

## 1 Donsker's Theorem for $(C[0, 1], \|\cdot\|_\infty)$

### Proposition 1.1

- Let  $\xi_1, \xi_2, \dots : \Omega \longrightarrow \mathbb{R}$  be a sequence of independent and identically distributed  $\mathbb{R}$ -valued random variables defined on the probability space  $(\Omega, \mathcal{A}, \mu)$ , with expectation value zero and common finite variance  $\sigma^2 > 0$ .
- Define the random variables:

$$\begin{cases} S_0 & : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto 0, & \text{and} \\ S_n & : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto \sum_{i=1}^n \xi_i(\omega), & \text{for each } n \in \mathbb{N}. \end{cases}$$

- For each  $n \in \mathbb{N}$ , define  $X^{(n)} : \Omega \longrightarrow C[0, 1]$  as follows:

$$X^{(n)}(\omega)(t) := \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) + n \left( t - \frac{i-1}{n} \right) \xi_i(\omega) \right\}, \text{ for each } \omega \in \Omega, t \in \left[ \frac{i-1}{n}, \frac{i}{n} \right], i = 1, 2, 3, \dots, n.$$

- For each  $n \in \mathbb{N}$  and each  $t \in [0, 1]$ , define  $X_t^{(n)} : \Omega \longrightarrow \mathbb{R}$  as follows:

$$X_t^{(n)}(\omega) := X^{(n)}(\omega)(t), \text{ for each } \omega \in \Omega.$$

Then, the following statements are true:

- (i) For each  $\omega \in \Omega$  and each  $n \in \mathbb{N}$ ,

$$X^{(n)}(\omega) \left( \frac{i}{n} \right) = \frac{1}{\sigma \cdot \sqrt{n}} \cdot S_i(\omega), \text{ for } i = 0, 1, 2, \dots, n.$$

- (ii) For each  $\omega \in \Omega$  and each  $n \in \mathbb{N}$ ,

$$X^{(n)}(\omega)(t) \text{ is the linear interpolation from } \frac{1}{\sigma \cdot \sqrt{n}} S_{i-1}(\omega) \text{ to } \frac{1}{\sigma \cdot \sqrt{n}} S_i(\omega) \text{ over } t \in \left[ \frac{i-1}{n}, \frac{i}{n} \right],$$

where  $i = 1, 2, \dots, n$ .

- (iii) For any  $0 \leq t_0 < t_1 < t_2 < \dots < t_k \leq 1$ ,

$$\left( X_{t_1}^{(n)} - X_{t_0}^{(n)}, \dots, X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \right) \xrightarrow{d} N \left( \mu = \mathbf{0}, \Sigma = \text{diag}(t_1 - t_0, \dots, t_k - t_{k-1}) \right), \text{ as } n \longrightarrow \infty.$$

- (iv) For any  $0 \leq t_1, t_2, \dots, t_k \leq 1$ ,

$$\left( X_{t_1}^{(n)}, X_{t_2}^{(n)}, \dots, X_{t_k}^{(n)} \right) \xrightarrow{d} N \left( \mu = \mathbf{0}, \Sigma = \left[ \min\{t_i, t_j\} \right]_{1 \leq i, j \leq k} \right), \text{ as } n \longrightarrow \infty.$$

PROOF

# Donsker's Theorems (Functional Central Limit Theorems)

Study Notes

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- (i) Obvious.
- (ii) Obvious.
- (iii) First, note that, for each  $\omega \in \Omega$ ,  $n \in \mathbb{N}$ , and  $t \in [0, 1]$ , we have

$$X_t^{(n)}(\omega) = \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{[nt]}(\omega) + (nt - [nt]) \cdot \xi_{[nt]+1}(\omega) \right\},$$

where  $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$ , defined by

$$\lfloor x \rfloor := \max \left\{ k \in \mathbb{Z} \mid k \leq x \right\}, \quad \text{for each } x \in \mathbb{R},$$

is the round-down function. We next state three Claims, whose proofs will be given below. We note that the desired conclusion follows readily from Claim 3 and the Cramér-Wold Theorem (Theorem 1.9, p.56, [4]); hence the present proof is complete once we establish the three Claims below.

**Claim 1:** If  $\{a_n\}_{n \in \mathbb{N}}$  is a sequence of non-negative integers and  $\{b_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$  a sequence of positive integers satisfying:

$$a_n < b_n, \text{ for sufficiently large } n \in \mathbb{N}, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{b_n - a_n}{n} = c > 0,$$

then

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \xrightarrow{d} \sqrt{c} \cdot Z, \quad \text{where } Z \sim N(0, 1).$$

**Claim 2:** For each fixed  $t \in [0, 1]$ ,

$$W(t)_n := \frac{1}{\sigma \cdot \sqrt{n}} \cdot (nt - [nt]) \cdot \xi_{[nt]+1} \xrightarrow{d} 0.$$

**Claim 3:** For  $0 \leq t_0 < t_1 < t_2 < \dots < t_k \leq 1$ , and arbitrary  $c_1, c_2, \dots, c_k \in \mathbb{R}$ ,

$$\sum_{i=1}^k c_i \left( X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \xrightarrow{d} N \left( 0, \sum_{i=1}^k c_i^2 (t_i - t_{i-1}) \right), \quad \text{as } n \rightarrow \infty.$$

Proof of Claim 1: Note that, for sufficiently large  $n \in \mathbb{N}$ , we may write

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i = \frac{\sqrt{b_n - a_n}}{\sqrt{n}} \cdot \left( \frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \right).$$

Since, by hypothesis, that

$$\lim_{n \rightarrow \infty} \frac{b_n - a_n}{n} = c > 0,$$

Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [3]), once we establish the following:

$$\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \xrightarrow{d} N(0, 1), \quad \text{as } n \rightarrow \infty.$$

# Donsker's Theorems (Functional Central Limit Theorems)

We establish the above convergence by invoking the Lindeberg Central Limit Theorem (Theorem 1.15, §1.5.5, p.67, [4]). In the present context, the Lindeberg Condition is the following:

$$\lim_{n \rightarrow \infty} \frac{1}{B_n^2} \cdot E \left[ \sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\{|\xi_i| \geq \varepsilon S_n\}} \right] = 0, \quad \text{for each } \varepsilon > 0,$$

where

$$B_n^2 := \text{Var} \left[ \sum_{i=1+a_n}^{b_n} \xi_i \right] = (b_n - a_n) \sigma^2 > 0.$$

The last equality used the hypothesis that  $\xi_1, \xi_2, \dots$  are independent and identically distributed with common finite variance  $0 < \sigma^2 < \infty$ . Hence, for each  $\varepsilon > 0$ ,

$$\begin{aligned} \frac{1}{B_n^2} \cdot E \left[ \sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\{|\xi_i| \geq \varepsilon B_n\}} \right] &= \frac{1}{(b_n - a_n) \sigma^2} \cdot (b_n - a_n) \cdot E \left[ \xi_1^2 \cdot I_{\{|\xi_1| \geq \varepsilon \sigma \sqrt{b_n - a_n}\}} \right] \\ &= \frac{1}{\sigma^2} \cdot E \left[ \xi_1^2 \cdot I_{\{|\xi_1| / \sigma \geq \sqrt{b_n - a_n}\}} \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

since  $\lim_{n \rightarrow \infty} \sqrt{b_n - a_n} = \infty$  and  $\sigma^2 = E[\xi_1^2]$  is finite. This verifies that the Lindeberg Condition indeed holds in the present context, and completes the proof of Claim 1.

Proof of Claim 2: First, note that  $E[W(t)_n] = 0$ . We now argue that  $W(t)_n \xrightarrow{p} 0$ . To this end, let  $\varepsilon > 0$  be given. Then,

$$\begin{aligned} \varepsilon^2 \cdot P(|W(t)_n| \geq \varepsilon) &\leq E[W(t)_n^2 \cdot I_{\{|W(t)_n| \geq \varepsilon\}}] \\ &\leq E[W(t)_n^2] = \text{Var}(W(t)_n) = \text{Var} \left[ \frac{1}{\sigma \cdot \sqrt{n}} \cdot (nt - \lfloor nt \rfloor) \cdot \xi_{\lfloor nt \rfloor + 1} \right] \\ &= \frac{1}{n \cdot \sigma^2} \cdot (nt - \lfloor nt \rfloor)^2 \cdot \text{Var}(\xi_{\lfloor nt \rfloor + 1}) = \frac{1}{n \cdot \sigma^2} \cdot (nt - \lfloor nt \rfloor)^2 \cdot \sigma^2 \\ &\leq \frac{1}{n}, \end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} P(|W(t)_n| \geq \varepsilon) = 0, \quad \text{for each } \varepsilon > 0,$$

i.e.  $W(t)_n \xrightarrow{p} 0$ , as  $n \rightarrow \infty$  (Definition 2, Chapter 1, [3]), which is equivalent to  $W(t)_n \xrightarrow{d} 0$ , as  $n \rightarrow \infty$  (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [3]). This proves Claim 2.

Proof of Claim 3: Let  $0 \leq t_0 < t_1 < t_2 < \dots < t_k \leq 1$ , and  $c_1, c_2, \dots, c_k \in \mathbb{R}$  be arbitrary. Observe that:

$$\begin{aligned} &\sum_{i=1}^k c_i \left( X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \\ &= \sum_{i=1}^k \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ S_{\lfloor nt_i \rfloor} - S_{\lfloor nt_{i-1} \rfloor} \right\} + \sum_{i=1}^k \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ (nt_i - \lfloor nt_i \rfloor) \cdot \xi_{\lfloor nt_i \rfloor + 1} - (nt_{i-1} - \lfloor nt_{i-1} \rfloor) \cdot \xi_{\lfloor nt_{i-1} \rfloor + 1} \right\} \\ &= \sum_{i=1}^k \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \right\} + \sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \\ &= \sum_{i=1}^k c_i Y_i^{(n)} + \sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \end{aligned}$$

# Donsker's Theorems (Functional Central Limit Theorems)

By Claim 2 and Slutsky's Theorem (Corollary, p.40, [3]),

$$\sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \xrightarrow{d} 0, \text{ as } n \rightarrow \infty. \quad (1.1)$$

Next, note that since  $\xi_1, \xi_2, \xi_3, \dots$  are independent, we see that, for each fixed  $n \in \mathbb{N}$ ,

$$Y_i^{(n)} := \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j, \quad i = 1, 2, 3, \dots, k,$$

are independent. Now, since  $0 \leq t_{i-1} < t_i \leq 1$ , it follows that  $\lfloor nt_{i-1} \rfloor < \lfloor nt_i \rfloor$  for sufficiently large  $n \in \mathbb{N}$ . In addition,

$$\begin{aligned} \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} &= \frac{\lfloor nt_i \rfloor}{n} - \frac{\lfloor nt_{i-1} \rfloor}{n} = \left( \frac{nt_i}{n} + \frac{\lfloor nt_i \rfloor - nt_i}{n} \right) - \left( \frac{nt_{i-1}}{n} + \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right) \\ &= t_i - t_{i-1} + \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n}, \end{aligned}$$

which implies

$$\left| \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} - (t_i - t_{i-1}) \right| = \left| \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right| \leq \frac{2}{n} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} = t_i - t_{i-1} > 0.$$

Thus, by Claim 1, we see that, for each  $i = 1, 2, \dots, k$ ,

$$Y_i^{(n)} := \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \xrightarrow{d} \sqrt{t_i - t_{i-1}} \cdot N(0, 1) = N\left(0, t_i - t_{i-1}\right), \text{ as } n \rightarrow \infty. \quad (1.2)$$

By (1.1), (1.2), Proposition A.1, and Slutsky's Theorem (Corollary, p.40, [3]), we now see that

$$\sum_{i=1}^k c_i \left( X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) = \sum_{i=1}^k c_i Y_i^{(n)} + \sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \xrightarrow{d} N\left(0, \sum_{i=1}^k c_i^2 (t_i - t_{i-1})\right).$$

This completes the proof of Claim 3. □

## A Technical Lemmas

Note that  $X_n \xrightarrow{d} X$  and  $Y_n \xrightarrow{d} Y$  does NOT in general imply  $X_n + Y_n \xrightarrow{d} X + Y$ . But the implication does hold if  $X_n$  and  $Y_n$  are independent for each  $n \in \mathbb{N}$ , and both  $X$  and  $Y$  are Gaussian random variables, as the following Proposition shows.

**Proposition A.1** *Let  $k \in \mathbb{N}$  be fixed. Suppose:*

- For each  $n \in \mathbb{N}$ ,

$$Y_1^{(n)}, Y_2^{(n)}, \dots, Y_k^{(n)} : \Omega^{(n)} \longrightarrow \mathbb{R}$$

*are independent  $\mathbb{R}$ -valued random variables defined on the probability space  $\Omega^{(n)}$ .*

- For each  $i = 1, 2, \dots, k$ ,

$$Y_i^{(n)} \xrightarrow{d} N(\mu_i, \sigma_i^2), \quad \text{as } n \longrightarrow \infty.$$

Then, for any  $c_1, c_2, \dots, c_k \in \mathbb{R}$ ,

$$\sum_{i=1}^k c_i Y_i^{(n)} \xrightarrow{d} N\left(\sum_{i=1}^k c_i \mu_i, \sum_{i=1}^k c_i^2 \sigma_i^2\right), \quad \text{as } n \longrightarrow \infty.$$

**PROOF** Let  $Y^{(n)} := \sum_{i=1}^k c_i Y_i^{(n)}$ . Let  $\varphi_X$  denote the characteristic function of a  $\mathbb{R}$ -valued random variable  $X$ . Then,

$$\begin{aligned} \varphi_{Y^{(n)}}(t) &= \varphi_{\sum_{i=1}^k c_i Y_i^{(n)}}(t) \\ &= \prod_{i=1}^k \varphi_{c_i Y_i^{(n)}}(t), \quad \text{since } Y_1^{(n)}, \dots, Y_k^{(n)} \text{ are independent} \\ &= \prod_{i=1}^k \varphi_{Y_i^{(n)}}(c_i t) \\ &\longrightarrow \prod_{i=1}^k \exp\left\{\sqrt{-1} \mu_i (c_i t) - \frac{1}{2} \sigma_i^2 (c_i t)^2\right\} \\ &= \exp\left\{\sqrt{-1} \left(\sum_{i=1}^k c_i \mu_i\right) t - \frac{1}{2} \left(\sum_{i=1}^k c_i^2 \sigma_i^2\right) t^2\right\}, \quad \text{as } n \longrightarrow \infty, \end{aligned}$$

where the second and third equalities follow from the properties of characteristic functions of random variables (see p.21, [3]), while the expression of the limit follows from the fact that the characteristic function  $\varphi_Z$  of a random variable  $Z$  with distribution  $N(\mu, \sigma^2)$  is

$$\varphi_Z = \exp\left\{\sqrt{-1} \mu t - \frac{1}{2} \sigma^2 t^2\right\}.$$

The Proposition now follows immediately from the Lévy-Cramér Continuity Theorem (Theorem 1.9(ii), p.56, [4]).  $\square$

## References

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- [4] SHAO, J. *Mathematical Statistics*, second ed. Springer, 2003.