose $k \in \mathbb{N}$ sufficiently large such that $\dfrac{1}{k} < \delta$. Hence, for any $f \in S$ and any $t \in [0,1]$, we have

$$\begin{aligned}
\Big| f(t) \Big| \;&=\; \left| f(t) - f\left(\frac{k-1}{k}\cdot t\right) + f\left(\frac{k-1}{k}\cdot t\right) - \;\cdots\; - f\left(\frac{1}{k}\cdot t\right) + f\left(\frac{1}{k}\cdot t\right) - f(0) + f(0) \right| \\
&\leq\; \Big| f(0) \Big| \;+\; \sum_{i=1}^{k} \left| f\left(\frac{i}{k}\cdot t\right) + f\left(\frac{i-1}{k}\cdot t\right) \right| \;\leq\; \Big| f(0) \Big| \;+\; k \cdot 1 \\
&\leq\; \sup_{f \in S}\left\{ |f(0)| \right\} \;+\; k,
\end{aligned}$$

where the last inequality follows from the first condition in (iii). Consequently, we see that, for each $f \in S$,

$$\| f \|_\infty \;:=\; \sup_{t \in [0,1]} \left\{ \, | \, f(t) \, | \, \right\} \;\leq\; \sup_{f \in S} \left\{ \, | \, f(0) \, | \, \right\} + k \;<\; \infty,$$

which in turn implies

$$\sup_{f \in S} \left\{ \, \| f \|_\infty \, \right\} \;\leq\; \sup_{f \in S} \left\{ \, | \, f(0) \, | \, \right\} + k \;<\; \infty.$$

This completes the proof that (ii) $\Longleftrightarrow$ (iii).

(iii) $\Longleftrightarrow$ (iv)

This follows trivially from the definition of the right-limit at zero of a $\mathbb{R}$-valued function defined on an interval $[0, \delta_0)$, for some $\delta_0 > 0$.

(iv) $\Longleftrightarrow$ (v)

Immediate by the definition of the modulus of continuity $w(f, \delta)$, for $f \in C[0,1]$ and $\delta \in (0,1]$.

$\square$

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