

## 1 Separating and convergence-determining classes

### Definition 1.1 (Separating class)

Suppose  $\Omega$  is a non-empty set,  $\mathcal{A}$  is a  $\sigma$ -algebra of subsets of  $\Omega$ ,  $(\Omega, \mathcal{A})$  is the corresponding measurable space, and  $\mathcal{M}_1(\Omega, \mathcal{A})$  is the set of all probability measures defined on  $\mathcal{A}$ . A **separating class** of subsets of  $(\Omega, \mathcal{A})$  is a collection  $\mathcal{S} \subset \mathcal{A}$  of subsets of  $\Omega$  which satisfies the following condition: For every two probability measures  $\mu, \nu \in \mathcal{M}_1(\Omega, \mathcal{A})$ ,

$$\mu(S) = \nu(S), \text{ for every } S \in \mathcal{S} \implies \mu(A) = \nu(A), \text{ for every } A \in \mathcal{A}$$

### Definition 1.2 (Convergence-determining class)

Suppose  $\Omega$  is a topological space,  $\mathcal{B}(\Omega)$  is its Borel  $\sigma$ -algebra,  $(\Omega, \mathcal{B}(\Omega))$  is the corresponding measurable space, and  $\mathcal{M}_1(\Omega, \mathcal{B}(\Omega))$  is the set of all probability measures defined on  $\mathcal{B}(\Omega)$ . A **convergence-determining class** of subsets of  $(\Omega, \mathcal{B}(\Omega))$  is a collection  $\mathcal{C} \subset \mathcal{B}(\Omega)$  of Borel subsets of  $\Omega$  which satisfies the following condition: For any  $\mu, \mu_1, \mu_2, \dots \in \mathcal{M}_1(\Omega, \mathcal{B}(\Omega))$ ,

$$\lim_{n \rightarrow \infty} \mu_n(C) = \mu(C), \text{ for every } C \in \mathcal{C} \implies \mu_n \xrightarrow{w} \mu.$$

### Theorem 1.3

Suppose  $\Omega$  is a non-empty set,  $\mathcal{A}$  is a  $\sigma$ -algebra of subsets of  $\Omega$ , and  $(\Omega, \mathcal{A})$  is the corresponding measurable space.

If

- $\mathcal{S} \subset \mathcal{A}$  is closed under finite intersections, and
- $\mathcal{S}$  generates  $\mathcal{A}$  (i.e.  $\sigma(\mathcal{S}) = \mathcal{A}$ ),

then  $\mathcal{S}$  is a separating class of subsets of  $(\Omega, \mathcal{A})$ .

PROOF Let  $\mu$  and  $\nu$  be two probability measures defined on  $(\Omega, \mathcal{A})$  such that  $\mu(S) = \nu(S)$  for each  $S \in \mathcal{S}$ . We need to show that  $\mu(A) = \nu(A)$  for each  $A \in \mathcal{A}$ . To this end, let

$$\mathcal{L} := \{ A \in \mathcal{A} = \sigma(\mathcal{S}) \mid \mu(A) = \nu(A) \}.$$

Note that  $\mathcal{S} \subset \mathcal{L}$ , by the hypothesis that  $\mu$  and  $\nu$  agree on  $\mathcal{S}$ , and  $\mathcal{L} \neq \emptyset$  since  $\Omega \in \mathcal{L}$ . By Corollary B.8, it suffices to establish that  $\mathcal{L}$  is a  $\lambda$ -system, since then it will follow that

$$\mathcal{A} = \sigma(\mathcal{S}) \subset \mathcal{L} := \{ A \in \mathcal{A} = \sigma(\mathcal{S}) \mid \mu(A) = \nu(A) \} \subset \sigma(\mathcal{S}) = \mathcal{A},$$

i.e.,  $\mathcal{A} = \sigma(\mathcal{S}) = \mathcal{L}$ , or equivalently,  $\mu$  and  $\nu$  agree on all of  $\mathcal{A} = \sigma(\mathcal{S})$ . Now, we have already noted that  $\Omega \in \mathcal{L}$ . For  $A \in \mathcal{L}$ , we have

$$\mu(\Omega \setminus A) = 1 - \mu(A) = 1 - \nu(A) = \nu(\Omega \setminus A),$$

hence  $\Omega \setminus A \in \mathcal{L}$ . Thus,  $\mathcal{L}$  is closed under complementations. Lastly, let  $A_1, A_2, \dots \in \mathcal{L}$  be pairwise disjoint. Then,

$$\mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \nu(A_n) = \nu\left(\bigsqcup_{n=1}^{\infty} A_n\right),$$

thus  $\bigsqcup_{n=1}^{\infty} A_n \in \mathcal{L}$ , which proves that  $\mathcal{L}$  is closed under countable disjoint unions.  $\mathcal{L}$  is therefore indeed a  $\lambda$ -system and the proof of the Theorem is complete.  $\square$

**Corollary 1.4** Suppose  $S$  is a topological space and  $\mathcal{B}(S)$  is its Borel  $\sigma$ -algebra (i.e. the  $\sigma$ -algebra generated by the collection of open subsets of  $S$ ). Then, the collection of open subsets of  $S$  is a separating class of subsets of the measurable space  $(S, \mathcal{B}(S))$ .

PROOF Recall that the collection of open sets are closed under finite intersections (by definition of topology), and they generate the Borel  $\sigma$ -algebras (by definition of Borel  $\sigma$ -algebras). Thus the Corollary follows immediately from Theorem 1.3.  $\square$

## 2 Examples of separating and convergence-determining classes of $\mathbb{R}^\infty$

**Definition 2.1** (The metric on  $\mathbb{R}^\infty$ , Example 1.2, [1])

Let  $\mathbb{R}^\infty$  denotes the set of all infinite sequences of real numbers, i.e.

$$\mathbb{R}^\infty := \{ (x_1, x_2, \dots) \mid x_i \in \mathbb{R}, \text{ for each } i \in \mathbb{N} \}.$$

Define  $\rho : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow [0, 1]$  as follows:

$$\rho(x, y) := \sum_{n=1}^{\infty} \frac{\min\{1, |x_n - y_n|\}}{2^n}.$$

**Remark 2.2** Recall that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2} \cdot \left( \frac{1}{1 - \frac{1}{2}} \right) = 1,$$

which proves indeed that  $0 \leq \rho(x, y) \leq 1$ , for any  $x, y \in \mathbb{R}^\infty$ .

**Theorem 2.3** (The metric space properties of  $\mathbb{R}^\infty$ )

- (i)  $(\mathbb{R}^\infty, \rho)$  is a metric space. Let  $\mathbb{R}^\infty$  denote also this metric space in the remainder of this Theorem.
- (ii) For  $x, x^{(1)}, x^{(2)}, x^{(3)}, \dots \in \mathbb{R}^\infty$ , we have:

$$\rho(x^{(n)}, x) \longrightarrow 0 \iff \text{for each } i \in \mathbb{N}, \lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0$$

- (iii) For each  $n \in \mathbb{N}$ , the “natural projection to the initial segment of length  $n$ ”

$$\pi_n : \mathbb{R}^\infty \longrightarrow \mathbb{R}^n : x \longmapsto (x_1, x_2, \dots, x_n)$$

is continuous, where  $\mathbb{R}^n$  has the usual Euclidean topology.

- (iv) For each  $x \in \mathbb{R}^\infty$ ,  $n \in \mathbb{N}$ , and  $\varepsilon > 0$ , let  $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)$  denote the open hypercube in  $\mathbb{R}^n$  of side length  $2\varepsilon$  centred at  $\pi_n(x) \in \mathbb{R}^n$ , i.e.

$$C_{\mathbb{R}^n}(\pi_n(x), \varepsilon) := \left\{ y \in \mathbb{R}^n \mid \begin{array}{l} |y_i - x_i| < \varepsilon, \\ i = 1, 2, \dots, n \end{array} \right\}$$

Then, its pre-image in  $\mathbb{R}^\infty$  under  $\pi_n$

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) = \left\{ y \in \mathbb{R}^\infty \mid \begin{array}{l} |y_i - x_i| < \varepsilon, \\ i = 1, 2, \dots, n \end{array} \right\}$$

is an open subset of  $\mathbb{R}^\infty$ .

- (v) For each  $x \in \mathbb{R}^\infty$ ,  $n \in \mathbb{N}$ , and  $\varepsilon > 0$ , we have:

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right),$$

where  $B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right)$  is the open ball in  $\mathbb{R}^\infty$  centred at  $x$  of radius  $\varepsilon + \frac{1}{2^n}$ , i.e.

$$B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right) := \left\{ y \in \mathbb{R}^\infty \mid \rho(y, x) < \varepsilon + \frac{1}{2^n} \right\}$$

(vi) *The collection*

$$\left\{ \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset \mathbb{R}^\infty \mid n \in \mathbb{N}, x \in \mathbb{R}^\infty, \varepsilon > 0 \right\}$$

*of all pre-images under  $\pi_n$  of open hypercubes in  $\mathbb{R}^n$ , for all  $n \in \mathbb{N}$ , forms a basis for the topology of  $\mathbb{R}^\infty$ .*

(vii)  $\mathbb{R}^\infty$  is a separable metric space.

(viii)  $\mathbb{R}^\infty$  is a complete metric space.

PROOF

(i) Clearly,  $\rho$  is non-negative and symmetric. We now show that, for any  $x, y \in \mathbb{R}^\infty$ , we have  $\rho(x, y) = 0$  implies  $x = y$ . Indeed,

$$\begin{aligned} \rho(x, y) = 0 &\iff \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - y_i|\}}{2^i} = 0 \\ &\iff \min\{1, |x_i - y_i|\} = 0, \text{ for each } i \in \mathbb{N} \\ &\iff |x_i - y_i| = 0, \text{ for each } i \in \mathbb{N} \\ &\iff x = y. \end{aligned}$$

In order to show that  $\rho$  is a metric, it remains only to establish the Triangle Inequality. By Lemma A.2, for any  $x, y, z \in \mathbb{R}^\infty$ , we have

$$\begin{aligned} \rho(x, y) &= \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - y_i|\}}{2^i} \\ &\leq \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - z_i|\} + \min\{1, |z_i - y_i|\}}{2^i} \\ &= \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - z_i|\}}{2^i} + \sum_{i=1}^{\infty} \frac{\min\{1, |z_i - y_i|\}}{2^i} \\ &= \rho(x, z) + \rho(z, y), \end{aligned}$$

where we have used the fact that  $0 \leq \rho \leq 1$  to split the infinite sum into two terms in second-to-last equality. This proves that  $\rho$  satisfies the Triangle Inequality, and it is thus a metric on  $\mathbb{R}^\infty$ .

(ii)  $\lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = 0 \implies \lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0$ , for each  $i \in \mathbb{N}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = 0 &\implies \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} = 0 \\ &\implies \lim_{n \rightarrow \infty} \min\{1, |x_i^{(n)} - x_i|\} = 0, \text{ for each } i \in \mathbb{N} \\ &\implies \lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0, \text{ for each } i \in \mathbb{N} \end{aligned}$$

$$\lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0, \text{ for each } i \in \mathbb{N} \implies \lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = 0$$

This follows from the Weierstrass  $M$ -test. Suppose  $\lim_{n \rightarrow \infty} |x_i^{(n)} - x_i| = 0$ , for each  $i \in \mathbb{N}$ . Then,

$$\lim_{n \rightarrow \infty} \frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} = 0 =: y_i, \text{ for each } i \in \mathbb{N}.$$

For each  $i \in \mathbb{N}$ , let  $M_i := \frac{1}{2^i}$ . Then,

$$\frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} \leq M_i \quad \text{and} \quad \sum_{i=1}^{\infty} M_i < \infty.$$

Hence, by the Weierstrass  $M$ -test (Lemma A.3), we have

$$\lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \frac{\min\{1, |x_i^{(n)} - x_i|\}}{2^i} = \sum_{i=1}^{\infty} y_i = 0.$$

(iii) Immediate by (ii).

(iv) Since  $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon) \subset \mathbb{R}^n$  is an open subset of  $\mathbb{R}^n$ , its pre-image under the continuous (by (iii)) map  $\pi_n : \mathbb{R}^\infty \rightarrow \mathbb{R}^n$  is an open subset of  $\mathbb{R}^\infty$ .

(v) For  $y \in \mathbb{R}^\infty$ , we have

$$\begin{aligned} y \in \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) &\implies |y_i - x_i| < \varepsilon, \text{ for each } i = 1, 2, \dots, n \\ &\implies \rho(x, y) := \sum_{i=1}^{\infty} \frac{\min\{1, |x_i - y_i|\}}{2^i} \leq \sum_{i=1}^n \frac{\varepsilon}{2^i} + \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \varepsilon + \frac{1}{2^n}. \end{aligned}$$

This proves:

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right).$$

(vi) It suffices to show that every open ball in  $B_{\mathbb{R}^\infty}(x, r) \subset \mathbb{R}^\infty$ ,  $r > 0$ , contains the pre-image of an open hypercube centred at  $\pi_n(x) \in \mathbb{R}^n$  under  $\pi_n$ . To this end, for  $r > 0$ , choose  $\varepsilon > 0$  sufficiently small and  $n \in \mathbb{N}$  sufficiently large such that  $\varepsilon + \frac{1}{2^n} < r$ . Then, for any  $x \in \mathbb{R}^\infty$ , by (v), we have:

$$x \in \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)) \subset B_{\mathbb{R}^\infty}\left(x, \varepsilon + \frac{1}{2^n}\right) \subset B_{\mathbb{R}^\infty}(x, r),$$

as required.

(vii) It suffices to exhibit a countable subset of  $\mathbb{R}^\infty$  that intersects every open ball in  $\mathbb{R}^\infty$ . To this end, let

$$D := \bigcup_{n=1}^{\infty} \left\{ x = (x_1, x_2, \dots) \in \mathbb{R}^\infty \mid \begin{array}{l} x_i \in \mathbb{Q}, \text{ for each } i \in \mathbb{N} \\ x_i = 0, \text{ for all } i \geq n \end{array} \right\}.$$

Clearly,  $D$  is a countable subset of  $\mathbb{R}^\infty$ . Now let  $B_{\mathbb{R}^\infty}(x, \varepsilon)$  be an arbitrary open ball in  $\mathbb{R}^\infty$ . Choose  $\delta > 0$  small enough and  $n \in \mathbb{N}$  large enough such that  $\delta + \frac{1}{2^n} < \varepsilon$ . Then,

$$\pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \delta)) \subset B_{\mathbb{R}^\infty}\left(x, \delta + \frac{1}{2^n}\right) \subset B_{\mathbb{R}^\infty}(x, \varepsilon),$$

Now, for each  $i = 1, 2, \dots, n$ , choose  $z_i \in \mathbb{Q} \cap (x_i - \delta, x_i + \delta)$ . Let  $z = (z_1, z_2, \dots, z_n, 0, 0, \dots) \in \mathbb{R}^\infty$ . Then, we have

$$z \in D \cap \left\{ y \in \mathbb{R}^\infty \mid \begin{array}{l} y_i \in (x_i - \delta, x_i + \delta), \\ \text{for each } i = 1, 2, \dots, n \end{array} \right\} = D \cap \pi_n^{-1}(C_{\mathbb{R}^n}(\pi_n(x), \delta)) \subset D \cap B_{\mathbb{R}^\infty}(x, \varepsilon).$$

This proves the countable subset  $D \subset \mathbb{R}^\infty$  has non-empty intersection with every open ball in  $\mathbb{R}^\infty$ , i.e.  $D$  is dense in  $\mathbb{R}^\infty$ . Hence,  $\mathbb{R}^\infty$  is separable.

(viii) We need to show that every Cauchy sequence in  $\mathbb{R}^\infty$  converges to any element in  $\mathbb{R}^\infty$ .

$$\begin{aligned}
 & \left\{ x^{(n)} \right\}_{n \in \mathbb{N}} \subset \mathbb{R}^\infty \text{ is a Cauchy sequence in } \mathbb{R}^\infty \\
 \iff & \text{ for each } \varepsilon > 0, \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that } \rho(x^{(m)}, x^{(n)}) < \varepsilon, \text{ for any } m, n > N_\varepsilon \\
 \implies & \text{ for each } i \in \mathbb{N}, \text{ we have:} \\
 & \text{ for each } \varepsilon > 0, \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that } \left| x_i^{(m)} - x_i^{(n)} \right| < \varepsilon, \text{ for any } m, n > N_\varepsilon \\
 \implies & \text{ for each } i \in \mathbb{N}, \left\{ x_i^{(n)} \right\}_{n \in \mathbb{N}} \subset \mathbb{R} \text{ is a Cauchy sequence in } \mathbb{R}; \text{ hence } x_i := \lim_{n \rightarrow \infty} x_i^{(n)} \in \mathbb{R} \text{ exists} \\
 \implies & \lim_{n \rightarrow \infty} \rho(x^{(n)}, x) = 0, \text{ where } x := (x_1, x_2, \dots) \in \mathbb{R}^\infty \quad (\text{by (ii)})
 \end{aligned}$$

This proves that  $\mathbb{R}^\infty$  indeed is a complete metric space.

□

## Definition 2.4

The **finite-dimensional class** of subsets of  $\mathbb{R}^\infty$  is, by definition, the following:

$$\mathcal{B}_f(\mathbb{R}^\infty) := \left\{ \pi_k^{-1}(B) \subset \mathbb{R}^\infty \mid \begin{array}{l} k \in \mathbb{N} \\ B \in \mathcal{B}(\mathbb{R}^k) \end{array} \right\},$$

where  $\pi_k : \mathbb{R}^\infty \longrightarrow \mathbb{R}^k : x = (x_1, x_2, \dots) \longmapsto (x_1, \dots, x_k)$  is the projection of  $\mathbb{R}^\infty$  onto  $\mathbb{R}^k$ .

## Theorem 2.5

- (i)  $\mathcal{B}_f(\mathbb{R}^\infty) \subset \mathcal{B}(\mathbb{R}^\infty)$ .
- (ii)  $\mathcal{B}_f(\mathbb{R}^\infty)$  is a separating class of Borel subsets of  $\mathbb{R}^\infty$ .
- (iii)  $\mathcal{B}_f(\mathbb{R}^\infty)$  is a convergence-determining class of Borel subsets of  $\mathbb{R}^\infty$ .

PROOF

- (i) Note that

$$\mathcal{B}_f(\mathbb{R}^\infty) := \left\{ \pi_k^{-1}(B) \subset \mathbb{R}^\infty \mid \begin{array}{l} k \in \mathbb{N} \\ B \in \mathcal{B}(\mathbb{R}^k) \end{array} \right\} = \bigcup_{k=1}^{\infty} \pi_k^{-1}(\mathcal{B}(\mathbb{R}^k)).$$

Thus, (i) is equivalent to the statement that each  $\pi_k : \mathbb{R}^\infty \longrightarrow \mathbb{R}^k$  is Borel measurable. But each  $\pi_k$  is continuous, hence Borel measurable (Corollary B.12). This proves (i).

- (ii) We apply Theorem 1.3 to  $\mathcal{B}_f(\mathbb{R}^\infty)$ .

$\mathcal{B}_f(\mathbb{R}^\infty)$  is closed under finite intersections

$\mathcal{B}_f(\mathbb{R}^\infty)$  generates  $\mathcal{B}(\mathbb{R}^\infty)$

Let  $\mathcal{O}(\mathbb{R}^\infty)$  denote the collection of open sets of  $\mathbb{R}^\infty$ . Hence  $\mathcal{B}(\mathbb{R}^\infty) := \sigma(\mathcal{O}(\mathbb{R}^\infty))$ . By (i), we have  $\mathcal{B}_f(\mathbb{R}^\infty) \subset \mathcal{B}(\mathbb{R}^\infty) = \sigma(\mathcal{O}(\mathbb{R}^\infty))$ , which implies  $\sigma(\mathcal{B}_f(\mathbb{R}^\infty)) \subset \sigma(\mathcal{O}(\mathbb{R}^\infty))$ . We need to establish the reverse inclusion, which will immediately follow from:

**Claim:**  $\mathcal{O}(\mathbb{R}^\infty) \subset \sigma(\mathcal{B}_f(\mathbb{R}^\infty))$ .

Proof of Claim: By Theorem 2.3(v), every open ball  $B_{\mathbb{R}^\infty}(x, \varepsilon)$  in  $\mathbb{R}^\infty$  contains the pre-image of an open hypercube from some finite-dimensional Euclidean space, where that pre-image itself contains  $x$ . We therefore see that every open set in  $\mathbb{R}^\infty$  can be expressed as a union of pre-images of open hypercubes from finite-dimensional Euclidean spaces. By Theorem 2.3(vii),  $\mathbb{R}^\infty$  is separable. Hence, by Theorem C.1, we see that every open set in  $\mathbb{R}^\infty$  can be expressed as a countable union of pre-images of open hypercubes from finite-dimensional Euclidean spaces. Since pre-images of open hypercubes from finite-dimensional Euclidean spaces belong to  $\mathcal{B}_f(\mathbb{R}^\infty)$ , we see that  $\mathcal{O}(\mathbb{R}^\infty) \subset \sigma(\mathcal{B}_f(\mathbb{R}^\infty))$ . This completes the proof of the Claim. □

## A Technical Lemmas

**Lemma A.1** *Define*

$$\phi : [0, \infty) \longrightarrow [0, 1] : t \longmapsto \min\{1, t\}.$$

*Then,  $\phi$  satisfies:*

$$\phi(s+t) \leq \phi(s) + \phi(t), \text{ for each } s, t \in [0, \infty).$$

PROOF For any  $s, t \in [0, \infty)$ , either  $s+t \geq 1$  or  $s+t < 1$ . If  $s+t \geq 1$ , then

$$\phi(s+t) = \min\{1, s+t\} = 1 < 2 = 1+1 \leq \min\{1, s\} + \min\{1, t\} = \phi(s) + \phi(t),$$

hence, the required inequality holds. On the other hand, if  $s+t < 1$ , then we must also have  $s < 1$  and  $t < 1$  (since  $s, t \geq 0$ ). Hence,

$$\phi(s+t) = \min\{1, s+t\} = s+t = \min\{1, s\} + \min\{1, t\} = \phi(s) + \phi(t),$$

thus, the required inequality also holds. □

**Lemma A.2** *For any  $x, y, z \in \mathbb{R}$ , we have:*

$$\min\{1, |x-y|\} \leq \min\{1, |x-z|\} + \min\{1, |z-y|\}.$$

PROOF Observe that  $|x-y| \leq |x-z| + |z-y|$  implies

$$\min\{1, |x-y|\} \leq |x-z| + |z-y|.$$

The above inequality, together with  $\min\{1, |x-y|\} \leq 1$ , thus in turn imply:

$$\min\{1, |x-y|\} \leq \min\{1, |x-z| + |z-y|\}.$$

By Lemma A.1, we therefore have:

$$\min\{1, |x-y|\} \leq \min\{1, |x-z| + |z-y|\} \leq \min\{1, |x-z|\} + \min\{1, |z-y|\},$$

which proves the present Lemma. □

**Lemma A.3 (The Weierstrass  $M$ -test, Theorem A.28, [2])**

*Suppose that  $\lim_{n \rightarrow \infty} x_i^{(n)} = x_i$ , for each  $i \in \mathbb{N}$ , and that  $|x_i^{(n)}| \leq M_i$ , where  $\sum_{i=1}^{\infty} M_i < \infty$ . Then,*

- (i)  $\sum_{i=1}^{\infty} x_i$  exists, and  $\sum_{i=1}^{\infty} x_i^{(n)}$  exists for each  $n \in \mathbb{N}$ .

(ii) Furthermore,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

PROOF

(i)  $\sum_{i=1}^{\infty} M_i < \infty$  and  $|x_i^{(n)}| \leq M_i \implies$  the series  $\sum_{i=1}^{\infty} x_i$  and  $\sum_{i=1}^{\infty} x_i^{(n)}$ ,  $n \in \mathbb{N}$ , converge absolutely.

(ii) Let  $\varepsilon > 0$  be given. Choose  $K \in \mathbb{N}$  sufficiently large such that  $\sum_{j=K+1}^{\infty} M_i < \frac{\varepsilon}{3}$ . Next, choose  $N \in \mathbb{N}$  sufficiently large such that

$$|x_i^{(n)} - x_i| < \frac{\varepsilon}{3K}, \text{ for any } n > N \text{ and } i = 1, 2, \dots, K.$$

Then, we have, for each  $n > N$ ,

$$\begin{aligned} \left| \sum_{i=1}^{\infty} x_i^{(n)} - \sum_{i=1}^{\infty} x_i \right| &= \left| \sum_{i=1}^K (x_i^{(n)} - x_i) + \sum_{i=K+1}^{\infty} x_i^{(n)} - \sum_{i=K+1}^{\infty} x_i \right| \\ &\leq \sum_{i=1}^K |x_i^{(n)} - x_i| + \sum_{i=K+1}^{\infty} |x_i^{(n)}| + \sum_{i=K+1}^{\infty} |x_i| \\ &\leq K \cdot \frac{\varepsilon}{3K} + \sum_{i=K+1}^{\infty} M_i + \sum_{i=K+1}^{\infty} M_i \leq \frac{\varepsilon}{3} + 2 \cdot \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since  $\varepsilon$  is arbitrary, this proves:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

□

## B $\sigma$ -algebras and $\lambda$ -systems

### Definition B.1

Suppose  $\Omega$  is a non-empty set. A  $\sigma$ -algebra of subsets of  $\Omega$  is a collection  $\mathcal{A}$  of subsets of  $\Omega$  which satisfies the following conditions:

- $\Omega \in \mathcal{A}$ .
- $\Omega \setminus A \in \mathcal{A}$ , for every  $A \in \mathcal{A}$ .
- $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ , whenever  $A_1, A_2, \dots \in \mathcal{A}$

### Definition B.2

Suppose  $\Omega$  is a non-empty set. A  $\lambda$ -system of subsets of  $\Omega$  is a collection  $\mathcal{L}$  of subsets of  $\Omega$  which satisfies the following conditions:

- $\Omega \in \mathcal{L}$ .
- $\Omega \setminus A \in \mathcal{L}$ , for every  $A \in \mathcal{L}$ .
- $\bigcap_{i=1}^{\infty} A_i \in \mathcal{L}$ , whenever  $A_1, A_2, \dots \in \mathcal{L}$  and  $A_i \cap A_j = \emptyset$ , for any  $i, j \in \mathbb{N}$  with  $i \neq j$ .

**Remark B.3** Clearly, every  $\sigma$ -algebra is also a  $\lambda$ -system.

**Theorem B.4**

Suppose  $\Omega$  is a non-empty set and  $\mathcal{L}$  is a  $\lambda$ -system of subsets of  $\Omega$ .

- (i)  $\mathcal{L}$  is closed under proper set-theoretic differences, i.e.  $A, B \in \mathcal{L}$  and  $A \subset B$  together imply  $B \setminus A \in \mathcal{L}$ .
- (ii) If  $\mathcal{L}$  is closed under finite intersections, then  $\mathcal{L}$  is a  $\sigma$ -algebra of subsets of  $\Omega$ .

PROOF For each  $X \subset \Omega$ , write  $\Omega \setminus X$  as  $X^c$ .

- (i) Suppose  $A, B \in \mathcal{L}$  with  $A \subset B$ . Then,  $B^c \cap A = \emptyset$ . Hence,  $B \setminus A = B \cap A^c = (B^c \cup A)^c = (B^c \sqcup A)^c \in \mathcal{L}$ , since  $\mathcal{L}$  is closed under complementations and finite disjoint unions.
- (ii) Since  $\mathcal{L}$  is a  $\lambda$ -system, we immediately have  $\Omega \in \mathcal{L}$ , and hence  $\Omega \setminus A \in \mathcal{L}$ , for every  $A \in \mathcal{L}$ . It remains to show that  $\mathcal{L}$  is closed under countable unions, i.e. for  $A_1, A_2, \dots \in \mathcal{L}$ , we need to show  $\bigcup_{i=1}^\infty A_i \in \mathcal{L}$ . To this end, define:

$$\begin{aligned} B_1 &:= A_1 \\ B_2 &:= A_2 \cap A_1^c \\ B_3 &:= A_3 \cap A_1^c \cap A_2^c \\ &\vdots \\ B_n &:= A_n \cap A_1^c \cap A_2^c \cap \dots \cap A_{n-1}^c \end{aligned}$$

Being a  $\lambda$ -system,  $\mathcal{L}$  is closed under complementations. By hypothesis,  $\mathcal{L}$  is furthermore closed under finite intersections. We thus see that  $B_n \in \mathcal{L}$ , for each  $n \in \mathbb{N}$ . Note also that the  $B_n$ 's are pairwise disjoint, and

$$\bigcup_{i=1}^n A_i = \bigsqcup_{i=1}^n B_i, \quad \text{for each } n \in \mathbb{N}.$$

Hence,

$$\bigcup_{i=1}^\infty A_i = \bigsqcup_{i=1}^\infty B_i \in \mathcal{L},$$

since  $\mathcal{L}$  is closed under countable pairwise disjoint unions ( $\mathcal{L}$  being a  $\lambda$ -system). This proves that  $\mathcal{L}$  is a  $\sigma$ -algebra of subsets of  $\Omega$ . □

**Theorem B.5** Let  $\Omega$  be a non-empty set.

- (i) The intersection of a non-empty collection of  $\sigma$ -algebras of subsets of  $\Omega$  is itself a  $\sigma$ -algebra of subsets of  $\Omega$ .
- (ii) The intersection of a non-empty collection of  $\lambda$ -systems of subsets of  $\Omega$  is itself a  $\lambda$ -system of subsets of  $\Omega$ .

PROOF

- (i) Suppose  $\Gamma$  is an (arbitrary) non-empty set, and, for each  $\gamma \in \Gamma$ ,  $\mathcal{A}_\gamma$  is a  $\sigma$ -algebra of subsets of  $\Omega$ . We need to prove that  $\mathcal{A} := \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma$  is itself a  $\sigma$ -algebra of subsets of  $\Omega$ .

$$\Omega \in \mathcal{A} := \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma$$

Since, for each  $\gamma \in \Gamma$ ,  $\mathcal{A}_\gamma$  is a  $\sigma$ -algebra of subsets of  $\Omega$ , we have  $\Omega \in \mathcal{A}_\gamma$ . Thus,  $\Omega \in \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma$ .



$$\underline{A \in \mathcal{A} \implies \Omega \setminus A \in \mathcal{A}}$$

$$A \in \mathcal{A} := \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma \iff A \in \mathcal{A}_\gamma, \forall \gamma \in \Gamma \implies \Omega \setminus A \in \mathcal{A}_\gamma, \forall \gamma \in \Gamma \implies \Omega \setminus A \in \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma =: \mathcal{A}$$

$$\underline{A_1, A_2, \dots \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}}$$

$$\begin{aligned} A_1, A_2, \dots \in \mathcal{A} &:= \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma \implies A_1, A_2, \dots \in \mathcal{A}_\gamma, \forall \gamma \in \Gamma \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_\gamma, \forall \gamma \in \Gamma \\ &\implies \bigcup_{i=1}^{\infty} A_i \in \bigcap_{\gamma \in \Gamma} \mathcal{A}_\gamma =: \mathcal{A} \end{aligned}$$

- (ii) Suppose  $\Gamma$  is an (arbitrary) non-empty set, and, for each  $\gamma \in \Gamma$ ,  $\mathcal{L}_\gamma$  is a  $\lambda$ -system of subsets of  $\Omega$ . We need to prove that  $\mathcal{L} := \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma$  is itself a  $\lambda$ -system of subsets of  $\Omega$ .

$$\underline{\Omega \in \mathcal{L} := \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma}$$

Since, for each  $\gamma \in \Gamma$ ,  $\mathcal{L}_\gamma$  is a  $\lambda$ -system of subsets of  $\Omega$ , we have  $\Omega \in \mathcal{L}_\gamma$ . Thus,  $\Omega \in \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma$ .

$$\underline{A \in \mathcal{L} \implies \Omega \setminus A \in \mathcal{L}}$$

$$A \in \mathcal{L} := \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma \iff A \in \mathcal{L}_\gamma, \forall \gamma \in \Gamma \implies \Omega \setminus A \in \mathcal{L}_\gamma, \forall \gamma \in \Gamma \implies \Omega \setminus A \in \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma =: \mathcal{L}$$

$$\underline{A_1, A_2, \dots \in \mathcal{L} \text{ and } A_i \cap A_j \text{ whenever } i \neq j \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{L}}$$

$$\begin{aligned} &A_1, A_2, \dots \in \mathcal{L} := \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma, \text{ and } A_i \cap A_j \text{ whenever } i \neq j \\ &\implies A_1, A_2, \dots \in \mathcal{L}_\gamma, \forall \gamma \in \Gamma, \text{ and } A_i \cap A_j \text{ whenever } i \neq j \\ &\implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{L}_\gamma, \forall \gamma \in \Gamma \\ &\implies \bigcup_{i=1}^{\infty} A_i \in \bigcap_{\gamma \in \Gamma} \mathcal{L}_\gamma =: \mathcal{L} \end{aligned}$$

□

**Theorem B.6** Suppose  $\Omega$  is a non-empty set,  $\mathcal{S}$  is non-empty collection of subsets of  $\Omega$ . Denote the power set of  $\Omega$  by  $\mathcal{P}(\Omega)$ . Define

$$\begin{aligned} \sigma(\mathcal{S}) &:= \bigcap_{\mathcal{A} \in \Sigma(\mathcal{S})} \mathcal{A}, \text{ where } \Sigma(\mathcal{S}) := \left\{ \mathcal{A} \subset \mathcal{P}(\Omega) \mid \begin{array}{l} \mathcal{A} \text{ is a } \sigma\text{-algebra of subsets of } \Omega, \\ \text{and } \mathcal{S} \subset \mathcal{A} \end{array} \right\}, \text{ and} \\ \lambda(\mathcal{S}) &:= \bigcap_{\mathcal{L} \in \Lambda(\mathcal{S})} \mathcal{L}, \text{ where } \Lambda(\mathcal{S}) := \left\{ \mathcal{L} \subset \mathcal{P}(\Omega) \mid \begin{array}{l} \mathcal{L} \text{ is a } \lambda\text{-system of subsets of } \Omega, \\ \text{and } \mathcal{S} \subset \mathcal{L} \end{array} \right\}. \end{aligned}$$

Then,  $\sigma(\mathcal{S})$  is the unique smallest  $\sigma$ -algebra of subsets of  $\Omega$  that contains  $\mathcal{S} \subset \mathcal{P}(\Omega)$ , and  $\lambda(\mathcal{S})$  is the unique smallest  $\lambda$ -system of subsets of  $\Omega$  that contains  $\mathcal{S} \subset \mathcal{P}(\Omega)$ . More precisely, we have

- $\mathcal{S} \subset \sigma(\mathcal{S})$ ,  $\mathcal{S} \subset \lambda(\mathcal{S})$ , and
- $\sigma(\mathcal{S})$  is a  $\sigma$ -algebra of subsets of  $\Omega$ , and  $\lambda(\mathcal{S})$  is a  $\lambda$ -system of subsets of  $\Omega$ , and
- if  $\mathcal{A} \subset \mathcal{P}(\Omega)$  is a  $\sigma$ -algebra and  $\mathcal{S} \subset \mathcal{A}$ , then  $\sigma(\mathcal{S}) \subset \mathcal{A}$ .
- if  $\mathcal{L} \subset \mathcal{P}(\Omega)$  is a  $\lambda$ -system and  $\mathcal{S} \subset \mathcal{L}$ , then  $\lambda(\mathcal{S}) \subset \mathcal{L}$ .

**PROOF** First, note that  $\Sigma(\mathcal{S}) \neq \emptyset$  since  $\mathcal{P}(\Omega) \in \Sigma(\mathcal{S})$ . Similarly,  $\Lambda(\mathcal{S}) \neq \emptyset$  since  $\mathcal{P}(\Omega) \in \Lambda(\mathcal{S})$ . It is immediate that  $\mathcal{S} \subset \sigma(\mathcal{S})$ , and  $\sigma(\mathcal{S})$  is contained in every  $\sigma$ -algebra which contains  $\mathcal{S}$ . Similarly,  $\mathcal{S} \subset \lambda(\mathcal{S})$ , and  $\lambda(\mathcal{S})$  is contained in every  $\lambda$ -system which contains  $\mathcal{S}$ . Since  $\sigma(\mathcal{S})$  is, by definition, an intersection of  $\sigma$ -algebras, it itself is a  $\sigma$ -algebra of subsets of  $\Omega$  by Theorem B.5. Similarly, since  $\lambda(\mathcal{S})$  is, by definition, an intersection of  $\lambda$ -systems, it itself is a  $\lambda$ -system of subsets of  $\Omega$  by Theorem B.5.  $\square$

**Theorem B.7** Suppose  $\Omega$  is a non-empty set and  $\mathcal{S}$  is a non-empty collection of subsets of  $\Omega$ . Then,

$$\mathcal{S} \text{ is closed under finite intersections} \implies \lambda(\mathcal{S}) \text{ is a } \sigma\text{-algebra of subsets of } \Omega,$$

where  $\lambda(\mathcal{S})$  is  $\lambda$ -system of subsets of  $\Omega$  generated by  $\mathcal{S}$ .

**PROOF** By Theorem B.4(ii), it suffices to show that  $\lambda(\mathcal{S})$  is closed under finite intersections. We establish the proof in the following series of claims:

**Claim 1:** For each  $A \in \lambda(\mathcal{S})$ ,

$$\mathcal{L}(A) := \{ B \subset \Omega \mid A \cap B \in \lambda(\mathcal{S}) \}$$

is a  $\lambda$ -system of subsets of  $\Omega$ .

**Proof of Claim 1:** Clearly,  $\Omega \in \mathcal{L}(A)$ , since  $A \cap \Omega = A \in \lambda(\mathcal{S})$ . Next, we prove that  $\mathcal{L}(A)$  is closed under complementations. Let  $B \in \mathcal{L}(A)$ . Then,  $A \cap B \in \lambda(\mathcal{S})$ . Note that  $A = (A \cap B) \sqcup (A \cap B^c)$ , hence  $A \cap B^c = A \setminus (A \cap B) \in \lambda(\mathcal{S})$ , since  $A, A \cap B \in \lambda(\mathcal{S})$  and  $\lambda(\mathcal{S})$  is closed under proper set-theoretic differences by Theorem B.4(i). This proves that  $\mathcal{L}(A)$  is indeed closed under complementations. We now prove that  $\mathcal{L}(A)$  is closed under countable disjoint unions. Let  $B_1, B_2, \dots \in \mathcal{L}(A)$  be pairwise disjoint. Then,  $A \cap B_1, A \cap B_2, \dots \in \lambda(\mathcal{S})$  are pairwise disjoint. Hence,

$$A \cap \left( \bigcup_{i=1}^{\infty} B_i \right) = \bigcup_{i=1}^{\infty} (A \cap B_i) \in \lambda(\mathcal{S}),$$

since  $\lambda(\mathcal{S})$  is closed under countable disjoint unions. This proves that  $\mathcal{L}(A)$  is a  $\lambda$ -system and thus completes the proof of the Claim 1.

**Claim 2:**  $\mathcal{S} \subset \mathcal{L}(A)$ , for each  $A \in \mathcal{S}$ . Consequently,  $\lambda(\mathcal{S}) \subset \mathcal{L}(A)$ , for each  $A \in \mathcal{S}$ .

**Proof of Claim 2:** Suppose  $A \in \mathcal{S}$ . Then,  $A \cap B \in \mathcal{S}$  for each  $B \in \mathcal{S}$ , by the hypothesis that  $\mathcal{S}$  is closed under finite intersections. Thus,  $A \cap B \in \lambda(\mathcal{S})$ , since  $\mathcal{S} \subset \lambda(\mathcal{S})$ . Hence,  $B \in \mathcal{L}(A)$ , for any  $A, B \in \mathcal{S}$ . This proves that  $\mathcal{S} \subset \mathcal{L}(A)$ , for each  $A \in \mathcal{S}$ . By Claim 1,  $\mathcal{L}(A)$  is a  $\lambda$ -system. Hence,  $\mathcal{L}(A) \supset \lambda(\mathcal{S})$ , the smallest  $\lambda$ -system containing  $\mathcal{S}$ . This proves Claim 2.

**Claim 3:**  $A \cap B \in \lambda(\mathcal{S})$ , for each  $A \in \mathcal{S}$  and  $B \in \lambda(\mathcal{S})$ .

**Proof of Claim 3:** Let  $A \in \mathcal{S}$  and  $B \in \lambda(\mathcal{S})$ . By Claim 2, we have  $\lambda(\mathcal{S}) \subset \mathcal{L}(A)$ . Thus we have  $B \in \mathcal{L}(A)$ , which is equivalent to  $A \cap B \in \lambda(\mathcal{S})$ . This proves Claim 3.

**Claim 4:**  $\mathcal{S} \subset \mathcal{L}(B)$ , for each  $B \in \lambda(\mathcal{S})$ . Consequently,  $\lambda(\mathcal{S}) \subset \mathcal{L}(B)$ , for each  $B \in \lambda(\mathcal{S})$ .

Proof of Claim 4: Suppose  $B \in \lambda(\mathcal{S})$ . Then,  $A \cap B \in \lambda(\mathcal{S})$  for each  $A \in \mathcal{S}$ , by Claim 3. This proves that  $\mathcal{S} \subset \mathcal{L}(B)$ . By Claim 1,  $\mathcal{L}(B)$  is a  $\lambda$ -system. Hence,  $\mathcal{L}(B) \supset \lambda(\mathcal{S})$ , the smallest  $\lambda$ -system containing  $\mathcal{S}$ . This proves Claim 4.

**Claim 5:**  $A \cap B \in \lambda(\mathcal{S})$ , for each  $A, B \in \lambda(\mathcal{S})$ .

Proof of Claim 5: Let  $A, B \in \lambda(\mathcal{S})$ . By Claim 4, we have  $\lambda(\mathcal{S}) \subset \mathcal{L}(B)$ . Thus we have  $A \in \mathcal{L}(B)$ , which is equivalent to  $A \cap B \in \lambda(\mathcal{S})$ . This proves Claim 5.

Claim 5 states precisely that  $\lambda(\mathcal{S})$  is closed under finite intersections, and completes the proof. □

**Corollary B.8** Suppose  $\Omega$  is a non-empty set and  $\mathcal{S}$  is a non-empty collection of subsets of  $\Omega$ . If  $\mathcal{S}$  is closed under finite intersections, then

- (i)  $\sigma(\mathcal{S}) \subset \lambda(\mathcal{S})$ , and
- (ii)  $\sigma(\mathcal{S}) \subset \mathcal{L}$ , for any  $\lambda$ -system  $\mathcal{L}$  of subsets of  $\Omega$  such that  $\mathcal{S} \subset \mathcal{L}$ ,

where  $\sigma(\mathcal{S})$  and  $\lambda(\mathcal{S})$  are, respectively, the  $\sigma$ -algebra and  $\lambda$ -system of subsets of  $\Omega$  generated by  $\mathcal{S}$ .

PROOF

- (i) By Theorem B.6,  $\lambda(\mathcal{S})$  is the smallest  $\lambda$ -system containing  $\mathcal{S}$ . Since  $\mathcal{S}$  is, by hypothesis, closed under finite intersections,  $\lambda(\mathcal{S})$  is furthermore a  $\sigma$ -algebra, by Theorem B.7. Thus, by Theorem B.6 again, we have  $\sigma(\mathcal{S}) \subset \lambda(\mathcal{S})$ .

- (ii) This is now immediate since

$$\sigma(\mathcal{S}) \subset \lambda(\mathcal{S}) \subset \mathcal{L},$$

where the first inclusion follows by (i), and the second inclusion follows by Theorem B.6. □

**Lemma B.9 (The pre-image of a  $\sigma$ -algebra is itself a  $\sigma$ -algebra.)**

Suppose  $\Omega$  is a non-empty set,  $(X, \mathcal{X})$  is a measurable space, and  $f : \Omega \rightarrow X$  is a map from  $\Omega$  into  $X$ . Then,

$$f^{-1}(\mathcal{X}) := \{ f^{-1}(V) \subset \Omega \mid V \in \mathcal{X} \}$$

is a  $\sigma$ -algebra of subsets of  $\Omega$ .

PROOF

$$\Omega \in f^{-1}(\mathcal{X}) \quad f(\Omega) \subset X \implies \Omega = f^{-1}(X) \in f^{-1}(\mathcal{X}).$$

$f^{-1}(\mathcal{X})$  is closed under complementations Let  $V \in \mathcal{X}$ . Then,  $X \setminus V \in \mathcal{X}$ , and

$$\Omega \setminus f^{-1}(V) = \{ \omega \in \Omega \mid f(\omega) \notin V \} = \{ \omega \in \Omega \mid f(\omega) \in X \setminus V \} = f^{-1}(X \setminus V) \in f^{-1}(\mathcal{X}),$$

which shows that  $f^{-1}(\mathcal{X})$  is indeed closed under complementations.

$f^{-1}(\mathcal{X})$  is closed countable unions Let  $V_1, V_2, \dots \in \mathcal{X}$ . Then,  $\bigcup_{i=1}^{\infty} V_i \in \mathcal{X}$ , and

$$\bigcup_{i=1}^{\infty} f^{-1}(V_i) = \left\{ \omega \in \Omega \mid \begin{array}{l} f(\omega) \in V_i \\ \text{for some } i \in \mathbb{N} \end{array} \right\} = f^{-1}\left(\bigcup_{i=1}^{\infty} V_i\right) \in f^{-1}(\mathcal{X}),$$

which proves that  $f^{-1}(\mathcal{X})$  is indeed closed under countable unions.

This concludes the proof that  $f^{-1}(\mathcal{X})$  is a  $\sigma$ -algebra of subsets of  $\Omega$ . □

## Lemma B.10

Suppose  $(\Omega, \mathcal{A})$  is a measurable space,  $X$  is a non-empty set, and  $f : \Omega \rightarrow X$  is a map from  $\Omega$  into  $X$ . Then,

$$\mathcal{F} := \{ V \subset X \mid f^{-1}(V) \in \mathcal{A} \}$$

is a  $\sigma$ -algebra of subsets of  $X$ .

PROOF

$$\underline{X \in \mathcal{F}} \quad f^{-1}(X) = \Omega \in \mathcal{A} \implies X \in \mathcal{F}.$$

$\mathcal{F}$  is closed under complementations  $V \in \mathcal{F} \implies f^{-1}(V) \in \mathcal{A} \implies f^{-1}(X \setminus V) = \Omega \setminus f^{-1}(V) \in \mathcal{A} \implies X \setminus V \in \mathcal{F}$ , which proves that  $\mathcal{F}$  is indeed closed under complementations.

$\mathcal{F}$  is closed under countable unions

$$\begin{aligned} V_1, V_2, \dots \in \mathcal{F} &\implies f^{-1}(V_1), f^{-1}(V_2), \dots \in \mathcal{A} \\ &\implies f^{-1}\left(\bigcup_{i=1}^{\infty} V_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(V_i) \in \mathcal{A} \\ &\implies \bigcup_{i=1}^{\infty} V_i \in \mathcal{F}, \end{aligned}$$

which proves that  $\mathcal{F}$  is indeed closed under countable unions. □

## Theorem B.11

Suppose  $(\Omega, \mathcal{A})$  and  $(X, \mathcal{X})$  are measurable spaces, and  $f : \Omega \rightarrow X$  is a map from  $\Omega$  into  $X$ . Then,  $f$  is  $(\mathcal{A}, \mathcal{X})$ -measurable if there exists  $\mathcal{S} \subset \mathcal{X}$  satisfying the following conditions:

- $\mathcal{S}$  generates  $\mathcal{X}$ , i.e.  $\sigma(\mathcal{S}) = \mathcal{X}$ , and
- $f^{-1}(\mathcal{S}) \subset \mathcal{A}$ .

PROOF By Lemma B.10,

$$\mathcal{F} := \{ V \subset X \mid f^{-1}(V) \in \mathcal{A} \}$$

is a  $\sigma$ -algebra of subsets of  $X$ . By hypothesis,  $\mathcal{S} \subset \mathcal{F}$ ; hence,  $\mathcal{X} = \sigma(\mathcal{S}) \subset \mathcal{F}$ . Thus,  $f^{-1}(\mathcal{X}) \subset \mathcal{A}$ ; equivalently,  $f$  is  $(\mathcal{A}, \mathcal{X})$ -measurable. □

## Corollary B.12 (Continuous maps are Borel measurable.)

Suppose  $X_1, X_2$  are topological spaces, and  $\mathcal{B}_1, \mathcal{B}_2$  are their respective Borel  $\sigma$ -algebras. Then, every continuous map  $f : X_1 \rightarrow X_2$  is  $(\mathcal{B}_1, \mathcal{B}_2)$ -measurable.

# C Topology

## Theorem C.1 (Appendix M3, [1])

Suppose  $S$  is a metric space. Then, the following conditions are equivalent:

- (i)  $S$  is separable.
- (ii) The topology of  $S$  has a countable basis.
- (iii) Every open cover of *each subset* of  $S$  has a countable subcover.

## References

- [1] BILLINGSLEY, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.
- [2] BILLINGSLEY, P. *Probability and Measure*, anniversary ed. John Wiley & Sons, 2012.