The Portmanteau Theorem

1   The Portmanteau Theorem

Theorem 1.1 (The Portmanteau Theorem, Theorem 2.1, [1])

Suppose:

- *(S, ρ) is a metric space, $\mathcal{B}(S)$ its the Borel σ-algebra, $(S, \mathcal{B}(S))$ is the corresponding measurable space.*

- *$P, P_1, P_2, \ldots \in \mathcal{M}_1(S, \mathcal{B}(S))$ are probability measures on $(S, \mathcal{B}(S))$.*

Then, the following are equivalent:

(i)   *P_n converges weakly to P, i.e. for each bounded continuous $\mathbb{R}$-valued function $f : S \longrightarrow \mathbb{R}$, we have*

$$\lim_{n \to \infty} \int_S f(s) \, \mathrm{d}P_n(s) \;=\; \int_S f(s) \, \mathrm{d}P(s).$$

(ii)   *For each closed set $F \subset S$, we have*

$$\limsup_{n \to \infty} P_n(F) \;\leq\; P(F).$$

(iii)   *For each open set $G \subset S$, we have*

$$\limsup_{n \to \infty} P_n(G) \;\geq\; P(G).$$

(iv)   *For each P-continuity set $A \in \mathcal{B}(S)$, i.e. $P(\partial A) = 0$, we have*

$$\lim_{n \to \infty} P_n(A) \;=\; P(A).$$

PROOF

(i) $\Longrightarrow$ (ii)

For each $\varepsilon > 0$, by Lemma A.2(ii), choose a bounded continuous functions $f_\varepsilon : S \longrightarrow [0,1]$ such that

$$I_F \;\leq\; f_\varepsilon \;\leq\; I_{F^\varepsilon}.$$

This implies, for each $\varepsilon > 0$, we have

$$P_n(F) \;=\; \int_S I_F(x) \, \mathrm{d}P_n(x) \;\leq\; \int_S f_\varepsilon(x) \, \mathrm{d}P_n(x).$$

By (i), we thus have

$$\limsup_{n \to \infty} P_n(F) \;\leq\; \lim_{n \to \infty} \int_S f_\varepsilon(x) \, \mathrm{d}P_n(x) \;=\; \int_S f_\varepsilon(x) \, \mathrm{d}P(x) \;\leq\; \int_S I_{F^\varepsilon}(x) \, \mathrm{d}P(x) \;=\; P(F^\varepsilon).$$

By Lemma A.2(i), we have $F^\varepsilon \downarrow F$ as $\varepsilon \downarrow 0$. Hence, $P(F^\varepsilon) \downarrow P(F)$ as $\varepsilon \downarrow 0$. We may now conclude:

$$\limsup_{n \to \infty} P_n(F) \;\leq\; \lim_{\varepsilon \to 0^+} P(F^\varepsilon) \;=\; P(F).$$

$\square$

A   Technical Lemmas

Lemma A.1   *Suppose (S, ρ) is a metric space, and $A \subset S$ is an arbitrary non-empty subset. Define*

$$\rho(\,\cdot\,, A) \ : \ S \ \longrightarrow \ \mathbb{R} \ : \ x \ \longmapsto \ \inf_{y \in A} \{\, \rho(x, y)\, \}$$

Then,

(i) $\rho(\,\cdot\,, A)$ *is a continuous $\mathbb{R}$-valued function on S.*

(ii)   *For each $x \in S$, $\rho(x, A) = 0$ if and only if $x \in \overline{A}$.*

PROOF

(i)   Suppose $x_n \longrightarrow x$. We need to prove $\rho(x_n, A) \longrightarrow \rho(x, A)$, which follows immediately from the following two Claims:

  Claim 1: $\rho(x, A) \ \leq \ \liminf_{n \to \infty} \rho(x_n, A).$

  Claim 2: $\limsup_{n \to \infty} \rho(x_n, A) \ \leq \ \rho(x, A).$

  Proof of Claim 1:   For each $y \in S$, we have:

$$\rho(x, y) \ \leq \ \rho(x, x_n) \ + \ \rho(x_n, y).$$

  Hence,

$$\rho(x, A) \ = \ \inf_{y \in A} \rho(x, y) \ \leq \ \rho(x, x_n) \ + \ \inf_{y \in A} \rho(x_n, y) \ = \ \rho(x, x_n) \ + \ \rho(x_n, A).$$

  Since $\rho(x, x_n) \longrightarrow 0$, the preceding inequality implies

$$\rho(x, A) \ \leq \ \liminf_{n \to \infty} \rho(x_n, A).$$

  This proves Claim 1.

  Proof of Claim 2:   For each $y \in S$, we have:

$$\rho(x_n, y) \ \leq \ \rho(x_n, x) \ + \ \rho(x, y).$$

  Hence,

$$\rho(x_n, A) \ = \ \inf_{y \in A} \rho(x_n, y) \ \leq \ \rho(x_n, x) \ + \ \inf_{y \in A} \rho(x, y) \ = \ \rho(x_n, x) \ + \ \rho(x, A).$$

  Since $\rho(x, x_n) \longrightarrow 0$, the preceding inequality implies

$$\limsup_{n \to \infty} \rho(x_n, A) \ \leq \ \rho(x, A).$$

  This proves Claim 2.

(ii)

$$
\begin{aligned}
\rho(x, A) = 0 \ &\Longleftrightarrow \ \inf_{y \in A} \rho(x, y) = 0 \\
&\Longleftrightarrow \ \text{For each } \varepsilon > 0, \text{ there exists } y \in A \text{ such that } \rho(x, y) < \varepsilon \\
&\Longleftrightarrow \ y \in \overline{A}
\end{aligned}
$$

$\square$

2

Lemma A.2   *Suppose (S, ρ) is a metric space, and $A \subset S$ is a non-empty subset. For each $\varepsilon > 0$, define*

$$A^\varepsilon \ := \ \{\, s \in S \ | \ \rho(s, A) < \varepsilon \,\}.$$

Then the following are true:

(i) $A^\varepsilon \downarrow \overline{A}$, *as* $\varepsilon \downarrow 0$.

(ii)   *There exists a bounded continuous $\mathbb{R}$-valued function $f : S \longrightarrow \mathbb{R}$ such that*

$$I_{\overline{A}}(x) \ \leq \ f(x) \ \leq \ I_{A^\varepsilon}(x), \quad \text{for each } x \in S.$$

PROOF

(i)

(ii)   Define $f : S \longrightarrow \mathbb{R}$ as follows:

$$f(x) \ := \ \max\left\{\, 0\,,\, 1 - \frac{\rho(x, A)}{\varepsilon} \,\right\}.$$

Then, by Lemma A.1(i), f is continuous $\mathbb{R}$-valued function on S. Clear, $0 \leq f(x) \leq 1$, for each $x \in S$. By Lemma A.1(ii), we have

$$x \in \overline{A} \quad \Longleftrightarrow \quad \rho(x, F) \ = \ 0 \quad \Longleftrightarrow \quad f(x) \ = \ 1.$$

This proves $I_{\overline{A}}(x) \leq 1 = f(x)$, for each $x \in \overline{A}$, and hence for each $x \in S$ (since $I_{\overline{A}}(x) = 0$ for $x \in S \setminus \overline{A}$, and the inequality holds trivially). On the other hand,

$$x \in S \setminus A^\varepsilon \quad \Longleftrightarrow \quad \varepsilon \ \leq \ \rho(x, A) \quad \Longleftrightarrow \quad 1 - \frac{\rho(x, A)}{\varepsilon} \ \leq \ 0 \quad \Longrightarrow \quad f(x) \ = \ 0.$$

This proves $f(x) = 0 \leq I_{A^\varepsilon}(x)$, for each $x \in S \setminus A^\varepsilon$, and hence for each $x \in S$ (since $I_{A^\varepsilon}(x) = 1$ for each $x \in A^\varepsilon$ and the inequality holds trivially). This completes the proof of (ii).

$\square$

References

[1] BILLINGSLEY, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.