

This set of notes aims to provide complete proofs of a number of asymptotic results regarding the Bootstrap [3] contained in Bickel and Freedman [1].

1 Bootstrap asymptotics for sample mean

Theorem 1.1 (Bootstrap Central Limit Theorem for I.I.D. sample mean, Theorem 2.1 [1])

Let $(\Omega, \mathcal{A}, \nu)$ be a probability space. Let $X, X_1, X_2, \dots : \Omega \rightarrow \mathbb{R}$ be a sequence of independent and identically distributed \mathbb{R} -valued random variables defined on Ω *with finite expectation value $\mu_X \in \mathbb{R}$ and variance $\sigma_X^2 < \infty$* . For each $n \in \mathbb{N}$ be fixed, define:

$$\bar{X}_n : \Omega \rightarrow \mathbb{R} : \omega \mapsto \frac{1}{n} \sum_{i=1}^n X_i(\omega).$$

For $n, m \in \mathbb{N}$, define $\mathcal{S}_m^{(n)}$ to be the set of all functions from $\{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$. Thus, each

$$s = (s(1), s(2), \dots, s(m)) \in \mathcal{S}_m^{(n)}$$

can be regarded as a length- m finite (ordered) sequence of positive integers between 1 and n , inclusive. Note that $\mathcal{S}_m^{(n)}$ is a finite set with $|\mathcal{S}_m^{(n)}| = n^m$. Endow $\mathcal{S}_m^{(n)}$ with the probability space structure induced by the uniform probability function:

$$P_{\mathcal{S}_m^{(n)}}(s) := \frac{1}{n^m}, \quad \text{for each } s \in \mathcal{S}_m^{(n)}.$$

Let $\Omega \times \mathcal{S}_m^{(n)}$ be the product probability space of Ω and $\mathcal{S}_m^{(n)}$. Define:

$$\bar{X}_m^{(n)} : \Omega \times \mathcal{S}_m^{(n)} \rightarrow \mathbb{R} : (\omega, s) \mapsto \frac{1}{m} \sum_{j=1}^m X_{s(j)}(\omega).$$

For each $\omega \in \Omega$, define:

$$\Phi_{m,\omega}^{(n)} : \mathcal{S}_m^{(n)} \rightarrow \mathbb{R} : s \mapsto \sqrt{m} \left(\bar{X}_m^{(n)}(\omega, s) - \bar{X}_n(\omega) \right)$$

Then,

$$P \left(\Phi_{m,\omega}^{(n)} \xrightarrow{d} N(0, \sigma_X^2), \text{ as } n, m \rightarrow \infty \right) = \nu \left(\left\{ \omega \in \Omega \mid \Phi_{m,\omega}^{(n)} \xrightarrow{d} N(0, \sigma_X^2), \text{ as } n, m \rightarrow \infty \right\} \right) = 1.$$

Remark 1.2

For each fixed $\omega \in \Omega$, $\left\{ \Phi_{m,\omega}^{(n)} : \mathcal{S}_m^{(n)} \rightarrow \mathbb{R} \right\}_{n,m \in \mathbb{N}}$ is a doubly indexed sequence of \mathbb{R} -valued random variables. Note that their respective domains $\mathcal{S}_m^{(n)}$ are pairwise distinct probability spaces. The **Bootstrap Central Limit Theorem for I.I.D. sample mean** asserts that for almost every $\omega \in \Omega$, the doubly indexed sequence $\left\{ \Phi_{m,\omega}^{(n)} \right\}$ of \mathbb{R} -valued random variables converges in distribution to $N(0, \sigma_X^2)$ as $n, m \rightarrow \infty$.

Remark 1.3 The following results are well known from classical asymptotic theory:

By the **Weak Law of Large Numbers**, \bar{X}_n converges in probability to μ_X , as $n \rightarrow \infty$; in other words,

$$\lim_{n \rightarrow \infty} P \left(|\bar{X}_n - \mu_X| > \varepsilon \right) = \lim_{n \rightarrow \infty} \nu \left(\left\{ \omega \in \Omega : |\bar{X}_n(\omega) - \mu_X| > \varepsilon \right\} \right) = 0, \quad \text{for each } \varepsilon > 0.$$

By the **Strong Law of Large Numbers**, \bar{X}_n converges almost surely to μ_X , as $n \rightarrow \infty$; in other words,

$$P \left(\lim_{n \rightarrow \infty} \bar{X}_n = \mu_X \right) = \nu \left(\left\{ \omega \in \Omega \mid \lim_{n \rightarrow \infty} \bar{X}_n(\omega) = \mu_X \right\} \right) = 1.$$

By the **Central Limit Theorem**, $\sqrt{n}(\bar{X}_n - \mu_X)$ converges in distribution to $N(0, \sigma_X^2)$.

A A stochastic process $\{X_t : \Omega \longrightarrow V\}_{t \in T}$ and its equivalent V^T -valued random variable $X : \Omega \longrightarrow V^T$

Let Ω , T , and V be non-empty sets. Let $\{X_t : \Omega \longrightarrow V\}_{t \in T}$ be a T -index family of maps, each of which maps from Ω into V . Note that this family of maps is set-theoretically equivalent (in the sense that either one completely determines the other) to the following (V^T) -valued map defined on Ω :

$$X : \Omega \longrightarrow V^T : \omega \longmapsto (t \longmapsto X_t(\omega)),$$

where $V^T = \prod_{t \in T} V$ denotes the set of all (arbitrary) V -valued functions defined on T . In this section, we aim to establish the following two results:

- Suppose (Ω, \mathcal{A}) and (V, \mathcal{F}) are measurable space structures on Ω and V , respectively. Then, $X : \Omega \longrightarrow V^T$ is $(\mathcal{A}, \sigma[(V, \mathcal{F})^T])$ -measurable if and only if $X_t : \Omega \longrightarrow V$ is $(\mathcal{A}, \mathcal{F})$ -measurable for each $t \in T$. Here, $\sigma[(V, \mathcal{F})^T]$ denotes the product σ -algebra on V^T , which is by definition the smallest σ -algebra on V^T such that, for each $t \in T$, the projection map (or evaluation map)

$$\pi_t : V^T \longrightarrow V : x \longmapsto x(t)$$

is $(\sigma[(V, \mathcal{F})^T], \mathcal{F})$ -measurable.

- An immediate corollary of the above result is that: Suppose $(\Omega, \mathcal{A}, \mu)$ is a probability space structure on Ω , (V, \mathcal{F}) is a measurable space structure on V , and $\sigma[(V, \mathcal{F})^T]$ is the product σ -algebra on V^T . Then, $X : (\Omega, \mathcal{A}, \mu) \longrightarrow (V^T, \sigma[(V, \mathcal{F})^T])$ is V^T -valued random variable if and only if $\{X_t : (\Omega, \mathcal{A}, \mu) \longrightarrow (V, \mathcal{F})\}_{t \in T}$ is a stochastic process.

Definition A.1 (The product σ -algebra of a Cartesian product of measurable spaces)

Let T be an arbitrary non-empty set. For each $t \in T$, let (V_t, \mathcal{F}_t) be a measurable space (in particular, $V_t \neq \emptyset$). Let $\prod_{t \in T} V_t$ be the Cartesian product of $\{V_t\}_{t \in T}$. In other words,

$$\prod_{t \in T} V_t := \left\{ v : T \longrightarrow \bigsqcup_{t \in T} V_t \mid v(t) \in V_t, \text{ for each } t \in T \right\}.$$

That $\prod_{t \in T} V_t \neq \emptyset$ follows from the Axiom of Choice. For each $t \in T$, let

$$\pi_t : \prod_{\tau \in T} V_\tau \longrightarrow V_t : v \longmapsto v(t)$$

be the projection map from $\prod_{\tau \in T} V_\tau$ onto V_t . The **product σ -algebra** on $\prod_{t \in T} V_t$ is the following:

$$\sigma\left(\left\{ \pi_t^{-1}(F) \subset \prod_{\tau \in T} V_\tau \mid F \in \mathcal{F}_t, t \in T \right\}\right) \subset \text{PowerSet}\left(\prod_{t \in T} V_t\right).$$

Clearly, it is the smallest σ -algebra on $\prod_{t \in T} V_t$ with respect to which each projection map $\pi_t : \prod_{t \in T} V_t \longrightarrow (V_t, \mathcal{F}_t)$ is measurable. We denote the product σ -algebra on $\prod_{t \in T} V_t$ by

$$\sigma\left(\prod_{t \in T} (V_t, \mathcal{F}_t)\right)$$

Theorem A.2

Suppose Ω , T , and V are non-empty sets. Let $\{X_t : \Omega \longrightarrow V\}_{t \in T}$ be a T -index family of maps, each of which maps from Ω into V . Then,

1. The family $\{X_t : \Omega \longrightarrow V\}_{t \in T}$ of maps is set-theoretically equivalent (in the sense that either completely determines the other) to the following (V^T) -valued map defined on Ω :

$$X : \Omega \longrightarrow V^T : \omega \longmapsto (t \longmapsto X_t(\omega)),$$

where $V^T = \prod_{t \in T} V$ denotes the set of all (arbitrary) V -valued functions defined on T .

2. Suppose (Ω, \mathcal{A}) and (V, \mathcal{F}) are measurable space structures on Ω and V respectively, and $\sigma[(V, \mathcal{F})^T]$ denotes the corresponding product σ -algebra on V^T . Then, $X : \Omega \longrightarrow V^T$ is $(\mathcal{A}, \sigma[(V, \mathcal{F})^T])$ -measurable if and only if $X_t : \Omega \longrightarrow V$ is $(\mathcal{A}, \mathcal{F})$ -measurable for each $t \in T$.

PROOF The proof of the first statement is routine and we omit it. We now prove the second statement. Suppose $X : \Omega \longrightarrow V^T$ is $(\mathcal{A}, \sigma[(V, \mathcal{F})^T])$ -measurable. Note that $X_t = \pi_t \circ X$, where $\pi_t : \prod_{t \in T} V \longrightarrow V$ is the projection from $V^T = \prod_{t \in T} V$ onto the t -th factor. By construction of the product σ -algebra $\sigma[(V, \mathcal{F})^T]$ on V^T , $\pi_t : V^T \longrightarrow V$ is $(\sigma[(V, \mathcal{F})^T], \mathcal{F})$ -measurable for each $t \in T$. This implies, for each $t \in T$, $X_t = \pi_t \circ X$ is $(\mathcal{A}, \mathcal{F})$ -measurable, being a composition of two measurable maps. Conversely, suppose X_t is $(\mathcal{A}, \mathcal{F})$ -measurable for each $t \in T$. Recall that the product σ -algebra on V^T is generated by sets of the form:

$$\pi_t^{-1}(F), \text{ for some } t \in T \text{ and } F \in \mathcal{F}.$$

It follows that, for each $t \in T$ and each $F \in \mathcal{F}$, we have

$$X^{-1}(\pi_t^{-1}(F)) = (X^{-1} \circ \pi_t^{-1})(F) = (\pi_t \circ X)^{-1}(F) = X_t^{-1}(F) \subset \Omega$$

is \mathcal{A} -measurable, since $X_t : (\Omega, \mathcal{A}) \longrightarrow (V, \mathcal{F})$ is $(\mathcal{A}, \mathcal{F})$ -measurable by hypothesis. This proves that $X : \Omega \longrightarrow U$ is $(\mathcal{A}, \sigma[(V, \mathcal{F})^T])$ -measurable. \square

Definition A.3 (Stochastic processes)

Suppose $(\Omega, \mathcal{A}, \mu)$ is a probability space, (V, \mathcal{F}) is a measurable space, and T is an arbitrary non-empty set. A **stochastic process indexed by T defined on $(\Omega, \mathcal{A}, \mu)$ with state space (V, \mathcal{F})** is a family

$$\{X_t : (\Omega, \mathcal{A}, \mu) \longrightarrow (V, \mathcal{F})\}_{t \in T}$$

indexed by T of V -valued $(\mathcal{A}, \mathcal{F})$ -measurable maps from Ω into V .

Corollary A.4

Suppose Ω, T, V are non-empty sets, $(\Omega, \mathcal{A}, \mu)$ is a probability space structure on Ω , (V, \mathcal{F}) is a measurable space structure on V , and $\sigma[(V, \mathcal{F})^T]$ denotes the corresponding product σ -algebra on $V^T = \prod_{t \in T} V$. Let $\{X_t : \Omega \longrightarrow V\}_{t \in T}$ be a T -index family of maps, each of which maps from Ω into V , and let

$$X : \Omega \longrightarrow V^T : \omega \longmapsto (t \longmapsto X_t(\omega))$$

be its set-theoretically equivalent (V^T) -valued map defined on Ω . Then,

$$\{X_t : (\Omega, \mathcal{A}, \mu) \longrightarrow (V, \mathcal{F})\}_{t \in T}$$

is a stochastic process if and only if

$$X : (\Omega, \mathcal{A}, \mu) \longrightarrow (V^T, \sigma[(V, \mathcal{F})^T])$$

is a (V^T) -valued random variable.

B Uniqueness of the “full distribution” of a stochastic process $\{X_t : \Omega \longrightarrow V\}_{t \in T}$ given its finite-dimensional distributions

In this section, we state the precise sense in which the finite-dimensional distributions of a stochastic process completely determine the “full distribution” of the stochastic process.

First, let Ω , T , and V be non-empty sets. Let $\{X_t : \Omega \longrightarrow V\}_{t \in T}$ be a T -index family of maps, each of which maps from Ω into V . Note that this family of maps is set-theoretically equivalent (in the sense that either completely determines the other) to the following (V^T) -valued map defined on Ω :

$$X : \Omega \longrightarrow V^T : \omega \longmapsto (t \longmapsto X_t(\omega)),$$

where $V^T = \prod_{t \in T} V$ denotes the set of all (arbitrary) V -valued functions defined on T .

Next, suppose $(\Omega, \mathcal{A}, \mu)$ is a probability space structure on Ω , and (V, \mathcal{F}) is a measurable space structure on V . It will be shown that $X : \Omega \longrightarrow V^T$ is $(\mathcal{A}, \sigma[(V, \mathcal{F})^T])$ -measurable if and only if $X_t : \Omega \longrightarrow V$ is $(\mathcal{A}, \mathcal{F})$ -measurable for each $t \in T$. Here, $\sigma[(V, \mathcal{F})^T]$ denotes the product σ -algebra on V^T , which is by definition the smallest σ -algebra on V^T such that, for each $t \in T$, the projection map (or evaluation map)

$$\pi_t : V^T \longrightarrow V : x \longmapsto x(t)$$

is $(\sigma[(V, \mathcal{F})^T], \mathcal{F})$ -measurable.

The result we aim to state precisely and prove is the following: The finite-dimensional distributions of a stochastic process $\{X_t : (\Omega, \mathcal{A}, \mu) \longrightarrow (V, \mathcal{F})\}_{t \in T}$ completely determines the distribution of its associated (V^T) -valued random variable $X : (\Omega, \mathcal{A}, \mu) \longrightarrow (V^T, \sigma[(V, \mathcal{F})^T])$.

Definition B.1 (Stochastic processes)

Suppose $(\Omega, \mathcal{A}, \mu)$ is a probability space, (V, \mathcal{F}) is a measurable space, and T is an arbitrary non-empty set. A **stochastic process indexed by T defined on $(\Omega, \mathcal{A}, \mu)$ with state space (V, \mathcal{F})** is a family

$$\{X_t : (\Omega, \mathcal{A}, \mu) \longrightarrow (V, \mathcal{F})\}_{t \in T}$$

indexed by T of V -valued $(\mathcal{A}, \mathcal{F})$ -measurable maps from Ω into V .

Definition B.2 (The product σ -algebra of a Cartesian product of measurable spaces)

Let T be an arbitrary non-empty set. For each $t \in T$, let (V_t, \mathcal{F}_t) be a measurable space (in particular, $V_t \neq \emptyset$). Let $\prod_{t \in T} V_t$ be the Cartesian product of $\{V_t\}_{t \in T}$. In other words,

$$\prod_{t \in T} V_t := \left\{ v : T \longrightarrow \bigsqcup_{t \in T} V_t \mid v(t) \in V_t, \text{ for each } t \in T \right\}.$$

That $\prod_{t \in T} V_t \neq \emptyset$ follows from the Axiom of Choice. For each $t \in T$, let

$$\pi_t : \prod_{\tau \in T} V_\tau \longrightarrow V_t : v \longmapsto v(t)$$

be the projection map from $\prod_{\tau \in T} V_\tau$ onto V_t . The **product σ -algebra** on $\prod_{t \in T} V_t$ is the following:

$$\sigma\left(\left\{ \pi_t^{-1}(F) \subset \prod_{\tau \in T} V_\tau \mid F \in \mathcal{F}_t, t \in T \right\}\right) \subset \text{PowerSet}\left(\prod_{t \in T} V_t\right).$$

Clearly, it is the smallest σ -algebra on $\prod_{t \in T} V_t$ with respect to which each projection map $\pi_t : \prod_{t \in T} V_t \longrightarrow (V_t, \mathcal{F}_t)$ is measurable. We denote the product σ -algebra on $\prod_{t \in T} V_t$ by

$$\sigma\left(\prod_{t \in T} (V_t, \mathcal{F}_t)\right)$$

Proposition B.3

Suppose Ω , T , and V are non-empty sets. Let $\{X_t : \Omega \rightarrow V\}_{t \in T}$ be a T -index family of maps, each of which maps from Ω into V . Then,

1. The family $\{X_t : \Omega \rightarrow V\}_{t \in T}$ of maps is set-theoretically equivalent (in the sense that either completely determines the other) to the following (V^T) -valued map defined on Ω :

$$X : \Omega \rightarrow V^T : \omega \mapsto (t \mapsto X_t(\omega)),$$

where $V^T = \prod_{t \in T} V$ denotes the set of all (arbitrary) V -valued functions defined on T .

2. Suppose $(\Omega, \mathcal{A}, \mu)$ is a probability space structure on Ω , (V, \mathcal{F}) is a measurable space structure on V , and $\sigma[(V, \mathcal{F})^T]$ denotes the corresponding product σ -algebra on V^T . Then, $X : \Omega \rightarrow V^T$ is $(\mathcal{A}, \sigma[(V, \mathcal{F})^T])$ -measurable if and only if $X_t : \Omega \rightarrow V$ is $(\mathcal{A}, \mathcal{F})$ -measurable for each $t \in T$.

PROOF The proof of the first statement is routine and we omit it. We now prove the second statement. Suppose $X : \Omega \rightarrow V^T$ is $(\mathcal{A}, \sigma[(V, \mathcal{F})^T])$ -measurable. Note that $X_t = \pi_t \circ X$, where $\pi_t : \prod_{t \in T} V \rightarrow V$ is the projection from $V^T = \prod_{t \in T} V$ onto the t -th factor. By construction of the product σ -algebra $\sigma[(V, \mathcal{F})^T]$ on V^T , $\pi_t : V^T \rightarrow V$ is $(\sigma[(V, \mathcal{F})^T], \mathcal{F})$ -measurable for each $t \in T$. This implies, for each $t \in T$, $X_t = \pi_t \circ X$ is $(\mathcal{A}, \mathcal{F})$ -measurable, being a composition of two measurable maps. Conversely, suppose X_t is $(\mathcal{A}, \mathcal{F})$ -measurable for each $t \in T$. Recall that the product σ -algebra on V^T is generated by sets of the form:

$$\pi_t^{-1}(F), \text{ for some } t \in T \text{ and } F \in \mathcal{F}.$$

It follows that, for each $t \in T$ and each $F \in \mathcal{F}$, we have

$$X^{-1}(\pi_t^{-1}(F)) = (X^{-1} \circ \pi_t^{-1})(F) = (\pi_t \circ X)^{-1}(F) = X_t^{-1}(F) \subset \Omega$$

is \mathcal{A} -measurable, since $X_t : (\Omega, \mathcal{A}) \rightarrow (V, \mathcal{F})$ is $(\mathcal{A}, \mathcal{F})$ -measurable by hypothesis. This proves that $X : \Omega \rightarrow U$ is $(\mathcal{A}, \sigma[(V, \mathcal{F})^T])$ -measurable. \square

Corollary B.4

Suppose Ω , T , V are non-empty sets, $(\Omega, \mathcal{A}, \mu)$ is a probability space structure on Ω , (V, \mathcal{F}) is a measurable space structure on V , and $\sigma[(V, \mathcal{F})^T]$ denotes the corresponding product σ -algebra on $V^T = \prod_{t \in T} V$. Let $\{X_t : \Omega \rightarrow V\}_{t \in T}$ be a T -index family of maps, each of which maps from Ω into V , and let

$$X : \Omega \rightarrow V^T : \omega \mapsto (t \mapsto X_t(\omega))$$

be its set-theoretically equivalent (V^T) -valued map defined on Ω . Then,

$$\{X_t : (\Omega, \mathcal{A}, \mu) \rightarrow (V, \mathcal{F})\}_{t \in T}$$

is a stochastic process if and only if

$$X : (\Omega, \mathcal{A}, \mu) \rightarrow (V^T, \sigma[(V, \mathcal{F})^T])$$

is a (V^T) -valued random variable.

Definition B.5 (Finite-dimensional distributions of a stochastic processes)

Let $\{X_t : (\Omega, \mathcal{A}, \mu) \rightarrow (V, \mathcal{F})\}_{t \in T}$ be a stochastic process. Let $n \in \mathbb{N}$ and $t_1, t_2, \dots, t_n \in T$ be distinct elements of T . The probability distribution induced on the product measurable space $(V^n, \mathcal{F}^{\otimes n})$ by $(X_{t_1}, X_{t_2}, \dots, X_{t_n}) : \Omega \rightarrow V^n$ is called a **finite-dimensional distribution** of the stochastic process.

C Existence of a stochastic process given its finite-dimensional distributions: Komolgorov's Existence Theorem

Definition C.1 (Stochastic processes)

Suppose $(\Omega, \mathcal{A}, \mu)$ is a probability space, (V, \mathcal{F}) is a measurable space, and T is an arbitrary non-empty set. A **stochastic process** indexed by T defined on Ω with codomain V is a family $\{X_t : \Omega \rightarrow V\}_{t \in T}$ indexed by T of V -valued random variables defined on Ω .

Definition C.2 (Finite-dimensional distributions of a stochastic processes)

Let $\{X_t : \Omega \rightarrow (V, \mathcal{F})\}_{t \in T}$ be a stochastic process. Let $n \in \mathbb{N}$ and $t_1, t_2, \dots, t_n \in T$ be distinct elements of T . The probability distribution induced on the product measurable space $(V^n, \mathcal{F}^{\otimes n})$ by $(X_{t_1}, X_{t_2}, \dots, X_{t_n}) : \Omega \rightarrow V^n$ is called a **finite-dimensional distribution** of the stochastic process.

Definition C.3 (Komolgorov systems of finite-dimensional distributions & Komolgorov consistency)

Let T be an arbitrary non-empty set, and $\mathcal{D}(T)$ the set of all finite ordered sequences of elements of T whose elements are pairwise distinct; in other words,

$$\mathcal{D}(T) := \left\{ (t_1, t_2, \dots, t_n) \in \bigcup_{k=1}^{\infty} T^k \mid n \in \mathbb{N}, t_i \neq t_j, \text{ whenever } i \neq j \right\}.$$

For each $n \in \mathbb{N}$, let $\mathcal{M}_1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ be the set of all probability measures defined on the product measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. A **Komolgorov system of finite-dimensional distributions** is a $\mathcal{D}(T)$ -indexed family \mathcal{P} of probability measures of the following form:

$$\mathcal{P} = \{ P_{(t_1, \dots, t_n)} \in \mathcal{M}_1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \mid (t_1, \dots, t_n) \in \mathcal{D}(T) \}.$$

Furthermore, \mathcal{P} is said to be **Komolgorov consistent** if it satisfies both of the following conditions:

- **permutation invariance:** For any $n \in \mathbb{N}$, any $(t_1, \dots, t_n) \in \mathcal{D}(T)$, any $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$, and any permutation $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$, the following equality holds:

$$P_{(t_1, \dots, t_n)}(B_1 \times \dots \times B_n) = P_{(t_{\pi(1)}, \dots, t_{\pi(n)})}(B_{\pi(1)} \times \dots \times B_{\pi(n)}).$$

- **projection invariance:** For any $n \in \mathbb{N}$, any $(t_1, \dots, t_{n+1}) \in \mathcal{D}(T)$, and any $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$, the following equality holds:

$$P_{(t_1, \dots, t_n, t_{n+1})}(B_1 \times \dots \times B_n \times \mathbb{R}) = P_{(t_1, \dots, t_n)}(B_1 \times \dots \times B_n).$$

Remark C.4

It is obvious that the collection of finite-dimensional distributions of any \mathbb{R} -valued stochastic process is a Komolgorov consistent Komolgorov system of finite-dimensional distributions.

Definition C.5

Let $\{X_t : \Omega \rightarrow \mathbb{R}\}_{t \in T}$ be an \mathbb{R} -valued stochastic process, and

$$\mathcal{P} = \{ P_{(t_1, \dots, t_n)} \in \mathcal{M}_1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \mid (t_1, \dots, t_n) \in \mathcal{D}(T) \}$$

be a Komolgorov system of finite-dimensional distributions. We say that **the stochastic process $\{X_t\}$ admits \mathcal{P} as its collection of finite-dimensional distributions** if, for each $n \in \mathbb{N}$ and any $(t_1, t_2, \dots, t_n) \in \mathcal{D}(T)$, the probability distribution induced on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ by the map

$$(X_{t_1}, \dots, X_{t_n}) : \Omega \rightarrow \mathbb{R}^n$$

equals $P_{(t_1, \dots, t_n)} \in \mathcal{P}$.

Theorem C.6 (Komolgorov's Existence Theorem, Theorem 36.2, [2])

Let

$$\mathcal{P} = \{ P_{(t_1, \dots, t_n)} \in \mathcal{M}_1(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \mid (t_1, \dots, t_n) \in \mathcal{D}(T) \}.$$

be a Komolgorov system of finite-dimensional distributions. Then, there exists a stochastic process

$$\{ X_t : (\Omega, \mathcal{A}, \mu) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) \}_{t \in T}$$

which admits \mathcal{P} as its collection of finite-dimensional distributions if and only if \mathcal{P} is Komolgorov consistent.

D Gaussian Processes

Definition D.1 (Gaussian processes)

An \mathbb{R} -valued stochastic process $\{ X_t : \Omega \longrightarrow \mathbb{R} \}_{t \in T}$ is said to be **Gaussian** if each of its finite-dimensional distribution is a (univariate or multivariate) Gaussian distribution.

Definition D.2 (Mean and covariance functions of \mathbb{R} -valued stochastic processes)

Let $\{ X_t : \Omega \longrightarrow \mathbb{R} \}_{t \in T}$ be an \mathbb{R} -valued stochastic process.

- If, for each $t \in T$, we have $E(X_t) \in \mathbb{R}$, then the function

$$a_X : T \longrightarrow \mathbb{R} : t \longmapsto E(X_t)$$

is called the **mean** function of the \mathbb{R} -valued stochastic process $\{ X_t \}$.

- In addition, if, for each $t_1, t_2 \in T$, we have $0 \leq \text{Cov}(X_{t_1}, X_{t_2}) < \infty$, then the function

$$\Sigma_X : T \times T \longrightarrow \mathbb{R} : (t_1, t_2) \longmapsto \text{Cov}(X_{t_1}, X_{t_2})$$

is called the **covariance** function of the \mathbb{R} -valued stochastic process $\{ X_t \}$.

Theorem D.3

Let T be an arbitrary non-empty set, $a : T \longrightarrow \mathbb{R}$ an arbitrary \mathbb{R} -valued function defined on T , and $\Sigma : T \times T \longrightarrow [0, \infty)$ a non-negative \mathbb{R} -valued function defined on $T \times T$. Then, there exists a Gaussian process whose mean and covariance functions are a and Σ , respectively.

Theorem D.4

The mean and covariance functions of a Gaussian process together completely determine its collection of finite-dimensional distributions.

Definition D.5 (Brownian motion, a.k.a. Wiener process)

A **Brownian motion**, or **Wiener process**, is a stochastic process $\{ W_t : (\Omega, \mathcal{A}, \mu) \longrightarrow \mathbb{R} \}_{t \geq 0}$ indexed by the non-negative real line satisfying the following conditions:

- At $t = 0$, the process takes value 0 with probability 1; more precisely:

$$P(W_0 = 0) = \mu(\{ \omega \in \Omega \mid W_0(\omega) = 0 \}) = 1.$$

- The process $\{ W_t \}$ has independent increments; more precisely: for any $0 \leq t_1 \leq t_2 \leq \dots \leq t_n < \infty$,

$$W_{t_n} - W_{t_{n-1}}, \quad W_{t_{n-1}} - W_{t_{n-2}}, \quad \dots, \quad W_{t_2} - W_{t_1} : \Omega \longrightarrow \mathbb{R}$$

are independent random variables.

- For $0 \leq t_1 < t_2 < \infty$, the increment $W_{t_2} - W_{t_1}$ follows a Gaussian distribution with mean 0 and variance $t_2 - t_1$.

Definition D.6 (Brownian bridge)

A **Brownian bridge** is a Gaussian process $\{W_t^\circ : (\Omega, \mathcal{A}, \mu) \longrightarrow \mathbb{R}\}_{t \in [0,1]}$ indexed by the closed unit interval in \mathbb{R} satisfying the following conditions:

- For each $t \in [0, 1]$, we have $E(W_t^\circ) = 0$.
- For any $t_1, t_2 \in [0, 1]$, we have $\text{Cov}(W_{t_1}^\circ, W_{t_2}^\circ) = \min\{t_1, t_2\} - t_1 t_2$.

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