Survival Analysis

1 Outline

Suppose:

- $(\Omega, \mathcal{A}, \mu)$ is a probability space.
- $n \in \mathbb{N}$ is an natural number (positive integer).
- $T_1, T_2, \ldots, T_n : \Omega \longrightarrow [0, \infty]$ are independent identically distributed extended $\mathbb{R}$-valued random variables.
- $U_1, U_2, \ldots, U_n : \Omega \longrightarrow [0, \infty]$ are independent identically distributed extended $\mathbb{R}$-valued random variables.
- For each $i = 1, 2, \ldots, n$, let $X_i := \min\{T_i, U_i\}$, and $C_i := I_{\{T_i \,\leq\, U_i\}}$.

For each subject $i = 1, 2, \ldots, n$, the random variable T_i is interpreted to be the "survival time" of subject i, while U_i is interpreted to be the "censoring time" of subject i.

We wish to make inference about the (common) *survival function*

$$S(t) \;\; := \;\; P(\,T > t\,) \;\; = \;\; \mu\Big(\Big\{\,\omega \in \Omega \;\Big\vert\; T(\omega) > t\,\Big\}\Big)$$

of $T_1, T_2, \ldots, T_n$. However, in survival analysis, the inference about $S(t)$ is made based on the *right-censored survival time data* $\{\,X_i,\, C_i\,\}$, $i = 1, 2, \ldots, n$ (rather than on the T_i's directly).

The *hazard function*:

$$\lambda(t) \;\; := \;\; \lim_{h \to 0^+} \frac{1}{h} \cdot P\Big(\,t \leq T < t + h \;\Big\vert\; t \leq T\,\Big)$$

The *cumulative hazard function*:

$$\Lambda(t) \;\; := \;\; \int_0^t \lambda(t)\,\mathrm{d}t$$

The *Nelson-Aalen estimator* for the cumulative hazard function $\Lambda(t)$:

$$\widehat{\Lambda}(\omega, t) \;\; := \;\; \sum_{\substack{C_i(\omega)\,=\,1 \\ T_i(\omega)\,\leq\,t}} \frac{1}{Y(\omega, T_i(\omega))}\,,$$

where

$$Y_i(\omega, t) \;\; := \;\; \begin{cases} 1, & t - h < X_i(\omega), \text{ for each } h > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$Y(\omega, t) \;\; := \;\; \sum_{i\,=\,1}^{n} Y_i(\omega, t)$$

The aggregated counting process for subject i:

$$N_i(\omega, t) \;\; := \;\; I_{\{X_i(\omega)\,\leq\,t\}}$$

The aggregated counting process:

$$N(\omega, t) \;\; := \;\; \sum_{i\,=\,1}^{n} N_i(\omega, t) \;\; = \;\; \sum_{i\,=\,1}^{n} I_{\{X_i(\omega)\,\leq\,t\}}$$

The aggregated intensity process:

$$\alpha(\omega,t) \;\;:=\;\; \lim_{h\to 0^+}\frac{1}{h}\cdot P\left(N(\omega,t+h)-N(\omega,t)=1 \;\middle|\; \mathcal{F}_t\right) \;\;=\;\; \lim_{h\to 0^+}\frac{1}{h}\cdot E\left[N(\omega,t+h)-N(\omega,t) \;\middle|\; \mathcal{F}_t\right]$$

The aggregated cumulative intensity process:

$$A(\omega,t) \;\;:=\;\; \int_0^t \alpha(\omega,t)\,\mathrm{d}t$$

Then, the process

$$M(\omega,t) \;\;:=\;\; N(\omega,t)-A(\omega,t) \;\;=\;\; N(\omega,t)-\int_0^t \alpha(\omega,t)\,\mathrm{d}t$$

is a martingale process. In particular, $M(\,\cdot\,,t)$ satisfies

$$E\left[\,M(\,\cdot\,,t+h)-M(\,\cdot\,,t)\;\middle|\;\mathcal{F}_t\right](\omega) \;\;=\;\; M(\omega,t)$$

A Integration on product measure spaces

Definition A.1 (Product σ-algebra)
Suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces. Define

$$\mathcal{A}_1 \times \mathcal{A}_2 \;:=\; \left\{\, A_1 \times A_2 \in \Omega_1 \times \Omega_2 \;\middle\vert\; A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2 \,\right\}.$$

We refer to $\mathcal{A}_1 \times \mathcal{A}_2$ as the collection of all measurable rectangles in $\Omega_1 \times \Omega_2$. The product σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ of $\mathcal{A}_1$ and $\mathcal{A}_2$ is, by definition, the following:

$$\mathcal{A}_1 \otimes \mathcal{A}_2 \;:=\; \sigma(\,\mathcal{A}_1 \times \mathcal{A}_2\,).$$

In other words, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the σ-algebra of subsets of $\Omega_1 \times \Omega_2$ containing all Cartesian products $A_1 \times A_2$, where $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.

Definition A.2 (Horizontal and vertical sections in a set-theoretic Cartesian product)
Suppose X and Y are two non-empty sets. For each $x \in X$, $y \in Y$, and $V \subset X \times Y$, we define:

$$\begin{aligned} V_{(x,\,\cdot\,)} &:= \left\{\, y \in Y \;\middle\vert\; (x,y) \in V \,\right\} \\ V_{(\,\cdot\,,y)} &:= \left\{\, x \in X \;\middle\vert\; (x,y) \in V \,\right\} \end{aligned}$$

Theorem A.3 (Sections of measurable subsets in a product measurable space are themselves measurable.)
Suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces. Then,

(i) $V_{(x,\,\cdot\,)} \in \mathcal{A}_2$, *for each $x \in \Omega_1$ and each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$, and*

(ii) $V_{(\,\cdot\,,y)} \in \mathcal{A}_1$, *for each $y \in \Omega_2$ and each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$.*

Proof We give only the proof of (i); that of (ii) is similar. Define $\mathcal{F} \subset \mathcal{P}(\Omega_1 \times \Omega_2)$ as follows:

$$\mathcal{F} \;:=\; \left\{\, V \in \Omega_1 \times \Omega_2 \;\middle\vert\; V_{(x,\,\cdot\,)} \in \mathcal{A}_2,\ \text{for each } x \in \Omega_1 \,\right\}.$$

Claim 1: $\mathcal{A}_1 \times \mathcal{A}_2 \;\subset\; \mathcal{F}$

Claim 2: $\mathcal{F}$ is a σ-algebra of subsets of $\Omega_1 \times \Omega_2$.

Proof of Claim 1: Suppose $x \in \Omega_1$ and $V = A_1 \times A_2$, where $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Then,

$$V_{(x,\,\cdot\,)} \;=\; \begin{cases} A_2, & \text{if } x \in A_1 \\ \varnothing, & \text{otherwise} \end{cases}$$

This proves that $V_{(x,\,\cdot\,)} = (A_1 \times A_2)_{(x,\,\cdot\,)} \subset \mathcal{F}$. Since $x \in \Omega_1$, $A_1 \in \mathcal{A}_1$, and $A_2 \in \mathcal{A}_2$ are arbitrary, Claim 1 follows.

Proof of Claim 2: First, note that, for each $x \in \Omega_1$, we have $(\Omega_1 \times \Omega_2)_{(x,\,\cdot\,)} := \left\{\, y \in \Omega \;\middle\vert\; (x,y) \in \Omega_1 \times \Omega_2 \,\right\} = \Omega_2 \in \mathcal{A}_2$. Hence, $\Omega_1 \times \Omega_2 \in \mathcal{F}$. Next, suppose $V \in \mathcal{F}$ and $V^c := (\Omega_1 \times \Omega_2) \setminus V$. Then, for each $x \in \Omega_1$,

$$\begin{aligned} (\,V^c\,)_{(x,\,\cdot\,)} &= \left\{\, y \in \Omega_2 \;\middle\vert\; (x,y) \in V^c \,\right\} \;=\; \left\{\, y \in \Omega_2 \;\middle\vert\; (x,y) \notin V \,\right\} \\ &= \Omega_2 \Big\backslash \left\{\, y \in \Omega_2 \;\middle\vert\; (x,y) \in V \,\right\} \;=\; \left(V_{(x,\,\cdot\,)}\right)^c \;\in\; \mathcal{A}_2, \end{aligned}$$

where the last containment follows from the fact that $\mathcal{A}_2$ is a σ-algebra (hence closed under complementation) and that $V \in \mathcal{F}$ (hence $V_{(x,\,\cdot\,)} \in \mathcal{A}_2$). This proves that $\mathcal{F}$ is closed under complementation. Lastly, suppose $V_1, V_2, \ldots, \in \mathcal{F}$. Then,

$$\left(\bigcup_{i=1}^{\infty} V_i \right)_{(x,\,\cdot\,)} = \left\{ y \in \Omega_2 \;\middle|\; (x,y) \in \bigcup_{i=1}^{\infty} V_i \right\} = \bigcup_{i=1}^{\infty} \left\{ y \in \Omega_2 \;\middle|\; (x,y) \in V_i \right\} = \bigcup_{i=1}^{\infty} (\, V_i \,)_{(x,\,\cdot\,)} \;\in\; \mathcal{A}_2,$$

where the last containment follows from the fact that $\mathcal{A}_2$ is a σ-algebra (hence closed under countable union) and that each $V_i \in \mathcal{F}$ (hence $(\, V_i \,)_{(x,\,\cdot\,)} \in \mathcal{A}_2$). This proves that $\mathcal{F}$ is closed under countable union. This completes the proof of Claim 2.

Claim 1 and Claim 2 together immediately imply that

$$\mathcal{A}_1 \otimes \mathcal{A}_2 \;:=\; \sigma\,(\,\mathcal{A}_1 \times \mathcal{A}_2\,) \;\subset\; \mathcal{F} \;:=\; \left\{ V \in \Omega_1 \times \Omega_2 \;\middle|\; \begin{array}{c} V_{(x,\,\cdot\,)} \in \mathcal{A}_2\,, \\ \text{for each } x \in \Omega_1 \end{array} \right\}.$$

This completes the proof of statement (i) in the present Theorem. $\square$

Theorem A.4 (Sections of measurable maps are themselves measurable.)
Suppose $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$, $(S, \mathcal{S})$ *are measurable spaces, and* $f : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \longrightarrow (S, \mathcal{S})$ *is an* $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{S})$*-measurable map. Then,*

(i) $f(x,\,\cdot\,) : \Omega_2 \longrightarrow S : y \longmapsto f(x,y)$ *is an* $(\mathcal{A}_2, \mathcal{S})$*-measurable map for each* $x \in \Omega_1$.

(ii) $f(\,\cdot\,,y) : \Omega_1 \longrightarrow S : x \longmapsto f(x,y)$ *is an* $(\mathcal{A}_1, \mathcal{S})$*-measurable map for each* $y \in \Omega_2$.

PROOF

(i) We need to show that $f(x,\,\cdot\,)^{-1}(V) \in \mathcal{A}_2$, for each $x \in \Omega_1$, and each $V \in \mathcal{S}$. To this end, note that

$$f(x,\,\cdot\,)^{-1}(V) \;=\; \left\{ y \in \Omega_2 \;\middle|\; f(x,y) \in V \right\} \;=\; \left\{ y \in \Omega_2 \;\middle|\; (x,y) \in f^{-1}(V) \right\} \;=\; f^{-1}(V)_{(x,\,\cdot\,)} \;\in\; \mathcal{A}_2\,,$$

where the last containment follows, by Theorem A.3, from the fact that $f^{-1}(V) \in \mathcal{A}_1 \otimes \mathcal{A}_2$ (since $V \in \mathcal{S}$ and f is $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{S})$-measurable).

(ii) The proof here is similar to that of (i). $\square$

Definition A.5 (Elementary subsets of the set-theoretic Cartesian product of two measurable spaces)
Suppose $(\Omega_1, \mathcal{A}_1)$ *and* $(\Omega_2, \mathcal{A}_2)$ *are two measurable spaces. The collection of* <u>elementary subsets</u> *of* $\Omega_1 \times \Omega_2$ *with respect to their respective* σ*-algebras* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *is, by definition, the following:*

$$\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2) \;:=\; \left\{ \bigsqcup_{i=1}^{n} A_1^{(i)} \times A_2^{(i)} \;\in\; \Omega_1 \times \Omega_2 \;\middle|\; \begin{array}{l} A_k^{(i)} \in \mathcal{A}_k, \;\; \text{for } k = 1,2, \\ \text{for each } i = 1,2,\ldots,n, \\ \text{for each } n \in \mathbb{N} \end{array} \right\}$$

Definition A.6 (Monotone class)
Suppose X *is a non-empty set. Then, a collection* $\mathcal{M}$ *of subsets of* X *is called a* <u>monotone class</u> *if* $\mathcal{M}$ *satisfies both of the following two conditions:*

(i)

$$A \;:=\; \bigcup_{i=1}^{\infty} A_i \;\in\; \mathcal{M}, \quad \text{whenever } \{\, A_i \,\}_{i \in \mathbb{N}} \text{ satistfies } A_i \in \mathcal{M} \text{ and } A_i \subset A_{i+1}, \text{ for each } i \in \mathbb{N}.$$

(ii)

$$B \;:=\; \bigcap_{i\,=\,1}^{\infty} B_i \,\in\, \mathcal{M}, \quad \text{whenever } \{\, B_i \,\}_{i\in\mathbb{N}} \text{ satistfies } B_i \in \mathcal{M} \text{ and } B_i \supset B_{i+1}, \text{ for each } i \in \mathbb{N}.$$

Lemma A.7 (An arbitrary intersection of monotone classes is itself a monotone class)
Suppose X is a non-empty set and $\{\, \mathcal{M}_t \,\}_{t\in T}$ is a family of monotone classes of subsets of X indexed by the non-empty set T. Then,

$$\mathcal{M} \;:=\; \bigcap_{t\,\in\,T} \mathcal{M}_t \;\subset\; \mathcal{P}(\,X\,)$$

is itself a monotone class of subsets of X.

PROOF Suppose $\{\, A_i \,\}_{i\in\mathbb{N}}$ satisfies $A_i \subset A_{i+1}$, for each $i \in \mathbb{N}$. Then, note the following implications:

$$\begin{aligned}
& A_i \,\in\, \mathcal{M} \,=\, \bigcap_{t\in T} M_t, \text{ for each } i \in \mathbb{N} \\
\Longleftrightarrow\quad & A_i \in \mathcal{M}_t, \;\; \text{for each } i \in \mathbb{N}, \text{ each } t \in T \\
\Longrightarrow\quad & A \;:=\; \bigcup_{i\,=\,1}^{\infty} A_i \,\in\, \mathcal{M}_t, \;\; \text{for each } t \in T \quad \text{(since each } \mathcal{M}_t \text{ is a monotone class)} \\
\Longrightarrow\quad & A \;:=\; \bigcup_{i\,=\,1}^{\infty} A_i \;\in\; \mathcal{M} \,:=\, \bigcap_{t\in T} M_t
\end{aligned}$$

Similarly, suppose $\{\, B_i \,\}_{i\in\mathbb{N}}$ satisfies $B_i \supset B_{i+1}$, for each $i \in \mathbb{N}$. Then, note the following implications:

$$\begin{aligned}
& B_i \,\in\, \mathcal{M} \,=\, \bigcap_{t\in T} M_t, \text{ for each } i \in \mathbb{N} \\
\Longleftrightarrow\quad & B_i \in \mathcal{M}_t, \;\; \text{for each } i \in \mathbb{N}, \text{ each } t \in T \\
\Longrightarrow\quad & B \;:=\; \bigcap_{i\,=\,1}^{\infty} B_i \,\in\, \mathcal{M}_t, \;\; \text{for each } t \in T \quad \text{(since each } \mathcal{M}_t \text{ is a monotone class)} \\
\Longrightarrow\quad & B \;:=\; \bigcap_{i\,=\,1}^{\infty} B_i \;\in\; \mathcal{M} \,:=\, \bigcap_{t\in T} M_t
\end{aligned}$$

This shows that $\mathcal{M} \;:=\; \bigcap_{t\,\in\,T} \mathcal{M}_t$ is indeed a monotone class, and completes the proof of the Theorem. □

Lemma A.8
Suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces.
Then, $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$ is closed under taking intersections, unions, and set-theoretic subtractions.

PROOF We prove this Lemma by proving the following series of claims:

Claim 1: $\mathcal{A}_1 \times \mathcal{A}_2$ is closed under finite intersections.

Proof of Claim 1: This claim follows immediately from the following set-theoretic identity

$$\bigcap_{i=1}^{n} \left(A_1^{(i)} \times A_2^{(i)} \right) \;\;=\;\; \left(\bigcap_{i=1}^{n} A_1^{(i)} \right) \times \left(\bigcap_{i=1}^{n} A_2^{(i)} \right),$$

and the fact that $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-algebras; hence, in particular they are closed under countable (hence finite) intersections.

Claim 2: For every $P, Q \in \mathcal{A}_1 \times \mathcal{A}_2$, there exist disjoint $R, S \in \mathcal{A}_1 \times \mathcal{A}_2$ such that $P \backslash Q = R \sqcup S$.

Proof of Claim 2: This claim follows immediately from the following set-theoretic identity

$$(A_1 \times A_2) \,\backslash\, (B_1 \times B_2) \;=\; \Big((A_1 \backslash B_1) \times A_2 \Big) \,\sqcup\, \Big((A_1 \cap B_1) \times (A_2 \backslash B_2) \Big),$$

and that fact that $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-algebras.

Claim 3:
For every $P, Q \in \mathcal{A}_1 \times \mathcal{A}_2$, there exist pairwise disjoint $R, S, T \in \mathcal{A}_1 \times \mathcal{A}_2$ such that $P \cup Q = R \sqcup S \sqcup T$.

Proof of Claim 3: This claim follows immediately from the following set-theoretic identity

$$\begin{aligned}(A_1 \times A_2) \cup (B_1 \times B_2) &= \Big((A_1 \times A_2) \backslash (B_1 \times B_2) \Big) \sqcup \Big(B_1 \times B_2 \Big) \\ &= \Big((A_1 \backslash B_1) \times A_2 \Big) \sqcup \Big((A_1 \cap B_1) \times (A_2 \backslash B_2) \Big) \sqcup \Big(B_1 \times B_2 \Big),\end{aligned}$$

and the fact that $\mathcal{A}_1$ and $\mathcal{A}_2$ are are σ-algebras.

Claim 4:
For every $P, Q, R, S \in \mathcal{A}_1 \times \mathcal{A}_2$ with $P \cap Q = \varnothing$ and $R \cap S = \varnothing$, there exist pairwise disjoint $T_1, T_2, T_3, T_4 \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$(\, P \sqcup Q \,) \bigcap (\, R \sqcup S \,) \;=\; T_1 \sqcup T_2 \sqcup T_3 \sqcup T_4.$$

Proof of Claim 4: This claim follows from Claim 1 and the following set-theoretic identity

$$\begin{aligned}(\, P \sqcup Q \,) \bigcap (\, R \sqcup S \,) &= \Big(P \cap (\, R \sqcup S \,) \Big) \bigsqcup \Big(Q \cap (\, R \sqcup S \,) \Big) \\ &= (\, P \cap R \,) \sqcup (\, P \cap S \,) \bigsqcup (\, Q \cap R \,) \sqcup (\, Q \cap S \,).\end{aligned}$$

Claim 5:
For every $P_1, \ldots, P_n, Q_1, \ldots, Q_n \in \mathcal{A}_1 \times \mathcal{A}_2$ with $P_i \cap Q_i = \varnothing$, for each $i = 1, 2, \ldots, n$, there exist pairwise disjoint $T_1, T_2, T_3, \ldots, T_{2^n} \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\bigcap_{i=1}^{n} (\, P_i \sqcup Q_i \,) \;=\; \bigsqcup_{k=1}^{2^n} T_k$$

Proof of Claim 5: This claim follows from Claim 4 and finite induction.

Claim 6: $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$ is closed under intersections.

Proof of Claim 6: This claim follows from the following set-theoretic identity:

$$\left(\bigsqcup_{i=1}^{n} A_1^{(i)} \times A_2^{(i)} \right) \bigcap \left(\bigsqcup_{k=1}^{m} B_1^{(k)} \times B_2^{(k)} \right) \;=\; \bigsqcup_{i=1}^{n} \bigsqcup_{k=1}^{m} \Big(A_1^{(i)} \cap B_1^{(k)} \Big) \times \Big(A_2^{(i)} \cap B_2^{(k)} \Big),$$

and the fact that $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-algebras.

Claim 7: $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$ is closed under unions.

Proof of Claim 7: Let $\bigsqcup_{i=1}^{n} A_1^{(i)} \times A_2^{(i)}$ and $\bigsqcup_{k=1}^{m} B_1^{(k)} \times B_2^{(k)}$ be two elements of $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$. Then,

$$\begin{aligned}
\left(\bigsqcup_{i=1}^{n} A_1^{(i)} \times A_2^{(i)}\right) \bigcap \left(\bigsqcup_{k=1}^{m} B_1^{(k)} \times B_2^{(k)}\right) &= \bigsqcup_{i=1}^{n}\left(\left(A_1^{(i)} \times A_2^{(i)}\right) \bigcap \left(\bigsqcup_{k=1}^{m} B_1^{(k)} \times B_2^{(k)}\right)\right) \\
&= \bigsqcup_{i=1}^{n}\left(\bigsqcup_{k=1}^{m}\left(A_1^{(i)} \times A_2^{(i)}\right) \bigcap \left(B_1^{(k)} \times B_2^{(k)}\right)\right) \\
&= \bigsqcup_{i=1}^{n}\left(\bigsqcup_{k=1}^{m}\left(A_1^{(i)} \cap B_1^{(k)}\right) \times \left(A_2^{(i)} \cap B_2^{(k)}\right)\right).
\end{aligned}$$

This completes the proof of Claim 7.

Claim 8: $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$ is closed under set-theoretic subtractions.

Proof of Claim 8: Let $\bigsqcup_{i=1}^{n} A_1^{(i)} \times A_2^{(i)}$ and $\bigsqcup_{k=1}^{m} B_1^{(k)} \times B_2^{(k)}$ be two elements of $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$. Then,

$$\begin{aligned}
\left(\bigsqcup_{i=1}^{n} A_1^{(i)} \times A_2^{(i)}\right) \Big\backslash \left(\bigsqcup_{k=1}^{m} B_1^{(k)} \times B_2^{(k)}\right) &= \bigsqcup_{i=1}^{n}\left(\left(A_1^{(i)} \times A_2^{(i)}\right) \Big\backslash \left(\bigsqcup_{k=1}^{m} B_1^{(k)} \times B_2^{(k)}\right)\right) \\
&= \bigsqcup_{i=1}^{n}\left(\left(A_1^{(i)} \times A_2^{(i)}\right) \bigcap \left(\bigsqcup_{k=1}^{m} B_1^{(k)} \times B_2^{(k)}\right)^c\right) \\
&= \bigsqcup_{i=1}^{n}\left(\left(A_1^{(i)} \times A_2^{(i)}\right) \bigcap \left(\bigcap_{k=1}^{m} \left(B_1^{(k)} \times B_2^{(k)}\right)^c\right)\right) \\
&= \bigsqcup_{i=1}^{n}\left(\bigcap_{k=1}^{m}\left(A_1^{(i)} \times A_2^{(i)}\right) \Big\backslash \left(B_1^{(k)} \times B_2^{(k)}\right)\right) \\
&= \bigsqcup_{i=1}^{n}\left(\bigcap_{k=1}^{m}\left(R^{(i,k)} \sqcup S^{(i,k)}\right)\right), \quad \text{by Claim 5} \\
&= \bigsqcup_{i=1}^{n}\left(\bigsqcup_{j=1}^{2^m} T^{(i,j)}\right), \quad \text{by Claim 2}
\end{aligned}$$

where the existence of $R^{(i,k)}, S^{(i,k)} \in \mathcal{A}_1 \times \mathcal{A}_2$ follows from Claim 2, while that of $T^{(i,j)} \in \mathcal{A}_1 \times \mathcal{A}_2$ from Claim 5. This completes the proof of Claim 8, as well as that of the present Lemma. □

Lemma A.9

Suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces.
Then, the smallest monotone class $\mathcal{M}$ containing $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$ is a σ-algebra of subsets of $\Omega_1 \times \Omega_2$.

Proof First, by Lemma A.7, $\mathcal{M}$ exists and equals the intersection of all monotone classes of subsets of $\Omega_1 \times \Omega_2$ which contain $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$. For every $P \subset \Omega_1 \times \Omega_2$, define:

$$\mathcal{M}\langle\, P \,\rangle \;:=\; \left\{\, Q \in \Omega_1 \times \Omega_2 \;\middle|\; P\backslash Q,\; Q\backslash P,\; P \cup Q \,\in\, \mathcal{M} \,\right\}$$

Clearly, we have

$$P \in \mathcal{M}\langle\, Q \,\rangle \iff Q \in \mathcal{M}\langle\, P \,\rangle\,, \quad \text{for every } P,\, Q \in \Omega_1 \times \Omega_2.$$

Claim 1: For each $P \subset \Omega_1 \times \Omega_2$, $\mathcal{M}\langle\, P \,\rangle$ is a monotone class.

Proof of Claim 1: First, let $Q_1, Q_2, \ldots \in \mathcal{M}\langle\, P \,\rangle$ with $Q_1 \subset Q_2 \subset \cdots$. We need to show that $Q := \bigcup_{i=1}^{\infty} Q_i \in \mathcal{M}\langle\, P \,\rangle$. In other words, we need to show that $P \backslash Q$, $Q \backslash P$, $P \cup Q \in \mathcal{M}$. To this end, observe that:

$$\begin{aligned}
P \backslash Q &= P \Bigg\backslash \left(\bigcup_{i=1}^{\infty} Q_i \right) = P \bigcap \left(\bigcup_{i=1}^{\infty} Q_i \right)^c = P \bigcap \left(\bigcap_{i=1}^{\infty} Q_i^c \right) = \bigcap_{i=1}^{\infty} \underbrace{(\, P \backslash Q_i \,)}_{\in \mathcal{M}} \in \mathcal{M}, \\
Q \backslash P &= \left(\bigcup_{i=1}^{\infty} Q_i \right) \Bigg\backslash P = \left(\bigcup_{i=1}^{\infty} Q_i \right) \bigcap P^c = \bigcup_{i=1}^{\infty} (\, Q_i \cap P^c \,) = \bigcup_{i=1}^{\infty} \underbrace{(\, Q_i \backslash P \,)}_{\in \mathcal{M}} \in \mathcal{M}, \\
P \cup Q &= P \bigcup \left(\bigcup_{i=1}^{\infty} Q_i \right) = \bigcup_{i=1}^{\infty} \underbrace{(\, P \cup Q_i \,)}_{\in \mathcal{M}} \in \mathcal{M},
\end{aligned}$$

where we have used the fact that $P \backslash Q_i \supset P \backslash Q_{i+1}$, $Q_i \backslash P \subset Q_{i+1} \backslash P$, $P \cup Q_i \subset P \cup Q_{i+1}$, and that $\mathcal{M}$ is a monotone class. This proves that we indeed have $Q := \bigcup_{i=1}^{\infty} Q_i \in \mathcal{M}\langle\, P \,\rangle$.

Next, let $R_1, R_2, \ldots \in \mathcal{M}\langle\, P \,\rangle$ with $R_1 \supset R_2 \supset \cdots$. We need to show that $R := \bigcap_{i=1}^{\infty} R_i \in \mathcal{M}\langle\, P \,\rangle$. Observe that:

$$\begin{aligned}
P \backslash R &= P \Bigg\backslash \left(\bigcap_{i=1}^{\infty} R_i \right) = P \bigcap \left(\bigcap_{i=1}^{\infty} R_i \right)^c = P \bigcap \left(\bigcup_{i=1}^{\infty} R_i^c \right) = \bigcup_{i=1}^{\infty} \underbrace{(\, P \backslash R_i \,)}_{\in \mathcal{M}} \in \mathcal{M}, \\
R \backslash P &= \left(\bigcap_{i=1}^{\infty} R_i \right) \Bigg\backslash P = \left(\bigcap_{i=1}^{\infty} R_i \right) \bigcap P^c = \bigcap_{i=1}^{\infty} (\, R_i \cap P^c \,) = \bigcap_{i=1}^{\infty} \underbrace{(\, R_i \backslash P \,)}_{\in \mathcal{M}} \in \mathcal{M}, \\
P \cup R &= P \bigcup \left(\bigcap_{i=1}^{\infty} R_i \right) = \bigcap_{i=1}^{\infty} \underbrace{(\, P \cup R_i \,)}_{\in \mathcal{M}} \in \mathcal{M},
\end{aligned}$$

where we have used the fact that $P \backslash R_i \subset P \backslash R_{i+1}$, $R_i \backslash P \supset R_{i+1} \backslash P$, $P \cup R_i \supset P \cup R_{i+1}$, and that $\mathcal{M}$ is a monotone class. This proves that we indeed have $R := \bigcap_{i=1}^{\infty} R_i \in \mathcal{M}\langle\, P \,\rangle$. This completes the proof of Claim 1.

Claim 2: For each $P \in \mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$, we have $\mathcal{M} \subset \mathcal{M}\langle\, P \,\rangle$.

Proof of Claim 2: Let $P, Q \in \mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$ be arbitrary. By Lemma A.8, we have $P \backslash Q$, $Q \backslash P$, $P \cup Q \in \mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{M}$. Hence, $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{M}\langle\, P \,\rangle$, for every $P \in \mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$. Claim 1 and Lemma A.7 together imply that, for every $P \in \mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$, we have $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{M} \subset \mathcal{M}\langle\, P \,\rangle$. This proves Claim 2.

Claim 3: For each $P \in \mathcal{M}$, we have $\mathcal{M} \subset \mathcal{M}\langle\, P \,\rangle$.

Proof of Claim 3: By Claim 2, $P \subset \mathcal{M}\langle\, Q \,\rangle$, for every $P \in \mathcal{M}$ and every $Q \in \mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$. But, recall that $U \in \mathcal{M}\langle\, V \,\rangle \Longleftrightarrow V \in \mathcal{M}\langle\, U \,\rangle$, for any $U, V \in \Omega_1 \times \Omega_2$. We thus see that $Q \subset \mathcal{M}\langle\, P \,\rangle$, for every $P \in \mathcal{M}$ and every $Q \in \mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$, which in turn implies that $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{M}\langle\, P \,\rangle$, for every $P \in \mathcal{M}$. Claim 1 and Lemma A.7 together imply that, for every $P \in \mathcal{M}$, we have $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{M} \subset \mathcal{M}\langle\, P \,\rangle$. This proves Claim 3.

Claim 4: For each $P, Q \in \mathcal{M}$, we have $P \backslash Q$, $P \cup Q \in \mathcal{M}$.

Proof of Claim 4: For any $P, Q \in \mathcal{M}$, we have, by Claim 3, that $Q \in \mathcal{M} \subset \mathcal{M}\langle\, P \,\rangle$, which immediately Claim 4.

Claim 5: $\Omega_1 \times \Omega_2 \in \mathcal{M}$.

Proof of Claim 5: This Claim follows immediately from the observation that: $\Omega_1 \times \Omega_2 \in \mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{M}$.

Claim 6: $\mathcal{M}$ is closed under complementation.

Proof of Claim 6: $W \in \mathcal{M} \implies (\Omega_1 \times \Omega_2)\backslash W \in \mathcal{M}$, by Claim 4 and Claim 5. This proves Claim 6.

Claim 7: $\mathcal{M}$ is closed under countable unions.

Proof of Claim 7: Let $W_1, W_2, \ldots \in \mathcal{M}$. We need to show $W := \bigcup_{i=1}^{\infty} W_i \in \mathcal{M}$. To this end, define $Q_n := \bigcup_{i=1}^{n} W_i$, for each $n \in \mathbb{N}$. Note that $W = \bigcup_{n=1}^{\infty} Q_n$. Note also that $Q_n \subset Q_{n+1}$, for each $n \in \mathbb{N}$. By Claim 4 and finite induction, we see that $Q_n \in \mathcal{M}$, for each $n \in \mathbb{N}$. Since $\mathcal{M}$ is a monotone class, we have that $W \in \mathcal{M}$. This proves Claim 7.

Claim 5, Claim 6, and Claim 7 together means precisely that $\mathcal{M}$ is a σ-algebra of subsets of $\Omega_1 \times \Omega_2$. This completes the proof of the present Lemma. □

Theorem A.10
Suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces. Then, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the smallest monotone class which satistifes $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{A}_1 \otimes \mathcal{A}_2$.

Proof First note that, since $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a σ-algebra, it is closed under countable intersections and countable unions. Hence, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is in particular a monotone class. It is also immediate that $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2) \subset \mathcal{A}_1 \otimes \mathcal{A}_2$, since $\mathcal{A}_1 \otimes \mathcal{A}_2$ is closed under finite disjoint unions (being closed under countable unions) and it contains $\mathcal{A}_1 \times \mathcal{A}_2$, i.e. the collection of all subsets of $\Omega_1 \times \Omega_2$ of the form $A_1 \times A_2$ with $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. So, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a monotone class of subsets of $\Omega_1 \times \Omega_2$ which contains $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$.

Let $\mathcal{M}$ be the smallest monotone class containing $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$. By Lemma A.7, $\mathcal{M}$ exists and equals the intersection of all monotone classes of subsets of $\Omega_1 \times \Omega_2$ which contain $\mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2)$. By the preceding paragraph, we therefore have $\mathcal{M} \subset \mathcal{A}_1 \otimes \mathcal{A}_2$, and hence the following series of containment:

$$\mathcal{A}_1 \times \mathcal{A}_2 \;\subset\; \mathcal{E}(\mathcal{A}_1 \times \mathcal{A}_2) \;\subset\; \mathcal{M} \;\subset\; \mathcal{A}_1 \otimes \mathcal{A}_2 \;:=\; \sigma(\,\mathcal{A}_1 \times \mathcal{A}_2\,) \;\subset\; \mathcal{P}(\,\Omega_1 \times \Omega_2\,).$$

But by Lemma A.9, $\mathcal{M}$ is itself a σ-algebra. Thus, we may now conclude $\mathcal{M} = \mathcal{A}_1 \otimes \mathcal{A}_2$. This completes the proof of the present Theorem. □

Theorem A.11 (Well-definition of the product measure of two σ-finite measures)
Suppose $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ are two σ-finite measure spaces. Let $(\mathbb{R}, \mathcal{B})$ be $\mathbb{R}$ equipped with its Borel σ-algebra $\mathcal{B} = \sigma(\mathcal{O}(\mathbb{R}))$. Then, for each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$, the following statements hold:

(i) *the map* $\Omega_1 \longrightarrow \mathbb{R} : x \longmapsto \mu_2\big(V_{(x,\,\cdot\,)}\big) = \int_{\Omega_2} 1_V(x,y)\,\mathrm{d}\mu_2(y)$ *is $(\mathcal{A}_1, \mathcal{B})$-measurable,*

(ii) *the map* $\Omega_2 \longrightarrow \mathbb{R} : y \longmapsto \mu_1\big(V_{(\,\cdot\,,y)}\big) = \int_{\Omega_1} 1_V(x,y)\,\mathrm{d}\mu_1(x)$ *is $(\mathcal{A}_2, \mathcal{B})$-measurable, and*

(iii) *the following equality of Lebesgue integrals (of measurable $\mathbb{R}$-valued functions) holds:*

$$\int_{\Omega_1} \mu_2\big(V_{(x,\,\cdot\,)}\big)\;\mathrm{d}\mu_1(x) \;=\; \int_{\Omega_2} \mu_1\big(V_{(\,\cdot\,,y)}\big)\;\mathrm{d}\mu_2(y),$$

or equivalently,

$$\int_{\Omega_1}\left(\int_{\Omega_2} 1_V(x,y)\,\mathrm{d}\mu_2(y)\right)\mathrm{d}\mu_1(x) \;=\; \int_{\Omega_2}\left(\int_{\Omega_1} 1_V(x,y)\,\mathrm{d}\mu_1(x)\right)\mathrm{d}\mu_2(y).$$

PROOF Define $\mathcal{C} \subset \mathcal{A}_1 \otimes \mathcal{A}_2$ as follows:

$$\mathcal{C} \;:=\; \left\{\, V \in \mathcal{A}_1 \otimes \mathcal{A}_2 \;\left|\; \int_{\Omega_1} \mu_2\big(V_{(x,\,\cdot\,)}\big)\,\mathrm{d}\mu_1(x) \;=\; \int_{\Omega_2} \mu_1\big(V_{(\,\cdot\,,y)}\big)\,\mathrm{d}\mu_2(y) \right.\,\right\}.$$

Claim 1: $A_1 \times A_2 \in \mathcal{C}$, for each $A_1 \in \mathcal{A}_1$ and each $A_2 \in \mathcal{A}_2$.

Claim 2: $V \;:=\; \bigcup_{i=1}^{\infty} V_i \;\in\; \mathcal{C}$, whenever $\{\,V_i\,\}_{i\in\mathbb{N}} \subset \mathcal{C}$ and $V_i \subset V_{i+1}$, for each $i \in \mathbb{N}$.

Claim 3: $V \;:=\; \bigsqcup_{i=1}^{\infty} V_i \;\in\; \mathcal{C}$, whenever $\{\,V_i\,\}_{i\in\mathbb{N}} \subset \mathcal{C}$ is a disjoint countable collection of members in $\mathcal{C}$.

Claim 4: Suppose $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$, with $\mu_1(A_1), \mu_2(A_2) < \infty$. Suppose also that $\{\,V_i\,\}_{i\in\mathbb{N}} \subset \mathcal{C}$ satisfies $A_1 \times A_2 \supset V_1 \supset V_2 \supset V_3 \supset \cdots$. Then, $V \;:=\; \bigcap_{i=1}^{\infty} V_i \;\in\; \mathcal{C}$.

Proof of Claim 1:

Proof of Claim 2:

Proof of Claim 3:

Proof of Claim 4:

Next, note that, since $(\Omega_1, \mathcal{A}_1, \mu_1)$ is a σ-finite measure space, there exist mutually disjoint $\Omega_1^{(1)}, \Omega_1^{(2)}, \ldots \in \mathcal{A}_1$ such that

$$\Omega_1 \;=\; \bigsqcup_{n=1}^{\infty} \Omega_1^{(n)}, \quad \text{and} \quad \mu_1\Big(\Omega_1^{(n)}\Big) \;<\; \infty, \;\; \text{for each } n \in \mathbb{N}.$$

Similarly, there exist mutually disjoint $\Omega_2^{(1)}, \Omega_2^{(2)}, \ldots \in \mathcal{A}_2$ such that

$$\Omega_2 \;=\; \bigsqcup_{n=1}^{\infty} \Omega_2^{(n)}, \quad \text{and} \quad \mu_2\Big(\Omega_2^{(n)}\Big) \;<\; \infty, \;\; \text{for each } n \in \mathbb{N}.$$

We now define

$$\mathcal{M} \;:=\; \left\{\, V \in \mathcal{A}_1 \otimes \mathcal{A}_2 \;\left|\; V \cap \Big(\Omega_1^{(m)} \times \Omega_2^{(n)}\Big) \;\in\; \mathcal{C}, \text{ for each } m, n \in \mathbb{N} \right.\,\right\}.$$

Claim 5: $\mathcal{M}$ is a monotone class.

Claim 6:

$$\mathcal{E} \;\subset\; \mathcal{M}$$

Proof of Claim 5: Suppose $V_1, V_2, \ldots \in \mathcal{M}$, with $V_1 \subset V_2 \subset V_3 \subset \cdots$. We need to show $V \;:=\; \bigcup_{i=1}^{\infty} V_i \;\in\; \mathcal{M}$. To this end, note that, for each $m, n \in \mathbb{N}$, we have

$$V \cap \Big(\Omega_1^{(m)} \times \Omega_2^{(n)}\Big) \;=\; \left(\bigcup_{i=1}^{\infty} V_i\right) \bigcap \Big(\Omega_1^{(m)} \times \Omega_2^{(n)}\Big) \;=\; \bigcup_{i=1}^{\infty} \underbrace{\Big(V_i \cap (\Omega_1^{(m)} \times \Omega_2^{(n)})\Big)}_{\in\,\mathcal{C}} \;\in\; \mathcal{C},$$

where the last containment follows from Claim 3. Thus, we see that we indeed have $V \in \mathcal{M}$. Next, suppose that $W_1, W_2, \ldots \in \mathcal{M}$, with $W_1 \supset W_2 \supset W_3 \supset \cdots$. We need to show $W := \bigcap_{i=1}^{\infty} W_i \in \mathcal{M}$. Now, for each $m, n \in \mathbb{N}$, we have:

$$W \cap \left(\Omega_1^{(m)} \times \Omega_2^{(n)}\right) \;=\; \left(\bigcap_{i=1}^{\infty} W_i\right) \bigcap \left(\Omega_1^{(m)} \times \Omega_2^{(n)}\right) \;=\; \bigcap_{i=1}^{\infty} \underbrace{\left(W_i \cap (\Omega_1^{(m)} \times \Omega_2^{(n)})\right)}_{\in\,\mathcal{C}} \;\in\; \mathcal{C}.$$

where the last containment follows from Claim 4. This proves that $\mathcal{M}$ is indeed a monotone class and completes the proof of Claim 5.

It follows from Claim 5, Claim 6 and Theorem **??** that $\mathcal{M} = \mathcal{A}_1 \otimes \mathcal{A}_2$, which in turn implies that $V \cap \left(\Omega_1^{(m)} \times \Omega_2^{(n)}\right) \in \mathcal{C}$, for each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$ and each $m, n \in \mathbb{N}$. Hence, for each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$, we have

$$V \;=\; V \cap (\Omega_1 \times \Omega_2) \;=\; V \bigcap \left(\bigsqcup_{m,n\,\in\,\mathbb{N}} \Omega_1^{(m)} \times \Omega_2^{(n)}\right) \;=\; \bigsqcup_{m,n\,\in\,\mathbb{N}} \underbrace{V \cap \left(\Omega_1^{(m)} \times \Omega_2^{(n)}\right)}_{\in\,\mathcal{C}} \;\in\; \mathcal{C},$$

where the last containment follows from Claim 3. Lastly, recall that $V \in \mathcal{C}$ is equivalent to

$$\int_{\Omega_1} \mu_2\big(V_{(x,\,\cdot\,)}\big)\,\mathrm{d}\mu_1(x) \;=\; \int_{\Omega_2} \mu_1\big(V_{(\,\cdot\,,y)}\big)\,\mathrm{d}\mu_2(y).$$

This completes the proof of the present Theorem. □

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