The Prokhorov Theorem

1 The Prokhorov Theorem

Definition 1.1 (Tightness and weak sequential compactness)

Suppose:

- (S, ρ) *is a metric space,* $\mathcal{B}(S)$ *its the Borel* σ*-algebra,* $(S, \mathcal{B}(S))$ *is the corresponding measurable space.*
- $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ *is a family of probability measures on* $(S, \mathcal{B}(S))$.

The family Π *is said to be:*

(i) ***tight*** *if, for each* $\varepsilon > 0$*, there exists a compact subset* $K_\varepsilon \subset S$ *such that*

$$1 - \epsilon \ < \ P(\, K_\varepsilon \,) \ \leq \ 1, \quad \text{for each } P \in \Pi.$$

(ii) ***weakly sequentially compact*** *if, for every sequence* $\{\, P_n \,\}_{n\in\mathbb{N}} \ \subset \ \Pi$*, there exists a probability measure* $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ *and subsequence* $\{\, P_{n(i)} \,\}_{i\in\mathbb{N}}$ *such that*

$$P_{n(i)} \ \overset{w}{\longrightarrow} \ P, \quad \text{as } i \longrightarrow \infty.$$

Theorem 1.2 (The Prokhorov Theorem, Theorems 5.1 & 5.2, [1])

Suppose:

- (S, ρ) *is a metric space,* $\mathcal{B}(S)$ *its the Borel* σ*-algebra,* $(S, \mathcal{B}(S))$ *is the corresponding measurable space.*
- $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ *is a collection of probability measures on* $(S, \mathcal{B}(S))$.

Then, the following statements hold:

(i) *Tightness of* Π *implies weak sequential compactness of* Π.

(ii) *Suppose further that* (S, ρ) *is complete and separable.*
Then, weak sequential compactness of Π *implies tightness of* Π.

Proof We first prove statement (ii), then statement (i).

Proof of (ii)

Suppose S is complete and separable. Let $\varepsilon > 0$ be fixed. We need to find a compact subset $K \subset S$ such that

$$1 - \varepsilon \quad < \quad P(\, K \,) \quad \leq \quad 1, \quad \text{for each } P \in \Pi.$$

Now, separability of S implies that every open cover of every subset of S admits a countable subcover (Appendix M3, [1]). Denote by $B(x, r) \subset S$ the open ball in S centred at $x \in S$ of radius $r > 0$. For each $k \in \mathbb{N}$, the open cover

$$\left\{\, B\!\left(x, \frac{1}{k}\right) \,\right\}_{x\in S}$$

of S admits a countable subcover, say,

$$\{\, A_{ki} \,\}_{i\in\mathbb{N}} \ \subset \ \left\{\, B\!\left(x, \frac{1}{k}\right) \,\right\}_{x\in S}.$$

Let $G_{kn} := \bigcup_{i=1}^{n} A_{ki}$. Then, each G_{kn} is an open subset of S and $G_{kn} \uparrow S$, as $n \longrightarrow \infty$. Hence, by the Claim below, there exists $n_k \in \mathbb{N}$ such that

$$1 - \frac{\varepsilon}{2^k} \;<\; P\left(\bigcup_{i=1}^{n_k} A_{ki}\right) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Now, let

$$K \;:=\; \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}}$$

Note that K, being a closed subset of the complete metric space S, is itself complete. Note also that the set $\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}$ is totally bounded; hence so is its closure K. Being complete and totally bounded, K is therefore compact (Appendix M5, [1]). It now remains only to show that $1-\varepsilon < P(K) \leq 1$, for each $P \in \Pi$; or equivalently, that $P(K^c) \leq \varepsilon$, for each $P \in \Pi$. To this end, write $B_k := \bigcup_{i=1}^{n_k} A_{ki}$. Then,

$$1 - \frac{\varepsilon}{2^k} \;<\; P(\,B_k\,) \;\leq\; 1\,; \quad \text{equivalently,} \quad P(\,B_k^c\,) \;\leq\; \frac{\varepsilon}{2^k}.$$

Also,

$$K \;:=\; \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}} \;:=\; \overline{\bigcap_{k=1}^{\infty} B_k} \;\supset\; \bigcap_{k=1}^{\infty} B_k.$$

Hence,

$$K^c \;\subset\; \left(\bigcap_{k=1}^{\infty} B_k\right)^c \;=\; \bigcup_{k=1}^{\infty} B_k^c,$$

which implies:

$$P(\,K^c\,) \;\leq\; \sum_{k=1}^{\infty} P(B_k^c) \;\leq\; \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} \;=\; \varepsilon.$$

Thus, the proof of (ii) will be complete once we prove the following:

> **Claim:** Let $\{\,G_n\,\}_{n\in\mathbb{N}}$ be a sequence of open subsets of S with $G_n \uparrow S$. Then, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that
>
> $$1 - \varepsilon \;<\; P(\,G_{n_\varepsilon}\,) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Proof of Claim: Suppose the Claim is false, and we derive a contradiction. The failure of the Claim implies that there exists some $0 < \varepsilon < 1$ such that for each $n \in \mathbb{N}$, there exists $P_n \in \Pi$ such that

$$P_n(\,G_n\,) \;\leq\; 1 - \varepsilon.$$

By the hypothesis of weak sequential compactness of Π, there exists some probability measure $Q \in \mathcal{M}_1(S, \mathcal{B}(S))$ and the subsequence $\{\,P_{n(i)}\,\}$ of $\{\,P_n\,\}$ such that $P_{n(i)} \xrightarrow{w} Q$, as $i \longrightarrow \infty$. Now, for each fixed $n \in \mathbb{N}$, we have:

$$\begin{aligned}
Q(\,G_n\,) \;&\leq\; \liminf_{i\to\infty} P_{n(i)}(\,G_n\,), \quad \text{by the Portmanteau Theorem} \\
&\leq\; \liminf_{i\to\infty} P_{n(i)}\big(\,G_{n(i)}\,\big), \quad \text{since } \{\,G_n\,\} \text{ is increasing} \\
&\leq\; 1 - \varepsilon, \quad \text{by choice of } P_n
\end{aligned}$$

But, by hypothesis, we also have $G_n \uparrow S$. Hence, we therefore have:

$$1 \;=\; Q(S) \;=\; \lim_{n\to\infty} Q(G_n) \;\leq\; 1 - \varepsilon,$$

which is the desired contradiction. This completes the proof of the Claim, hence that of (ii).

Proof of (i)

□

References

[1] Billingsley, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.