Separating and convergence-determining classes of $\mathbb{R}^n$, $\mathbb{R}^\infty$ and $C([0,1], \mathbb{R})$

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1 Separating and convergence-determining classes

Definition 1.1 (Separating class)

*Suppose Ω is a non-empty set, $\mathcal{A}$ is a σ-algebra of subsets of Ω, $(S, \mathcal{A})$ is the corresponding measurable space, and $\mathcal{M}_1(\Omega, \mathcal{A})$ is the set of all probability measures defined on $\mathcal{A}$. A **separating class** of subsets of $(\Omega, \mathcal{A})$ is a collection $\mathcal{S} \subset \mathcal{A}$ of subsets of Ω which satisfies the following condition: For every two probability measures $\mu, \nu \in \mathcal{M}_1(\Omega, \mathcal{A})$,*

$$\mu(S) = \nu(S), \text{ for every } S \in \mathcal{S} \quad \Longrightarrow \quad \mu(A) = \nu(A), \text{ for every } A \in \mathcal{A}$$

Definition 1.2 (Convergence-determining class)

*Suppose Ω is a topological space, $\mathcal{B}(\Omega)$ is its Borel σ-algebra, $(\Omega, \mathcal{B}(\Omega))$ is the corresponding measurable space, and $\mathcal{M}_1(\Omega, \mathcal{B}(S))$ is the set of all probability measures defined on $\mathcal{B}(\Omega)$. A **convergence-determining class** of subsets of $(\Omega, \mathcal{B}(\Omega))$ is a collection $\mathcal{C} \subset \mathcal{B}(\Omega)$ of Borel subsets of Ω which satisfies the following condition: For any $\mu, \mu_1, \mu_2, \ldots \in \mathcal{M}_1(\Omega, \mathcal{B}(\Omega))$,*

$$\lim_{n \to \infty} \mu_n(C) = \mu(C), \text{ for every } C \in \mathcal{C} \quad \Longrightarrow \quad \mu_n \xrightarrow{w} \mu.$$

Theorem 1.3

Suppose Ω is a non-empty set, $\mathcal{A}$ is a σ-algebra of subsets of Ω, and $(\Omega, \mathcal{A})$ is the corresponding measurable space.

If

- *$\mathcal{S} \subset \mathcal{A}$ is closed under finite intersections, and*

- *$\mathcal{S}$ generates $\mathcal{A}$ (i.e. $\sigma(\mathcal{S}) = \mathcal{A}$),*

then $\mathcal{S}$ is a separating class of subsets of $(\Omega, \mathcal{A})$.

PROOF   Let μ and ν be two probability measures defined on $(\Omega, \mathcal{A})$ such that $\mu(S) = \nu(S)$ for each $S \in \mathcal{S}$. We need to show that $\mu(A) = \nu(A)$ for each $A \in \mathcal{A}$. To this end, let

$$\mathcal{L} := \{ A \in \mathcal{A} = \sigma(\mathcal{S}) \mid \mu(A) = \nu(A) \}.$$

Note that $\mathcal{S} \subset \mathcal{L}$, by the hypothesis that μ and ν agree on $\mathcal{S}$, and $\mathcal{L} \neq \varnothing$ since $\Omega \in \mathcal{L}$. By Corollary B.8, it suffices to establish that $\mathcal{L}$ is a λ-system, since then it will follow that

$$\mathcal{A} = \sigma(\mathcal{S}) \subset \mathcal{L} := \{ A \in \mathcal{A} = \sigma(\mathcal{S}) \mid \mu(A) = \nu(A) \} \subset \sigma(\mathcal{S}) = \mathcal{A},$$

i.e., $\mathcal{A} = \sigma(\mathcal{S}) = \mathcal{L}$, or equivalently, μ and ν agree on all of $\mathcal{A} = \sigma(\mathcal{S})$. Now, we have already noted that $\Omega \in \mathcal{L}$. For $A \in \mathcal{L}$, we have

$$\mu(\Omega \setminus A) = 1 - \mu(A) = 1 - \nu(A) = \nu(\Omega \setminus A),$$

hence $\Omega \setminus A \in \mathcal{L}$. Thus, $\mathcal{L}$ is closed under complementations. Lastly, let $A_1, A_2, \ldots \in \mathcal{L}$ be pairwise disjoint. Then,

$$\mu\left(\bigsqcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \nu(A_n) = \nu\left(\bigsqcup_{n=1}^{\infty} A_n \right),$$

thus $\bigsqcup_{n=1}^{\infty} A_n \in \mathcal{L}$, which proves that $\mathcal{L}$ is closed under countable disjoint unions. $\mathcal{L}$ is therefore indeed a λ-system and the proof of the Theorem is complete.   □

Corollary 1.4   *Suppose S is a topological space and $\mathcal{B}(S)$ is its Borel σ-algebra (i.e. the σ-algebra generated by the collection of open subsets of S). Then, the collection of open subsets of S is a separating class of subsets of the measurable space $(S, \mathcal{B}(S))$.*

PROOF   Recall that the collection of open sets are closed under finite intersections (by definition of topology), and they generate the Borel σ-algebras (by definition of Borel σ-algebras). Thus the Corollary follows immediately from Theorem 1.3.   □

2   Examples of separating and convergence-determining classes of $\mathbb{R}^\infty$

Definition 2.1 (The metric on $\mathbb{R}^\infty$, Example 1.2, [1])

Let $\mathbb{R}^\infty$ denotes the set of all infinite sequences of real numbers, i.e.

$$\mathbb{R}^\infty \;:=\; \{\,(x_1, x_2, \dots)\mid x_i \in \mathbb{R},\text{ for each } i \in \mathbb{N}\,\}.$$

Define $\rho : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow [\,0,1\,]$ as follows:

$$\rho(x,y) \;:=\; \sum_{n=1}^{\infty} \frac{\min\{\,1\,,\,|\,x_n - y_n\,|\,\}}{2^n}.$$

Remark 2.2 *Recall that*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \;=\; \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \;=\; \frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}\right) \;=\; 1,$$

which proves indeed that $0 \le \rho(x,y) \le 1$, for any $x, y \in \mathbb{R}^\infty$.

Theorem 2.3 (The metric space properties of $\mathbb{R}^\infty$)

(i)   *$(\mathbb{R}^\infty, \rho)$ is a metric space. Let $\mathbb{R}^\infty$ denote also this metric space in the remainder of this Theorem.*

(ii)   *For $x, x^{(1)}, x^{(2)}, x^{(3)}, \dots, \in \mathbb{R}^\infty$, we have:*

$$\rho\left(x^{(n)},\, x\right) \;\longrightarrow\; 0 \quad\Longleftrightarrow\quad \text{for each } i \in \mathbb{N},\;\; \lim_{n\to\infty}\left|x_i^{(n)} - x_i\right| \;=\; 0$$

(iii)   *For each $n \in \mathbb{N}$, the "natural projection to the initial segment of length n"*

$$\pi_n \;:\; \mathbb{R}^\infty \;\longrightarrow\; \mathbb{R}^n \;:\; x \;\longmapsto\; (x_1, x_2, \dots, x_n)$$

is continuous, where $\mathbb{R}^n$ has the usual Euclidean topology.

(iv)   *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, let $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)$ denote the open hypercube in $\mathbb{R}^n$ of side length 2ε centred at $\pi_n(x) \in \mathbb{R}^n$, i.e.*

$$C_{\mathbb{R}^n}(\pi_n(x), \varepsilon) \;:=\; \left\{\, y \in \mathbb{R}^n \;\middle|\; \begin{array}{c}|\,y_i - x_i\,| < \varepsilon,\\ i = 1, 2, \dots, n\end{array} \right\}$$

Then, its pre-image in $\mathbb{R}^\infty$ under π_n

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)\,) \;=\; \left\{\, y \in \mathbb{R}^\infty \;\middle|\; \begin{array}{c}|\,y_i - x_i\,| < \varepsilon,\\ i = 1, 2, \dots, n\end{array} \right\}$$

is an open subset of $\mathbb{R}^\infty$.

(v)   *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, we have:*

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)\,) \;\subset\; B_{\mathbb{R}^\infty}\left(x\,,\, \varepsilon + \frac{1}{2^n}\right),$$

where $B_{\mathbb{R}^\infty}\left(x\,,\, \varepsilon + \dfrac{1}{2^n}\right)$ is the open ball in $\mathbb{R}^\infty$ centred at x of radius $\varepsilon + \dfrac{1}{2^n}$, i.e.

$$B_{\mathbb{R}^\infty}\left(x\,,\, \varepsilon + \frac{1}{2^n}\right) \;:=\; \left\{\, y \in \mathbb{R}^\infty \;\middle|\; \rho(y,x) < \varepsilon + \frac{1}{2^n}\right\}$$

(vi)   *The collection*

$$\left\{ \, \pi_n^{-1}\big(\, C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)\,\big) \subset \mathbb{R}^\infty \,\,\middle|\,\, n \in \mathbb{N},\, x \in \mathbb{R}^\infty,\, \varepsilon > 0 \,\right\}$$

of all pre-images under π_n of open hypercubes in $\mathbb{R}^n$, for all $n \in \mathbb{N}$, forms a basis for the topology of $\mathbb{R}^\infty$.

(vii)   *$\mathbb{R}^\infty$ is a separable metric space.*

(viii)   *$\mathbb{R}^\infty$ is a complete metric space.*

PROOF

(i)   Clearly, ρ is non-negative and symmetric. We now show that, for any $x, y \in \mathbb{R}^\infty$, we have $\rho(x, y) = 0$ implies $x = y$. Indeed,

$$\rho(x, y) = 0 \iff \sum_{i=1}^\infty \frac{\min\{\,1\,,\,|\,x_i - y_i\,|\,\}}{2^i} = 0$$
$$\iff \min\{\,1\,,\,|\,x_i - y_i\,|\,\} = 0, \text{ for each } i \in \mathbb{N}$$
$$\iff |\,x_i - y_i\,| = 0, \text{ for each } i \in \mathbb{N}$$
$$\iff x = y.$$

In order to show that ρ is a metric, it remains only to establish the Triangle Inequality. By Lemma A.2, for any $x, y, z \in \mathbb{R}^\infty$, we have

$$\rho(x, y) = \sum_{i=1}^\infty \frac{\min\{\,1\,,\,|\,x_i - y_i\,|\,\}}{2^i}$$
$$\leq \sum_{i=1}^\infty \frac{\min\{\,1\,,\,|\,x_i - z_i\,|\,\} + \min\{\,1\,,\,|\,z_i - y_i\,|\,\}}{2^i}$$
$$= \sum_{i=1}^\infty \frac{\min\{\,1\,,\,|\,x_i - z_i\,|\,\}}{2^i} + \sum_{i=1}^\infty \frac{\min\{\,1\,,\,|\,z_i - y_i\,|\,\}}{2^i}$$
$$= \rho(x, z) + \rho(z, y),$$

where we have used the fact that $0 \leq \rho \leq 1$ to split the infinite sum into two terms in second-to-last equality. This proves that ρ satisfies the Triangle Inequality, and it is thus a metric on $\mathbb{R}^\infty$.

(ii) $\underline{\lim_{n\to\infty} \rho\big(x^{(n)}, x\big) = 0 \implies \lim_{n\to\infty} \left|\,x_i^{(n)} - x_i\,\right| = 0, \text{ for each } i \in \mathbb{N}}$

$$\lim_{n\to\infty} \rho\big(x^{(n)}, x\big) = 0 \implies \lim_{n\to\infty} \sum_{i=1}^\infty \frac{\min\{\,1\,,\,|\,x_i^{(n)} - x_i\,|\,\}}{2^i} = 0$$
$$\implies \lim_{n\to\infty} \min\{\,1\,,\,|\,x_i^{(n)} - x_i\,|\,\} = 0, \text{ for each } i \in \mathbb{N}$$
$$\implies \lim_{n\to\infty} \left|\,x_i^{(n)} - x_i\,\right| = 0, \text{ for each } i \in \mathbb{N}$$

$\underline{\lim_{n\to\infty} \left|\,x_i^{(n)} - x_i\,\right| = 0, \text{ for each } i \in \mathbb{N} \implies \lim_{n\to\infty} \rho\big(x^{(n)}, x\big) = 0}$

This follows from the Weierstrass M-test. Suppose $\lim_{n\to\infty} \left|\,x_i^{(n)} - x_i\,\right| = 0$, for each $i \in \mathbb{N}$. Then,

$$\lim_{n\to\infty} \frac{\min\{\,1\,,\,|\,x_i^{(n)} - x_i\,|\,\}}{2^i} = 0 =: y_i, \text{ for each } i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$, let $M_i := \dfrac{1}{2^i}$. Then,

$$\frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} \leq M_i \quad \text{and} \quad \sum_{i=1}^{\infty} M_i < \infty.$$

Hence, by the Weierstrass M-test (Lemma A.3), we have

$$\lim_{n \to \infty} \rho\left(x^{(n)}, x\right) = \lim_{n \to \infty} \sum_{i=1}^{\infty} \frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} = \sum_{i=1}^{\infty} y_i = 0.$$

(iii)   Immediate by (ii).

(iv)   Since $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon) \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, its pre-image under the continuous (by (iii)) map $\pi_n : \mathbb{R}^\infty \longrightarrow \mathbb{R}^n$ is an open subset of $\mathbb{R}^\infty$.

(v)   For $y \in \mathbb{R}^\infty$, we have

$$y \in \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \implies |y_i - x_i| < \varepsilon, \text{ for each } i = 1, 2, \ldots, n$$

$$\implies \rho(x, y) := \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - y_i|\}}{2^i} \leq \sum_{i=1}^{n} \frac{\varepsilon}{2^i} + \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \varepsilon + \frac{1}{2^n}.$$

This proves:

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right).$$

(vi)   It suffices to show that every open ball in $B_{\mathbb{R}^\infty}(x, r) \subset \mathbb{R}^\infty$, $r > 0$, contains the pre-image of an open hypercube centred at $\pi_n(x) \in \mathbb{R}^n$ under π_n. To this end, for $r > 0$, choose $\varepsilon > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large such that $\varepsilon + \dfrac{1}{2^n} < r$. Then, for any $x \in \mathbb{R}^\infty$, by (v), we have:

$$x \in \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right) \subset B_{\mathbb{R}^\infty}(x, r),$$

as required.

(vii)   It suffices to exhibit a countable subset of $\mathbb{R}^\infty$ that intersects every open ball in $\mathbb{R}^\infty$. To this end, let

$$D := \bigcup_{n=1}^{\infty} \left\{ x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty \;\middle|\; \begin{array}{l} x_i \in \mathbb{Q}, \text{ for each } i \in \mathbb{N} \\ x_i = 0, \text{ for all } i \geq n \end{array} \right\}.$$

Clearly, D is a countable subset of $\mathbb{R}^\infty$. Now let $B_{\mathbb{R}^\infty}(x, \varepsilon)$ be an arbitrary open ball in $\mathbb{R}^\infty$. Choose $\delta > 0$ small enough and $n \in \mathbb{N}$ large enough such that $\delta + \dfrac{1}{2^n} < \varepsilon$. Then,

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \delta)) \subset B_{\mathbb{R}^\infty}\left(x, \delta + \frac{1}{2^n}\right) \subset B_{\mathbb{R}^\infty}(x, \varepsilon),$$

Now, for each $i = 1, 2, \ldots, n$, choose $z_i \in \mathbb{Q} \cap (x_i - \delta, x_i + \delta)$. Let $z = (z_1, z_2, \ldots, z_n, 0, 0, \ldots) \in \mathbb{R}^\infty$. Then, we have

$$z \in D \bigcap \left\{ y \in \mathbb{R}^\infty \;\middle|\; \begin{array}{l} y_i \in (x_i - \delta, x_i + \delta), \\ \text{for each } i = 1, 2, \ldots, n \end{array} \right\} = D \bigcap \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \delta)) \subset D \bigcap B_{\mathbb{R}^\infty}(x, \varepsilon).$$

This proves the the countable subset $D \subset \mathbb{R}^\infty$ has non-empty intersection with every open ball in $\mathbb{R}^\infty$, i.e. D is dense in $\mathbb{R}^\infty$. Hence, $\mathbb{R}^\infty$ is separable.

(viii)   We need to show that every Cauchy sequence in $\mathbb{R}^\infty$ converges to any element in $\mathbb{R}^\infty$.

$$\left\{ x^{(n)} \right\}_{n \in \mathbb{N}} \subset \mathbb{R}^\infty \text{ is a Cauchy sequence in } \mathbb{R}^\infty$$

$\Longleftrightarrow$ for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $\rho\left(x^{(m)}, x^{(n)}\right) < \varepsilon$, for any $m, n > N_\varepsilon$

$\Longrightarrow$ for each $i \in \mathbb{N}$, we have:

for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $\left| x_i^{(m)} - x_i^{(n)} \right| < \varepsilon$, for any $m, n > N_\varepsilon$

$\Longrightarrow$ for each $i \in \mathbb{N}$, $\left\{ x_i^{(n)} \right\}_{n \in \mathbb{N}} \subset \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$; hence $x_i := \lim_{n \to \infty} x_i^{(n)} \in \mathbb{R}$ exists

$\Longrightarrow$ $\lim_{n \to \infty} \rho\left(x^{(n)}, x\right) = 0$, where $x := (x_1, x_2, \dots) \in \mathbb{R}^\infty$ (by (ii))

This proves that $\mathbb{R}^\infty$ indeed is a complete metric space.

$\square$

Definition 2.4

*The **finite-dimensional class** of subsets of $\mathbb{R}^\infty$ is, by definition, the following:*

$$\mathcal{B}_f(\mathbb{R}^\infty) := \left\{ \pi_k^{-1}(B) \subset \mathbb{R}^\infty \ \middle| \ \begin{array}{c} k \in \mathbb{N} \\ B \in \mathcal{B}(\mathbb{R}^k) \end{array} \right\},$$

where $\pi_k : \mathbb{R}^\infty \longrightarrow \mathbb{R}^k : x = (x_1, x_2, \dots) \longmapsto (x_1, \dots, x_k)$ is the projection of $\mathbb{R}^\infty$ onto $\mathbb{R}^k$.

Theorem 2.5

(i) $\mathcal{B}_f(\mathbb{R}^\infty) \subset \mathcal{B}(\mathbb{R}^\infty)$.

(ii) $\mathcal{B}_f(\mathbb{R}^\infty)$ *is a separating class of Borel subsets of $\mathbb{R}^\infty$.*

(iii) $\mathcal{B}_f(\mathbb{R}^\infty)$ *is a convergence-determining class of Borel subsets of $\mathbb{R}^\infty$.*

PROOF

(i)   Note that

$$\mathcal{B}_f(\mathbb{R}^\infty) := \left\{ \pi_k^{-1}(B) \subset \mathbb{R}^\infty \ \middle| \ \begin{array}{c} k \in \mathbb{N} \\ B \in \mathcal{B}(\mathbb{R}^k) \end{array} \right\} = \bigcup_{k=1}^{\infty} \pi_k^{-1}\left(\mathcal{B}(\mathbb{R}^k)\right).$$

Thus, (i) is equivalent to the statement that each $\pi_k : \mathbb{R}^\infty \longrightarrow \mathbb{R}^k$ is Borel measurable. But each π_k is continuous, hence Borel measurable (Corollary B.12). This proves (i).

(ii)   We apply Theorem 1.3 to $\mathcal{B}_f(\mathbb{R}^\infty)$.

$\underline{\mathcal{B}_f(\mathbb{R}^\infty) \text{ is closed under finite intersections}}$

Let $\pi_k^{-1}(A)$ and $\pi_l^{-1}(B) \in \mathcal{B}_f(\mathbb{R}^\infty)$. Note that this implies $A \in \mathcal{B}(\mathbb{R}^k)$ and $B \in \mathcal{B}(\mathbb{R}^l)$. We need to show that $\pi_k^{-1}(A) \cap \pi_l^{-1}(B) \in \mathcal{B}_f(\mathbb{R}^\infty)$. Now, if $k = l$, this is immediately, since then $A \cap B \in \mathcal{B}(\mathbb{R}^k)$, and

$$\pi_k^{-1}(A) \cap \pi_l^{-1}(B) = \pi_k^{-1}(A) \cap \pi_k^{-1}(B) = \pi_k^{-1}(A \cap B) \in \mathcal{B}_f(\mathbb{R}^\infty).$$

For the case $k \neq l$, without loss of generality, assume $k < l$. Then, note that

$$\pi_k^{-1}(A) = \left\{ y = (y_1, y_2, \dots) \in \mathbb{R}^\infty \ \middle| \ (y_1, \dots, y_k) \in A \right\}$$

$$= \left\{ y = (y_1, y_2, \dots) \in \mathbb{R}^\infty \ \middle| \ (y_1, \dots, y_k, y_{k+1}, \dots, y_l) \in A \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{k \text{ factors}} \right\}$$

$$= \pi_l^{-1}(A \times \mathbb{R} \times \cdots \times \mathbb{R}).$$

Since $(A \times \mathbb{R} \cdots \times \mathbb{R}) \cap B \in \mathcal{B}(\mathbb{R}^l)$, we now see that

$$\pi_k^{-1}(A) \cap \pi_l^{-1}(B) \; = \; \pi_l^{-1}(A \times \mathbb{R} \cdots \times \mathbb{R}) \cap \pi_l^{-1}(B) \; = \; \pi_l^{-1}((A \times \mathbb{R} \cdots \times \mathbb{R}) \cap B) \; \in \; \mathcal{B}_f(\mathbb{R}^\infty).$$

This proves that $\mathcal{B}_f(\mathbb{R}^\infty)$ is indeed closed under finite intersections.

$\mathcal{B}_f(\mathbb{R}^\infty)$ generates $\mathcal{B}(\mathbb{R}^\infty)$

Let $\mathcal{O}(\mathbb{R}^\infty)$ denote the collection of open sets of $\mathbb{R}^\infty$. Hence $\mathcal{B}(\mathbb{R}^\infty) := \sigma(\mathcal{O}(\mathbb{R}^\infty))$. By (i), we have $\mathcal{B}_f(\mathbb{R}^\infty) \subset \mathcal{B}(\mathbb{R}^\infty) = \sigma(\mathcal{O}(\mathbb{R}^\infty))$, which implies $\sigma(\mathcal{B}_f(\mathbb{R}^\infty)) \subset \sigma(\mathcal{O}(\mathbb{R}^\infty))$. We need to establish the reverse inclusion, which will immediately follow from:

Claim: $\mathcal{O}(\mathbb{R}^\infty) \subset \sigma(\mathcal{B}_f(\mathbb{R}^\infty))$.

Proof of Claim: By Theorem 2.3(v), every open ball $B_{\mathbb{R}^\infty}(x, \varepsilon)$ in $\mathbb{R}^\infty$ contains the pre-image of an open hypercube from some finite-dimensional Euclidean space, where that pre-image itself contains x. We therefore see that every open set in $\mathbb{R}^\infty$ can be expressed as a union of pre-images of open hypercubes from finite-dimensional Euclidean spaces. By Theorem 2.3(vii), $\mathbb{R}^\infty$ is separable. Hence, by Theorem C.1, we see that every open set in $\mathbb{R}^\infty$ can be expressed as a countable union of pre-images of open hypercubes from finite-dimensional Euclidean spaces. Since pre-images of open hypercubes from finite-dimensional Euclidean spaces belong to $\mathcal{B}_f(\mathbb{R}^\infty)$, we see that $\mathcal{O}(\mathbb{R}^\infty) \subset \sigma(\mathcal{B}_f(\mathbb{R}^\infty))$. This completes the proof of the Claim.

We have established that $\mathcal{B}_f(\mathbb{R}^\infty)$ is contained in $\mathcal{B}(\mathbb{R}^\infty)$, is closed under finite intersections, and $\sigma(\mathcal{B}_f(\mathbb{R}^\infty)) = \mathcal{B}_f(\mathbb{R}^\infty)$. Therefore, by Theorem 1.3, $\mathcal{B}_f(\mathbb{R}^\infty)$ is a separating class for the measurable space $(\mathbb{R}^\infty, \mathcal{B}_f(\mathbb{R}^\infty))$.

$\square$

A   Technical Lemmas

Lemma A.1 *Define*

$$\phi \, : \, [0, \infty) \; \longrightarrow \; [0, 1] \, : \, t \, \longmapsto \, \min\{1, t\}.$$

Then, ϕ satisfies:

$$\phi(s + t) \; \leq \; \phi(s) + \phi(t), \; \text{ for each } s, t \in [0, \infty).$$

PROOF   For any $s, t \in [0, \infty)$, either $s + t \geq 1$ or $s + t < 1$. If $s + t \geq 1$, then

$$\phi(s + t) \; = \; \min\{1, s + t\} \; = \; 1 \; < \; 2 \; = \; 1 + 1 \; \leq \; \min\{1, s\} + \min\{1, t\} \; = \; \phi(s) + \phi(t),$$

hence, the required inequality holds. On the other hand, if $s + t < 1$, then we must also have $s < 1$ and $t < 1$ (since $s, t \geq 0$). Hence,

$$\phi(s + t) \; = \; \min\{1, s + t\} \; = \; s + t \; = \; \min\{1, s\} + \min\{1, t\} \; = \; \phi(s) + \phi(t),$$

thus, the required inequality also holds. $\square$

Lemma A.2 *For any $x, y, z \in \mathbb{R}$, we have:*

$$\min\{1, |x - y|\} \; \leq \; \min\{1, |x - z|\} \; + \; \min\{1, |z - y|\}.$$

PROOF   Observe that $|x - y| \leq |x - z| + |z - y|$ implies

$$\min\{1, |x - y|\} \; \leq \; |x - z| + |z - y|.$$

The above inequality, together with $\min\{\,1\,,|\,x-y\,|\,\} \leq 1$, thus in turn imply:

$$\min\{\,1\,,|\,x-y\,|\,\} \leq \min\{\,1\,,|\,x-z\,|+|\,z-y\,|\,\}.$$

By Lemma A.1, we therefore have:

$$\min\{\,1\,,|\,x-y\,|\,\} \leq \min\{\,1\,,|\,x-z\,|+|\,z-y\,|\,\}. \leq \min\{\,1\,,|\,x-z\,|\,\} + \min\{\,1\,,|\,z-y\,|\,\},$$

which proves the present Lemma. $\qquad\square$

Lemma A.3 (The Weierstrass M-test, Theorem A.28, [2])

Suppose that $\lim\limits_{n\to\infty} x_i^{(n)} = x_i$, for each $i \in \mathbb{N}$, and that $\left| x_i^{(n)} \right| \leq M_i$, where $\sum\limits_{i=1}^{\infty} M_i < \infty$. Then,

(i) $\sum\limits_{i=1}^{\infty} x_i$ *exists, and* $\sum\limits_{i=1}^{\infty} x_i^{(n)}$ *exists for each $n \in \mathbb{N}$.*

(ii)   *Furthermore,*

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

PROOF

(i) $\sum\limits_{i=1}^{\infty} M_i < \infty$ and $\left| x_i^{(n)} \right| \leq M_i$ $\implies$ the series $\sum\limits_{i=1}^{\infty} x_i$ and $\sum\limits_{i=1}^{\infty} x_i^{(n)}$, $n \in \mathbb{N}$, converge absolutely.

(ii)   Let $\varepsilon > 0$ be given. Choose $K \in \mathbb{N}$ sufficiently large such that $\sum\limits_{j=K+1}^{\infty} M_i < \dfrac{\varepsilon}{3}$. Next, choose $N \in \mathbb{N}$ sufficiently large such that

$$\left| x_i^{(n)} - x_i \right| < \frac{\varepsilon}{3K}, \text{ for any } n > N \text{ and } i = 1, 2, \ldots, K.$$

Then, we have, for each $n > N$,

$$\left| \sum_{i=1}^{\infty} x_i^{(n)} - \sum_{i=1}^{\infty} x_i \right| = \left| \sum_{i=1}^{K} \left(x_i^{(n)} - x_i \right) + \sum_{i=K+1}^{\infty} x_i^{(n)} - \sum_{i=K+1}^{\infty} x_i \right|$$

$$\leq \sum_{i=1}^{K} \left| x_i^{(n)} - x_i \right| + \sum_{i=K+1}^{\infty} \left| x_i^{(n)} \right| + \sum_{i=K+1}^{\infty} |\,x_i\,|$$

$$\leq K \cdot \frac{\varepsilon}{3K} + \sum_{i=K+1}^{\infty} M_i + \sum_{i=K+1}^{\infty} M_i \leq \frac{\varepsilon}{3} + 2 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Since ε is arbitrary, this proves:

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

$\qquad\square$

B σ-algebras and λ-systems

Definition B.1

Suppose Ω is a non-empty set. A σ-algebra of subsets of Ω is a collection $\mathcal{A}$ of subsets of Ω which satisfies the following conditions:

- $\Omega \in \mathcal{A}$.

- $\Omega \setminus A \in \mathcal{A}$, *for every* $A \in \mathcal{A}$.

- $\displaystyle\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, *whenever* $A_1, A_2, \ldots \in \mathcal{A}$

Definition B.2

Suppose Ω is a non-empty set. A λ-system of subsets of Ω is a collection $\mathcal{L}$ of subsets of Ω which satisfies the following conditions:

- $\Omega \in \mathcal{L}$.

- $\Omega \setminus A \in \mathcal{L}$, *for every* $A \in \mathcal{L}$.

- $\displaystyle\bigsqcup_{i=1}^{\infty} A_i \in \mathcal{L}$, *whenever* $A_1, A_2, \ldots \in \mathcal{L}$ *and* $A_i \cap A_j = \varnothing$, *for any* $i, j \in \mathbb{N}$ *with* $i \neq j$.

Remark B.3   *Clearly, every σ-algebra is also a λ-system.*

Theorem B.4

Suppose Ω is a non-empty set and $\mathcal{L}$ is a λ-system of subsets of Ω.

(i)   *$\mathcal{L}$ is closed under proper set-theoretic differences, i.e. $A, B \in \mathcal{L}$ and $A \subset B$ together imply $B \setminus A \in \mathcal{L}$.*

(ii)   *If $\mathcal{L}$ is closed under finite intersections, then $\mathcal{L}$ is a σ-algebra of subsets of Ω.*

PROOF   For each $X \subset \Omega$, write $\Omega \setminus X$ as X^c.

(i)   Suppose $A, B \in \mathcal{L}$ with $A \subset B$. Then, $B^c \cap A = \varnothing$. Hence, $B \setminus A = B \cap A^c = (B^c \cup A)^c = (B^c \sqcup A)^c \in \mathcal{L}$, since $\mathcal{L}$ is closed under complementations and finite disjoint unions.

(ii)   Since $\mathcal{L}$ is a λ-system, we immediately have $\Omega \in \mathcal{L}$, and hence $\Omega \setminus A \in \mathcal{L}$, for every $A \in \mathcal{L}$. It remains to show that $\mathcal{L}$ closed under countable unions, i.e. for $A_1, A_2, \ldots \in \mathcal{L}$, we need to show $\bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$. To this end, define:

$$
\begin{aligned}
B_1 &:= A_1 \\
B_2 &:= A_2 \cap A_1^c \\
B_3 &:= A_3 \cap A_1^c \cap A_2^c \\
&\vdots \\
B_n &:= A_n \cap A_1^c \cap A_2^c \cap \cdots \cap A_n^c
\end{aligned}
$$

Being a λ-system, $\mathcal{L}$ is closed under complementations. By hypothesis, $\mathcal{L}$ is furthermore closed under finite intersections. We thus see that $B_n \in \mathcal{L}$, for each $n \in \mathbb{N}$. Note also that the B_n's are pairwise disjoint, and

$$
\bigcup_{i=1}^{n} A_i = \bigsqcup_{i=1}^{n} B_i, \quad \text{for each } n \in \mathbb{N}.
$$

Hence,

$$\bigcup_{i=1}^{\infty} A_i \;=\; \biguplus_{i=1}^{\infty} B_i \;\in\; \mathcal{L},$$

since $\mathcal{L}$ is closed under countable pairwise disjoint unions ($\mathcal{L}$ being a λ-system). This proves that $\mathcal{L}$ is a σ-algebra of subsets of Ω.

$\square$

Theorem B.5 $\quad$ *Let Ω be a non-empty set.*

(i) $\quad$ *The intersection of a non-empty collection of σ-algebras of subsets of Ω is itself a σ-algebra of subsets of Ω.*

(ii) $\quad$ *The intersection of a non-empty collection of λ-systems of subsets of Ω is itself a λ-system of subsets of Ω.*

PROOF

(i) $\quad$ Suppose Γ is an (arbitrary) non-empty set, and, for each $\gamma \in \Gamma$, $\mathcal{A}_\gamma$ is a σ-algebra of subsets of Ω. We need to prove that $\mathcal{A} := \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma$ is itself a σ-algebra of subsets of Ω.

$\underline{\Omega \;\in\; \mathcal{A} := \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma}$

Since, for each $\gamma \in \Gamma$, $\mathcal{A}_\gamma$ is a σ-algebra of subsets of Ω, we have $\Omega \in \mathcal{A}_\gamma$. Thus, $\Omega \in \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma$.

$\underline{A \in \mathcal{A} \implies \Omega \setminus A \in \mathcal{A}}$

$$A \in \mathcal{A} := \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma \;\Longleftrightarrow\; A \in \mathcal{A}_\gamma,\, \forall\, \gamma \in \Gamma \;\implies\; \Omega \setminus A \in \mathcal{A}_\gamma,\, \forall\, \gamma \in \Gamma \;\implies\; \Omega \setminus A \in \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma =: \mathcal{A}$$

$\underline{A_1, A_2, \ldots \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}}$

$$A_1, A_2, \ldots \in \mathcal{A} := \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma \;\implies\; A_1, A_2, \ldots \in \mathcal{A}_\gamma,\, \forall\, \gamma \in \Gamma \;\implies\; \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_\gamma,\, \forall\, \gamma \in \Gamma$$

$$\implies\; \bigcup_{i=1}^{\infty} A_i \in \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma =: \mathcal{A}$$

(ii) $\quad$ Suppose Γ is an (arbitrary) non-empty set, and, for each $\gamma \in \Gamma$, $\mathcal{L}_\gamma$ is a λ-system of subsets of Ω. We need to prove that $\mathcal{L} := \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma$ is itself a λ-system of subsets of Ω.

$\underline{\Omega \;\in\; \mathcal{L} := \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma}$

Since, for each $\gamma \in \Gamma$, $\mathcal{L}_\gamma$ is a λ-system of subsets of Ω, we have $\Omega \in \mathcal{L}_\gamma$. Thus, $\Omega \in \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma$.

$$\underline{A \in \mathcal{L} \implies \Omega \setminus L \in \mathcal{L}}$$

$$A \in \mathcal{L} := \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma \iff A \in \mathcal{L}_\gamma, \, \forall \, \gamma \in \Gamma \implies \Omega \setminus A \in \mathcal{L}_\gamma, \, \forall \, \gamma \in \Gamma \implies \Omega \setminus A \in \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma =: \mathcal{L}$$

$$\underline{A_1, A_2, \ldots \in \mathcal{L} \text{ and } A_i \cap A_j \text{ whenever } i \neq j \implies \bigsqcup_{i=1}^{\infty} A_i \in \mathcal{L}}$$

$$A_1, A_2, \ldots \in \mathcal{L} := \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma, \text{ and } A_i \cap A_j \text{ whenever } i \neq j$$

$$\implies A_1, A_2, \ldots \in \mathcal{L}_\gamma, \, \forall \, \gamma \in \Gamma, \text{ and } A_i \cap A_j \text{ whenever } i \neq j$$

$$\implies \bigsqcup_{i=1}^{\infty} A_i \in \mathcal{L}_\gamma, \, \forall \, \gamma \in \Gamma$$

$$\implies \bigsqcup_{i=1}^{\infty} A_i \in \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma =: \mathcal{L}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem B.6   *Suppose Ω is a non-empty set, $\mathcal{S}$ is non-empty collection of subsets of Ω. Denote the power set of Ω by $\mathcal{P}(\Omega)$. Define*

$$\sigma(\mathcal{S}) := \bigcap_{\mathcal{A} \in \Sigma(\mathcal{S})} \mathcal{A}, \quad \text{where} \quad \Sigma(\mathcal{S}) := \left\{ \mathcal{A} \subset \mathcal{P}(\Omega) \, \middle| \, \begin{array}{c} \mathcal{A} \text{ is a } \sigma\text{-algebra of subsets of } \Omega, \\ \text{and } \mathcal{S} \subset \mathcal{A} \end{array} \right\}, \text{ and}$$

$$\lambda(\mathcal{S}) := \bigcap_{\mathcal{L} \in \Lambda(\mathcal{S})} \mathcal{L}, \quad \text{where} \quad \Lambda(\mathcal{S}) := \left\{ \mathcal{L} \subset \mathcal{P}(\Omega) \, \middle| \, \begin{array}{c} \mathcal{L} \text{ is a } \lambda\text{-system of subsets of } \Omega, \\ \text{and } \mathcal{S} \subset \mathcal{L} \end{array} \right\}.$$

Then, $\sigma(\mathcal{S})$ is the unique smallest σ-algebra of subsets of Ω that contains $\mathcal{S} \subset \mathcal{P}(\Omega)$, and $\lambda(\mathcal{S})$ is the unique smallest λ-system of subsets of Ω that contains $\mathcal{S} \subset \mathcal{P}(\Omega)$. More precisely, we have

- *$\mathcal{S} \subset \sigma(\mathcal{S})$, $\mathcal{S} \subset \lambda(\mathcal{S})$, and*

- *$\sigma(\mathcal{S})$ is a σ-algebra of subsets of Ω, and $\lambda(\mathcal{S})$ is a λ-system of subsets of Ω, and*

- *if $\mathcal{A} \subset \mathcal{P}(\Omega)$ is a σ-algebra and $\mathcal{S} \subset \mathcal{A}$, then $\sigma(\mathcal{S}) \subset \mathcal{A}$.*

- *if $\mathcal{L} \subset \mathcal{P}(\Omega)$ is a λ-system and $\mathcal{S} \subset \mathcal{L}$, then $\lambda(\mathcal{S}) \subset \mathcal{L}$.*

PROOF   First, note that $\Sigma(\mathcal{S}) \neq \varnothing$ since $\mathcal{P}(\Omega) \in \Sigma(\mathcal{S})$. Similarly, $\Lambda(\mathcal{S}) \neq \varnothing$ since $\mathcal{P}(\Omega) \in \Lambda(\mathcal{S})$. It is immediate that $\mathcal{S} \subset \sigma(\mathcal{S})$, and $\sigma(\mathcal{S})$ is contained in every σ-algebra which contains $\mathcal{S}$. Similarly, $\mathcal{S} \subset \lambda(\mathcal{S})$, and $\lambda(\mathcal{S})$ is contained in every λ-system which contains $\mathcal{S}$. Since $\sigma(\mathcal{S})$ is, by definition, an intersection of σ-algebras, it itself is a σ-algebra of subsets of Ω by Theorem B.5. Similarly, since $\lambda(\mathcal{S})$ is, by definition, an intersection of λ-systems, it itself is a λ-system of subsets of Ω by Theorem B.5. $\qquad\qquad$ $\square$

Theorem B.7   *Suppose Ω is a non-empty set and $\mathcal{S}$ is a non-empty collection of subsets of Ω. Then,*

$$\mathcal{S} \text{ is closed under finite intersections} \implies \lambda(\mathcal{S}) \text{ is a } \sigma\text{-algebra of subsets of } \Omega,$$

where $\lambda(\mathcal{S})$ is λ-system of subsets of Ω generated by $\mathcal{S}$.

PROOF   By Theorem B.4(ii), it suffices to show that $\lambda(\mathcal{S})$ is closed under finite intersections. We establish the proof in the following series of claims:

Claim 1: For each $A \in \lambda(\mathcal{S})$,

$$\mathcal{L}(A) := \{ B \subset \Omega \mid A \cap B \in \lambda(\mathcal{S}) \}$$

is a λ-system of subsets of Ω.

<u>Proof of Claim 1:</u> Clearly, $\Omega \in \mathcal{L}(A)$, since $A \cap \Omega = A \in \lambda(\mathcal{S})$. Next, we prove that $\mathcal{L}(A)$ is closed under complementations. Let $B \in \mathcal{L}(A)$. Then, $A \cap B \in \lambda(\mathcal{S})$. Note that $A = (A \cap B) \bigsqcup (A \cap B^c)$, hence $A \cap B^c = A \setminus (A \cap B) \in \lambda(\mathcal{S})$, since $A, A \cap B \in \lambda(\mathcal{S})$ and $\lambda(\mathcal{S})$ is closed under proper set-theoretic differences by Theorem B.4(i). This proves that $\mathcal{L}(A)$ is indeed closed under complementations. We now prove that $\mathcal{L}(A)$ is closed under countable disjoint unions. Let $B_1, B_2, \ldots \in \mathcal{L}(A)$ be pairwise disjoint. Then, $A \cap B_1, A \cap B_2, \ldots \subset \lambda(\mathcal{S})$ are pairwise disjoint. Hence,

$$A \bigcap \left(\bigsqcup_{i=1}^{\infty} B_i \right) = \bigsqcup_{i=1}^{\infty} (A \cap B_i) \in \lambda(\mathcal{S}),$$

since $\lambda(\mathcal{S})$ is closed under countable disjoint unions. This proves that $\mathcal{L}(A)$ is a λ-system and thus completes the proof of the Claim 1.

Claim 2: $\mathcal{S} \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$. Consequently, $\lambda(\mathcal{S}) \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$.

<u>Proof of Claim 2:</u> Suppose $A \in \mathcal{S}$. Then, $A \cap B \in \mathcal{S}$ for each $B \in \mathcal{S}$, by the hypothesis that $\mathcal{S}$ is closed under finite intersections. Thus, $A \cap B \in \lambda(\mathcal{S})$, since $\mathcal{S} \subset \lambda(\mathcal{S})$. Hence, $B \in \mathcal{L}(A)$, for any $A, B \in \mathcal{S}$. This proves that $\mathcal{S} \subset \mathcal{L}(A)$, for each $A \in \mathcal{S}$. By Claim 1, $\mathcal{L}(A)$ is a λ-system. Hence, $\mathcal{L}(A) \supset \lambda(\mathcal{S})$, the smallest λ-system containing $\mathcal{S}$. This proves Claim 2.

Claim 3: $A \cap B \in \lambda(\mathcal{S})$, for each $A \in \mathcal{S}$ and $B \in \lambda(\mathcal{S})$.

<u>Proof of Claim 3:</u> Let $A \in \mathcal{S}$ and $B \in \lambda(\mathcal{S})$. By Claim 2, we have $\lambda(\mathcal{S}) \subset \mathcal{L}(A)$. Thus we have $B \in \mathcal{L}(A)$, which is equivalent to $A \cap B \in \lambda(S)$. This proves Claim 3.

Claim 4: $\mathcal{S} \subset \mathcal{L}(B)$, for each $B \in \lambda(\mathcal{S})$. Consequently, $\lambda(\mathcal{S}) \subset \mathcal{L}(B)$, for each $B \in \lambda(\mathcal{S})$.

<u>Proof of Claim 4:</u> Suppose $B \in \lambda(\mathcal{S})$. Then, $A \cap B \in \lambda(\mathcal{S})$ for each $A \in \mathcal{S}$, by Claim 3. This proves that $\mathcal{S} \subset \mathcal{L}(B)$. By Claim 1, $\mathcal{L}(B)$ is a λ-system. Hence, $\mathcal{L}(B) \supset \lambda(\mathcal{S})$, the smallest λ-system containing $\mathcal{S}$. This proves Claim 4.

Claim 5: $A \cap B \in \lambda(\mathcal{S})$, for each $A, B \in \lambda(\mathcal{S})$.

<u>Proof of Claim 5:</u> Let $A, B \in \lambda(\mathcal{S})$. By Claim 4, we have $\lambda(\mathcal{S}) \subset \mathcal{L}(B)$. Thus we have $A \in \mathcal{L}(B)$, which is equivalent to $A \cap B \in \lambda(S)$. This proves Claim 5.

Claim 5 states precisely that $\lambda(\mathcal{S})$ is closed under finite intersections, and completes the proof. $\square$

Corollary B.8 *Suppose Ω is a non-empty set and $\mathcal{S}$ is a non-empty collection of subsets of Ω.*
If $\mathcal{S}$ is closed under finite intersections, then

(i) *$\sigma(S) \subset \lambda(\mathcal{S})$, and*

(ii) *$\sigma(S) \subset \mathcal{L}$, for any λ-system $\mathcal{L}$ of subsets of Ω such that $\mathcal{S} \subset \mathcal{L}$,*

where $\sigma(\mathcal{S})$ and $\lambda(\mathcal{S})$ are, respectively, the σ-algebra and λ-system of subsets of Ω generated by $\mathcal{S}$.

PROOF

(i)   By Theorem B.6, $\lambda(\mathcal{S})$ is the smallest λ-system containing $\mathcal{S}$. Since $\mathcal{S}$ is, by hypothesis, closed under finite intersections, $\lambda(\mathcal{S})$ is furthermore a σ-algebra, by Theorem B.7. Thus, by Theorem B.6 again, we have $\sigma(S) \subset \lambda(\mathcal{S})$.

(ii)   This is now immediate since
$$\sigma(S) \ \subset \ \lambda(\mathcal{S}) \ \subset \ \mathcal{L},$$
where the first inclusion follows by (i), and the second inclusion follows by Theorem B.6.

$\square$

Lemma B.9 (The pre-image of a σ-algebra is itself a σ-algebra.)
Suppose Ω is a non-empty set, $(X, \mathcal{X})$ is a measurable space, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then,
$$f^{-1}(\mathcal{X}) \ := \ \left\{ \ f^{-1}(V) \subset \Omega \ | \ V \in \mathcal{X} \ \right\}$$
is a σ-algebra of subsets of Ω.

PROOF

$\underline{\Omega \in f^{-1}(\mathcal{X})}$ $f(\Omega) \subset X \implies \Omega = f^{-1}(X) \in f^{-1}(\mathcal{X})$.

$\underline{f^{-1}(\mathcal{X}) \text{ is closed under complementations}}$ Let $V \in \mathcal{X}$. Then, $X \setminus V \in \mathcal{X}$, and
$$\Omega \setminus f^{-1}(V) \ = \ \left\{ \ \omega \in \Omega \ | \ f(\omega) \notin V \ \right\} \ = \ \left\{ \ \omega \in \Omega \ | \ f(\omega) \in X \setminus V \ \right\} \ = \ f^{-1}(X \setminus V) \ \in \ f^{-1}(\mathcal{X}),$$
which shows that $f^{-1}(\mathcal{X})$ is indeed closed under complementations.

$\underline{f^{-1}(\mathcal{X}) \text{ is closed countable unions}}$ Let $V_1, V_2, \ldots \in \mathcal{X}$. Then, $\bigcup_{i=1}^{\infty} V_i \in \mathcal{X}$, and
$$\bigcup_{i=1}^{\infty} f^{-1}(V_i) \ = \ \left\{ \ \omega \in \Omega \ \Bigg| \ \begin{array}{c} f(\omega) \in V_i \\ \text{for some } i \in \mathbb{N} \end{array} \ \right\} \ = \ f^{-1}\left(\bigcup_{i=1}^{\infty} V_i \right) \ \in \ f^{-1}(\mathcal{X}),$$
which proves that $f^{-1}(\mathcal{X})$ is indeed closed under countable unions.

This concludes the proof that that $f^{-1}(\mathcal{X})$ is a σ-algebra of subsets of Ω. $\square$

Lemma B.10
Suppose $(\Omega, \mathcal{A})$ is a measurable space, X is a non-empty set, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then,
$$\mathcal{F} \ := \ \left\{ \ V \subset X \ \big| \ f^{-1}(V) \in \mathcal{A} \ \right\}$$
is a σ-algebra of subsets of X.

PROOF

$\underline{X \in \mathcal{F}}$ $f^{-1}(X) = \Omega \in \mathcal{A} \implies X \in \mathcal{F}$.

$\underline{\mathcal{F} \text{ is closed under complementations}}$ $V \in \mathcal{F} \implies f^{-1}(V) \in \mathcal{A} \implies f^{-1}(X \setminus V) = \Omega \setminus f^{-1}(V) \in \mathcal{A} \implies X \setminus V \in \mathcal{F}$,
which proves that $\mathcal{F}$ is indeed closed under complementations.

$\mathcal{F}$ is closed under countable unions

$$
\begin{aligned}
V_1, V_2, \ldots \in \mathcal{F} &\implies f^{-1}(V_1), f^{-1}(V_2), \ldots \in \mathcal{A} \\
&\implies f^{-1}\left(\bigcup_{i=1}^{\infty} V_i \right) = \bigcup_{i=1}^{\infty} f^{-1}(V_i) \in \mathcal{A} \\
&\implies \bigcup_{i=1}^{\infty} V_i \in \mathcal{F},
\end{aligned}
$$

which proves that $\mathcal{F}$ is indeed closed under countable unions.   □

Theorem B.11

Suppose $(\Omega, \mathcal{A})$ and $(X, \mathcal{X})$ are measurable spaces, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then, f is $(\mathcal{A}, \mathcal{X})$-measurable if there exists $\mathcal{S} \subset \mathcal{X}$ satisfying the following conditions:

- *$\mathcal{S}$ generates $\mathcal{X}$, i.e. $\sigma(\mathcal{S}) = \mathcal{X}$, and*

- *$f^{-1}(\mathcal{S}) \subset \mathcal{A}$.*

PROOF   By Lemma B.10,
$$
\mathcal{F} := \left\{ V \subset X \mid f^{-1}(V) \in \mathcal{A} \right\}
$$
is a σ-algebra of subsets of X. By hypothesis, $\mathcal{S} \subset \mathcal{F}$; hence, $\mathcal{X} = \sigma(\mathcal{S}) \subset \mathcal{F}$. Thus, $f^{-1}(\mathcal{X}) \subset \mathcal{A}$; equivalently, f is $(\mathcal{A}, \mathcal{X})$-measurable.   □

Corollary B.12 (Continuous maps are Borel measurable.)

Suppose X_1, X_2 are topological spaces, and $\mathcal{B}_1$, $\mathcal{B}_2$ are their respective Borel σ-algebras. Then, every continuous map $f : X_1 \longrightarrow X_2$ is $(\mathcal{B}_1, \mathcal{B}_2)$-measurable.

C   Topology

Theorem C.1 (Appendix M3, [1])

Suppose S is a metric space. Then, the following conditions are equivalent:

(i)   *S is separable.*

(ii)   *The topology of S has a countable basis.*

(iii)   *Every open cover of each subset of S has a countable subcover.*

References

[1] BILLINGSLEY, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.

[2] BILLINGSLEY, P. *Probability and Measure*, anniversary ed. John Wiley & Sons, 2012.