

1 The metric space structure on \mathbb{R}^∞

Definition 1.1 (The metric on \mathbb{R}^∞ , Example 1.2, [1])

Let \mathbb{R}^∞ denotes the set of all infinite sequences of real numbers, i.e.

$$\mathbb{R}^\infty := \{ (x_1, x_2, \dots) \mid x_i \in \mathbb{R}, \text{ for each } i \in \mathbb{N} \}.$$

Define $\rho : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow [0, 1]$ as follows:

$$\rho(x, y) := \sum_{n=1}^{\infty} \frac{\min\{1, |x_n - y_n|\}}{2^n}.$$

Remark 1.2 Recall that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2} \cdot \left(\frac{1}{1 - \frac{1}{2}} \right) = 1,$$

which proves indeed that $0 \leq \rho(x, y) \leq 1$, for any $x, y \in \mathbb{R}^\infty$.

Theorem 1.3

- (i) $(\mathbb{R}^\infty, \rho)$ is a metric space. Let \mathbb{R}^∞ denote also this metric space in the remainder of this Theorem.
- (ii) For $x, x^{(1)}, x^{(2)}, x^{(3)}, \dots \in \mathbb{R}^\infty$, we have:

$$\rho(x^{(n)}, x) \longrightarrow 0 \iff \text{for each } i \in \mathbb{N}, \lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0$$

- (iii) For each $n \in \mathbb{N}$, the “natural projection to the initial segment of length n ”

$$\pi_n : \mathbb{R}^\infty \longrightarrow \mathbb{R}^n : x \longmapsto (x_1, x_2, \dots, x_n)$$

is continuous, where \mathbb{R}^n has the usual Euclidean topology.

- (iv) For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, let $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)$ denote the open hypercube in \mathbb{R}^n of side length 2ε centred at $\pi_n(x) \in \mathbb{R}^n$, i.e.

$$C_{\mathbb{R}^n}(\pi_n(x), \varepsilon) := \left\{ y \in \mathbb{R}^n \mid \begin{array}{l} |y_i - x_i| < \varepsilon, \\ i = 1, 2, \dots, n \end{array} \right\}$$

Then, its pre-image in \mathbb{R}^∞ under π_n

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) = \left\{ y \in \mathbb{R}^\infty \mid \begin{array}{l} |y_i - x_i| < \varepsilon, \\ i = 1, 2, \dots, n \end{array} \right\}$$

is an open subset of \mathbb{R}^∞ .

- (v) For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, we have:

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right),$$

where $B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right)$ is the open ball in \mathbb{R}^∞ centred at x of radius $\varepsilon + \frac{1}{2^n}$, i.e.

$$B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right) := \left\{ y \in \mathbb{R}^\infty \mid \rho(y, x) < \varepsilon + \frac{1}{2^n} \right\}$$

(vi) *The collection*

$$\left\{ \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset \mathbb{R}^\infty \mid n \in \mathbb{N}, x \in \mathbb{R}^\infty, \varepsilon > 0 \right\}$$

of all pre-images under π_n of open hypercubes in \mathbb{R}^n , for all $n \in \mathbb{N}$, forms a basis for the topology of \mathbb{R}^∞ .

(vii) \mathbb{R}^∞ is a separable and complete metric space. Hence, every probability measure on \mathbb{R}^∞ is tight.

PROOF

(i) Clearly, ρ is non-negative and symmetric. We now show that, for any $x, y \in \mathbb{R}^\infty$, we have $\rho(x, y) = 0$ implies $x = y$. Indeed,

$$\begin{aligned} \rho(x, y) = 0 &\iff \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - y_i|\}}{2^i} = 0 \\ &\iff \min\{1, |x_i - y_i|\} = 0, \text{ for each } i \in \mathbb{N} \\ &\iff |x_i - y_i| = 0, \text{ for each } i \in \mathbb{N} \\ &\iff x = y. \end{aligned}$$

In order to show that ρ is a metric, it remains only to establish the Triangle Inequality. By Lemma A.2, for any $x, y, z \in \mathbb{R}^\infty$, we have

$$\begin{aligned} \rho(x, y) &= \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - y_i|\}}{2^i} \\ &\leq \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - z_i|\} + \min\{1, |z_i - y_i|\}}{2^i} \\ &= \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - z_i|\}}{2^i} + \sum_{i=1}^{\infty} \frac{\min\{1, |z_i - y_i|\}}{2^i} \\ &= \rho(x, z) + \rho(z, y), \end{aligned}$$

where we have used the fact that $0 \leq \rho \leq 1$ to split the infinite sum into two terms in second-to-last equality. This proves that ρ satisfies the Triangle Inequality, and it is thus a metric on \mathbb{R}^∞ .

(ii) $\lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = 0 \implies \lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0, \text{ for each } i \in \mathbb{N}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = 0 &\implies \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} = 0 \\ &\implies \lim_{n \rightarrow \infty} \min\{1, |x_i^{(n)} - x_i|\} = 0, \text{ for each } i \in \mathbb{N} \\ &\implies \lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0, \text{ for each } i \in \mathbb{N} \end{aligned}$$

$$\lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0, \text{ for each } i \in \mathbb{N} \implies \lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = 0$$

This follows from the Weierstrass M -test. Suppose $\lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0, \text{ for each } i \in \mathbb{N}$. Then,

$$\lim_{n \rightarrow \infty} \frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} = 0 =: y_i, \text{ for each } i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$, let $M_i := \frac{1}{2^i}$. Then,

$$\frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} \leq M_i \quad \text{and} \quad \sum_{i=1}^{\infty} M_i < \infty.$$

Hence, by the Weierstrass M -test (Lemma A.3), we have

$$\lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} = \sum_{i=1}^{\infty} y_i = 0.$$

(iii) Immediate by (ii).

(iv) Since $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon) \subset \mathbb{R}^n$ is an open subset of \mathbb{R}^n , its pre-image under the continuous (by (iii)) map $\pi_n : \mathbb{R}^\infty \rightarrow \mathbb{R}^n$ is an open subset of \mathbb{R}^∞ .

(v) For $y \in \mathbb{R}^\infty$, we have

$$\begin{aligned} y \in \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) &\implies |y_i - x_i| < \varepsilon, \text{ for each } i = 1, 2, \dots, n \\ \implies \rho(x, y) &:= \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - y_i|\}}{2^i} \leq \sum_{i=1}^n \frac{\varepsilon}{2^i} + \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \varepsilon + \frac{1}{2^n}. \end{aligned}$$

This proves:

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right).$$

(vi) It suffices to show that every open ball in $B_{\mathbb{R}^\infty}(x, r) \subset \mathbb{R}^\infty$, $r > 0$, contains the pre-image of an open hypercube centred at $\pi_n(x) \in \mathbb{R}^n$ under π_n . To this end, for $r > 0$, choose $\varepsilon > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large such that $\varepsilon + \frac{1}{2^n} < r$. Then, for any $x \in \mathbb{R}^\infty$, by (v), we have:

$$x \in \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right) \subset B_{\mathbb{R}^\infty}(x, r),$$

as required. □

A Technical Lemmas

Lemma A.1 *Define*

$$\phi : [0, \infty) \rightarrow [0, 1] : t \mapsto \min\{1, t\}.$$

Then, ϕ satisfies:

$$\phi(s+t) \leq \phi(s) + \phi(t), \text{ for each } s, t \in [0, \infty).$$

PROOF For any $s, t \in [0, \infty)$, either $s+t \geq 1$ or $s+t < 1$. If $s+t \geq 1$, then

$$\phi(s+t) = \min\{1, s+t\} = 1 < 2 = 1+1 \leq \min\{1, s\} + \min\{1, t\} = \phi(s) + \phi(t),$$

hence, the required inequality holds. On the other hand, if $s+t < 1$, then we must also have $s < 1$ and $t < 1$ (since $s, t \geq 0$). Hence,

$$\phi(s+t) = \min\{1, s+t\} = s+t = \min\{1, s\} + \min\{1, t\} = \phi(s) + \phi(t),$$

thus, the required inequality also holds. □

Lemma A.2 For any $x, y, z \in \mathbb{R}$, we have:

$$\min\{1, |x - y|\} \leq \min\{1, |x - z|\} + \min\{1, |z - y|\}.$$

PROOF Observe that $|x - y| \leq |x - z| + |z - y|$ implies

$$\min\{1, |x - y|\} \leq |x - z| + |z - y|.$$

The above inequality, together with $\min\{1, |x - y|\} \leq 1$, thus in turn imply:

$$\min\{1, |x - y|\} \leq \min\{1, |x - z| + |z - y|\}.$$

By Lemma A.1, we therefore have:

$$\min\{1, |x - y|\} \leq \min\{1, |x - z| + |z - y|\} \leq \min\{1, |x - z|\} + \min\{1, |z - y|\},$$

which proves the present Lemma. □

Lemma A.3 (The Weierstrass M -test, Theorem A.28, [2])

Suppose that $\lim_{n \rightarrow \infty} x_i^{(n)} = x_i$, for each $i \in \mathbb{N}$, and that $|x_i^{(n)}| \leq M_i$, where $\sum_{i=1}^{\infty} M_i < \infty$. Then,

(i) $\sum_{i=1}^{\infty} x_i$ exists, and $\sum_{i=1}^{\infty} x_i^{(n)}$ exists for each $n \in \mathbb{N}$.

(ii) Furthermore,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

PROOF

(i) $\sum_{i=1}^{\infty} M_i < \infty$ and $|x_i^{(n)}| \leq M_i \implies$ the series $\sum_{i=1}^{\infty} x_i$ and $\sum_{i=1}^{\infty} x_i^{(n)}$, $n \in \mathbb{N}$, converge absolutely.

(ii) Let $\varepsilon > 0$ be given. Choose $K \in \mathbb{N}$ sufficiently large such that $\sum_{j=K+1}^{\infty} M_i < \frac{\varepsilon}{3}$. Next, choose $N \in \mathbb{N}$ sufficiently large such that

$$|x_i^{(n)} - x_i| < \frac{\varepsilon}{3K}, \text{ for any } n > N \text{ and } i = 1, 2, \dots, K.$$

Then, we have, for each $n > N$,

$$\begin{aligned} \left| \sum_{i=1}^{\infty} x_i^{(n)} - \sum_{i=1}^{\infty} x_i \right| &= \left| \sum_{i=1}^K (x_i^{(n)} - x_i) + \sum_{i=K+1}^{\infty} x_i^{(n)} - \sum_{i=K+1}^{\infty} x_i \right| \\ &\leq \sum_{i=1}^K |x_i^{(n)} - x_i| + \sum_{i=K+1}^{\infty} |x_i^{(n)}| + \sum_{i=K+1}^{\infty} |x_i| \\ &\leq K \cdot \frac{\varepsilon}{3K} + \sum_{i=K+1}^{\infty} M_i + \sum_{i=K+1}^{\infty} M_i \leq \frac{\varepsilon}{3} + 2 \cdot \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

□

References

- [1] BILLINGSLEY, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.
- [2] BILLINGSLEY, P. *Probability and Measure*, anniversary ed. John Wiley & Sons, 2012.