A Technical Lemmas

Lemma A.1 (p.343, [1])

$$\left| \, e^{\mathbf{i}x} \; - \; \sum_{k=0}^{n} \dfrac{(\mathbf{i}\,x)^k}{k!} \, \right| \;\; \leq \;\; \min\left\{ \, \dfrac{|\,x\,|^{n+1}}{(n+1)!}, \; \dfrac{2\,|\,x\,|^n}{n!} \, \right\}, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof

Claim 1:

$$\int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s \;\; = \;\; \dfrac{x^{n+1}}{n+1} \; + \; \dfrac{\mathbf{i}}{n+1} \int_0^x (x-s)^{n+1} \, e^{\mathbf{i}s} \, \mathrm{d}s, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof of Claim 1: We proceed by integration by parts. Let $u = e^{\mathbf{i}s}$ and $\mathrm{d}v = (x-s)^n \, \mathrm{d}s$. Then, $\mathrm{d}u = \mathbf{i}\, e^{\mathbf{i}s}$ and $v = -(x-s)^{n+1}/(n+1)$. Hence,

$$\begin{aligned}
\int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s \;\; &= \;\; \int u \, \mathrm{d}v \;\; = \;\; uv \; - \; \int v \, \mathrm{d}u \\
&= \;\; \left[\, e^{\mathbf{i}s} \cdot \dfrac{(-1)(x-s)^{n+1}}{n+1} \, \right]_{s=0}^{s=x} \; - \; \int_0^x \dfrac{(-1)\,(x-s)^{n+1}}{n+1} \cdot \mathbf{i}e^{\mathbf{i}s} \, \mathrm{d}s, \\
&= \;\; \dfrac{x^{n+1}}{n+1} \; + \; \dfrac{\mathbf{i}}{n+1} \int_0^x (x-s)^{n+1} \; e^{\mathbf{i}s} \, \mathrm{d}s.
\end{aligned}$$

This proves Claim 1.

Claim 2:

$$e^{\mathbf{i}x} \;\; = \;\; \sum_{k=0}^{n} \dfrac{(\mathbf{i}\,x)^k}{k!} \; + \; \dfrac{\mathbf{i}^{n+1}}{n!} \int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof of Claim 2: We proceed by induction. For $n = 0$, we have:

$$\begin{aligned}
\mathrm{RHS}(n=0) \;\; &= \;\; \sum_{k=0}^{0} \dfrac{(\mathbf{i}\,x)^k}{k!} \; + \; \dfrac{\mathbf{i}^{0+1}}{0!} \int_0^x (x-s)^0 \, e^{\mathbf{i}s} \, \mathrm{d}s \;\; = \;\; 1 \; + \; \mathbf{i} \int_0^x e^{\mathbf{i}s} \, \mathrm{d}s \;\; = \;\; 1 \; + \; \mathbf{i} \left[\, \dfrac{e^{\mathbf{i}s}}{\mathbf{i}} \, \right]_{s=0}^{s=x} \\
&= \;\; 1 + \left(e^{\mathbf{i}x} - 1 \right) \;\; = \;\; e^{\mathbf{i}x}.
\end{aligned}$$

Thus, Claim 2 is indeed true for $n = 0$. Next, by induction hypothesis, assume Claim 2 is true for n, and we verify that Claim 2 is also true for $n + 1$.

$$\begin{aligned}
\mathrm{RHS}(n+1) \;\; &= \;\; \sum_{k=0}^{n+1} \dfrac{(\mathbf{i}\,x)^k}{k!} \; + \; \dfrac{\mathbf{i}^{n+2}}{(n+1)!} \int_0^x (x-s)^{n+1} \, e^{\mathbf{i}s} \, \mathrm{d}s \\
&= \;\; \sum_{k=0}^{n} \dfrac{(\mathbf{i}\,x)^k}{k!} \; + \; \dfrac{(\mathbf{i}\,x)^{n+1}}{(n+1)!} \; + \; \dfrac{\mathbf{i}^{n+2}}{(n+1)!} \cdot \dfrac{n+1}{\mathbf{i}} \left[\, \int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s \; - \; \dfrac{x^{n+1}}{n+1} \, \right] \\
&= \;\; \sum_{k=0}^{n} \dfrac{(\mathbf{i}\,x)^k}{k!} \; + \; \dfrac{\mathbf{i}^{n+1}}{n!} \int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s \; + \; \dfrac{(\mathbf{i}\,x)^{n+1}}{(n+1)!} \; - \; \dfrac{\mathbf{i}^{n+1}}{n!} \cdot \dfrac{x^{n+1}}{n+1} \;\; = \;\; e^{\mathbf{i}x},
\end{aligned}$$

where the second equality follows from Claim 1 and the last equality follows from the induction hypothesis (that Claim 2 holds for n). This proves Claim 2.

□

Lemma A.2 (§7.1, [2])

Let $\{\,\theta_{nj} \in \mathbb{C} \mid 1 \leq j \leq k_n, n \in \mathbb{N}\,\}$ be doubly indexed array of complex numbers. If all the following three conditions are true:

(a) *there exists $M > 0$ such that*

$$\sum_{j=1}^{k_n} |\,\theta_{nj}\,| \;\leq\; M, \quad \text{for each } n \in \mathbb{N},$$

(b) $\displaystyle\lim_{n\to\infty} \max_{1\leq j\leq k_n} |\,\theta_{nj}\,| \;=\; 0$, *and*

(c) *there exists $\theta \in \mathbb{C}$ such that*

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \theta_{nj} \;=\; \theta,$$

then

$$\lim_{n\to\infty} \prod_{j=1}^{k_n} (1+\theta_{nj}) \;=\; e^{\theta}.$$

Proof First, note that hypothesis (b) immediately implies that there exists some $n_0 \in \mathbb{N}$ such that

$$|\,\theta_{nj}\,| \;\leq\; \frac{1}{2}, \quad \text{for each } n \geq n_0, \text{ for each } 1 \leq j \leq k_n.$$

Thus, without loss of generality, we may assume that:

$$|\,\theta_{nj}\,| \;\leq\; \frac{1}{2}, \quad \text{for each } n \in \mathbb{N}, \text{ for each } 1 \leq j \leq k_n.$$

We denote by $\log(1+\theta_{nj})$ the (unique) complex logarithm[1] of $1+\theta_{nj}$ with argument in $(-\pi,\pi]$. Next, recall the MacLaurin Series for $\log(1+x)$:

$$\log(1+x) \;=\; \sum_{m=1}^{\infty} (-1)^{n+1}\,\frac{x^m}{m}, \quad \text{for any } x \in \mathbb{C} \text{ with } |\,x\,| < 1.$$

Hence, we have the following inequality: for each $n \in \mathbb{N}$ and for each $1 \leq j \leq k_n$,

$$\begin{aligned}
|\,\log(1+\theta_{nj}) - \theta_{nj}\,| \;&=\; \left|\,\sum_{m=2}^{\infty} (-1)^{n+1}\,\frac{(\theta_{nj})^m}{m}\,\right| \;\leq\; \sum_{m=2}^{\infty} \frac{|\,\theta_{nj}\,|^m}{m} \;\leq\; \frac{|\,\theta_{nj}\,|^2}{2} \sum_{m=2}^{\infty} |\,\theta_{nj}\,|^{m-2} \\
&\leq\; \frac{|\,\theta_{nj}\,|^2}{2} \sum_{m=2}^{\infty} \left(\frac{1}{2}\right)^{m-2} \;=\; \frac{|\,\theta_{nj}\,|^2}{2} \sum_{i=2}^{\infty} \left(\frac{1}{2}\right)^{i-2} \;=\; \frac{|\,\theta_{nj}\,|^2}{2}\cdot 2 \;=\; |\,\theta_{nj}\,|^2.
\end{aligned}$$

This in turn implies: for each $n \in \mathbb{N}$,

$$\left|\,\sum_{j=1}^{k_n} \log(1+\theta_{nj}) \;-\; \sum_{j=1}^{k_n} \theta_{nj}\,\right| \;=\; \left|\,\sum_{j=1}^{k_n} \left(\log(1+\theta_{nj}) - \theta_{nj}\right)\,\right| \;\leq\; \sum_{j=1}^{k_n} |\,\log(1+\theta_{nj}) \;-\; \theta_{nj}\,| \;\leq\; \sum_{j=1}^{k_n} |\,\theta_{nj}\,|^2.$$

[1] Recall that the complex exponential function is defined by $\exp : \mathbb{C} \to \mathbb{C} : x + \mathbf{i}\,y \mapsto e^x \cdot e^{\mathbf{i}\,y} = e^x\,(\cos y + \mathbf{i}\sin y)$. Clearly, exp is not injective. More precisely, for $x_1 + \mathbf{i}\,y_1,\, x_2 + \mathbf{i}\,y_2 \in \mathbb{C}\backslash\{\,0\,\}$, we have $e^{x_1+\mathbf{i}y_1} = e^{x_2+\mathbf{i}y_2}$ if and only if $x_1 = x_2 \in \mathbb{R}\backslash\{\,0\,\}$ and $y_1 - y_2 \in 2\pi\mathbb{Z}$. For $z = re^{\mathbf{i}\theta} \in \mathbb{C}\backslash\{\,0\,\}$, a complex logarithm of z is any $w = x + \mathbf{i}\,y \in \mathbb{C}\backslash\{\,0\,\}$ such that $e^{x+\mathbf{i}y} = e^w = z = re^{\mathbf{i}\theta}$, i.e. $x = \log r$ and $y = \theta + 2\pi\mathbb{Z}$. In particular, let $\mathcal{D} := \{\,x + \mathbf{i}\,y \in \mathbb{C} \mid x \in \mathbb{R}, y \in (-\pi,\pi]\,\}$. Then, the restriction $\exp : \mathcal{D} \to \mathbb{C}\backslash\{\,0\,\}$ is bijective.

Thus, for each $n \in \mathbb{N}$, there exists $\Lambda_n \in \mathbb{C}$ with $|\,\Lambda_n\,| \leq 1$ such that

$$\sum_{j=1}^{k_n} \log(1+\theta_{nj}) \;\;=\;\; \sum_{j=1}^{k_n} \theta_{nj} \;+\; \Lambda_n \cdot \sum_{j=1}^{k_n} |\,\theta_{nj}\,|^2.$$

(Since for any $z \in \mathbb{C}$, $|\,z\,| \leq A \implies z = A \cdot w$, for some $w \in \mathbb{C}$ with $|\,w\,| \leq 1$.) Next note that, hypotheses (a) and (b) together imply:

$$\sum_{j=1}^{k_n} |\,\theta_{nj}\,|^2 \;\;\leq\;\; \left(\max_{1\leq j\leq k_n} |\,\theta_{nj}\,|\right)\left(\sum_{j=1}^{k_n} |\,\theta_{nj}\,|\right) \;\;\leq\;\; M \cdot \left(\max_{1\leq j\leq k_n} |\,\theta_{nj}\,|\right) \;\;\longrightarrow\;\; 0, \quad \text{as } n \longrightarrow \infty.$$

Therefore, since $|\,\Lambda_n\,| \leq 1$ for each $n \in \mathbb{N}$, we now see that

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \log(1+\theta_{nj}) \;\;=\;\; \lim_{n\to\infty} \sum_{j=1}^{k_n} \theta_{nj} \;+\; \lim_{n\to\infty} \left(\Lambda_n \cdot \sum_{j=1}^{k_n} |\,\theta_{nj}\,|^2 \right) \;\;=\;\; \theta + 0 \;\;=\;\; \theta.$$

We may now conclude, by continuity of the exponential function $\exp(\,\cdot\,)$:

$$\begin{aligned}
\lim_{n\to\infty} \prod_{j=1}^{k_n} (\,1+\theta_{nj}\,) \;\;&=\;\; \lim_{n\to\infty} \exp\left(\log \prod_{j=1}^{k_n} (\,1+\theta_{nj}\,) \right) \;\;=\;\; \lim_{n\to\infty} \exp\left(\sum_{j=1}^{k_n} \log\,(\,1+\theta_{nj}\,) \right) \\
&=\;\; \exp\left(\lim_{n\to\infty} \sum_{j=1}^{k_n} \log\,(\,1+\theta_{nj}\,) \right) \;\;=\;\; \exp\,(\,\theta\,)
\end{aligned}$$

This completes the proof of the Lemma. □

B The Central Limit Theorems

Theorem B.1 (Lindeberg's Central Limit Theorem, Theorem 1.15, [3])

Suppose:

- $\{\,k_n\,\}_{n\in\mathbb{N}} \subset \mathbb{N}$ *is a sequence of natural numbers such that* $k_n \to \infty$ *as* $n \to \infty$, *and*
- *for each* $n \in \mathbb{N}$, $X_1^{(n)}, X_2^{(n)}, \ldots, X_{k_n}^{(n)} : \Omega_n \longrightarrow \mathbb{R}$ *are independent (but not necessarily identically distributed)* $\mathbb{R}$*-valued random variables defined on a common probability space* $(\,\Omega_n, \mathcal{A}_n, \mu_n\,)$ *such that*

$$\mu_j^{(n)} \;:=\; E\Big[\,X_j^{(n)}\,\Big] \;\in\; \mathbb{R} \;\text{ exists, for each } 1 \leq j \leq k_n, \quad \text{and} \quad 0 \;<\; \sigma_n^2 \;:=\; \mathrm{Var}\left[\,\sum_{j=1}^{k_n} X_j^{(n)}\,\right] \;<\; \infty.$$

Let $N(0,1)$ *denote the standard Gaussian distribution on* $\mathbb{R}$. *Then, the following implication holds: If*

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{j=1}^{k_n} E\left[\,\left(X_j^{(n)} - \mu_j^{(n)}\right)^2 \cdot I_{\left\{|X_j^{(n)}-\mu_j^{(n)}|\,>\,\epsilon\sigma_n\right\}}\,\right] \;\;=\;\; 0, \quad \text{for each } \epsilon > 0,$$

then

$$\frac{1}{\sigma_n} \sum_{j=1}^{k_n} \left(\,X_j^{(n)} - \mu_j^{(n)}\,\right) \;\;\xrightarrow{\;\mathcal{L}\;}\;\; N(0,1).$$

Proof Considering $\left(X_j^{(n)} - \mu_j^{(n)} \right) \Big/ \sigma_n$, we may assume, without loss of generality, that

$$E\Big[\, X_j^{(n)} \,\Big] = 0, \quad \text{and} \quad \sigma_n^2 \;:=\; \mathrm{Var}\left[\, \sum_{j=1}^{k_n} X_j^{(n)} \,\right] = 1.$$

□

Lemma A.2.

References

[1] Billingsley, P. *Probability and Measure*, third ed. John Wiley & Sons, 1995.

[2] Chung, K. L. *A Course in Probability Theory*, third ed. Academic Press, 2001.

[3] Shao, J. *Mathematical Statistics*, second ed. Springer, 2003.