The Prokhorov Theorem

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1   The Prokhorov Theorem

Definition 1.1 (Tightness and weak sequential compactness)

Suppose:

- *(S, ρ) is a metric space, $\mathcal{B}(S)$ its the Borel σ-algebra, $(S, \mathcal{B}(S))$ is the corresponding measurable space.*

- *$\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is a family of probability measures on $(S, \mathcal{B}(S))$.*

The family Π is said to be:

(i)   **tight** *if, for each $\varepsilon > 0$, there exists a compact subset $K_\varepsilon \subset S$ such that*

$$1 - \epsilon \; < \; P(\, K_\varepsilon \,) \; \leq \; 1, \quad \text{for each } P \in \Pi.$$

(ii)   **weakly sequentially compact** *if, for every sequence $\{\, P_n \,\}_{n \in \mathbb{N}} \subset \Pi$, there exists a probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ and subsequence $\{\, P_{n(i)} \,\}_{i \in \mathbb{N}}$ such that*

$$P_{n(i)} \; \xrightarrow{w} \; P, \quad \text{as } i \longrightarrow \infty.$$

Theorem 1.2 (The Prokhorov Theorem, Theorems 5.1 & 5.2, [2])

Suppose:

- *(S, ρ) is a metric space, $\mathcal{B}(S)$ its the Borel σ-algebra, $(S, \mathcal{B}(S))$ is the corresponding measurable space.*

- *$\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is a collection of probability measures on $(S, \mathcal{B}(S))$.*

Then, the following statements hold:

(i)   *Tightness of Π implies weak sequential compactness of Π.*

(ii)   *Suppose further that (S, ρ) is complete and separable.*
Then, weak sequential compactness of Π implies tightness of Π.

PROOF   We first prove statement (ii), then statement (i).

Proof of (ii)

Suppose S is complete and separable. Let $\varepsilon > 0$ be fixed. We need to find a compact subset $K \subset S$ such that

$$1 - \varepsilon \; < \; P(\, K \,) \; \leq \; 1, \quad \text{for each } P \in \Pi.$$

Now, separability of S implies that every open cover of every subset of S admits a countable subcover (Appendix M3, [2]). Denote by $B(x, r) \subset S$ the open ball in S centred at $x \in S$ of radius $r > 0$. For each $k \in \mathbb{N}$, the open cover

$$\left\{\, B\left(x, \frac{1}{k} \right) \,\right\}_{x \in S}$$

of S admits a countable subcover, say,

$$\{\, A_{ki} \,\}_{i \in \mathbb{N}} \; \subset \; \left\{\, B\left(x, \frac{1}{k} \right) \,\right\}_{x \in S}.$$

Let $G_{kn} := \bigcup_{i=1}^{n} A_{ki}$. Then, each G_{kn} is an open subset of S and $G_{kn} \uparrow S$, as $n \longrightarrow \infty$. Hence, by the Claim below, there exists $n_k \in \mathbb{N}$ such that

$$1 - \frac{\varepsilon}{2^k} \;<\; P\left(\bigcup_{i=1}^{n_k} A_{ki} \right) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Now, let

$$K \;:=\; \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}}$$

Note that K, being a closed subset of the complete metric space S, is itself complete. Note also that the set $\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}$ is totally bounded; hence so is its closure K. Being complete and totally bounded, K is therefore compact (Appendix M5, [2]). It now remains only to show that $1 - \varepsilon < P(K) \leq 1$, for each $P \in \Pi$; or equivalently, that $P(K^c) \leq \varepsilon$, for each $P \in \Pi$. To this end, write $B_k := \bigcup_{i=1}^{n_k} A_{ki}$. Then,

$$1 - \frac{\varepsilon}{2^k} \;<\; P(\, B_k \,) \;\leq\; 1; \quad \text{equivalently,} \;\; P(\, B_k^c \,) \;\leq\; \frac{\varepsilon}{2^k}.$$

Also,

$$K \;:=\; \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}} \;:=\; \overline{\bigcap_{k=1}^{\infty} B_k} \;\supset\; \bigcap_{k=1}^{\infty} B_k.$$

Hence,

$$K^c \;\subset\; \left(\bigcap_{k=1}^{\infty} B_k \right)^c \;=\; \bigcup_{k=1}^{\infty} B_k^c,$$

which implies:

$$P(\, K^c \,) \;\leq\; \sum_{k=1}^{\infty} P(B_k^c) \;\leq\; \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} \;=\; \varepsilon.$$

Thus, the proof of (ii) will be complete once we prove the following:

> **Claim:** Let $\{\, G_n \,\}_{n \in \mathbb{N}}$ be a sequence of open subsets of S with $G_n \uparrow S$. Then, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that
>
> $$1 - \varepsilon \;<\; P(\, G_{n_\varepsilon} \,) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Proof of Claim: Suppose the Claim is false, and we derive a contradiction. The failure of the Claim implies that there exists some $0 < \varepsilon < 1$ such that for each $n \in \mathbb{N}$, there exists $P_n \in \Pi$ such that

$$P_n(\, G_n \,) \;\leq\; 1 - \varepsilon.$$

By the hypothesis of weak sequential compactness of Π, there exists some probability measure $Q \in \mathcal{M}_1(S, \mathcal{B}(S))$ and the subsequence $\{\, P_{n(i)} \,\}$ of $\{\, P_n \,\}$ such that $P_{n(i)} \xrightarrow{w} Q$, as $i \longrightarrow \infty$. Now, for each fixed $n \in \mathbb{N}$, we have:

$$\begin{aligned}
Q(\, G_n \,) \;&\leq\; \liminf_{i \to \infty} P_{n(i)}(\, G_n \,), \quad \text{by the Portmanteau Theorem} \\
&\leq\; \liminf_{i \to \infty} P_{n(i)}\big(\, G_{n(i)} \,\big), \quad \text{since } \{\, G_n \,\} \text{ is increasing} \\
&\leq\; 1 - \varepsilon, \quad \text{by choice of } P_n
\end{aligned}$$

But, by hypothesis, we also have $G_n \uparrow S$. Hence, we therefore have:

$$1 \;=\; Q(S) \;=\; \lim_{n \to \infty} Q(G_n) \;\leq\; 1 - \varepsilon,$$

which is the desired contradiction. This completes the proof of the Claim, hence that of (ii).

Proof of (i)

Suppose $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is tight. We need to establish that Π is weakly sequentially compact. In other words, if $\{P_n\} \subset \Pi$ is a sequence of probability measures contained in Π, we need to establish that there exists a Borel probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ and a subsequence $\{P_{n(i)}\} \subset \{P_n\}$ such that $P_{n(i)} \xrightarrow{w} P$, as $i \longrightarrow \infty$.

So, let $\{P_n\} \subset \Pi$. We prove the Theorem by establishing the following series of Claims. Note that the proof of the Theorem is complete once we establish Claim 7.

Claim 1:   There exists an increasing sequence of compact subsets $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ such that

$$1 - \frac{1}{m} \ < \ P_n(K_m) \ \leq \ 1, \quad \text{for every } m, \, n \in \mathbb{N}.$$

Claim 2:   Let $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ be one such sequence of compact subsets of S as in Claim 1. Then, the following statements are true:

(a) $\Sigma := \bigcup_{m=1}^{\infty} K_m$ is a separable subset of S.

(b)   There exists a countable collection $\mathcal{A}$ of open subsets of S satisfying the following property:
For each $x \in S$ and for each open subset G of S,

$$x \in G \bigcap \left(\bigcup_{m=1}^{\infty} K_m \right) \quad \Longrightarrow \quad x \in A \subset \overline{A} \subset G, \text{ for some } A \in \mathcal{A}.$$

(c)   The collection $\mathcal{A}$ is an open cover of Σ.

Claim 3:   Define:

$$\mathcal{H} \ := \ \{\varnothing\} \bigcup \left\{ \begin{array}{c} \text{all finite unions of sets of the form} \\ \overline{A} \cap K_m, \text{ where } A \in \mathcal{A} \text{ and } m \in \mathbb{N} \end{array} \right\}.$$

Then, the following statements are true:

(a) $K_m \in \mathcal{H}$, for each $m \in \mathbb{N}$.

(b)   There exists a subsequence $\{P_{n(i)}\} \subset \{P_n\}$ such that the limit

$$\lim_{i \to \infty} P_{n(i)}(H) \ \text{ exists, } \ \text{for each } H \in \mathcal{H}.$$

We may therefore define the following function:

$$\alpha \ : \ \mathcal{H} \ \longrightarrow \ [0,1] \ : \ H \ \longmapsto \ \lim_{i \to \infty} P_{n(i)}(H).$$

Claim 4: The function $\alpha : \mathcal{H} \longrightarrow [0,1]$ satisfies the following properties:

(a) $\alpha(\varnothing) = 0$.

(b)   monotonicity: $\alpha(H_1) \leq \alpha(H_2)$, for any $H_1, H_2 \in \mathcal{H}$ with $H_1 \subset H_2$.

(c)   finite additivity for disjoint sets:

   $\alpha(H_1 \cup H_2) = \alpha(H_1) + \alpha(H_2)$, for any $H_1, H_2 \in \mathcal{H}$ with $H_1 \cap H_2 = \varnothing$.

(d)   finite sub-additivity: $\alpha(H_1 \cup H_2) \leq \alpha(H_1) + \alpha(H_2)$, for any $H_1, H_2 \in \mathcal{H}$.

Claim 5: Let $\mathcal{O}(S)$ denote the collection of all open subsets of S. Define the following function:

$$\beta \; : \; \mathcal{O}(S) \; \longrightarrow \; [0,1] \; : \; G \; \longmapsto \; \sup \left\{ \alpha(H) \in [0,1] \;\middle|\; \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset G \end{array} \right\}.$$

(Note that the supremum above is always taken over a non-empty set: For each open $G \subset S$, the set $\{ H \in \mathcal{H} \mid H \subset G \}$ is non-empty, since $\varnothing \in \mathcal{H}$.)

Next, let $\mathcal{P}(S)$ be the power set of S, i.e. the collection of all subsets of S. Define the following function:

$$\gamma \; : \; \mathcal{P}(S) \; \longrightarrow \; [0,1] \; : \; W \; \longmapsto \; \inf \left\{ \beta(G) \in [0,1] \;\middle|\; \begin{array}{l} G \in \mathcal{O}(S), \text{ and} \\ W \subset G \end{array} \right\}.$$

Then, the function $\gamma : \mathcal{P}(S) \longrightarrow [0,1]$ is an outer measure defined on S.

Claim 6: The σ-algebra $\mathcal{A}(\gamma)$ of γ-measurable subsets of S contains the Borel σ-algebra $\mathcal{B}(S)$ of S.

Claim 7: The restriction $P := \gamma \mid_{\mathcal{B}(S)}$ of γ to $\mathcal{B}(S)$ is a Borel probability measure which statisfies:

$$P(G) \;\; = \;\; \beta(G) \;\; := \;\; \sup \left\{ \alpha(H) \in [0,1] \;\middle|\; \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset G \end{array} \right\}, \;\; \text{ for each open subset } G \subset S.$$

Claim 8: $P_{n(i)} \xrightarrow{\; w \;} P$, as $i \longrightarrow \infty$.

<u>Proof of Claim 1:</u>   By tightness hypothesis on Π, for each $m \in \mathbb{N}$, there exists a compact subset $L_m \subset S$ such that

$$1 - \frac{1}{m} \;\; < \;\; P(L_m) \;\; \leq \;\; 1, \quad \text{for each } P \in \Pi.$$

Define, for each $m \in \mathbb{N}$, $K_m := \bigcup_{i=1}^{m} L_i$. Then, each K_m is compact (since finite unions of compact subsets are themselves compact). Next, we trivially have $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$. Also,

$$P(K_m) \;\; = \;\; P\left(\bigcup_{i=1}^{m} L_i \right) \;\; \geq \;\; L_m \;\; > \;\; 1 - \frac{1}{m}, \quad \text{for each } P \in \Pi.$$

In particular, the above inequality holds for each P_n. This proves Claim 1.

<u>Proof of Claim 2:</u>   Separability of $\Sigma := \bigcup\limits_{m=1}^{\infty} K_m$ is an immediate consequence of Lemma A.1 and Lemma A.2. Then, the existence of $\mathcal{A}$ follows immediately from the separability of Σ and Lemma A.3.

It remains to show that $\mathcal{A}$ is an open cover of Σ. To this end, first note that the collection $\mathcal{O}_S$ of open subsets of S forms an open cover of S, hence $\mathcal{O}_S$ is also an open cover of Σ. Therefore, for each $x \in \Sigma$, we may choose $G_x \in \mathcal{O}_S$ such that $x \in G_x$. Thus, $x \in \Sigma \cap G_x$, and by properties of $\mathcal{A}$, we may furthermore choose $A_x \in \mathcal{A}$ such that $x \in A_x \subset \overline{A_x} \subset G_x$. We thus see that the collection

$$\left\{ \, A_x \in \mathcal{A} \;\middle|\; x \in \Sigma \, \right\} \;\subset\; \mathcal{A}$$

is an open cover of Σ consisting of subsets in $\mathcal{A}$. This completes the proof of Claim 2.

<u>Proof of Claim 3:</u>

(a)   By Claim 2, $\mathcal{A}$ is an open cover of $\Sigma := \bigcup\limits_{m=1}^{\infty} K_m$. In particular, $\mathcal{A}$ is an open cover of K_m for each $m \in \mathbb{N}$. Compactness of K_m implies that $\mathcal{A}$ admits a finite subcover of K_m. Thus we have

$$K_m \;\subset\; \bigcup_{i=1}^{J_m} A_i^{(m)} \;\subset\; \bigcup_{i=1}^{J_m} \overline{A_i^{(m)}}, \quad \text{for some } A_1^{(m)}, \, A_2^{(m)}, \, \ldots, \, A_{J_m}^{(m)} \in \mathcal{A},$$

which implies

$$K_m \;=\; K_m \cap \left(\bigcup_{i=1}^{J_m} \overline{A_i^{(m)}} \right) \;=\; \bigcup_{i=1}^{J_m} \left(K_m \cap \overline{A_i^{(m)}} \right) \;\in\; \mathcal{H}.$$

(b)   Note that $\mathcal{H}$ is a countable collection of subsets of S. Let $\mathcal{H} = \{\, H_1, H_2, H_3, \ldots \,\}$ be an enumeration of $\mathcal{H}$. Consider the following array of real numbers:

$$
\begin{array}{cccc}
P_1(H_1) & P_2(H_1) & P_3(H_1) & \cdots \\
P_1(H_2) & P_2(H_2) & P_3(H_2) & \cdots \\
P_1(H_3) & P_2(H_3) & P_3(H_3) & \cdots \\
\vdots & \vdots & \vdots & \vdots
\end{array}
$$

Note that each row of the above array is bounded between 0 and 1. Hence, by Theorem A.4, there exists an increasing sequence

$$n(1) \;<\; n(2) \;<\; n(3) \;<\; \cdots \;\in \mathbb{N}$$

of natural numbers such that the limit

$$\lim_{k\to\infty} P_{n(k)}(H_r), \quad \text{exists for each } r \in \mathbb{N}.$$

This completes the proof of Claim 3.

<u>Proof of Claim 4:</u>

<u>Proof of Claim 5:</u>

<u>Proof of Claim 6:</u>

<u>Proof of Claim 7:</u>

<u>Proof of Claim 8:</u> Let $G \subset S$ be an arbitrary open subset of S. Then, for each $H \in \mathcal{H}$ with $H \subset G$, we have

$$\alpha(H) \quad := \quad \lim_{i \to \infty} P_{n(i)}(H) \quad \leq \quad \liminf_{i \to \infty} P_{n(i)}(G).$$

The preceding inequality and Claim 7 together imply:

$$P(G) \;=\; \sup\left\{ \alpha(H) \,\in\, [0,1] \;\middle|\; \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset G \end{array} \right\} \;\leq\; \liminf_{i \to \infty} P_{n(i)}(G), \text{ for each open subset } G \subset S,$$

which is equivalent to the weak convergence $P_{n(i)} \xrightarrow{w} P$, as $i \longrightarrow \infty$, by the Portmanteau Theorem (Theorem 2.1, [2]). This completes the proof of Claim 8. $\qquad\square$

A Technical Lemmas

Lemma A.1

Every compact subset of a metric space is also a separable subset of that metric space.

PROOF   Let (X, ρ) be a metric space and $K \subset X$ be a compact subset of X. For each $x \in X$ and positive $r > 0$, let

$$B(x, r) \ := \ \{ \, y \in X \mid \rho(x, y) < r \, \} \ \subset \ X,$$

i.e. $B(x, r)$ is the open ball in X centred at x with radius $r > 0$. For each $n \in \mathbb{N}$, the following forms an open cover of K:

$$\mathcal{C}_n \ := \ \left\{ \, B\left(x, \frac{1}{n} \right) \subset X \ \middle| \ x \in K \, \right\}.$$

Since K is compact, each $\mathcal{C}_n$ admits a finite subcover:

$$\mathcal{F}_n \ := \ \left\{ \, B\left(x_i^{(n)}, \frac{1}{n} \right) \subset X \ \middle| \ x_i^{(n)} \in K, \ i = 1, 2, \ldots, J_n \, \right\}.$$

Let

$$\mathcal{D}_n \ := \ \left\{ \, x_i^{(n)} \in K \ \middle| \ i = 1, 2, \ldots, J_n \, \right\} \ \subset \ K,$$

and let $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset K$. We claim that $\mathcal{D}$ is dense in K. Indeed, let $y \in K$. Since each $\mathcal{F}_n$ is a (finite) open cover of K, we have:

$$y \ \in \ K \ \subset \ \bigcup_{i=1}^{J_n} B\left(x_i^{(n)}, \frac{1}{n} \right), \quad \text{for each } n \in \mathbb{N}.$$

Since $x_i^{(n)} \in \mathcal{D}$, for each $i = 1, 2, \ldots, J_n$ and for each $n \in \mathbb{N}$, the above inclusion shows that, for each $n \in \mathbb{N}$, there exists some $x \in \mathcal{D}$ such that $\rho(y, x) < \frac{1}{n}$. In particular, $\mathcal{D}$ contains a sequence that converges to $y \in K$. Since $y \in K$ is an arbitrary element of K, we see that $\overline{\mathcal{D}} \supset K$. Since $\mathcal{D} \subset K$ and K is compact, hence closed, we trivially have $\overline{\mathcal{D}} \subset K$. We may now conclude that $\overline{\mathcal{D}} = K$. This completes the proof of the Lemma. $\square$

Lemma A.2

Every countable union of separable subsets of a metric space is itself a separable subset of that metric space.

PROOF   Let $S := \bigcup_{i=1}^{\infty} S_i \subset X$ be a countable union of separable subsets S_i of a metric space X. For each fixed $i \in \mathbb{N}$, since S_i is separable, there exists countable $D_i \subset S_i$ which is dense in S_i. Let $D := \bigcup_{i=1}^{\infty} D_i$. Then, D is a countable subset of S. The Lemma is proved once we establish that D is dense in S. To this end, let $x \in S = \bigcup_{i=1}^{\infty} S_i$. Then, $x \in S_i$ for some $i \in \mathbb{N}$. Since D_i is dense in S_i, there exists a sequence $\{ y_k \} \subset D_i \subset D$ such that $y_k \longrightarrow x$, as $k \longrightarrow \infty$. This proves that D is indeed dense in S, and completes the proof of the Lemma. $\square$

Lemma A.3 (second theorem in Appendix M3, [2])

Let (S, ρ) be a metric space and $\Sigma \subset S$ a separable subset of S. Then, there exists a countable collection $\mathcal{A}$ of open subsets of S satisfying the following property: For each $x \in S$ and each open subset G of S,

$$x \in G \bigcap \Sigma \ \implies \ x \in A \subset \overline{A} \subset G, \ \text{for some } A \in \mathcal{A}.$$

PROOF   Let $D \subset \Sigma$ be a countable dense subset of Σ. Let

$$\mathcal{A} \ := \ \left\{ \ B(d,r) \ \subset \ S \ \bigg| \ \begin{array}{c} d \in D, \\ r \in \mathbb{Q}, \ r > 0 \end{array} \ \right\}.$$

Then, $\mathcal{A}$ is a countable collection of open balls in S. Now, let $G \subset S$ be an arbitrary open subset of S and $x \in G \bigcap \Sigma$. First, choose $\varepsilon > 0$ such that $B(x, \varepsilon) \subset G$. Next, since $x \in \Sigma$ and D is dense in Σ, we may choose $d \in D$ such that $d \in B(x, \varepsilon/2)$, or equivalently $\rho(x, d) < \varepsilon/2$. Finally choose positive rational $r > 0$ such that $\rho(x, d) < r < \varepsilon/2$.

Now, note that $\overline{B(d,r)} \subset B(x, \varepsilon)$; indeed,

$$y \in \overline{B(d,r)} \ \iff \ \rho(y,d) \leq r \ \implies \ \rho(x,y) \ \leq \ \rho(x,d) + \rho(d,y) \ < \ \varepsilon/2 + r \ < \ \varepsilon/2 + \varepsilon/2 \ \implies \ y \in B(x,\varepsilon).$$

Thus, we have

$$x \ \in \ B(d,r) \ \subset \ \overline{B(d,r)} \ \subset \ B(x,\varepsilon) \ \subset \ G.$$

This completes the proof of the Lemma. $\square$

Theorem A.4 (The Diagonal Method, Appendix A.14, [1])

Suppose that each row of the array

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

is a bounded sequence of real numbers. Then, there exists an increasing sequence

$$n_1 \ < \ n_2 \ < \ n_3 \ < \cdots \ \in \ \mathbb{N}$$

of positive integers such that the limit

$$\lim_{k \to \infty} \ x_{r,n_k} \ \ \text{exists,} \ \ \text{for each } r = 1, 2, 3, \ldots$$

PROOF   From the first row, select a convergent subsequence

$$x_{1,n(1,1)}, \ \ x_{1,n(1,2)}, \ \ x_{1,n(1,3)}, \ \ \cdots\cdots$$

Here, we have $n(1,1) < n(1,2) < n(1,3) < \cdots \ \in \ \mathbb{N}$, and $\lim_{k \to \infty} x_{1,n(1,k)} \in \mathbb{R}$ exists. Next, note that the following subsequence of the second row:

$$x_{2,n(1,1)}, \ \ x_{2,n(1,2)}, \ \ x_{2,n(1,3)}, \ \ \cdots\cdots$$

is still a bounded sequence of real numbers, and we may thus select a convergent subsequence:

$$x_{2,n(2,1)}, \ \ x_{2,n(2,2)}, \ \ x_{2,n(2,3)}, \ \ \cdots\cdots$$

Here, we have $n(2,1) < n(2,2) < n(2,3) < \cdots \ \in \ \{ \, n(1,k) \, \}_{k \in \mathbb{N}}$, and $\lim_{k \to \infty} x_{2,n(2,k)} \in \mathbb{R}$ exists. Continuing inductively, we obtain an array of positive integers

$$\begin{array}{cccc} n(1,1) & n(1,2) & n(1,3) & \cdots \\ n(2,1) & n(2,2) & n(2,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

which satisfies: For each $r \in \mathbb{N}$, we have

- each row is an increasing sequence of positive integers, i.e. $n(r,1) \ < \ n(r,2) \ < \ n(r,3) \ < \cdots$,

- the $(r+1)^{\text{th}}$ row is a subsequence of the rth row, i.e. $\{\, n(r+1,k)\,\}_{k\in\mathbb{N}} \subset \{\, n(r,k)\,\}_{k\in\mathbb{N}}$, and

- $\lim\limits_{k\to\infty} x_{r,n(r,k)} \in \mathbb{R}$ exists.

Note that the first two properties together imply:

$$n(k,k) \;<\; n(k,k+1) \;\leq\; n(k+1,k+1), \quad \text{for each } k\in\mathbb{N}.$$

Now, define $n_k := n(k,k)$, for $k\in\mathbb{N}$. We then see that

$$n_k \;:=\; n(k,k) \;< n(k+1,k+1) \;=:\; n_{k+1},$$

i.e., $\{\, n_k\,\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. Lastly, for each $r\in\mathbb{N}$, consider the sequence

$$x_{r,n_1},\;\; x_{r,n_2},\;\; x_{r,n_3},\;\; \cdots$$

Note that, for each $r\in\mathbb{N}$,

$$x_{r,n_r},\;\; x_{r,n_{r+1}},\;\; x_{r,n_{r+2}},\;\; \cdots$$

is a subsequence of $\{\, x_{r,n(r,k)}\,\}_{k\in\mathbb{N}}$. We saw above that $\lim\limits_{k\to\infty} x_{r,n(r,k)}$ exists, which in turn implies that $\lim\limits_{k\to\infty} x_{r,n_k}$ exists. Since $r\in\mathbb{N}$ is arbitrary, the proof of the Theorem is now complete. $\qquad\square$

References

[1] BILLINGSLEY, P. *Probability and Measure*, third ed. John Wiley & Sons, 1995.

[2] BILLINGSLEY, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.