

## 1 The Prokhorov Theorem

### Definition 1.1 (Tightness and weak sequential compactness)

Suppose:

- $(S, \rho)$  is a metric space,  $\mathcal{B}(S)$  its the Borel  $\sigma$ -algebra,  $(S, \mathcal{B}(S))$  is the corresponding measurable space.
- $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$  is a family of probability measures on  $(S, \mathcal{B}(S))$ .

The family  $\Pi$  is said to be:

- (i) **tight** if, for each  $\varepsilon > 0$ , there exists a compact subset  $K_\varepsilon \subset S$  such that

$$1 - \varepsilon < P(K_\varepsilon) \leq 1, \quad \text{for each } P \in \Pi.$$

- (ii) **weakly sequentially compact** if, for every sequence  $\{P_n\}_{n \in \mathbb{N}} \subset \Pi$ , there exists a probability measure  $P \in \mathcal{M}_1(S, \mathcal{B}(S))$  and subsequence  $\{P_{n(i)}\}_{i \in \mathbb{N}}$  such that

$$P_{n(i)} \xrightarrow{w} P, \quad \text{as } i \rightarrow \infty.$$

### Theorem 1.2 (The Prokhorov Theorem, Theorems 5.1 & 5.2, [2])

Suppose:

- $(S, \rho)$  is a metric space,  $\mathcal{B}(S)$  its the Borel  $\sigma$ -algebra,  $(S, \mathcal{B}(S))$  is the corresponding measurable space.
- $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$  is a collection of probability measures on  $(S, \mathcal{B}(S))$ .

Then, the following statements hold:

- (i) Tightness of  $\Pi$  implies weak sequential compactness of  $\Pi$ .
- (ii) Suppose further that  $(S, \rho)$  is complete and separable.  
Then, weak sequential compactness of  $\Pi$  implies tightness of  $\Pi$ .

PROOF We first prove statement (ii), then statement (i).

Proof of (ii)

Suppose  $S$  is complete and separable. Let  $\varepsilon > 0$  be fixed. We need to find a compact subset  $K \subset S$  such that

$$1 - \varepsilon < P(K) \leq 1, \quad \text{for each } P \in \Pi.$$

Now, separability of  $S$  implies that every open cover of every subset of  $S$  admits a countable subcover (Appendix M3, [2]). Denote by  $B(x, r) \subset S$  the open ball in  $S$  centred at  $x \in S$  of radius  $r > 0$ . For each  $k \in \mathbb{N}$ , the open cover

$$\left\{ B\left(x, \frac{1}{k}\right) \right\}_{x \in S}$$

of  $S$  admits a countable subcover, say,

$$\{A_{ki}\}_{i \in \mathbb{N}} \subset \left\{ B\left(x, \frac{1}{k}\right) \right\}_{x \in S}.$$

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Let  $G_{kn} := \bigcup_{i=1}^n A_{ki}$ . Then, each  $G_{kn}$  is an open subset of  $S$  and  $G_{kn} \uparrow S$ , as  $n \rightarrow \infty$ . Hence, by the Claim below, there exists  $n_k \in \mathbb{N}$  such that

$$1 - \frac{\varepsilon}{2^k} < P\left(\bigcup_{i=1}^{n_k} A_{ki}\right) \leq 1, \quad \text{for each } P \in \Pi.$$

Now, let

$$K := \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}}$$

Note that  $K$ , being a closed subset of the complete metric space  $S$ , is itself complete. Note also that the set  $\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}$  is totally bounded; hence so is its closure  $K$ . Being complete and totally bounded,  $K$  is therefore compact (Appendix M5, [2]). It now remains only to show that  $1 - \varepsilon < P(K) \leq 1$ , for each  $P \in \Pi$ ; or equivalently, that  $P(K^c) \leq \varepsilon$ , for each  $P \in \Pi$ . To this end, write  $B_k := \bigcup_{i=1}^{n_k} A_{ki}$ . Then,

$$1 - \frac{\varepsilon}{2^k} < P(B_k) \leq 1; \quad \text{equivalently, } P(B_k^c) \leq \frac{\varepsilon}{2^k}.$$

Also,

$$K := \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}} := \overline{\bigcap_{k=1}^{\infty} B_k} \supset \bigcap_{k=1}^{\infty} B_k.$$

Hence,

$$K^c \subset \left(\bigcap_{k=1}^{\infty} B_k\right)^c = \bigcup_{k=1}^{\infty} B_k^c,$$

which implies:

$$P(K^c) \leq \sum_{k=1}^{\infty} P(B_k^c) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Thus, the proof of (ii) will be complete once we prove the following:

**Claim:** Let  $\{G_n\}_{n \in \mathbb{N}}$  be a sequence of open subsets of  $S$  with  $G_n \uparrow S$ . Then, for each  $\varepsilon > 0$ , there exists  $n_\varepsilon \in \mathbb{N}$  such that

$$1 - \varepsilon < P(G_{n_\varepsilon}) \leq 1, \quad \text{for each } P \in \Pi.$$

**Proof of Claim:** Suppose the Claim is false, and we derive a contradiction. The failure of the Claim implies that there exists some  $0 < \varepsilon < 1$  such that for each  $n \in \mathbb{N}$ , there exists  $P_n \in \Pi$  such that

$$P_n(G_n) \leq 1 - \varepsilon.$$

By the hypothesis of weak sequential compactness of  $\Pi$ , there exists some probability measure  $Q \in \mathcal{M}_1(S, \mathcal{B}(S))$  and the subsequence  $\{P_{n(i)}\}$  of  $\{P_n\}$  such that  $P_{n(i)} \xrightarrow{w} Q$ , as  $i \rightarrow \infty$ . Now, for each fixed  $n \in \mathbb{N}$ , we have:

$$\begin{aligned} Q(G_n) &\leq \liminf_{i \rightarrow \infty} P_{n(i)}(G_n), \quad \text{by the Portmanteau Theorem} \\ &\leq \liminf_{i \rightarrow \infty} P_{n(i)}(G_{n(i)}), \quad \text{since } \{G_n\} \text{ is increasing} \\ &\leq 1 - \varepsilon, \quad \text{by choice of } P_n \end{aligned}$$

But, by hypothesis, we also have  $G_n \uparrow S$ . Hence, we therefore have:

$$1 = Q(S) = \lim_{n \rightarrow \infty} Q(G_n) \leq 1 - \varepsilon,$$

which is the desired contradiction. This completes the proof of the Claim, hence that of (ii).

### Proof of (i)

Suppose  $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$  is tight. We need to establish that  $\Pi$  is weakly sequentially compact. In other words, if  $\{P_n\} \subset \Pi$  is a sequence of probability measures contained in  $\Pi$ , we need to establish that there exists a Borel probability measure  $P \in \mathcal{M}_1(S, \mathcal{B}(S))$  and a subsequence  $\{P_{n(i)}\} \subset \{P_n\}$  such that  $P_{n(i)} \xrightarrow{w} P$ , as  $i \rightarrow \infty$ .

So, let  $\{P_n\} \subset \Pi$ . We prove the Theorem by establishing the following series of Claims. Note that the proof of the Theorem is complete once we establish Claim 5.

**Claim 1:** There exists an increasing sequence of compact subsets  $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$  such that

$$1 - \frac{1}{m} < P_n(K_m) \leq 1, \quad \text{for every } m, n \in \mathbb{N}.$$

**Claim 2:** Let  $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$  be one such sequence of compact subsets of  $S$  as in Claim 1. Then,  $\bigcup_{m=1}^{\infty} K_m$  is a separable subset of  $S$ , and there exists a countable collection  $\mathcal{A}$  of open subsets of  $S$  satisfying the following property: For each  $x \in S$  and for each open subset  $G$  of  $S$ ,

$$x \in G \cap \left( \bigcup_{m=1}^{\infty} K_m \right) \implies x \in A \subset \bar{A} \subset G, \quad \text{for some } A \in \mathcal{A}.$$

**Claim 3:** Define:

$$\mathcal{H} := \{\emptyset\} \cup \left\{ \begin{array}{l} \text{all finite unions of sets of the form} \\ \bar{A} \cap K_m, \text{ where } A \in \mathcal{A} \text{ and } m \in \mathbb{N} \end{array} \right\}.$$

Then, there exists a subsequence  $\{P_{n(i)}\} \subset \{P_n\}$  such that the limit

$$\alpha(H) := \lim_{i \rightarrow \infty} P_{n(i)}(H) \text{ exists, for each } H \in \mathcal{H}.$$

**Claim 4:** There exists a Borel probability measure  $P \in \mathcal{M}_1(S, \mathcal{B}(S))$  such that

$$P(G) = \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset G \end{array} \right\}, \quad \text{for each open subset } G \subset S.$$

**Claim 5:**  $P_{n(i)} \xrightarrow{w} P$ , as  $i \rightarrow \infty$ .

Proof of Claim 1: By tightness hypothesis on  $\Pi$ , for each  $m \in \mathbb{N}$ , there exists a compact subset  $L_m \subset S$  such that

$$1 - \frac{1}{m} < P(L_m) \leq 1, \quad \text{for each } P \in \Pi.$$

Define, for each  $m \in \mathbb{N}$ ,  $K_m := \bigcup_{i=1}^m L_i$ . Then, each  $K_m$  is compact (since finite unions of compact subsets are

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themselves compact). Next, we trivially have  $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ . Also,

$$P(K_m) = P\left(\bigcup_{i=1}^m L_i\right) \geq L_m > 1 - \frac{1}{m}, \quad \text{for each } P \in \Pi.$$

In particular, the above inequality holds for each  $P_n$ . This proves Claim 1.

Proof of Claim 2: Separability of  $\bigcup_{m=1}^{\infty} K_m$  is an immediate consequence of Lemma A.1 and Lemma A.2. Then, the existence of  $\mathcal{A}$  follows immediately from the separability of  $\bigcup_{m=1}^{\infty} K_m$  and Lemma A.3. This proves Claim 2.

Proof of Claim 3: Note that  $\mathcal{H}$  is a countable collection of subsets of  $S$ . Let  $\mathcal{H} = \{H_1, H_2, H_3, \dots\}$  be an enumeration of  $\mathcal{H}$ . Consider the following array of real numbers:

$$\begin{array}{cccc} P_1(H_1) & P_2(H_1) & P_3(H_1) & \cdots \\ P_1(H_2) & P_2(H_2) & P_3(H_2) & \cdots \\ P_1(H_3) & P_2(H_3) & P_3(H_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Note that each row of the above array is bounded between 0 and 1. Hence, by Theorem A.4, there exists an increasing sequence

$$n(1) < n(2) < n(3) < \cdots \in \mathbb{N}$$

of natural numbers such that the limit

$$\lim_{k \rightarrow \infty} P_{n(k)}(H_r), \quad \text{exists for each } r \in \mathbb{N}.$$

This completes the proof of Claim 3.

Proof of Claim 4:

Proof of Claim 5: Let  $G \subset S$  be an arbitrary open subset of  $S$ . Then, for each  $H \in \mathcal{H}$  with  $H \subset G$ , we have

$$\alpha(H) := \lim_{i \rightarrow \infty} P_{n(i)}(H) \leq \liminf_{i \rightarrow \infty} P_{n(i)}(G).$$

This implies:

$$P(G) = \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset G \end{array} \right\} \leq \liminf_{i \rightarrow \infty} P_{n(i)}(G), \quad \text{for each open subset } G \subset S,$$

which is equivalent to the weak convergence  $P_{n(i)} \xrightarrow{w} P$ , as  $i \rightarrow \infty$ , by the Portmanteau Theorem (Theorem 2.1, [2]). This completes the proof of Claim 5.  $\square$

## A Technical Lemmas

### Lemma A.1

*Every compact subset of a metric space is also a separable subset of that metric space.*

PROOF Let  $(X, \rho)$  be a metric space and  $K \subset X$  be a compact subset of  $X$ . For each  $x \in X$  and positive  $r > 0$ , let

$$B(x, r) := \{ y \in X \mid \rho(x, y) < r \} \subset X,$$

i.e.  $B(x, r)$  is the open ball in  $X$  centred at  $x$  with radius  $r > 0$ . For each  $n \in \mathbb{N}$ , the following forms an open cover of  $K$ :

$$\mathcal{C}_n := \left\{ B\left(x, \frac{1}{n}\right) \subset X \mid x \in K \right\}.$$

Since  $K$  is compact, each  $\mathcal{C}_n$  admits a finite subcover:

$$\mathcal{F}_n := \left\{ B\left(x_i^{(n)}, \frac{1}{n}\right) \subset X \mid x_i^{(n)} \in K, i = 1, 2, \dots, J_n \right\}.$$

Let

$$\mathcal{D}_n := \left\{ x_i^{(n)} \in K \mid i = 1, 2, \dots, J_n \right\} \subset K,$$

and let  $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset K$ . We claim that  $\mathcal{D}$  is dense in  $K$ . Indeed, let  $y \in K$ . Since each  $\mathcal{F}_n$  is a (finite) open cover of  $K$ , we have:

$$y \in K \subset \bigcup_{i=1}^{J_n} B\left(x_i^{(n)}, \frac{1}{n}\right), \quad \text{for each } n \in \mathbb{N}.$$

Since  $x_i^{(n)} \in \mathcal{D}$ , for each  $i = 1, 2, \dots, J_n$  and for each  $n \in \mathbb{N}$ , the above inclusion shows that, for each  $n \in \mathbb{N}$ , there exists some  $x \in \mathcal{D}$  such that  $\rho(y, x) < \frac{1}{n}$ . In particular,  $\mathcal{D}$  contains a sequence that converges to  $y \in K$ . Since  $y \in K$  is an arbitrary element of  $K$ , we see that  $\overline{\mathcal{D}} \supset K$ . Since  $\mathcal{D} \subset K$  and  $K$  is compact, hence closed, we trivially have  $\overline{\mathcal{D}} \subset K$ . We may now conclude that  $\overline{\mathcal{D}} = K$ . This completes the proof of the Lemma.  $\square$

### Lemma A.2

*Every countable union of separable subsets of a metric space is itself a separable subset of that metric space.*

PROOF Let  $S := \bigcup_{i=1}^{\infty} S_i \subset X$  be a countable union of separable subsets  $S_i$  of a metric space  $X$ . For each fixed  $i \in \mathbb{N}$ , since  $S_i$  is separable, there exists countable  $D_i \subset S_i$  which is dense in  $S_i$ . Let  $D := \bigcup_{i=1}^{\infty} D_i$ . Then,  $D$  is a countable subset of  $S$ . The Lemma is proved once we establish that  $D$  is dense in  $S$ . To this end, let  $x \in S = \bigcup_{i=1}^{\infty} S_i$ . Then,  $x \in S_i$  for some  $i \in \mathbb{N}$ . Since  $D_i$  is dense in  $S_i$ , there exists a sequence  $\{y_k\} \subset D_i \subset D$  such that  $y_k \rightarrow x$ , as  $k \rightarrow \infty$ . This proves that  $D$  is indeed dense in  $S$ , and completes the proof of the Lemma.  $\square$

### Lemma A.3 (second theorem in Appendix M3, [2])

*Let  $(S, \rho)$  be a metric space and  $\Sigma \subset S$  a separable subset of  $S$ . Then, there exists a countable collection  $\mathcal{A}$  of open subsets of  $S$  satisfying the following property: For each  $x \in S$  and each open subset  $G$  of  $S$ ,*

$$x \in G \cap \Sigma \implies x \in A \subset \overline{A} \subset G, \quad \text{for some } A \in \mathcal{A}.$$

PROOF Let  $D \subset \Sigma$  be a countable dense subset of  $\Sigma$ . Let

$$\mathcal{A} := \left\{ B(d, r) \subset S \mid \begin{array}{l} d \in D, \\ r \in \mathbb{Q}, r > 0 \end{array} \right\}.$$

Then,  $\mathcal{A}$  is a countable collection of open balls in  $S$ . Now, let  $G \subset S$  be an arbitrary open subset of  $S$  and  $x \in G \cap \Sigma$ . First, choose  $\varepsilon > 0$  such that  $B(x, \varepsilon) \subset G$ . Next, since  $x \in \Sigma$  and  $D$  is dense in  $\Sigma$ , we may choose  $d \in D$  such that  $d \in B(x, \varepsilon/2)$ , or equivalently  $\rho(x, d) < \varepsilon/2$ . Finally choose positive rational  $r > 0$  such that  $\rho(x, d) < r < \varepsilon/2$ .

Now, note that  $\overline{B(d, r)} \subset B(x, \varepsilon)$ ; indeed,

$$y \in \overline{B(d, r)} \iff \rho(y, d) \leq r \implies \rho(x, y) \leq \rho(x, d) + \rho(d, y) < \varepsilon/2 + r < \varepsilon/2 + \varepsilon/2 \implies y \in B(x, \varepsilon).$$

Thus, we have

$$x \in B(d, r) \subset \overline{B(d, r)} \subset B(x, \varepsilon) \subset G.$$

This completes the proof of the Lemma. □

## Theorem A.4 (The Diagonal Method, Appendix A.14, [1])

Suppose that each row of the array

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

is a bounded sequence of real numbers. Then, there exists an increasing sequence

$$n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$$

of positive integers such that the limit

$$\lim_{k \rightarrow \infty} x_{r, n_k} \text{ exists, for each } r = 1, 2, 3, \dots$$

PROOF From the first row, select a convergent subsequence

$$x_{1, n(1,1)}, x_{1, n(1,2)}, x_{1, n(1,3)}, \dots$$

Here, we have  $n(1,1) < n(1,2) < n(1,3) < \cdots \in \mathbb{N}$ , and  $\lim_{k \rightarrow \infty} x_{1, n(1,k)} \in \mathbb{R}$  exists. Next, note that the following subsequence of the second row:

$$x_{2, n(1,1)}, x_{2, n(1,2)}, x_{2, n(1,3)}, \dots$$

is still a bounded sequence of real numbers, and we may thus select a convergent subsequence:

$$x_{2, n(2,1)}, x_{2, n(2,2)}, x_{2, n(2,3)}, \dots$$

Here, we have  $n(2,1) < n(2,2) < n(2,3) < \cdots \in \{n(1,k)\}_{k \in \mathbb{N}}$ , and  $\lim_{k \rightarrow \infty} x_{2, n(2,k)} \in \mathbb{R}$  exists. Continuing inductively, we obtain an array of positive integers

$$\begin{array}{cccc} n(1,1) & n(1,2) & n(1,3) & \cdots \\ n(2,1) & n(2,2) & n(2,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

which satisfies: For each  $r \in \mathbb{N}$ , we have

- each row is an increasing sequence of positive integers, i.e.  $n(r,1) < n(r,2) < n(r,3) < \cdots$ ,

- the  $(r + 1)^{\text{th}}$  row is a subsequence of the  $r^{\text{th}}$  row, i.e.  $\{n(r + 1, k)\}_{k \in \mathbb{N}} \subset \{n(r, k)\}_{k \in \mathbb{N}}$ , and
- $\lim_{k \rightarrow \infty} x_{r, n(r, k)} \in \mathbb{R}$  exists.

Note that the first two properties together imply:

$$n(k, k) < n(k, k + 1) \leq n(k + 1, k + 1), \quad \text{for each } k \in \mathbb{N}.$$

Now, define  $n_k := n(k, k)$ , for  $k \in \mathbb{N}$ . We then see that

$$n_k := n(k, k) < n(k + 1, k + 1) =: n_{k+1},$$

i.e.,  $\{n_k\}_{k \in \mathbb{N}}$  is a strictly increasing sequence of positive integers. Lastly, for each  $r \in \mathbb{N}$ , consider the sequence

$$x_{r, n_1}, \quad x_{r, n_2}, \quad x_{r, n_3}, \quad \dots$$

Note that, for each  $r \in \mathbb{N}$ ,

$$x_{r, n_r}, \quad x_{r, n_{r+1}}, \quad x_{r, n_{r+2}}, \quad \dots$$

is a subsequence of  $\{x_{r, n(r, k)}\}_{k \in \mathbb{N}}$ . We saw above that  $\lim_{k \rightarrow \infty} x_{r, n(r, k)}$  exists, which in turn implies that  $\lim_{k \rightarrow \infty} x_{r, n_k}$  exists. Since  $r \in \mathbb{N}$  is arbitrary, the proof of the Theorem is now complete.  $\square$

## References

- [1] BILLINGSLEY, P. *Probability and Measure*, third ed. John Wiley & Sons, 1995.
- [2] BILLINGSLEY, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.