1 Chapter 1

Exercise 1.1(a)

Let X be the sum of the two number obtained.
Let X_1 be the number obtained on Die 1.
Let X_2 be the number obtained on Die 2.
Thus, $X = X_1 + X_2$, and

$$E_x = \{X = x\} = \{X_1 + X_2 = x\} = \{\ X_1 = x_1, X_2 = x - x_1 \mid 1 \leq x_1, x - x_1 \leq 6\ \}$$

Now,

$$1 \leq x - x_1 \leq 6 \quad \Longleftrightarrow \quad -1 \geq x_1 - x \geq -6 \quad \Longleftrightarrow \quad x - 1 \geq x_1 \geq x - 6 \quad \Longleftrightarrow \quad x - 6 \leq x_1 \leq x - 1$$

Hence,

$$E_x = \{X = x\} = \{X_1 + X_2 = x\} = \{\ X_1 = x_1, X_2 = x - x_1 \mid \max\{1, x-6\} \leq x_1 \leq \min\{6, x-1\}\ \}$$

$$\begin{aligned} P(E_x) &= \sum_{x_1=\max\{1,x-6\}}^{\min\{6,x-1\}} P(X_1 = x_1, X_2 = x - x_1) \;=\; \sum_{x_1=\max\{1,x-6\}}^{\min\{6,x-1\}} \frac{1}{6^2} \\ &= \frac{1}{6^2}\left(\min\{6, x-1\} - \max\{1, x-6\} + 1\right) \end{aligned}$$

Next, note that

$$\min\{6, x-1\} = \left\{\begin{array}{ll} x-1, & \text{if } x = 2, 3, \ldots, 6 \\ 6, & \text{if } x = 7, 8, \ldots, 12 \end{array}\right. \quad \text{and} \quad \max\{1, x-6\} = \left\{\begin{array}{ll} 1, & \text{if } x = 2, 3, \ldots, 6 \\ x-6, & \text{if } x = 7, 8, \ldots, 12 \end{array}\right.$$

Hence,

$$\begin{aligned} P(E_x) &= \frac{1}{6^2}\left(\min\{6, x-1\} - \max\{1, x-6\} + 1\right) \;=\; \frac{1}{36}\left\{\begin{array}{ll} (x-1) - 1 + 1, & \text{if } x = 2, 3, \ldots, 6 \\ 6 - (x-6) + 1, & \text{if } x = 7, 8, \ldots, 12 \end{array}\right. \\ &= \frac{1}{36}\left\{\begin{array}{ll} x-1, & \text{if } x = 2, 3, \ldots, 6 \\ 13 - x, & \text{if } x = 7, 8, \ldots, 12 \end{array}\right. \end{aligned}$$

$\square$

Exercise 1.18

Recapitulation of the rules of craps: *Let x be the number obtained on the first roll. If $x \in \{7, 11\}$, then the player wins. If $x \in \{2, 3, 12\}$, then the player loses. If $x \in \{4, 5, 6, 8, 9, 10\}$, then the player keeps rolling, until either 7 is rolled or x is rolled. If x is rolled first (before 7 is rolled), then the player wins. If 7 is rolled first (before x is rolled), then the player loses.*

Let W be the $\{0, 1\}$-valued random variable such that $W = 1$ if the player wins, and $W = 0$ if the player loses. We thus seek to compute $P(W = 1)$.

Let X be (the random variable of) the sum of the two numbers obtained on the first roll. Note that $\text{Range}(X) = \{\,2, 3, 4, \ldots, 12\,\}$. Then,

$$\begin{aligned} P(W = 1) &= \sum_{x=2}^{12} P(W = 1|X = x) \cdot P(X = x) \\ &= P(W = 1|X = 7)\, P(X = 7) + P(W = 1|X = 11)\, P(X = 11) + \sum_{x \in \{4,5,6,8,9,10\}} P(W = 1|X = x) \cdot P(X = x) \end{aligned}$$

Now, note that $P(W=1|X=7) = P(W=1|X=11) = 1$, $P(X=7) = \dfrac{6}{36} = \dfrac{1}{6}$, and $P(X=11) = \dfrac{2}{36} = \dfrac{1}{18}$.

From Exercise 1.1(a), we have:

$$\begin{aligned}
P(X=x) &= \frac{1}{6^2}\left(\min\{6, x-1\} - \max\{1, x-6\} + 1\right) = \frac{1}{36}\left\{\begin{array}{ll} (x-1)-1+1, & \text{if } x = 2, 3, \ldots, 6 \\ 6-(x-6)+1, & \text{if } x = 7, 8, \ldots, 12 \end{array}\right. \\
&= \frac{1}{36}\left\{\begin{array}{ll} x-1, & \text{if } x = 2, 3, \ldots, 6 \\ 13-x, & \text{if } x = 7, 8, \ldots, 12 \end{array}\right.
\end{aligned}$$

Next, let Y_n be the random variable of the sum of the two numbers obtained on the $(n+1)$st roll. Then,

$$\begin{aligned}
P(W=1|X=x) &= \sum_{n=1}^{\infty}\left[1 - P(Y_n=7) - P(Y_n=x)\right]^{n-1} \cdot P(X=x) \\
&= P(X=x) \cdot \sum_{n=1}^{\infty}\left[1 - P(Y_n=7) - P(Y_n=x)\right]^{n-1} \\
&= P(X=x) \cdot \frac{1}{1 - [1 - P(Y=7) - P(Y=x)]} \\
&= \frac{P(X=x)}{P(Y=7) + P(Y=x)} \\
&= \frac{P(X=x)}{\frac{1}{6} + P(Y=x)}
\end{aligned}$$

Hence,

$$\begin{aligned}
P(W=1) &= \sum_{x=2}^{12} P(W=1|X=x) \cdot P(X=x) \\
&= P(W=1|X=7)\,P(X=7) + P(W=1|X=11)\,P(X=11) + \sum_{x\in\{4,5,6,8,9,10\}} P(W=1|X=x) \cdot P(X=x) \\
&= \frac{6}{36} + \frac{2}{36} + \sum_{x\in\{4,5,6,8,9,10\}} \frac{P(X=x)^2}{\frac{1}{6} + P(Y=x)} \\
&= \frac{6}{36} + \frac{2}{36} + \frac{(\frac{4-1}{36})^2}{\frac{1}{6} + \frac{4-1}{36}} + \frac{(\frac{5-1}{36})^2}{\frac{1}{6} + \frac{5-1}{36}} + \frac{(\frac{6-1}{36})^2}{\frac{1}{6} + \frac{6-1}{36}} + \frac{(\frac{13-8}{36})^2}{\frac{1}{6} + \frac{13-8}{36}} + \frac{(\frac{13-9}{36})^2}{\frac{1}{6} + \frac{13-9}{36}} + \frac{(\frac{13-10}{36})^2}{\frac{1}{6} + \frac{13-10}{36}} \\
&= \frac{6}{36} + \frac{2}{36} + \frac{(1/36)^2}{1/36}\left(\frac{3^2}{6+3} + \frac{4^2}{6+4} + \frac{5^2}{6+5} + \frac{5^2}{6+5} + \frac{4^2}{6+4} + \frac{3^2}{6+3}\right) \\
&= \frac{6}{36} + \frac{2}{36} + \frac{2}{36}\left(\frac{3^2}{6+3} + \frac{4^2}{6+4} + \frac{5^2}{6+5}\right) = \frac{1}{36}\left[6 + 2 + 2\left(\frac{9}{9} + \frac{16}{10} + \frac{25}{11}\right)\right] \\
&= \frac{1}{36}\left[8 + 2\left(\frac{536}{110}\right)\right] = \frac{1}{36}\left[\frac{1952}{110}\right] = \frac{1}{2^2\cdot 3^2}\left[\frac{2^5\cdot 61}{2\cdot 5\cdot 11}\right] \\
&= \frac{2^2\cdot 61}{3^2\cdot 5\cdot 11} \approx 0.4929293
\end{aligned}$$

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References