# 1 The Hájek Central Limit Theorem for Simple Random Sampling without Replacement

Suppose we are given a sequence of finite populations, on each of which is defined an $\mathbb{R}$-valued population characteristic. Suppose on each of the finite populations, we use SRSWOR (of fixed sample size) to select a sample, observe the values of the corresponding population characteristics on the selected elements, and use the Horvitz-Thompson estimator for the population mean. We seek to determine a necessary and sufficient condition for the (associated sequence of) "standardized deviations from the mean" of the Horvitz-Thompson estimator for population mean to converge in distribution to the standard Gaussian distribution.

**Theorem 1.1 (The Hájek Central Limit Theorem for SRSWOR)**
*Suppose we have the following:*

- *Let* $\{\,U_\nu\,\}_{\nu\in\mathbb{N}}$ *be a sequence of finite populations, and* $N_\nu = |\,U_\nu\,|$ *be the population size of* $U_\nu$*. Let the elements of* $U_\nu$ *be indexed by* $1,2,3,\ldots,N_\nu$.
- *For each* $\nu\in\mathbb{N}$*, let* $y^{(\nu)} : U_\nu \longrightarrow \mathbb{R}$ *be an* $\mathbb{R}$*-valued population characteristic. For each* $i \in U_\nu$*, let* $y_i^{(\nu)}$ *denote* $y^{(\nu)}(i)$*, the value of* $y^{(\nu)}$ *evaluated at the* $i^{\text{th}}$ *element of* $U_\nu$.
- *For each* $\nu\in\mathbb{N}$*, let* $n_\nu \in \{1,2,3,\ldots,N_\nu\}$ *be given, and let* $\mathcal{S}_\nu$ *be the set of all* $n_\nu$*-element subsets of* $U_\nu$*. Let* $\mathcal{S}_\nu$ *be endowed with the uniform probability function, namely*

$$P(s) \;=\; \frac{1}{\begin{pmatrix} N_\nu \\ n_\nu \end{pmatrix}}, \;\; \text{for each } s\in\mathcal{S}_\nu.$$

- *For each* $\nu\in\mathbb{N}$*, let* $\widehat{\overline{Y}}_\nu : \mathcal{S}_\nu \longrightarrow \mathbb{R}$ *be the random variable defined as follows:*

$$\widehat{\overline{Y}}_\nu(s) \;:=\; \frac{1}{n_\nu}\sum_{i\in s} y_i^{(\nu)}, \;\; \text{for each } s\in\mathcal{S}_\nu$$

*Let*

$$\mu_\nu \;:=\; E\!\left[\,\widehat{\overline{Y}}_\nu\,\right] \;=\; \frac{1}{N_\nu}\sum_{i\in U_\nu} y_i^{(\nu)} \quad\text{and}\quad \sigma_\nu^2 \;:=\; \operatorname{Var}\!\left[\,\widehat{\overline{Y}}_\nu\,\right] \;=\; \left(1-\frac{n_\nu}{N_\nu}\right)\frac{S_\nu^2}{n_\nu},$$

*where*

$$S_\nu^2 \;:=\; \frac{1}{N_\nu-1}\sum_{i\in U_\nu}\left(y_i^{(\nu)}-\mu_\nu\right)^2$$

- *For each* $\nu\in\mathbb{N}$ *and each* $\delta>0$ *define:*

$$U_\nu(\delta) \;:=\; \left\{\, i\in U_\nu \;\middle|\; |y_i^{(\nu)}-\mu_\nu| > \delta\,\sqrt{\sigma_\nu^2}\,\right\} \;\subset\; U_\nu.$$

*Suppose* $n_\nu \longrightarrow \infty$ *and* $N_\nu - n_\nu \longrightarrow \infty$*. Then,*

$$\lim_{\nu\to\infty} P\left\{\, s\in\mathcal{S}_\nu \;\middle|\; \frac{\widehat{\overline{Y}}_\nu(s)-\mu_\nu}{\sqrt{\sigma_\nu^2}} < x \,\right\} \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t$$

*if and only if*

$$\lim_{\nu\to\infty} \frac{\displaystyle\sum_{i\in U_\nu(\delta)} \left(y_i^{(\nu)} - \mu_\nu\right)^2}{\displaystyle\sum_{i\in U_\nu} \left(y_i^{(\nu)} - \mu_\nu\right)^2} \;=\; 0, \;\; \text{for every } \delta > 0.$$

**Lemma 1.2**
*Bernoulli sampling from a finite population $U$ of size $N$ with individual selection probability $n/N$, where $n = 1, 2, \ldots, N$, is equivalent to the following two-step sampling scheme:*

- ***Step 1:*** *Sample $k$ from* $\text{Binomial}\,(N, n/N)$.
- ***Step 2:*** *Take an SRSWOR sample $s$ of size $k$ from $U$.*

PROOF Note that the collection of possible samples for both schemes is the power set $\mathcal{P}(U)$ of $U$, i.e. all possible subsets of $U$. Let $P_{\text{B}}$ and $P_1$ be the probability functions defined on $\mathcal{P}(U)$ under Bernoulli sampling and the two-step scheme, respectively. Then,

$$P_{\text{B}}(s) \;=\; \left(\frac{n}{N}\right)^k \left(1 - \frac{n}{N}\right)^{N-k}, \quad \text{for each } \; s \in \mathcal{P}(U), \;\; \text{where } \; k = |\,s\,|.$$

On the other hand,

$$\begin{aligned} P_1(s) &= P(\; S = s \mid S \sim \text{SRSWOR}(k, N) \;) \cdot P(\; K = k \mid K \sim \text{Binomial}(N, n/N) \;) \\ &= \frac{1}{\binom{N}{k}} \cdot \binom{N}{k} \left(\frac{n}{N}\right)^k \left(1 - \frac{n}{N}\right)^{N-k} \\ &= \left(\frac{n}{N}\right)^k \left(1 - \frac{n}{N}\right)^{N-k}, \quad \text{for each } \; s \in \mathcal{P}(U), \;\; \text{where } \; k = |\,s\,|. \end{aligned}$$

Thus, $P_{\text{B}} = P_1$ as (probability) functions on $\mathcal{P}(U)$. Hence, the two sampling schemes are equivalent. □

**Definition 1.3 (The Hájek Sampling Design of size $n$)**
*Suppose $U$ is a finite population of size $N \in \mathbb{N}$ with $N \geq 3$. Let $n \in \{2, \ldots, N\}$ be fixed. Let $\mathcal{P}(U)$ be the power set of $U$. Let $\mathcal{S}(U, n)$ be the collection of all subsets of $U$ with exactly $n$ elements. The* ***Hájek Sampling Design of size $n$ on*** *$U$, by definition, selects an ordered pair of samples $\left(s^{(0)}, s^{(1)}\right) \in \mathcal{S}(U, n) \times \mathcal{P}(U)$ as follows:*

- *First, select $k \in \{0, 1, 2, \ldots, N\}$ based on the binomial distribution* $\text{Binomial}(N, n/N)$.

  *More precisely, let $K \sim$* $\text{Binomial}(N, n/N)$*, i.e. let $K$ be a random variable following the binomial distribution with number of trials $N$ and probability of success $n/N$. In other words,*

  $$P(K = k) \;=\; \binom{N}{k} \cdot \left(\frac{n}{N}\right)^k \cdot \left(1 - \frac{n}{N}\right)^{N-k}, \quad \text{for each } \; k = 0, 1, 2, \ldots, N.$$

  *Let $k \in \{0, 1, 2, \ldots, N\}$ be a realization of the random variable $K \sim$* $\text{Binomial}(N, n/N)$.

- *If $k = n$, take an SRSWOR sample $s^{(0)} \subset U$ of size $n$, and let $s^{(1)} = s^{(0)}$.*
- *If $k > n$, take an SRSWOR sample $s^{(1)} \subset U$ of size $k$. Then, select an SRSWOR sample $s^{(0)}$ of $s^{(1)}$ of size $n$.*
- *If $k < n$, take an SRSWOR sample $s^{(0)} \subset U$ of size $n$. Then, select an SRSWOR sample $s^{(1)}$ of $s^{(0)}$ of size $k$.*

**Remark 1.4**

*Note that the Hájek Sampling Design defines implicitly a probability function $P_{\mathrm{H}}$ on $\mathcal{S}(U,n)\times\mathcal{P}(U)$, making it a finite probability space. More explicitly, for each $\left(s^{(0)},s^{(1)}\right)\in\mathcal{S}(U,n)\times\mathcal{P}(U)$, writing $k=\vert\,s^{(1)}\,\vert$, we have*

$$P_{\mathrm{H}}\left(s^{(0)},s^{(1)}\right) \;=\; \begin{cases} \left(\begin{array}{c} N \\ n \end{array}\right)\left(\dfrac{n}{N}\right)^{n}\left(1-\dfrac{n}{N}\right)^{N-n}\cdot\dfrac{1}{\left(\begin{array}{c} N \\ n \end{array}\right)}, & \text{if } s^{(0)}=s^{(1)} \\ \left(\begin{array}{c} N \\ k \end{array}\right)\left(\dfrac{n}{N}\right)^{k}\left(1-\dfrac{n}{N}\right)^{N-k}\cdot\dfrac{1}{\left(\begin{array}{c} N \\ k \end{array}\right)}\cdot\dfrac{1}{\left(\begin{array}{c} k \\ n \end{array}\right)}, & \text{if } s^{(0)}\subsetneq s^{(1)} \\ \left(\begin{array}{c} N \\ k \end{array}\right)\left(\dfrac{n}{N}\right)^{k}\left(1-\dfrac{n}{N}\right)^{N-k}\cdot\dfrac{1}{\left(\begin{array}{c} N \\ n \end{array}\right)}\cdot\dfrac{1}{\left(\begin{array}{c} n \\ k \end{array}\right)}, & \text{if } s^{(0)}\supsetneq s^{(1)} \\ 0, & \text{otherwise} \end{cases}$$

**Lemma 1.5 (Properties of the Hájek Sampling Design)**

*Suppose $U$ is a finite population of size $N\in\mathbb{N}$ with $N\geq 3$. Let $n\in\{2,\ldots,N\}$ be fixed. Let $P_{\mathrm{H}}:\mathcal{S}(U,n)\times\mathcal{P}(U)\longrightarrow[0,1]$ be the Hájek Sampling Design. Then, the following statements are true:*

(a) *The marginal sampling design induced on $\mathcal{S}(U,n)$ by $P_{\mathrm{H}}$ is* $\mathrm{SRSWOR}(U,n)$.

(b) *The marginal sampling design induced on $\mathcal{P}(U)$ by $P_{\mathrm{H}}$ is Bernoulli Sampling from $U$ with unit selection probability $n/N$.*

(c) *For each fixed $k\in\{n+1,n+2,\ldots,N\}$, the sampling design induced on $\mathcal{S}(U,k-n)$ by pushing forward the conditional sampling design of $P_{\mathrm{H}}\vert_{\vert S^{(1)}\vert=k}$ via the following map:*

$$\left\{\left(s^{(0)},s^{(1)}\right)\in\mathcal{S}(U,n)\times\mathcal{P}(U)\;\middle\vert\;\vert\,s^{(1)}\,\vert=k\right\}\longrightarrow\mathcal{S}(U,k-n):\left(s^{(0)},s^{(1)}\right)\longmapsto s^{(1)}\backslash\,s^{(0)}$$

*is equivalent to* $\mathrm{SRSWOR}(U,k-n)$.

(d) *For each fixed $k\in\{0,1,2,\ldots,n-1\}$, the sampling design induced on $\mathcal{S}(U,n-k)$ by pushing forward the pertinent restriction of $P_{\mathrm{H}}$ via the following map:*

$$\left\{\left(s^{(0)},s^{(1)}\right)\in\mathcal{S}(U,n)\times\mathcal{P}(U)\;\middle\vert\;\vert\,s^{(1)}\,\vert=k\right\}\longrightarrow\mathcal{S}(U,n-k):\left(s^{(0)},s^{(1)}\right)\longmapsto s^{(0)}\backslash\,s^{(1)}$$

*is equivalent to* $\mathrm{SRSWOR}(U,n-k)$.

PROOF

(a) For each $s^{(0)}\in\mathcal{S}(U,n)$, it suffices to show that the marginal probability $P_{\mathrm{H}}\left(s^{(0)},\,\cdot\,\right)$ is given by:

$$P_{\mathrm{H}}\left(s^{(0)},\,\cdot\,\right) \;=\; \frac{1}{\left(\begin{array}{c} N \\ n \end{array}\right)}$$

To this end,

$$\begin{aligned}
P_{\mathrm{H}}\left(s^{(0)},\,\cdot\,\right) &= \sum_{s^{(1)}=s^{(0)}} P_{\mathrm{H}}\left(s^{(0)},s^{(1)}\right) + \sum_{s^{(1)}\supsetneq s^{(0)}} P_{\mathrm{H}}\left(s^{(0)},s^{(1)}\right) + \sum_{s^{(1)}\subsetneq s^{(0)}} P_{\mathrm{H}}\left(s^{(0)},s^{(1)}\right) \\
&= \binom{N}{n}\left(\frac{n}{N}\right)^{n}\left(1-\frac{n}{N}\right)^{N-n}\cdot\frac{1}{\binom{N}{n}} \\
&\quad + \sum_{k=n+1}^{N}\binom{N}{k}\left(\frac{n}{N}\right)^{k}\left(1-\frac{n}{N}\right)^{N-k}\cdot\frac{1}{\binom{N}{k}}\cdot\frac{1}{\binom{k}{n}}\cdot\binom{N-n}{k-n} \\
&\quad + \sum_{k=0}^{n-1}\binom{N}{k}\left(\frac{n}{N}\right)^{k}\left(1-\frac{n}{N}\right)^{N-k}\cdot\frac{1}{\binom{N}{n}}\cdot\frac{1}{\binom{n}{k}}\cdot\binom{n}{k}
\end{aligned}$$

We remark that, for a given $s^{(0)}\in\mathcal{S}(U,n)$ and $k>n$, the quantity $\binom{N-n}{k-n}$ is the number of elements in $\mathcal{P}(U)$ (i.e. number of subsets of $U$) of size $k$ containing $s^{(0)}$ as a proper subset. Note also that, for $k>n$,

$$\frac{1}{\binom{N}{k}}\cdot\frac{1}{\binom{k}{n}}\cdot\binom{N-n}{k-n} = \frac{k!(N-k)!}{N!}\cdot\frac{n!(k-n)!}{k!}\cdot\frac{(N-n)!}{(k-n)!(N-k)!} = \frac{n!(N-n)!}{N!} = \frac{1}{\binom{N}{n}}.$$

Hence, we have

$$P_{\mathrm{H}}\left(s^{(0)},\,\cdot\,\right) = \frac{1}{\binom{N}{n}}\cdot\sum_{k=0}^{N}\binom{N}{k}\left(\frac{n}{N}\right)^{k}\left(1-\frac{n}{N}\right)^{N-k} = \frac{1}{\binom{N}{n}}\cdot 1 = \frac{1}{\binom{N}{n}}$$

(b) For each $s^{(1)}\in\mathcal{P}(U)$, it suffices to show that the marginal probability $P_{\mathrm{H}}\left(\,\cdot\,,s^{(1)}\right)$ is given by:

$$P_{\mathrm{H}}\left(\,\cdot\,,s^{(1)}\right) = \left(\frac{n}{N}\right)^{k}\cdot\left(1-\frac{n}{N}\right)^{N-k},\quad \text{where } k=\left|\,s^{(1)}\,\right|.$$

To this end, first note that either $k=\left|\,s^{(1)}\,\right|\geq n$ holds, or $k=\left|\,s^{(1)}\,\right|<n$ holds. In the first case, i.e. $k=\left|\,s^{(1)}\,\right|\geq n$, we have

$$\begin{aligned}
P_{\mathrm{H}}\left(\,\cdot\,,s^{(1)}\right) &= P\left(S^{(1)}=s^{(1)}\,\middle|\,K=k\right)\cdot P(K=k) \\
&= \frac{1}{\binom{N}{k}}\cdot\binom{N}{k}\left(\frac{n}{N}\right)^{k}\left(1-\frac{n}{N}\right)^{N-k} \\
&= \left(\frac{n}{N}\right)^{k}\cdot\left(1-\frac{n}{N}\right)^{N-k}.
\end{aligned}$$

In the second case, i.e. $k = |\,s^{(1)}\,| < n$, we have

$$\begin{aligned}
P_{\mathrm{H}}\Big(\,\cdot\,,s^{(1)}\Big) &= \sum_{s^{(0)}\supsetneq s^{(1)}} P_{\mathrm{H}}\Big(s^{(0)},s^{(1)}\Big) \;=\; \sum_{s^{(0)}\supsetneq s^{(1)}} \binom{N}{k}\left(\frac{n}{N}\right)^k\left(1-\frac{n}{N}\right)^{N-k}\cdot\frac{1}{\binom{N}{n}}\cdot\frac{1}{\binom{n}{k}} \\
&= \binom{N-k}{n-k}\cdot\binom{N}{k}\left(\frac{n}{N}\right)^k\left(1-\frac{n}{N}\right)^{N-k}\cdot\frac{1}{\binom{N}{n}}\cdot\frac{1}{\binom{n}{k}} \\
&= \binom{N}{k}\left(\frac{n}{N}\right)^k\left(1-\frac{n}{N}\right)^{N-k}\cdot\frac{(N-k)!}{(n-k)!(N-n)!}\cdot\frac{n!(N-n)!}{N!}\cdot\frac{k!(n-k)!}{n!} \\
&= \binom{N}{k}\left(\frac{n}{N}\right)^k\left(1-\frac{n}{N}\right)^{N-k}\cdot\frac{k!(N-k)!}{N!} \;=\; \binom{N}{k}\left(\frac{n}{N}\right)^k\left(1-\frac{n}{N}\right)^{N-k}\cdot\frac{1}{\binom{N}{k}} \\
&= \left(\frac{n}{N}\right)^k\cdot\left(1-\frac{n}{N}\right)^{N-k}
\end{aligned}$$

We remark that, for a given $s^{(1)}\in\mathcal{P}(U)$ with $|\,s^{(1)}\,| = k < n$, the quantity $\binom{N-k}{n-k}$ is the number of elements in $\mathcal{S}(U,n)$ containing $s^{(1)}$ as a proper subset.

(c) Let $\widetilde{P}:\mathcal{S}(U,k-n)$ be the induced sampling design on $\mathcal{S}(U,k-n)$. Then, for each $s^{(2)}\in\mathcal{S}(U,k-n)$, we have

$$\begin{aligned}
\widetilde{P}\Big(s^{(2)}\Big) &= \sum_{s^{(1)}\setminus s^{(0)}=s^{(2)}} P_{\mathrm{H}}\Big(s^{(0)},s^{(1)}\,\Big|\,K=k\Big) \;=\; \sum_{s^{(1)}\setminus s^{(0)}=s^{(2)}} \frac{1}{\binom{N}{k}}\cdot\frac{1}{\binom{k}{n}} \\
&= \binom{N-k+n}{n}\cdot\frac{1}{\binom{N}{k}}\cdot\frac{1}{\binom{k}{n}} \;=\; \frac{(N-k+n)!}{n!(N-k)!}\cdot\frac{k!(N-k)!}{N!}\cdot\frac{n!(k-n)!}{k!} \\
&= \frac{(k-n)!(N-k+n)!}{N!} \;=\; 1\Big/\binom{N}{k-n}
\end{aligned}$$

This proves that $\widetilde{P}$ is indeed equivalent to $\mathrm{SRSWOR}(U,k-n)$.

(d) Let $P':\mathcal{S}(U,n-k)$ be the induced sampling design on $\mathcal{S}(U,n-k)$. Then, for each $s^{(2)}\in\mathcal{S}(U,n-k)$, we have

$$\begin{aligned}
P'\Big(s^{(2)}\Big) &= \sum_{s^{(0)}\setminus s^{(1)}=s^{(2)}} P_{\mathrm{H}}\Big(s^{(0)},s^{(1)}\,\Big|\,K=k\Big) \;=\; \sum_{s^{(0)}\setminus s^{(1)}=s^{(2)}} \frac{1}{\binom{N}{n}}\cdot\frac{1}{\binom{n}{k}} \\
&= \binom{N-n+k}{k}\cdot\frac{1}{\binom{N}{n}}\cdot\frac{1}{\binom{n}{k}} \;=\; \frac{(N-n+k)!}{k!(N-n)!}\cdot\frac{n!(N-n)!}{N!}\cdot\frac{k!(n-k)!}{n!} \\
&= \frac{(n-k)!(N-n+k)!}{N!} \;=\; 1\Big/\binom{N}{n-k}
\end{aligned}$$

This proves that $P'$ is indeed equivalent to $\mathrm{SRSWOR}(U,n-k)$.

The proof of this Lemma is complete. $\square$

**Theorem 1.6 (The Hájek Fundamental Lemma)**

*Suppose $U$ is a finite population of size $N\in\mathbb{N}$ with $N\geq 3$, and $y:U\longrightarrow\mathbb{R}$ is a population characteristic. Let $n\in\{2,\ldots,N\}$ be fixed. Let $\overline{y}_U := \frac{1}{N}\sum_{i\in U} y_i$. Let $\mathcal{S}(U,n)\times\mathcal{P}(U)$ be endowed with the probability function $P_{\mathrm{H}}$ defined* *by the Hájek Sampling Design. Define the $\mathbb{R}^2$-valued random variable $Y = \left(Y^{(0)}, Y^{(1)}\right) : \mathcal{S}(U,n) \times \mathcal{P}(U) \longrightarrow \mathbb{R}^2$ as follows: For any $\left(s^{(0)}, s^{(1)}\right) \in \mathcal{S}(U,n) \times \mathcal{P}(U)$,*

$$Y^{(0)}\left(s^{(0)}\right) \;:=\; \frac{1}{n}\sum_{i\in s^{(0)}}\left(y_i - \overline{y}_U\right), \quad \text{and} \quad Y^{(1)}\left(s^{(1)}\right) \;:=\; \frac{1}{n}\sum_{i\in s^{(1)}}\left(y_i - \overline{y}_U\right).$$

*Then,*

$$E\left[\left(\frac{Y^{(0)}}{\sqrt{\mathrm{Var}\left[\,Y^{(1)}\,\right]}} - \frac{Y^{(1)}}{\sqrt{\mathrm{Var}\left[\,Y^{(1)}\,\right]}}\right)^2\right] \;=\; \frac{E\left[\left(Y^{(0)} - Y^{(1)}\right)^2\right]}{\mathrm{Var}\left[\,Y^{(1)}\,\right]} \;\leq\; \sqrt{\frac{1}{n} + \frac{1}{N-n}}$$

Proof We write $k := \left|\,s^{(1)}\,\right|$. First, observed that

$$Y^{(0)} - Y^{(1)} \;=\; \begin{cases} 0, & \text{if } k = n \\[2ex] \dfrac{|\,k-n\,|}{n}\cdot\dfrac{1}{|\,k-n\,|}\cdot\displaystyle\sum_{i\in s^{(0)}\backslash s^{(1)}}\left(y_i - \overline{y}_U\right), & \text{if } k < n \\[2ex] \dfrac{|\,k-n\,|}{n}\cdot\dfrac{1}{|\,k-n\,|}\cdot\displaystyle\sum_{i\in s^{(1)}\backslash s^{(0)}}\left(y_i - \overline{y}_U\right), & \text{if } k > n \end{cases}$$

By Lemma 1.5(c,d), for $k := \left|\,s^{(1)}\,\right|$ fixed, we may regard $s^0 \setminus s^{(1)}$ and $s^1 \setminus s^{(0)}$ as realizations from SRSWOR$(U, |\,k-n\,|)$. Hence,

$$E\left[\;\left(Y^{(0)} - Y^{(1)}\right)\;\middle|\;\left|s^{(1)}\right| = k\;\right] \;=\; \frac{|\,k-n\,|}{n}\cdot E\left[\,\widehat{\overline{T}}^{\,\mathrm{HT}}_{\mathrm{SRSWOR}}\,\right] \;=\; 0$$

Hence,

$$\begin{aligned}
E\left[\;\left(Y^{(0)} - Y^{(1)}\right)^2\;\middle|\;\left|s^{(1)}\right| = k\;\right] \;&=\; \mathrm{Var}\left[\;Y^{(0)} - Y^{(1)}\;\middle|\;\left|s^{(1)}\right| = k\;\right] \\
&=\; \frac{|\,k-n\,|^2}{n^2}\left(1 - \frac{|\,k-n\,|}{N}\right)\frac{1}{|\,k-n\,|}\frac{\sum_{i\in U}\left(y_i - \overline{y}_U\right)^2}{N-1} \\
&=\; \frac{|\,k-n\,|}{n^2}\left(\frac{N - |\,k-n\,|}{N-1}\right)\frac{\sum_{i\in U}\left(y_i - \overline{y}_U\right)^2}{N} \\
&\leq\; \frac{|\,k-n\,|}{n^2}\frac{\sum_{i\in U}\left(y_i - \overline{y}_U\right)^2}{N}
\end{aligned}$$

Consequently,

$$\begin{aligned}
E\left\{\;\left(Y^{(0)} - Y^{(1)}\right)^2\;\right\} \;&=\; E\left\{\;E\left[\;\left(Y^{(0)} - Y^{(1)}\right)^2\;\middle|\;\left|s^{(1)}\right| = k\;\right]\;\right\} \\
&\leq\; E\left\{\;E\left[\;\frac{|\,k-n\,|}{n^2}\frac{\sum_{i\in U}\left(y_i - \overline{y}_U\right)^2}{N}\;\middle|\;\left|s^{(1)}\right| = k\;\right]\;\right\} \\
&=\; \frac{1}{n^2}\frac{\sum_{i\in U}\left(y_i - \overline{y}_U\right)^2}{N}\cdot E\{\,|\,k-n\,|\,\} \;\leq\; \frac{1}{n^2}\frac{\sum_{i\in U}\left(y_i - \overline{y}_U\right)^2}{N}\cdot\sqrt{E\left\{\,|\,k-n\,|^2\,\right\}} \\
&\leq\; \frac{1}{n^2}\frac{\sum_{i\in U}\left(y_i - \overline{y}_U\right)^2}{N}\cdot\sqrt{n\left(1 - \frac{n}{N}\right)},
\end{aligned}$$

where we used the Cauchy-Schwarz inequality (Theorem 9.3, [2]) for the second last inequality. Next, we compute $\mathrm{Var}\big[\,Y^{(1)}\,\big]$. To this end, note that

$$Y^{(1)} \;=\; \sum_{i\in U} Z_i,$$

where, for each $i \in U$,

$$Z_i : \mathcal{S}(U,n)\times\mathcal{P}(U) \longrightarrow \mathbb{R} : \left(s^{(0)}, s^{(1)}\right) \longmapsto \begin{cases} \dfrac{1}{n}\,(y_i \,-\, \overline{y}_U)\,, & \text{if } i \in s^{(1)} \\ \\ 0, & \text{if } i \notin s^{(1)} \end{cases}$$

Note that, since $Z_i$ depends only on $s^{(1)}$, which can be regarded as a Bernoulli sample from $U$, by Lemma 1.5, we see that the $Z_i, i \in U$, are independent, and

$$P\!\left(\,Z_i = \frac{1}{n}\,(y_i \,-\, \overline{y}_U)\right) \;=\; \frac{n}{N}, \quad \text{and} \quad P(\,Z_i = 0\,) \;=\; 1 - \frac{n}{N}.$$

Thus,

$$\mathrm{Var}[\,Z_i\,] \;=\; \left(\frac{y_i - \overline{y}_U}{n}\right)^2 \cdot \frac{n}{N}\left(1 - \frac{n}{N}\right),$$

which in turn imples

$$\mathrm{Var}\Big[\,Y^{(1)}\Big] \;=\; \sum_{i\in U}\mathrm{Var}[\,Z_i\,] \;=\; \sum_{i\in U}\left(\frac{y_i - \overline{y}_U}{n}\right)^2 \cdot \frac{n}{N}\left(1 - \frac{n}{N}\right) \;=\; \cdots \;=\; \frac{1}{n^2}\,\frac{\sum\limits_{i\in U}(y_i - \overline{y}_U)^2}{N}\cdot n\left(1 - \frac{n}{N}\right)$$

Thus, we see that

$$\frac{E\Big[\,\left(Y^{(0)} - Y^{(1)}\right)^2\,\Big]}{\mathrm{Var}\big[\,Y^{(1)}\,\big]} \;\leq\; \frac{\dfrac{1}{n^2}\,\dfrac{\sum\limits_{i\in U}(y_i - \overline{y}_U)^2}{N}\cdot\sqrt{n\left(1 - \dfrac{n}{N}\right)}}{\dfrac{1}{n^2}\,\dfrac{\sum\limits_{i\in U}(y_i - \overline{y}_U)^2}{N}\cdot n\left(1 - \dfrac{n}{N}\right)} \;=\; \frac{1}{\sqrt{n\left(1 - \dfrac{n}{N}\right)}} \;=\; \cdots \;=\; \sqrt{\frac{1}{n} + \frac{1}{N-n}}.$$

This completes the proof of Hájek's Fundamental Lemma. □

**Corollary 1.7**

*Suppose we have the following:*

- *Let* $\{\,U_\nu\,\}_{\nu\in\mathbb{N}}$ *be a sequence of finite populations, and* $N_\nu = |\,U_\nu\,|$ *be the population size of* $U_\nu$. *Let the elements of* $U_\nu$ *be indexed by* $1, 2, 3, \ldots, N_\nu$.
- *For each* $\nu \in \mathbb{N}$, *let* $y^{(\nu)} : U_\nu \longrightarrow \mathbb{R}$ *be an* $\mathbb{R}$*-valued population characteristic. For each* $i \in U_\nu$, *let* $y_i^{(\nu)}$ *denote* $y^{(\nu)}(i)$, *the value of* $y^{(\nu)}$ *evaluated at the* $i^{\text{th}}$ *element of* $U_\nu$. *Let* $\overline{y}_{U_\nu} := \dfrac{1}{N_\nu}\cdot\sum\limits_{i\in U_\nu} y_i^{(\nu)}$.
- *For each* $\nu \in \mathbb{N}$, *let* $n_\nu \in \{1, 2, 3, \ldots, N_\nu\}$ *be given, and let* $p_\nu : \mathcal{S}(U_\nu, n_\nu)\times\mathcal{P}(U_\nu) \longrightarrow [0,1]$ *be the Hájek Sampling Design of size* $n_\nu$ *on* $U_\nu$.
- *For each* $\nu \in \mathbb{N}$, *let* $Y_\nu = \left(Y_\nu^{(0)}, Y_\nu^{(1)}\right) : \mathcal{S}(U_\nu, n_\nu)\times\mathcal{P}(U_\nu) \longrightarrow \mathbb{R}^2$ *be the* $\mathbb{R}^2$*-valued random variable defined as follows: For any* $\left(s_\nu^{(0)}, s_\nu^{(1)}\right) \in \mathcal{S}(U_\nu, n_\nu)\times\mathcal{P}(U_\nu)$,

$$Y_\nu^{(0)}\!\left(s_\nu^{(0)}\right) \;:=\; \frac{1}{n_\nu}\sum_{i\in s_\nu^{(0)}}\left(y_i^{(\nu)} - \overline{y}_{U_\nu}\right), \quad \text{and} \quad Y_\nu^{(1)}\!\left(s_\nu^{(1)}\right) \;:=\; \frac{1}{n_\nu}\sum_{i\in s_\nu^{(1)}}\left(y_i^{(\nu)} - \overline{y}_{U_\nu}\right).$$

*Then, the following implication holds:*

$$\left.\begin{array}{ccc} n_\nu & \longrightarrow & \infty \\ N_\nu - n_\nu & \longrightarrow & \infty \\ \dfrac{Y_\nu^{(1)}}{\sqrt{\operatorname{Var}\left[\,Y_\nu^{(1)}\,\right]}} & \overset{\mathcal{L}}{\longrightarrow} & N(0,1) \end{array}\right\} \quad\Longrightarrow\quad \frac{Y_\nu^{(0)}}{\sqrt{\operatorname{Var}\left[\,Y_\nu^{(0)}\,\right]}} \;\overset{\mathcal{L}}{\longrightarrow}\; N(0,1).$$

Proof By the Hájek Fundamental Lemma (Theorem 1.6), we have for each $\nu \in \mathbb{N}$,

$$E\left[\left(\frac{Y_\nu^{(0)}}{\sqrt{\operatorname{Var}\left[\,Y_\nu^{(1)}\,\right]}} - \frac{Y_\nu^{(1)}}{\sqrt{\operatorname{Var}\left[\,Y_\nu^{(1)}\,\right]}}\right)^2\right] \;\leq\; \sqrt{\frac{1}{n_\nu} + \frac{1}{N_\nu - n_\nu}}.$$

Thus, the hypotheses $n_\nu \longrightarrow \infty$ and $N_\nu - n_\nu \longrightarrow \infty$ together imply that

$$\frac{Y_\nu^{(0)}}{\sqrt{\operatorname{Var}\left[\,Y_\nu^{(1)}\,\right]}} - \frac{Y_\nu^{(1)}}{\sqrt{\operatorname{Var}\left[\,Y_\nu^{(1)}\,\right]}}$$

converges to 0 in the second mean (Definition 3, p.4, [1]), hence also in probability (by Theorem 1(b), p.4, [1]). This convergence to 0 in probability and the hypothesis $Y_\nu^{(1)} \Big/ \sqrt{\operatorname{Var}\left[\,Y_\nu^{(1)}\,\right]} \overset{\mathcal{L}}{\longrightarrow} N(0,1)$ then together imply, by Slutsky's Theorem (Theorem 6(b), p.39, [1]),

$$\frac{Y_\nu^{(0)}}{\sqrt{\operatorname{Var}\left[\,Y_\nu^{(1)}\,\right]}} \;\overset{\mathcal{L}}{\longrightarrow}\; N(0,1).$$

Next, recall from the proof of the Hájek Fundamental Lemma (Theorem 1.6) that

$$\operatorname{Var}\left[\,Y_\nu^{(1)}\,\right] \;=\; \cdots \;=\; \frac{1}{n_\nu}\left(1 - \frac{n_\nu}{N_\nu}\right)\frac{\sum_{i\in U_\nu}\left(y_i^{(\nu)} - \overline{y}_{U_\nu}\right)^2}{N_\nu}.$$

On the other hand,

$$\operatorname{Var}\left[\,Y_\nu^{(0)}\,\right] \;=\; \operatorname{Var}\left[\frac{1}{n_\nu}\sum_{i\in s_\nu^{(0)}}\left(y_i^{(\nu)} - \overline{y}_{U_\nu}\right)\right] \;=\; \operatorname{Var}\left[\frac{1}{n_\nu}\sum_{i\in s_\nu^{(0)}} y_i^{(\nu)}\right] \;=\; \frac{1}{n_\nu}\left(1 - \frac{n_\nu}{N_\nu}\right)\frac{\sum_{i\in U_\nu}\left(y_i^{(\nu)} - \overline{y}_{U_\nu}\right)^2}{N_\nu - 1}$$

Hence,

$$\frac{\sqrt{\operatorname{Var}\left[\,Y_\nu^{(1)}\,\right]}}{\sqrt{\operatorname{Var}\left[\,Y_\nu^{(0)}\,\right]}} \;=\; \left(\frac{\frac{1}{n_\nu}\cdot\left(1 - \frac{n_\nu}{N_\nu}\right)\cdot\sum_{i\in U_\nu}\left(y_i^{(\nu)} - \overline{y}_{U_\nu}\right)^2 \Big/ N_\nu}{\frac{1}{n_\nu}\cdot\left(1 - \frac{n_\nu}{N_\nu}\right)\cdot\sum_{i\in U_\nu}\left(y_i^{(\nu)} - \overline{y}_{U_\nu}\right)^2 \Big/ (N_\nu - 1)}\right)^{1/2} \;=\; \sqrt{\frac{N_\nu - 1}{N_\nu}} \;\longrightarrow\; 1, \text{ as } \nu \longrightarrow \infty.$$

Note that $\left\{ \sqrt{\mathrm{Var}\!\left[\, Y_\nu^{(1)} \,\right]} \Big/ \sqrt{\mathrm{Var}\!\left[\, Y_\nu^{(0)} \,\right]} \right\}_{\nu\in\mathbb{N}}$ is a sequence of real numbers; we may regard it as a sequence of (constant) $\mathbb{R}$-valued random variables (defined on $\mathcal{S}(U_\nu) \times \mathcal{P}(U_\nu)$). Its convergence (as a sequence of real numbers) to 1, as we have established above, implies that it converges (as constant $\mathbb{R}$-valued random variables) almost surely to 1, hence also in probability as well as in distribution (see Theorem 1, p.4, [1]). By a corollary of Slutkey's Theorem (Corollary and Example 6, p.40, [1]), we therefore have

$$\frac{Y_\nu^{(0)}}{\sqrt{\mathrm{Var}\!\left[\, Y_\nu^{(0)} \,\right]}} \;=\; \frac{\sqrt{\mathrm{Var}\!\left[\, Y_\nu^{(1)} \,\right]}}{\sqrt{\mathrm{Var}\!\left[\, Y_\nu^{(0)} \,\right]}} \cdot \frac{Y_\nu^{(0)}}{\sqrt{\mathrm{Var}\!\left[\, Y_\nu^{(1)} \,\right]}} \;\xrightarrow{\mathcal{L}}\; 1 \cdot N(0,1) \;=\; N(0,1).$$

This completes the proof of the Corollary. □

# References

[1] Ferguson, T. S. *A Course in Large Sample Theory*, first ed. Texts in Statistical Science. CRC Press, 1996.

[2] Jacod, J., and Protter, P. *Probability Essentials*. Springer-Verlag, New York, 2004. Universitext.