A Technical Lemmas

Lemma A.1 (p.343, [1])

$$\left| e^{\mathbf{i}x} - \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \right| \;\leq\; \min\left\{ \frac{|\,x\,|^{n+1}}{(n+1)!}\,,\; \frac{2\,|\,x\,|^{n}}{n!} \right\}, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof We first establish a number of Claims, which will easily imply the Lemma.

Claim 1:

$$\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s \;=\; \frac{x^{n+1}}{n+1} \;+\; \frac{\mathbf{i}}{n+1}\int_0^x (x-s)^{n+1}\, e^{\mathbf{i}s}\,\mathrm{d}s, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof of Claim 1: We proceed by integration by parts. Let $u = e^{\mathbf{i}s}$ and $\mathrm{d}v = (x-s)^n\,\mathrm{d}s$. Then, $\mathrm{d}u = \mathbf{i}\,e^{\mathbf{i}s}$ and $v = -(x-s)^{n+1}/(n+1)$. Hence,

$$\begin{aligned}
\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s \;&=\; \int u\,\mathrm{d}v \;=\; uv \;-\; \int v\,\mathrm{d}u \\
&=\; \left[\, e^{\mathbf{i}s}\cdot \frac{(-1)(x-s)^{n+1}}{n+1} \,\right]_{s=0}^{s=x} \;-\; \int_0^x \frac{(-1)\,(x-s)^{n+1}}{n+1}\cdot \mathbf{i}e^{\mathbf{i}s}\,\mathrm{d}s, \\
&=\; \frac{x^{n+1}}{n+1} \;+\; \frac{\mathbf{i}}{n+1}\int_0^x (x-s)^{n+1}\; e^{\mathbf{i}s}\,\mathrm{d}s.
\end{aligned}$$

This proves Claim 1.

Claim 2:

$$e^{\mathbf{i}x} \;=\; \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \;+\; \frac{\mathbf{i}^{n+1}}{n!}\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof of Claim 2: We proceed by induction. For $n = 0$, we have:

$$\begin{aligned}
\mathrm{RHS}(n=0) \;&=\; \sum_{k=0}^{0} \frac{(\mathbf{i}\,x)^k}{k!} \;+\; \frac{\mathbf{i}^{0+1}}{0!}\int_0^x (x-s)^0\, e^{\mathbf{i}s}\,\mathrm{d}s \;=\; 1 \;+\; \mathbf{i}\int_0^x e^{\mathbf{i}s}\,\mathrm{d}s \;=\; 1 \;+\; \mathbf{i}\left[\,\frac{e^{\mathbf{i}s}}{\mathbf{i}}\,\right]_{s=0}^{s=x} \\
&=\; 1 + \left(e^{\mathbf{i}x} - 1\right) \;=\; e^{\mathbf{i}x}.
\end{aligned}$$

Thus, Claim 2 is indeed true for $n = 0$. Next, by induction hypothesis, assume Claim 2 is true for n, and we verify that Claim 2 is also true for $n+1$.

$$\begin{aligned}
\mathrm{RHS}(n+1) \;&=\; \sum_{k=0}^{n+1} \frac{(\mathbf{i}\,x)^k}{k!} \;+\; \frac{\mathbf{i}^{n+2}}{(n+1)!}\int_0^x (x-s)^{n+1}\, e^{\mathbf{i}s}\,\mathrm{d}s \\
&=\; \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \;+\; \frac{(\mathbf{i}\,x)^{n+1}}{(n+1)!} \;+\; \frac{\mathbf{i}^{n+2}}{(n+1)!}\cdot\frac{n+1}{\mathbf{i}}\left[\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s \;-\; \frac{x^{n+1}}{n+1}\right] \\
&=\; \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \;+\; \frac{\mathbf{i}^{n+1}}{n!}\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s \;+\; \frac{(\mathbf{i}\,x)^{n+1}}{(n+1)!} \;-\; \frac{\mathbf{i}^{n+1}}{n!}\cdot\frac{x^{n+1}}{n+1} \;=\; e^{\mathbf{i}x},
\end{aligned}$$

where the second equality follows from Claim 1 and the last equality follows from the induction hypothesis (that Claim 2 holds for n). This proves Claim 2.

Claim 3:

$$e^{\mathbf{i}x} \;=\; \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \;+\; \frac{\mathbf{i}^n}{(n-1)!}\int_0^x (x-s)^{n-1}\left(e^{\mathbf{i}s}-1\right)\mathrm{d}s, \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1.$$

Proof of Claim 3: By Claim 1, we have (replacing n with $n-1$):

$$\int_0^x (x-s)^{n-1}\,e^{\mathbf{i}s}\,\mathrm{d}s \;=\; \frac{x^n}{n} \;+\; \frac{\mathbf{i}}{n}\int_0^x (x-s)^n\,e^{\mathbf{i}s}\,\mathrm{d}s, \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1.$$

Isolating the integral on the right-hand-side, we have:

$$\int_0^x (x-s)^n\,e^{\mathbf{i}s}\,\mathrm{d}s \;=\; \frac{n}{\mathbf{i}}\left[\int_0^x (x-s)^{n-1}\,e^{\mathbf{i}s}\,\mathrm{d}s \;-\; \frac{x^n}{n}\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1.$$

Next, note that, for any $x\in\mathbb{R}$ and any $n\geq 1$,

$$\int_0^x (\,x-s\,)^{n-1}\,\mathrm{d}s \;=\; -\left[\frac{(x-s)^n}{n}\right]_{s=0}^{s=x} \;=\; -\left[\,0\;-\;\frac{x^n}{n}\,\right] \;=\; \frac{x^n}{n}$$

Hence, we have:

$$\begin{aligned}
\int_0^x (x-s)^n\,e^{\mathbf{i}s}\,\mathrm{d}s \;&=\; \frac{n}{\mathbf{i}}\left[\int_0^x (x-s)^{n-1}\,e^{\mathbf{i}s}\,\mathrm{d}s \;-\; \frac{x^n}{n}\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1.\\
&=\; \frac{n}{\mathbf{i}}\left[\int_0^x (x-s)^{n-1}\,e^{\mathbf{i}s}\,\mathrm{d}s \;-\; \int_0^x (\,x-s\,)^{n-1}\,\mathrm{d}s\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1.\\
&=\; \frac{n}{\mathbf{i}}\left[\int_0^x (x-s)^{n-1}\left(\,e^{\mathbf{i}s}\;-\;1\,\right)\,\mathrm{d}s\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1.
\end{aligned}$$

Substituting the above into the right-hand-side of Claim 2, we have:

$$\begin{aligned}
e^{\mathbf{i}x} \;&=\; \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \;+\; \frac{\mathbf{i}^{n+1}}{n!}\int_0^x (x-s)^n\,e^{\mathbf{i}s}\,\mathrm{d}s, \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 0\\
&=\; \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \;+\; \frac{\mathbf{i}^{n+1}}{n!}\cdot\frac{n}{\mathbf{i}}\cdot\left[\int_0^x (x-s)^{n-1}\left(e^{\mathbf{i}s}\;-\;1\right)\,\mathrm{d}s\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1\\
&=\; \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \;+\; \frac{\mathbf{i}^{n}}{(n-1)!}\cdot\left[\int_0^x (x-s)^{n-1}\left(e^{\mathbf{i}s}\;-\;1\right)\,\mathrm{d}s\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1
\end{aligned}$$

This proves Claim 3.

Claim 4:
$$\left| \int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s \right| \;\leq\; \frac{|\,x\,|^{n+1}}{n+1}, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof of Claim 4: First, consider $x \geq 0$, in which case, we have, for any $n \geq 0$,

$$\left| \int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s \right| \;\leq\; \int_0^x |\,x-s\,|^n \, \mathrm{d}s \;\leq\; \int_0^x (\,x-s\,)^n \, \mathrm{d}s \;=\; \cdots \;=\; \frac{x^{n+1}}{n+1} \;=\; \frac{|\,x\,|^{n+1}}{n+1}$$

Next, for $x < 0$, let $y := -x > 0$. Then,

$$\begin{aligned} \left| \int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s \right| \;&=\; \left| \int_0^{-y} (-y-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s \right| \;=\; \left| \int_0^y (-y+t)^n \, e^{-\mathbf{i}t} \, \mathrm{d}t \right| \\ &\leq\; \int_0^y |\,y-t\,|^n \, \mathrm{d}t \;=\; \int_0^y (\,y-t\,)^n \, \mathrm{d}t \;=\; \cdots \;=\; \frac{y^{n+1}}{n+1} \;=\; \frac{|\,x\,|^{n+1}}{n+1} \end{aligned}$$

This completes the proof Claim 4.

Claim 5:
$$\left| \int_0^x (x-s)^{n-1} \left(e^{\mathbf{i}s} - 1 \right) \mathrm{d}s \right| \;\leq\; \frac{2\,|\,x\,|^n}{n}, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 1.$$

Proof of Claim 5: First, consider $x \geq 0$, in which case, we have, for any $n \geq 1$,

$$\left| \int_0^x (x-s)^{n-1} \left(e^{\mathbf{i}s} - 1 \right) \mathrm{d}s \right| \;\leq\; \int_0^x \left| (x-s)^{n-1} \left(e^{\mathbf{i}s} - 1 \right) \right| \mathrm{d}s \;\leq\; 2 \int_0^x (x-s)^{n-1} \, \mathrm{d}s \;=\; \frac{2\,x^n}{n} \;=\; \frac{2\,|\,x\,|^n}{n},$$

where the second last equality follows from the simple calculation:

$$\int_0^x (\,x-s\,)^{n-1} \, \mathrm{d}s \;=\; -\left[\frac{(x-s)^n}{n} \right]_{s=0}^{s=x} \;=\; -\left[0 \;-\; \frac{x^n}{n} \right] \;=\; \frac{x^n}{n}.$$

Next, for $x < 0$, let $y := -x > 0$. Then,

$$\begin{aligned} \left| \int_0^x (x-s)^{n-1} \left(e^{\mathbf{i}s} - 1 \right) \mathrm{d}s \right| \;&=\; \left| \int_0^{-y} (-y-s)^{n-1} \left(e^{\mathbf{i}s} - 1 \right) \mathrm{d}s \right| \;=\; \left| -\int_0^y (-y+t)^{n-1} \left(e^{-\mathbf{i}t} - 1 \right) \mathrm{d}t \right| \\ &\leq\; 2 \int_0^y |\,t-y\,|^{n-1} \, \mathrm{d}t \;=\; 2 \int_0^y (y-t)^{n-1} \, \mathrm{d}t \;=\; \frac{2\,y^n}{n} \;=\; \frac{2\,|\,x\,|^n}{n}. \end{aligned}$$

This completes the proof of Claim 5.

The proof of the Lemma now follows readily from the preceding Claims.

$$\begin{aligned} & \left| e^{\mathbf{i}x} \;-\; \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \right| \\ \leq\;& \min\left\{ \left| \frac{\mathbf{i}^{n+1}}{n!} \int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s \right|, \left| \frac{\mathbf{i}^n}{(n-1)!} \int_0^x (x-s)^{n-1} \left(e^{\mathbf{i}s} - 1 \right) \mathrm{d}s \right| \right\}, \text{ by Claims 2 and 3} \\ \leq\;& \min\left\{ \frac{1}{n!} \cdot \frac{|\,x\,|^{n+1}}{n+1}, \frac{1}{(n-1)!} \cdot \frac{2\,|\,x\,|^n}{n} \right\}, \text{ by Claims 4 and 5} \\ \leq\;& \min\left\{ \frac{|\,x\,|^{n+1}}{(n+1)!}, \frac{2\,|\,x\,|^n}{n!} \right\} \end{aligned}$$

This completes the proof of the Lemma. $\square$

Lemma A.2 (§7.1, [2])

Let $\{ \theta_{nj} \in \mathbb{C} \mid 1 \leq j \leq k_n, n \in \mathbb{N} \}$ be doubly indexed array of complex numbers. If all the following three conditions are true:

(a) *there exists $M > 0$ such that*

$$\sum_{j=1}^{k_n} |\, \theta_{nj} \,| \;\leq\; M, \quad \text{for each } n \in \mathbb{N},$$

(b) $\displaystyle\lim_{n\to\infty} \max_{1\leq j\leq k_n} |\, \theta_{nj} \,| \;=\; 0$, *and*

(c) *there exists $\theta \in \mathbb{C}$ such that*

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \theta_{nj} \;=\; \theta,$$

then

$$\lim_{n\to\infty} \prod_{j=1}^{k_n} (1 + \theta_{nj}) \;=\; e^{\theta}.$$

PROOF First, note that hypothesis (b) immediately implies that there exists some $n_0 \in \mathbb{N}$ such that

$$|\, \theta_{nj} \,| \;\leq\; \frac{1}{2}, \quad \text{for each } n \geq n_0, \text{ for each } 1 \leq j \leq k_n.$$

Thus, without loss of generality, we may assume that:

$$|\, \theta_{nj} \,| \;\leq\; \frac{1}{2}, \quad \text{for each } n \in \mathbb{N}, \text{ for each } 1 \leq j \leq k_n.$$

We denote by $\log(1+\theta_{nj})$ the (unique) complex logarithm[1] of $1+\theta_{nj}$ with argument in $(-\pi, \pi]$. Next, recall the MacLaurin Series for $\log(1+x)$:

$$\log(1+x) \;=\; \sum_{m=1}^{\infty} (-1)^{n+1} \frac{x^m}{m}, \quad \text{for any } x \in \mathbb{C} \text{ with } |\, x \,| < 1.$$

Hence, we have the following inequality: for each $n \in \mathbb{N}$ and for each $1 \leq j \leq k_n$,

$$\begin{aligned} |\, \log(1+\theta_{nj}) - \theta_{nj} \,| \;&=\; \left| \sum_{m=2}^{\infty} (-1)^{n+1} \frac{(\theta_{nj})^m}{m} \right| \;\leq\; \sum_{m=2}^{\infty} \frac{|\, \theta_{nj} \,|^m}{m} \;\leq\; \frac{|\, \theta_{nj} \,|^2}{2} \sum_{m=2}^{\infty} |\, \theta_{nj} \,|^{m-2} \\ &\leq\; \frac{|\, \theta_{nj} \,|^2}{2} \sum_{m=2}^{\infty} \left(\frac{1}{2}\right)^{m-2} \;=\; \frac{|\, \theta_{nj} \,|^2}{2} \sum_{i=2}^{\infty} \left(\frac{1}{2}\right)^{i-2} \;=\; \frac{|\, \theta_{nj} \,|^2}{2} \cdot 2 \;=\; |\, \theta_{nj} \,|^2. \end{aligned}$$

This in turn implies: for each $n \in \mathbb{N}$,

$$\left| \sum_{j=1}^{k_n} \log(1+\theta_{nj}) \;-\; \sum_{j=1}^{k_n} \theta_{nj} \right| \;=\; \left| \sum_{j=1}^{k_n} (\log(1+\theta_{nj}) - \theta_{nj}) \right| \;\leq\; \sum_{j=1}^{k_n} |\, \log(1+\theta_{nj}) \;-\; \theta_{nj} \,| \;\leq\; \sum_{j=1}^{k_n} |\, \theta_{nj} \,|^2.$$

[1] Recall that the complex exponential function is defined by $\exp : \mathbb{C} \to \mathbb{C} : x + \mathbf{i}\, y \mapsto e^x \cdot e^{\mathbf{i}\, y} = e^x (\cos y + \mathbf{i} \sin y)$. Clearly, exp is not injective. More precisely, for $x_1 + \mathbf{i}\, y_1,\, x_2 + \mathbf{i}\, y_2 \in \mathbb{C}\backslash\{0\}$, we have $e^{x_1+\mathbf{i}y_1} = e^{x_2+\mathbf{i}y_2}$ if and only if $x_1 = x_2 \in \mathbb{R}\backslash\{0\}$ and $y_1 - y_2 \in 2\pi\mathbb{Z}$. For $z = re^{\mathbf{i}\theta} \in \mathbb{C}\backslash\{0\}$, a complex logarithm of z is any $w = x + \mathbf{i}\, y \in \mathbb{C}\backslash\{0\}$ such that $e^{x+\mathbf{i}y} = e^w = z = re^{\mathbf{i}\theta}$, i.e. $x = \log r$ and $y = \theta + 2\pi\mathbb{Z}$. In particular, let $\mathcal{D} := \{ x + \mathbf{i}\, y \in \mathbb{C} \mid x \in \mathbb{R}, y \in (-\pi, \pi] \}$. Then, the restriction $\exp : \mathcal{D} \to \mathbb{C}\backslash\{0\}$ is bijective.

Thus, for each $n \in \mathbb{N}$, there exists $\Lambda_n \in \mathbb{C}$ with $|\,\Lambda_n\,| \leq 1$ such that

$$\sum_{j=1}^{k_n} \log(1+\theta_{nj}) \;=\; \sum_{j=1}^{k_n} \theta_{nj} \;+\; \Lambda_n \cdot \sum_{j=1}^{k_n} |\,\theta_{nj}\,|^2.$$

(Since for any $z \in \mathbb{C}$, $|\,z\,| \leq A \implies z = A \cdot w$, for some $w \in \mathbb{C}$ with $|\,w\,| \leq 1$.) Next note that, hypotheses (a) and (b) together imply:

$$\sum_{j=1}^{k_n} |\,\theta_{nj}\,|^2 \;\leq\; \left(\max_{1\leq j\leq k_n} |\,\theta_{nj}\,|\right)\left(\sum_{j=1}^{k_n} |\,\theta_{nj}\,|\right) \;\leq\; M \cdot \left(\max_{1\leq j\leq k_n} |\,\theta_{nj}\,|\right) \quad\longrightarrow\quad 0, \quad \text{as } n \longrightarrow \infty.$$

Therefore, since $|\,\Lambda_n\,| \leq 1$ for each $n \in \mathbb{N}$, we now see that

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \log(1+\theta_{nj}) \;=\; \lim_{n\to\infty} \sum_{j=1}^{k_n} \theta_{nj} \;+\; \lim_{n\to\infty} \left(\Lambda_n \cdot \sum_{j=1}^{k_n} |\,\theta_{nj}\,|^2\right) \;=\; \theta + 0 \;=\; \theta.$$

We may now conclude, by continuity of the exponential function $\exp(\,\cdot\,)$:

$$\begin{aligned}\lim_{n\to\infty} \prod_{j=1}^{k_n} (\,1+\theta_{nj}\,) \;&=\; \lim_{n\to\infty} \exp\left(\log \prod_{j=1}^{k_n} (\,1+\theta_{nj}\,)\right) \;=\; \lim_{n\to\infty} \exp\left(\sum_{j=1}^{k_n} \log\,(\,1+\theta_{nj}\,)\right) \\ &=\; \exp\left(\lim_{n\to\infty} \sum_{j=1}^{k_n} \log\,(\,1+\theta_{nj}\,)\right) \;=\; \exp\,(\,\theta\,)\end{aligned}$$

This completes the proof of the Lemma. □

B The Central Limit Theorems

Theorem B.1 (Lindeberg's Central Limit Theorem, Theorem 1.15, [3])

Suppose:

- $\{\,k_n\,\}_{n\in\mathbb{N}} \subset \mathbb{N}$ *is a sequence of natural numbers such that* $k_n \to \infty$ *as* $n \to \infty$, *and*
- *for each* $n \in \mathbb{N}$, $X_1^{(n)}, X_2^{(n)}, \ldots, X_{k_n}^{(n)} : \Omega_n \longrightarrow \mathbb{R}$ *are independent (but not necessarily identically distributed)* $\mathbb{R}$*-valued random variables defined on a common probability space* $(\,\Omega_n, \mathcal{A}_n, \mu_n\,)$ *such that*

$$\mu_j^{(n)} \;:=\; E\Big[\,X_j^{(n)}\,\Big] \;\in\; \mathbb{R} \;\text{ exists, for each } 1 \leq j \leq k_n, \quad \text{and} \quad 0 \;<\; \sigma_n^2 \;:=\; \mathrm{Var}\left[\,\sum_{j=1}^{k_n} X_j^{(n)}\,\right] \;<\; \infty.$$

Let $N(0,1)$ *denote the standard Gaussian distribution on* $\mathbb{R}$*. Then, the following implication holds: If*

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{j=1}^{k_n} E\left[\,\left(X_j^{(n)} - \mu_j^{(n)}\right)^2 \cdot I_{\left\{|X_j^{(n)} - \mu_j^{(n)}| \,>\, \epsilon\sigma_n\right\}}\,\right] \;=\; 0, \quad \text{for each } \epsilon > 0,$$

then

$$\frac{1}{\sigma_n} \sum_{j=1}^{k_n} \left(\,X_j^{(n)} - \mu_j^{(n)}\,\right) \;\xrightarrow{\;\mathcal{L}\;}\; N(0,1).$$

PROOF Considering $\left(X_j^{(n)} - \mu_j^{(n)} \right) \Big/ \sigma_n$, we may assume, without loss of generality, that

$$E\left[\, X_j^{(n)} \,\right] = 0, \quad \text{and} \quad \sigma_n^2 \;:=\; \mathrm{Var}\left[\, \sum_{j=1}^{k_n} X_j^{(n)} \,\right] = 1.$$

□

Lemma A.2.

References

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[3] SHAO, J. *Mathematical Statistics*, second ed. Springer, 2003.