

1 Variance estimation for multi-stage sampling

Let U be a finite population and $\mathcal{P}(U)$ the power set of U . Let $p : \mathcal{S} \rightarrow (0, 1]$ be a r -stage sampling design ($r \geq 2$), where $\mathcal{S} \subset \mathcal{P}(U)$ is the set of all admissible samples under the design p . We express the hierarchical structure of the population U , with respect to the r -stage design p , as follows:

$$U = \bigsqcup_{i \in U^{(1)}} U_i^{(2+)} = \bigsqcup_{i \in U^{(1)}} \bigsqcup_{a \in U_i^{(2)}} U_{ia}^{(3)} = \dots = \bigsqcup_{i \in U^{(1)}} \dots \bigsqcup_{b \in U_{i \dots}^{(r-1)}} U_{i \dots b}^{(r)} \quad (1.1)$$

where $U^{(1)}$ is the set of all primary sampling units (PSU), and for each PSU $i \in U^{(1)}$, $U_i^{(2)}$ denotes the set of all secondary sampling units (SSU) contained in $i \in U^{(1)}$, and for each SSU $a \in U_i^{(2)}$, $U_{ia}^{(3)}$ denotes the set of all tertiary sampling units (TSU) contained in $a \in U_i^{(2)}$, and so on. Similarly, we express the hierarchical structure of every admissible sample $s \in \mathcal{S}$ as follows:

$$s = \bigsqcup_{i \in s^{(1)}} s_i^{(2+)} = \bigsqcup_{i \in s^{(1)}} \bigsqcup_{a \in s_i^{(2)}} s_{ia}^{(3)} = \dots \quad (1.2)$$

Let $y : U \rightarrow \mathbb{R}$ be a population characteristic. Let T be its population total, i.e.

$$T = \sum_{u \in U} y_u = \sum_{i \in U^{(1)}} T_i = \sum_{i \in U^{(1)}} \sum_{a \in U_{ia}^{(2)}} T_{ia} = \dots = \sum_{i \in U^{(1)}} \dots \sum_{u \in U_{i \dots}^{(r)}} y_u \quad (1.3)$$

Theorem 1.1

If $\hat{T}_i : \mathcal{S}_i^{(2+)} \rightarrow \mathbb{R}$ is an unbiased estimator for T_i , for each PSU $i \in U^{(1)}$, then the random variable $\hat{T} : \mathcal{S} \rightarrow \mathbb{R}$ defined as follows:

$$\hat{T}(s) := \sum_{i \in s^{(1)}} \frac{\hat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \quad (1.4)$$

is a design-unbiased estimator for

$$T = \sum_{i \in U^{(1)}} T_i$$

If the r -stage sampling design has invariant and independent subsampling, then the design-variance of \hat{T} is given by:

$$\text{Var}[\hat{T}] = \underbrace{\text{Var}^{(1)}\left(E^{(2+)}(\hat{T} \mid s^{(1)})\right)}_{V_{\text{PSU}}} + \underbrace{E^{(1)}\left(\text{Var}^{(2+)}(\hat{T} \mid s^{(1)})\right)}_{V_{\text{subsampling}}}, \quad (1.5)$$

where

$$\begin{aligned} \text{Var}^{(1)}\left(E^{(2+)}(\hat{T} \mid s^{(1)})\right) &= \sum_{i \in U^{(1)}} \sum_{j \in U^{(1)}} \Delta_{ij}^{(1)} \frac{T_i}{\pi_i^{(1)}} \frac{T_j}{\pi_j^{(1)}}, \quad \text{and} \\ E^{(1)}\left(\text{Var}^{(2+)}(\hat{T} \mid s^{(1)})\right) &= \sum_{i \in U^{(1)}} \frac{V_i}{\pi_i^{(1)}}, \end{aligned}$$

with

$$V_i := \text{Var}^{(2+)}[\hat{T}_i] \quad \text{and} \quad \Delta_{ij}^{(1)} := \begin{cases} \pi_i^{(1)}(1 - \pi_i^{(1)}), & \text{if } i = j \\ \pi_{ij}^{(1)} - \pi_i^{(1)}\pi_j^{(1)}, & \text{if } i \neq j \end{cases} \quad (1.6)$$

Furthermore, if $\hat{V}_i : \mathcal{S}_i^{(2+)} \rightarrow \mathbb{R}$ is an unbiased estimator for $V_i := \text{Var}^{(2+)}[\hat{T}_i]$, and $\pi_i^{(1)} > 0$, $\pi_{ij}^{(1)} > 0$ for any PSU $s, i, j \in U^{(1)}$, then

$$\begin{aligned} \widehat{\text{Var}}[\hat{T}](s) &:= \sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{\hat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \cdot \frac{\hat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} + \sum_{i \in s^{(1)}} \frac{\hat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}}, \\ \widehat{\text{Var}}^{(1)}[\hat{T}](s) &:= \sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{\hat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \cdot \frac{\hat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} + \sum_{i \in s^{(1)}} \left(1 - \frac{1}{\pi_i^{(1)}}\right) \frac{\hat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}}, \quad \text{and} \\ \widehat{\text{Var}}^{(2+)}[\hat{T}](s) &:= \sum_{i \in s^{(1)}} \frac{\hat{V}_i(s_i^{(2+)})}{(\pi_i^{(1)})^2} \end{aligned}$$

are unbiased estimators for $\text{Var}[\hat{T}]$, $V_{\text{PSU}} := \text{Var}^{(1)}(E^{(2+)}(\hat{T} | s^{(1)}))$, and $V_{\text{subsampling}} := E^{(1)}(\text{Var}^{(2+)}(\hat{T} | s^{(1)}))$, respectively.

Corollary 1.2

$$\widehat{\text{Var}}^{(1)}[\hat{T}](s) = \widehat{\text{Var}}[\hat{T}](s) - \widehat{\text{Var}}^{(2+)}[\hat{T}](s) \quad (1.7)$$

PROOF

$$\begin{aligned} \text{Var}^{(1)}[E^{(2+)}[\hat{T} | s^{(1)}]] &= \text{Var}^{(1)}\left[E^{(2+)}\left[\sum_{i \in s^{(1)}} \frac{\hat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \mid s^{(1)}\right]\right] \\ &= \text{Var}^{(1)}\left[\sum_{i \in s^{(1)}} \frac{E^{(2+)}[\hat{T}_i(s_i^{(2+)})]}{\pi_i^{(1)}}\right] \\ &= \text{Var}^{(1)}\left[\sum_{i \in s^{(1)}} \frac{T_i}{\pi_i^{(1)}}\right] \\ &= \sum_{i \in U^{(1)}} \sum_{j \in U^{(1)}} \Delta_{ij}^{(1)} \frac{T_i}{\pi_i^{(1)}} \frac{T_j}{\pi_j^{(1)}} \end{aligned}$$

$$\begin{aligned}
 E^{(1)} \left[\text{Var}^{(2+)} \left[\hat{T} \mid s^{(1)} \right] \right] &= E^{(1)} \left[\text{Var}^{(2+)} \left[\sum_{i \in s^{(1)}} \frac{\hat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \mid s^{(1)} \right] \right] \\
 &= E^{(1)} \left[\sum_{i \in s^{(1)}} \text{Var}^{(2+)} \left[\frac{\hat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \right] \right] \\
 &= E^{(1)} \left[\sum_{i \in s^{(1)}} \frac{\text{Var}^{(2+)} \left[\hat{T}_i(s_i^{(2+)}) \right]}{\left(\pi_i^{(1)} \right)^2} \right] \\
 &= E^{(1)} \left[\sum_{i \in s^{(1)}} \frac{V_i / \pi_i^{(1)}}{\pi_i^{(1)}} \right] \\
 &= \sum_{i \in U^{(1)}} \frac{V_i}{\pi_i^{(1)}}
 \end{aligned}$$

$$\begin{aligned}
 E \left(\widehat{\text{Var}}^{(2+)}(\hat{T}) \right) &= E \left(\sum_{i \in s^{(1)}} \frac{\hat{V}_i(s_i^{(2+)})}{\left(\pi_i^{(1)} \right)^2} \right) = E^{(1)} \left(E^{(2+)} \left(\sum_{i \in s^{(1)}} \frac{\hat{V}_i(s_i^{(2+)}) / \pi_i^{(1)}}{\pi_i^{(1)}} \mid s^{(1)} \right) \right) \\
 &= E^{(1)} \left(\sum_{i \in s^{(1)}} \frac{E^{(2+)} \left[\hat{V}_i(s_i^{(2+)}) \mid s^{(1)} \right] / \pi_i^{(1)}}{\pi_i^{(1)}} \right) = E^{(1)} \left(\sum_{i \in s^{(1)}} \frac{V_i / \pi_i^{(1)}}{\pi_i^{(1)}} \right) = \sum_{i \in U^{(1)}} \frac{V_i}{\pi_i^{(1)}} \\
 &= E^{(1)} \left(\text{Var}^{(2+)}(\hat{T} \mid s^{(1)}) \right) = V_{\text{PSU}}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 E \left(\sum_{i \in s^{(1)}} \frac{\hat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}} \right) &= E^{(1)} \left(E^{(2+)} \left(\sum_{i \in s^{(1)}} \frac{\hat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}} \mid s^{(1)} \right) \right) = E^{(1)} \left(\sum_{i \in s^{(1)}} \frac{E^{(2+)} \left[\hat{V}_i(s_i^{(2+)}) \mid s^{(1)} \right]}{\pi_i^{(1)}} \right) \\
 &= E^{(1)} \left(\sum_{i \in s^{(1)}} \frac{V_i}{\pi_i^{(1)}} \right) = \sum_{i \in U^{(1)}} V_i
 \end{aligned}$$

Next, observe that

$$\begin{aligned}
 E \left[\sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{\hat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \cdot \frac{\hat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} \right] &= E^{(1)} \left(E^{(2+)} \left(\sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{\hat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \cdot \frac{\hat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} \mid s^{(1)} \right) \right) \\
 &= E^{(1)} \left(\sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{E^{(2+)} \left[\hat{T}_i(s_i^{(2+)}) \cdot \hat{T}_j(s_j^{(2+)}) \mid s^{(1)} \right]}{\pi_i^{(1)} \pi_j^{(1)}} \right)
 \end{aligned}$$

Now, observe (the key technical observation) that

$$E^{(2+)} \left[\widehat{T}_i(s_i^{(2+)}) \cdot \widehat{T}_j(s_j^{(2+)}) \mid s^{(1)} \right] = E^{(2+)} \left[\widehat{T}_i(s_i^{(2+)}) \cdot \widehat{T}_j(s_j^{(2+)}) \right] = \begin{cases} \text{Var}^{(2+)}(\widehat{T}_i) + E^{(2+)}(\widehat{T}_i)^2, & \text{if } i = j, \\ E^{(2+)}(\widehat{T}_i) \cdot E^{(2+)}(\widehat{T}_j), & \text{if } i \neq j \end{cases}$$

Hence,

$$\begin{aligned} E \left[\sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \cdot \frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} \right] &= E^{(1)} \left(\sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{E^{(2+)} \left[\widehat{T}_i(s_i^{(2+)}) \cdot \widehat{T}_j(s_j^{(2+)}) \mid s^{(1)} \right]}{\pi_i^{(1)} \pi_j^{(1)}} \right) \\ &= E^{(1)} \left(\sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{T_i}{\pi_i^{(1)}} \cdot \frac{T_j}{\pi_j^{(1)}} + \sum_{i \in s^{(1)}} \frac{(1 - \pi_i^{(1)}) V_i / \pi_i^{(1)}}{\pi_i^{(1)}} \right) \\ &= \sum_{i \in U^{(1)}} \sum_{j \in U^{(1)}} \Delta_{ij}^{(1)} \cdot \frac{T_i}{\pi_i^{(1)}} \cdot \frac{T_j}{\pi_j^{(1)}} + \sum_{i \in U^{(1)}} \frac{(1 - \pi_i^{(1)})}{\pi_i^{(1)}} \cdot V_i \\ &= \sum_{i \in U^{(1)}} \sum_{j \in U^{(1)}} \Delta_{ij}^{(1)} \cdot \frac{T_i}{\pi_i^{(1)}} \cdot \frac{T_j}{\pi_j^{(1)}} + \sum_{i \in U^{(1)}} \frac{V_i}{\pi_i^{(1)}} - \sum_{i \in U^{(1)}} V_i \end{aligned}$$

We may now establish that

$$\begin{aligned} E \left[\widehat{\text{Var}}(\widehat{T}) \right] &= E \left[\sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \cdot \frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} + \sum_{i \in s^{(1)}} \frac{\widehat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}} \right] \\ &= \left\{ \sum_{i \in U^{(1)}} \sum_{j \in U^{(1)}} \Delta_{ij}^{(1)} \cdot \frac{T_i}{\pi_i^{(1)}} \cdot \frac{T_j}{\pi_j^{(1)}} + \sum_{i \in U^{(1)}} \frac{V_i}{\pi_i^{(1)}} - \sum_{i \in U^{(1)}} V_i \right\} + \left\{ \sum_{i \in U^{(1)}} V_i \right\} \\ &= \sum_{i \in U^{(1)}} \sum_{j \in U^{(1)}} \Delta_{ij}^{(1)} \cdot \frac{T_i}{\pi_i^{(1)}} \cdot \frac{T_j}{\pi_j^{(1)}} + \sum_{i \in U^{(1)}} \frac{V_i}{\pi_i^{(1)}} \\ &= \text{Var}(\widehat{T}) \end{aligned}$$

Lastly, note that

$$\widehat{\text{Var}}^{(1)}[\widehat{T}](s) = \widehat{\text{Var}}[\widehat{T}](s) - \widehat{\text{Var}}^{(2+)}[\widehat{T}](s)$$

Hence,

$$\begin{aligned} E \left[\widehat{\text{Var}}^{(1)}(\widehat{T}) \right] &= E \left[\widehat{\text{Var}}(\widehat{T}) \right] - E \left[\widehat{\text{Var}}^{(2+)}(\widehat{T}) \right] \\ &= \text{Var}[\widehat{T}] - E^{(1)} \left(\text{Var}^{(2+)}(\widehat{T} \mid s^{(1)}) \right) \\ &= \text{Var}^{(1)} \left(E^{(2+)}(\widehat{T} \mid s^{(1)}) \right) + E^{(1)} \left(\text{Var}^{(2+)}(\widehat{T} \mid s^{(1)}) \right) - E^{(1)} \left(\text{Var}^{(2+)}(\widehat{T} \mid s^{(1)}) \right) \\ &= \text{Var}^{(1)} \left(E^{(2+)}(\widehat{T} \mid s^{(1)}) \right) = V_{\text{subsampling}} \end{aligned}$$

□

2 Variance estimation for three-stage sampling

Let U be a finite population and $\mathcal{P}(U)$ the power set of U . Let $p : \mathcal{S} \rightarrow (0, 1]$ be a three-stage sampling design, where $\mathcal{S} \subset \mathcal{P}(U)$ is the set of all admissible samples under the design p . We express the three-stage structure of the population U , with respect to the three-stage design p , as follows:

$$U = \bigsqcup_{i \in U^{(1)}} U_i^{(2+)} = \bigsqcup_{i \in U^{(1)}} \bigsqcup_{a \in U_i^{(2)}} U_{ia}^{(3)}, \quad (2.1)$$

where $U^{(1)}$ is the set of all primary sampling units (PSU), and for each PSU $i \in U^{(1)}$, $U_i^{(2)}$ denotes the set of all secondary sampling units (SSU) contained in $i \in U^{(1)}$, and for each SSU $a \in U_i^{(2)}$, $U_{ia}^{(3)}$ denotes the set of all tertiary sampling units (TSU) contained in $a \in U_i^{(2)}$. Similarly, we express the three-stage structure of every admissible sample $s \in \mathcal{S}$ as follows:

$$s = \bigsqcup_{i \in s^{(1)}} s_i^{(2+)} = \bigsqcup_{i \in s^{(1)}} \bigsqcup_{a \in s_i^{(2)}} s_{ia}^{(3)} \quad (2.2)$$

Let $y : U \rightarrow \mathbb{R}$ be a population characteristic. Let T be its population total, i.e.

$$T = \sum_{u \in U} y_u = \sum_{i \in U^{(1)}} T_i = \sum_{i \in U^{(1)}} \sum_{a \in U_i^{(2)}} T_{ia} = \sum_{i \in U^{(1)}} \sum_{a \in U_i^{(2)}} \sum_{u \in U_{ia}^{(3)}} y_u \quad (2.3)$$

Theorem 2.1

For a three-stage sampling design with invariant and independent subsampling,

1. The random variable $\hat{T} : \mathcal{S} \rightarrow \mathbb{R}$, defined as follows:

$$\hat{T}(s) := \sum_{i \in s^{(1)}} \frac{\hat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} = \sum_{i \in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a \in s_i^{(2)}} \frac{\hat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \right) = \sum_{i \in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a \in s_i^{(2)}} \frac{1}{\pi_{i|a}^{(2)}} \left(\sum_{u \in s_{ia}^{(3)}} \frac{y_u}{\pi_{ia|u}^{(3)}} \right) \right)$$

is a design-unbiased estimator for T , i.e. $E[\hat{T}] = T$.

2. If the three-stage sampling design has invariant and independent subsampling, then the design-variance of \hat{T} can be given by:

$$\begin{aligned} & \text{Var}[\hat{T}] \\ &= \text{Var}^{(1)}\left(E^{(2+)}\left(\hat{T} \mid s^{(1)}\right)\right) + E^{(1)}\left(\text{Var}^{(2+)}\left(\hat{T} \mid s^{(1)}\right)\right) \\ &= \text{Var}^{(1)}\left(E^{(2+)}\left(\hat{T} \mid s^{(1)}\right)\right) + E^{(1)}\left\{\text{Var}^{(2)}\left(E^{(3)}\left(\hat{T} \mid s^{(2)}\right) \mid s^{(1)}\right) + E^{(2)}\left(\text{Var}^{(3)}\left(\hat{T} \mid s^{(2)}\right) \mid s^{(1)}\right)\right\} \\ &= \underbrace{\text{Var}^{(1)}\left(E^{(2+)}\left(\hat{T} \mid s^{(1)}\right)\right)}_{V_{\text{PSU}}} + \underbrace{E^{(1)}\left\{\text{Var}^{(2)}\left(E^{(3)}\left(\hat{T} \mid s^{(2)}\right) \mid s^{(1)}\right)\right\}}_{V_{\text{SSU}}} + \underbrace{E^{(1)}\left\{E^{(2)}\left(\text{Var}^{(3)}\left(\hat{T} \mid s^{(2)}\right) \mid s^{(1)}\right)\right\}}_{V_{\text{TSU}}} \end{aligned}$$

where

$$\begin{aligned}
 V_{\text{PSU}} &:= \text{Var}^{(1)}\left(E^{(2+)}\left(\hat{T} \mid s^{(1)}\right)\right) = \sum_{i \in U^{(1)}} \sum_{j \in U^{(1)}} \Delta_{ij}^{(1)} \frac{T_i}{\pi_i^{(1)}} \frac{T_j}{\pi_j^{(1)}}, \\
 V_{\text{SSU}} &:= E^{(1)}\left\{\text{Var}^{(2)}\left(E^{(3)}\left(\hat{T} \mid s^{(2)}\right) \mid s^{(1)}\right)\right\} = \sum_{i \in U^{(1)}} \frac{V_i^{(2)}}{\pi_i^{(1)}}, \quad \text{and} \\
 V_{\text{TSU}} &:= E^{(1)}\left\{E^{(2)}\left(\text{Var}^{(3)}\left(\hat{T} \mid s^{(2)}\right) \mid s^{(1)}\right)\right\} = \sum_{i \in U^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a \in U_i^{(2)}} \frac{V_{ia}^{(3)}}{\pi_{i|a}^{(2)}} \right),
 \end{aligned}$$

with

$$\begin{aligned}
 \Delta_{ij}^{(1)} &:= \begin{cases} \pi_i^{(1)} (1 - \pi_i^{(1)}), & \text{if } i = j \\ \pi_{ij}^{(1)} - \pi_i^{(1)} \pi_j^{(1)}, & \text{if } i \neq j \end{cases} \\
 V_i^{(2)} &:= \text{Var}^{(2)}\left[\sum_{a \in s_i^{(2)}} \frac{T_{ia}}{\pi_{i|a}^{(2)}}\right] = \sum_{a \in U_i^{(2)}} \sum_{b \in U_i^{(2)}} \Delta_{i|ab}^{(2)} \cdot \frac{T_{ia}}{\pi_{i|a}^{(2)}} \cdot \frac{T_{ib}}{\pi_{i|b}^{(2)}} \\
 \Delta_{i|ab}^{(2)} &:= \begin{cases} \pi_{i|a}^{(2)} (1 - \pi_{i|a}^{(2)}), & \text{if } a = b \\ \pi_{i|ab}^{(2)} - \pi_{i|a}^{(2)} \pi_{i|b}^{(2)}, & \text{if } a \neq b \end{cases} \\
 V_{ia}^{(3)} &:= \text{Var}^{(3)}\left(\hat{T}_{ia} \mid s^{(1)}, s^{(2)}\right)
 \end{aligned}$$

Theorem 2.2

For a three-stage sampling design $p : \mathcal{S} \subset \mathcal{P}(U) \rightarrow \mathbb{R}$ with invariant and independent subsampling, let $\hat{T} : \mathcal{S} \rightarrow \mathbb{R}$ be the random variable defined as follows:

$$\hat{T}(s) := \sum_{i \in s^{(1)}} \frac{\hat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} = \sum_{i \in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a \in s_i^{(2)}} \frac{\hat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \right) = \sum_{i \in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a \in s_i^{(2)}} \frac{1}{\pi_{i|a}^{(2)}} \left(\sum_{u \in s_{ia}^{(3)}} \frac{y_u}{\pi_{ia|u}^{(3)}} \right) \right)$$

Recall that \hat{T} is an unbiased estimator of the population total

$$T := \sum_{i \in U^{(1)}} T_i := \sum_{i \in U^{(1)}} \sum_{a \in U_{ia}^{(2)}} T_{ia} := \sum_{i \in U^{(1)}} \sum_{a \in U_i^{(2)}} \sum_{u \in U_{ia}^{(3)}} y_u$$

Let $\widehat{\text{Var}}[\widehat{T}] : \mathcal{S} \rightarrow \mathbb{R}$ be the random variable defined in a recursive manner as follows:

$$\begin{aligned}\widehat{\text{Var}}[\widehat{T}](s) &:= \sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \cdot \frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} + \sum_{i \in s^{(1)}} \frac{\widehat{\text{Var}}[\widehat{T}_i](s_i^{(2+)})}{\pi_i^{(1)}} \\ \widehat{\text{Var}}[\widehat{T}_i](s_i^{(2+)}) &:= \sum_{a \in s_i^{(2)}} \sum_{b \in s_i^{(2)}} \Delta_{i|ab}^{(2)} \cdot \frac{\widehat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \cdot \frac{\widehat{T}_{ib}(s_{ib}^{(3)})}{\pi_{i|b}^{(2)}} + \sum_{a \in s_i^{(2)}} \frac{\widehat{\text{Var}}[\widehat{T}_{ia}](s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \\ \widehat{\text{Var}}[\widehat{T}_{ia}](s_{ia}^{(3)}) &:= \sum_{u \in s_{ia}^{(3)}} \sum_{v \in s_{ia}^{(3)}} \Delta_{ia|uv}^{(3)} \cdot \frac{y_u}{\pi_{ia|u}^{(3)}} \cdot \frac{y_v}{\pi_{ia|v}^{(3)}}\end{aligned}$$

where

$$\begin{aligned}\Delta_{ij}^{(1)} &:= \begin{cases} \pi_i^{(1)} (1 - \pi_i^{(1)}), & \text{if } i = j \\ \pi_{ij}^{(1)} - \pi_i^{(1)} \pi_j^{(1)}, & \text{if } i \neq j \end{cases} \\ \Delta_{i|ab}^{(2)} &:= \begin{cases} \pi_{i|a}^{(2)} (1 - \pi_{i|a}^{(2)}), & \text{if } a = b \\ \pi_{i|ab}^{(2)} - \pi_{i|a}^{(2)} \pi_{i|b}^{(2)}, & \text{if } a \neq b \end{cases} \\ \Delta_{ia|uv}^{(3)} &:= \begin{cases} \pi_{ia|u}^{(3)} (1 - \pi_{ia|u}^{(3)}), & \text{if } u = v \\ \pi_{ia|uv}^{(3)} - \pi_{ia|u}^{(3)} \pi_{ia|v}^{(3)}, & \text{if } u \neq v \end{cases}\end{aligned}$$

Then, $\widehat{\text{Var}}[\widehat{T}]$ is a design-unbiased estimator of the design variance $\text{Var}[\widehat{T}]$ of the \widehat{T} .

Corollary 2.3

For a three-stage sampling design with invariant and independent subsampling, the fully expanded expression for

$\widehat{\text{Var}}[\widehat{T}]$ is as follows:

$$\begin{aligned}
 \widehat{\text{Var}}[\widehat{T}](s) &:= \sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \cdot \frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} + \sum_{i \in s^{(1)}} \frac{\widehat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}} \\
 &= \sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \cdot \frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} \\
 &\quad + \sum_{i \in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left\{ \sum_{a \in s_i^{(2)}} \sum_{b \in s_i^{(2)}} \Delta_{i|ab}^{(2)} \cdot \frac{\widehat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \cdot \frac{\widehat{T}_{ib}(s_{ib}^{(3)})}{\pi_{i|b}^{(2)}} + \sum_{a \in s_i^{(2)}} \frac{\widehat{V}_{ia}}{\pi_{i|a}^{(2)}} \right\} \\
 &= \sum_{i \in s^{(1)}} \sum_{j \in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(i)}} \cdot \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \cdot \frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} \\
 &\quad + \sum_{i \in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left\{ \sum_{a \in s_i^{(2)}} \sum_{b \in s_i^{(2)}} \Delta_{i|ab}^{(2)} \cdot \frac{\widehat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \cdot \frac{\widehat{T}_{ib}(s_{ib}^{(3)})}{\pi_{i|b}^{(2)}} + \sum_{a \in s_i^{(2)}} \frac{1}{\pi_{i|a}^{(2)}} \left(\sum_{u \in s_{ia}^{(3)}} \sum_{v \in s_{ia}^{(3)}} \Delta_{ia|uv}^{(3)} \frac{y_u}{\pi_{ia|u}^{(3)}} \frac{y_v}{\pi_{ia|v}^{(3)}} \right) \right\},
 \end{aligned}$$

where

$$\begin{aligned}
 \widehat{T}_{ia}(s_{ia}^{(3)}) &= \sum_{u \in s_{ia}^{(3)}} \frac{y_u}{\pi_{ia|u}^{(3)}} \quad \text{and} \quad \widehat{T}_i(s_i^{(2+)}) = \sum_{a \in s_i^{(2)}} \frac{\widehat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} = \sum_{a \in s_i^{(2)}} \frac{1}{\pi_{i|a}^{(2)}} \left(\sum_{u \in s_{ia}^{(3)}} \frac{y_u}{\pi_{ia|u}^{(3)}} \right) \\
 \Delta_{ia|uv}^{(3)} &:= \begin{cases} \pi_{ia|u}^{(3)} (1 - \pi_{ia|u}^{(3)}), & \text{if } u = v \\ \pi_{ia|uv}^{(3)} - \pi_{ia|u}^{(3)} \pi_{ia|v}^{(3)}, & \text{if } u \neq v \end{cases}
 \end{aligned}$$