1 Separating and convergence-determining classes

Definition 1.1 (Separating class)
Suppose S is a non-empty set, $\mathcal{B}$ is a σ-algebra of subsets of S, $(S,\mathcal{B})$ is the corresponding measurable space, and $\mathcal{M}_1(S,\mathcal{B})$ is the set of all probability measures defined on $\mathcal{B}$. A ***separating class*** *of subsets of $(S,\mathcal{B})$ is a collection $\mathcal{A} \subset \mathcal{B}$ of subsets of S which satisfies the following condition: For every two probability measures $\mu,\nu \in \mathcal{M}_1(S,\mathcal{B})$,*

$$\mu(A) = \nu(A),\ \text{for every } A \in \mathcal{A} \quad \Longrightarrow \quad \mu(B) = \nu(B),\ \text{for every } B \in \mathcal{B}$$

Definition 1.2 (Convergence-determining class)
Suppose S is a topological space, $\mathcal{B}(S)$ is its Borel σ-algebra, $(S,\mathcal{B}(S))$ is the corresponding measurable space, and $\mathcal{M}_1(S,\mathcal{B}(S))$ is the set of all probability measures defined on $\mathcal{B}(S)$. A ***convergence-determining class*** *of subsets of $(S,\mathcal{B}(S))$ is a collection $\mathcal{A} \subset \mathcal{B}(S)$ of subsets of S which satisfies the following condition: For any $\mu,\mu_1,\mu_2,\ldots \in \mathcal{M}_1(S,\mathcal{B})$,*

$$\lim_{n\to\infty} \mu_n(A) \;=\; \mu(A),\ \text{for every } A \in \mathcal{A} \quad \Longrightarrow \quad \mu_n \xrightarrow{\;w\;} \mu.$$

2 Examples of separating and convergence-determining classes of $\mathbb{R}^\infty$

Definition 2.1 (The metric on $\mathbb{R}^\infty$, Example 1.2, [1])
Let $\mathbb{R}^\infty$ denotes the set of all infinite sequences of real numbers, i.e.

$$\mathbb{R}^\infty \;:=\; \{\ (x_1,x_2,\ldots)\ \mid x_i \in \mathbb{R},\ \text{for each } i \in \mathbb{N}\ \}.$$

Define $\rho : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow [\,0,1\,]$ as follows:

$$\rho(x,y) \;:=\; \sum_{n=1}^{\infty} \frac{\min\{\,1\,,\,|\,x_n - y_n\,|\,\}}{2^n}.$$

Remark 2.2 *Recall that*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \;=\; \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^{n-1}} \;=\; \frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}\right) \;=\; 1,$$

which proves indeed that $0 \leq \rho(x,y) \leq 1$, for any $x,y \in \mathbb{R}^\infty$.

Theorem 2.3 (The metric space properties of $\mathbb{R}^\infty$)

(i) *$(\,\mathbb{R}^\infty,\rho\,)$ is a metric space. Let $\mathbb{R}^\infty$ denote also this metric space in the remainder of this Theorem.*

(ii) *For $x, x^{(1)}, x^{(2)}, x^{(3)}, \ldots, \in \mathbb{R}^\infty$, we have:*

$$\rho\Big(\,x^{(n)},\,x\,\Big) \;\longrightarrow\; 0 \quad \Longleftrightarrow \quad \text{for each } i \in \mathbb{N},\ \lim_{n\to\infty}\left|\,x_i^{(n)} - x_i\,\right| \;=\; 0$$

(iii) *For each $n \in \mathbb{N}$, the "natural projection to the initial segment of length n"*

$$\pi_n \;:\; \mathbb{R}^\infty \;\longrightarrow\; \mathbb{R}^n \;:\; x \;\longmapsto\; (x_1,x_2,\ldots,x_n)$$

is continuous, where $\mathbb{R}^n$ has the usual Euclidean topology.

(iv) *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, let $C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)$ denote the open hypercube in $\mathbb{R}^n$ of side length 2ε centred at $\pi_n(x) \in \mathbb{R}^n$, i.e.*

$$C_{\mathbb{R}^n}(\pi_n(x),\varepsilon) \;:=\; \left\{\, y \in \mathbb{R}^n \;\middle|\; \begin{array}{c} |\,y_i - x_i\,| \,<\, \varepsilon, \\ i = 1,2,\ldots,n \end{array} \right\}$$

Then, its pre-image in $\mathbb{R}^\infty$ under π_n

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;=\; \left\{\, y \in \mathbb{R}^\infty \;\middle|\; \begin{array}{c} |\,y_i - x_i\,| \,<\, \varepsilon, \\ i = 1,2,\ldots,n \end{array} \right\}$$

is an open subset of $\mathbb{R}^\infty$.

(v) *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, we have:*

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\subset\; B_{\mathbb{R}^\infty}\!\left(x\,,\,\varepsilon + \frac{1}{2^n}\right),$$

where $B_{\mathbb{R}^\infty}\!\left(x\,,\,\varepsilon + \frac{1}{2^n}\right)$ is the open ball in $\mathbb{R}^\infty$ centred at x of radius $\varepsilon + \frac{1}{2^n}$, i.e.

$$B_{\mathbb{R}^\infty}\!\left(x\,,\,\varepsilon + \frac{1}{2^n}\right) \;:=\; \left\{\, y \in \mathbb{R}^\infty \;\middle|\; \rho(y,x) \,<\, \varepsilon + \frac{1}{2^n} \,\right\}$$

(vi) *The collection*

$$\left\{\; \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \subset \mathbb{R}^\infty \;\middle|\; n \in \mathbb{N},\, x \in \mathbb{R}^\infty,\, \varepsilon > 0 \;\right\}$$

of all pre-images under π_n of open hypercubes in $\mathbb{R}^n$, for all $n \in \mathbb{N}$, forms a basis for the topology of $\mathbb{R}^\infty$.

(vii) *$\mathbb{R}^\infty$ is a separable metric space.*

(viii) *$\mathbb{R}^\infty$ is a complete metric space.*

Proof

(i) Clearly, ρ is non-negative and symmetric. We now show that, for any $x, y \in \mathbb{R}^\infty$, we have $\rho(x,y) = 0$ implies $x = y$. Indeed,

$$\begin{aligned}
\rho(x,y) \,=\, 0 \quad &\Longleftrightarrow \quad \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - y_i\,|\,\}}{2^i} \,=\, 0 \\
&\Longleftrightarrow \quad \min\{\,1\,,|\,x_i - y_i\,|\,\} \,=\, 0, \;\; \text{for each } i \in \mathbb{N} \\
&\Longleftrightarrow \quad |\,x_i - y_i\,| \,=\, 0, \;\; \text{for each } i \in \mathbb{N} \\
&\Longleftrightarrow \quad x \,=\, y.
\end{aligned}$$

In order to show that ρ is a metric, it remains only to establish the Triangle Inequality. By Lemma A.2, for any $x, y, z \in \mathbb{R}^\infty$, we have

$$\begin{aligned}
\rho(x,y) \;&=\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - y_i\,|\,\}}{2^i} \\
&\leq\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - z_i\,|\,\} + \min\{\,1\,,|\,z_i - y_i\,|\,\}}{2^i} \\
&=\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - z_i\,|\,\}\,\}}{2^i} + \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,z_i - y_i\,|\,\}}{2^i} \\
&=\; \rho(x,z) + \rho(z,y),
\end{aligned}$$

where we have used the fact that $0 \leq \rho \leq 1$ to split the infinite sum into two terms in second-to-last equality. This proves that ρ satisfies the Triangle Inequality, and it is thus a metric on $\mathbb{R}^\infty$.

(ii) $\underline{\lim_{n\to\infty} \rho\big(x^{(n)},x\big) = 0 \implies \lim_{n\to\infty} \left| x_i^{(n)} - x_i \right| = 0, \text{ for each } i \in \mathbb{N}}$

$$\begin{aligned}
\lim_{n\to\infty} \rho\Big(x^{(n)},x\Big) = 0 \quad &\implies \quad \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i^{(n)} - x_i\,|\,\}}{2^i} = 0 \\
&\implies \quad \lim_{n\to\infty} \min\{\,1\,,|\,x_i^{(n)} - x_i\,|\,\} = 0, \;\; \text{for each } i \in \mathbb{N} \\
&\implies \quad \lim_{n\to\infty} \left| x_i^{(n)} - x_i \right| = 0, \;\; \text{for each } i \in \mathbb{N}
\end{aligned}$$

$\underline{\lim_{n\to\infty} \left| x_i^{(n)} - x_i \right| = 0, \text{ for each } i \in \mathbb{N} \implies \lim_{n\to\infty} \rho\big(x^{(n)},x\big) = 0}$

This follows from the Weierstrass M-test. Suppose $\lim_{n\to\infty} \left| x_i^{(n)} - x_i \right| = 0$, for each $i \in \mathbb{N}$. Then,

$$\lim_{n\to\infty} \frac{\min\{\,1\,,|\,x_i^{(n)} - x_i\,|\,\}}{2^i} \; = \; 0 \; =: \; y_i, \;\; \text{for each } i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$, let $M_i \; := \; \dfrac{1}{2^i}$. Then,

$$\frac{\min\{1\,,|\,x_i^{(n)} - x_i\,|\}}{2^i} \;\leq\; M_i \quad \text{and} \quad \sum_{i=1}^{\infty} M_i \;<\; \infty.$$

Hence, by the Weierstrass M-test (Lemma A.3), we have

$$\lim_{n\to\infty} \rho\Big(x^{(n)},x\Big) \;\; = \;\; \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i^{(n)} - x_i\,|\,\}}{2^i} \;\; = \;\; \sum_{i=1}^{\infty} y_i \;\; = \;\; 0.$$

(iii) Immediate by (ii).

(iv) Since $C_{\mathbb{R}^n}(\pi_n(x),\varepsilon) \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, its pre-image under the continuous (by (iii)) map $\pi_n : \mathbb{R}^\infty \longrightarrow \mathbb{R}^n$ is an open subset of $\mathbb{R}^\infty$.

(v) For $y \in \mathbb{R}^\infty$, we have

$$\begin{aligned}
y \in \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \quad &\implies \quad |\,y_i - x_i\,| < \varepsilon, \text{ for each } i = 1,2,\ldots,n \\
&\implies \quad \rho(x,y) \;\; := \;\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - y_i\,|\,\}}{2^i} \;\; \leq \;\; \sum_{i=1}^{n} \frac{\varepsilon}{2^i} \; + \; \sum_{i=n+1}^{\infty} \frac{1}{2^i} \;\; = \;\; \varepsilon + \frac{1}{2^n}.
\end{aligned}$$

This proves:

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\; \subset \;\; B_{\mathbb{R}^\infty}\left(\,x\,,\,\varepsilon + \frac{1}{2^n}\,\right).$$

(vi) It suffices to show that every open ball in $B_{\mathbb{R}^\infty}(x,r) \subset \mathbb{R}^\infty$, $r > 0$, contains the pre-image of an open hypercube centred at $\pi_n(x) \in \mathbb{R}^n$ under π_n. To this end, for $r > 0$, choose $\varepsilon > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large such that $\varepsilon + \dfrac{1}{2^n} \, < \, r$. Then, for any $x \in \mathbb{R}^\infty$, by (v), we have:

$$x \;\; \in \;\; \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\; \subset \;\; B_{\mathbb{R}^\infty}\left(\,x\,,\,\varepsilon + \frac{1}{2^n}\,\right) \;\; \subset \;\; B_{\mathbb{R}^\infty}(\,x\,,r\,),$$

as required.

(vii) It suffices to exhibit a countable subset of $\mathbb{R}^\infty$ that intersects every open ball in $\mathbb{R}^\infty$. To this end, let

$$D \;:=\; \bigcup_{n=1}^{\infty}\left\{\, x=(x_1,x_2,\dots)\in\mathbb{R}^\infty \;\middle|\; \begin{array}{c} x_i\in\mathbb{Q},\ \text{for each } i\in\mathbb{N} \\ x_i=0,\ \text{for all } i\geq n \end{array}\right\}.$$

Clearly, D is a countable subset of $\mathbb{R}^\infty$. Now let $B_{\mathbb{R}^\infty}(x,\varepsilon)$ be an arbitrary open ball in $\mathbb{R}^\infty$. Choose $\delta>0$ small enough and $n\in\mathbb{N}$ large enough such that $\delta+\dfrac{1}{2^n}<\varepsilon$. Then,

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\delta)\,)\ \subset\ B_{\mathbb{R}^\infty}\!\left(x\,,\,\delta+\frac{1}{2^n}\right)\ \subset\ B_{\mathbb{R}^\infty}(x\,,\varepsilon\,)\,,$$

Now, for each $i=1,2,\dots,n$, choose $z_i\in\mathbb{Q}\cap(x_i-\delta,x_i+\delta)$. Let $z=(z_1,z_2,\dots,z_n,\,0,\,0,\,\dots)\in\mathbb{R}^\infty$. Then, we have

$$z\ \in\ D\bigcap\left\{\, y\in\mathbb{R}^\infty \;\middle|\; \begin{array}{c} y_i\in(x_i-\delta,x_i+\delta), \\ \text{for each } i=1,2,\dots,n \end{array}\right\}\ =\ D\bigcap\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\delta)\,)\ \subset\ D\bigcap B_{\mathbb{R}^\infty}(x\,,\varepsilon\,)\,.$$

This proves the the countable subset $D\subset\mathbb{R}^\infty$ has non-empty intersection with every open ball in $\mathbb{R}^\infty$, i.e. D is dense in $\mathbb{R}^\infty$. Hence, $\mathbb{R}^\infty$ is separable.

(viii) We need to show that every Cauchy sequence in $\mathbb{R}^\infty$ converges to any element in $\mathbb{R}^\infty$.

$$\begin{aligned}
& \left\{x^{(n)}\right\}_{n\in\mathbb{N}}\subset\mathbb{R}^\infty \text{ is a Cauchy sequence in } \mathbb{R}^\infty \\
\Longleftrightarrow\quad & \text{for each } \varepsilon>0, \text{ there exists } N_\varepsilon\in\mathbb{N} \text{ such that } \rho\!\left(x^{(m)},x^{(n)}\right)\ <\ \varepsilon,\ \text{for any } m,n>N_\varepsilon \\
\Longrightarrow\quad & \text{for each } i\in\mathbb{N}, \text{ we have:} \\
& \text{for each } \varepsilon>0, \text{ there exists } N_\varepsilon\in\mathbb{N} \text{ such that } \left|\,x_i^{(m)}-x_i^{(n)}\,\right|\ <\ \varepsilon,\ \text{for any } m,n>N_\varepsilon \\
\Longrightarrow\quad & \text{for each } i\in\mathbb{N},\ \left\{x_i^{(n)}\right\}_{n\in\mathbb{N}}\subset\mathbb{R} \text{ is a Cauchy sequence in } \mathbb{R};\ \text{hence } x_i\ :=\ \lim_{n\to\infty}x_i^{(n)}\in\mathbb{R} \text{ exists} \\
\Longrightarrow\quad & \lim_{n\to\infty}\rho\!\left(x^{(n)},x\right)\ =\ 0,\ \text{ where } x:=(x_1,x_2,\dots)\in\mathbb{R}^\infty \quad \text{(by (ii))}
\end{aligned}$$

This proves that $\mathbb{R}^\infty$ indeed is a complete metric space.

□

Definition 2.4
*The **finite-dimensional class** of subsets of $\mathbb{R}^\infty$ is, by definition, the following:*

$$\mathcal{B}_f(\mathbb{R}^\infty)\ \ :=\ \left\{\,\pi_k^{-1}(B)\subset\mathbb{R}^\infty \;\middle|\; \begin{array}{c} k\in\mathbb{N} \\ B\in\mathcal{B}\big(\mathbb{R}^k\big) \end{array}\right\},$$

where $\pi_k:\mathbb{R}^\infty\longrightarrow\mathbb{R}^k : x=(x_1,x_2,\dots)\longmapsto(x_1,\dots,x_k)$ is the projection of $\mathbb{R}^\infty$ onto $\mathbb{R}^k$.

Theorem 2.5

(i) $\mathcal{B}_f(\mathbb{R}^\infty)\subset\mathcal{B}(\mathbb{R}^\infty)$.

(ii) $\mathcal{B}_f(\mathbb{R}^\infty)$ *is a separating class of Borel subsets of* $\mathbb{R}^\infty$.

(iii) $\mathcal{B}_f(\mathbb{R}^\infty)$ *is a convergence-determining class of Borel subsets of* $\mathbb{R}^\infty$.

A Technical Lemmas

Lemma A.1 *Define*

$$\phi : [0,\infty) \longrightarrow [0,1] : t \longmapsto \min\{1, t\}.$$

Then, ϕ satisfies:

$$\phi(s+t) \le \phi(s) + \phi(t), \text{ for each } s,t \in [0,\infty).$$

PROOF For any $s,t \in [0,\infty)$, either $s+t \ge 1$ or $s+t < 1$. If $s+t \ge 1$, then

$$\phi(s+t) = \min\{1, s+t\} = 1 < 2 = 1+1 \le \min\{1,s\} + \min\{1,t\} = \phi(s) + \phi(t),$$

hence, the required inequality holds. On the other hand, if $s+t<1$, then we must also have $s<1$ and $t<1$ (since $s,t \ge 0$). Hence,

$$\phi(s+t) = \min\{1, s+t\} = s+t = \min\{1,s\} + \min\{1,t\} = \phi(s) + \phi(t),$$

thus, the required inequality also holds. $\square$

Lemma A.2 *For any $x,y,z \in \mathbb{R}$, we have:*

$$\min\{1, |x-y|\} \le \min\{1, |x-z|\} + \min\{1, |z-y|\}.$$

PROOF Observe that $|x-y| \le |x-z| + |z-y|$ implies

$$\min\{1, |x-y|\} \le |x-z| + |z-y|.$$

The above inequality, together with $\min\{1, |x-y|\} \le 1$, thus in turn imply:

$$\min\{1, |x-y|\} \le \min\{1, |x-z| + |z-y|\}.$$

By Lemma A.1, we therefore have:

$$\min\{1, |x-y|\} \le \min\{1, |x-z| + |z-y|\}. \le \min\{1, |x-z|\} + \min\{1, |z-y|\},$$

which proves the present Lemma. $\square$

Lemma A.3 (The Weierstrass M-test, Theorem A.28, [2])
Suppose that $\lim_{n\to\infty} x_i^{(n)} = x_i$, for each $i \in \mathbb{N}$, and that $\left| x_i^{(n)} \right| \le M_i$, where $\sum_{i=1}^\infty M_i < \infty$. Then,

(i) $\sum_{i=1}^\infty x_i$ *exists, and* $\sum_{i=1}^\infty x_i^{(n)}$ *exists for each $n \in \mathbb{N}$.*

(ii) *Furthermore,*

$$\lim_{n\to\infty} \sum_{i=1}^\infty x_i^{(n)} = \sum_{i=1}^\infty x_i.$$

PROOF

(i) $\sum_{i=1}^\infty M_i < \infty$ and $\left| x_i^{(n)} \right| \le M_i \implies$ the series $\sum_{i=1}^\infty x_i$ and $\sum_{i=1}^\infty x_i^{(n)}$, $n \in \mathbb{N}$, converge absolutely.

(ii) Let $\varepsilon > 0$ be given. Choose $K \in \mathbb{N}$ sufficiently large such that $\sum_{j=K+1}^{\infty} M_i < \frac{\varepsilon}{3}$. Next, choose $N \in \mathbb{N}$ sufficiently large such that

$$\left| x_i^{(n)} - x_i \right| < \frac{\varepsilon}{3K}, \quad \text{for any } n > N \text{ and } i = 1, 2, \ldots, K.$$

Then, we have, for each $n > N$,

$$\begin{aligned}
\left| \sum_{i=1}^{\infty} x_i^{(n)} - \sum_{i=1}^{\infty} x_i \right| &= \left| \sum_{i=1}^{K} \left(x_i^{(n)} - x_i \right) + \sum_{i=K+1}^{\infty} x_i^{(n)} - \sum_{i=K+1}^{\infty} x_i \right| \\
&\leq \sum_{i=1}^{K} \left| x_i^{(n)} - x_i \right| + \sum_{i=K+1}^{\infty} \left| x_i^{(n)} \right| + \sum_{i=K+1}^{\infty} \left| x_i \right| \\
&\leq K \cdot \frac{\varepsilon}{3K} + \sum_{i=K+1}^{\infty} M_i + \sum_{i=K+1}^{\infty} M_i \leq \frac{\varepsilon}{3} + 2 \cdot \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since ε is arbitrary, this proves:

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

□

References

[1] Billingsley, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.

[2] Billingsley, P. *Probability and Measure*, anniversary ed. John Wiley & Sons, 2012.