

1 The Prokhorov Theorem

Definition 1.1 (Tightness and weak sequential compactness)

Suppose:

- (S, ρ) is a metric space, $\mathcal{B}(S)$ its the Borel σ -algebra, $(S, \mathcal{B}(S))$ is the corresponding measurable space.
- $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is a family of probability measures on $(S, \mathcal{B}(S))$.

The family Π is said to be:

- (i) **tight** if, for each $\varepsilon > 0$, there exists a compact subset $K_\varepsilon \subset S$ such that

$$1 - \varepsilon < P(K_\varepsilon) \leq 1, \quad \text{for each } P \in \Pi.$$

- (ii) **weakly sequentially compact** if, for every sequence $\{P_n\}_{n \in \mathbb{N}} \subset \Pi$, there exists a probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ and subsequence $\{P_{n(i)}\}_{i \in \mathbb{N}}$ such that

$$P_{n(i)} \xrightarrow{w} P, \quad \text{as } i \rightarrow \infty.$$

Theorem 1.2 (The Prokhorov Theorem, Theorems 5.1 & 5.2, [2])

Suppose:

- (S, ρ) is a metric space, $\mathcal{B}(S)$ its the Borel σ -algebra, $(S, \mathcal{B}(S))$ is the corresponding measurable space.
- $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is a collection of probability measures on $(S, \mathcal{B}(S))$.

Then, the following statements hold:

- (i) Tightness of Π implies weak sequential compactness of Π .
- (ii) Suppose further that (S, ρ) is complete and separable.
Then, weak sequential compactness of Π implies tightness of Π .

PROOF We first prove statement (ii), then statement (i).

Proof of (ii)

Suppose S is complete and separable. Let $\varepsilon > 0$ be fixed. We need to find a compact subset $K \subset S$ such that

$$1 - \varepsilon < P(K) \leq 1, \quad \text{for each } P \in \Pi.$$

Now, separability of S implies that every open cover of every subset of S admits a countable subcover (Appendix M3, [2]). Denote by $B(x, r) \subset S$ the open ball in S centred at $x \in S$ of radius $r > 0$. For each $k \in \mathbb{N}$, the open cover

$$\left\{ B\left(x, \frac{1}{k}\right) \right\}_{x \in S}$$

of S admits a countable subcover, say,

$$\{A_{ki}\}_{i \in \mathbb{N}} \subset \left\{ B\left(x, \frac{1}{k}\right) \right\}_{x \in S}.$$

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Let $G_{kn} := \bigcup_{i=1}^n A_{ki}$. Then, each G_{kn} is an open subset of S and $G_{kn} \uparrow S$, as $n \rightarrow \infty$. Hence, by the Claim below, there exists $n_k \in \mathbb{N}$ such that

$$1 - \frac{\varepsilon}{2^k} < P\left(\bigcup_{i=1}^{n_k} A_{ki}\right) \leq 1, \quad \text{for each } P \in \Pi.$$

Now, let

$$K := \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}}$$

Note that K , being a closed subset of the complete metric space S , is itself complete. Note also that the set $\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}$ is totally bounded; hence so is its closure K . Being complete and totally bounded, K is therefore compact (Appendix M5, [2]). It now remains only to show that $1 - \varepsilon < P(K) \leq 1$, for each $P \in \Pi$; or equivalently, that $P(K^c) \leq \varepsilon$, for each $P \in \Pi$. To this end, write $B_k := \bigcup_{i=1}^{n_k} A_{ki}$. Then,

$$1 - \frac{\varepsilon}{2^k} < P(B_k) \leq 1; \quad \text{equivalently, } P(B_k^c) \leq \frac{\varepsilon}{2^k}.$$

Also,

$$K := \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}} := \overline{\bigcap_{k=1}^{\infty} B_k} \supset \bigcap_{k=1}^{\infty} B_k.$$

Hence,

$$K^c \subset \left(\bigcap_{k=1}^{\infty} B_k\right)^c = \bigcup_{k=1}^{\infty} B_k^c,$$

which implies:

$$P(K^c) \leq \sum_{k=1}^{\infty} P(B_k^c) \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Thus, the proof of (ii) will be complete once we prove the following:

Claim: Let $\{G_n\}_{n \in \mathbb{N}}$ be a sequence of open subsets of S with $G_n \uparrow S$. Then, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$1 - \varepsilon < P(G_{n_\varepsilon}) \leq 1, \quad \text{for each } P \in \Pi.$$

Proof of Claim: Suppose the Claim is false, and we derive a contradiction. The failure of the Claim implies that there exists some $0 < \varepsilon < 1$ such that for each $n \in \mathbb{N}$, there exists $P_n \in \Pi$ such that

$$P_n(G_n) \leq 1 - \varepsilon.$$

By the hypothesis of weak sequential compactness of Π , there exists some probability measure $Q \in \mathcal{M}_1(S, \mathcal{B}(S))$ and the subsequence $\{P_{n(i)}\}$ of $\{P_n\}$ such that $P_{n(i)} \xrightarrow{w} Q$, as $i \rightarrow \infty$. Now, for each fixed $n \in \mathbb{N}$, we have:

$$\begin{aligned} Q(G_n) &\leq \liminf_{i \rightarrow \infty} P_{n(i)}(G_n), \quad \text{by the Portmanteau Theorem} \\ &\leq \liminf_{i \rightarrow \infty} P_{n(i)}(G_{n(i)}), \quad \text{since } \{G_n\} \text{ is increasing} \\ &\leq 1 - \varepsilon, \quad \text{by choice of } P_n \end{aligned}$$

But, by hypothesis, we also have $G_n \uparrow S$. Hence, we therefore have:

$$1 = Q(S) = \lim_{n \rightarrow \infty} Q(G_n) \leq 1 - \varepsilon,$$

which is the desired contradiction. This completes the proof of the Claim, hence that of (ii).

Proof of (i)

Suppose $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is tight. We need to establish that Π is weakly sequentially compact. In other words, if $\{P_n\} \subset \Pi$ is a sequence of probability measures contained in Π , we need to establish that there exists a Borel probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ and a subsequence $\{P_{n(i)}\} \subset \{P_n\}$ such that $P_{n(i)} \xrightarrow{w} P$, as $i \rightarrow \infty$.

So, let $\{P_n\} \subset \Pi$. We prove the Theorem by establishing the following series of Claims. Note that the proof of the Theorem is complete once we establish Claim 5.

Claim 1: There exists an increasing sequence of compact subsets $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ such that

$$1 - \frac{1}{m} < P_n(K_m) \leq 1, \quad \text{for every } m, n \in \mathbb{N}.$$

Claim 2: Let $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ be one such sequence of compact subsets of S as in Claim 1. Then, the following statements are true:

- (i) $\Sigma := \bigcup_{m=1}^{\infty} K_m$ is a separable subset of S .
- (ii) There exists a countable collection \mathcal{A} of open subsets of S satisfying the following property:
For each $x \in S$ and for each open subset G of S ,

$$x \in G \cap \left(\bigcup_{m=1}^{\infty} K_m \right) \implies x \in A \subset \bar{A} \subset G, \quad \text{for some } A \in \mathcal{A}.$$

- (iii) The collection \mathcal{A} is an open cover of Σ .

Claim 3: Define:

$$\mathcal{H} := \{\emptyset\} \cup \left\{ \begin{array}{l} \text{all finite unions of sets of the form} \\ \bar{A} \cap K_m, \text{ where } A \in \mathcal{A} \text{ and } m \in \mathbb{N} \end{array} \right\}.$$

Then, there exists a subsequence $\{P_{n(i)}\} \subset \{P_n\}$ such that the limit

$$\alpha(H) := \lim_{i \rightarrow \infty} P_{n(i)}(H) \text{ exists, for each } H \in \mathcal{H}.$$

Claim 4: There exists a Borel probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ such that

$$P(G) = \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset G \end{array} \right\}, \quad \text{for each open subset } G \subset S.$$

(Note that the supremum above is always taken over a non-empty set: For each open $G \subset S$, the set $\{H \in \mathcal{H} \mid H \subset G\}$ is non-empty, since $\emptyset \in \mathcal{H}$.)

Claim 5: $P_{n(i)} \xrightarrow{w} P$, as $i \rightarrow \infty$.

Proof of Claim 1: By tightness hypothesis on Π , for each $m \in \mathbb{N}$, there exists a compact subset $L_m \subset S$ such that

$$1 - \frac{1}{m} < P(L_m) \leq 1, \quad \text{for each } P \in \Pi.$$

Define, for each $m \in \mathbb{N}$, $K_m := \bigcup_{i=1}^m L_i$. Then, each K_m is compact (since finite unions of compact subsets are themselves compact). Next, we trivially have $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$. Also,

$$P(K_m) = P\left(\bigcup_{i=1}^m L_i\right) \geq P(L_m) > 1 - \frac{1}{m}, \quad \text{for each } P \in \Pi.$$

In particular, the above inequality holds for each P_n . This proves Claim 1.

Proof of Claim 2: Separability of $\Sigma := \bigcup_{m=1}^{\infty} K_m$ is an immediate consequence of Lemma A.1 and Lemma A.2. Then, the existence of \mathcal{A} follows immediately from the separability of Σ and Lemma A.3.

It remains to show that \mathcal{A} is an open cover of Σ . To this end, first note that the collection \mathcal{O}_S of open subsets of S forms an open cover of S , hence \mathcal{O}_S is also an open cover of Σ . Therefore, for each $x \in \Sigma$, we may choose $G_x \in \mathcal{O}_S$ such that $x \in G_x$. Thus, $x \in \Sigma \cap G_x$, and by properties of \mathcal{A} , we may furthermore choose $A_x \in \mathcal{A}$ such that $x \in A_x \subset \overline{A_x} \subset G_x$. We thus see that the collection

$$\left\{ A_x \in \mathcal{A} \mid x \in \Sigma \right\} \subset \mathcal{A}$$

is an open cover of Σ consisting of subsets in \mathcal{A} . This completes the proof of Claim 2.

Proof of Claim 3: Note that \mathcal{H} is a countable collection of subsets of S . Let $\mathcal{H} = \{H_1, H_2, H_3, \dots\}$ be an enumeration of \mathcal{H} . Consider the following array of real numbers:

$$\begin{array}{cccc} P_1(H_1) & P_2(H_1) & P_3(H_1) & \cdots \\ P_1(H_2) & P_2(H_2) & P_3(H_2) & \cdots \\ P_1(H_3) & P_2(H_3) & P_3(H_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Note that each row of the above array is bounded between 0 and 1. Hence, by Theorem A.4, there exists an increasing sequence

$$n(1) < n(2) < n(3) < \cdots \in \mathbb{N}$$

of natural numbers such that the limit

$$\lim_{k \rightarrow \infty} P_{n(k)}(H_r), \quad \text{exists for each } r \in \mathbb{N}.$$

This completes the proof of Claim 3.

Proof of Claim 4:

Proof of Claim 5: Let $G \subset S$ be an arbitrary open subset of S . Then, for each $H \in \mathcal{H}$ with $H \subset G$, we have

$$\alpha(H) := \lim_{i \rightarrow \infty} P_{n(i)}(H) \leq \liminf_{i \rightarrow \infty} P_{n(i)}(G).$$

The preceding inequality and Claim 4 together imply:

$$P(G) = \sup \left\{ \alpha(H) \in [0, 1] \mid \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset G \end{array} \right\} \leq \liminf_{i \rightarrow \infty} P_{n(i)}(G), \quad \text{for each open subset } G \subset S,$$

which is equivalent to the weak convergence $P_{n(i)} \xrightarrow{w} P$, as $i \rightarrow \infty$, by the Portmanteau Theorem (Theorem 2.1, [2]). This completes the proof of Claim 5. \square

A Technical Lemmas

Lemma A.1

Every compact subset of a metric space is also a separable subset of that metric space.

PROOF Let (X, ρ) be a metric space and $K \subset X$ be a compact subset of X . For each $x \in X$ and positive $r > 0$, let

$$B(x, r) := \{ y \in X \mid \rho(x, y) < r \} \subset X,$$

i.e. $B(x, r)$ is the open ball in X centred at x with radius $r > 0$. For each $n \in \mathbb{N}$, the following forms an open cover of K :

$$\mathcal{C}_n := \left\{ B\left(x, \frac{1}{n}\right) \subset X \mid x \in K \right\}.$$

Since K is compact, each \mathcal{C}_n admits a finite subcover:

$$\mathcal{F}_n := \left\{ B\left(x_i^{(n)}, \frac{1}{n}\right) \subset X \mid x_i^{(n)} \in K, i = 1, 2, \dots, J_n \right\}.$$

Let

$$\mathcal{D}_n := \left\{ x_i^{(n)} \in K \mid i = 1, 2, \dots, J_n \right\} \subset K,$$

and let $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset K$. We claim that \mathcal{D} is dense in K . Indeed, let $y \in K$. Since each \mathcal{F}_n is a (finite) open cover of K , we have:

$$y \in K \subset \bigcup_{i=1}^{J_n} B\left(x_i^{(n)}, \frac{1}{n}\right), \quad \text{for each } n \in \mathbb{N}.$$

Since $x_i^{(n)} \in \mathcal{D}$, for each $i = 1, 2, \dots, J_n$ and for each $n \in \mathbb{N}$, the above inclusion shows that, for each $n \in \mathbb{N}$, there exists some $x \in \mathcal{D}$ such that $\rho(y, x) < \frac{1}{n}$. In particular, \mathcal{D} contains a sequence that converges to $y \in K$. Since $y \in K$ is an arbitrary element of K , we see that $\overline{\mathcal{D}} \supset K$. Since $\mathcal{D} \subset K$ and K is compact, hence closed, we trivially have $\overline{\mathcal{D}} \subset K$. We may now conclude that $\overline{\mathcal{D}} = K$. This completes the proof of the Lemma. \square

Lemma A.2

Every countable union of separable subsets of a metric space is itself a separable subset of that metric space.

PROOF Let $S := \bigcup_{i=1}^{\infty} S_i \subset X$ be a countable union of separable subsets S_i of a metric space X . For each fixed $i \in \mathbb{N}$, since S_i is separable, there exists countable $D_i \subset S_i$ which is dense in S_i . Let $D := \bigcup_{i=1}^{\infty} D_i$. Then, D is a countable subset of S . The Lemma is proved once we establish that D is dense in S . To this end, let $x \in S = \bigcup_{i=1}^{\infty} S_i$. Then, $x \in S_i$ for some $i \in \mathbb{N}$. Since D_i is dense in S_i , there exists a sequence $\{y_k\} \subset D_i \subset D$ such that $y_k \rightarrow x$, as $k \rightarrow \infty$. This proves that D is indeed dense in S , and completes the proof of the Lemma. \square

Lemma A.3 (second theorem in Appendix M3, [2])

Let (S, ρ) be a metric space and $\Sigma \subset S$ a separable subset of S . Then, there exists a countable collection \mathcal{A} of open subsets of S satisfying the following property: For each $x \in S$ and each open subset G of S ,

$$x \in G \cap \Sigma \implies x \in A \subset \overline{A} \subset G, \quad \text{for some } A \in \mathcal{A}.$$

PROOF Let $D \subset \Sigma$ be a countable dense subset of Σ . Let

$$\mathcal{A} := \left\{ B(d, r) \subset S \mid \begin{array}{l} d \in D, \\ r \in \mathbb{Q}, r > 0 \end{array} \right\}.$$

Then, \mathcal{A} is a countable collection of open balls in S . Now, let $G \subset S$ be an arbitrary open subset of S and $x \in G \cap \Sigma$. First, choose $\varepsilon > 0$ such that $B(x, \varepsilon) \subset G$. Next, since $x \in \Sigma$ and D is dense in Σ , we may choose $d \in D$ such that $d \in B(x, \varepsilon/2)$, or equivalently $\rho(x, d) < \varepsilon/2$. Finally choose positive rational $r > 0$ such that $\rho(x, d) < r < \varepsilon/2$.

Now, note that $\overline{B(d, r)} \subset B(x, \varepsilon)$; indeed,

$$y \in \overline{B(d, r)} \iff \rho(y, d) \leq r \implies \rho(x, y) \leq \rho(x, d) + \rho(d, y) < \varepsilon/2 + r < \varepsilon/2 + \varepsilon/2 \implies y \in B(x, \varepsilon).$$

Thus, we have

$$x \in B(d, r) \subset \overline{B(d, r)} \subset B(x, \varepsilon) \subset G.$$

This completes the proof of the Lemma. □

Theorem A.4 (The Diagonal Method, Appendix A.14, [1])

Suppose that each row of the array

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

is a bounded sequence of real numbers. Then, there exists an increasing sequence

$$n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$$

of positive integers such that the limit

$$\lim_{k \rightarrow \infty} x_{r, n_k} \text{ exists, for each } r = 1, 2, 3, \dots$$

PROOF From the first row, select a convergent subsequence

$$x_{1, n(1,1)}, x_{1, n(1,2)}, x_{1, n(1,3)}, \dots$$

Here, we have $n(1,1) < n(1,2) < n(1,3) < \cdots \in \mathbb{N}$, and $\lim_{k \rightarrow \infty} x_{1, n(1,k)} \in \mathbb{R}$ exists. Next, note that the following subsequence of the second row:

$$x_{2, n(1,1)}, x_{2, n(1,2)}, x_{2, n(1,3)}, \dots$$

is still a bounded sequence of real numbers, and we may thus select a convergent subsequence:

$$x_{2, n(2,1)}, x_{2, n(2,2)}, x_{2, n(2,3)}, \dots$$

Here, we have $n(2,1) < n(2,2) < n(2,3) < \cdots \in \{n(1,k)\}_{k \in \mathbb{N}}$, and $\lim_{k \rightarrow \infty} x_{2, n(2,k)} \in \mathbb{R}$ exists. Continuing inductively, we obtain an array of positive integers

$$\begin{array}{cccc} n(1,1) & n(1,2) & n(1,3) & \cdots \\ n(2,1) & n(2,2) & n(2,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

which satisfies: For each $r \in \mathbb{N}$, we have

- each row is an increasing sequence of positive integers, i.e. $n(r,1) < n(r,2) < n(r,3) < \cdots$,

- the $(r + 1)^{\text{th}}$ row is a subsequence of the r^{th} row, i.e. $\{n(r + 1, k)\}_{k \in \mathbb{N}} \subset \{n(r, k)\}_{k \in \mathbb{N}}$, and
- $\lim_{k \rightarrow \infty} x_{r, n(r, k)} \in \mathbb{R}$ exists.

Note that the first two properties together imply:

$$n(k, k) < n(k, k + 1) \leq n(k + 1, k + 1), \quad \text{for each } k \in \mathbb{N}.$$

Now, define $n_k := n(k, k)$, for $k \in \mathbb{N}$. We then see that

$$n_k := n(k, k) < n(k + 1, k + 1) =: n_{k+1},$$

i.e., $\{n_k\}_{k \in \mathbb{N}}$ is a strictly increasing sequence of positive integers. Lastly, for each $r \in \mathbb{N}$, consider the sequence

$$x_{r, n_1}, \quad x_{r, n_2}, \quad x_{r, n_3}, \quad \dots$$

Note that, for each $r \in \mathbb{N}$,

$$x_{r, n_r}, \quad x_{r, n_{r+1}}, \quad x_{r, n_{r+2}}, \quad \dots$$

is a subsequence of $\{x_{r, n(r, k)}\}_{k \in \mathbb{N}}$. We saw above that $\lim_{k \rightarrow \infty} x_{r, n(r, k)}$ exists, which in turn implies that $\lim_{k \rightarrow \infty} x_{r, n_k}$ exists. Since $r \in \mathbb{N}$ is arbitrary, the proof of the Theorem is now complete. \square

References

- [1] BILLINGSLEY, P. *Probability and Measure*, third ed. John Wiley & Sons, 1995.
- [2] BILLINGSLEY, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.