Separating and convergence-determining classes of $\mathbb{R}^n$, $\mathbb{R}^\infty$ and $C([0,1],\mathbb{R})$

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1 Separating and convergence-determining classes

Definition 1.1 (Separating class)
Suppose Ω is a non-empty set, $\mathcal{A}$ is a σ-algebra of subsets of Ω, $(S,\mathcal{A})$ is the corresponding measurable space, and $\mathcal{M}_1(\Omega,\mathcal{A})$ is the set of all probability measures defined on $\mathcal{A}$. A ***separating class*** *of subsets of $(\Omega,\mathcal{A})$ is a collection $\mathcal{S}\subset\mathcal{A}$ of subsets of Ω which satisfies the following condition: For every two probability measures $\mu,\nu\in\mathcal{M}_1(\Omega,\mathcal{A})$,*

$$\mu(S)=\nu(S),\ \text{for every } S\in\mathcal{S} \quad\Longrightarrow\quad \mu(A)=\nu(A),\ \text{for every } A\in\mathcal{A}$$

Definition 1.2 (Convergence-determining class)
Suppose Ω is a topological space, $\mathcal{B}(\Omega)$ is its Borel σ-algebra, $(\Omega,\mathcal{B}(\Omega))$ is the corresponding measurable space, and $\mathcal{M}_1(\Omega,\mathcal{B}(S))$ is the set of all probability measures defined on $\mathcal{B}(\Omega)$. A ***convergence-determining class*** *of subsets of $(\Omega,\mathcal{B}(\Omega))$ is a collection $\mathcal{C}\subset\mathcal{B}(\Omega)$ of Borel subsets of Ω which satisfies the following condition: For any $\mu,\mu_1,\mu_2,\ldots\in\mathcal{M}_1(\Omega,\mathcal{B}(\Omega))$,*

$$\lim_{n\to\infty}\mu_n(C)\ =\ \mu(C),\ \text{for every } C\in\mathcal{C}_\mu \quad\Longrightarrow\quad \mu_n\xrightarrow{w}\mu,$$

where

$$\mathcal{C}_\mu\ :=\ \{\ A\in\mathcal{C}\ \mid\ \mu(\partial A)\ =\ 0\ \},$$

*and $\mathcal{C}_\mu$ is called the collection of μ-**continuity sets** in $\mathcal{C}$.*

Theorem 1.3
Suppose Ω is a non-empty set, $\mathcal{A}$ is a σ-algebra of subsets of Ω, and $(\Omega,\mathcal{A})$ is the corresponding measurable space.
If

- *$\mathcal{S}\subset\mathcal{A}$ is closed under finite intersections, and*
- *$\mathcal{S}$ generates $\mathcal{A}$ (i.e. $\sigma(\mathcal{S})=\mathcal{A}$),*

then $\mathcal{S}$ is a separating class of subsets of $(\Omega,\mathcal{A})$.

PROOF Let μ and ν be two probability measures defined on $(\Omega,\mathcal{A})$ such that $\mu(S)=\nu(S)$ for each $S\in\mathcal{S}$. We need to show that $\mu(A)=\nu(A)$ for each $A\in\mathcal{A}$. To this end, let

$$\mathcal{L}\ :=\ \{\ A\in\mathcal{A}=\sigma(\mathcal{S})\ \mid\ \mu(A)=\nu(A)\ \}.$$

Note that $\mathcal{S}\subset\mathcal{L}$, by the hypothesis that μ and ν agree on $\mathcal{S}$, and $\mathcal{L}\neq\varnothing$ since $\Omega\in\mathcal{L}$. By Corollary B.8, it suffices to establish that $\mathcal{L}$ is a λ-system, since then it will follow that

$$\mathcal{A}\ =\ \sigma(\mathcal{S})\ \subset\ \mathcal{L}\ :=\ \{\ A\in\mathcal{A}=\sigma(\mathcal{S})\ \mid\ \mu(A)=\nu(A)\ \}\ \subset\ \sigma(\mathcal{S})\ =\ \mathcal{A},$$

i.e., $\mathcal{A}=\sigma(\mathcal{S})=\mathcal{L}$, or equivalently, μ and ν agree on all of $\mathcal{A}=\sigma(\mathcal{S})$. Now, we have already noted that $\Omega\in\mathcal{L}$. For $A\in\mathcal{L}$, we have

$$\mu(\Omega\setminus A)\ =\ 1-\mu(A)\ =\ 1-\nu(A)\ =\ \nu(\Omega\setminus A),$$

hence $\Omega\setminus A\in\mathcal{L}$. Thus, $\mathcal{L}$ is closed under complementations. Lastly, let $A_1,A_2,\ldots\in\mathcal{L}$ be pairwise disjoint. Then,

$$\mu\left(\bigsqcup_{n=1}^\infty A_n\right)\ =\ \sum_{n=1}^\infty\mu(A_n)\ =\ \sum_{n=1}^\infty\nu(A_n)\ =\ \nu\left(\bigsqcup_{n=1}^\infty A_n\right),$$

thus $\bigsqcup_{n=1}^\infty A_n\in\mathcal{L}$, which proves that $\mathcal{L}$ is closed under countable disjoint unions. $\mathcal{L}$ is therefore indeed a λ-system and the proof of the Theorem is complete. $\square$

Corollary 1.4 *Suppose S is a topological space and $\mathcal{B}(S)$ is its Borel σ-algebra (i.e. the σ-algebra generated by the collection of open subsets of S). Then, the collection of open subsets of S is a separating class of subsets of the measurable space $(S, \mathcal{B}(S))$.*

Proof Recall that the collection of open sets are closed under finite intersections (by definition of topology), and they generate the Borel σ-algebras (by definition of Borel σ-algebras). Thus the Corollary follows immediately from Theorem 1.3. □

2 Examples of separating and convergence-determining classes of $\mathbb{R}^\infty$

Definition 2.1 (The metric on $\mathbb{R}^\infty$, Example 1.2, [1])
Let $\mathbb{R}^\infty$ denotes the set of all infinite sequences of real numbers, i.e.

$$\mathbb{R}^\infty := \{ (x_1, x_2, \dots) \mid x_i \in \mathbb{R}, \text{ for each } i \in \mathbb{N} \}.$$

Define $\rho : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow [0,1]$ as follows:

$$\rho(x,y) := \sum_{n=1}^{\infty} \frac{\min\{1, |x_n - y_n|\}}{2^n}.$$

Remark 2.2 *Recall that*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}\right) = 1,$$

which proves indeed that $0 \leq \rho(x,y) \leq 1$, for any $x, y \in \mathbb{R}^\infty$.

Theorem 2.3 (The metric space properties of $\mathbb{R}^\infty$)

(i) *$(\mathbb{R}^\infty, \rho)$ is a metric space. Let $\mathbb{R}^\infty$ denote also this metric space in the remainder of this Theorem.*

(ii) *For $x, x^{(1)}, x^{(2)}, x^{(3)}, \dots, \in \mathbb{R}^\infty$, we have:*

$$\rho\left(x^{(n)}, x\right) \longrightarrow 0 \quad \Longleftrightarrow \quad \text{for each } i \in \mathbb{N}, \lim_{n\to\infty} \left| x_i^{(n)} - x_i \right| = 0$$

(iii) *For each $n \in \mathbb{N}$, the "natural projection to the initial segment of length n"*

$$\pi_n : \mathbb{R}^\infty \longrightarrow \mathbb{R}^n : x \longmapsto (x_1, x_2, \dots, x_n)$$

is continuous, where $\mathbb{R}^n$ has the usual Euclidean topology.

(iv) *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, let $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)$ denote the open hypercube in $\mathbb{R}^n$ of side length 2ε centred at $\pi_n(x) \in \mathbb{R}^n$, i.e.*

$$C_{\mathbb{R}^n}(\pi_n(x), \varepsilon) := \left\{ y \in \mathbb{R}^n \;\middle|\; \begin{array}{c} |y_i - x_i| < \varepsilon, \\ i = 1, 2, \dots, n \end{array} \right\}$$

Then, its pre-image in $\mathbb{R}^\infty$ under π_n

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) = \left\{ y \in \mathbb{R}^\infty \;\middle|\; \begin{array}{c} |y_i - x_i| < \varepsilon, \\ i = 1, 2, \dots, n \end{array} \right\}$$

is an open subset of $\mathbb{R}^\infty$.

(v) *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, we have:*

$$\pi_n^{-1}(\, C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\subset\; B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon+\frac{1}{2^n}\right),$$

where $B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon+\frac{1}{2^n}\right)$ is the open ball in $\mathbb{R}^\infty$ centred at x of radius $\varepsilon+\frac{1}{2^n}$, i.e.

$$B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon+\frac{1}{2^n}\right) \;:=\; \left\{\, y\in\mathbb{R}^\infty \;\middle|\; \rho(y,x)\;<\;\varepsilon+\frac{1}{2^n}\,\right\}$$

(vi) *The collection*

$$\left\{\, \pi_n^{-1}(\, C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,)\subset\mathbb{R}^\infty \;\middle|\; n\in\mathbb{N},\, x\in\mathbb{R}^\infty,\, \varepsilon>0 \,\right\}$$

of all pre-images under π_n of open hypercubes in $\mathbb{R}^n$, for all $n\in\mathbb{N}$, forms a basis for the topology of $\mathbb{R}^\infty$.

(vii) *$\mathbb{R}^\infty$ is a separable metric space.*

(viii) *$\mathbb{R}^\infty$ is a complete metric space.*

PROOF

(i) Clearly, ρ is non-negative and symmetric. We now show that, for any $x,y\in\mathbb{R}^\infty$, we have $\rho(x,y)=0$ implies $x=y$. Indeed,

$$\begin{aligned}
\rho(x,y)\;=\;0 \quad&\Longleftrightarrow\quad \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-y_i\,|\,\}}{2^i}\;=\;0\\
&\Longleftrightarrow\quad \min\{\,1\,,|\,x_i-y_i\,|\,\}\;=\;0,\;\text{ for each } i\in\mathbb{N}\\
&\Longleftrightarrow\quad |\,x_i-y_i\,|\;=\;0,\;\text{ for each } i\in\mathbb{N}\\
&\Longleftrightarrow\quad x\;=\;y.
\end{aligned}$$

In order to show that ρ is a metric, it remains only to establish the Triangle Inequality. By Lemma A.2, for any $x,y,z\in\mathbb{R}^\infty$, we have

$$\begin{aligned}
\rho(x,y)\;&=\;\sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-y_i\,|\,\}}{2^i}\\
&\leq\;\sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-z_i\,|\,\}\,+\,\min\{\,1\,,|\,z_i-y_i\,|\,\}}{2^i}\\
&=\;\sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-z_i\,|\,\}\,\}}{2^i}\;+\;\sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,z_i-y_i\,|\,\}}{2^i}\\
&=\;\rho(x,z)+\rho(z,y),
\end{aligned}$$

where we have used the fact that $0\leq\rho\leq1$ to split the infinite sum into two terms in second-to-last equality. This proves that ρ satisfies the Triangle Inequality, and it is thus a metric on $\mathbb{R}^\infty$.

(ii) $\underline{\displaystyle\lim_{n\to\infty}\rho\big(x^{(n)},x\big)=0 \;\Longrightarrow\; \lim_{n\to\infty}\left|\,x_i^{(n)}-x_i\,\right|=0,\text{ for each } i\in\mathbb{N}}$

$$\begin{aligned}
\lim_{n\to\infty}\rho\Big(x^{(n)},x\Big)=0\quad&\Longrightarrow\quad \lim_{n\to\infty}\sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i^{(n)}-x_i\,|\,\}}{2^i}=0\\
&\Longrightarrow\quad \lim_{n\to\infty}\min\{\,1\,,|\,x_i^{(n)}-x_i\,|\,\}=0,\;\text{ for each } i\in\mathbb{N}\\
&\Longrightarrow\quad \lim_{n\to\infty}\left|\,x_i^{(n)}-x_i\,\right|=0,\;\text{ for each } i\in\mathbb{N}
\end{aligned}$$

$\underline{\lim_{n\to\infty}\left|x_i^{(n)} - x_i\right| = 0, \text{ for each } i\in\mathbb{N} \implies \lim_{n\to\infty}\rho\big(x^{(n)},x\big) = 0}$

This follows from the Weierstrass M-test. Suppose $\lim_{n\to\infty}\left|x_i^{(n)} - x_i\right| = 0$, for each $i\in\mathbb{N}$. Then,

$$\lim_{n\to\infty}\frac{\min\{\,1\,,|\,x_i^{(n)}-x_i\,|\,\}}{2^i} \;=\; 0 \;=:\; y_i, \text{ for each } i\in\mathbb{N}.$$

For each $i\in\mathbb{N}$, let $M_i \;:=\; \dfrac{1}{2^i}$. Then,

$$\frac{\min\{1\,,|\,x_i^{(n)}-x_i\,|\}}{2^i} \;\leq\; M_i \quad\text{and}\quad \sum_{i=1}^{\infty}M_i \;<\;\infty.$$

Hence, by the Weierstrass M-test (Lemma A.3), we have

$$\lim_{n\to\infty}\rho\Big(x^{(n)},x\Big) \;=\; \lim_{n\to\infty}\sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i^{(n)}-x_i\,|\,\}}{2^i} \;=\; \sum_{i=1}^{\infty}y_i \;=\; 0.$$

(iii) Immediate by (ii).

(iv) Since $C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\subset\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, its pre-image under the continuous (by (iii)) map $\pi_n:\mathbb{R}^\infty\longrightarrow\mathbb{R}^n$ is an open subset of $\mathbb{R}^\infty$.

(v) For $y\in\mathbb{R}^\infty$, we have

$$\begin{aligned} y\in\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) &\implies |\,y_i-x_i\,|<\varepsilon, \text{ for each } i=1,2,\ldots,n \\ &\implies \rho(x,y) \;:=\; \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-y_i\,|\,\}}{2^i} \;\leq\; \sum_{i=1}^{n}\frac{\varepsilon}{2^i} \;+\; \sum_{i=n+1}^{\infty}\frac{1}{2^i} \;=\; \varepsilon+\frac{1}{2^n}. \end{aligned}$$

This proves:

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\subset\; B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon+\frac{1}{2^n}\right).$$

(vi) It suffices to show that every open ball in $B_{\mathbb{R}^\infty}(x,r)\subset\mathbb{R}^\infty$, $r>0$, contains the pre-image of an open hypercube centred at $\pi_n(x)\in\mathbb{R}^n$ under π_n. To this end, for $r>0$, choose $\varepsilon>0$ sufficiently small and $n\in\mathbb{N}$ sufficiently large such that $\varepsilon+\dfrac{1}{2^n}<r$. Then, for any $x\in\mathbb{R}^\infty$, by (v), we have:

$$x \;\in\; \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\subset\; B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon+\frac{1}{2^n}\right) \;\subset\; B_{\mathbb{R}^\infty}(\,x\,,r\,),$$

as required.

(vii) It suffices to exhibit a countable subset of $\mathbb{R}^\infty$ that intersects every open ball in $\mathbb{R}^\infty$. To this end, let

$$D \;:=\; \bigcup_{n=1}^{\infty}\left\{\,x=(x_1,x_2,\ldots\,)\in\mathbb{R}^\infty \;\middle|\; \begin{array}{l} x_i\in\mathbb{Q}, \text{ for each } i\in\mathbb{N} \\ x_i=0, \text{ for all } i\geq n \end{array}\right\}.$$

Clearly, D is a countable subset of $\mathbb{R}^\infty$. Now let $B_{\mathbb{R}^\infty}(x,\varepsilon)$ be an arbitrary open ball in $\mathbb{R}^\infty$. Choose $\delta>0$ small enough and $n\in\mathbb{N}$ large enough such that $\delta+\dfrac{1}{2^n}<\varepsilon$. Then,

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\delta)\,) \;\subset\; B_{\mathbb{R}^\infty}\left(x\,,\,\delta+\frac{1}{2^n}\right) \;\subset\; B_{\mathbb{R}^\infty}(\,x\,,\varepsilon\,),$$

Now, for each $i = 1, 2, \ldots, n$, choose $z_i \in \mathbb{Q} \cap (x_i - \delta, x_i + \delta)$. Let $z = (z_1, z_2, \ldots, z_n, 0, 0, \ldots) \in \mathbb{R}^\infty$. Then, we have

$$z \;\in\; D \bigcap \left\{ y \in \mathbb{R}^\infty \;\middle|\; \begin{array}{c} y_i \in (x_i - \delta, x_i + \delta), \\ \text{for each } i = 1, 2, \ldots, n \end{array} \right\} \;=\; D \bigcap \pi_n^{-1}\left(C_{\mathbb{R}^n}(\pi_n(x), \delta) \right) \;\subset\; D \bigcap B_{\mathbb{R}^\infty}(x\,,\varepsilon)\,.$$

This proves the the countable subset $D \subset \mathbb{R}^\infty$ has non-empty intersection with every open ball in $\mathbb{R}^\infty$, i.e. D is dense in $\mathbb{R}^\infty$. Hence, $\mathbb{R}^\infty$ is separable.

(viii) We need to show that every Cauchy sequence in $\mathbb{R}^\infty$ converges to any element in $\mathbb{R}^\infty$.

$$\begin{aligned}
& \left\{ x^{(n)} \right\}_{n\in\mathbb{N}} \subset \mathbb{R}^\infty \text{ is a Cauchy sequence in } \mathbb{R}^\infty \\
\Longleftrightarrow\quad & \text{for each } \varepsilon > 0, \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that } \rho\left(x^{(m)}, x^{(n)}\right) \;<\; \varepsilon, \text{ for any } m, n > N_\varepsilon \\
\Longrightarrow\quad & \text{for each } i \in \mathbb{N}, \text{ we have:} \\
& \text{for each } \varepsilon > 0, \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that } \left| x_i^{(m)} - x_i^{(n)} \right| \;<\; \varepsilon, \text{ for any } m, n > N_\varepsilon \\
\Longrightarrow\quad & \text{for each } i \in \mathbb{N}, \left\{ x_i^{(n)} \right\}_{n\in\mathbb{N}} \subset \mathbb{R} \text{ is a Cauchy sequence in } \mathbb{R}; \text{ hence } x_i := \lim_{n\to\infty} x_i^{(n)} \in \mathbb{R} \text{ exists} \\
\Longrightarrow\quad & \lim_{n\to\infty} \rho\left(x^{(n)}, x\right) \;=\; 0, \;\text{ where } x := (x_1, x_2, \ldots) \in \mathbb{R}^\infty \quad \text{(by (ii))}
\end{aligned}$$

This proves that $\mathbb{R}^\infty$ indeed is a complete metric space.

□

Definition 2.4

*The **finite-dimensional class** of subsets of $\mathbb{R}^\infty$ is, by definition, the following:*

$$\mathcal{B}_f(\mathbb{R}^\infty) \;:=\; \left\{ \pi_k^{-1}(B) \subset \mathbb{R}^\infty \;\middle|\; \begin{array}{c} k \in \mathbb{N} \\ B \in \mathcal{B}(\mathbb{R}^k) \end{array} \right\},$$

where $\pi_k : \mathbb{R}^\infty \longrightarrow \mathbb{R}^k : x = (x_1, x_2, \ldots) \longmapsto (x_1, \ldots, x_k)$ is the projection of $\mathbb{R}^\infty$ onto $\mathbb{R}^k$.

Theorem 2.5

(i) $\mathcal{B}_f(\mathbb{R}^\infty) \subset \mathcal{B}(\mathbb{R}^\infty)$.

(ii) $\mathcal{B}_f(\mathbb{R}^\infty)$ *is a separating class of Borel subsets of* $\mathbb{R}^\infty$.

(iii) $\mathcal{B}_f(\mathbb{R}^\infty)$ *is a convergence-determining class of Borel subsets of* $\mathbb{R}^\infty$.

Proof

(i) Note that

$$\mathcal{B}_f(\mathbb{R}^\infty) \;:=\; \left\{ \pi_k^{-1}(B) \subset \mathbb{R}^\infty \;\middle|\; \begin{array}{c} k \in \mathbb{N} \\ B \in \mathcal{B}(\mathbb{R}^k) \end{array} \right\} \;=\; \bigcup_{k=1}^{\infty} \pi_k^{-1}\left(\mathcal{B}(\mathbb{R}^k)\right).$$

Thus, (i) is equivalent to the statement that each $\pi_k : \mathbb{R}^\infty \longrightarrow \mathbb{R}^k$ is Borel measurable. But each π_k is continuous, hence Borel measurable (Corollary B.12). This proves (i).

(ii) We apply Theorem 1.3 to $\mathcal{B}_f(\mathbb{R}^\infty)$.

$\mathcal{B}_f(\mathbb{R}^\infty)$ is closed under finite intersections

Let $\pi_k^{-1}(A)$ and $\pi_l^{-1}(B) \in \mathcal{B}_f(\mathbb{R}^\infty)$. Note that this implies $A \in \mathcal{B}(\mathbb{R}^k)$ and $B \in \mathcal{B}(\mathbb{R}^l)$. We need to show that $\pi_k^{-1}(A) \cap \pi_l^{-1}(B) \in \mathcal{B}_f(\mathbb{R}^\infty)$. Now, if $k = l$, this is immediately, since then $A \cap B \in \mathcal{B}(\mathbb{R}^k)$, and

$$\pi_k^{-1}(A) \cap \pi_l^{-1}(B) \;=\; \pi_k^{-1}(A) \cap \pi_k^{-1}(B) \;=\; \pi_k^{-1}(A \cap B) \;\in\; \mathcal{B}_f(\mathbb{R}^\infty).$$

For the case $k \neq l$, without loss of generality, assume $k < l$. Then, note that

$$\begin{aligned}
\pi_k^{-1}(A) &= \left\{ y = (y_1, y_2, \ldots) \in \mathbb{R}^\infty \;\middle|\; (y_1, \ldots, y_k) \in A \right\} \\
&= \left\{ y = (y_1, y_2, \ldots) \in \mathbb{R}^\infty \;\middle|\; (y_1, \ldots, y_k, y_{k+1}, \ldots, y_l) \in A \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{k \text{ factors}} \right\} \\
&= \pi_l^{-1}(\, A \times \mathbb{R} \times \cdots \times \mathbb{R}\,).
\end{aligned}$$

Since $(A \times \mathbb{R} \cdots \times \mathbb{R}) \cap B \in \mathcal{B}(\mathbb{R}^l)$, we now see that

$$\pi_k^{-1}(A) \cap \pi_l^{-1}(B) \;=\; \pi_l^{-1}(A \times \mathbb{R} \cdots \times \mathbb{R}) \cap \pi_l^{-1}(B) \;=\; \pi_l^{-1}(\,(A \times \mathbb{R} \cdots \times \mathbb{R}) \cap B\,) \;\in\; \mathcal{B}_f(\mathbb{R}^\infty).$$

This proves that $\mathcal{B}_f(\mathbb{R}^\infty)$ is indeed closed under finite intersections.

$\mathcal{B}_f(\mathbb{R}^\infty)$ generates $\mathcal{B}(\mathbb{R}^\infty)$

Let $\mathcal{O}(\mathbb{R}^\infty)$ denote the collection of open sets of $\mathbb{R}^\infty$. Hence $\mathcal{B}(\mathbb{R}^\infty) := \sigma(\mathcal{O}(\mathbb{R}^\infty))$. By (i), we have $\mathcal{B}_f(\mathbb{R}^\infty) \subset \mathcal{B}(\mathbb{R}^\infty) = \sigma(\mathcal{O}(\mathbb{R}^\infty))$, which implies $\sigma(\mathcal{B}_f(\mathbb{R}^\infty)) \subset \sigma(\mathcal{O}(\mathbb{R}^\infty))$. We need to establish the reverse inclusion, which will immediately follow from:

> **Claim:** $\mathcal{O}(\mathbb{R}^\infty) \subset \sigma(\mathcal{B}_f(\mathbb{R}^\infty))$.
>
> Proof of Claim: By Theorem 2.3(v), every open ball $B_{\mathbb{R}^\infty}(x, \varepsilon)$ in $\mathbb{R}^\infty$ contains the pre-image of an open hypercube from some finite-dimensional Euclidean space, where that pre-image itself contains x. We therefore see that every open set in $\mathbb{R}^\infty$ can be expressed as a union of pre-images of open hypercubes from finite-dimensional Euclidean spaces. By Theorem 2.3(vii), $\mathbb{R}^\infty$ is separable. Hence, by Theorem C.1, we see that every open set in $\mathbb{R}^\infty$ can be expressed as a countable union of pre-images of open hypercubes from finite-dimensional Euclidean spaces. Since pre-images of open hypercubes from finite-dimensional Euclidean spaces belong to $\mathcal{B}_f(\mathbb{R}^\infty)$, we see that $\mathcal{O}(\mathbb{R}^\infty) \subset \sigma(\mathcal{B}_f(\mathbb{R}^\infty))$. This completes the proof of the Claim.

We have established that $\mathcal{B}_f(\mathbb{R}^\infty)$ is contained in $\mathcal{B}(\mathbb{R}^\infty)$, is closed under finite intersections, and $\sigma(\mathcal{B}_f(\mathbb{R}^\infty)) = \mathcal{B}_f(\mathbb{R}^\infty)$. Therefore, by Theorem 1.3, $\mathcal{B}_f(\mathbb{R}^\infty)$ is a separating class for the measurable space $(\mathbb{R}^\infty, \mathcal{B}_f(\mathbb{R}^\infty))$.

(iii) Since $\mathbb{R}^\infty$ is separable, by Theorem D.4, it suffices to show that $\mathcal{B}_f(\mathbb{R}^\infty)$ is closed under finite intersections, and for each $x \in \mathbb{R}^\infty$ and $\varepsilon > 0$, the collection

$$\partial\,\mathcal{B}_f(\mathbb{R}^\infty)(x,\varepsilon) \;:=\; \left\{\; \partial A \,\subset\, S \;\middle|\; A \,\in\, \mathcal{B}_f(\mathbb{R}^\infty)(x,\varepsilon) \;\right\}$$

contains uncountably many disjoint sets, where

$$\mathcal{B}_f(\mathbb{R}^\infty)(x,\varepsilon) \;:=\; \left\{\; A \in \mathcal{B}_f(\mathbb{R}^\infty) \;\middle|\; x \,\in\, A^\circ \,\subset\, A \,\subset\, B(x,\varepsilon) \;\right\}.$$

Now, we have already proved that $\mathcal{B}_f(\mathbb{R}^\infty)$ is closed under finite intersections in the proof of statement (ii). Next, let $x \in \mathbb{R}^\infty$ and $\varepsilon > 0$ be given. For any $k \in \mathbb{N}$ with $\dfrac{1}{2^k} < \dfrac{\varepsilon}{2}$ and $0 < \delta < \dfrac{\varepsilon}{2}$, define

$$A_{k,\delta} \;:=\; \left\{\; y = (y_1, y_2, \ldots) \in \mathbb{R}^\infty \;\middle|\; \begin{array}{c} |\,y_i - x_i\,| \,<\, \delta, \\ i = 1,2,\ldots,k \end{array} \right\}.$$

Then, by Theorem 2.3(v), we have

$$x \;\in\; (A_{k,\delta})^\circ \;=\; A_{k,\delta} \;\subset\; B\!\left(x, \delta + \frac{1}{2^k}\right) \;\subset\; B(x,\varepsilon).$$

Clearly, each $A_{k,\delta} \in \mathcal{B}_f(\mathbb{R}^\infty)$. Thus, for each fixed $k \in \mathbb{N}$ with $\dfrac{1}{2^k} < \dfrac{\varepsilon}{2}$, we have

$$\left\{\; A_{k,\delta} \;\middle|\; 0 \,<\, \delta \,<\, \frac{\varepsilon}{2} \;\right\} \;\subset\; \mathcal{B}_f(\mathbb{R}^\infty).$$

Now, note that

$$\partial A_{k,\delta} \;=\; \left\{\; y = (y_1, y_2, \ldots) \in \mathbb{R}^\infty \;\middle|\; \begin{array}{ll} |\,y_i - x_i\,| \,\leq\, \delta, & \text{for each } i = 1,2,\ldots,k \\ |\,y_i - x_i\,| \,=\, \delta, & \text{for at least one } i \in \{1,2,\ldots,k\} \end{array} \right\},$$

which in particular implies

$$\partial A_{k,\delta} \,\cap\, \partial A_{k,\delta'} \;=\; \varnothing, \;\; \text{whenever } 0 < \delta \neq \delta' < \frac{\varepsilon}{2}.$$

This proves that $\partial\,\mathcal{B}_f(\mathbb{R}^\infty)(x,\varepsilon)$ indeed contains uncountably many disjoint sets, and completes and the proof of (iii).

$\square$

A Technical Lemmas

Lemma A.1 *Define*

$$\phi \,:\, [\,0,\infty\,) \longrightarrow [\,0,1\,] \,:\, t \longmapsto \min\{\,1\,,t\,\}.$$

Then, ϕ satisfies:

$$\phi(s+t) \;\leq\; \phi(s) \,+\, \phi(t), \;\; \text{for each } s,t \in [0,\infty).$$

PROOF For any $s,t \in [0,\infty)$, either $s+t \geq 1$ or $s+t < 1$. If $s+t \geq 1$, then

$$\phi(s+t) \;=\; \min\{1, s+t\} \;=\; 1 \;<\; 2 \;=\; 1+1 \;\leq\; \min\{1,s\} + \min\{1,t\} \;=\; \phi(s) + \phi(t),$$

hence, the required inequality holds. On the other hand, if $s+t<1$, then we must also have $s<1$ and $t<1$ (since $s,t \geq 0$). Hence,

$$\phi(s+t) \;=\; \min\{1, s+t\} \;=\; s+t \;=\; \min\{1,s\} + \min\{1,t\} \;=\; \phi(s) + \phi(t),$$

thus, the required inequality also holds. $\square$

Lemma A.2 *For any $x,y,z \in \mathbb{R}$, we have:*

$$\min\{1, |\,x-y\,|\} \;\leq\; \min\{1, |\,x-z\,|\} \;+\; \min\{1, |\,z-y\,|\}.$$

Proof Observe that $|\,x-y\,| \;\leq\; |\,x-z\,|+|\,z-y\,|$ implies

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; |\,x-z\,|+|\,z-y\,|.$$

The above inequality, together with $\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; 1$, thus in turn imply:

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; \min\{\,1\,,|\,x-z\,|+|\,z-y\,|\,\}.$$

By Lemma A.1, we therefore have:

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; \min\{\,1\,,|\,x-z\,|+|\,z-y\,|\,\}. \;\leq\; \min\{\,1\,,|\,x-z\,|\,\} \;+\; \min\{\,1\,,|\,z-y\,|\,\},$$

which proves the present Lemma. □

Lemma A.3 (The Weierstrass M-test, Theorem A.28, [2])
Suppose that $\lim_{n\to\infty} x_i^{(n)} \;=\; x_i$, *for each* $i\in\mathbb{N}$, *and that* $\left|\,x_i^{(n)}\,\right| \;\leq\; M_i$, *where* $\sum_{i=1}^{\infty} M_i < \infty$. *Then,*

(i) $\sum_{i=1}^{\infty} x_i$ *exists, and* $\sum_{i=1}^{\infty} x_i^{(n)}$ *exists for each* $n\in\mathbb{N}$.

(ii) *Furthermore,*

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} x_i^{(n)} \;=\; \sum_{i=1}^{\infty} x_i.$$

Proof

(i) $\sum_{i=1}^{\infty} M_i < \infty$ and $\left|\,x_i^{(n)}\,\right| \leq M_i \implies$ the series $\sum_{i=1}^{\infty} x_i$ and $\sum_{i=1}^{\infty} x_i^{(n)}$, $n\in\mathbb{N}$, converge absolutely.

(ii) Let $\varepsilon>0$ be given. Choose $K\in\mathbb{N}$ sufficiently large such that $\sum_{j=K+1}^{\infty} M_i < \frac{\varepsilon}{3}$. Next, choose $N\in\mathbb{N}$ sufficiently large such that

$$\left|\,x_i^{(n)} - x_i\,\right| \;<\; \frac{\varepsilon}{3K}, \quad \text{for any } n>N \text{ and } i=1,2,\ldots,K.$$

Then, we have, for each $n>N$,

$$\begin{aligned}
\left|\,\sum_{i=1}^{\infty} x_i^{(n)} - \sum_{i=1}^{\infty} x_i\,\right| &= \left|\,\sum_{i=1}^{K}\left(x_i^{(n)} - x_i\right) + \sum_{i=K+1}^{\infty} x_i^{(n)} - \sum_{i=K+1}^{\infty} x_i\,\right| \\
&\leq \sum_{i=1}^{K}\left|\,x_i^{(n)} - x_i\,\right| + \sum_{i=K+1}^{\infty}\left|\,x_i^{(n)}\,\right| + \sum_{i=K+1}^{\infty}|\,x_i\,| \\
&\leq K\cdot\frac{\varepsilon}{3K} + \sum_{i=K+1}^{\infty} M_i + \sum_{i=K+1}^{\infty} M_i \;\leq\; \frac{\varepsilon}{3} + 2\cdot\frac{\varepsilon}{3} \;=\; \varepsilon.
\end{aligned}$$

Since ε is arbitrary, this proves:

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} x_i^{(n)} \;=\; \sum_{i=1}^{\infty} x_i.$$

□

B σ-algebras and λ-systems

Definition B.1

Suppose Ω is a non-empty set. A σ-algebra of subsets of Ω is a collection $\mathcal{A}$ of subsets of Ω which satisfies the following conditions:

- $\Omega \in \mathcal{A}$.
- $\Omega \setminus A \in \mathcal{A}$, *for every* $A \in \mathcal{A}$.
- $\displaystyle\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, *whenever* $A_1, A_2, \ldots \in \mathcal{A}$

Definition B.2

Suppose Ω is a non-empty set. A λ-system of subsets of Ω is a collection $\mathcal{L}$ of subsets of Ω which satisfies the following conditions:

- $\Omega \in \mathcal{L}$.
- $\Omega \setminus A \in \mathcal{L}$, *for every* $A \in \mathcal{L}$.
- $\displaystyle\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{L}$, *whenever* $A_1, A_2, \ldots \in \mathcal{L}$ *and* $A_i \cap A_j = \varnothing$, *for any* $i, j \in \mathbb{N}$ *with* $i \neq j$.

Remark B.3 *Clearly, every σ-algebra is also a λ-system.*

Theorem B.4

Suppose Ω is a non-empty set and $\mathcal{L}$ is a λ-system of subsets of Ω.

(i) *$\mathcal{L}$ is closed under proper set-theoretic differences, i.e. $A, B \in \mathcal{L}$ and $A \subset B$ together imply $B \setminus A \in \mathcal{L}$.*

(ii) *If $\mathcal{L}$ is closed under finite intersections, then $\mathcal{L}$ is a σ-algebra of subsets of Ω.*

PROOF For each $X \subset \Omega$, write $\Omega \setminus X$ as X^c.

(i) Suppose $A, B \in \mathcal{L}$ with $A \subset B$. Then, $B^c \cap A = \varnothing$. Hence, $B \setminus A = B \cap A^c = (B^c \cup A)^c = (B^c \sqcup A)^c \in \mathcal{L}$, since $\mathcal{L}$ is closed under complementations and finite disjoint unions.

(ii) Since $\mathcal{L}$ is a λ-system, we immediately have $\Omega \in \mathcal{L}$, and hence $\Omega \setminus A \in \mathcal{L}$, for every $A \in \mathcal{L}$. It remains to show that $\mathcal{L}$ closed under countable unions, i.e. for $A_1, A_2, \ldots \in \mathcal{L}$, we need to show $\bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$. To this end, define:

$$\begin{aligned}
B_1 &:= A_1 \\
B_2 &:= A_2 \cap A_1^c \\
B_3 &:= A_3 \cap A_1^c \cap A_2^c \\
&\vdots \\
B_n &:= A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_n^c
\end{aligned}$$

Being a λ-system, $\mathcal{L}$ is closed under complementations. By hypothesis, $\mathcal{L}$ is furthermore closed under finite intersections. We thus see that $B_n \in \mathcal{L}$, for each $n \in \mathbb{N}$. Note also that the B_n's are pairwise disjoint, and

$$\bigcup_{i=1}^{n} A_i = \bigsqcup_{i=1}^{n} B_i, \quad \text{for each } n \in \mathbb{N}.$$

Hence,

$$\bigcup_{i=1}^{\infty} A_i \;=\; \bigsqcup_{i=1}^{\infty} B_i \;\in\; \mathcal{L},$$

since $\mathcal{L}$ is closed under countable pairwise disjoint unions ($\mathcal{L}$ being a λ-system). This proves that $\mathcal{L}$ is a σ-algebra of subsets of Ω.

$\square$

Theorem B.5 *Let Ω be a non-empty set.*

(i) *The intersection of a non-empty collection of σ-algebras of subsets of Ω is itself a σ-algebra of subsets of Ω.*

(ii) *The intersection of a non-empty collection of λ-systems of subsets of Ω is itself a λ-system of subsets of Ω.*

PROOF

(i) Suppose Γ is an (arbitrary) non-empty set, and, for each $\gamma \in \Gamma$, $\mathcal{A}_\gamma$ is a σ-algebra of subsets of Ω. We need to prove that $\mathcal{A} := \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma$ is itself a σ-algebra of subsets of Ω.

$\underline{\Omega \;\in\; \mathcal{A} := \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma}$

Since, for each $\gamma \in \Gamma$, $\mathcal{A}_\gamma$ is a σ-algebra of subsets of Ω, we have $\Omega \in \mathcal{A}_\gamma$. Thus, $\Omega \in \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma$.

$\underline{A \in \mathcal{A} \implies \Omega \setminus A \in \mathcal{A}}$

$$A \;\in\; \mathcal{A} := \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma \quad\Longleftrightarrow\quad A \;\in\; \mathcal{A}_\gamma,\, \forall\, \gamma\in\Gamma \quad\Longrightarrow\quad \Omega\setminus A \;\in\; \mathcal{A}_\gamma,\, \forall\,\gamma\in\Gamma \quad\Longrightarrow\quad \Omega\setminus A \;\in\; \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma \;=:\; \mathcal{A}$$

$\underline{A_1, A_2, \ldots \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}}$

$$\begin{aligned} A_1, A_2, \ldots \;\in\; \mathcal{A} := \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma \quad&\Longrightarrow\quad A_1, A_2, \ldots \;\in\; \mathcal{A}_\gamma,\, \forall\,\gamma\in\Gamma \;\Longrightarrow\; \bigcup_{i=1}^{\infty} A_i \;\in\; \mathcal{A}_\gamma,\, \forall\,\gamma\in\Gamma \\ &\Longrightarrow\quad \bigcup_{i=1}^{\infty} A_i \;\in\; \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma \;=:\; \mathcal{A} \end{aligned}$$

(ii) Suppose Γ is an (arbitrary) non-empty set, and, for each $\gamma \in \Gamma$, $\mathcal{L}_\gamma$ is a λ-system of subsets of Ω. We need to prove that $\mathcal{L} := \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma$ is itself a λ-system of subsets of Ω.

$\underline{\Omega \;\in\; \mathcal{L} := \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma}$

Since, for each $\gamma \in \Gamma$, $\mathcal{L}_\gamma$ is a λ-system of subsets of Ω, we have $\Omega \in \mathcal{L}_\gamma$. Thus, $\Omega \in \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma$.

$\underline{A \in \mathcal{L} \implies \Omega \setminus L \in \mathcal{L}}$

$$A \in \mathcal{L} := \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma \iff A \in \mathcal{L}_\gamma,\, \forall\, \gamma \in \Gamma \implies \Omega \setminus A \in \mathcal{L}_\gamma,\, \forall\, \gamma \in \Gamma \implies \Omega \setminus A \in \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma =: \mathcal{L}$$

$\underline{A_1, A_2, \ldots \in \mathcal{L} \text{ and } A_i \cap A_j \text{ whenever } i \neq j \implies \bigsqcup_{i=1}^{\infty} A_i \in \mathcal{L}}$

$$\begin{aligned}
& A_1, A_2, \ldots \in \mathcal{L} := \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma\,, \text{ and } A_i \cap A_j \text{ whenever } i \neq j \\
\implies & A_1, A_2, \ldots \in \mathcal{L}_\gamma,\, \forall\, \gamma \in \Gamma\,, \text{ and } A_i \cap A_j \text{ whenever } i \neq j \\
\implies & \bigsqcup_{i=1}^{\infty} A_i \in \mathcal{L}_\gamma,\, \forall\, \gamma \in \Gamma \\
\implies & \bigsqcup_{i=1}^{\infty} A_i \in \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma =: \mathcal{L}
\end{aligned}$$

□

Theorem B.6 *Suppose Ω is a non-empty set, $\mathcal{S}$ is non-empty collection of subsets of Ω. Denote the power set of Ω by $\mathcal{P}(\Omega)$. Define*

$$\begin{aligned}
\sigma(\mathcal{S}) &:= \bigcap_{\mathcal{A}\,\in\,\Sigma(\mathcal{S})} \mathcal{A}\,, \quad \text{where} \quad \Sigma(\mathcal{S}) := \left\{ \mathcal{A} \subset \mathcal{P}(\Omega) \;\middle|\; \begin{array}{c} \mathcal{A} \text{ is a } \sigma\text{-algebra of subsets of } \Omega, \\ \text{and } \mathcal{S} \subset \mathcal{A} \end{array} \right\}, \text{ and} \\
\lambda(\mathcal{S}) &:= \bigcap_{\mathcal{L}\,\in\,\Lambda(\mathcal{S})} \mathcal{L}\,, \quad \text{where} \quad \Lambda(\mathcal{S}) := \left\{ \mathcal{L} \subset \mathcal{P}(\Omega) \;\middle|\; \begin{array}{c} \mathcal{L} \text{ is a } \lambda\text{-system of subsets of } \Omega, \\ \text{and } \mathcal{S} \subset \mathcal{L} \end{array} \right\}.
\end{aligned}$$

Then, $\sigma(\mathcal{S})$ is the unique smallest σ-algebra of subsets of Ω that contains $\mathcal{S} \subset \mathcal{P}(\Omega)$, and $\lambda(\mathcal{S})$ is the unique smallest λ-system of subsets of Ω that contains $\mathcal{S} \subset \mathcal{P}(\Omega)$. More precisely, we have

- *$\mathcal{S} \subset \sigma(\mathcal{S})$, $\mathcal{S} \subset \lambda(\mathcal{S})$, and*
- *$\sigma(\mathcal{S})$ is a σ-algebra of subsets of Ω, and $\lambda(\mathcal{S})$ is a λ-system of subsets of Ω, and*
- *if $\mathcal{A} \subset \mathcal{P}(\Omega)$ is a σ-algebra and $\mathcal{S} \subset \mathcal{A}$, then $\sigma(\mathcal{S}) \subset \mathcal{A}$.*
- *if $\mathcal{L} \subset \mathcal{P}(\Omega)$ is a λ-system and $\mathcal{S} \subset \mathcal{L}$, then $\lambda(\mathcal{S}) \subset \mathcal{L}$.*

Proof First, note that $\Sigma(\mathcal{S}) \neq \varnothing$ since $\mathcal{P}(\Omega) \in \Sigma(\mathcal{S})$. Similarly, $\Lambda(\mathcal{S}) \neq \varnothing$ since $\mathcal{P}(\Omega) \in \Lambda(\mathcal{S})$. It is immediate that $\mathcal{S} \subset \sigma(\mathcal{S})$, and $\sigma(\mathcal{S})$ is contained in every σ-algebra which contains $\mathcal{S}$. Similarly, $\mathcal{S} \subset \lambda(\mathcal{S})$, and $\lambda(\mathcal{S})$ is contained in every λ-system which contains $\mathcal{S}$. Since $\sigma(\mathcal{S})$ is, by definition, an intersection of σ-algebras, it itself is a σ-algebra of subsets of Ω by Theorem B.5. Similarly, since $\lambda(\mathcal{S})$ is, by definition, an intersection of λ-systems, it itself is a λ-system of subsets of Ω by Theorem B.5. □

Theorem B.7 *Suppose Ω is a non-empty set and $\mathcal{S}$ is a non-empty collection of subsets of Ω. Then,*

$$\mathcal{S} \text{ is closed under finite intersections} \implies \lambda(\mathcal{S}) \text{ is a } \sigma\text{-algebra of subsets of } \Omega,$$

where $\lambda(\mathcal{S})$ is λ-system of subsets of Ω generated by $\mathcal{S}$.

Proof By Theorem B.4(ii), it suffices to show that $\lambda(\mathcal{S})$ is closed under finite intersections. We establish the proof in the following series of claims:

Claim 1: For each $A \in \lambda(\mathcal{S})$,

$$\mathcal{L}(A) \;:=\; \{\, B \subset \Omega \;\vert\; A \cap B \in \lambda(\mathcal{S}) \,\}$$

is a λ-system of subsets of Ω.

Proof of Claim 1: Clearly, $\Omega \in \mathcal{L}(A)$, since $A \cap \Omega = A \in \lambda(\mathcal{S})$. Next, we prove that $\mathcal{L}(A)$ is closed under complementations. Let $B \in \mathcal{L}(A)$. Then, $A \cap B \in \lambda(\mathcal{S})$. Note that $A = (A \cap B) \bigsqcup (A \cap B^c)$, hence $A \cap B^c = A \backslash (A \cap B) \in \lambda(\mathcal{S})$, since $A, A \cap B \in \lambda(\mathcal{S})$ and $\lambda(\mathcal{S})$ is closed under proper set-theoretic differences by Theorem B.4(i). This proves that $\mathcal{L}(A)$ is indeed closed under complementations. We now prove that $\mathcal{L}(A)$ is closed under countable disjoint unions. Let $B_1, B_2, \ldots \in \mathcal{L}(A)$ be pairwise disjoint. Then, $A \cap B_1, A \cap B_2, \ldots \subset \lambda(\mathcal{S})$ are pairwise disjoint. Hence,

$$A \bigcap \left(\bigsqcup_{i=1}^{\infty} B_i \right) \;=\; \bigsqcup_{i=1}^{\infty} (\, A \cap B_i \,) \;\in\; \lambda(\mathcal{S})\,,$$

since $\lambda(\mathcal{S})$ is closed under countable disjoint unions. This proves that $\mathcal{L}(A)$ is a λ-system and thus completes the proof of the Claim 1.

Claim 2: $\mathcal{S} \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$. Consequently, $\lambda(\mathcal{S}) \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$.

Proof of Claim 2: Suppose $A \in \mathcal{S}$. Then, $A \cap B \in \mathcal{S}$ for each $B \in \mathcal{S}$, by the hypothesis that $\mathcal{S}$ is closed under finite intersections. Thus, $A \cap B \in \lambda(\mathcal{S})$, since $\mathcal{S} \subset \lambda(\mathcal{S})$. Hence, $B \in \mathcal{L}(A)$, for any $A, B \in \mathcal{S}$. This proves that $\mathcal{S} \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$. By Claim 1, $\mathcal{L}(A)$ is a λ-system. Hence, $\mathcal{L}(A) \supset \lambda(\mathcal{S})$, the smallest λ-system containing $\mathcal{S}$. This proves Claim 2.

Claim 3: $A \cap B \in \lambda(\mathcal{S})$, for each $A \in \mathcal{S}$ and $B \in \lambda(\mathcal{S})$.

Proof of Claim 3: Let $A \in \mathcal{S}$ and $B \in \lambda(\mathcal{S})$. By Claim 2, we have $\lambda(\mathcal{S}) \subset \mathcal{L}(A)$. Thus we have $B \in \mathcal{L}(A)$, which is equivalent to $A \cap B \in \lambda(S)$. This proves Claim 3.

Claim 4: $\mathcal{S} \subset \mathcal{L}(B)$, for each $B \in \lambda(\mathcal{S})$. Consequently, $\lambda(\mathcal{S}) \subset \mathcal{L}(B)$, for each $B \in \lambda(\mathcal{S})$.

Proof of Claim 4: Suppose $B \in \lambda(\mathcal{S})$. Then, $A \cap B \in \lambda(\mathcal{S})$ for each $A \in \mathcal{S}$, by Claim 3. This proves that $\mathcal{S} \subset \mathcal{L}(B)$. By Claim 1, $\mathcal{L}(B)$ is a λ-system. Hence, $\mathcal{L}(B) \supset \lambda(\mathcal{S})$, the smallest λ-system containing $\mathcal{S}$. This proves Claim 4.

Claim 5: $A \cap B \in \lambda(\mathcal{S})$, for each $A, B \in \lambda(\mathcal{S})$.

Proof of Claim 5: Let $A, B \in \lambda(\mathcal{S})$. By Claim 4, we have $\lambda(\mathcal{S}) \subset \mathcal{L}(B)$. Thus we have $A \in \mathcal{L}(B)$, which is equivalent to $A \cap B \in \lambda(S)$. This proves Claim 5.

Claim 5 states precisely that $\lambda(\mathcal{S})$ is closed under finite intersections, and completes the proof. □

Corollary B.8 *Suppose Ω is a non-empty set and $\mathcal{S}$ is a non-empty collection of subsets of Ω. If $\mathcal{S}$ is closed under finite intersections, then*

(i) $\sigma(S) \subset \lambda(\mathcal{S})$, *and*

(ii) $\sigma(S) \subset \mathcal{L}$, *for any λ-system $\mathcal{L}$ of subsets of Ω such that $\mathcal{S} \subset \mathcal{L}$,*

where $\sigma(\mathcal{S})$ and $\lambda(\mathcal{S})$ are, respectively, the σ-algebra and λ-system of subsets of Ω generated by $\mathcal{S}$.

Proof

(i) By Theorem B.6, $\lambda(\mathcal{S})$ is the smallest λ-system containing $\mathcal{S}$. Since $\mathcal{S}$ is, by hypothesis, closed under finite intersections, $\lambda(\mathcal{S})$ is furthermore a σ-algebra, by Theorem B.7. Thus, by Theorem B.6 again, we have $\sigma(S) \subset \lambda(\mathcal{S})$.

(ii) This is now immediate since

$$\sigma(S) \;\subset\; \lambda(\mathcal{S}) \;\subset\; \mathcal{L},$$

where the first inclusion follows by (i), and the second inclusion follows by Theorem B.6.

□

Lemma B.9 (The pre-image of a σ-algebra is itself a σ-algebra.)
Suppose Ω is a non-empty set, $(X,\mathcal{X})$ is a measurable space, and $f:\Omega\longrightarrow X$ is a map from Ω into X. Then,

$$f^{-1}(\mathcal{X}) \;:=\; \left\{\, f^{-1}(V)\subset\Omega \;\middle|\; V\in\mathcal{X}\,\right\}$$

is a σ-algebra of subsets of Ω.

Proof

$\underline{\Omega\in f^{-1}(\mathcal{X})}$ $\quad f(\Omega)\subset X \implies \Omega = f^{-1}(X)\in f^{-1}(\mathcal{X})$.

$\underline{f^{-1}(\mathcal{X}) \text{ is closed under complementations}}$ Let $V\in\mathcal{X}$. Then, $X\setminus V\in\mathcal{X}$, and

$$\Omega\setminus f^{-1}(V) \;=\; \{\,\omega\in\Omega \mid f(\omega)\notin V\,\} \;=\; \{\,\omega\in\Omega \mid f(\omega)\in X\setminus V\,\} \;=\; f^{-1}(X\setminus V) \;\in\; f^{-1}(\mathcal{X}),$$

which shows that $f^{-1}(\mathcal{X})$ is indeed closed under complementations.

$\underline{f^{-1}(\mathcal{X}) \text{ is closed countable unions}}$ Let $V_1, V_2, \ldots\in\mathcal{X}$. Then, $\bigcup_{i=1}^{\infty} V_i\in\mathcal{X}$, and

$$\bigcup_{i=1}^{\infty} f^{-1}(V_i) \;=\; \left\{\,\omega\in\Omega \;\middle|\; \begin{array}{c} f(\omega)\in V_i \\ \text{for some } i\in\mathbb{N}\end{array}\right\} \;=\; f^{-1}\!\left(\bigcup_{i=1}^{\infty} V_i\right) \;\in\; f^{-1}(\mathcal{X}),$$

which proves that $f^{-1}(\mathcal{X})$ is indeed closed under countable unions.

This concludes the proof that that $f^{-1}(\mathcal{X})$ is a σ-algebra of subsets of Ω. □

Lemma B.10
Suppose $(\Omega,\mathcal{A})$ is a measurable space, X is a non-empty set, and $f:\Omega\longrightarrow X$ is a map from Ω into X. Then,

$$\mathcal{F} \;:=\; \left\{\, V\subset X \;\middle|\; f^{-1}(V)\in\mathcal{A}\,\right\}$$

is a σ-algebra of subsets of X.

Proof

$\underline{X\in\mathcal{F}}$ $\quad f^{-1}(X)=\Omega\in\mathcal{A} \implies X\in\mathcal{F}$.

$\underline{\mathcal{F} \text{ is closed under complementations}}$ $\quad V\in\mathcal{F} \implies f^{-1}(V)\in\mathcal{A} \implies f^{-1}(X\setminus V)=\Omega\setminus f^{-1}(V)\in\mathcal{A} \implies X\setminus V\in\mathcal{F}$, which proves that $\mathcal{F}$ is indeed closed under complementations.

$\mathcal{F}$ is closed under countable unions

$$\begin{aligned} V_1, V_2, \ldots \in \mathcal{F} &\implies f^{-1}(V_1), f^{-1}(V_2), \ldots \in \mathcal{A} \\ &\implies f^{-1}\left(\bigcup_{i=1}^{\infty} V_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(V_i) \in \mathcal{A} \\ &\implies \bigcup_{i=1}^{\infty} V_i \in \mathcal{F}, \end{aligned}$$

which proves that $\mathcal{F}$ is indeed closed under countable unions. □

Theorem B.11
Suppose $(\Omega,\mathcal{A})$ and $(X,\mathcal{X})$ are measurable spaces, and $f:\Omega\longrightarrow X$ is a map from Ω into X. Then, f is $(\mathcal{A},\mathcal{X})$-measurable if there exists $\mathcal{S}\subset\mathcal{X}$ satisfying the following conditions:

- *$\mathcal{S}$ generates $\mathcal{X}$, i.e. $\sigma(\mathcal{S})=\mathcal{X}$, and*
- *$f^{-1}(\mathcal{S})\subset\mathcal{A}$.*

Proof By Lemma B.10,
$$\mathcal{F} := \left\{ V\subset X \;\middle|\; f^{-1}(V)\in\mathcal{A} \right\}$$
is a σ-algebra of subsets of X. By hypothesis, $\mathcal{S}\subset\mathcal{F}$; hence, $\mathcal{X}=\sigma(\mathcal{S})\subset\mathcal{F}$. Thus, $f^{-1}(\mathcal{X})\subset\mathcal{A}$; equivalently, f is $(\mathcal{A},\mathcal{X})$-measurable. □

Corollary B.12 (Continuous maps are Borel measurable.)
Suppose X_1, X_2 are topological spaces, and $\mathcal{B}_1$, $\mathcal{B}_2$ are their respective Borel σ-algebras. Then, every continuous map $f:X_1\longrightarrow X_2$ is $(\mathcal{B}_1,\mathcal{B}_2)$-measurable.

C Topology

Theorem C.1 (Appendix M3, [1])
Suppose S is a metric space. Then, the following conditions are equivalent:

(i) *S is separable.*

(ii) *The topology of S has a countable basis.*

(iii) *Every open cover of each subset of S has a countable subcover.*

D The Portmanteau Theorem and its corollaries (criteria for weak convergence of measures)

Theorem D.1 (The Portmanteau Theorem, Theorem 2.1, [1])
Suppose:

- *(S,ρ) is a metric space, $\mathcal{B}(S)$ its the Borel σ-algebra, $(S,\mathcal{B}(S))$ is the corresponding measurable space.*
- *$P,P_1,P_2,\ldots\in\mathcal{M}_1(S,\mathcal{B}(S))$ are probability measures on $(S,\mathcal{B}(S))$.*

Then, the following are equivalent:

(i) *P_n converges weakly to P, i.e. for each bounded continuous $\mathbb{R}$-valued function $f : S \longrightarrow \mathbb{R}$, we have*

$$\lim_{n\to\infty} \int_S f(s)\,\mathrm{d}P_n(s) \;=\; \int_S f(s)\,\mathrm{d}P(s).$$

(ii) *For each closed set $F \subset S$, we have*

$$\limsup_{n\to\infty} P_n(F) \;\leq\; P(F).$$

(iii) *For each open set $G \subset S$, we have*

$$\liminf_{n\to\infty} P_n(G) \;\geq\; P(G).$$

(iv) *For each $A \in \mathcal{B}(S)$, we have*

$$P(A^\circ) \;\leq\; \liminf_{n\to\infty} P_n(A^\circ) \;\leq\; \liminf_{n\to\infty} P_n(A) \;\leq\; \limsup_{n\to\infty} P_n(A) \;\leq\; \limsup_{n\to\infty} P_n\big(\overline{A}\big) \;\leq\; P\big(\overline{A}\big).$$

(v) *For each P-continuity set $A \in \mathcal{B}(S)$, i.e. $P(\partial A) = 0$, we have*

$$\lim_{n\to\infty} P_n(A) \;=\; P(A).$$

Theorem D.2 (Theorem 2.2, [1])
Suppose (S,ρ) is a metric space, and $P, P_1, P_2, \ldots \mathcal{M}_1(S,\mathcal{B}(S))$ are probability measures on the measurable space $(S,\mathcal{B}(S))$. Then, $P_n \overset{w}{\longrightarrow} P$ if there exists a sub-collection $\mathcal{A} \subset \mathcal{B}(S)$ satisfying the following conditions:

(i) *$\mathcal{A}$ is closed under finite intersections,*

(ii) *$\lim_{n\to\infty} P_n(A) = P(A)$, for each $A \in \mathcal{A}$, and*

(iii) *each open subset of S is a countable union of sets in $\mathcal{A}$.*

Proof
By the Portmanteau Theorem (Theorem D.1), it suffices to establish the following:

$$P(G) \;\leq\; \liminf_{n\to\infty} P_n(G), \quad \text{for each open subset } G \subset S.$$

By hypothesis, $G = \bigcup_{i=1}^\infty A_i$, where $A_i \in \mathcal{A}$ for each $i \in \mathbb{N}$. For each $\varepsilon > 0$, choose $r \in \mathbb{N}$ sufficiently large such that

$$P(G) - \varepsilon \;<\; P\left(\bigcup_{i=1}^r A_i\right) \;\leq\; P(G).$$

Now, observe that:

$$\begin{aligned}
P_n\left(\bigcup_{i=1}^r A_i\right) \;&=\; \sum_{i=1}^r P_n(A_i) - \sum_{i=1}^r \sum_{j=i+1}^r P_n(A_i \cap A_j) + \sum_{i=1}^r \sum_{j=i+1}^r \sum_{k=j+1}^r P_n(A_i \cap A_j \cap A_k) - \cdots \\
&\longrightarrow\; \sum_{i=1}^r P(A_i) - \sum_{i=1}^r \sum_{j=i+1}^r P(A_i \cap A_j) + \sum_{i=1}^r \sum_{j=i+1}^r \sum_{k=j+1}^r P(A_i \cap A_j \cap A_k) - \cdots \\
&=\; P\left(\bigcup_{i=1}^r A_i\right),
\end{aligned}$$

where we have used the hypotheses (i) and (ii) and the fact the ellipses above represent sums of finitely many terms. Thus we have:

$$P(G) - \varepsilon \;\leq\; P\left(\bigcup_{i=1}^{r} A_i\right) \;=\; \lim_{n\to\infty} P_n\left(\bigcup_{i=1}^{r} A_i\right) \;\leq\; \liminf_{n\to\infty} P_n(G).$$

Since $\varepsilon > 0$ is arbitrary, it follows that:

$$P(G) \;\leq\; \liminf_{n\to\infty} P_n(G),$$

which completes the proof the present Theorem. □

Theorem D.3 (Theorem 2.3, [1])

Suppose (S,ρ) is a separable metric space, and $P, P_1, P_2, \ldots \mathcal{M}_1(S,\mathcal{B}(S))$ are probability measures on the measurable space $(S,\mathcal{B}(S))$. Then, $P_n \xrightarrow{w} P$ if there exists a sub-collection $\mathcal{A} \subset \mathcal{B}(S)$ satisfying the following conditions:

(i) *$\mathcal{A}$ is closed under finite intersections,*

(ii) *$\lim_{n\to\infty} P_n(A) = P(A)$, for each $A \in \mathcal{A}$, and*

(iii) *for each $x \in S$ and $\varepsilon > 0$, the set*

$$\mathcal{A}(x,\varepsilon) \;:=\; \left\{\; A \in \mathcal{A} \;\middle|\; x \,\in\, A^\circ \,\subset\, A \,\subset\, B(x,\varepsilon) \;\right\} \;\neq\; \varnothing.$$

PROOF

By the preceding Theorem, it suffices to establish that each open subset $G \subset S$ can be expressed as a countable union of sets in $\mathcal{A}$. But this follows from the separability of S and hypothesis (iii). Indeed, let $G \subset S$ be an open subset of S. For each $x \in G$, choose $\epsilon_x > 0$ such that $B(x,\varepsilon_x) \subset G$. Next, by hypothesis (iii), we may choose $A_x \in \mathcal{A}$ such that

$$x \;\in\; A_x^\circ \;\subset\; A_x \;\subset\; B(x,\varepsilon_x) \;\subset\; G.$$

Thus,

$$G \;=\; \bigcup_{x\in G} A_x^\circ.$$

Since S is separable, by Theorem C.1, there exists $x_1, x_2, \ldots \in G$ such that $G \;=\; \bigcup_{i=1}^{\infty} A_{x_i}^\circ$. But then

$$G \;=\; \bigcup_{i=1}^{\infty} A_{x_i}^\circ \;\subset\; \bigcup_{i=1}^{\infty} A_{x_i} \;\subset\; \bigcup_{i=1}^{\infty} B(x_i,\varepsilon_{x_i}) \;\subset\; G,$$

which implies

$$G \;=\; \bigcup_{i=1}^{\infty} A_{x_i}.$$

This completes the proof of the present Theorem. □

Theorem D.4 (Theorem 2.4, [1])

Suppose (S,ρ) is a separable metric space. Then, a sub-collection $\mathcal{A} \subset \mathcal{B}(S)$ is a convergence-determining class of Borel subsets of $(S,\mathcal{B}(S))$ if $\mathcal{A}$ satisfies the following conditions:

(i) *$\mathcal{A}$ is closed under finite intersections, and*

(ii) *for each $x \in S$ and $\varepsilon > 0$, the set*

$$\partial\mathcal{A}(x,\varepsilon) \;:=\; \left\{\; \partial A \subset S \;\middle|\; A \in \mathcal{A}(x,\varepsilon) \;\right\}$$

either contains $\varnothing$ or contains uncountably many disjoint sets, where

$$\mathcal{A}(x,\varepsilon) \;:=\; \left\{\; A \in \mathcal{A} \;\middle|\; x \in A^\circ \subset A \subset B(x,\varepsilon) \;\right\}.$$

Proof We need to prove that the following implication holds:

$$\left.\begin{array}{r} P,\, P_1,\, P_2,\, \ldots \in \mathcal{M}_1(S,\mathcal{B}(S))\,, \text{ and} \\ \lim\limits_{n\to\infty} P_n(A) \,=\, P(A), \text{ for each } A \in \mathcal{A}_P \end{array}\right\} \quad\Longrightarrow\quad P_n \xrightarrow{w} P,$$

where $\mathcal{A}_P := \{\, A \in \mathcal{A} \mid P(\partial A) = 0 \,\}$ is the collection of P-continuity sets in $\mathcal{A}$.
By the preceding Theorem, it suffices to establish that $\mathcal{A}_P$ is closed under finite intersections and that

$$\mathcal{A}_P(x,\varepsilon) := \left\{\; A \in \mathcal{A}_P \;\middle|\; x \in A^\circ \subset A \subset B(x,\varepsilon) \;\right\} \;=\; \mathcal{A}_P \cap \mathcal{A}(x,\varepsilon) \;\neq\; \varnothing, \text{ for each } x \in S \text{ and } \varepsilon > 0.$$

<u>$\mathcal{A}_P$ is closed under finite intersections</u>
For any $A, B \subset S$, note that

$$\begin{aligned}
\partial(A \cap B) \;&:=\; \left\{\; x \in S \;\middle|\; \begin{array}{l} \text{for each } \varepsilon > 0: \\ B(x,\varepsilon) \cap (A \cap B) \;\neq \varnothing, \text{ and} \\ B(x,\varepsilon) \cap (A \cap B)^c \neq \varnothing \end{array} \right\} \\
&=\; \left\{\; x \in S \;\middle|\; \begin{array}{l} \text{for each } \varepsilon > 0: \\ B(x,\varepsilon) \cap (A \cap B) \;\neq \varnothing, \text{ and} \\ B(x,\varepsilon) \cap (A^c \cup B^c) \neq \varnothing \end{array} \right\} \\
&=\; \left\{\; x \in S \;\middle|\; \begin{array}{l} \text{for each } \varepsilon > 0: \\ B(x,\varepsilon) \cap (A \cap B) \neq \varnothing, \text{ and} \\ (B(x,\varepsilon) \cap A^c) \cup (B(x,\varepsilon) \cap B^c) \neq \varnothing \end{array} \right\} \\
&\subset\; \left\{\; x \in S \;\middle|\; \begin{array}{l} \text{for each } \varepsilon > 0: \\ B(x,\varepsilon) \cap A \;\neq \varnothing, \text{ and} \\ B(x,\varepsilon) \cap A^c \neq \varnothing \end{array} \right\} \bigcup \left\{\; x \in S \;\middle|\; \begin{array}{l} \text{for each } \varepsilon > 0: \\ B(x,\varepsilon) \cap B \;\neq \varnothing, \text{ and} \\ B(x,\varepsilon) \cap B^c \neq \varnothing \end{array} \right\} \\
&=\; (\partial A) \cup (\partial B),
\end{aligned}$$

which immediately implies that $A \cap B \in \mathcal{A}_P$ whenever $A, B \in \mathcal{A}_P$. Thus, $\mathcal{A}_P$ is closed under finite intersections.

<u>$\mathcal{A}_P(x,\varepsilon) \neq \varnothing$, for each $x \in S$ and $\varepsilon > 0$</u>

$$\begin{aligned}
\text{(ii)} \;&\Longrightarrow\; \partial\mathcal{A}(x,\varepsilon) \text{ contains a set of } P\text{-measure zero} \\
&\Longrightarrow\; \text{there exists } B \in \partial\mathcal{A}(x,\varepsilon) \text{ such that } P(B) = 0 \\
&\Longrightarrow\; \text{there exists } A \in \mathcal{A}(x,\varepsilon) \text{ such that } P(\partial A) = 0 \\
&\Longrightarrow\; \text{there exists } A \in \mathcal{A}(x,\varepsilon) \cap \mathcal{A}_P \;=\; \mathcal{A}_P(x,\varepsilon) \\
&\Longrightarrow\; \mathcal{A}_P(x,\varepsilon) \;\neq\; \varnothing,
\end{aligned}$$

where the first implication follows from the general fact that, for an arbitrary finite measure space $(\Omega, \mathcal{F}, \mu)$, $\mu(\varnothing) = 0$, and in every uncountable collection of disjoint $\mathcal{F}$-measurable sets, at most countably many of these sets can have positive μ-measures.

The proof of the present Theorem is now complete. $\square$

References

[1] Billingsley, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.

[2] Billingsley, P. *Probability and Measure*, anniversary ed. John Wiley & Sons, 2012.