1 Variance estimation for multi-stage sampling

Let U be a finite population and $\mathcal{P}(U)$ the power set of U. Let $p : \mathcal{S} \longrightarrow (0,1]$ be a r-stage sampling design ($r \geq 2$), where $\mathcal{S} \subset \mathcal{P}(U)$ is the set of all admissible samples under the design p. We express the hierarchical structure of the population U, with respect to the r-stage design p, as follows:

$$U \;=\; \bigsqcup_{i\in U^{(1)}} U_i^{(2+)} \;=\; \bigsqcup_{i\in U^{(1)}} \bigsqcup_{a\in U_i^{(2)}} U_{ia}^{(3)} \;=\; \cdots \;=\; \bigsqcup_{i\in U^{(1)}} \cdots \bigsqcup_{b\in U_{i\cdots}^{(r-1)}} U_{i\cdots b}^{(r)} \tag{1.1}$$

where $U^{(1)}$ is the set of all primary sampling units (PSU), and for each PSU $i \in U^{(1)}$, $U_i^{(2)}$ denotes the set of all secondary sampling units (SSU) contained in $i \in U^{(1)}$, and for each SSU $a \in U_i^{(2)}$, $U_{ia}^{(3)}$ denotes the set of all tertiary sampling units (TSU) contained in $a \in U_i^{(2)}$, and so on. Similarly, we express the hierachical structure of every admissible sample $s \in \mathcal{S}$ as follows:

$$s \;=\; \bigsqcup_{i\in s^{(1)}} s_i^{(2+)} \;=\; \bigsqcup_{i\in s^{(1)}} \bigsqcup_{a\in s_i^{(2)}} s_{ia}^{(3)} \;=\; \cdots \tag{1.2}$$

Let $y : U \longrightarrow \mathbb{R}$ be a population characteristic. Let T be its population total, i.e.

$$T \;=\; \sum_{u\in U} y_u \;=\; \sum_{i\in U^{(1)}} T_i \;=\; \sum_{i\in U^{(1)}} \sum_{a\in U_{ia}^{(2)}} T_{ia} \;=\; \cdots \;=\; \sum_{i\in U^{(1)}} \cdots \sum_{u\in U_{i\cdots}^{(r)}} y_u \tag{1.3}$$

Theorem 1.1
If $\widehat{T}_i : \mathcal{S}_i^{(2+)} \longrightarrow \mathbb{R}$ is an unbiased estimator for T_i, for each PSU *$i \in U^{(1)}$, then the random variable $\widehat{T} : \mathcal{S} \longrightarrow \mathbb{R}$ defined as follows:*

$$\widehat{T}(s) \;:=\; \sum_{i\in s^{(1)}} \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \tag{1.4}$$

is a design-unbiased estimator for

$$T \;=\; \sum_{i\in U^{(1)}} T_i$$

If the r-stage sampling design has invariant and independent subsampling, then the design-variance of $\widehat{T}$ is given by:

$$\mathrm{Var}\Big[\,\widehat{T}\,\Big] \;=\; \underbrace{\mathrm{Var}^{(1)}\Big(E^{(2+)}\big(\,\widehat{T}\,\big|\,s^{(1)}\big)\Big)}_{\mathrm{V_{PSU}}} + \underbrace{E^{(1)}\Big(\mathrm{Var}^{(2+)}\big(\,\widehat{T}\,\big|\,s^{(1)}\big)\Big)}_{\mathrm{V_{subsampling}}}, \tag{1.5}$$

where

$$\begin{aligned}
\mathrm{Var}^{(1)}\Big(E^{(2+)}\big(\,\widehat{T}\,\big|\,s^{(1)}\big)\Big) &= \sum_{i\in U^{(1)}} \sum_{j\in U^{(1)}} \Delta_{ij}^{(1)} \frac{T_i}{\pi_i^{(1)}} \frac{T_j}{\pi_j^{(1)}}, \quad \text{and} \\
E^{(1)}\Big(\mathrm{Var}^{(2+)}\big(\,\widehat{T}\,\big|\,s^{(1)}\big)\Big) &= \sum_{i\in U^{(1)}} \frac{V_i}{\pi_i^{(1)}},
\end{aligned}$$

with

$$V_i \;:=\; \mathrm{Var}^{(2+)}\Big[\,\widehat{T}_i\,\Big] \quad\text{and}\quad \Delta_{ij}^{(1)} \;:=\; \begin{cases} \pi_i^{(1)}\left(1-\pi_i^{(1)}\right), & \text{if } i = j \\[2ex] \pi_{ij}^{(1)} - \pi_i^{(1)}\pi_j^{(1)}, & \text{if } i \neq j \end{cases} \tag{1.6}$$

Furthermore, if $\widehat{V}_i : \mathcal{S}_i^{(2+)} \longrightarrow \mathbb{R}$ *is an unbiased estimator for* $V_i := \mathrm{Var}^{(2+)}\left[\,\widehat{T}_i\,\right]$, *and* $\pi_i^{(1)} > 0$, $\pi_{ij}^{(1)} > 0$ *for any* PSUs $i, j \in U^{(1)}$, *then*

$$\begin{aligned}
\widehat{\mathrm{Var}}\left[\,\widehat{T}\,\right](s) &:= \sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\cdot\frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} + \sum_{i\in s^{(1)}}\frac{\widehat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}}, \\
\widehat{\mathrm{Var}}^{(1)}\left[\,\widehat{T}\,\right](s) &:= \sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\cdot\frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} + \sum_{i\in s^{(1)}}\left(1-\frac{1}{\pi_i^{(1)}}\right)\frac{\widehat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}}, \text{ and} \\
\widehat{\mathrm{Var}}^{(2+)}\left[\,\widehat{T}\,\right](s) &:= \sum_{i\in s^{(1)}}\frac{\widehat{V}_i(s_i^{(2+)})}{\left(\pi_i^{(1)}\right)^2}
\end{aligned}$$

are unbiased estimators for $\mathrm{Var}\left[\,\widehat{T}\,\right]$, $\mathrm{V}_{\mathrm{PSU}} := \mathrm{Var}^{(1)}\Big(E^{(2+)}(\,\widehat{T}\,\big|\,s^{(1)})\Big)$, *and* $\mathrm{V}_{\mathrm{subsampling}} := E^{(1)}\Big(\mathrm{Var}^{(2+)}(\,\widehat{T}\,\big|\,s^{(1)})\Big)$, *respectively.*

Corollary 1.2

$$\widehat{\mathrm{Var}}^{(1)}\left[\,\widehat{T}\,\right](s) = \widehat{\mathrm{Var}}\left[\,\widehat{T}\,\right](s) - \widehat{\mathrm{Var}}^{(2+)}\left[\,\widehat{T}\,\right](s) \tag{1.7}$$

PROOF

$$\begin{aligned}
\mathrm{Var}^{(1)}\left[E^{(2+)}\left[\,\widehat{T}\,\middle|\,s^{(1)}\right]\right] &= \mathrm{Var}^{(1)}\left[E^{(2+)}\left[\sum_{i\in s^{(1)}}\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\,\middle|\,s^{(1)}\right]\right] \\
&= \mathrm{Var}^{(1)}\left[\sum_{i\in s^{(1)}}\frac{E^{(2+)}\left[\,\widehat{T}_i(s_i^{(2+)})\,\right]}{\pi_i^{(1)}}\right] \\
&= \mathrm{Var}^{(1)}\left[\sum_{i\in s^{(1)}}\frac{T_i}{\pi_i^{(1)}}\right] \\
&= \sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}}\Delta_{ij}^{(1)}\frac{T_i}{\pi_i^{(1)}}\frac{T_j}{\pi_j^{(1)}}
\end{aligned}$$

$$\begin{aligned}
E^{(1)}\left[\mathrm{Var}^{(2+)}\left[\,\widehat{T}\,\middle|\,s^{(1)}\right]\right] &= E^{(1)}\left[\mathrm{Var}^{(2+)}\left[\sum_{i\in s^{(1)}}\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\,\middle|\,s^{(1)}\right]\right] \\
&= E^{(1)}\left[\sum_{i\in s^{(1)}}\mathrm{Var}^{(2+)}\left[\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\right]\right] \\
&= E^{(1)}\left[\sum_{i\in s^{(1)}}\frac{\mathrm{Var}^{(2+)}\left[\widehat{T}_i(s_i^{(2+)})\,\right]}{\left(\pi_i^{(1)}\right)^2}\right] \\
&= E^{(1)}\left[\sum_{i\in s^{(1)}}\frac{V_i/\pi_i^{(1)}}{\pi_i^{(1)}}\right] \\
&= \sum_{i\in U^{(1)}}\frac{V_i}{\pi_i^{(1)}}
\end{aligned}$$

$$
\begin{aligned}
E\left(\widehat{\mathrm{Var}}^{(2+)}\left(\widehat{T}\right)\right) &= E\left(\sum_{i\in s^{(1)}} \frac{\widehat{V}_i(s_i^{(2+)})}{\left(\pi_i^{(1)}\right)^2}\right) = E^{(1)}\left(E^{(2+)}\left(\sum_{i\in s^{(1)}} \frac{\widehat{V}_i(s_i^{(2+)})/\pi_i^{(1)}}{\pi_i^{(1)}}\,\middle|\, s^{(1)}\right)\right) \\
&= E^{(1)}\left(\sum_{i\in s^{(1)}} \frac{E^{(2+)}\left[\,\widehat{V}_i(s_i^{(2+)})\,\middle|\, s^{(1)}\right]\Big/\pi_i^{(1)}}{\pi_i^{(1)}}\right) = E^{(1)}\left(\sum_{i\in s^{(1)}} \frac{V_i/\pi_i^{(1)}}{\pi_i^{(1)}}\right) = \sum_{i\in U^{(1)}} \frac{V_i}{\pi_i^{(1)}} \\
&= E^{(1)}\left(\mathrm{Var}^{(2+)}(\,\widehat{T}\,\middle|\, s^{(1)})\right) = \mathrm{V}_{\mathrm{PSU}}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E\left(\sum_{i\in s^{(1)}} \frac{\widehat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}}\right) &= E^{(1)}\left(E^{(2+)}\left(\sum_{i\in s^{(1)}} \frac{\widehat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}}\,\middle|\, s^{(1)}\right)\right) = E^{(1)}\left(\sum_{i\in s^{(1)}} \frac{E^{(2+)}\left[\,\widehat{V}_i(s_i^{(2+)})\,\middle|\, s^{(1)}\right]}{\pi_i^{(1)}}\right) \\
&= E^{(1)}\left(\sum_{i\in s^{(1)}} \frac{V_i}{\pi_i^{(1)}}\right) = \sum_{i\in U^{(1)}} V_i
\end{aligned}
$$

Next, observe that

$$
\begin{aligned}
E\left[\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\cdot\frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}}\right] &= E^{(1)}\left(E^{(2+)}\left(\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\cdot\frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}}\,\middle|\, s^{(1)}\right)\right) \\
&= E^{(1)}\left(\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{E^{(2+)}\left[\,\widehat{T}_i(s_i^{(2+)})\cdot\widehat{T}_j(s_j^{(2+)})\,\middle|\, s^{(1)}\right]}{\pi_i^{(1)}\,\pi_j^{(1)}}\right)
\end{aligned}
$$

Now, observe (the key technical observation) that

$$
E^{(2+)}\left[\,\widehat{T}_i(s_i^{(2+)})\cdot\widehat{T}_j(s_j^{(2+)})\,\middle|\, s^{(1)}\right] = E^{(2+)}\left[\widehat{T}_i(s_i^{(2+)})\cdot\widehat{T}_j(s_j^{(2+)})\right] = \begin{cases} \mathrm{Var}^{(2+)}\left(\widehat{T}_i\right) + E^{(2+)}\left(\widehat{T}_i\right)^2, & \text{if } i=j, \\ E^{(2+)}\left(\widehat{T}_i\right)\cdot E^{(2+)}\left(\widehat{T}_i\right), & \text{if } i\neq j \end{cases}
$$

Hence,

$$
\begin{aligned}
E\left[\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\cdot\frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}}\right] &= E^{(1)}\left(\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{E^{(2+)}\left[\,\widehat{T}_i(s_i^{(2+)})\cdot\widehat{T}_j(s_j^{(2+)})\,\middle|\, s^{(1)}\right]}{\pi_i^{(1)}\,\pi_j^{(1)}}\right) \\
&= E^{(1)}\left(\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{T_i}{\pi_i^{(1)}}\cdot\frac{Tj}{\pi_j^{(1)}} + \sum_{i\in s^{(1)}} \frac{(1-\pi_i^{(1)})\,V_i\Big/\pi_i^{(1)}}{\pi_i^{(1)}}\right) \\
&= \sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}} \Delta_{ij}^{(1)}\cdot\frac{T_i}{\pi_i^{(1)}}\cdot\frac{Tj}{\pi_j^{(1)}} + \sum_{i\in U^{(1)}} \frac{(1-\pi_i^{(1)})}{\pi_i^{(1)}}\cdot V_i \\
&= \sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}} \Delta_{ij}^{(1)}\cdot\frac{T_i}{\pi_i^{(1)}}\cdot\frac{Tj}{\pi_j^{(1)}} + \sum_{i\in U^{(1)}} \frac{V_i}{\pi_i^{(1)}} - \sum_{i\in U^{(1)}} V_i
\end{aligned}
$$

We may now establish that

$$\begin{aligned}
E\left[\,\widehat{\mathrm{Var}}\left(\widehat{T}\right)\,\right] &= E\left[\,\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}}\frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\cdot\frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} \;+\; \sum_{i\in s^{(1)}}\frac{\widehat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}}\,\right] \\
&= \left\{\sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}}\Delta_{ij}^{(1)}\cdot\frac{T_i}{\pi_i^{(1)}}\cdot\frac{Tj}{\pi_j^{(1)}} \;+\; \sum_{i\in U^{(1)}}\frac{V_i}{\pi_i^{(1)}} \;-\; \sum_{i\in U^{(1)}}V_i\right\} \;+\; \left\{\sum_{i\in U^{(1)}}V_i\right\} \\
&= \sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}}\Delta_{ij}^{(1)}\cdot\frac{T_i}{\pi_i^{(1)}}\cdot\frac{Tj}{\pi_j^{(1)}} \;+\; \sum_{i\in U^{(1)}}\frac{V_i}{\pi_i^{(1)}} \\
&= \mathrm{Var}\left(\widehat{T}\right)
\end{aligned}$$

Lastly, note that

$$\widehat{\mathrm{Var}}^{(1)}\left[\widehat{T}\right](s) \;=\; \widehat{\mathrm{Var}}\left[\widehat{T}\right](s) \;-\; \widehat{\mathrm{Var}}^{(2+)}\left[\widehat{T}\right](s)$$

Hence,

$$\begin{aligned}
E\left[\widehat{\mathrm{Var}}^{(1)}\left(\widehat{T}\right)\right] &= E\left[\,\widehat{\mathrm{Var}}\left(\widehat{T}\right)\right] \;-\; E\left[\widehat{\mathrm{Var}}^{(2+)}\left(\widehat{T}\right)\right] \\
&= \mathrm{Var}\left[\,\widehat{T}\,\right] \;-\; E^{(1)}\left(\mathrm{Var}^{(2+)}\left(\,\widehat{T}\,\middle|\,s^{(1)}\right)\right) \\
&= \mathrm{Var}^{(1)}\left(E^{(2+)}\left(\,\widehat{T}\,\middle|\,s^{(1)}\right)\right) \;+\; E^{(1)}\left(\mathrm{Var}^{(2+)}\left(\,\widehat{T}\,\middle|\,s^{(1)}\right)\right) \;-\; E^{(1)}\left(\mathrm{Var}^{(2+)}\left(\,\widehat{T}\,\middle|\,s^{(1)}\right)\right) \\
&= \mathrm{Var}^{(1)}\left(E^{(2+)}\left(\,\widehat{T}\,\middle|\,s^{(1)}\right)\right) \;=\; \mathrm{V}_{\text{subsampling}}
\end{aligned}$$

□

2 Variance estimation for three-stage sampling

Let U be a finite population and $\mathcal{P}(U)$ the power set of U. Let $p:\mathcal{S}\longrightarrow(0,1]$ be a three-stage sampling design, where $\mathcal{S}\subset\mathcal{P}(U)$ is the set of all admissible samples under the design p. We express the three-stage structure of the population U, with respect to the three-stage design p, as follows:

$$U \;=\; \bigsqcup_{i\in U^{(1)}} U_i^{(2+)} \;=\; \bigsqcup_{i\in U^{(1)}}\;\bigsqcup_{a\in U_i^{(2)}} U_{ia}^{(3)}, \tag{2.1}$$

where $U^{(1)}$ is the set of all primary sampling units (PSU), and for each PSU $i\in U^{(1)}$, $U_i^{(2)}$ denotes the set of all secondary sampling units (SSU) contained in $i\in U^{(1)}$, and for each SSU $a\in U_i^{(2)}$, $U_{ia}^{(3)}$ denotes the set of all tertiary sampling units (TSU) contained in $a\in U_i^{(2)}$. Similarly, we express the three-stage structure of every admissible sample $s\in\mathcal{S}$ as follows:

$$s \;=\; \bigsqcup_{i\in s^{(1)}} s_i^{(2+)} \;=\; \bigsqcup_{i\in s^{(1)}}\;\bigsqcup_{a\in s_i^{(2)}} s_{ia}^{(3)} \tag{2.2}$$

Let $y:U\longrightarrow\mathbb{R}$ be a population characteristic. Let T be its population total, i.e.

$$T \;=\; \sum_{u\in U} y_u \;=\; \sum_{i\in U^{(1)}} T_i \;=\; \sum_{i\in U^{(1)}}\sum_{a\in U_{ia}^{(2)}} T_{ia} \;=\; \sum_{i\in U^{(1)}}\sum_{a\in U_i^{(2)}}\sum_{u\in U_{ia}^{(3)}} y_u \tag{2.3}$$

Theorem 2.1

For a three-stage sampling design with invariant and independent subsampling,

1. *The random variable $\widehat{T} : \mathcal{S} \longrightarrow \mathbb{R}$, defined as follows:*

$$\widehat{T}(s) \;:=\; \sum_{i\in s^{(1)}} \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \;=\; \sum_{i\in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a\in s_i^{(2)}} \frac{\widehat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \right) \;=\; \sum_{i\in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a\in s_i^{(2)}} \frac{1}{\pi_{i|a}^{(2)}} \left(\sum_{u\in s_{ia}^{(3)}} \frac{y_u}{\pi_{ia|u}^{(3)}} \right)\right)$$

   is a design-unbiased estimator for T, i.e. $E\left[\,\widehat{T}\,\right] = T$.

2. *If the three-stage sampling design has invariant and independent subsampling, then the design-variance of $\widehat{T}$ can be given by:*

$$\begin{aligned}
& \mathrm{Var}\left[\,\widehat{T}\,\right] \\
=\; & \mathrm{Var}^{(1)}\left(E^{(2+)}\left(\,\widehat{T}\;\middle|\; s^{(1)}\right)\right) \;+\; E^{(1)}\left(\mathrm{Var}^{(2+)}\left(\,\widehat{T}\;\middle|\; s^{(1)}\right)\right) \\
=\; & \mathrm{Var}^{(1)}\left(E^{(2+)}\left(\,\widehat{T}\;\middle|\; s^{(1)}\right)\right) \;+\; E^{(1)}\left\{\mathrm{Var}^{(2)}\left(E^{(3)}\left(\widehat{T}\;\middle|\; s^{(2)}\right)\;\middle|\; s^{(1)}\right) \;+\; E^{(2)}\left(\mathrm{Var}^{(3)}\left(\widehat{T}\;\middle|\; s^{(2)}\right)\;\middle|\; s^{(1)}\right)\right\} \\
=\; & \underbrace{\mathrm{Var}^{(1)}\left(E^{(2+)}\left(\,\widehat{T}\;\middle|\; s^{(1)}\right)\right)}_{\mathrm{V}_{\mathrm{PSU}}} \;+\; \underbrace{E^{(1)}\left\{\mathrm{Var}^{(2)}\left(E^{(3)}\left(\widehat{T}\;\middle|\; s^{(2)}\right)\;\middle|\; s^{(1)}\right)\right\}}_{\mathrm{V}_{\mathrm{SSU}}} \;+\; \underbrace{E^{(1)}\left\{E^{(2)}\left(\mathrm{Var}^{(3)}\left(\widehat{T}\;\middle|\; s^{(2)}\right)\;\middle|\; s^{(1)}\right)\right\}}_{\mathrm{V}_{\mathrm{TSU}}}
\end{aligned}$$

   where

$$\begin{aligned}
\mathrm{V}_{\mathrm{PSU}} \;&:=\; \mathrm{Var}^{(1)}\left(E^{(2+)}\left(\,\widehat{T}\;\middle|\; s^{(1)}\right)\right) \;=\; \sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}} \Delta_{ij}^{(1)} \frac{T_i}{\pi_i^{(1)}}\frac{T_j}{\pi_j^{(1)}}, \\
\mathrm{V}_{\mathrm{SSU}} \;&:=\; E^{(1)}\left\{\mathrm{Var}^{(2)}\left(\,E^{(3)}(\,\widehat{T}\;|\; s^{(2)}\,)\;\middle|\; s^{(1)}\right)\right\} \;=\; \sum_{i\in U^{(1)}} \frac{V_i^{(2)}}{\pi_i^{(1)}}, \quad \text{and} \\
\mathrm{V}_{\mathrm{TSU}} \;&:=\; E^{(1)}\left\{E^{(2)}\left(\,\mathrm{Var}^{(3)}(\,\widehat{T}\;|\; s^{(2)}\,)\;\middle|\; s^{(1)}\right)\right\} \;=\; \sum_{i\in U^{(1)}} \frac{1}{\pi_i^{(1)}}\left(\sum_{a\in U_i^{(2)}} \frac{V_{ia}^{(3)}}{\pi_{i|a}^{(2)}}\right),
\end{aligned}$$

   with

$$\begin{aligned}
\Delta_{ij}^{(1)} \;&:=\; \begin{cases} \pi_i^{(1)}\left(1-\pi_i^{(1)}\right), & \text{if } i=j \\[2ex] \pi_{ij}^{(1)} - \pi_i^{(1)}\pi_j^{(1)}, & \text{if } i\neq j \end{cases} \\
V_i^{(2)} \;&:=\; \mathrm{Var}^{(2)}\left[\sum_{a\in s_i^{(2)}} \frac{T_{ia}}{\pi_{i|a}^{(2)}}\right] \;=\; \sum_{a\in U_i^{(2)}}\sum_{b\in U_i^{(2)}} \Delta_{i|ab}^{(2)}\cdot\frac{T_{ia}}{\pi_{i|a}^{(2)}}\cdot\frac{T_{ib}}{\pi_{i|b}^{(2)}} \\
\Delta_{i|ab}^{(2)} \;&:=\; \begin{cases} \pi_{i|a}^{(2)}\left(1-\pi_{i|a}^{(2)}\right), & \text{if } a=b \\[2ex] \pi_{i|ab}^{(2)} - \pi_{i|a}^{(2)}\pi_{i|b}^{(2)}, & \text{if } a\neq b \end{cases} \\
V_{ia}^{(3)} \;&:=\; \mathrm{Var}^{(3)}\left(\,\widehat{T}_{ia}\;\middle|\; s^{(1)}, s^{(2)}\right)
\end{aligned}$$

Theorem 2.2

For a three-stage sampling design $p : \mathcal{S} \subset \mathcal{P}(U) \longrightarrow \mathbb{R}$ *with invariant and independent subsampling, let* $\widehat{T} : \mathcal{S} \longrightarrow \mathbb{R}$ *be the random variable defined as follows:*

$$\widehat{T}(s) \;:=\; \sum_{i\in s^{(1)}} \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \;=\; \sum_{i\in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a\in s_i^{(2)}} \frac{\widehat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \right) \;=\; \sum_{i\in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a\in s_i^{(2)}} \frac{1}{\pi_{i|a}^{(2)}} \left(\sum_{u\in s_{ia}^{(3)}} \frac{y_u}{\pi_{ia|u}^{(3)}} \right) \right)$$

Recall that $\widehat{T}$ *is an unbiased estimator of the population total*

$$T \;:=\; \sum_{i\in U^{(1)}} T_i \;:=\; \sum_{i\in U^{(1)}} \sum_{a\in U_{ia}^{(2)}} T_{ia} \;:=\; \sum_{i\in U^{(1)}} \sum_{a\in U_i^{(2)}} \sum_{u\in U_{ia}^{(3)}} y_u$$

Let $\widehat{\mathrm{Var}}\left[\,\widehat{T}\,\right] : \mathcal{S} \longrightarrow \mathbb{R}$ *be the random variable defined in a recursive manner as follows:*

$$\widehat{\mathrm{Var}}\left[\,\widehat{T}\,\right](s) \;:=\; \sum_{i\in s^{(1)}} \sum_{j\in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}} \cdot \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \cdot \frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} \;+\; \sum_{i\in s^{(1)}} \frac{\widehat{\mathrm{Var}}\left[\widehat{T}_i\right](s_i^{(2+)})}{\pi_i^{(1)}}$$

$$\widehat{\mathrm{Var}}\left[\widehat{T}_i\right](s_i^{(2+)}) \;:=\; \sum_{a\in s_i^{(2)}} \sum_{b\in s_i^{(2)}} \Delta_{i|ab}^{(2)} \cdot \frac{\widehat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \cdot \frac{\widehat{T}_{ib}(s_{ib}^{(3)})}{\pi_{i|b}^{(2)}} \;+\; \sum_{a\in s_i^{(2)}} \frac{\widehat{\mathrm{Var}}\left[\widehat{T}_{ia}\right](s_{ia}^{(3)})}{\pi_{i|a}^{(2)}}$$

$$\widehat{\mathrm{Var}}\left[\widehat{T}_{ia}\right](s_{ia}^{(3)}) \;:=\; \sum_{u\in s_{ia}^{(3)}} \sum_{v\in s_{ia}^{(3)}} \Delta_{ia|uv}^{(3)} \cdot \frac{y_u}{\pi_{ia|u}^{(3)}} \cdot \frac{y_v}{\pi_{ia|v}^{(3)}}$$

where

$$\Delta_{ij}^{(1)} \;:=\; \begin{cases} \pi_i^{(1)}\left(1-\pi_i^{(1)}\right), & \text{if } i=j \\ \pi_{ij}^{(1)} - \pi_i^{(1)}\pi_j^{(1)}, & \text{if } i\neq j \end{cases}$$

$$\Delta_{i|ab}^{(2)} \;:=\; \begin{cases} \pi_{i|a}^{(2)}\left(1-\pi_{i|a}^{(2)}\right), & \text{if } a=b \\ \pi_{i|ab}^{(2)} - \pi_{i|a}^{(2)}\pi_{i|b}^{(2)}, & \text{if } a\neq b \end{cases}$$

$$\Delta_{ia|uv}^{(3)} \;:=\; \begin{cases} \pi_{ia|u}^{(3)}\left(1-\pi_{ia|u}^{(3)}\right), & \text{if } u=v \\ \pi_{ia|uv}^{(3)} - \pi_{ia|u}^{(3)}\pi_{ia|v}^{(3)}, & \text{if } u\neq v \end{cases}$$

Then, $\widehat{\mathrm{Var}}\left[\,\widehat{T}\,\right]$ *is a design-unbiased estimator of the design variance* $\mathrm{Var}\left[\,\widehat{T}\,\right]$ *of the* $\widehat{T}$*.*

Corollary 2.3

For a three-stage sampling design with invariant and independent subsampling, the fully expanded expression for

$\widehat{\mathrm{Var}}\left[\,\widehat{T}\,\right]$ *is as follows:*

$$\begin{aligned}
\widehat{\mathrm{Var}}\left[\,\widehat{T}\,\right](s) \;&:=\; \sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta^{(1)}_{ij}}{\pi^{(1)}_{ij}}\cdot\frac{\widehat{T}_i(s^{(2+)}_i)}{\pi^{(1)}_i}\cdot\frac{\widehat{T}_j(s^{(2+)}_j)}{\pi^{(1)}_j} \;+\; \sum_{i\in s^{(1)}}\frac{\widehat{V}_i(s^{(2+)}_i)}{\pi^{(1)}_i} \\
&=\; \sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta^{(1)}_{ij}}{\pi^{(1)}_{ij}}\cdot\frac{\widehat{T}_i(s^{(2+)}_i)}{\pi^{(1)}_i}\cdot\frac{\widehat{T}_j(s^{(2+)}_j)}{\pi^{(1)}_j} \\
&\quad\; + \sum_{i\in s^{(1)}}\frac{1}{\pi^{(1)}_i}\left\{\sum_{a\in s^{(2)}_i}\sum_{b\in s^{(2)}_i}\Delta^{(2)}_{i|ab}\cdot\frac{\widehat{T}_{ia}(s^{(3)}_{ia})}{\pi^{(2)}_{i|a}}\cdot\frac{\widehat{T}_{ib}(s^{(3)}_{ib})}{\pi^{(2)}_{i|b}} + \sum_{a\in s^{(2)}_i}\frac{\widehat{V}_{ia}}{\pi^{(2)}_{i|a}}\right\} \\
&=\; \sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta^{(1)}_{ij}}{\pi^{(i)}_{ij}}\cdot\frac{\widehat{T}_i(s^{(2+)}_i)}{\pi^{(1)}_i}\cdot\frac{\widehat{T}_j(s^{(2+)}_j)}{\pi^{(1)}_j} \\
&\quad\; + \sum_{i\in s^{(1)}}\frac{1}{\pi^{(1)}_i}\left\{\sum_{a\in s^{(2)}_i}\sum_{b\in s^{(2)}_i}\Delta^{(2)}_{i|ab}\cdot\frac{\widehat{T}_{ia}(s^{(3)}_{ia})}{\pi^{(2)}_{i|a}}\cdot\frac{\widehat{T}_{ib}(s^{(3)}_{ib})}{\pi^{(2)}_{i|b}} + \sum_{a\in s^{(2)}_i}\frac{1}{\pi^{(2)}_{i|a}}\left(\sum_{u\in s^{(3)}_{ia}}\sum_{v\in s^{(3)}_{ia}}\Delta^{(3)}_{ia|uv}\frac{y_u}{\pi^{(3)}_{ia|u}}\frac{y_v}{\pi^{(3)}_{ia|v}}\right)\right\},
\end{aligned}$$

where

$$\widehat{T}_{ia}(s^{(3)}_{ia}) \;=\; \sum_{u\in s^{(3)}_{ia}}\frac{y_u}{\pi^{(3)}_{ia|u}} \quad\text{and}\quad \widehat{T}_i(s^{(2+)}_i) \;=\; \sum_{a\in s^{(2)}_i}\frac{\widehat{T}_{ia}(s^{(3)}_{ia})}{\pi^{(2)}_{i|a}} \;=\; \sum_{a\in s^{(2)}_i}\frac{1}{\pi^{(2)}_{i|a}}\left(\sum_{u\in s^{(3)}_{ia}}\frac{y_u}{\pi^{(3)}_{ia|u}}\right)$$

$$\Delta^{(3)}_{ia|uv} \;:=\; \begin{cases} \pi^{(3)}_{ia|u}\left(1-\pi^{(3)}_{ia|u}\right), & \text{if } u = v \\[2ex] \pi^{(3)}_{ia|uv} - \pi^{(3)}_{ia|u}\,\pi^{(3)}_{ia|v}, & \text{if } u \neq v \end{cases}$$