

1 Equivalence of $(C[0, 1], \|\cdot\|_\infty)$ -valued random variables and stochastic processes indexed by $[0, 1]$ with state space \mathbb{R} and continuous sample paths

Proposition 1.1 (The “one-dimensional subsets” of $C[0, 1]$ generate its Borel σ -algebra)

Let $(C[0, 1], \|\cdot\|_\infty)$ be the metric space of continuous \mathbb{R} -valued functions defined on the closed unit interval equipped with the supremum norm. For each $t \in [0, 1]$, let $\text{ev}_t : C[0, 1] \rightarrow \mathbb{R} : x \mapsto x(t)$. Define:

$$\mathcal{S} := \left\{ \text{ev}_t^{-1}(H) \subset C[0, 1] \mid \begin{array}{l} t \in [0, 1] \\ H \in \mathcal{O} \end{array} \right\} \subset \mathcal{P}(C[0, 1]).$$

Then, \mathcal{S} generates the Borel σ -algebra $\mathcal{B} := \mathcal{B}(C[0, 1], \|\cdot\|_\infty)$ of the metric space $(C[0, 1], \|\cdot\|_\infty)$; in other words,

$$\sigma(\mathcal{S}) = \mathcal{B}.$$

PROOF First, note that $\sigma(\mathcal{S}) \subset \mathcal{B}$. Indeed, recall that, for each $t \in [0, 1]$, $\text{ev}_t : C[0, 1] \rightarrow \mathbb{R}$ is continuous, hence $(\mathcal{B}, \mathcal{O})$ -measurable, by Corollary B.4. In particular, $\text{ev}_t^{-1}(H) \in \mathcal{B}$, for each $t \in [0, 1]$ and $H \in \mathcal{O}$. Thus, $\mathcal{S} \subset \mathcal{B}$; hence, $\sigma(\mathcal{S}) \subset \mathcal{B}$.

It remains to establish the reverse inclusion. To this end, first observe that, for each $x \in C[0, 1]$ and each $\varepsilon > 0$, we have

$$\overline{B(x, \varepsilon)} = \bigcap_{r \in \mathbb{Q} \cap [0, 1]} \left\{ y \in C[0, 1] \mid |y(r) - x(r)| \leq \varepsilon \right\} = \bigcap_{r \in \mathbb{Q} \cap [0, 1]} \text{ev}_r^{-1}([x(r) - \varepsilon, x(r) + \varepsilon]),$$

which shows that $\sigma(\mathcal{S})$ contains all the closed balls in $C[0, 1]$. On the other hand, recall that, in any metric space, every open ball can be expressed as a countable union of closed balls; indeed, for any y in the given metric space, and any $\delta > 0$, we have:

$$B(y, \delta) = \bigcup_{n \in \mathbb{N}} \overline{B\left(y, \delta - \frac{1}{n}\right)}.$$

We thus see that $\sigma(\mathcal{S})$ contains all the open balls in $C[0, 1]$. By the separability of $C[0, 1]$ and Theorem C.1, we see that every open subset of $C[0, 1]$ can be expressed as a countable union of open balls. Hence, $\sigma(\mathcal{S})$ in fact contains all the open subsets of $C[0, 1]$, which immediately yields $\mathcal{B} \subset \sigma(\mathcal{S})$. This proves $\sigma(\mathcal{S}) = \mathcal{B}$. \square

Theorem 1.2

Suppose:

- (Ω, \mathcal{A}) is a measurable space, and \mathcal{O} is the Borel σ -algebra of \mathbb{R} (equipped with usual Euclidean metric).
- $(C[0, 1], \|\cdot\|_\infty)$ is the metric space of continuous \mathbb{R} -valued functions defined on the compact unit interval equipped with the supremum norm, and $\mathcal{B} := \mathcal{B}(C[0, 1], \|\cdot\|_\infty)$ is its Borel σ -algebra.
- $X : \Omega \rightarrow C[0, 1]$ is a function with domain Ω and codomain $C[0, 1]$, but otherwise arbitrary.
- For each $t \in [0, 1]$, let $\text{ev}_t : C[0, 1] \rightarrow \mathbb{R} : x \mapsto x(t)$.
- For each $t \in [0, 1]$, let $X_t := \text{ev}_t \circ X$. In other words, $X_t : \Omega \rightarrow \mathbb{R} : \omega \mapsto \text{ev}_t(X(\omega)) = X(\omega)(t)$.

Then, X is $(\mathcal{A}, \mathcal{B})$ -measurable if and only if, for each $t \in [0, 1]$, X_t is $(\mathcal{A}, \mathcal{O})$ -measurable.

Donsker's Theorems (Functional Central Limit Theorems)

Study Notes

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PROOF

(\Rightarrow)

It is trivial to see that, for each $t \in [0, 1]$, $\text{ev}_t : (C[0, 1], \|\cdot\|_\infty) \rightarrow (\mathbb{R}, |\cdot|) : x \mapsto x(t)$ is continuous. Recall that continuous maps are necessarily Borel measurable; see Corollary B.4. Hence, $\text{ev}_t : (C[0, 1], \|\cdot\|_\infty) \rightarrow (\mathbb{R}, |\cdot|)$ is $(\mathcal{B}, \mathcal{O})$ -measurable, for each $t \in [0, 1]$. Now, suppose $X : \Omega \rightarrow C[0, 1]$ is $(\mathcal{A}, \mathcal{B})$ -measurable. Then, for each $t \in [0, 1]$, the composition $X_t := \text{ev}_t \circ X$ is $(\mathcal{A}, \mathcal{O})$ -measurable, as required.

(\Leftarrow)

Suppose that, for each $t \in [0, 1]$, $X_t := \text{ev}_t \circ X$ is $(\mathcal{A}, \mathcal{O})$ -measurable. We seek to establish that $X : (\Omega, \mathcal{A}) \rightarrow (C[0, 1], \mathcal{B})$ is $(\mathcal{A}, \mathcal{B})$ -measurable. To this end, let

$$\mathcal{S} := \left\{ \text{ev}_t^{-1}(H) \subset C[0, 1] \mid \begin{array}{l} t \in [0, 1] \\ H \in \mathcal{O} \end{array} \right\} \subset \mathcal{P}(C[0, 1]).$$

Then, note that the $(\mathcal{A}, \mathcal{B})$ -measurability of X follows immediately from Theorem B.3, Proposition 1.1, and the following

Claim: $X^{-1}(\mathcal{S}) \subset \mathcal{A}$.

Proof of Claim: Every set in \mathcal{S} has the form $\text{ev}_t^{-1}(H)$, for some $t \in [0, 1]$ and some $H \in \mathcal{O}$. Note that

$$X^{-1}(\text{ev}_t^{-1}(H)) = (\text{ev}_t \circ X)^{-1}(H) = X_t^{-1}(H) \in \mathcal{A},$$

where the last containment follows immediately from the $(\mathcal{A}, \mathcal{O})$ -measurability hypothesis on X_t , for each $t \in [0, 1]$. This shows that $X^{-1}(\mathcal{S}) \subset \mathcal{A}$ and completes the proof of the Claim.

The proof of the Theorem is now complete. □

Theorem 1.3

Suppose:

- (Ω, \mathcal{A}) is a measurable space, and \mathcal{O} is the Borel σ -algebra of \mathbb{R} (equipped with usual Euclidean metric).
- $(C[0, 1], \|\cdot\|_\infty)$ is the metric space of continuous \mathbb{R} -valued functions defined on the closed unit interval equipped with the supremum norm, and $\mathcal{B} := \mathcal{B}(C[0, 1], \|\cdot\|_\infty)$ is its Borel σ -algebra.
- $X : \Omega \rightarrow C[0, 1]$ is a function with domain Ω and codomain $C[0, 1]$, but otherwise arbitrary.
- For each $t \in [0, 1]$, let $\text{ev}_t : C[0, 1] \rightarrow \mathbb{R} : x \mapsto x(t)$.
- For each $t \in [0, 1]$, let $X_t := \text{ev}_t \circ X$. In other words, $X_t : \Omega \rightarrow \mathbb{R} : \omega \mapsto \text{ev}_t(X(\omega)) = X(\omega)(t)$.

Then, the following are equivalent:

- (i) X is a $(C[0, 1], \|\cdot\|_\infty)$ -valued random variable (in other words, X is $(\mathcal{A}, \mathcal{B})$ -measurable).
- (ii) For each $t \in [0, 1]$, X_t is an \mathbb{R} -valued random variable (in other words, each X_t is $(\mathcal{A}, \mathcal{O})$ -measurable).
- (iii) $\{X_t : \Omega \rightarrow \mathbb{R}\}_{t \in [0, 1]}$ is a stochastic process indexed by the closed unit interval defined on the probability space $(\Omega, \mathcal{A}, \mu)$ with state space \mathbb{R} and continuous sample paths.

PROOF The equivalence of (i) and (ii) is immediate by the preceding Theorem. The equivalence of (ii) and (iii) is immediate by the definition of stochastic processes. □

2 Donsker's Theorem for $(C[0, 1], \|\cdot\|_\infty)$

Proposition 2.1

- Let $\xi_1, \xi_2, \dots : \Omega \rightarrow \mathbb{R}$ be a sequence of independent and identically distributed \mathbb{R} -valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mu)$, with expectation value zero and common finite variance $\sigma^2 > 0$.
- Define the random variables:

$$\begin{cases} S_0 & : \Omega \rightarrow \mathbb{R} : \omega \mapsto 0, & \text{and} \\ S_n & : \Omega \rightarrow \mathbb{R} : \omega \mapsto \sum_{i=1}^n \xi_i(\omega), & \text{for each } n \in \mathbb{N}. \end{cases}$$

- For each $n \in \mathbb{N}$, define $X^{(n)} : \Omega \rightarrow C[0, 1]$ as follows:

$$X^{(n)}(\omega)(t) := \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) + n \left(t - \frac{i-1}{n} \right) \xi_i(\omega) \right\}, \text{ for each } \omega \in \Omega, t \in \left[\frac{i-1}{n}, \frac{i}{n} \right], i = 1, 2, 3, \dots, n.$$

- For each $n \in \mathbb{N}$ and each $t \in [0, 1]$, define $X_t^{(n)} : \Omega \rightarrow \mathbb{R}$ as follows:

$$X_t^{(n)}(\omega) := X^{(n)}(\omega)(t), \text{ for each } \omega \in \Omega.$$

Then, the following statements are true:

- (i) For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,

$$X^{(n)}(\omega) \left(\frac{i}{n} \right) = \frac{1}{\sigma \cdot \sqrt{n}} \cdot S_i(\omega), \text{ for } i = 0, 1, 2, \dots, n.$$

- (ii) For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,

$$X^{(n)}(\omega)(t) \text{ is the linear interpolation from } \frac{1}{\sigma \cdot \sqrt{n}} S_{i-1}(\omega) \text{ to } \frac{1}{\sigma \cdot \sqrt{n}} S_i(\omega) \text{ over } t \in \left[\frac{i-1}{n}, \frac{i}{n} \right],$$

where $i = 1, 2, \dots, n$.

- (iii) For any $0 \leq t_0 < t_1 < t_2 < \dots < t_k \leq 1$,

$$\left(X_{t_1}^{(n)} - X_{t_0}^{(n)}, \dots, X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \right) \xrightarrow{d} N \left(\mu = \mathbf{0}, \Sigma = \text{diag}(t_1 - t_0, \dots, t_k - t_{k-1}) \right), \text{ as } n \rightarrow \infty.$$

- (iv) For any $0 \leq t_1, t_2, \dots, t_k \leq 1$,

$$\left(X_{t_1}^{(n)}, X_{t_2}^{(n)}, \dots, X_{t_k}^{(n)} \right) \xrightarrow{d} N \left(\mu = \mathbf{0}, \Sigma = \left[\min\{t_i, t_j\} \right]_{1 \leq i, j \leq k} \right), \text{ as } n \rightarrow \infty.$$

PROOF

- (i) Obvious.
 (ii) Obvious.

Donsker's Theorems (Functional Central Limit Theorems)

(iii) First, note that, for each $\omega \in \Omega$, $n \in \mathbb{N}$, and $t \in [0, 1]$, we have

$$X_t^{(n)}(\omega) = \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{[nt]}(\omega) + \left(nt - [nt] \right) \cdot \xi_{[nt]+1}(\omega) \right\},$$

where $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$, defined by

$$\lfloor x \rfloor := \max \left\{ k \in \mathbb{Z} \mid k \leq x \right\}, \quad \text{for each } x \in \mathbb{R},$$

is the round-down function. We next state three Claims, whose proofs will be given below. We note that the desired conclusion follows readily from Claim 3 and the Cramér-Wold Theorem (Theorem 1.9(iii), p.56, [5]); hence the present proof is complete once we establish the three Claims below.

Claim 1: If $\{a_n\}_{n \in \mathbb{N}}$ is a sequence of non-negative integers and $\{b_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ a sequence of positive integers satisfying:

$$a_n < b_n, \text{ for sufficiently large } n \in \mathbb{N}, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{b_n - a_n}{n} = c > 0,$$

then

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \xrightarrow{d} \sqrt{c} \cdot Z, \quad \text{where } Z \sim N(0, 1).$$

Claim 2: For each fixed $t \in [0, 1]$,

$$W(t)_n := \frac{1}{\sigma \cdot \sqrt{n}} \cdot \left(nt - [nt] \right) \cdot \xi_{[nt]+1} \xrightarrow{d} 0.$$

Claim 3: For $0 \leq t_0 < t_1 < t_2 < \dots < t_k \leq 1$, and arbitrary $c_1, c_2, \dots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^k c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \xrightarrow{d} N \left(0, \sum_{i=1}^k c_i^2 (t_i - t_{i-1}) \right), \quad \text{as } n \rightarrow \infty.$$

Proof of Claim 1: Note that, for sufficiently large $n \in \mathbb{N}$, we may write

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i = \frac{\sqrt{b_n - a_n}}{\sqrt{n}} \cdot \left(\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \right).$$

Since, by hypothesis, that

$$\lim_{n \rightarrow \infty} \frac{b_n - a_n}{n} = c > 0,$$

Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [4]), once we establish the following:

$$\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \xrightarrow{d} N(0, 1), \quad \text{as } n \rightarrow \infty.$$

Donsker's Theorems (Functional Central Limit Theorems)

We establish the above convergence by invoking the Lindeberg Central Limit Theorem (Theorem 1.15, §1.5.5, p.67, [5]). In the present context, the Lindeberg Condition is the following:

$$\lim_{n \rightarrow \infty} \frac{1}{B_n^2} \cdot E \left[\sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\{|\xi_i| \geq \varepsilon S_n\}} \right] = 0, \quad \text{for each } \varepsilon > 0,$$

where

$$B_n^2 := \text{Var} \left[\sum_{i=1+a_n}^{b_n} \xi_i \right] = (b_n - a_n) \sigma^2 > 0.$$

The last equality used the hypothesis that ξ_1, ξ_2, \dots are independent and identically distributed with common finite variance $0 < \sigma^2 < \infty$. Hence, for each $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{B_n^2} \cdot E \left[\sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\{|\xi_i| \geq \varepsilon B_n\}} \right] &= \frac{1}{(b_n - a_n) \sigma^2} \cdot (b_n - a_n) \cdot E \left[\xi_1^2 \cdot I_{\{|\xi_1| \geq \varepsilon \sigma \sqrt{b_n - a_n}\}} \right] \\ &= \frac{1}{\sigma^2} \cdot E \left[\xi_1^2 \cdot I_{\{|\xi_1| / \sigma \geq \sqrt{b_n - a_n}\}} \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

since $\lim_{n \rightarrow \infty} \sqrt{b_n - a_n} = \infty$ and $\sigma^2 = E[\xi_1^2]$ is finite. This verifies that the Lindeberg Condition indeed holds in the present context, and completes the proof of Claim 1.

Proof of Claim 2: First, note that $E[W(t)_n] = 0$. We now argue that $W(t)_n \xrightarrow{p} 0$. To this end, let $\varepsilon > 0$ be given. Then,

$$\begin{aligned} \varepsilon^2 \cdot P(|W(t)_n| \geq \varepsilon) &\leq E[W(t)_n^2 \cdot I_{\{|W(t)_n| \geq \varepsilon\}}] \\ &\leq E[W(t)_n^2] = \text{Var}(W(t)_n) = \text{Var} \left[\frac{1}{\sigma \cdot \sqrt{n}} \cdot (nt - \lfloor nt \rfloor) \cdot \xi_{\lfloor nt \rfloor + 1} \right] \\ &= \frac{1}{n \cdot \sigma^2} \cdot (nt - \lfloor nt \rfloor)^2 \cdot \text{Var}(\xi_{\lfloor nt \rfloor + 1}) = \frac{1}{n \cdot \sigma^2} \cdot (nt - \lfloor nt \rfloor)^2 \cdot \sigma^2 \\ &\leq \frac{1}{n}, \end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} P(|W(t)_n| \geq \varepsilon) = 0, \quad \text{for each } \varepsilon > 0,$$

i.e. $W(t)_n \xrightarrow{p} 0$, as $n \rightarrow \infty$ (Definition 2, Chapter 1, [4]), which is equivalent to $W(t)_n \xrightarrow{d} 0$, as $n \rightarrow \infty$ (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [4]). This proves Claim 2.

Proof of Claim 3: Let $0 \leq t_0 < t_1 < t_2 < \dots < t_k \leq 1$, and $c_1, c_2, \dots, c_k \in \mathbb{R}$ be arbitrary. Observe that:

$$\begin{aligned} &\sum_{i=1}^k c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \\ &= \sum_{i=1}^k \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ S_{\lfloor nt_i \rfloor} - S_{\lfloor nt_{i-1} \rfloor} \right\} + \sum_{i=1}^k \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ (nt_i - \lfloor nt_i \rfloor) \cdot \xi_{\lfloor nt_i \rfloor + 1} - (nt_{i-1} - \lfloor nt_{i-1} \rfloor) \cdot \xi_{\lfloor nt_{i-1} \rfloor + 1} \right\} \\ &= \sum_{i=1}^k \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \right\} + \sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \\ &= \sum_{i=1}^k c_i Y_i^{(n)} + \sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \end{aligned}$$

Donsker's Theorems (Functional Central Limit Theorems)

By Claim 2 and Slutsky's Theorem (Corollary, p.40, [4]),

$$\sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \xrightarrow{d} 0, \text{ as } n \rightarrow \infty. \quad (2.1)$$

Next, note that since $\xi_1, \xi_2, \xi_3, \dots$ are independent, we see that, for each fixed $n \in \mathbb{N}$,

$$Y_i^{(n)} := \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j, \quad i = 1, 2, 3, \dots, k,$$

are independent. Now, since $0 \leq t_{i-1} < t_i \leq 1$, it follows that $\lfloor nt_{i-1} \rfloor < \lfloor nt_i \rfloor$ for sufficiently large $n \in \mathbb{N}$. In addition,

$$\begin{aligned} \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} &= \frac{\lfloor nt_i \rfloor}{n} - \frac{\lfloor nt_{i-1} \rfloor}{n} = \left(\frac{nt_i}{n} + \frac{\lfloor nt_i \rfloor - nt_i}{n} \right) - \left(\frac{nt_{i-1}}{n} + \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right) \\ &= t_i - t_{i-1} + \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n}, \end{aligned}$$

which implies

$$\left| \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} - (t_i - t_{i-1}) \right| = \left| \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right| \leq \frac{2}{n} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} = t_i - t_{i-1} > 0.$$

Thus, by Claim 1, we see that, for each $i = 1, 2, \dots, k$,

$$Y_i^{(n)} := \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \xrightarrow{d} \sqrt{t_i - t_{i-1}} \cdot N(0, 1) = N\left(0, t_i - t_{i-1}\right), \text{ as } n \rightarrow \infty. \quad (2.2)$$

By (2.1), (2.2), Proposition A.1, and Slutsky's Theorem (Corollary, p.40, [4]), we now see that

$$\sum_{i=1}^k c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) = \sum_{i=1}^k c_i Y_i^{(n)} + \sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \xrightarrow{d} N\left(0, \sum_{i=1}^k c_i^2 (t_i - t_{i-1})\right).$$

This completes the proof of Claim 3.

(iv) Let $t_0 := 0$, hence, $X_{t_0}^{(n)} \equiv 0$ for each $n \in \mathbb{N}$. We thus have, for each $n \in \mathbb{N}$,

$$\begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}}_T \cdot \begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} - X_{t_1}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \end{pmatrix}.$$

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By (iii), we know that

$$\begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} - X_{t_1}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \end{pmatrix} \xrightarrow{d} \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix} \sim N\left(\mu = \mathbf{0}, \Sigma = \text{diag}(t_1, t_2 - t_1, \dots, t_k - t_{k-1})\right), \text{ as } n \rightarrow \infty.$$

Since the map $\mathbb{R}^k \rightarrow \mathbb{R}^k : x \mapsto T \cdot x$ is continuous, we see immediately by Slutsky's Theorem (Theorem 6(a), p.39, [4]) that

$$\begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} \end{pmatrix} \xrightarrow{d} T \cdot \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix}, \text{ as } n \rightarrow \infty.$$

Since the map $\mathbb{R}^k \rightarrow \mathbb{R}^k : x \mapsto T \cdot x$ is an invertible linear automorphism on \mathbb{R}^k , we see that

$$L = \begin{pmatrix} L_{t_1} \\ L_{t_2} \\ \vdots \\ L_{t_k} \end{pmatrix} := T \cdot \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix}$$

is still an \mathbb{R}^k -valued Gaussian random variable, and it clearly has expectation value $\mathbf{0} \in \mathbb{R}^k$, since each of $Z_{t_1}, Z_{t_2-t_1}, \dots, Z_{t_k-t_{k-1}}$ has expectation value $0 \in \mathbb{R}$. It remains only to compute the covariance matrix of the \mathbb{R}^k -valued Gaussian random variable L . To this end, consider $1 \leq i \leq j \leq k$, i.e. $t_i \leq t_j$. Then, using the alternative notation $Z_{t_i-t_0} := Z_{t_i}$, we have

$$\begin{aligned} \text{Cov}(L_{t_i}, L_{t_j}) &= \text{Cov}(Z_{t_1} + Z_{t_2-t_1} + \dots + Z_{t_i-t_{i-1}}, Z_{t_1} + Z_{t_2-t_1} + \dots + Z_{t_j-t_{j-1}}) \\ &= \text{Cov}\left(\sum_{a=1}^i Z_{t_a-t_{a-1}}, \sum_{b=1}^j Z_{t_b-t_{b-1}}\right) = \sum_{a=1}^i \sum_{b=1}^j \text{Cov}(Z_{t_a-t_{a-1}}, Z_{t_b-t_{b-1}}) \\ &= \sum_{a=1}^i \text{Cov}(Z_{t_a-t_{a-1}}, Z_{t_a-t_{a-1}}) = \sum_{a=1}^i \text{Var}(Z_{t_a-t_{a-1}}) = \sum_{a=1}^i (t_a - t_{a-1}) \\ &= (t_1 - t_0) + (t_2 - t_1) + \dots + (t_{i-1} - t_{i-2}) + (t_i - t_{i-1}) \\ &= t_i = \min\{t_i, t_j\}, \end{aligned}$$

as required. □

A Technical Lemmas

Note that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ does NOT in general imply $X_n + Y_n \xrightarrow{d} X + Y$. But the implication does hold if X_n and Y_n are independent for each $n \in \mathbb{N}$, and both X and Y are Gaussian random variables, as the following Proposition shows.

Proposition A.1 *Let $k \in \mathbb{N}$ be fixed. Suppose:*

- For each $n \in \mathbb{N}$,

$$Y_1^{(n)}, Y_2^{(n)}, \dots, Y_k^{(n)} : \Omega^{(n)} \longrightarrow \mathbb{R}$$

are independent \mathbb{R} -valued random variables defined on the probability space $\Omega^{(n)}$.

- For each $i = 1, 2, \dots, k$,

$$Y_i^{(n)} \xrightarrow{d} N(\mu_i, \sigma_i^2), \quad \text{as } n \longrightarrow \infty.$$

Then, for any $c_1, c_2, \dots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^k c_i Y_i^{(n)} \xrightarrow{d} N\left(\sum_{i=1}^k c_i \mu_i, \sum_{i=1}^k c_i^2 \sigma_i^2\right), \quad \text{as } n \longrightarrow \infty.$$

PROOF Let $Y^{(n)} := \sum_{i=1}^k c_i Y_i^{(n)}$. Let φ_X denote the characteristic function of a \mathbb{R} -valued random variable X . Then,

$$\begin{aligned} \varphi_{Y^{(n)}}(t) &= \varphi_{\sum_{i=1}^k c_i Y_i^{(n)}}(t) \\ &= \prod_{i=1}^k \varphi_{c_i Y_i^{(n)}}(t), \quad \text{since } Y_1^{(n)}, \dots, Y_k^{(n)} \text{ are independent} \\ &= \prod_{i=1}^k \varphi_{Y_i^{(n)}}(c_i t) \\ &\longrightarrow \prod_{i=1}^k \exp\left\{\sqrt{-1} \mu_i (c_i t) - \frac{1}{2} \sigma_i^2 (c_i t)^2\right\} \\ &= \exp\left\{\sqrt{-1} \left(\sum_{i=1}^k c_i \mu_i\right) t - \frac{1}{2} \left(\sum_{i=1}^k c_i^2 \sigma_i^2\right) t^2\right\}, \quad \text{as } n \longrightarrow \infty, \end{aligned}$$

where the second and third equalities follow from the properties of characteristic functions of random variables (see p.21, [4]), while the expression of the limit follows from the fact that the characteristic function φ_Z of a random variable Z with distribution $N(\mu, \sigma^2)$ is

$$\varphi_Z = \exp\left\{\sqrt{-1} \mu t - \frac{1}{2} \sigma^2 t^2\right\}.$$

The Proposition now follows immediately from the Lévy-Cramér Continuity Theorem (Theorem 1.9(ii), p.56, [5]). \square

B Continuous maps are Borel measurable

Lemma B.1 (The pre-image of a σ -algebra is itself a σ -algebra.)

Suppose Ω is a non-empty set, (X, \mathcal{X}) is a measurable space, and $f : \Omega \longrightarrow X$ is a map from Ω into X . Then,

$$f^{-1}(\mathcal{X}) := \{f^{-1}(V) \subset \Omega \mid V \in \mathcal{X}\}$$

is a σ -algebra of subsets of Ω .

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PROOF

$$\Omega \in f^{-1}(\mathcal{X}) \quad f(\Omega) \subset X \implies \Omega = f^{-1}(X) \in f^{-1}(\mathcal{X}).$$

$f^{-1}(\mathcal{X})$ is closed under complementations Let $V \in \mathcal{X}$. Then, $X \setminus V \in \mathcal{X}$, and

$$\Omega \setminus f^{-1}(V) = \{ \omega \in \Omega \mid f(\omega) \notin V \} = \{ \omega \in \Omega \mid f(\omega) \in X \setminus V \} = f^{-1}(X \setminus V) \in f^{-1}(\mathcal{X}),$$

which shows that $f^{-1}(\mathcal{X})$ is indeed closed under complementations.

$f^{-1}(\mathcal{X})$ is closed countable unions Let $V_1, V_2, \dots \in \mathcal{X}$. Then, $\bigcup_{i=1}^{\infty} V_i \in \mathcal{X}$, and

$$\bigcup_{i=1}^{\infty} f^{-1}(V_i) = \left\{ \omega \in \Omega \mid \begin{array}{c} f(\omega) \in V_i \\ \text{for some } i \in \mathbb{N} \end{array} \right\} = f^{-1}\left(\bigcup_{i=1}^{\infty} V_i\right) \in f^{-1}(\mathcal{X}),$$

which proves that $f^{-1}(\mathcal{X})$ is indeed closed under countable unions.

This concludes the proof that that $f^{-1}(\mathcal{X})$ is a σ -algebra of subsets of Ω . □

Lemma B.2 (The push-forward of a σ -algebra is itself a σ -algebra.)

Suppose (Ω, \mathcal{A}) is a measurable space, X is a non-empty set, and $f : \Omega \rightarrow X$ is a map from Ω into X . Then,

$$\mathcal{F} := \{ V \subset X \mid f^{-1}(V) \in \mathcal{A} \}$$

is a σ -algebra of subsets of X .

PROOF

$$\underline{X \in \mathcal{F}} \quad f^{-1}(X) = \Omega \in \mathcal{A} \implies X \in \mathcal{F}.$$

\mathcal{F} is closed under complementations $V \in \mathcal{F} \implies f^{-1}(V) \in \mathcal{A} \implies f^{-1}(X \setminus V) = \Omega \setminus f^{-1}(V) \in \mathcal{A} \implies X \setminus V \in \mathcal{F}$, which proves that \mathcal{F} is indeed closed under complementations.

\mathcal{F} is closed under countable unions

$$\begin{aligned} V_1, V_2, \dots \in \mathcal{F} &\implies f^{-1}(V_1), f^{-1}(V_2), \dots \in \mathcal{A} \\ &\implies f^{-1}\left(\bigcup_{i=1}^{\infty} V_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(V_i) \in \mathcal{A} \\ &\implies \bigcup_{i=1}^{\infty} V_i \in \mathcal{F}, \end{aligned}$$

which proves that \mathcal{F} is indeed closed under countable unions. □

Theorem B.3

Suppose (Ω, \mathcal{A}) and (X, \mathcal{X}) are measurable spaces, and $f : \Omega \rightarrow X$ is a map from Ω into X . Then, f is $(\mathcal{A}, \mathcal{X})$ -measurable if there exists $\mathcal{S} \subset \mathcal{X}$ satisfying the following conditions:

- \mathcal{S} generates \mathcal{X} , i.e. $\sigma(\mathcal{S}) = \mathcal{X}$, and
- $f^{-1}(S) \in \mathcal{A}$.

PROOF By Lemma B.2,

$$\mathcal{F} := \{ V \subset X \mid f^{-1}(V) \in \mathcal{A} \}$$

is a σ -algebra of subsets of X . By hypothesis, $\mathcal{S} \subset \mathcal{F}$; hence, $\mathcal{X} = \sigma(\mathcal{S}) \subset \mathcal{F}$. Thus, $f^{-1}(\mathcal{X}) \subset \mathcal{A}$; equivalently, f is $(\mathcal{A}, \mathcal{X})$ -measurable. \square

Corollary B.4 (Continuous maps are Borel measurable.)

Suppose X_1, X_2 are topological spaces, and $\mathcal{B}_1, \mathcal{B}_2$ are their respective Borel σ -algebras. Then, every continuous map $f : X_1 \rightarrow X_2$ is $(\mathcal{B}_1, \mathcal{B}_2)$ -measurable.

C Topology

Theorem C.1 (Appendix M3, [3])

Suppose S is a metric space. Then, the following conditions are equivalent:

- (i) S is separable.
- (ii) The topology of S has a countable basis.
- (iii) Every open cover of *each subset* of S has a countable subcover.

D The Arzelà-Ascoli Theorem

Recall that the space $C(X)$ of continuous \mathbb{R} -valued functions defined on a compact topological space X equipped with the supremum norm is a complete metric space (see Theorem 9.3, [1]). The Arzelà-Ascoli Theorem characterizes compactness of subsets of $C(X)$.

Definition D.1 (Equicontinuity)

Let X be a topological space and (Y, d) a metric space. Let Y^X denote the set of arbitrary functions from X into Y .

- A subset $S \subset Y^X$ is said to be **equicontinuous at** $x_0 \in X$ if, for each $\varepsilon > 0$, there exists an open subset $V \subset X$ satisfying:

$$x_0 \in V, \quad \text{and} \quad \sup_{(x,f) \in V \times S} \left\{ d(f(x), f(x_0)) \right\} \leq \varepsilon.$$

- A subset $S \subset Y^X$ is said to be **equicontinuous** if it is equicontinuous at each $x_0 \in X$.

Definition D.2 (Uniform equicontinuity)

Let (X, ρ) and (Y, d) two metric spaces. Let Y^X denote the set of arbitrary functions from X into Y . A subset $S \subset Y^X$ is said to be **uniformly equicontinuous** if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\sup \left\{ d(f(x_1), f(x_2)) \mid \begin{array}{l} f \in S, \ x_1, x_2 \in X, \\ \rho(x_1, x_2) < \delta \end{array} \right\} \leq \varepsilon.$$

Proposition D.3

Let (X, ρ) and (Y, d) two metric spaces and suppose that (X, ρ) is compact. Let Y^X denote the set of arbitrary functions from X into Y . Then, a subset $S \subset Y^X$ is uniformly equicontinuous if and only if it is equicontinuous.

Theorem D.4 (Arzelà-Ascoli, Theorem 9.10, [1])

Suppose X is a compact topological space and $C(X)$ is the space of continuous \mathbb{R} -valued functions defined on X equipped with the supremum norm. Then, for each $S \subset C(X)$, the following conditions are equivalent:

- (i) S is a compact subset of $C(X)$.

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(ii) S is closed, bounded, and equicontinuous subset of $C(X)$.

PROOF

(i) \implies (ii)

Recall that every compact subset in a metric space is closed and bounded. Thus, it remains only to show that $S \subset C(X)$ is equicontinuous. To this end, let $\varepsilon > 0$ be given. Recall that a metric space is compact if and only if it is complete and totally bounded (Theorem 7.8, [1]). Thus, the compactness hypothesis on S implies S is totally bounded; in particular, there exist $f_1, \dots, f_n \in S$ such that

$$S \subset \bigcup_{i=1}^n B\left(f_i, \frac{\varepsilon}{3}\right).$$

Hence, for each $x_0 \in X$, we may define

$$V(x_0) := \bigcap_{i=1}^n f_i^{-1}\left(\left(f_i(x_0) - \frac{\varepsilon}{3}, f_i(x_0) + \frac{\varepsilon}{3}\right)\right).$$

Note that $V(x_0)$ is open and $x_0 \in V(x_0)$. Now, let $f \in S$ and $x \in V(x_0)$ be given. We may choose $i \in \{1, 2, \dots, n\}$ such that $f \in B\left(f_i, \frac{\varepsilon}{3}\right)$, i.e. $\|f - f_i\|_\infty \leq \frac{\varepsilon}{3}$. Hence,

$$\left|f(x) - f(x_0)\right| \leq \left|f(x) - f_i(x)\right| + \left|f_i(x) - f_i(x_0)\right| + \left|f_i(x_0) - f(x_0)\right| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus,

$$\sup_{(x,f) \in V(x_0) \times S} \left\{ \left|f(x) - f(x_0)\right| \right\} \leq \varepsilon.$$

This shows equicontinuity of S at $x_0 \in X$. Since $x_0 \in X$ is arbitrary, we may conclude that S is equicontinuous.

(ii) \implies (i)

Suppose $S \subset C(X)$ is closed, bounded, and equicontinuous. We need to show that S is a compact subset of $C(X)$. Recall that every subset of a metric space is compact if and only if it is sequentially compact (Theorem 7.3, [1]). Thus, it suffices to show that every sequence $\{f_n\}_{n \in \mathbb{N}} \subset S$ has a convergent subsequence with limit in S . We start by stating and proving the following:

Claim 1:

For each $k \in \mathbb{N}$, there exists a finite subset $F_k \subset X$ and open neighbourhoods $\{V_y\}_{y \in F_k}$ such that

$$X = \bigcup_{y \in F_k} V_y, \quad \text{and} \quad \sup \left\{ \left|f(x) - f(y)\right| \mid \begin{array}{l} x \in V_y, y \in F_k \\ f \in S \end{array} \right\} \leq \frac{1}{3k}.$$

Proof of Claim 1: By equicontinuity of S , for each $y \in X$, there exists an open neighbourhood $V_y \subset X$ of $y \in X$ such that

$$\sup \left\{ \left|f(x) - f(y)\right| \mid \begin{array}{l} x \in V_y \\ f \in S \end{array} \right\} \leq \frac{1}{3k}.$$

Thus, $X = \bigcup_{y \in X} V_y$ is an open cover of X . Compactness of X now implies that this open cover of X admits a finite subcover, i.e.

$$X = \bigcup_{y \in F_k} V_y, \quad \text{for some finite subset } F_k \subset X.$$

Lastly, note that

$$\sup \left\{ \left|f(x) - f(y)\right| \mid \begin{array}{l} x \in V_y, y \in F_k \\ f \in S \end{array} \right\} = \sup_{y \in F_k} \left\{ \sup \left\{ \left|f(x) - f(y)\right| \mid \begin{array}{l} x \in V_y \\ f \in S \end{array} \right\} \right\} \leq \frac{1}{3k}.$$

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This completes the proof of Claim 1.

Next, let $F := \bigcup_{k=1}^{\infty} F_k$. Note that F is a countably infinite set. Let $F = \{x_1, x_2, \dots\}$ be an enumeration of F . Recall that we wish to prove that every sequence $\{f_n\}_{n \in \mathbb{N}} \subset S$ contains a convergent subsequence with limit in S . Now, consider the array of real numbers:

$$\begin{array}{cccc} f_1(x_1) & f_2(x_1) & f_3(x_1) & \cdots \\ f_1(x_2) & f_2(x_2) & f_3(x_2) & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

Since $S \subset C(X)$ is bounded (with respect to the $\|\cdot\|_{\infty}$ norm on $C(X)$), there exists $M > 0$ such that $\sup_{f \in S} \|f\|_{\infty} \leq M$.

In particular, every row in the above array is bounded. By Theorem A.14, p.538, [2], there exists an increasing sequence of positive integers $n(1), n(2), n(3), \dots$ such that the limit

$$\lim_{i \rightarrow \infty} f_{n(i)}(x_k) \text{ exists, for each } k = 1, 2, \dots$$

Claim 2: $\{f_{n(i)}\}_{i \in \mathbb{N}}$ is a Cauchy sequence in $(C(X), \|\cdot\|_{\infty})$.

Proof of Claim 2: For each $k \in \mathbb{N}$, the convergence of $\{f_{n(i)}(x_k)\}_{i \in \mathbb{N}}$ in \mathbb{R} implies that each $\{f_{n(i)}(x_k)\}_{i \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} . Since the set F_k is finite, we see that there exists $m_k \in \mathbb{N}$ such that

$$|f_{n(i)}(y) - f_{n(j)}(y)| < \frac{1}{3k}, \quad \text{for any } i, j \geq m_k, \text{ and each } y \in F_k.$$

Now, for each $x \in X$, there exists $y \in F_k$ such that $x \in V_y$. Hence, for any $i, j \geq m_k$ and any $x \in X$, we have

$$\begin{aligned} |f_{n(i)}(x) - f_{n(j)}(x)| &\leq |f_{n(i)}(x) - f_{n(i)}(y)| + |f_{n(i)}(y) - f_{n(j)}(y)| + |f_{n(j)}(y) - f_{n(j)}(x)| \\ &\leq \frac{1}{3k} + \frac{1}{3k} + \frac{1}{3k} = \frac{1}{k}. \end{aligned}$$

In other words,

$$\|f_{n(i)} - f_{n(j)}\|_{\infty} \leq \frac{1}{k}, \quad \text{for any } i, j \geq m_k.$$

This shows that $\{f_{n(i)}\}_{i \in \mathbb{N}}$ is indeed a Cauchy sequence in $(C(X), \|\cdot\|_{\infty})$ and completes the proof of Claim 2.

Lastly, by Theorem 9.3, [1], $(C(X), \|\cdot\|_{\infty})$ is a complete metric space. Thus, the Cauchy sequence $\{f_{n(i)}\}_{i \in \mathbb{N}} \subset C(X)$ converges to some element $f_0 \in C(X)$. Since $S \subset C(X)$ is, by hypothesis, a closed subset of $C(X)$, we see furthermore that $f_0 \in S$. This proves the sequential compactness of S and completes the proof of the Arzelà-Ascoli Theorem. \square

Theorem D.5 (Theorem 7.2, p.81, [3])

Let $C[0, 1]$ denote the space of continuous \mathbb{R} -valued functions defined on the closed unit interval $[0, 1]$ equipped with the supremum norm. Then, for each subset $S \subset C[0, 1]$, the following are equivalent:

- (i) S is a relatively compact subset of $C[0, 1]$ (i.e. the closure of S is a compact subset of $C[0, 1]$).
- (ii) S satisfies: $\sup_{f \in S} |f(0)| < \infty$, and for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup \left\{ |f(t_1) - f(t_2)| \mid \begin{array}{l} f \in S, \, t_1, t_2 \in [0, 1], \\ |t_1 - t_2| < \delta \end{array} \right\} \leq \varepsilon.$$

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(iii) S satisfies: $\sup_{f \in S} \left\{ |f(0)| \right\} < \infty$, and

$$\lim_{\delta \rightarrow 0} \sup_{f \in S} \left(\sup_{|s-t| < \delta} \left\{ |f(s) - f(t)| \right\} \right) = 0.$$

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