# Lindeberg's and Lyapunov's Central Limit Theorems

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## 1 Lindeberg's Central Limit Theorem

**Theorem 1.1 (Lindeberg's Central Limit Theorem, Theorem 1.15, [5])**

*Suppose:*

- $\{ k_n \}_{n\in\mathbb{N}} \subset \mathbb{N}$ *is a sequence of natural numbers, and*
- *for each* $n \in \mathbb{N}$, $X_1^{(n)}, X_2^{(n)}, \ldots, X_{k_n}^{(n)} : \Omega_n \longrightarrow \mathbb{R}$ *are independent (but not necessarily identically distributed)* $\mathbb{R}$*-valued random variables defined on a common probability space* $(\Omega_n, \mathcal{A}_n, \mu_n)$ *such that*

$$\mu_j^{(n)} \;:=\; E\left[\,X_j^{(n)}\,\right] \;\in\; \mathbb{R} \;\text{ exists, for each } 1 \leq j \leq k_n, \quad \text{and} \quad 0 \;<\; \sigma_n^2 \;:=\; \mathrm{Var}\left[\,\sum_{j=1}^{k_n} X_j^{(n)}\,\right] \;<\; \infty.$$

*Then, Lindeberg's condition implies*

$$Z_n \;:=\; \frac{1}{\sigma_n}\sum_{j=1}^{k_n}\left(X_j^{(n)} - \mu_j^{(n)}\right) \;\xrightarrow{\;\mathcal{L}\;}\; N(0,1),$$

*where* $N(0,1)$ *denotes the standard Gaussian distribution on* $\mathbb{R}$*, and* ***Lindeberg's condition*** *is the following condition:*

$$\lim_{n\to\infty}\frac{1}{\sigma_n^2}\sum_{j=1}^{k_n} E\left[\,\left(X_j^{(n)} - \mu_j^{(n)}\right)^2 \cdot I_{\left\{\left|X_j^{(n)}-\mu_j^{(n)}\right| \,\geq\, \varepsilon\sigma_n\right\}}\,\right] \;=\; 0, \quad \text{for each } \varepsilon > 0.$$

PROOF Considering $\left(X_j^{(n)} - \mu_j^{(n)}\right)\Big/\sigma_n$, we may assume, without loss of generality, that

$$E\left[\,X_j^{(n)}\,\right] = 0, \quad \text{and} \quad \sigma_n^2 \;:=\; \mathrm{Var}\left[\,\sum_{j=1}^{k_n} X_j^{(n)}\,\right] = 1.$$

By Lévy's Continuity Theorem (Theorem 3(e), p.16, [3]), it suffices to show that

$$\lim_{n\to\infty}\varphi_{Z_n}(t) \;=\; \varphi_{N(0,1)}(t) \;=\; e^{-t^2/2}, \quad \text{for each } t \in \mathbb{R},$$

where $\varphi_{Z_n}(t)$ is the characteristic function of $Z_n$, and $\varphi_{N(0,1)}$ is the characteristic function of the standard Gaussian distribution (with mean zero and variance one). See Example 5, Chapter 13, p.107 in [4] for the proof that

$$\varphi_{N(0,1)}(t) \;=\; e^{-t^2/2}.$$

Define $\sigma_{nj}^2 \;:=\; \mathrm{Var}\left[\,X_j^{(n)}\,\right]$. Note that $\sum_{j=1}^{k_n}\sigma_{nj}^2 \;=\; \sigma_n^2 \;=\; 1$.

We now proceed with the main argument of the proof of Lindeberg's Central Limit Theorem, temporarily taking for granted the validity of a number of Claims (Claims 1 through 6; see below). These Claims are stated and proved after the main argument.

Let $t \in \mathbb{R}$ be fixed. Then, for each sufficiently large $n \in \mathbb{N}$, we have (assuming validity of Claims 1 through 6):

$$\begin{aligned} 0 \;\leq\; \left|\,\varphi_{Z_n}(t) - e^{-t^2/2}\,\right| &= \left|\,\prod_{j=1}^{k_n}\varphi_{X_j^{(n)}}(t) - \exp\left(-\sum_{j=1}^{k_n}\frac{t^2\sigma_{nj}^2}{2}\right)\right| = \left|\,\prod_{j=1}^{k_n}\varphi_{X_j^{(n)}}(t) - \prod_{j=1}^{k_n} e^{-t^2\sigma_{nj}^2/2}\,\right| \\ &\leq \left|\,\prod_{j=1}^{k_n}\varphi_{X_j^{(n)}}(t) - \prod_{j=1}^{k_n}\left(1-\frac{t^2\sigma_{nj}^2}{2}\right)\right| + \left|\,\prod_{j=1}^{k_n}\left(1-\frac{t^2\sigma_{nj}^2}{2}\right) - \prod_{j=1}^{k_n} e^{-t^2\sigma_{nj}^2/2}\,\right| \\ &\leq \sum_{j=1}^{k_n}\left|\,\varphi_{X_j^{(n)}}(t) - \left(1-\frac{t^2\sigma_{nj}^2}{2}\right)\right| + \sum_{j=1}^{k_n}\left|\,e^{-t^2\sigma_{nj}^2/2} - \left(1-\frac{t^2\sigma_{nj}^2}{2}\right)\right|, \end{aligned}$$

where the last inequality follows from Claim 3 and Claim 4. By Claim 5 and Claim 6, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}\left|\,\varphi_{Z_n}(t) - e^{-t^2/2}\,\right| \\ &\leq \lim_{n\to\infty}\sum_{j=1}^{k_n}\left|\,\varphi_{X_j^{(n)}}(t) - \left(1-\frac{t^2\sigma_{nj}^2}{2}\right)\right| + \lim_{n\to\infty}\sum_{j=1}^{k_n}\left|\,e^{-t^2\sigma_{nj}^2/2} - \left(1-\frac{t^2\sigma_{nj}^2}{2}\right)\right| = 0. \end{aligned}$$

This proves

$$\lim_{n\to\infty}\varphi_{Z_n}(t) = e^{-t^2/2}, \quad \text{for each } t \in \mathbb{R},$$

and hence $Z_n \xrightarrow{\mathcal{L}} N(0,1)$, as required. We now state and prove Claims 1 through 6.

**Claim 1:**

$$\lim_{n\to\infty}\max_{1\leq j\leq k_n}\left\{\,\sigma_{nj}^2\,\right\} = 0.$$

Proof of Claim 1: First, note that, for an arbitrary $\varepsilon > 0$, we have

$$\begin{aligned} 0 \leq \sigma_{nj}^2 &= \mathrm{Var}\left[X_j^{(n)}\right] = \int x^2\,\mathrm{d}\mu_{X_j^{(n)}}(x) \leq \int_{\left\{\left|X_j^{(n)}\right|<\varepsilon\right\}} x^2\,\mathrm{d}\mu_{X_j^{(n)}}(x) + \int_{\left\{\left|X_j^{(n)}\right|\geq\varepsilon\right\}} x^2\,\mathrm{d}\mu_{X_j^{(n)}}(x) \\ &\leq \varepsilon^2 + E\left[\left(X_j^{(n)}\right)^2\cdot I_{\left\{\left|X_j^{(n)}\right|\geq\varepsilon\right\}}\right] \leq \varepsilon^2 + \sum_{i=1}^{k_n}E\left[\left(X_i^{(n)}\right)^2\cdot I_{\left\{\left|X_i^{(n)}\right|\geq\varepsilon\right\}}\right]. \end{aligned}$$

It follows that

$$0 \leq \max_{1\leq j\leq k_n}\left\{\,\sigma_{nj}^2\,\right\} \leq \varepsilon^2 + \sum_{i=1}^{k_n}E\left[\left(X_i^{(n)}\right)^2\cdot I_{\left\{\left|X_i^{(n)}\right|\geq\varepsilon\right\}}\right].$$

Hence, Lindeberg's condition implies:

$$0 \leq \limsup_{n\to\infty}\max_{1\leq j\leq k_n}\left\{\,\sigma_{nj}^2\,\right\} \leq \varepsilon^2 + \lim_{n\to\infty}\sum_{i=1}^{k_n}E\left[\left(X_i^{(n)}\right)^2\cdot I_{\left\{\left|X_i^{(n)}\right|\geq\varepsilon\right\}}\right] = \varepsilon^2.$$

Since $\varepsilon > 0$ is arbitrary, we see that:

$$0 \leq \limsup_{n\to\infty}\max_{1\leq j\leq k_n}\left\{\,\sigma_{nj}^2\,\right\} = 0,$$

which implies

$$\lim_{n\to\infty}\max_{1\leq j\leq k_n}\left\{\,\sigma_{nj}^2\,\right\} = 0$$

This proves Claim 1.

**Claim 2:** For each sufficiently large $n \in \mathbb{N}$, we have:

$$0 \leq 1-\frac{t^2\sigma_{nj}^2}{2} \leq 1, \quad \text{for each } 1 \leq j \leq k_n.$$

Proof of Claim 2: Recall that $t \in \mathbb{R}$ is fixed in this argument. Hence, Claim 2 follows immediately from Claim 1.

**Claim 3:** For each sufficiently large $n \in \mathbb{N}$, we have:

$$\left| \prod_{j=1}^{k_n} \varphi_{X_j^{(n)}}(t) \;-\; \prod_{j=1}^{k_n} \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| \;\leq\; \sum_{j=1}^{k_n} \left| \varphi_{X_j^{(n)}}(t) \;-\; \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right|.$$

Proof of Claim 3: This follows immediately from Lemma A.2, Claim 2, and the fact that characteristic functions of $\mathbb{R}$-valued random variables (or probability measures defined on $\mathbb{R}$) always map into the closed unit disk in the complex plane.

**Claim 4:** For each sufficiently large $n \in \mathbb{N}$, we have:

$$\left| \prod_{j=1}^{k_n} e^{-t^2 \sigma_{nj}^2/2} \;-\; \prod_{j=1}^{k_n} \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| \;\leq\; \sum_{j=1}^{k_n} \left| e^{-t^2 \sigma_{nj}^2/2} \;-\; \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right|.$$

Proof of Claim 4: This follows immediately from Lemma A.2, Claim 2, and the fact that $\exp\{\,(-\infty, 0]\,\} \subset [0,1]$.

**Claim 5:**

$$\lim_{n \to \infty} \sum_{j=1}^{k_n} \left| \varphi_{X_j^{(n)}}(t) \;-\; \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| \;=\; 0.$$

Proof of Claim 5: By Lemma A.3, we have

$$\left| e^{\mathbf{i}tx} - \left( 1 + \mathbf{i}\, tx - \frac{t^2 x^2}{2} \right) \right| \;\leq\; \min\left\{ \,|\, tx \,|^2 \,,\, |\, tx \,|^3 \,\right\}.$$

Next, note that

$$\begin{aligned}
\int e^{\mathbf{i}tx} - \left( 1 + \mathbf{i}\, tx - \frac{t^2x^2}{2} \right) \mathrm{d}\mu_{X_j^{(n)}}(x) \;&=\; \int e^{\mathbf{i}tx}\, \mathrm{d}\mu_{X_j^{(n)}}(x) \;-\; \int \left( 1 + \mathbf{i}\, tx - \frac{t^2x^2}{2} \right) \mathrm{d}\mu_{X_j^{(n)}}(x) \\
&=\; \varphi_{X_j^{(n)}}(t) \;-\; \left( 1 \,+\, \mathbf{i}t \cdot E\!\left[ X_j^{(n)} \right] \;-\; \frac{t^2}{2} \cdot E\!\left[ \left( X_j^{(n)} \right)^2 \right] \right) \\
&=\; \varphi_{X_j^{(n)}}(t) \;-\; \left( 1 \;-\; \frac{t^2 \sigma_{nj}^2}{2} \right)
\end{aligned}$$

Hence, for an arbitrary $\varepsilon > 0$,

$$\begin{aligned}
\left| \varphi_{X_j^{(n)}}(t) \;-\; \left( 1 \;-\; \frac{t^2 \sigma_{nj}^2}{2} \right) \right| \;&=\; \left| \int e^{\mathbf{i}tx} - \left( 1 + \mathbf{i}\, tx - \frac{t^2x^2}{2} \right) \mathrm{d}\mu_{X_j^{(n)}}(x) \right| \;\leq\; \int \left| e^{\mathbf{i}tx} - \left( 1 + \mathbf{i}\, tx - \frac{t^2x^2}{2} \right) \right| \mathrm{d}\mu_{X_j^{(n)}}(x) \\
&\leq\; \int \min\left\{ \,|\, tx \,|^2 \,,\, |\, tx \,|^3 \,\right\} \mathrm{d}\mu_{X_j^{(n)}}(x) \\
&=\; \int_{\left\{ \left| X_j^{(n)} \right| \,<\, \varepsilon \right\}} \min\left\{ |\, tx \,|^2, |\, tx \,|^3 \right\} \mathrm{d}\mu_{X_j^{(n)}}(x) \;+\; \int_{\left\{ \left| X_j^{(n)} \right| \,\geq\, \varepsilon \right\}} \min\left\{ |\, tx \,|^2, |\, tx \,|^3 \right\} \mathrm{d}\mu_{X_j^{(n)}}(x) \\
&\leq\; \int_{\left\{ \left| X_j^{(n)} \right| \,<\, \varepsilon \right\}} |\, tx \,|^3\, \mathrm{d}\mu_{X_j^{(n)}}(x) \;+\; \int_{\left\{ \left| X_j^{(n)} \right| \,\geq\, \varepsilon \right\}} |\, tx \,|^2\, \mathrm{d}\mu_{X_j^{(n)}}(x) \\
&\leq\; \varepsilon |\, t \,|^3 \int_{\left\{ \left| X_j^{(n)} \right| \,<\, \varepsilon \right\}} |\, x \,|^2\, \mathrm{d}\mu_{X_j^{(n)}}(x) \;+\; |\, t \,|^2 \int_{\left\{ \left| X_j^{(n)} \right| \,\geq\, \varepsilon \right\}} |\, x \,|^2\, \mathrm{d}\mu_{X_j^{(n)}}(x) \\
&\leq\; \varepsilon |\, t \,|^3 \cdot \sigma_{nj}^2 \;+\; |\, t \,|^2 \cdot E\!\left[ \left( X_j^{(n)} \right)^2 \cdot I_{\left\{ \left| X_j^{(n)} \right| \,\geq\, \varepsilon \right\}} \right]
\end{aligned}$$

Thus, for an arbitrary $\varepsilon > 0$,

$$\sum_{j=1}^{k_n} \left| \varphi_{X_j^{(n)}}(t) - \left(1 - \frac{t^2\sigma_{nj}^2}{2}\right) \right| \;\leq\; \varepsilon|\,t\,|^3 \cdot \sum_{j=1}^{k_n} \sigma_{nj}^2 \;+\; |\,t\,|^2 \cdot \sum_{j=1}^{k_n} E\left[\,\left(X_j^{(n)}\right)^2 \cdot I_{\left\{\left|X_j^{(n)}\right| \,\geq\, \varepsilon\right\}}\,\right].$$

Recall that $t \in \mathbb{R}$ is fixed and $\sum_{j=1}^{k_n} \sigma_{nj}^2 = 1$. Lindeberg's condition therefore implies:

$$0 \;\leq\; \limsup_{n\to\infty} \sum_{j=1}^{k_n} \left| \varphi_{X_j^{(n)}}(t) - \left(1 - \frac{t^2\sigma_{nj}^2}{2}\right) \right| \;\leq\; \varepsilon|\,t\,|^3 \;+\; |\,t\,|^2 \cdot \lim_{n\to\infty} \sum_{j=1}^{k_n} E\left[\,\left(X_j^{(n)}\right)^2 \cdot I_{\left\{\left|X_j^{(n)}\right| \,\geq\, \varepsilon\right\}}\,\right] \;=\; \varepsilon|\,t\,|^3.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \left| \varphi_{X_j^{(n)}}(t) - \left(1 - \frac{t^2\sigma_{nj}^2}{2}\right) \right| \;=\; 0.$$

This proves Claim 5.

**Claim 6:**

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \left| e^{-t^2\sigma_{nj}^2/2} \;-\; \left(1 - \frac{t^2\sigma_{nj}^2}{2}\right) \right| \;=\; 0$$

Proof of Claim 6: Since $t \in \mathbb{R}$ is fixed, by Claim 1, we have that, for each sufficiently large $n \in \mathbb{N}$,

$$\left| \frac{t^2\sigma_{nj}^2}{2} \right| \;\leq\; \frac{1}{2}, \quad \text{for each } 1 \leq j \leq k_n.$$

Thus, Lemma A.1 implies that, for each sufficiently large $n \in \mathbb{N}$,

$$\left| e^{-t^2\sigma_{nj}^2/2} \;-\; \left(1 - \frac{t^2\sigma_{nj}^2}{2}\right) \right| \;=\; \left| e^{-t^2\sigma_{nj}^2/2} \;-\; 1 - \left(-\frac{t^2\sigma_{nj}^2}{2}\right) \right| \;\leq\; \left| \frac{t^2\sigma_{nj}^2}{2} \right|^2 \;\leq\; t^4\sigma_{nj}^4$$

Summing over $j$, we have: for each sufficiently large $n$,

$$\begin{aligned}
0 \;\leq\; \sum_{j=1}^{k_n} \left| e^{-t^2\sigma_{nj}^2/2} \;-\; \left(1 - \frac{t^2\sigma_{nj}^2}{2}\right) \right| \;&\leq\; t^4 \cdot \sum_{j=1}^{k_n} \sigma_{nj}^4 \\
&\leq\; t^4 \cdot \sum_{j=1}^{k_n} \left( \sigma_{nj}^2 \cdot \max_{1\leq i\leq k_n} \left\{\, \sigma_{ni}^2 \,\right\} \right) \\
&=\; t^4 \cdot \left( \sum_{j=1}^{k_n} \sigma_{nj}^2 \right) \left( \max_{1\leq i\leq k_n} \left\{\, \sigma_{ni}^2 \,\right\} \right) \\
&=\; t^4 \cdot \left( \max_{1\leq i\leq k_n} \left\{\, \sigma_{ni}^2 \,\right\} \right).
\end{aligned}$$

Claim 1 now implies

$$0 \;\leq\; \limsup_{n\to\infty} \sum_{j=1}^{k_n} \left| e^{-t^2\sigma_{nj}^2/2} \;-\; \left(1 - \frac{t^2\sigma_{nj}^2}{2}\right) \right| \;=\; t^4 \cdot \lim_{n\to\infty} \left( \max_{1\leq i\leq k_n} \left\{\, \sigma_{ni}^2 \,\right\} \right) \;=\; 0,$$

which in turn implies

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \left| e^{-t^2\sigma_{nj}^2/2} \;-\; \left(1 - \frac{t^2\sigma_{nj}^2}{2}\right) \right| \;=\; 0.$$

This proves Claim 6. This completes the proof of Lindeberg's Central Limit Theorem. $\square$

## 2 Lyapunov's Central Limit Theorem

**Theorem 2.1 (Lyapunov's Central Limit Theorem)**

*Suppose:*

- $\{ k_n \}_{n \in \mathbb{N}} \subset \mathbb{N}$ *is a sequence of natural numbers, and*
- *for each* $n \in \mathbb{N}$, $X_1^{(n)}, X_2^{(n)}, \ldots, X_{k_n}^{(n)} : \Omega_n \longrightarrow \mathbb{R}$ *are independent (but not necessarily identically distributed)* $\mathbb{R}$*-valued random variables defined on a common probability space* $(\Omega_n, \mathcal{A}_n, \mu_n)$ *such that*

$$\mu_j^{(n)} \;:=\; E\left[ X_j^{(n)} \right] \;\in\; \mathbb{R} \text{ exists, for each } 1 \leq j \leq k_n, \quad \text{and} \quad 0 \;<\; \sigma_n^2 \;:=\; \text{Var}\left[ \sum_{j=1}^{k_n} X_j^{(n)} \right] \;<\; \infty.$$

*Then,* ***Lyapunov's condition:***

$$\text{there exists } \delta > 0 \text{ such that } \lim_{n \to \infty} \frac{1}{\sigma_n^{2+\delta}} \sum_{j=1}^{k_n} E\left( \left| X_j^{(n)} - \mu_j^{(n)} \right|^{2+\delta} \right) \;=\; 0$$

*implies* ***Lindeberg's condition:***

$$\lim_{n \to \infty} \frac{1}{\sigma_n^2} \sum_{j=1}^{k_n} E\left[ \left( X_j^{(n)} - \mu_j^{(n)} \right)^2 \cdot I_{\left\{ \left| X_j^{(n)} - \mu_j^{(n)} \right| \geq \varepsilon \sigma_n \right\}} \right] \;=\; 0, \quad \text{for each } \varepsilon > 0.$$

*Consequently, Lyapunov's Condition implies*

$$Z_n \;:=\; \frac{1}{\sigma_n} \sum_{j=1}^{k_n} \left( X_j^{(n)} - \mu_j^{(n)} \right) \;\overset{\mathcal{L}}{\longrightarrow}\; N(0,1),$$

*where* $N(0,1)$ *denotes the standard Gaussian distribution on* $\mathbb{R}$.

PROOF Suppose Lyapunov's condition holds. Let $\varepsilon > 0$ be given. Note that:

$$\left| X_j^{(n)} - \mu_j^{(n)} \right| \;\geq\; \varepsilon\,\sigma_n \;\Longrightarrow\; \left| \frac{X_j^{(n)} - \mu_j^{(n)}}{\varepsilon\,\sigma_n} \right|^{\delta} \;\geq\; 1,$$

where $\delta > 0$ is as in Lyapunov's condition. Then,

$$\begin{aligned}
0 \;&\leq\; \frac{1}{\sigma_n^2} \cdot \sum_{j=1}^{k_n} E\left[ \left( X_i^{(n)} - \mu_i^{(n)} \right)^2 \cdot I_{\left\{ \left| X_j^{(n)} - \mu_j^{(n)} \right| \geq \varepsilon \sigma_n \right\}} \right] \\
&\leq\; \frac{1}{\sigma_n^2} \cdot \sum_{j=1}^{k_n} E\left[ \left( X_i^{(n)} - \mu_i^{(n)} \right)^2 \cdot \left| \frac{X_j^{(n)} - \mu_j^{(n)}}{\varepsilon\,\sigma_n} \right|^{\delta} \cdot I_{\left\{ \left| X_j^{(n)} - \mu_j^{(n)} \right| \geq \varepsilon \sigma_n \right\}} \right] \\
&=\; \frac{1}{\varepsilon^{\delta}\,\sigma_n^{2+\delta}} \cdot \sum_{j=1}^{k_n} E\left[ \left| X_j^{(n)} - \mu_j^{(n)} \right|^{2+\delta} \cdot I_{\left\{ \left| X_j^{(n)} - \mu_j^{(n)} \right| \geq \varepsilon \sigma_n \right\}} \right] \\
&\leq\; \frac{1}{\varepsilon^{\delta}\,\sigma_n^{2+\delta}} \cdot \sum_{j=1}^{k_n} E\left( \left| X_j^{(n)} - \mu_j^{(n)} \right|^{2+\delta} \right) \;\longrightarrow\; 0, \;\; \text{as } n \longrightarrow \infty.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we see that Lindeberg's condition indeed holds. By Lindeberg's Central Limit Theorem, we thus have $Z_n \overset{\mathcal{L}}{\longrightarrow} N(0,1)$. □

# A Technical Lemmas

**Lemma A.1**

$$|\, e^z - 1 - z \,| \;\leq\; z^2, \quad \text{for each } |\, z \,| \leq \frac{1}{2}.$$

PROOF First, note that $g(z) := e^z - 1 - z \geq 0$, for each $z \in \mathbb{R}$. Indeed, $g'(z) = e^z - 1$ and $g''(z) = e^z > 0$. So, $g$ is strictly convex. Next, note that $g'(z) = 0 \Longleftrightarrow z = 0$. So, $g$ achieves its unique minimum at $z = 0$. Since $g(0) = 0$, we see that $g(z) \geq 0$, for each $z \in \mathbb{R}$. Thus, $|\, e^z - 1 - z \,| = e^z - 1 - z$, for each $z \in \mathbb{R}$. Hence, to prove the Lemma, it suffices to prove that $h(z) := z^2 - (e^z - 1 - z) = z^2 + z + 1 - e^z \geq 0$, for each $z \in [-1/2, 1/2]$. Now, $h'(z) = 2z + 1 - e^z$ and $h''(z) = 2 - e^z$. So, $h''(z) = 0 \Longleftrightarrow z = \log(2) \approx 0.6931$, and $h''(z) > 0$ for each $z \in (-\infty, \log(2)) \supset [-1/2, 1/2]$. So, $h$ is strictly convex on the interval $[-1/2, 1/2]$. But $h(0) = h'(0) = 0$. Hence, $z = 0$ is the unique minimum of $h$ on $[-1/2, 1/2]$, and we may now conclude that $h(z) \geq 0$, for each $z \in [-1/2, 1/2]$, as required. □

**Lemma A.2 (p.358, [1])** *Let $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in \mathbb{C}$. Then,*

$$|\, a_i \,|, |\, b_i \,| \leq 1, \text{ for each } i = 1, 2, \ldots, m \quad \Longrightarrow \quad |\, a_1 a_2 \cdots a_m - b_1 b_2 \cdots b_m \,| \;\leq\; \sum_{i=1}^{m} |\, a_i - b_i \,|$$

PROOF Equality holds trivially for $m = 1$. We first prove the inequality for $m = 2$.

$$\begin{aligned}
|\, a_1 a_2 - b_1 b_2 \,| &= |\, a_1 a_2 - b_1 a_2 + b_1 a_2 - b_1 b_2 \,| \;\leq\; |\, a_1 a_2 - b_1 a_2 \,| + |\, b_1 a_2 - b_1 b_2 \,| \\
&\leq |\, a_1 a_2 - b_1 a_2 \,| + |\, b_1 a_2 - b_1 b_2 \,| \;\leq\; |\, a_1 - b_1 \,|\,|\, a_2 \,| + |\, b_1 \,|\,|\, a_2 - b_2 \,| \\
&\leq |\, a_1 - b_1 \,| + |\, a_2 - b_2 \,|, \quad \text{since } |\, a_2 \,|, |\, b_1 \,| \leq 1, \text{ by hypothesis.}
\end{aligned}$$

The general case now follows by induction: Assume the Lemma is valid for $1, 2, \ldots, m$, and we prove that it is also valid for $m + 1$.

$$\begin{aligned}
|\, a_1 \cdots a_m a_{m+1} - b_1 \cdots b_m b_{m+1} \,| &\leq |\, a_1 \cdots a_m - b_1 \cdots b_m \,| + |\, a_{m+1} - b_{m+1} \,| \\
&\leq \sum_{i=1}^{m} |\, a_i - b_i \,| + |\, a_{m+1} - b_{m+1} \,| \;=\; \sum_{i=1}^{m+1} |\, a_i - b_i \,|
\end{aligned}$$

The proof of the Lemma is complete. □

**Lemma A.3 (p.343, [1])**

$$\left|\, e^{\mathbf{i}x} - \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \,\right| \;\leq\; \min\left\{ \frac{|\,x\,|^{n+1}}{(n+1)!} \,,\, \frac{2\,|\,x\,|^n}{n!} \right\}, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

PROOF We first establish a number of Claims, which will easily imply the Lemma.

**Claim 1:**

$$\int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s \;=\; \frac{x^{n+1}}{n+1} + \frac{\mathbf{i}}{n+1} \int_0^x (x-s)^{n+1} \, e^{\mathbf{i}s} \, \mathrm{d}s, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof of Claim 1: We proceed by integration by parts. Let $u = e^{\mathbf{i}s}$ and $\mathrm{d}v = (x-s)^n \, \mathrm{d}s$. Then, $\mathrm{d}u = \mathbf{i}\, e^{\mathbf{i}s}$ and $v = -(x-s)^{n+1}/(n+1)$. Hence,

$$\begin{aligned}
\int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s &= \int u \, \mathrm{d}v \;=\; uv - \int v \, \mathrm{d}u \\
&= \left[ e^{\mathbf{i}s} \cdot \frac{(-1)(x-s)^{n+1}}{n+1} \right]_{s=0}^{s=x} - \int_0^x \frac{(-1)\,(x-s)^{n+1}}{n+1} \cdot \mathbf{i} e^{\mathbf{i}s} \, \mathrm{d}s, \\
&= \frac{x^{n+1}}{n+1} + \frac{\mathbf{i}}{n+1} \int_0^x (x-s)^{n+1} \, e^{\mathbf{i}s} \, \mathrm{d}s.
\end{aligned}$$

This proves Claim 1.

**Claim 2:**

$$e^{\mathbf{i}x} \;=\; \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \;+\; \frac{\mathbf{i}^{n+1}}{n!}\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s, \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 0.$$

Proof of Claim 2: We proceed by induction. For $n=0$, we have:

$$\begin{aligned}\mathrm{RHS}(n=0) &= \sum_{k=0}^{0} \frac{(\mathbf{i}\,x)^k}{k!} + \frac{\mathbf{i}^{0+1}}{0!}\int_0^x (x-s)^0\, e^{\mathbf{i}s}\,\mathrm{d}s \;=\; 1 + \mathbf{i}\int_0^x e^{\mathbf{i}s}\,\mathrm{d}s \;=\; 1 + \mathbf{i}\left[\frac{e^{\mathbf{i}s}}{\mathbf{i}}\right]_{s=0}^{s=x} \\ &= 1 + \left(e^{\mathbf{i}x}-1\right) \;=\; e^{\mathbf{i}x}.\end{aligned}$$

Thus, Claim 2 is indeed true for $n=0$. Next, by induction hypothesis, assume Claim 2 is true for $n$, and we verify that Claim 2 is also true for $n+1$.

$$\begin{aligned}\mathrm{RHS}(n+1) &= \sum_{k=0}^{n+1} \frac{(\mathbf{i}\,x)^k}{k!} + \frac{\mathbf{i}^{n+2}}{(n+1)!}\int_0^x (x-s)^{n+1}\, e^{\mathbf{i}s}\,\mathrm{d}s \\ &= \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} + \frac{(\mathbf{i}\,x)^{n+1}}{(n+1)!} + \frac{\mathbf{i}^{n+2}}{(n+1)!}\cdot\frac{n+1}{\mathbf{i}}\left[\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s - \frac{x^{n+1}}{n+1}\right] \\ &= \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} + \frac{\mathbf{i}^{n+1}}{n!}\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s + \frac{(\mathbf{i}\,x)^{n+1}}{(n+1)!} - \frac{\mathbf{i}^{n+1}}{n!}\cdot\frac{x^{n+1}}{n+1} \;=\; e^{\mathbf{i}x},\end{aligned}$$

where the second equality follows from Claim 1 and the last equality follows from the induction hypothesis (that Claim 2 holds for $n$). This proves Claim 2.

**Claim 3:**

$$e^{\mathbf{i}x} \;=\; \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \;+\; \frac{\mathbf{i}^{n}}{(n-1)!}\int_0^x (x-s)^{n-1}\left(e^{\mathbf{i}s}-1\right)\mathrm{d}s, \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1.$$

Proof of Claim 3: By Claim 1, we have (replacing $n$ with $n-1$):

$$\int_0^x (x-s)^{n-1}\, e^{\mathbf{i}s}\,\mathrm{d}s \;=\; \frac{x^n}{n} + \frac{\mathbf{i}}{n}\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s, \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1.$$

Isolating the integral on the right-hand-side, we have:

$$\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s \;=\; \frac{n}{\mathbf{i}}\left[\int_0^x (x-s)^{n-1}\, e^{\mathbf{i}s}\,\mathrm{d}s - \frac{x^n}{n}\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1.$$

Next, note that, for any $x\in\mathbb{R}$ and any $n\geq 1$,

$$\int_0^x (\,x-s\,)^{n-1}\,\mathrm{d}s \;=\; -\left[\frac{(x-s)^n}{n}\right]_{s=0}^{s=x} \;=\; -\left[0 - \frac{x^n}{n}\right] \;=\; \frac{x^n}{n}$$

Hence, we have:

$$\begin{aligned}\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s &= \frac{n}{\mathbf{i}}\left[\int_0^x (x-s)^{n-1}\, e^{\mathbf{i}s}\,\mathrm{d}s - \frac{x^n}{n}\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1. \\ &= \frac{n}{\mathbf{i}}\left[\int_0^x (x-s)^{n-1}\, e^{\mathbf{i}s}\,\mathrm{d}s - \int_0^x (\,x-s\,)^{n-1}\,\mathrm{d}s\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1. \\ &= \frac{n}{\mathbf{i}}\left[\int_0^x (x-s)^{n-1}\left(e^{\mathbf{i}s}-1\right)\mathrm{d}s\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1.\end{aligned}$$

Substituting the above into the right-hand-side of Claim 2, we have:

$$
\begin{aligned}
e^{\mathbf{i}x} &= \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} + \frac{\mathbf{i}^{n+1}}{n!}\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s, \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 0 \\
&= \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} + \frac{\mathbf{i}^{n+1}}{n!}\cdot\frac{n}{\mathbf{i}}\cdot\left[\int_0^x (x-s)^{n-1}\left(e^{\mathbf{i}s}-1\right)\mathrm{d}s\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1 \\
&= \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} + \frac{\mathbf{i}^{n}}{(n-1)!}\cdot\left[\int_0^x (x-s)^{n-1}\left(e^{\mathbf{i}s}-1\right)\mathrm{d}s\right], \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1
\end{aligned}
$$

This proves Claim 3.

**Claim 4:**

$$
\left|\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s\right| \leq \frac{|\,x\,|^{n+1}}{n+1}, \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 0.
$$

Proof of Claim 4: First, consider $x\geq 0$, in which case, we have, for any $n\geq 0$,

$$
\left|\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s\right| \leq \int_0^x |\,x-s\,|^n\,\mathrm{d}s \leq \int_0^x (\,x-s\,)^n\,\mathrm{d}s = \cdots = \frac{x^{n+1}}{n+1} = \frac{|\,x\,|^{n+1}}{n+1}
$$

Next, for $x<0$, let $y:=-x>0$. Then,

$$
\begin{aligned}
\left|\int_0^x (x-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s\right| &= \left|\int_0^{-y} (-y-s)^n\, e^{\mathbf{i}s}\,\mathrm{d}s\right| = \left|\int_0^{y} (-y+t)^n\, e^{-\mathbf{i}t}\,\mathrm{d}t\right| \\
&\leq \int_0^y |\,y-t\,|^n\,\mathrm{d}t = \int_0^y (\,y-t\,)^n\,\mathrm{d}t = \cdots = \frac{y^{n+1}}{n+1} = \frac{|\,x\,|^{n+1}}{n+1}
\end{aligned}
$$

This completes the proof Claim 4.

**Claim 5:**

$$
\left|\int_0^x (x-s)^{n-1}\left(e^{\mathbf{i}s}-1\right)\mathrm{d}s\right| \leq \frac{2\,|\,x\,|^n}{n}, \quad \text{for any } x\in\mathbb{R} \text{ and any } n\geq 1.
$$

Proof of Claim 5: First, consider $x\geq 0$, in which case, we have, for any $n\geq 1$,

$$
\left|\int_0^x (x-s)^{n-1}\left(e^{\mathbf{i}s}-1\right)\mathrm{d}s\right| \leq \int_0^x \left|(x-s)^{n-1}\left(e^{\mathbf{i}s}-1\right)\right|\mathrm{d}s \leq 2\int_0^x (x-s)^{n-1}\,\mathrm{d}s = \frac{2\,x^n}{n} = \frac{2\,|\,x\,|^n}{n},
$$

where the second last equality follows from the simple calculation:

$$
\int_0^x (\,x-s\,)^{n-1}\,\mathrm{d}s = -\left[\frac{(x-s)^n}{n}\right]_{s=0}^{s=x} = -\left[0-\frac{x^n}{n}\right] = \frac{x^n}{n}.
$$

Next, for $x<0$, let $y:=-x>0$. Then,

$$
\begin{aligned}
\left|\int_0^x (x-s)^{n-1}\left(e^{\mathbf{i}s}-1\right)\mathrm{d}s\right| &= \left|\int_0^{-y} (-y-s)^{n-1}\left(e^{\mathbf{i}s}-1\right)\mathrm{d}s\right| = \left|-\int_0^{y} (-y+t)^{n-1}\left(e^{-\mathbf{i}t}-1\right)\mathrm{d}t\right| \\
&\leq 2\int_0^y |\,t-y\,|^{n-1}\,\mathrm{d}t = 2\int_0^y (y-t)^{n-1}\,\mathrm{d}t = \frac{2\,y^n}{n} = \frac{2\,|\,x\,|^n}{n}.
\end{aligned}
$$

This completes the proof of Claim 5.

The proof of the Lemma now follows readily from the preceding Claims.

$$\begin{aligned}
& \left| \, e^{\mathbf{i}x} \; - \; \sum_{k=0}^{n} \frac{(\mathbf{i}\,x)^k}{k!} \, \right| \\
\leq \quad & \min\left\{ \, \left| \, \frac{\mathbf{i}^{n+1}}{n!} \int_0^x (x-s)^n \, e^{\mathbf{i}s} \, \mathrm{d}s \, \right| , \left| \, \frac{\mathbf{i}^{n}}{(n-1)!} \int_0^x (x-s)^{n-1} \left( e^{\mathbf{i}s} - 1 \right) \, \mathrm{d}s \, \right| \, \right\}, \quad \text{by Claims 2 and 3} \\
\leq \quad & \min\left\{ \, \frac{1}{n!} \cdot \frac{|\,x\,|^{n+1}}{n+1} \, , \, \frac{1}{(n-1)!} \cdot \frac{2\,|\,x\,|^{n}}{n} \, \right\}, \quad \text{by Claims 4 and 5} \\
\leq \quad & \min\left\{ \, \frac{|\,x\,|^{n+1}}{(n+1)!} \, , \, \frac{2\,|\,x\,|^{n}}{n!} \, \right\}
\end{aligned}$$

This completes the proof of the Lemma. □

**Lemma A.4 (§7.1, [2])**

*Let* $\{\, \theta_{nj} \in \mathbb{C} \;|\; 1 \leq j \leq k_n, n \in \mathbb{N} \,\}$ *be doubly indexed array of complex numbers. If all the following three conditions are true:*

(a) *there exists* $M > 0$ *such that*

$$\sum_{j=1}^{k_n} |\,\theta_{nj}\,| \;\leq\; M, \quad \text{for each } n \in \mathbb{N},$$

(b) $\displaystyle\lim_{n\to\infty} \max_{1 \leq j \leq k_n} |\,\theta_{nj}\,| \;=\; 0$, *and*

(c) *there exists* $\theta \in \mathbb{C}$ *such that*

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \theta_{nj} \;=\; \theta,$$

*then*

$$\lim_{n\to\infty} \prod_{j=1}^{k_n} (1 + \theta_{nj}) \;=\; e^{\theta}.$$

Proof First, note that hypothesis (b) immediately implies that there exists some $n_0 \in \mathbb{N}$ such that

$$|\,\theta_{nj}\,| \;\leq\; \frac{1}{2}, \quad \text{for each } n \geq n_0, \text{ for each } 1 \leq j \leq k_n.$$

Thus, without loss of generality, we may assume that:

$$|\,\theta_{nj}\,| \;\leq\; \frac{1}{2}, \quad \text{for each } n \in \mathbb{N}, \text{ for each } 1 \leq j \leq k_n.$$

We denote by $\log(1+\theta_{nj})$ the (unique) complex logarithm[1] of $1+\theta_{nj}$ with argument in $(-\pi,\pi]$. Next, recall the MacLaurin Series for $\log(1+x)$:

$$\log(1+x) \;=\; \sum_{m=1}^{\infty} (-1)^{n+1} \frac{x^m}{m}, \quad \text{for any } x \in \mathbb{C} \text{ with } |\,x\,| < 1.$$

[1] Recall that the complex exponential function is defined by $\exp : \mathbb{C} \to \mathbb{C} : x + \mathbf{i}\,y \mapsto e^x \cdot e^{\mathbf{i}\,y} = e^x(\cos y + \mathbf{i}\sin y)$. Clearly, exp is not injective. More precisely, for $x_1 + \mathbf{i}\,y_1,\, x_2 + \mathbf{i}\,y_2 \in \mathbb{C}\backslash\{\,0\,\}$, we have $e^{x_1 + \mathbf{i}y_1} = e^{x_2 + \mathbf{i}y_2}$ if and only if $x_1 = x_2 \in \mathbb{R}\backslash\{\,0\,\}$ and $y_1 - y_2 \in 2\pi\mathbb{Z}$. For $z = re^{\mathbf{i}\theta} \in \mathbb{C}\backslash\{\,0\,\}$, a complex logarithm of $z$ is any $w = x + \mathbf{i}\,y \in \mathbb{C}\backslash\{\,0\,\}$ such that $e^{x+\mathbf{i}y} = e^w = z = re^{\mathbf{i}\theta}$, i.e. $x = \log r$ and $y = \theta + 2\pi\mathbb{Z}$. In particular, let $\mathcal{D} := \{\, x + \mathbf{i}\,y \in \mathbb{C} \mid x \in \mathbb{R}, y \in (-\pi,\pi] \,\}$. Then, the restriction $\exp : \mathcal{D} \to \mathbb{C}\backslash\{\,0\,\}$ is bijective.

Hence, we have the following inequality: for each $n \in \mathbb{N}$ and for each $1 \leq j \leq k_n$,

$$\begin{aligned} \left|\, \log(1+\theta_{nj}) - \theta_{nj} \,\right| &= \left| \sum_{m=2}^{\infty} (-1)^{n+1} \frac{(\theta_{nj})^m}{m} \right| \leq \sum_{m=2}^{\infty} \frac{|\,\theta_{nj}\,|^m}{m} \leq \frac{|\,\theta_{nj}\,|^2}{2} \sum_{m=2}^{\infty} |\,\theta_{nj}\,|^{m-2} \\ &\leq \frac{|\,\theta_{nj}\,|^2}{2} \sum_{m=2}^{\infty} \left(\frac{1}{2}\right)^{m-2} = \frac{|\,\theta_{nj}\,|^2}{2} \sum_{i=2}^{\infty} \left(\frac{1}{2}\right)^{i-2} = \frac{|\,\theta_{nj}\,|^2}{2} \cdot 2 = |\,\theta_{nj}\,|^2 . \end{aligned}$$

This in turn implies: for each $n \in \mathbb{N}$,

$$\left| \sum_{j=1}^{k_n} \log(1+\theta_{nj}) - \sum_{j=1}^{k_n} \theta_{nj} \right| = \left| \sum_{j=1}^{k_n} (\log(1+\theta_{nj}) - \theta_{nj}) \right| \leq \sum_{j=1}^{k_n} |\log(1+\theta_{nj}) - \theta_{nj}| \leq \sum_{j=1}^{k_n} |\,\theta_{nj}\,|^2.$$

Thus, for each $n \in \mathbb{N}$, there exists $\Lambda_n \in \mathbb{C}$ with $|\,\Lambda_n\,| \leq 1$ such that

$$\sum_{j=1}^{k_n} \log(1+\theta_{nj}) = \sum_{j=1}^{k_n} \theta_{nj} + \Lambda_n \cdot \sum_{j=1}^{k_n} |\,\theta_{nj}\,|^2.$$

(Since for any $z \in \mathbb{C}$, $|\,z\,| \leq A \implies z = A \cdot w$, for some $w \in \mathbb{C}$ with $|\,w\,| \leq 1$.) Next note that, hypotheses (a) and (b) together imply:

$$\sum_{j=1}^{k_n} |\,\theta_{nj}\,|^2 \leq \left( \max_{1\leq j \leq k_n} |\,\theta_{nj}\,| \right) \left( \sum_{j=1}^{k_n} |\,\theta_{nj}\,| \right) \leq M \cdot \left( \max_{1\leq j \leq k_n} |\,\theta_{nj}\,| \right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Therefore, since $|\,\Lambda_n\,| \leq 1$ for each $n \in \mathbb{N}$, we now see that

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \log(1+\theta_{nj}) = \lim_{n\to\infty} \sum_{j=1}^{k_n} \theta_{nj} + \lim_{n\to\infty} \left( \Lambda_n \cdot \sum_{j=1}^{k_n} |\,\theta_{nj}\,|^2 \right) = \theta + 0 = \theta.$$

We may now conclude, by continuity of the exponential function $\exp(\,\cdot\,)$:

$$\begin{aligned} \lim_{n\to\infty} \prod_{j=1}^{k_n} (\,1+\theta_{nj}\,) &= \lim_{n\to\infty} \exp\left( \log \prod_{j=1}^{k_n} (\,1+\theta_{nj}\,) \right) = \lim_{n\to\infty} \exp\left( \sum_{j=1}^{k_n} \log(\,1+\theta_{nj}\,) \right) \\ &= \exp\left( \lim_{n\to\infty} \sum_{j=1}^{k_n} \log(\,1+\theta_{nj}\,) \right) = \exp(\,\theta\,) \end{aligned}$$

This completes the proof of the Lemma. □

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