1 Separating and convergence-determining classes

Definition 1.1 (Separating class)

Suppose Ω is a non-empty set, $\mathcal{A}$ is a σ-algebra of subsets of Ω, $(S,\mathcal{A})$ is the corresponding measurable space, and $\mathcal{M}_1(\Omega,\mathcal{A})$ is the set of all probability measures defined on $\mathcal{A}$. A ***separating class*** *of subsets of $(\Omega,\mathcal{A})$ is a collection $\mathcal{S} \subset \mathcal{A}$ of subsets of Ω which satisfies the following condition: For every two probability measures $\mu,\nu \in \mathcal{M}_1(\Omega,\mathcal{A})$,*

$$\mu(S) = \nu(S), \text{ for every } S \in \mathcal{S} \quad \Longrightarrow \quad \mu(A) = \nu(A), \text{ for every } A \in \mathcal{A}$$

Definition 1.2 (Convergence-determining class)

Suppose Ω is a topological space, $\mathcal{B}(\Omega)$ is its Borel σ-algebra, $(\Omega,\mathcal{B}(\Omega))$ is the corresponding measurable space, and $\mathcal{M}_1(\Omega,\mathcal{B}(S))$ is the set of all probability measures defined on $\mathcal{B}(\Omega)$. A ***convergence-determining class*** *of subsets of $(\Omega,\mathcal{B}(\Omega))$ is a collection $\mathcal{C} \subset \mathcal{B}(\Omega)$ of Borel subsets of Ω which satisfies the following condition: For any $\mu,\mu_1,\mu_2,\ldots \in \mathcal{M}_1(\Omega,\mathcal{B}(\Omega))$,*

$$\lim_{n\to\infty} \mu_n(C) \,=\, \mu(C), \text{ for every } C \in \mathcal{C} \quad \Longrightarrow \quad \mu_n \xrightarrow{w} \mu.$$

Theorem 1.3

Suppose Ω is a non-empty set, $\mathcal{A}$ is a σ-algebra of subsets of Ω, and $(\Omega,\mathcal{A})$ is the corresponding measurable space. If

- *$\mathcal{S} \subset \mathcal{A}$ is closed under finite intersections, and*
- *$\mathcal{S}$ generates $\mathcal{A}$ (i.e. $\sigma(\mathcal{S}) = \mathcal{A}$),*

then $\mathcal{S}$ is a separating class of subsets of $(\Omega,\mathcal{A})$.

PROOF Let μ and ν be two probability measures defined on $(\Omega,\mathcal{A})$ such that $\mu(S) = \nu(S)$ for each $S \in \mathcal{S}$. We need to show that $\mu(A) = \nu(A)$ for each $A \in \mathcal{A}$. To this end, let

$$\mathcal{L} \,:=\, \{\, A \in \mathcal{A} = \sigma(\mathcal{S}) \mid \mu(A) = \nu(A) \,\}\,.$$

Note that $\mathcal{S} \subset \mathcal{L}$, by the hypothesis that μ and ν agree on $\mathcal{S}$, and $\mathcal{L} \neq \varnothing$ since $\Omega \in \mathcal{L}$. By Corollary B.8, it suffices to establish that $\mathcal{L}$ is a λ-system, since then it will follow that

$$\mathcal{A} \,=\, \sigma(\mathcal{S}) \,\subset\, \mathcal{L} \,:=\, \{\, A \in \mathcal{A} = \sigma(\mathcal{S}) \mid \mu(A) = \nu(A) \,\} \,\subset\, \sigma(\mathcal{S}) \,=\, \mathcal{A}\,,$$

i.e., $\mathcal{A} = \sigma(\mathcal{S}) = \mathcal{L}$, or equivalently, μ and ν agree on all of $\mathcal{A} = \sigma(\mathcal{S})$. Now, we have already noted that $\Omega \in \mathcal{L}$. For $A \in \mathcal{L}$, we have

$$\mu(\Omega\setminus A) \;=\; 1-\mu(A) \;=\; 1-\nu(A) \;=\; \nu(\Omega\setminus A),$$

hence $\Omega\setminus A \in \mathcal{L}$. Thus, $\mathcal{L}$ is closed under complementations. Lastly, let $A_1, A_2, \ldots \in \mathcal{L}$ be pairwise disjoint. Then,

$$\mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) \;=\; \sum_{n=1}^{\infty} \mu(A_n) \;=\; \sum_{n=1}^{\infty} \nu(A_n) \;=\; \nu\left(\bigsqcup_{n=1}^{\infty} A_n\right),$$

thus $\displaystyle\bigsqcup_{n=1}^{\infty} A_n \in \mathcal{L}$, which proves that $\mathcal{L}$ is closed under countable disjoint unions. $\mathcal{L}$ is therefore indeed a λ-system and the proof of the Theorem is complete. $\square$

Corollary 1.4 *Suppose S is a topological space and $\mathcal{B}(S)$ is its Borel σ-algebra (i.e. the σ-algebra generated by the collection of open subsets of S). Then, the collection of open subsets of S is a separating class of subsets of the measurable space $(S,\mathcal{B}(S))$.*

PROOF Recall that the collection of open sets are closed under finite intersections (by definition of topology), and they generate the Borel σ-algebras (by definition of Borel σ-algebras). Thus the Corollary follows immediately from Theorem 1.3. $\square$

2 Examples of separating and convergence-determining classes of $\mathbb{R}^\infty$

Definition 2.1 (The metric on $\mathbb{R}^\infty$, Example 1.2, [1])
Let $\mathbb{R}^\infty$ denotes the set of all infinite sequences of real numbers, i.e.

$$\mathbb{R}^\infty \;:=\; \{\, (x_1, x_2, \dots) \mid x_i \in \mathbb{R},\ \text{for each } i \in \mathbb{N} \,\}.$$

Define $\rho : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow [\,0,1\,]$ as follows:

$$\rho(x,y) \;:=\; \sum_{n=1}^{\infty} \frac{\min\{\,1\,,\,|\,x_n - y_n\,|\,\}}{2^n}.$$

Remark 2.2 *Recall that*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \;=\; \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^{n-1}} \;=\; \frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}\right) \;=\; 1,$$

which proves indeed that $0 \leq \rho(x,y) \leq 1$, for any $x, y \in \mathbb{R}^\infty$.

Theorem 2.3 (The metric space properties of $\mathbb{R}^\infty$)

(i) *$(\,\mathbb{R}^\infty, \rho\,)$ is a metric space. Let $\mathbb{R}^\infty$ denote also this metric space in the remainder of this Theorem.*

(ii) *For $x, x^{(1)}, x^{(2)}, x^{(3)}, \dots, \in \mathbb{R}^\infty$, we have:*

$$\rho\Big(\,x^{(n)},\,x\,\Big) \;\longrightarrow\; 0 \quad\Longleftrightarrow\quad \text{for each } i \in \mathbb{N},\ \lim_{n\to\infty}\left|\,x_i^{(n)} - x_i\,\right| \;=\; 0$$

(iii) *For each $n \in \mathbb{N}$, the "natural projection to the initial segment of length n"*

$$\pi_n \;:\; \mathbb{R}^\infty \;\longrightarrow\; \mathbb{R}^n \;:\; x \;\longmapsto\; (x_1, x_2, \dots, x_n)$$

is continuous, where $\mathbb{R}^n$ has the usual Euclidean topology.

(iv) *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, let $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)$ denote the open hypercube in $\mathbb{R}^n$ of side length 2ε centred at $\pi_n(x) \in \mathbb{R}^n$, i.e.*

$$C_{\mathbb{R}^n}(\pi_n(x),\varepsilon) \;\;:=\;\; \left\{\; y \in \mathbb{R}^n \;\middle|\; \begin{array}{c} |\,y_i - x_i\,| \;<\; \varepsilon, \\ i = 1, 2, \dots, n \end{array} \;\right\}$$

Then, its pre-image in $\mathbb{R}^\infty$ under π_n

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\;=\;\; \left\{\; y \in \mathbb{R}^\infty \;\middle|\; \begin{array}{c} |\,y_i - x_i\,| \;<\; \varepsilon, \\ i = 1, 2, \dots, n \end{array} \;\right\}$$

is an open subset of $\mathbb{R}^\infty$.

(v) *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, we have:*

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\;\subset\;\; B_{\mathbb{R}^\infty}\!\left(x\,,\,\varepsilon + \frac{1}{2^n}\right),$$

where $B_{\mathbb{R}^\infty}\!\left(x\,,\,\varepsilon + \frac{1}{2^n}\right)$ is the open ball in $\mathbb{R}^\infty$ centred at x of radius $\varepsilon + \frac{1}{2^n}$, i.e.

$$B_{\mathbb{R}^\infty}\!\left(x\,,\,\varepsilon + \frac{1}{2^n}\right) \;\;:=\;\; \left\{\; y \in \mathbb{R}^\infty \;\middle|\; \rho(y,x) \;<\; \varepsilon + \frac{1}{2^n} \;\right\}$$

(vi) *The collection*

$$\left\{\ \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \subset \mathbb{R}^\infty \ \middle|\ n\in\mathbb{N},\, x\in\mathbb{R}^\infty,\, \varepsilon>0\ \right\}$$

of all pre-images under π_n of open hypercubes in $\mathbb{R}^n$, for all $n\in\mathbb{N}$, forms a basis for the topology of $\mathbb{R}^\infty$.

(vii) $\mathbb{R}^\infty$ *is a separable metric space.*

(viii) $\mathbb{R}^\infty$ *is a complete metric space.*

PROOF

(i) Clearly, ρ is non-negative and symmetric. We now show that, for any $x, y\in\mathbb{R}^\infty$, we have $\rho(x,y)=0$ implies $x=y$. Indeed,

$$\begin{aligned}
\rho(x,y) \;=\; 0 \quad &\Longleftrightarrow\quad \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-y_i\,|\,\}}{2^i} \;=\; 0\\
&\Longleftrightarrow\quad \min\{\,1\,,|\,x_i-y_i\,|\,\} \;=\; 0,\ \text{ for each } i\in\mathbb{N}\\
&\Longleftrightarrow\quad |\,x_i-y_i\,| \;=\; 0,\ \text{ for each } i\in\mathbb{N}\\
&\Longleftrightarrow\quad x \;=\; y.
\end{aligned}$$

In order to show that ρ is a metric, it remains only to establish the Triangle Inequality. By Lemma A.2, for any $x, y, z\in\mathbb{R}^\infty$, we have

$$\begin{aligned}
\rho(x,y) \quad &= \quad \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-y_i\,|\,\}}{2^i}\\
&\leq \quad \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-z_i\,|\,\}\,+\,\min\{\,1\,,|\,z_i-y_i\,|\,\}}{2^i}\\
&= \quad \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-z_i\,|\,\}\,\}}{2^i} \;+\; \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,z_i-y_i\,|\,\}}{2^i}\\
&= \quad \rho(x,z)+\rho(z,y),
\end{aligned}$$

where we have used the fact that $0\leq\rho\leq 1$ to split the infinite sum into two terms in second-to-last equality. This proves that ρ satisfies the Triangle Inequality, and it is thus a metric on $\mathbb{R}^\infty$.

(ii) $\underline{\lim\limits_{n\to\infty}\rho\big(x^{(n)},x\big)=0 \ \Longrightarrow\ \lim\limits_{n\to\infty}\left|\,x_i^{(n)}-x_i\,\right|=0, \text{ for each } i\in\mathbb{N}}$

$$\begin{aligned}
\lim_{n\to\infty}\rho\Big(x^{(n)},x\Big)=0 \quad &\Longrightarrow\quad \lim_{n\to\infty}\sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i^{(n)}-x_i\,|\,\}}{2^i}=0\\
&\Longrightarrow\quad \lim_{n\to\infty}\,\min\{\,1\,,|\,x_i^{(n)}-x_i\,|\,\}=0,\ \text{ for each } i\in\mathbb{N}\\
&\Longrightarrow\quad \lim_{n\to\infty}\left|\,x_i^{(n)}-x_i\,\right|=0,\ \text{ for each } i\in\mathbb{N}
\end{aligned}$$

$\underline{\lim\limits_{n\to\infty}\left|\,x_i^{(n)}-x_i\,\right|=0, \text{ for each } i\in\mathbb{N} \ \Longrightarrow\ \lim\limits_{n\to\infty}\rho\big(x^{(n)},x\big)=0}$

This follows from the Weierstrass M-test. Suppose $\lim\limits_{n\to\infty}\left|\,x_i^{(n)}-x_i\,\right|=0$, for each $i\in\mathbb{N}$. Then,

$$\lim_{n\to\infty}\frac{\min\{\,1\,,|\,x_i^{(n)}-x_i\,|\,\}}{2^i} \;=\; 0 \;=:\; y_i,\ \text{ for each } i\in\mathbb{N}.$$

For each $i \in \mathbb{N}$, let $M_i := \dfrac{1}{2^i}$. Then,

$$\frac{\min\{1\,,|\,x_i^{(n)} - x_i\,|\}}{2^i} \;\leq\; M_i \quad \text{and} \quad \sum_{i=1}^{\infty} M_i \;<\; \infty.$$

Hence, by the Weierstrass M-test (Lemma A.3), we have

$$\lim_{n\to\infty} \rho\Big(x^{(n)}, x\Big) \;=\; \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i^{(n)} - x_i\,|\,\}}{2^i} \;=\; \sum_{i=1}^{\infty} y_i \;=\; 0.$$

(iii) Immediate by (ii).

(iv) Since $C_{\mathbb{R}^n}(\pi_n(x),\varepsilon) \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, its pre-image under the continuous (by (iii)) map $\pi_n : \mathbb{R}^\infty \longrightarrow \mathbb{R}^n$ is an open subset of $\mathbb{R}^\infty$.

(v) For $y \in \mathbb{R}^\infty$, we have

$$\begin{aligned} y \in \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \quad &\Longrightarrow \quad |\,y_i - x_i\,| < \varepsilon, \text{ for each } i = 1,2,\ldots,n \\ &\Longrightarrow \quad \rho(x,y) \;:=\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - y_i\,|\,\}}{2^i} \;\leq\; \sum_{i=1}^{n} \frac{\varepsilon}{2^i} \;+\; \sum_{i=n+1}^{\infty} \frac{1}{2^i} \;=\; \varepsilon + \frac{1}{2^n}. \end{aligned}$$

This proves:

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\subset\; B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon + \frac{1}{2^n}\right).$$

(vi) It suffices to show that every open ball in $B_{\mathbb{R}^\infty}(x,r) \subset \mathbb{R}^\infty$, $r > 0$, contains the pre-image of an open hypercube centred at $\pi_n(x) \in \mathbb{R}^n$ under π_n. To this end, for $r > 0$, choose $\varepsilon > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large such that $\varepsilon + \dfrac{1}{2^n} < r$. Then, for any $x \in \mathbb{R}^\infty$, by (v), we have:

$$x \;\in\; \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\subset\; B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon + \frac{1}{2^n}\right) \;\subset\; B_{\mathbb{R}^\infty}(\,x\,,r\,),$$

as required.

(vii) It suffices to exhibit a countable subset of $\mathbb{R}^\infty$ that intersects every open ball in $\mathbb{R}^\infty$. To this end, let

$$D \;:=\; \bigcup_{n=1}^{\infty} \left\{\, x = (x_1, x_2, \ldots\,) \in \mathbb{R}^\infty \;\middle|\; \begin{array}{l} x_i \in \mathbb{Q}, \text{ for each } i \in \mathbb{N} \\ x_i = 0, \text{ for all } i \geq n \end{array} \right\}.$$

Clearly, D is a countable subset of $\mathbb{R}^\infty$. Now let $B_{\mathbb{R}^\infty}(x,\varepsilon)$ be an arbitrary open ball in $\mathbb{R}^\infty$. Choose $\delta > 0$ small enough and $n \in \mathbb{N}$ large enough such that $\delta + \dfrac{1}{2^n} < \varepsilon$. Then,

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\delta)\,) \;\subset\; B_{\mathbb{R}^\infty}\left(x\,,\,\delta + \frac{1}{2^n}\right) \;\subset\; B_{\mathbb{R}^\infty}(\,x\,,\varepsilon\,),$$

Now, for each $i = 1,2,\ldots,n$, choose $z_i \in \mathbb{Q} \cap (x_i - \delta, x_i + \delta)$. Let $z = (z_1, z_2, \ldots, z_n, 0, 0, \ldots) \in \mathbb{R}^\infty$. Then, we have

$$z \;\in\; D \bigcap \left\{\, y \in \mathbb{R}^\infty \;\middle|\; \begin{array}{c} y_i \in (x_i - \delta, x_i + \delta), \\ \text{for each } i = 1,2,\ldots,n \end{array} \right\} \;=\; D \bigcap \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\delta)\,) \;\subset\; D \bigcap B_{\mathbb{R}^\infty}(\,x\,,\varepsilon\,).$$

This proves the the countable subset $D \subset \mathbb{R}^\infty$ has non-empty intersection with every open ball in $\mathbb{R}^\infty$, i.e. D is dense in $\mathbb{R}^\infty$. Hence, $\mathbb{R}^\infty$ is separable.

(viii) We need to show that every Cauchy sequence in $\mathbb{R}^\infty$ converges to any element in $\mathbb{R}^\infty$.

$$\begin{aligned}
& \left\{ x^{(n)} \right\}_{n\in\mathbb{N}} \subset \mathbb{R}^\infty \text{ is a Cauchy sequence in } \mathbb{R}^\infty \\
\Longleftrightarrow\;\; & \text{for each } \varepsilon > 0, \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that } \rho\Big(x^{(m)}, x^{(n)}\Big) \;<\; \varepsilon, \text{ for any } m,n > N_\varepsilon \\
\Longrightarrow\;\; & \text{for each } i \in \mathbb{N}, \text{ we have:} \\
& \text{for each } \varepsilon > 0, \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that } \left| x_i^{(m)} - x_i^{(n)} \right| \;<\; \varepsilon, \text{ for any } m,n > N_\varepsilon \\
\Longrightarrow\;\; & \text{for each } i \in \mathbb{N}, \left\{ x_i^{(n)} \right\}_{n\in\mathbb{N}} \subset \mathbb{R} \text{ is a Cauchy sequence in } \mathbb{R}; \text{ hence } x_i := \lim_{n\to\infty} x_i^{(n)} \in \mathbb{R} \text{ exists} \\
\Longrightarrow\;\; & \lim_{n\to\infty} \rho\Big(x^{(n)}, x\Big) \;=\; 0, \text{ where } x := (x_1, x_2, \dots) \in \mathbb{R}^\infty \quad \text{(by (ii))}
\end{aligned}$$

This proves that $\mathbb{R}^\infty$ indeed is a complete metric space.

□

Definition 2.4
*The **finite-dimensional class** of subsets of $\mathbb{R}^\infty$ is, by definition, the following:*

$$\mathcal{B}_f(\mathbb{R}^\infty) \;:=\; \left\{ \pi_k^{-1}(B) \subset \mathbb{R}^\infty \;\middle|\; \begin{array}{c} k \in \mathbb{N} \\ B \in \mathcal{B}\big(\mathbb{R}^k\big) \end{array} \right\},$$

where $\pi_k : \mathbb{R}^\infty \longrightarrow \mathbb{R}^k : x = (x_1, x_2, \dots) \longmapsto (x_1, \dots, x_k)$ is the projection of $\mathbb{R}^\infty$ onto $\mathbb{R}^k$.

Theorem 2.5

(i) $\mathcal{B}_f(\mathbb{R}^\infty) \subset \mathcal{B}(\mathbb{R}^\infty)$.

(ii) $\mathcal{B}_f(\mathbb{R}^\infty)$ *is a separating class of Borel subsets of* $\mathbb{R}^\infty$.

(iii) $\mathcal{B}_f(\mathbb{R}^\infty)$ *is a convergence-determining class of Borel subsets of* $\mathbb{R}^\infty$.

Proof

(i) Note that

$$\mathcal{B}_f(\mathbb{R}^\infty) \;:=\; \left\{ \pi_k^{-1}(B) \subset \mathbb{R}^\infty \;\middle|\; \begin{array}{c} k \in \mathbb{N} \\ B \in \mathcal{B}\big(\mathbb{R}^k\big) \end{array} \right\} \;=\; \bigcup_{k=1}^{\infty} \pi_k^{-1}\big(\mathcal{B}(\mathbb{R}^k)\big).$$

Thus, (i) is equivalent to the statement that each $\pi_k : \mathbb{R}^\infty \longrightarrow \mathbb{R}^k$ is Borel measurable. But each π_k is continuous, hence Borel measurable (Corollary B.12). This proves (i).

(ii) We apply Theorem 1.3 to $\mathcal{B}_f(\mathbb{R}^\infty)$.

$\underline{\mathcal{B}_f(\mathbb{R}^\infty) \text{ is closed under finite intersections}}$

$\underline{\mathcal{B}_f(\mathbb{R}^\infty) \text{ generates } \mathcal{B}(\mathbb{R}^\infty)}$

Let $\mathcal{O}(\mathbb{R}^\infty)$ denote the collection of open sets of $\mathbb{R}^\infty$. Hence $\mathcal{B}(\mathbb{R}^\infty) := \sigma(\mathcal{O}(\mathbb{R}^\infty))$. By (i), we have $\mathcal{B}_f(\mathbb{R}^\infty) \subset \mathcal{B}(\mathbb{R}^\infty) = \sigma(\mathcal{O}(\mathbb{R}^\infty))$, which implies $\sigma(\mathcal{B}_f(\mathbb{R}^\infty)) \subset \sigma(\mathcal{O}(\mathbb{R}^\infty))$. We need to establish the reverse inclusion, which will immediately follows from:

Claim: $\mathcal{O}(\mathbb{R}^\infty) \subset \sigma(\mathcal{B}_f(\mathbb{R}^\infty))$.

Proof of Claim: By Theorem 2.3(v), every open ball $B_{\mathbb{R}^\infty}(x,\varepsilon)$ in $\mathbb{R}^\infty$ contains the pre-image of an open hypercube from some finite-dimensional Euclidean space, where that pre-image itself contains x. We therefore see that every open set in $\mathbb{R}^\infty$ can be expressed as a union of pre-images of open hypercubes from finite-dimensional Euclidean spaces. By Theorem 2.3(vii), $\mathbb{R}^\infty$ is separable. Hence, by Theorem C.1, we see that every open set in $\mathbb{R}^\infty$ can be expressed as a countable union of pre-images of open hypercubes from finite-dimensional Euclidean spaces. Since pre-images of open hypercubes from finite-dimensional Euclidean spaces belong to $\mathcal{B}_f(\mathbb{R}^\infty)$, we see that $\mathcal{O}(\mathbb{R}^\infty) \subset \sigma(\mathcal{B}_f(\mathbb{R}^\infty))$. This completes the proof of the Claim.

□

A Technical Lemmas

Lemma A.1 *Define*

$$\phi \,:\, [\,0,\infty\,) \longrightarrow [\,0,1\,] \,:\, t \longmapsto \min\{\,1\,,t\,\}.$$

Then, ϕ satisfies:

$$\phi(s+t) \;\leq\; \phi(s) \,+\, \phi(t), \quad \text{for each } s,t \in [0,\infty).$$

PROOF For any $s,t \in [0,\infty)$, either $s+t \geq 1$ or $s+t < 1$. If $s+t \geq 1$, then

$$\phi(s+t) \;=\; \min\{1,s+t\} \;=\; 1 \;<\; 2 \;=\; 1+1 \;\leq\; \min\{1,s\}+\min\{1,t\} \;=\; \phi(s)+\phi(t),$$

hence, the required inequality holds. On the other hand, if $s+t<1$, then we must also have $s<1$ and $t<1$ (since $s,t \geq 0$). Hence,

$$\phi(s+t) \;=\; \min\{1,s+t\} \;=\; s+t \;=\; \min\{1,s\}+\min\{1,t\} \;=\; \phi(s)+\phi(t),$$

thus, the required inequality also holds. □

Lemma A.2 *For any $x,y,z \in \mathbb{R}$, we have:*

$$\min\{1,|\,x-y\,|\} \;\leq\; \min\{1,|\,x-z\,|\} \;+\; \min\{1,|\,z-y\,|\}.$$

PROOF Observe that $|\,x-y\,| \;\leq\; |\,x-z\,|+|\,z-y\,|$ implies

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; |\,x-z\,|+|\,z-y\,|.$$

The above inequality, together with $\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; 1$, thus in turn imply:

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; \min\{\,1\,,|\,x-z\,|+|\,z-y\,|\,\}.$$

By Lemma A.1, we therefore have:

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; \min\{\,1\,,|\,x-z\,|+|\,z-y\,|\,\}. \;\leq\; \min\{\,1\,,|\,x-z\,|\,\} \;+\; \min\{\,1\,,|\,z-y\,|\,\},$$

which proves the present Lemma. □

Lemma A.3 (The Weierstrass M-test, Theorem A.28, [2])

Suppose that $\lim\limits_{n\to\infty} x_i^{(n)} = x_i$*, for each* $i \in \mathbb{N}$*, and that* $\left|\, x_i^{(n)} \,\right| \leq M_i$*, where* $\sum\limits_{i=1}^{\infty} M_i < \infty$*. Then,*

(i) $\sum\limits_{i=1}^{\infty} x_i$ *exists, and* $\sum\limits_{i=1}^{\infty} x_i^{(n)}$ *exists for each* $n \in \mathbb{N}$.

(ii) *Furthermore,*

$$\lim_{n\to\infty}\sum_{i=1}^{\infty} x_i^{(n)} \;=\; \sum_{i=1}^{\infty} x_i.$$

Proof

(i) $\sum_{i=1}^{\infty} M_i < \infty$ and $\left| x_i^{(n)} \right| \leq M_i \implies$ the series $\sum_{i=1}^{\infty} x_i$ and $\sum_{i=1}^{\infty} x_i^{(n)}$, $n\in\mathbb{N}$, converge absolutely.

(ii) Let $\varepsilon > 0$ be given. Choose $K\in\mathbb{N}$ sufficiently large such that $\sum_{j=K+1}^{\infty} M_i < \dfrac{\varepsilon}{3}$. Next, choose $N\in\mathbb{N}$ sufficiently large such that

$$\left| x_i^{(n)} - x_i \right| \;<\; \frac{\varepsilon}{3K},\quad \text{for any } n > N \text{ and } i = 1,2,\ldots,K.$$

Then, we have, for each $n > N$,

$$\begin{aligned}
\left| \sum_{i=1}^{\infty} x_i^{(n)} - \sum_{i=1}^{\infty} x_i \right| \;&=\; \left| \sum_{i=1}^{K}\left(x_i^{(n)} - x_i \right) + \sum_{i=K+1}^{\infty} x_i^{(n)} - \sum_{i=K+1}^{\infty} x_i \right| \\
&\leq\; \sum_{i=1}^{K}\left| x_i^{(n)} - x_i \right| + \sum_{i=K+1}^{\infty}\left| x_i^{(n)} \right| + \sum_{i=K+1}^{\infty} |\, x_i \,| \\
&\leq\; K\cdot\frac{\varepsilon}{3K} + \sum_{i=K+1}^{\infty} M_i + \sum_{i=K+1}^{\infty} M_i \;\leq\; \frac{\varepsilon}{3} + 2\cdot\frac{\varepsilon}{3} \;=\; \varepsilon.
\end{aligned}$$

Since ε is arbitrary, this proves:

$$\lim_{n\to\infty}\sum_{i=1}^{\infty} x_i^{(n)} \;=\; \sum_{i=1}^{\infty} x_i.$$

□

B σ-algebras and λ-systems

Definition B.1

Suppose Ω is a non-empty set. A σ-algebra of subsets of Ω is a collection $\mathcal{A}$ of subsets of Ω which satisfies the following conditions:

- $\Omega \in \mathcal{A}$.
- $\Omega \setminus A \in \mathcal{A}$, *for every* $A \in \mathcal{A}$.
- $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, *whenever* $A_1, A_2, \ldots \in \mathcal{A}$

Definition B.2

Suppose Ω is a non-empty set. A λ-system of subsets of Ω is a collection $\mathcal{L}$ of subsets of Ω which satisfies the following conditions:

- $\Omega \in \mathcal{L}$.
- $\Omega \setminus A \in \mathcal{L}$, *for every* $A \in \mathcal{L}$.
- $\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{L}$, *whenever* $A_1, A_2, \ldots \in \mathcal{L}$ *and* $A_i \cap A_j = \varnothing$, *for any* $i, j \in \mathbb{N}$ *with* $i \neq j$.

Remark B.3 *Clearly, every σ-algebra is also a λ-system.*

Theorem B.4
Suppose Ω is a non-empty set and $\mathcal{L}$ is a λ-system of subsets of Ω.

(i) *$\mathcal{L}$ is closed under proper set-theoretic differences, i.e. $A, B \in \mathcal{L}$ and $A \subset B$ together imply $B \setminus A \in \mathcal{L}$.*

(ii) *If $\mathcal{L}$ is closed under finite intersections, then $\mathcal{L}$ is a σ-algebra of subsets of Ω.*

PROOF For each $X \subset \Omega$, write $\Omega \setminus X$ as X^c.

(i) Suppose $A, B \in \mathcal{L}$ with $A \subset B$. Then, $B^c \cap A = \varnothing$. Hence, $B \setminus A = B \cap A^c = (B^c \cup A)^c = (B^c \sqcup A)^c \in \mathcal{L}$, since $\mathcal{L}$ is closed under complementations and finite disjoint unions.

(ii) Since $\mathcal{L}$ is a λ-system, we immediately have $\Omega \in \mathcal{L}$, and hence $\Omega \setminus A \in \mathcal{L}$, for every $A \in \mathcal{L}$. It remains to show that $\mathcal{L}$ closed under countable unions, i.e. for $A_1, A_2, \ldots \in \mathcal{L}$, we need to show $\bigcup_{i=1}^\infty A_i \in \mathcal{L}$. To this end, define:

$$\begin{aligned}
B_1 &:= A_1 \\
B_2 &:= A_2 \cap A_1^c \\
B_3 &:= A_3 \cap A_1^c \cap A_2^c \\
&\vdots \\
B_n &:= A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_n^c
\end{aligned}$$

Being a λ-system, $\mathcal{L}$ is closed under complementations. By hypothesis, $\mathcal{L}$ is furthermore closed under finite intersections. We thus see that $B_n \in \mathcal{L}$, for each $n \in \mathbb{N}$. Note also that the B_n's are pairwise disjoint, and

$$\bigcup_{i=1}^n A_i \;=\; \bigsqcup_{i=1}^n B_i, \quad \text{for each } n \in \mathbb{N}.$$

Hence,

$$\bigcup_{i=1}^\infty A_i \;=\; \bigsqcup_{i=1}^\infty B_i \;\in\; \mathcal{L},$$

since $\mathcal{L}$ is closed under countable pairwise disjoint unions ($\mathcal{L}$ being a λ-system). This proves that $\mathcal{L}$ is a σ-algebra of subsets of Ω.

□

Theorem B.5 *Let Ω be a non-empty set.*

(i) *The intersection of a non-empty collection of σ-algebras of subsets of Ω is itself a σ-algebra of subsets of Ω.*

(ii) *The intersection of a non-empty collection of λ-systems of subsets of Ω is itself a λ-system of subsets of Ω.*

PROOF

(i) Suppose Γ is an (arbitrary) non-empty set, and, for each $\gamma \in \Gamma$, $\mathcal{A}_\gamma$ is a σ-algebra of subsets of Ω. We need to prove that $\mathcal{A} := \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma$ is itself a σ-algebra of subsets of Ω.

$\underline{\Omega \in \mathcal{A} := \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma}$

Since, for each $\gamma \in \Gamma$, $\mathcal{A}_\gamma$ is a σ-algebra of subsets of Ω, we have $\Omega \in \mathcal{A}_\gamma$. Thus, $\Omega \in \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma$.

$\underline{A \in \mathcal{A} \implies \Omega \backslash A \in \mathcal{A}}$

$$A \in \mathcal{A} := \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma \iff A \in \mathcal{A}_\gamma,\, \forall\, \gamma \in \Gamma \implies \Omega \backslash A \in \mathcal{A}_\gamma,\, \forall\, \gamma \in \Gamma \implies \Omega \backslash A \in \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma =: \mathcal{A}$$

$\underline{A_1, A_2, \ldots \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}}$

$$\begin{aligned} A_1, A_2, \ldots \in \mathcal{A} := \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma \implies A_1, A_2, \ldots \in \mathcal{A}_\gamma,\, \forall\, \gamma \in \Gamma \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_\gamma,\, \forall\, \gamma \in \Gamma \\ \implies \bigcup_{i=1}^{\infty} A_i \in \bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma =: \mathcal{A} \end{aligned}$$

(ii) Suppose Γ is an (arbitrary) non-empty set, and, for each $\gamma \in \Gamma$, $\mathcal{L}_\gamma$ is a λ-system of subsets of Ω. We need to prove that $\mathcal{L} := \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma$ is itself a λ-system of subsets of Ω.

$\underline{\Omega \in \mathcal{L} := \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma}$

Since, for each $\gamma \in \Gamma$, $\mathcal{L}_\gamma$ is a λ-system of subsets of Ω, we have $\Omega \in \mathcal{L}_\gamma$. Thus, $\Omega \in \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma$.

$\underline{A \in \mathcal{L} \implies \Omega \backslash L \in \mathcal{L}}$

$$A \in \mathcal{L} := \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma \iff A \in \mathcal{L}_\gamma,\, \forall\, \gamma \in \Gamma \implies \Omega \backslash A \in \mathcal{L}_\gamma,\, \forall\, \gamma \in \Gamma \implies \Omega \backslash A \in \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma =: \mathcal{L}$$

$\underline{A_1, A_2, \ldots \in \mathcal{L} \text{ and } A_i \cap A_j \text{ whenever } i \neq j \implies \coprod_{i=1}^{\infty} A_i \in \mathcal{L}}$

$$\begin{aligned} & A_1, A_2, \ldots \in \mathcal{L} := \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma\,, \text{ and } A_i \cap A_j \text{ whenever } i \neq j \\ \implies & A_1, A_2, \ldots \in \mathcal{L}_\gamma,\, \forall\, \gamma \in \Gamma\,, \text{ and } A_i \cap A_j \text{ whenever } i \neq j \\ \implies & \coprod_{i=1}^{\infty} A_i \in \mathcal{L}_\gamma,\, \forall\, \gamma \in \Gamma \\ \implies & \coprod_{i=1}^{\infty} A_i \in \bigcap_{\gamma\in\Gamma} \mathcal{L}_\gamma =: \mathcal{L} \end{aligned}$$

$\square$

Theorem B.6 *Suppose Ω is a non-empty set, $\mathcal{S}$ is non-empty collection of subsets of Ω. Denote the power set of Ω by $\mathcal{P}(\Omega)$. Define*

$$\begin{aligned} \sigma(\mathcal{S}) &:= \bigcap_{\mathcal{A}\,\in\,\Sigma(\mathcal{S})} \mathcal{A}\,, \quad \text{where} \quad \Sigma(\mathcal{S}) := \left\{ \mathcal{A} \subset \mathcal{P}(\Omega) \;\middle|\; \begin{array}{c} \mathcal{A} \text{ is a } \sigma\text{-algebra of subsets of } \Omega, \\ \text{and } \mathcal{S} \subset \mathcal{A} \end{array} \right\}, \text{ and} \\ \lambda(\mathcal{S}) &:= \bigcap_{\mathcal{L}\,\in\,\Lambda(\mathcal{S})} \mathcal{L}\,, \quad \text{where} \quad \Lambda(\mathcal{S}) := \left\{ \mathcal{L} \subset \mathcal{P}(\Omega) \;\middle|\; \begin{array}{c} \mathcal{L} \text{ is a } \lambda\text{-system of subsets of } \Omega, \\ \text{and } \mathcal{S} \subset \mathcal{L} \end{array} \right\}. \end{aligned}$$

Then, $\sigma(\mathcal{S})$ is the unique smallest σ-algebra of subsets of Ω that contains $\mathcal{S} \subset \mathcal{P}(\Omega)$, and $\lambda(\mathcal{S})$ is the unique smallest λ-system of subsets of Ω that contains $\mathcal{S} \subset \mathcal{P}(\Omega)$. More precisely, we have

- *$\mathcal{S} \subset \sigma(\mathcal{S})$, $\mathcal{S} \subset \lambda(\mathcal{S})$, and*
- *$\sigma(\mathcal{S})$ is a σ-algebra of subsets of Ω, and $\lambda(\mathcal{S})$ is a λ-system of subsets of Ω, and*
- *if $\mathcal{A} \subset \mathcal{P}(\Omega)$ is a σ-algebra and $\mathcal{S} \subset \mathcal{A}$, then $\sigma(\mathcal{S}) \subset \mathcal{A}$.*
- *if $\mathcal{L} \subset \mathcal{P}(\Omega)$ is a λ-system and $\mathcal{S} \subset \mathcal{L}$, then $\lambda(\mathcal{S}) \subset \mathcal{L}$.*

Proof First, note that $\Sigma(\mathcal{S}) \neq \varnothing$ since $\mathcal{P}(\Omega) \in \Sigma(\mathcal{S})$. Similarly, $\Lambda(\mathcal{S}) \neq \varnothing$ since $\mathcal{P}(\Omega) \in \Lambda(\mathcal{S})$. It is immediate that $\mathcal{S} \subset \sigma(\mathcal{S})$, and $\sigma(\mathcal{S})$ is contained in every σ-algebra which contains $\mathcal{S}$. Similarly, $\mathcal{S} \subset \lambda(\mathcal{S})$, and $\lambda(\mathcal{S})$ is contained in every λ-system which contains $\mathcal{S}$. Since $\sigma(\mathcal{S})$ is, by definition, an intersection of σ-algebras, it itself is a σ-algebra of subsets of Ω by Theorem B.5. Similarly, since $\lambda(\mathcal{S})$ is, by definition, an intersection of λ-systems, it itself is a λ-system of subsets of Ω by Theorem B.5. □

Theorem B.7 *Suppose Ω is a non-empty set and $\mathcal{S}$ is a non-empty collection of subsets of Ω. Then,*

$$\mathcal{S} \text{ is closed under finite intersections} \quad \Longrightarrow \quad \lambda(\mathcal{S}) \text{ is a } \sigma\text{-algebra of subsets of } \Omega,$$

where $\lambda(\mathcal{S})$ is λ-system of subsets of Ω generated by $\mathcal{S}$.

Proof By Theorem B.4(ii), it suffices to show that $\lambda(\mathcal{S})$ is closed under finite intersections. We establish the proof in the following series of claims:

Claim 1: For each $A \in \lambda(\mathcal{S})$,

$$\mathcal{L}(A) \;:=\; \{\, B \subset \Omega \;\mid A \cap B \in \lambda(\mathcal{S}) \,\}$$

is a λ-system of subsets of Ω.

Proof of Claim 1: Clearly, $\Omega \in \mathcal{L}(A)$, since $A \cap \Omega = A \in \lambda(\mathcal{S})$. Next, we prove that $\mathcal{L}(A)$ is closed under complementations. Let $B \in \mathcal{L}(A)$. Then, $A \cap B \in \lambda(\mathcal{S})$. Note that $A = (A \cap B) \bigsqcup (A \cap B^c)$, hence $A \cap B^c = A \backslash (A \cap B) \in \lambda(\mathcal{S})$, since $A, A \cap B \in \lambda(\mathcal{S})$ and $\lambda(\mathcal{S})$ is closed under proper set-theoretic differences by Theorem B.4(i). This proves that $\mathcal{L}(A)$ is indeed closed under complementations. We now prove that $\mathcal{L}(A)$ is closed under countable disjoint unions. Let $B_1, B_2, \ldots \in \mathcal{L}(A)$ be pairwise disjoint. Then, $A \cap B_1, A \cap B_2, \ldots \subset \lambda(\mathcal{S})$ are pairwise disjoint. Hence,

$$A \bigcap \left(\bigsqcup_{i=1}^{\infty} B_i \right) \;=\; \bigsqcup_{i=1}^{\infty} (\, A \cap B_i \,) \;\in\; \lambda(\mathcal{S}) \,,$$

since $\lambda(\mathcal{S})$ is closed under countable disjoint unions. This proves that $\mathcal{L}(A)$ is a λ-system and thus completes the proof of the Claim 1.

Claim 2: $\mathcal{S} \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$. Consequently, $\lambda(\mathcal{S}) \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$.

Proof of Claim 2: Suppose $A \in \mathcal{S}$. Then, $A \cap B \in \mathcal{S}$ for each $B \in \mathcal{S}$, by the hypothesis that $\mathcal{S}$ is closed under finite intersections. Thus, $A \cap B \in \lambda(\mathcal{S})$, since $\mathcal{S} \subset \lambda(\mathcal{S})$. Hence, $B \in \mathcal{L}(A)$, for any $A, B \in \mathcal{S}$. This proves that $\mathcal{S} \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$. By Claim 1, $\mathcal{L}(A)$ is a λ-system. Hence, $\mathcal{L}(A) \supset \lambda(\mathcal{S})$, the smallest λ-system containing $\mathcal{S}$. This proves Claim 2.

Claim 3: $A \cap B \in \lambda(\mathcal{S})$, for each $A \in \mathcal{S}$ and $B \in \lambda(\mathcal{S})$.

Proof of Claim 3: Let $A \in \mathcal{S}$ and $B \in \lambda(\mathcal{S})$. By Claim 2, we have $\lambda(\mathcal{S}) \subset \mathcal{L}(A)$. Thus we have $B \in \mathcal{L}(A)$, which is equivalent to $A \cap B \in \lambda(S)$. This proves Claim 3.

Claim 4: $\mathcal{S} \subset \mathcal{L}(B)$, for each $B \in \lambda(\mathcal{S})$. Consequently, $\lambda(\mathcal{S}) \subset \mathcal{L}(B)$, for each $B \in \lambda(\mathcal{S})$.

Proof of Claim 4: Suppose $B \in \lambda(\mathcal{S})$. Then, $A \cap B \in \lambda(\mathcal{S})$ for each $A \in \mathcal{S}$, by Claim 3. This proves that $\mathcal{S} \subset \mathcal{L}(B)$. By Claim 1, $\mathcal{L}(B)$ is a λ-system. Hence, $\mathcal{L}(B) \supset \lambda(\mathcal{S})$, the smallest λ-system containing $\mathcal{S}$. This proves Claim 4.

Claim 5: $A \cap B \in \lambda(\mathcal{S})$, for each $A, B \in \lambda(\mathcal{S})$.

Proof of Claim 5: Let $A, B \in \lambda(\mathcal{S})$. By Claim 4, we have $\lambda(\mathcal{S}) \subset \mathcal{L}(B)$. Thus we have $A \in \mathcal{L}(B)$, which is equivalent to $A \cap B \in \lambda(S)$. This proves Claim 5.

Claim 5 states precisely that $\lambda(\mathcal{S})$ is closed under finite intersections, and completes the proof. $\square$

Corollary B.8 *Suppose Ω is a non-empty set and $\mathcal{S}$ is a non-empty collection of subsets of Ω. If $\mathcal{S}$ is closed under finite intersections, then*

(i) $\sigma(S) \subset \lambda(\mathcal{S})$, *and*

(ii) $\sigma(S) \subset \mathcal{L}$, *for any λ-system $\mathcal{L}$ of subsets of Ω such that $\mathcal{S} \subset \mathcal{L}$,*

where $\sigma(\mathcal{S})$ and $\lambda(\mathcal{S})$ are, respectively, the σ-algebra and λ-system of subsets of Ω generated by $\mathcal{S}$.

Proof

(i) By Theorem B.6, $\lambda(\mathcal{S})$ is the smallest λ-system containing $\mathcal{S}$. Since $\mathcal{S}$ is, by hypothesis, closed under finite intersections, $\lambda(\mathcal{S})$ is furthermore a σ-algebra, by Theorem B.7. Thus, by Theorem B.6 again, we have $\sigma(S) \subset \lambda(\mathcal{S})$.

(ii) This is now immediate since
$$\sigma(S) \;\subset\; \lambda(\mathcal{S}) \;\subset\; \mathcal{L},$$
where the first inclusion follows by (i), and the second inclusion follows by Theorem B.6.

$\square$

Lemma B.9 (The pre-image of a σ-algebra is itself a σ-algebra.)
Suppose Ω is a non-empty set, $(X, \mathcal{X})$ is a measurable space, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then,
$$f^{-1}(\mathcal{X}) \;:=\; \left\{\; f^{-1}(V) \subset \Omega \;\mid\; V \in \mathcal{X} \;\right\}$$
is a σ-algebra of subsets of Ω.

Proof

$\Omega \in f^{-1}(\mathcal{X})$ $f(\Omega) \subset X \implies \Omega = f^{-1}(X) \in f^{-1}(\mathcal{X})$.

$f^{-1}(\mathcal{X})$ is closed under complementations Let $V \in \mathcal{X}$. Then, $X \setminus V \in \mathcal{X}$, and
$$\Omega \setminus f^{-1}(V) \;=\; \{\, \omega \in \Omega \mid f(\omega) \notin V \,\} \;=\; \{\, \omega \in \Omega \mid f(\omega) \in X \setminus V \,\} \;=\; f^{-1}(X \setminus V) \;\in\; f^{-1}(\mathcal{X}),$$
which shows that $f^{-1}(\mathcal{X})$ is indeed closed under complementations.

$f^{-1}(\mathcal{X})$ is closed countable unions Let $V_1, V_2, \ldots \in \mathcal{X}$. Then, $\displaystyle\bigcup_{i=1}^{\infty} V_i \in \mathcal{X}$, and
$$\bigcup_{i=1}^{\infty} f^{-1}(V_i) \;=\; \left\{\, \omega \in \Omega \;\middle|\; \begin{array}{c} f(\omega) \in V_i \\ \text{for some } i \in \mathbb{N} \end{array} \right\} \;=\; f^{-1}\!\left(\bigcup_{i=1}^{\infty} V_i\right) \;\in\; f^{-1}(\mathcal{X}),$$

which proves that $f^{-1}(\mathcal{X})$ is indeed closed under countable unions.

This concludes the proof that that $f^{-1}(\mathcal{X})$ is a σ-algebra of subsets of Ω. □

Lemma B.10
Suppose $(\Omega,\mathcal{A})$ is a measurable space, X is a non-empty set, and $f:\Omega\longrightarrow X$ is a map from Ω into X. Then,

$$\mathcal{F} \;:=\; \left\{\, V\subset X \;\middle|\; f^{-1}(V)\in\mathcal{A} \,\right\}$$

is a σ-algebra of subsets of X.

Proof
$\underline{X\in\mathcal{F}}$ $f^{-1}(X)=\Omega\in\mathcal{A} \implies X\in\mathcal{F}$.

$\underline{\mathcal{F}\text{ is closed under complementations}}$ $V\in\mathcal{F}\implies f^{-1}(V)\in\mathcal{A}\implies f^{-1}(X\setminus V)=\Omega\setminus f^{-1}(V)\in\mathcal{A}\implies X\setminus V\in\mathcal{F}$, which proves that $\mathcal{F}$ is indeed closed under complementations.

$\underline{\mathcal{F}\text{ is closed under countable unions}}$

$$\begin{aligned} V_1,V_2,\ldots \;\in\;\mathcal{F} &\implies f^{-1}(V_1),f^{-1}(V_2),\ldots \;\in\;\mathcal{A} \\ &\implies f^{-1}\left(\bigcup_{i=1}^{\infty}V_i\right) \;=\; \bigcup_{i=1}^{\infty}f^{-1}(V_i)\;\in\;\mathcal{A} \\ &\implies \bigcup_{i=1}^{\infty}V_i\;\in\;\mathcal{F}, \end{aligned}$$

which proves that $\mathcal{F}$ is indeed closed under countable unions. □

Theorem B.11
Suppose $(\Omega,\mathcal{A})$ and $(X,\mathcal{X})$ are measurable spaces, and $f:\Omega\longrightarrow X$ is a map from Ω into X. Then, f is $(\mathcal{A},\mathcal{X})$-measurable if there exists $\mathcal{S}\subset\mathcal{X}$ satisfying the following conditions:

- *$\mathcal{S}$ generates $\mathcal{X}$, i.e. $\sigma(\mathcal{S})=\mathcal{X}$, and*
- *$f^{-1}(\mathcal{S})\subset\mathcal{A}$.*

Proof By Lemma B.10,

$$\mathcal{F} \;:=\; \left\{\, V\subset X \;\middle|\; f^{-1}(V)\in\mathcal{A} \,\right\}$$

is a σ-algebra of subsets of X. By hypothesis, $\mathcal{S}\subset\mathcal{F}$; hence, $\mathcal{X}=\sigma(\mathcal{S})\subset\mathcal{F}$. Thus, $f^{-1}(\mathcal{X})\subset\mathcal{A}$; equivalently, f is $(\mathcal{A},\mathcal{X})$-measurable. □

Corollary B.12 (Continuous maps are Borel measurable.)
Suppose X_1, X_2 are topological spaces, and $\mathcal{B}_1$, $\mathcal{B}_2$ are their respective Borel σ-algebras. Then, every continuous map $f:X_1\longrightarrow X_2$ is $(\mathcal{B}_1,\mathcal{B}_2)$-measurable.

C Topology

Theorem C.1 (Appendix M3, [1])
Suppose S is a metric space. Then, the following conditions are equivalent:

(i) *S is separable.*

(ii) *The topology of S has a countable basis.*

(iii) *Every open cover of each subset of S has a countable subcover.*

References

[1] Billingsley, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.

[2] Billingsley, P. *Probability and Measure*, anniversary ed. John Wiley & Sons, 2012.