

1 The Prokhorov Theorem

Definition 1.1 (Tightness and weak sequential compactness)

Suppose:

- (S, ρ) is a metric space, $\mathcal{B}(S)$ its the Borel σ -algebra, $(S, \mathcal{B}(S))$ is the corresponding measurable space.
- $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is a family of probability measures on $(S, \mathcal{B}(S))$.

The family Π is said to be:

- (i) **tight** if, for each $\varepsilon > 0$, there exists a compact subset $K_\varepsilon \subset S$ such that

$$1 - \varepsilon < P(K_\varepsilon) \leq 1, \quad \text{for each } P \in \Pi.$$

- (ii) **weakly sequentially compact** if, for every sequence $\{P_n\}_{n \in \mathbb{N}} \subset \Pi$, there exists a probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ and subsequence $\{P_{n(i)}\}_{i \in \mathbb{N}}$ such that

$$P_{n(i)} \xrightarrow{w} P, \quad \text{as } i \rightarrow \infty.$$

Theorem 1.2 (The Prokhorov Theorem, Theorems 5.1 & 5.2, [2])

Suppose:

- (S, ρ) is a metric space, $\mathcal{B}(S)$ its the Borel σ -algebra, $(S, \mathcal{B}(S))$ is the corresponding measurable space.
- $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is a collection of probability measures on $(S, \mathcal{B}(S))$.

Then, the following statements hold:

- (i) Tightness of Π implies weak sequential compactness of Π .
- (ii) Suppose further that (S, ρ) is complete and separable.
Then, weak sequential compactness of Π implies tightness of Π .

PROOF We first prove statement (ii), then statement (i).

Proof of (ii)

Suppose S is complete and separable. Let $\varepsilon > 0$ be fixed. We need to find a compact subset $K \subset S$ such that

$$1 - \varepsilon < P(K) \leq 1, \quad \text{for each } P \in \Pi.$$

Now, separability of S implies that every open cover of every subset of S admits a countable subcover (Appendix M3, [2]). Denote by $B(x, r) \subset S$ the open ball in S centred at $x \in S$ of radius $r > 0$. For each $k \in \mathbb{N}$, the open cover

$$\left\{ B\left(x, \frac{1}{k}\right) \right\}_{x \in S}$$

of S admits a countable subcover, say,

$$\{A_{ki}\}_{i \in \mathbb{N}} \subset \left\{ B\left(x, \frac{1}{k}\right) \right\}_{x \in S}.$$

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Let $G_{kn} := \bigcup_{i=1}^n A_{ki}$. Then, each G_{kn} is an open subset of S and $G_{kn} \uparrow S$, as $n \rightarrow \infty$. Hence, by the Claim below, there exists $n_k \in \mathbb{N}$ such that

$$1 - \frac{\varepsilon}{2^k} < P\left(\bigcup_{i=1}^{n_k} A_{ki}\right) \leq 1, \quad \text{for each } P \in \Pi.$$

Now, let

$$K := \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}}$$

Note that K , being a closed subset of the complete metric space S , is itself complete. Note also that the set $\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}$ is totally bounded; hence so is its closure K . Being complete and totally bounded, K is therefore compact (Appendix M5, [2]). It now remains only to show that $1 - \varepsilon < P(K) \leq 1$, for each $P \in \Pi$; or equivalently, that $P(K^c) \leq \varepsilon$, for each $P \in \Pi$. To this end, write $B_k := \bigcup_{i=1}^{n_k} A_{ki}$. Then,

$$1 - \frac{\varepsilon}{2^k} < P(B_k) \leq 1; \quad \text{equivalently, } P(B_k^c) \leq \frac{\varepsilon}{2^k}.$$

Also,

$$K := \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}} := \overline{\bigcap_{k=1}^{\infty} B_k} \supset \bigcap_{k=1}^{\infty} B_k.$$

Hence,

$$K^c \subset \left(\bigcap_{k=1}^{\infty} B_k\right)^c = \bigcup_{k=1}^{\infty} B_k^c,$$

which implies:

$$P(K^c) \leq \sum_{k=1}^{\infty} P(B_k^c) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Thus, the proof of (ii) will be complete once we prove the following:

Claim: Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of open subsets of S with $G_n \uparrow S$. Then, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$1 - \varepsilon < P(G_{n_\varepsilon}) \leq 1, \quad \text{for each } P \in \Pi.$$

Proof of Claim: Suppose the Claim is false, and we derive a contradiction. The failure of the Claim implies that there exists some $0 < \varepsilon < 1$ such that for each $n \in \mathbb{N}$, there exists $P_n \in \Pi$ such that

$$P_n(G_n) \leq 1 - \varepsilon.$$

By the hypothesis of weak sequential compactness of Π , there exists some probability measure $Q \in \mathcal{M}_1(S, \mathcal{B}(S))$ and the subsequence $\{P_{n(i)}\}$ of $\{P_n\}$ such that $P_{n(i)} \xrightarrow{w} Q$, as $i \rightarrow \infty$. Now, for each fixed $n \in \mathbb{N}$, we have:

$$\begin{aligned} Q(G_n) &\leq \liminf_{i \rightarrow \infty} P_{n(i)}(G_n), \quad \text{by the Portmanteau Theorem} \\ &\leq \liminf_{i \rightarrow \infty} P_{n(i)}(G_{n(i)}), \quad \text{since } \{G_n\} \text{ is increasing} \\ &\leq 1 - \varepsilon, \quad \text{by choice of } P_n \end{aligned}$$

But, by hypothesis, we also have $G_n \uparrow S$. Hence, we therefore have:

$$1 = Q(S) = \lim_{n \rightarrow \infty} Q(G_n) \leq 1 - \varepsilon,$$

which is the desired contradiction. This completes the proof of the Claim, hence that of (ii).

Proof of (i)

Suppose $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is tight. We need to establish that Π is weakly sequentially compact. In other words, if $\{P_n\} \subset \Pi$ is a sequence of probability measures contained in Π , we need to establish that there exists a Borel probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ and a subsequence $\{P_{n(i)}\} \subset \{P_n\}$ such that $P_{n(i)} \xrightarrow{w} P$, as $i \rightarrow \infty$.

So, let $\{P_n\} \subset \Pi$. We prove the Theorem by establishing the following series of Claims. Note that the proof of the Theorem is complete once we establish Claim 8.

Claim 1: There exists an increasing sequence of compact subsets $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ such that

$$1 - \frac{1}{m} < P_n(K_m) \leq 1, \quad \text{for every } m, n \in \mathbb{N}.$$

Claim 2: Let $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ be one such sequence of compact subsets of S as in Claim 1. Then, the following statements are true:

- (a) $\Sigma := \bigcup_{m=1}^{\infty} K_m$ is a separable subset of S .
- (b) There exists a countable collection \mathcal{A} of open subsets of S satisfying the following property:
For each $x \in S$ and for each open subset G of S ,

$$x \in G \cap \left(\bigcup_{m=1}^{\infty} K_m \right) \implies x \in A \subset \bar{A} \subset G, \text{ for some } A \in \mathcal{A}.$$

- (c) The collection \mathcal{A} is an open cover of Σ .

Claim 3: Define:

$$\mathcal{H} := \{\emptyset\} \cup \left\{ \begin{array}{l} \text{all finite unions of sets of the form} \\ \bar{A} \cap K_m, \text{ where } A \in \mathcal{A} \text{ and } m \in \mathbb{N} \end{array} \right\}.$$

Then, the following statements are true:

- (a) $K_m \in \mathcal{H}$, for each $m \in \mathbb{N}$.
- (b) There exists a subsequence $\{P_{n(i)}\} \subset \{P_n\}$ such that the limit

$$\lim_{i \rightarrow \infty} P_{n(i)}(H) \text{ exists, for each } H \in \mathcal{H}.$$

We may therefore define the following function:

$$\alpha : \mathcal{H} \longrightarrow [0, 1] : H \longmapsto \lim_{i \rightarrow \infty} P_{n(i)}(H).$$

Claim 4: The function $\alpha : \mathcal{H} \rightarrow [0, 1]$ satisfies the following properties:

- (a) $\alpha(\emptyset) = 0$.
- (b) monotonicity: $\alpha(H_1) \leq \alpha(H_2)$, for any $H_1, H_2 \in \mathcal{H}$ with $H_1 \subset H_2$.
- (c) finite additivity for disjoint sets:
 $\alpha(H_1 \cup H_2) = \alpha(H_1) + \alpha(H_2)$, for any $H_1, H_2 \in \mathcal{H}$ with $H_1 \cap H_2 = \emptyset$.
- (d) finite sub-additivity: $\alpha(H_1 \cup H_2) \leq \alpha(H_1) + \alpha(H_2)$, for any $H_1, H_2 \in \mathcal{H}$.

Claim 5: Let $\mathcal{O}(S)$ denote the collection of all open subsets of S . Define the following function:

$$\beta : \mathcal{O}(S) \rightarrow [0, 1] : G \mapsto \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset G \end{array} \right\}.$$

(Note that the supremum above is always taken over a non-empty set: For each open $G \subset S$, the set $\{H \in \mathcal{H} \mid H \subset G\}$ is non-empty, since $\emptyset \in \mathcal{H}$.)

Next, let $\mathcal{P}(S)$ be the power set of S , i.e. the collection of all subsets of S . Define the following function:

$$\gamma : \mathcal{P}(S) \rightarrow [0, 1] : W \mapsto \inf \left\{ \beta(G) \in [0, 1] \mid \begin{array}{l} G \in \mathcal{O}(S), \text{ and} \\ W \subset G \end{array} \right\}.$$

Then, the function $\gamma : \mathcal{P}(S) \rightarrow [0, 1]$ is an outer measure defined on S .

Claim 6: The σ -algebra $\mathcal{A}(\gamma)$ of γ -measurable subsets of S contains the Borel σ -algebra $\mathcal{B}(S)$ of S .

Claim 7: The restriction $P := \gamma|_{\mathcal{B}(S)}$ of γ to $\mathcal{B}(S)$ is a Borel probability measure which satisfies:

$$P(G) = \gamma(G) = \beta(G) := \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H}, \\ H \subset G \end{array} \right\}, \text{ for each open subset } G \subset S.$$

Claim 8: $P_{n(i)} \xrightarrow{w} P$, as $i \rightarrow \infty$.

Proof of Claim 1: By tightness hypothesis on Π , for each $m \in \mathbb{N}$, there exists a compact subset $L_m \subset S$ such that

$$1 - \frac{1}{m} < P(L_m) \leq 1, \text{ for each } P \in \Pi.$$

Define, for each $m \in \mathbb{N}$, $K_m := \bigcup_{i=1}^m L_i$. Then, each K_m is compact (since finite unions of compact subsets are themselves compact). Next, we trivially have $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$. Also,

$$P(K_m) = P\left(\bigcup_{i=1}^m L_i\right) \geq P(L_m) > 1 - \frac{1}{m}, \text{ for each } P \in \Pi.$$

In particular, the above inequality holds for each P_n . This proves Claim 1.

Proof of Claim 2: Separability of $\Sigma := \bigcup_{m=1}^{\infty} K_m$ is an immediate consequence of Lemma A.4 and Lemma A.5. Then, the existence of \mathcal{A} follows immediately from the separability of Σ and Lemma A.6.

It remains to show that \mathcal{A} is an open cover of Σ . To this end, first note that the collection $\mathcal{O}(S)$ of open subsets of S forms an open cover of S , hence $\mathcal{O}(S)$ is also an open cover of Σ . Therefore, for each $x \in \Sigma$, we may choose $G_x \in \mathcal{O}(S)$ such that $x \in G_x$. Thus, $x \in \Sigma \cap G_x$, and by properties of \mathcal{A} , we may furthermore choose $A_x \in \mathcal{A}$ such that $x \in A_x \subset \overline{A_x} \subset G_x$. We thus see that the collection

$$\left\{ A_x \in \mathcal{A} \mid x \in \Sigma \right\} \subset \mathcal{A}$$

is an open cover of Σ consisting of subsets in \mathcal{A} . This completes the proof of Claim 2.

Proof of Claim 3:

- (a) By Claim 2, \mathcal{A} is an open cover of $\Sigma := \bigcup_{m=1}^{\infty} K_m$. In particular, \mathcal{A} is an open cover of K_m for each $m \in \mathbb{N}$. Compactness of K_m implies that \mathcal{A} admits a finite subcover of K_m . Thus we have

$$K_m \subset \bigcup_{i=1}^{J_m} A_i^{(m)} \subset \bigcup_{i=1}^{J_m} \overline{A_i^{(m)}}, \quad \text{for some } A_1^{(m)}, A_2^{(m)}, \dots, A_{J_m}^{(m)} \in \mathcal{A},$$

which implies

$$K_m = K_m \cap \left(\bigcup_{i=1}^{J_m} \overline{A_i^{(m)}} \right) = \bigcup_{i=1}^{J_m} \left(K_m \cap \overline{A_i^{(m)}} \right) \in \mathcal{H}.$$

- (b) Note that \mathcal{H} is a countable collection of subsets of S . Let $\mathcal{H} = \{H_1, H_2, H_3, \dots\}$ be an enumeration of \mathcal{H} . Consider the following array of real numbers:

$$\begin{array}{cccc} P_1(H_1) & P_2(H_1) & P_3(H_1) & \cdots \\ P_1(H_2) & P_2(H_2) & P_3(H_2) & \cdots \\ P_1(H_3) & P_2(H_3) & P_3(H_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Note that each row of the above array is bounded between 0 and 1. Hence, by Theorem A.7, there exists an increasing sequence

$$n(1) < n(2) < n(3) < \cdots \in \mathbb{N}$$

of natural numbers such that the limit

$$\lim_{k \rightarrow \infty} P_{n(k)}(H_r), \quad \text{exists for each } r \in \mathbb{N}.$$

This completes the proof of Claim 3.

Proof of Claim 4:

- (a) Obviously, $P_{n(i)}(\emptyset) = 0$, for each $i \in \mathbb{N}$. Hence,

$$\alpha(\emptyset) = \lim_{i \rightarrow \infty} P_{n(i)}(\emptyset) = \lim_{i \rightarrow \infty} (0) = 0.$$

- (b) For $H_1, H_2 \in \mathcal{H}$ with $H_1 \subset H_2$, we have $P_{n(i)}(H_1) \leq P_{n(i)}(H_2)$, for each $i \in \mathbb{N}$. Hence,

$$\alpha(H_1) := \lim_{i \rightarrow \infty} P_{n(i)}(H_1) \leq \lim_{i \rightarrow \infty} P_{n(i)}(H_2) =: \alpha(H_2).$$

(c) For $H_1, H_2 \in \mathcal{H}$ with $H_1 \cap H_2 = \emptyset$, we have $P_{n(i)}(H_1 \cup H_2) = P_{n(i)}(H_1) + P_{n(i)}(H_2)$, for each $i \in \mathbb{N}$. Hence,

$$\begin{aligned} \alpha(H_1 \cup H_2) &= \lim_{i \rightarrow \infty} P_{n(i)}(H_1 \cup H_2) \\ &= \lim_{i \rightarrow \infty} [P_{n(i)}(H_1) + P_{n(i)}(H_2)] \\ &= \lim_{i \rightarrow \infty} P_{n(i)}(H_1) + \lim_{i \rightarrow \infty} P_{n(i)}(H_2) = \alpha(H_1) + \alpha(H_2). \end{aligned}$$

(d) For $H_1, H_2 \in \mathcal{H}$, we have $P_{n(i)}(H_1 \cup H_2) \leq P_{n(i)}(H_1) + P_{n(i)}(H_2)$, for each $i \in \mathbb{N}$. Hence,

$$\begin{aligned} \alpha(H_1 \cup H_2) &= \lim_{i \rightarrow \infty} P_{n(i)}(H_1 \cup H_2) \\ &\leq \lim_{i \rightarrow \infty} [P_{n(i)}(H_1) + P_{n(i)}(H_2)] \\ &= \lim_{i \rightarrow \infty} P_{n(i)}(H_1) + \lim_{i \rightarrow \infty} P_{n(i)}(H_2) = \alpha(H_1) + \alpha(H_2). \end{aligned}$$

Proof of Claim 5:

Claim 5A: For any subsets $F, G \subset S$, where F is closed and G is open, we have

$$\left. \begin{array}{l} F \subset G, \text{ and} \\ \text{there exists } H \in \mathcal{H} \text{ such that } F \subset H \end{array} \right\} \implies \text{there exists } H_0 \in \mathcal{H} \text{ such that } F \subset H_0 \subset G$$

Proof of Claim 5A: For each $x \in F$, choose $A_x \in \mathcal{A}$ such that $x \in A_x \subset \overline{A_x} \subset G$. Recall that $H \in \mathcal{H}$ is compact, hence F is also compact (being a closed subset of the compact set H). Therefore, the open cover $\{A_x\}_{x \in F}$ of F admits a finite sub-cover, say $\{A_{x_1}, A_{x_2}, \dots, A_{x_k}\}$. Secondly, note that $F \subset K_m$, for some $m \in \mathbb{N}$. Let $H_0 := \bigcup_{i=1}^k (\overline{A_{x_i}} \cap K_m)$. It is clear that $H_0 \in \mathcal{H}$. Furthermore,

$$F \subset \left(\bigcup_{i=1}^k \overline{A_{x_i}} \right) \cap K_m = \underbrace{\bigcup_{i=1}^k (\overline{A_{x_i}} \cap K_m)}_{H_0} \subset \left(\bigcup_{i=1}^k \overline{A_{x_i}} \right) \subset G$$

This proves Claim 5A.

Claim 5B: $\beta : \mathcal{O}(S) \rightarrow [0, 1]$ is finitely subadditive.

Proof of Claim 5B: Suppose $G_1, G_2 \subset S$ are two arbitrary open subsets of S . We need to show:

$$\beta(G_1 \cup G_2) \leq \beta(G_1) + \beta(G_2).$$

Recall that

$$\beta(G_1 \cup G_2) := \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset G_1 \cup G_2 \end{array} \right\}.$$

Now, for each $H \in \mathcal{H}$ with $H \subset G_1 \cup G_2$, define

$$F_1 := \left\{ x \in H \mid \rho(x, G_1^c) \geq \rho(x, G_2^c) \right\}, \text{ and } F_2 := \left\{ x \in H \mid \rho(x, G_1^c) \leq \rho(x, G_2^c) \right\}.$$

We immediately have $H = F_1 \cup F_2$. Note that $F_1 \subset G_1$. Indeed,

$$x \in F_1 \cap G_1^c \implies \left\{ \begin{array}{l} x \in H \cap G_1^c \subset G_2, \text{ and} \\ \rho(x, G_2^c) \leq \rho(x, G_1^c) = 0 \end{array} \right. \implies 0 < \rho(x, G_2^c) \leq \rho(x, G_1^c) = 0, \text{ a contradiction.}$$

This proves indeed that $F_1 \subset G_1$. Similarly, we have $F_2 \subset G_2$. Now,

$$\left. \begin{array}{l} F_1 \subset H, H \in \mathcal{H} \\ \text{Claim 5A} \end{array} \right\} \implies F_1 \subset H_1 \subset G_1, \text{ for some } H_1 \in \mathcal{H}.$$

Similarly, $F_2 \subset H_2 \subset G_2$, for some $H_2 \in \mathcal{H}$. We thus have $H = F_1 \cup F_2 \subset H_1 \cup H_2$. By Claim 4 and definition of $\beta : \mathcal{O}(S) \rightarrow [0, 1]$, we therefore have

$$\alpha(H) \leq \alpha(H_1 \cup H_2) \leq \alpha(H_1) + \alpha(H_2) \leq \beta(G_1) + \beta(G_2).$$

Recall that $H \subset G_1 \cup G_2$ and $H \in \mathcal{H}$, but otherwise H is arbitrary; hence, we have

$$\beta(G_1 \cup G_2) := \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset G_1 \cup G_2 \end{array} \right\} \leq \beta(G_1) + \beta(G_2).$$

This proves Claim 5B.

Claim 5C: $\beta : \mathcal{O}(S) \rightarrow [0, 1]$ is countably subadditive.

Proof of Claim 5C: Let $G_1, G_2, \dots \subset S$ be open subsets of S . We need to show:

$$\beta\left(\bigcup_{i=1}^{\infty} G_i\right) \leq \sum_{i=1}^{\infty} \beta(G_i).$$

Recall that

$$\beta\left(\bigcup_{i=1}^{\infty} G_i\right) := \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset \bigcup_{i=1}^{\infty} G_i \end{array} \right\}.$$

Now, for any $H \in \mathcal{H}$ with $H \subset \bigcup_{i=1}^{\infty} G_i$, compactness of H implies $H \subset \bigcup_{i=1}^k G_i$, for some $k \in \mathbb{N}$. Therefore,

$$\alpha(H) \leq \sup \left\{ \alpha(H') \in [0, 1] \mid \begin{array}{l} H' \in \mathcal{H}, \text{ and} \\ H' \subset \bigcup_{i=1}^k G_i \end{array} \right\} =: \beta\left(\bigcup_{i=1}^k G_i\right) \leq \sum_{i=1}^k \beta(G_i) \leq \sum_{i=1}^{\infty} \beta(G_i),$$

which in turn immediately implies

$$\beta\left(\bigcup_{i=1}^{\infty} G_i\right) := \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset \bigcup_{i=1}^{\infty} G_i \end{array} \right\} \leq \sum_{i=1}^{\infty} \beta(G_i).$$

This proves Claim 5C.

Claim 5D: $\gamma : \mathcal{P}(S) \rightarrow [0, 1]$ is outer measure on S .

It is trivial to see that $\gamma(\emptyset) = 0$. We next establish the monotonicity of γ . Let W_1 and W_2 be arbitrary subsets of S with $W_1 \subset W_2 \subset S$. Then,

$$\{G \in \mathcal{O}(S) \mid W_1 \subset G\} \supset \{G \in \mathcal{O}(S) \mid W_2 \subset G\},$$

which implies

$$\gamma(W_1) := \inf \left\{ \beta(G) \in [0, 1] \mid \begin{array}{l} G \in \mathcal{O}(S) \\ W_1 \subset G \end{array} \right\} \leq \inf \left\{ \beta(G) \in [0, 1] \mid \begin{array}{l} G \in \mathcal{O}(S) \\ W_2 \subset G \end{array} \right\} =: \gamma(W_2).$$

This proves the monotonicity of γ . It remains only to show that γ is countably subadditive. To this end, let $W_1, W_2, \dots \subset S$ be arbitrary subsets of S . We need to show that

$$\gamma\left(\bigcup_{i=1}^{\infty} W_i\right) \leq \sum_{i=1}^{\infty} \gamma(W_i).$$

Now, let $\varepsilon > 0$ be given. By definition γ , we may choose, for each $i \in \mathbb{N}$, an open subset $G_i \subset S$ such that

$$W_i \subset G_i \quad \text{and} \quad \gamma(W_i) \leq \beta(G_i) < \gamma(W_i) + \frac{\varepsilon}{2^i}.$$

The fact that $\bigcup_{i=1}^{\infty} W_i \subset \bigcup_{i=1}^{\infty} G_i$ and the countable subadditivity of β (Claim 5C) together imply

$$\begin{aligned} \gamma\left(\bigcup_{i=1}^{\infty} W_i\right) &:= \inf \left\{ \beta(G) \in [0, 1] \mid \begin{array}{l} G \in \mathcal{O}(S), \text{ and} \\ \bigcup_{i=1}^{\infty} W_i \subset G \end{array} \right\} \leq \beta\left(\bigcup_{i=1}^{\infty} G_i\right) \\ &\leq \sum_{i=1}^{\infty} \beta(G_i) \leq \sum_{i=1}^{\infty} \left(\gamma(W_i) + \frac{\varepsilon}{2^i} \right) \leq \left(\sum_{i=1}^{\infty} \gamma(W_i) \right) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, countable subadditivity of γ now follows. This completes the proof of Claim 5D.

Proof of Claim 6: First, note that

$$\gamma(F \cap G) + \gamma(F^c \cap G) \leq \beta(G), \quad \text{for each closed } F \subset S \text{ and each open } G \subset S. \quad (1.1)$$

Indeed, let $\varepsilon > 0$ be given. Since $F^c \cap G \in \mathcal{O}(S)$, by definition of β as a supremum, we may choose $H_1 \in \mathcal{H}$ such that

$$H_1 \subset F^c \cap G \quad \text{and} \quad \beta(F^c \cap G) - \varepsilon < \alpha(H_1) \leq \beta(F^c \cap G).$$

The first inclusion immediately implies that $F \cap G \subset H_1^c \cap G$. Now, recall that $H_1 \subset S$ is a closed subset; hence, $H_1^c \cap G$ is open. Thus, we may choose $H_0 \in \mathcal{H}$ such that

$$H_0 \subset H_1^c \cap G \quad \text{and} \quad \beta(H_1^c \cap G) - \varepsilon < \alpha(H_0) \leq \beta(H_1^c \cap G).$$

Since $H_0 \cap H_1 = \emptyset$, $H_0 \cup H_1 \subset G$, and $F \cap G \subset H_1^c \cap G$, we have

$$\begin{aligned} \beta(G) &\geq \alpha(H_0 \cup H_1) = \alpha(H_0) + \alpha(H_1) \\ &> \beta(H_1^c \cap G) - \varepsilon + \beta(F^c \cap G) - \varepsilon \\ &\geq \gamma(F \cap G) + \gamma(F^c \cap G) - 2\varepsilon, \end{aligned}$$

which implies $\beta(G) \geq \gamma(F \cap G) + \gamma(F^c \cap G)$, since $\varepsilon > 0$ is arbitrary.

Now, let $F \subset S$ be an arbitrary closed subset of S , and $W \subset S$ be an arbitrary subset of S . Then, by (1.1), we have

$$\gamma(F \cap W) + \gamma(F^c \cap W) \leq \gamma(F \cap G) + \gamma(F^c \cap G) \leq \beta(G), \quad \text{for each open } G \subset S \text{ with } W \subset G.$$

Hence,

$$\gamma(F \cap W) + \gamma(F^c \cap W) \leq \inf \left\{ \beta(G) \mid \begin{array}{l} G \in \mathcal{O}(S) \\ W \subset G \end{array} \right\} =: \gamma(W).$$

This proves the γ -measurability of each closed subset $F \subset S$, and hence the γ -measurability of each open subset of S , i.e. $\mathcal{O}(S) \subset \mathcal{A}(\gamma)$. It now follows immediately that

$$\mathcal{B}(S) = \sigma(\mathcal{O}(S)) \subset \mathcal{A}(\gamma).$$

This completes the proof of Claim 6.

Proof of Claim 7: By Claim 3, $K_m \in \mathcal{H}$ for each $m \in \mathbb{N}$. Thus,

$$\alpha(K_m) := \lim_{i \rightarrow \infty} P_{n(i)}(K_m) \geq 1 - \frac{1}{m}.$$

We therefore have

$$1 \geq P(S) := \beta(S) := \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H} \\ H \subset S \end{array} \right\} \geq \sup_{m \in \mathbb{N}} \left\{ \alpha(K_m) \right\} \geq \sup_{m \in \mathbb{N}} \left\{ 1 - \frac{1}{m} \right\} = 1,$$

which proves that P is indeed a probability measure on $(S, \mathcal{B}(S))$. Lastly, note that it is immediate by the definitions of $P : \mathcal{B}(S) \rightarrow [0, 1]$, $\gamma : \mathcal{P}(S) \rightarrow [0, 1]$, and $\beta : \mathcal{O}(S) \rightarrow [0, 1]$ that $P(G) = \gamma(G) = \beta(G)$, for each open subset G of S . This completes the proof of Claim 7.

Proof of Claim 8: Let $G \subset S$ be an arbitrary open subset of S . Then, we have

$$\alpha(H) := \lim_{i \rightarrow \infty} P_{n(i)}(H) \leq \liminf_{i \rightarrow \infty} P_{n(i)}(G), \quad \text{for each } H \in \mathcal{H} \text{ with } H \subset G.$$

The preceding inequality and Claim 7 together imply:

$$P(G) = \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H} \\ H \subset G \end{array} \right\} \leq \liminf_{i \rightarrow \infty} P_{n(i)}(G), \quad \text{for each open subset } G \subset S,$$

which is equivalent to the weak convergence $P_{n(i)} \xrightarrow{w} P$, as $i \rightarrow \infty$, by the Portmanteau Theorem (Theorem 2.1, [2]). This completes the proof of Claim 8. \square

A Technical Lemmas

Definition A.1

Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω . An **outer measure** on Ω is a function $\varphi : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ satisfying the following conditions:

- $\varphi(\emptyset) = 0$.
- *monotonicity*: $\varphi(A) \leq \varphi(B)$, for every $A, B \in \mathcal{P}(\Omega)$ with $A \subset B$.
- *countable sub-additivity*:

$$\varphi\left(\bigcap_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \varphi(A_i), \quad \text{for any } A_1, A_2, \dots \in \mathcal{P}(\Omega).$$

Definition A.2

Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω . Let $\varphi : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ be an outer measure on Ω . A subset $A \subset \Omega$ is said to be φ -measurable if

$$\varphi(E) = \varphi(A \cap E) + \varphi(A^c \cap E), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

Theorem A.3

Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω . Let $\varphi : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ be an outer measure on Ω .

- (i) A subset $A \subset \Omega$ is φ -measurable if and only if

$$\varphi(E) \geq \varphi(A \cap E) + \varphi(A^c \cap E), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

- (ii) The collection $\mathcal{A}(\varphi)$ of φ -measurable subsets of Ω forms a σ -algebra of subsets of Ω .
- (iii) The restriction $\varphi|_{\mathcal{A}(\varphi)}$ of the outer measure φ to the σ -algebra $\mathcal{A}(\varphi)$ is a (countably additive) complete measure on the measurable space $(\Omega, \mathcal{A}(\varphi))$.

Lemma A.4

Every compact subset of a metric space is also a separable subset of that metric space.

PROOF Let (X, ρ) be a metric space and $K \subset X$ be a compact subset of X . For each $x \in X$ and positive $r > 0$, let

$$B(x, r) := \{y \in X \mid \rho(x, y) < r\} \subset X,$$

i.e. $B(x, r)$ is the open ball in X centred at x with radius $r > 0$. For each $n \in \mathbb{N}$, the following forms an open cover of K :

$$\mathcal{C}_n := \left\{ B\left(x, \frac{1}{n}\right) \subset X \mid x \in K \right\}.$$

Since K is compact, each \mathcal{C}_n admits a finite subcover:

$$\mathcal{F}_n := \left\{ B\left(x_i^{(n)}, \frac{1}{n}\right) \subset X \mid x_i^{(n)} \in K, \ i = 1, 2, \dots, J_n \right\}.$$

Let

$$\mathcal{D}_n := \left\{ x_i^{(n)} \in K \mid i = 1, 2, \dots, J_n \right\} \subset K,$$

and let $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset K$. We claim that \mathcal{D} is dense in K . Indeed, let $y \in K$. Since each \mathcal{F}_n is a (finite) open cover of K , we have:

$$y \in K \subset \bigcup_{i=1}^{J_n} B\left(x_i^{(n)}, \frac{1}{n}\right), \quad \text{for each } n \in \mathbb{N}.$$

Since $x_i^{(n)} \in \mathcal{D}$, for each $i = 1, 2, \dots, J_n$ and for each $n \in \mathbb{N}$, the above inclusion shows that, for each $n \in \mathbb{N}$, there exists some $x \in \mathcal{D}$ such that $\rho(y, x) < \frac{1}{n}$. In particular, \mathcal{D} contains a sequence that converges to $y \in K$. Since $y \in K$ is an arbitrary element of K , we see that $\overline{\mathcal{D}} \supset K$. Since $\mathcal{D} \subset K$ and K is compact, hence closed, we trivially have $\overline{\mathcal{D}} \subset K$. We may now conclude that $\overline{\mathcal{D}} = K$. This completes the proof of the Lemma. \square

Lemma A.5

Every countable union of separable subsets of a metric space is itself a separable subset of that metric space.

PROOF Let $S := \bigcup_{i=1}^{\infty} S_i \subset X$ be a countable union of separable subsets S_i of a metric space X . For each fixed $i \in \mathbb{N}$, since S_i is separable, there exists countable $D_i \subset S_i$ which is dense in S_i . Let $D := \bigcup_{i=1}^{\infty} D_i$. Then, D is a countable subset of S . The Lemma is proved once we establish that D is dense in S . To this end, let $x \in S = \bigcup_{i=1}^{\infty} S_i$. Then, $x \in S_i$ for some $i \in \mathbb{N}$. Since D_i is dense in S_i , there exists a sequence $\{y_k\} \subset D_i \subset D$ such that $y_k \rightarrow x$, as $k \rightarrow \infty$. This proves that D is indeed dense in S , and completes the proof of the Lemma. \square

Lemma A.6 (second theorem in Appendix M3, [2])

Let (S, ρ) be a metric space and $\Sigma \subset S$ a separable subset of S . Then, there exists a countable collection \mathcal{A} of open subsets of S satisfying the following property: For each $x \in S$ and each open subset G of S ,

$$x \in G \cap \Sigma \implies x \in A \subset \overline{A} \subset G, \quad \text{for some } A \in \mathcal{A}.$$

PROOF Let $D \subset \Sigma$ be a countable dense subset of Σ . Let

$$\mathcal{A} := \left\{ B(d, r) \subset S \mid \begin{array}{l} d \in D, \\ r \in \mathbb{Q}, r > 0 \end{array} \right\}.$$

Then, \mathcal{A} is a countable collection of open balls in S . Now, let $G \subset S$ be an arbitrary open subset of S and $x \in G \cap \Sigma$. First, choose $\varepsilon > 0$ such that $B(x, \varepsilon) \subset G$. Next, since $x \in \Sigma$ and D is dense in Σ , we may choose $d \in D$ such that $d \in B(x, \varepsilon/2)$, or equivalently $\rho(x, d) < \varepsilon/2$. Finally choose positive rational $r > 0$ such that $\rho(x, d) < r < \varepsilon/2$.

Now, note that $\overline{B(d, r)} \subset B(x, \varepsilon)$; indeed,

$$y \in \overline{B(d, r)} \iff \rho(y, d) \leq r \implies \rho(x, y) \leq \rho(x, d) + \rho(d, y) < \varepsilon/2 + r < \varepsilon/2 + \varepsilon/2 \implies y \in B(x, \varepsilon).$$

Thus, we have

$$x \in B(d, r) \subset \overline{B(d, r)} \subset B(x, \varepsilon) \subset G.$$

This completes the proof of the Lemma. \square

Theorem A.7 (The Diagonal Method, Appendix A.14, [1])

The Prokhorov Theorem

Suppose that each row of the array

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

is a bounded sequence of real numbers. Then, there exists an increasing sequence

$$n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$$

of positive integers such that the limit

$$\lim_{k \rightarrow \infty} x_{r,n_k} \text{ exists, for each } r = 1, 2, 3, \dots$$

PROOF From the first row, select a convergent subsequence

$$x_{1,n(1,1)}, x_{1,n(1,2)}, x_{1,n(1,3)}, \dots$$

Here, we have $n(1,1) < n(1,2) < n(1,3) < \cdots \in \mathbb{N}$, and $\lim_{k \rightarrow \infty} x_{1,n(1,k)} \in \mathbb{R}$ exists. Next, note that the following subsequence of the second row:

$$x_{2,n(1,1)}, x_{2,n(1,2)}, x_{2,n(1,3)}, \dots$$

is still a bounded sequence of real numbers, and we may thus select a convergent subsequence:

$$x_{2,n(2,1)}, x_{2,n(2,2)}, x_{2,n(2,3)}, \dots$$

Here, we have $n(2,1) < n(2,2) < n(2,3) < \cdots \in \{n(1,k)\}_{k \in \mathbb{N}}$, and $\lim_{k \rightarrow \infty} x_{2,n(2,k)} \in \mathbb{R}$ exists. Continuing inductively, we obtain an array of positive integers

$$\begin{array}{cccc} n(1,1) & n(1,2) & n(1,3) & \cdots \\ n(2,1) & n(2,2) & n(2,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

which satisfies: For each $r \in \mathbb{N}$, we have

- each row is an increasing sequence of positive integers, i.e. $n(r,1) < n(r,2) < n(r,3) < \cdots$,
- the $(r+1)^{\text{th}}$ row is a subsequence of the r^{th} row, i.e. $\{n(r+1,k)\}_{k \in \mathbb{N}} \subset \{n(r,k)\}_{k \in \mathbb{N}}$, and
- $\lim_{k \rightarrow \infty} x_{r,n(r,k)} \in \mathbb{R}$ exists.

Note that the first two properties together imply:

$$n(k,k) < n(k,k+1) \leq n(k+1,k+1), \text{ for each } k \in \mathbb{N}.$$

Now, define $n_k := n(k,k)$, for $k \in \mathbb{N}$. We then see that

$$n_k := n(k,k) < n(k+1,k+1) =: n_{k+1},$$

i.e., $\{n_k\}_{k \in \mathbb{N}}$ is a strictly increasing sequence of positive integers. Lastly, for each $r \in \mathbb{N}$, consider the sequence

$$x_{r,n_1}, x_{r,n_2}, x_{r,n_3}, \dots$$

Note that, for each $r \in \mathbb{N}$,

$$x_{r,n_r}, x_{r,n_{r+1}}, x_{r,n_{r+2}}, \dots$$

is a subsequence of $\{x_{r,n(r,k)}\}_{k \in \mathbb{N}}$. We saw above that $\lim_{k \rightarrow \infty} x_{r,n(r,k)}$ exists, which in turn implies that $\lim_{k \rightarrow \infty} x_{r,n_k}$ exists. Since $r \in \mathbb{N}$ is arbitrary, the proof of the Theorem is now complete. \square

References

- [1] BILLINGSLEY, P. *Probability and Measure*, third ed. John Wiley & Sons, 1995.
- [2] BILLINGSLEY, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.