

1 Outline

Suppose:

- $(\Omega, \mathcal{A}, \mu)$ is a probability space.
- $n \in \mathbb{N}$ is a natural number (positive integer).
- $T_1, T_2, \dots, T_n : \Omega \longrightarrow [0, \infty]$ are independent identically distributed extended \mathbb{R} -valued random variables.
- $U_1, U_2, \dots, U_n : \Omega \longrightarrow [0, \infty]$ are independent identically distributed extended \mathbb{R} -valued random variables.
- For each $i = 1, 2, \dots, n$, let $X_i := \min\{T_i, U_i\}$, and $C_i := I_{\{T_i \leq U_i\}}$.

For each subject $i = 1, 2, \dots, n$, the random variable T_i is interpreted to be the “survival time” of subject i , while U_i is interpreted to be the “censoring time” of subject i .

We wish to make inference about the (common) *survival function*

$$S(t) := P(T > t) = \mu\left(\left\{\omega \in \Omega \mid T(\omega) > t\right\}\right)$$

of T_1, T_2, \dots, T_n . However, in survival analysis, the inference about $S(t)$ is made based on the *right-censored survival time data* $\{X_i, C_i\}$, $i = 1, 2, \dots, n$ (rather than on the T_i ’s directly).

The *hazard function*:

$$\lambda(t) := \lim_{h \rightarrow 0^+} \frac{1}{h} \cdot P\left(t \leq T < t + h \mid t \leq T\right)$$

The *cumulative hazard function*:

$$\Lambda(t) := \int_0^t \lambda(t) dt$$

The *Nelson-Aalen estimator* for the cumulative hazard function $\Lambda(t)$:

$$\hat{\Lambda}(\omega, t) := \sum_{\substack{C_i(\omega)=1 \\ T_i(\omega) \leq t}} \frac{1}{Y(\omega, T_i(\omega))},$$

where

$$Y_i(\omega, t) := \begin{cases} 1, & t - h < X_i(\omega), \text{ for each } h > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$Y(\omega, t) := \sum_{i=1}^n Y_i(\omega, t)$$

The aggregated counting process for subject i :

$$N_i(\omega, t) := I_{\{X_i(\omega) \leq t\}}$$

The aggregated counting process:

$$N(\omega, t) := \sum_{i=1}^n N_i(\omega, t) = \sum_{i=1}^n I_{\{X_i(\omega) \leq t\}}$$

The aggregated intensity process:

$$\alpha(\omega, t) := \lim_{h \rightarrow 0^+} \frac{1}{h} \cdot P\left(N(\omega, t+h) - N(\omega, t) = 1 \mid \mathcal{F}_t\right) = \lim_{h \rightarrow 0^+} \frac{1}{h} \cdot E\left[N(\omega, t+h) - N(\omega, t) \mid \mathcal{F}_t\right]$$

The aggregated cumulative intensity process:

$$A(\omega, t) := \int_0^t \alpha(\omega, t) dt$$

Then, the process

$$M(\omega, t) := N(\omega, t) - A(\omega, t) = N(\omega, t) - \int_0^t \alpha(\omega, t) dt$$

is a martingale process. In particular, $M(\cdot, t)$ satisfies

$$E\left[M(\cdot, t+h) - M(\cdot, t) \mid \mathcal{F}_t\right](\omega) = 0$$

A Integration on product measure spaces

Definition A.1 (Product σ -algebra)

Suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces. The product σ -algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ of \mathcal{A}_1 and \mathcal{A}_2 is, by definition, the following:

$$\mathcal{A}_1 \otimes \mathcal{A}_2 := \sigma \left(\left\{ A_1 \times A_2 \in \Omega_1 \times \Omega_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2 \right\} \right).$$

In other words, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the σ -algebra of subsets of $\Omega_1 \times \Omega_2$ containing all Cartesian products $A_1 \times A_2$, where $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.

Definition A.2 (Horizontal and vertical sections in a set-theoretic Cartesian product)

Suppose X and Y are two non-empty sets. For each $x \in X$, $y \in Y$, and $V \subset X \times Y$, we define:

$$\begin{aligned} V_{(x, \cdot)} &:= \left\{ y \in Y \mid (x, y) \in V \right\} \\ V_{(\cdot, y)} &:= \left\{ x \in X \mid (x, y) \in V \right\} \end{aligned}$$

Theorem A.3 (Sections of measurable subsets in a product measurable space are themselves measurable.)

Suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces. Then,

- (i) $V_{(x, \cdot)} \in \mathcal{A}_2$, for each $x \in \Omega_1$ and each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$, and
- (ii) $V_{(\cdot, y)} \in \mathcal{A}_1$, for each $y \in \Omega_2$ and each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$.

PROOF We give only the proof of (i); that of (ii) is similar. Define $\mathcal{F} \subset \mathcal{P}(\Omega_1 \times \Omega_2)$ as follows:

$$\mathcal{F} := \left\{ V \in \Omega_1 \times \Omega_2 \mid V_{(x, \cdot)} \in \mathcal{A}_2, \text{ for each } x \in \Omega_1 \right\}.$$

Claim 1: $\left\{ A_1 \times A_2 \in \Omega_1 \times \Omega_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2 \right\} \subset \mathcal{F}$

Claim 2: \mathcal{F} is a σ -algebra of subsets of $\Omega_1 \times \Omega_2$.

Proof of Claim 1: Suppose $x \in \Omega_1$ and $V = A_1 \times A_2$, where $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Then,

$$V_{(x, \cdot)} = \begin{cases} A_2, & \text{if } x \in A_1 \\ \emptyset, & \text{otherwise} \end{cases}$$

This proves that $V_{(x, \cdot)} = (A_1 \times A_2)_{(x, \cdot)} \subset \mathcal{F}$. Since $x \in \Omega_1$, $A_1 \in \mathcal{A}_1$, and $A_2 \in \mathcal{A}_2$ are arbitrary, Claim 1 follows.

Proof of Claim 2: First, note that, for each $x \in \Omega_1$, we have $(\Omega_1 \times \Omega_2)_{(x, \cdot)} := \left\{ y \in \Omega_2 \mid (x, y) \in \Omega_1 \times \Omega_2 \right\} = \Omega_2 \in \mathcal{A}_2$. Hence, $\Omega_1 \times \Omega_2 \in \mathcal{F}$. Next, suppose $V \in \mathcal{F}$ and $V^c := (\Omega_1 \times \Omega_2) \setminus V$. Then, for each $x \in \Omega_1$,

$$\begin{aligned} (V^c)_{(x, \cdot)} &= \left\{ y \in \Omega_2 \mid (x, y) \in V^c \right\} = \left\{ y \in \Omega_2 \mid (x, y) \notin V \right\} \\ &= \Omega_2 \setminus \left\{ y \in \Omega_2 \mid (x, y) \in V \right\} = (V_{(x, \cdot)})^c \in \mathcal{A}_2, \end{aligned}$$

where the last containment follows from the fact that \mathcal{A}_2 is a σ -algebra (hence closed under complementation) and that $V \in \mathcal{F}$ (hence $V_{(x, \cdot)} \in \mathcal{A}_2$). This proves that \mathcal{F} is closed under complementation. Lastly, suppose $V_1, V_2, \dots \in \mathcal{F}$. Then,

$$\left(\bigcup_{i=1}^{\infty} V_i \right)_{(x, \cdot)} = \left\{ y \in \Omega_2 \mid (x, y) \in \bigcup_{i=1}^{\infty} V_i \right\} = \bigcup_{i=1}^{\infty} \left\{ y \in \Omega_2 \mid (x, y) \in V_i \right\} = \bigcup_{i=1}^{\infty} (V_i)_{(x, \cdot)} \in \mathcal{A}_2,$$

where the last containment follows from the fact that \mathcal{A}_2 is a σ -algebra (hence closed under countable union) and that each $V_i \in \mathcal{F}$ (hence $(V_i)_{(x, \cdot)} \in \mathcal{A}_2$). This proves that \mathcal{F} is closed under countable union. This completes the proof of Claim 2.

Claim 1 and Claim 2 together immediately imply that

$$\mathcal{A}_1 \otimes \mathcal{A}_2 := \sigma \left(\left\{ A_1 \times A_2 \in \Omega_1 \times \Omega_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2 \right\} \right) \subset \mathcal{F} := \left\{ V \in \Omega_1 \times \Omega_2 \mid \begin{array}{l} V_{(x, \cdot)} \in \mathcal{A}_2, \\ \text{for each } x \in \Omega_1 \end{array} \right\}.$$

This completes the proof of statement (i) in the present Theorem. \square

Theorem A.4 (Sections of measurable maps are themselves measurable.)

Suppose $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$, (S, \mathcal{S}) are measurable spaces, and $f : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \rightarrow (S, \mathcal{S})$ is an $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{S})$ -measurable map. Then,

- (i) $f(x, \cdot) : \Omega_2 \rightarrow S : y \mapsto f(x, y)$ is an $(\mathcal{A}_2, \mathcal{S})$ -measurable map for each $x \in \Omega_1$.
- (ii) $f(\cdot, y) : \Omega_1 \rightarrow S : x \mapsto f(x, y)$ is an $(\mathcal{A}_1, \mathcal{S})$ -measurable map for each $y \in \Omega_2$.

PROOF

- (i) We need to show that $f(x, \cdot)^{-1}(V) \in \mathcal{A}_2$, for each $x \in \Omega_1$, and each $V \in \mathcal{S}$. To this end, note that

$$f(x, \cdot)^{-1}(V) = \left\{ y \in \Omega_2 \mid f(x, y) \in V \right\} = \left\{ y \in \Omega_2 \mid (x, y) \in f^{-1}(V) \right\} = f^{-1}(V)_{(x, \cdot)} \in \mathcal{A}_2,$$

where the last containment follows, by Theorem A.3, from the fact that $f^{-1}(V) \in \mathcal{A}_1 \otimes \mathcal{A}_2$ (since $V \in \mathcal{S}$ and f is $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{S})$ -measurable).

- (ii) The proof here is similar to that of (i). \square

Theorem A.5 (Well-definition of the product measure of two σ -finite measures)

Suppose $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ are two σ -finite measure spaces. Let $(\mathbb{R}, \mathcal{B})$ be \mathbb{R} equipped with its Borel σ -algebra $\mathcal{B} = \sigma(\mathcal{O}(\mathbb{R}))$. Then, for each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$, the following statements hold:

- (i) the map $\Omega_1 \rightarrow \mathbb{R} : x \mapsto \mu_2(V_{(x, \cdot)}) = \int_{\Omega_2} 1_V(x, y) d\mu_2(y)$ is $(\mathcal{A}_1, \mathcal{B})$ -measurable,
- (ii) the map $\Omega_2 \rightarrow \mathbb{R} : y \mapsto \mu_1(V_{(\cdot, y)}) = \int_{\Omega_1} 1_V(x, y) d\mu_1(x)$ is $(\mathcal{A}_2, \mathcal{B})$ -measurable, and
- (iii) the following equality of Lebesgue integrals (of measurable \mathbb{R} -valued functions) holds:

$$\int_{\Omega_1} \mu_2(V_{(x, \cdot)}) d\mu_1(x) = \int_{\Omega_2} \mu_1(V_{(\cdot, y)}) d\mu_2(y),$$

or equivalently,

$$\int_{\Omega_1} \left(\int_{\Omega_2} 1_V(x, y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_2} \left(\int_{\Omega_1} 1_V(x, y) d\mu_1(x) \right) d\mu_2(y).$$

PROOF Define $\mathcal{C} \subset \mathcal{A}_1 \otimes \mathcal{A}_2$ as follows:

$$\mathcal{C} := \left\{ V \in \mathcal{A}_1 \otimes \mathcal{A}_2 \mid \int_{\Omega_1} \mu_2(V_{(x, \cdot)}) d\mu_1(x) = \int_{\Omega_2} \mu_1(V_{(\cdot, y)}) d\mu_2(y) \right\}.$$

Claim 1: $A_1 \times A_2 \in \mathcal{C}$, for each $A_1 \in \mathcal{A}_1$ and each $A_2 \in \mathcal{A}_2$.

Claim 2: $V := \bigcup_{i=1}^{\infty} V_i \in \mathcal{C}$, whenever $\{V_i\}_{i \in \mathbb{N}} \subset \mathcal{C}$ and $V_i \subset V_{i+1}$, for each $i \in \mathbb{N}$.

Claim 3: $V := \bigsqcup_{i=1}^{\infty} V_i \in \mathcal{C}$, whenever $\{V_i\}_{i \in \mathbb{N}} \subset \mathcal{C}$ is a disjoint countable collection of members in \mathcal{C} .

Claim 4: Suppose $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$, with $\mu_1(A_1), \mu_2(A_2) < \infty$. Suppose also that $\{V_i\}_{i \in \mathbb{N}} \subset \mathcal{C}$ satisfies $A_1 \times A_2 \supset V_1 \supset V_2 \supset V_3 \supset \dots$. Then, $V := \bigcap_{i=1}^{\infty} V_i \in \mathcal{C}$.

Proof of Claim 1:

Proof of Claim 2:

Proof of Claim 3:

Proof of Claim 4:

Next, note that, since $(\Omega_1, \mathcal{A}_1, \mu_1)$ is a σ -finite measure space, there exist mutually disjoint $\Omega_1^{(1)}, \Omega_1^{(2)}, \dots \in \mathcal{A}_1$ such that

$$\Omega_1 = \bigsqcup_{n=1}^{\infty} \Omega_1^{(n)}, \quad \text{and} \quad \mu_1(\Omega_1^{(n)}) < \infty, \quad \text{for each } n \in \mathbb{N}.$$

Similarly, there exist mutually disjoint $\Omega_2^{(1)}, \Omega_2^{(2)}, \dots \in \mathcal{A}_2$ such that

$$\Omega_2 = \bigsqcup_{n=1}^{\infty} \Omega_2^{(n)}, \quad \text{and} \quad \mu_2(\Omega_2^{(n)}) < \infty, \quad \text{for each } n \in \mathbb{N}.$$

We now define

$$\mathcal{M} := \left\{ V \in \mathcal{A}_1 \otimes \mathcal{A}_2 \mid V \cap (\Omega_1^{(m)} \times \Omega_2^{(n)}) \in \mathcal{C}, \text{ for each } m, n \in \mathbb{N} \right\}.$$

Claim 5: \mathcal{M} is a monotone class.

Claim 6:

Proof of Claim 5: Suppose $V_1, V_2, \dots \in \mathcal{M}$, with $V_1 \subset V_2 \subset V_3 \subset \dots$. We need to show $V := \bigcup_{i=1}^{\infty} V_i \in \mathcal{M}$. To this end, note that, for each $m, n \in \mathbb{N}$, we have

$$V \cap (\Omega_1^{(m)} \times \Omega_2^{(n)}) = \left(\bigcup_{i=1}^{\infty} V_i \right) \cap (\Omega_1^{(m)} \times \Omega_2^{(n)}) = \bigcup_{i=1}^{\infty} \underbrace{(V_i \cap (\Omega_1^{(m)} \times \Omega_2^{(n)}))}_{\in \mathcal{C}} \in \mathcal{C},$$

where the last containment follows from Claim 3. Thus, we see that we indeed have $V \in \mathcal{M}$. Next, suppose that $W_1, W_2, \dots \in \mathcal{M}$, with $W_1 \supset W_2 \supset W_3 \supset \dots$. We need to show $W := \bigcap_{i=1}^{\infty} W_i \in \mathcal{M}$. Now, for each $m, n \in \mathbb{N}$, we

have:

$$W \cap (\Omega_1^{(m)} \times \Omega_2^{(n)}) = \left(\bigcap_{i=1}^{\infty} W_i \right) \cap (\Omega_1^{(m)} \times \Omega_2^{(n)}) = \bigcap_{i=1}^{\infty} \underbrace{(W_i \cap (\Omega_1^{(m)} \times \Omega_2^{(n)}))}_{\in \mathcal{C}} \in \mathcal{C}.$$

where the last containment follows from Claim 4. This proves that \mathcal{M} is indeed a monotone class and completes the proof of Claim 5.

It follows from Claim 5, Claim 6 and Theorem ?? that $\mathcal{M} = \mathcal{A}_1 \otimes \mathcal{A}_2$, which in turn implies that $V \cap (\Omega_1^{(m)} \times \Omega_2^{(n)}) \in \mathcal{C}$, for each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$ and each $m, n \in \mathbb{N}$. But,

$$V = V \cap (\Omega_1 \times \Omega_2) = V \cap \left(\bigsqcup_{m,n \in \mathbb{N}} \Omega_1^{(m)} \times \Omega_2^{(n)} \right) = \bigsqcup_{m,n \in \mathbb{N}} \underbrace{V \cap (\Omega_1^{(m)} \times \Omega_2^{(n)})}_{\in \mathcal{C}} \in \mathcal{C},$$

or equivalently,

$$\int_{\Omega_1} \mu_2(V_{(x, \cdot)}) \, d\mu_1(x) = \int_{\Omega_2} \mu_1(V_{(\cdot, y)}) \, d\mu_2(y).$$

This completes the proof of the present Theorem. □

References

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