Donsker's Theorems (Functional Central Limit Theorems)

1 Donsker's Theorem for $(\,C[0,1]\,,\,\|\cdot\|_\infty\,)$

Proposition 1.1

- *Let $\xi_1, \xi_2, \ldots : \Omega \longrightarrow \mathbb{R}$ be a sequence of independent and identically distributed $\mathbb{R}$-valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mu)$, with expectation value zero and common finite variance $\sigma^2 > 0$.*
- *Define the random variables:*

$$\begin{cases} S_0 \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto 0, & \text{and} \\ S_n \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto \displaystyle\sum_{i=1}^{n} \xi_i(\omega), & \text{for each } n \in \mathbb{N}. \end{cases}$$

- *For each $n \in \mathbb{N}$, define $X^{(n)} : \Omega \longrightarrow C[0,1]$ as follows:*

$$X^{(n)}(\omega)(t) \;:=\; \frac{1}{\sigma\cdot\sqrt{n}}\left\{ S_{i-1}(\omega) \;+\; n\left(t - \frac{i-1}{n}\right)\xi_i(\omega)\right\}, \;\text{for each } \omega\in\Omega,\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],\; i = 1,2,3,\ldots,n.$$

- *For each $n \in \mathbb{N}$ and each $t \in [0,1]$, define $X_t^{(n)} : \Omega \longrightarrow \mathbb{R}$ as follows:*

$$X_t^{(n)}(\omega) \;:=\; X^{(n)}(\omega)(t), \quad \text{for each } \omega \in \Omega.$$

Then, the following statements are true:

(i) *For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,*

$$X^{(n)}(\omega)\left(\frac{i}{n}\right) \;=\; \frac{1}{\sigma\cdot\sqrt{n}}\cdot S_i(\omega), \quad \text{for } i = 0,1,2,\ldots,n.$$

(ii) *For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,*

$$X^{(n)}(\omega)(t) \;\text{ is the linear interpolation from }\; \frac{1}{\sigma\cdot\sqrt{n}}S_{i-1}(\omega) \;\text{ to }\; \frac{1}{\sigma\cdot\sqrt{n}}S_i(\omega) \;\text{ over }\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],$$

where $i = 1,2,\ldots,n$.

(iii) *For each $t \in [0,1]$,*

$$X_t^{(n)} \;\overset{d}{\longrightarrow}\; \sqrt{t}\cdot N(0,1), \;\text{ as }\; n \longrightarrow \infty.$$

(iv) *For any $0 \leq t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq 1$,*

$$\left(\,X_{t_1}^{(n)} - X_{t_0}^{(n)},\;\ldots\;,\;X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)}\,\right) \;\overset{d}{\longrightarrow}\; N\left(\,\mu = \mathbf{0}\,,\,\Sigma = \mathrm{diag}(\,t_1 - t_0,\;\ldots\;,\;t_k - t_{k-1}\,)\,\right), \;\text{ as }\; n \longrightarrow \infty.$$

(v) *For any $0 \;\leq\; t_1,\; t_2,\; \cdots\;,\; t_k \leq 1$,*

$$\left(\,X_{t_1}^{(n)},\;X_{t_2}^{(n)},\;\ldots\;,\;X_{t_k}^{(n)}\,\right) \;\overset{d}{\longrightarrow}\; N\left(\,\mu = \mathbf{0}\,,\,\Sigma = \Big[\,\min\{t_i, t_j\}\,\Big]_{1\leq i,j\leq k}\,\right), \;\text{ as }\; n \longrightarrow \infty.$$

Proof

(i) Obvious.

(ii) Obvious.

(iii) The statement holds trivially for $t = 0$. We prove the statement for $t \in (0,1]$. Now, for each $t \in (0,1]$, note that

$$X_t^{(n)}(\omega) \;=\; \frac{1}{\sigma\cdot\sqrt{n}}\left\{\,S_{\lfloor nt\rfloor}(\omega) \;+\; \left(nt - \lfloor nt\rfloor\right)\cdot\xi_{\lfloor nt\rfloor+1}(\omega)\,\right\},$$

where $\lfloor\cdot\rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$, defined by

$$\lfloor x\rfloor \;:=\; \max\left\{k\in\mathbb{Z}\;\middle|\;k\leq x\right\},\quad \text{for each } x\in\mathbb{R},$$

is the round-down function.

Claim 1: For each fixed $t \in (0,1]$,

$$\frac{1}{\sigma\cdot\sqrt{n}}\cdot S_{\lfloor nt\rfloor} \;\xrightarrow{d}\; \sqrt{t}\cdot Z,\quad \text{where } Z \sim N(0,1).$$

Claim 2: For each fixed $t \in (0,1]$,

$$B_n \;:=\; \frac{1}{\sigma\cdot\sqrt{n}}\cdot\left(nt-\lfloor nt\rfloor\right)\cdot\xi_{\lfloor nt\rfloor+1} \;\xrightarrow{d}\; 0.$$

The desired statement now follows by Slutsky's Theorem (Corollary, p.40, [3]).

Proof of Claim 1: Note that

$$\frac{1}{\sigma\cdot\sqrt{n}}\cdot S_{\lfloor nt\rfloor} \;=\; \frac{\sqrt{\lfloor nt\rfloor}}{\sqrt{n}}\left(\frac{1}{\sigma\cdot\sqrt{\lfloor nt\rfloor}}\cdot S_{\lfloor nt\rfloor}\right),$$

and

$$\frac{\sqrt{\lfloor nt\rfloor}}{\sqrt{n}} \;\longrightarrow\; \sqrt{t},\quad \text{as } n\longrightarrow\infty.$$

Hence, Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [3]), once we establish the following:

$$\frac{1}{\sigma\cdot\sqrt{\lfloor nt\rfloor}}\cdot S_{\lfloor nt\rfloor} \;\xrightarrow{d}\; N(0,1),\quad \text{as } n\longrightarrow\infty.$$

By Theorem 2.6, p.20, [2], it suffices to show that:

$$\begin{array}{l}\text{Every subsequence } \{A_{n_i}\}_{i\in\mathbb{N}} \text{ of } \left\{A_n \;:=\; \dfrac{1}{\sigma\cdot\sqrt{\lfloor nt\rfloor}}\cdot S_{\lfloor nt\rfloor}\right\}_{n\in\mathbb{N}} \text{ contains a further} \\ \text{subsequence } \left\{A_{n_{i_k}}\right\}_{j\in\mathbb{N}} \text{ that converges in distribution to } N(0,1), \text{ as } k\longrightarrow\infty.\end{array} \tag{1.1}$$

To this end, first recall that by the Central Limit Theorem,

$$\frac{1}{\sigma\cdot\sqrt{m}}\cdot S_m \;\xrightarrow{d}\; N(0,1),\quad \text{as } m\longrightarrow\infty.$$

By Theorem 2.6, p.20, [2], this is equivalent to:

$$\text{Every subsequence of } \left\{\, \frac{1}{\sigma\cdot\sqrt{m}}\cdot S_m \,\right\}_{m\in\mathbb{N}} \text{ contains a further subsequence which converges in distribution to } N(0,1). \tag{1.2}$$

Next, note that, for each fixed $t\in(0,1]$, $\{\,\lfloor nt\rfloor\,\}_{n\in\mathbb{N}}$ is a sequence of positive integers non-decreasing in $n\in\mathbb{N}$ and satisfying $\lim\limits_{n\to\infty}\lfloor nt\rfloor=\infty$. Thus, $\{\,\lfloor nt\rfloor\,\}_{n\in\mathbb{N}}$ is a subsequence of $\mathbb{N}=\{\,1,2,3,\ldots\}$. Hence, for every subsequence $\{\,n_i\,\}_{i\in\mathbb{N}}$ of $\mathbb{N}=\{\,1,2,3,\ldots\}$, $\{\,\lfloor n_i\cdot t\rfloor\,\}_{i\in\mathbb{N}}$ is itself a subsequence of $\mathbb{N}=\{\,1,2,3,\ldots\}$. Therefore, by (1.2), $\left\{\, A_{n_i} \;:=\; \dfrac{1}{\sigma\cdot\sqrt{\lfloor n_i\cdot t\rfloor}}\cdot S_{\lfloor n_i\cdot t\rfloor}\,\right\}_{i\in\mathbb{N}}$ contains a further subsequence which converges in distribution to $N(0,1)$; in other words, (1.1) holds. This proves Claim 1.

<u>Proof of Claim 2:</u> First, note that $E[\,B_n\,]=0$. We now argue that $B_n\xrightarrow{p}0$. To this end, let $\varepsilon>0$ be given. Then,

$$\begin{aligned}
\varepsilon^2\cdot P(\,|\,B_n\,|\,\geq\,\varepsilon\,) &\leq E\big[\,B_n^2\cdot I_{\{\,|\,B_n\,|\,\geq\,\varepsilon\,\}}\,\big] \\
&\leq E\big[\,B_n^2\,\big] \;=\; \mathrm{Var}(\,B_n\,) \;=\; \mathrm{Var}\left[\,\frac{1}{\sigma\cdot\sqrt{n}}\cdot\Big(nt-\lfloor nt\rfloor\Big)\cdot\xi_{\lfloor nt\rfloor+1}\,\right] \\
&= \frac{1}{n\cdot\sigma^2}\cdot\Big(nt-\lfloor\,nt\,\rfloor\Big)^2\cdot\mathrm{Var}\big(\,\xi_{\lfloor nt\rfloor+1}\,\big) \;=\; \frac{1}{n\cdot\sigma^2}\cdot\Big(nt-\lfloor\,nt\,\rfloor\Big)^2\cdot\sigma^2 \\
&\leq \frac{1}{n},
\end{aligned}$$

which implies

$$\lim_{n\to\infty} P(\,|\,B_n\,|\,\geq\,\varepsilon\,)\;=\;0,\quad\text{for each }\varepsilon>0,$$

i.e. $B_n\xrightarrow{p}0$, as $n\longrightarrow\infty$ (Definition 2, Chapter 1, [3]), which is equivalent to $B_n\xrightarrow{d}0$, as $n\longrightarrow\infty$ (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [3]). This proves Claim 2.

First, recall that, by the Central Limit Theorem, we have:

$$\frac{1}{\sigma\sqrt{n}}\cdot S_n \;\xrightarrow{d}\; N(0,1),\quad\text{as } n\longrightarrow\infty.$$

By Theorem 2.6, [2], the above convergence is equivalent to:

For each subsequence $\{\,n_i\}_{i\in\mathbb{N}}$ of $\{\,1,2,\ldots\}$, there exists a further subsequence $\big\{\,n_{i(k)}\big\}_{k\in\mathbb{N}}$ such that

$$\frac{1}{\sigma\sqrt{n_{i(k)}}}\cdot S_{n_{i(k)}} \;\xrightarrow{d}\; N(0,1),\quad\text{as } k\longrightarrow\infty.$$

Note that $\lfloor\cdot\rfloor$ is non-decreasing over all of $\mathbb{R}$. Hence, for each fixed $t\in(0,1]$, $\{\,\lfloor\,nt\,\rfloor\,\}_{n\in\mathbb{N}}$ is a non-decreasing sequence of non-negative integers satisfying $\lfloor\,nt\,\rfloor\longrightarrow\infty$ as $n\longrightarrow\infty$.

□

Remark 1.2
By the Central Limit Theorem,

$$X_t^{(n)}$$

A Technical Lemmas

Definition A.1
Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω. An ***outer measure*** *on Ω is a function $\varphi : \mathcal{P}(\Omega) \longrightarrow [0,\infty]$ satisfying the following conditions:*

- $\varphi(\,\varnothing\,) = 0$.
- *monotonicity: $\varphi(A) \;\leq\; \varphi(B)$, for every $A, B \in \mathcal{P}(\Omega)$ with $A \subset B$.*
- *countable sub-additivity:*

$$\varphi\left(\bigcap_{i=1}^{\infty} A_i\right) \;\leq\; \sum_{i=1}^{\infty} \varphi(\,A_i\,), \quad \text{for any } A_1, A_2, \ldots \in \mathcal{P}(\Omega).$$

Definition A.2
Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω. Let $\varphi : \mathcal{P}(\Omega) \longrightarrow [0,\infty]$ be an outer measure on Ω. A subset $A \subset \Omega$ is said to be φ-measurable if

$$\varphi(E) \;=\; \varphi(\,A \cap E\,) \;+\; \varphi(\,A^c \cap E\,), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

Theorem A.3
Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω. Let $\varphi : \mathcal{P}(\Omega) \longrightarrow [0,\infty]$ be an outer measure on Ω.

(i) *A subset $A \subset \Omega$ is φ-measurable if and only if*

$$\varphi(E) \;\geq\; \varphi(\,A \cap E\,) \;+\; \varphi(\,A^c \cap E\,), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

(ii) *The collection $\mathcal{A}(\varphi)$ of φ-measurable subsets of Ω forms a σ-algebra of subsets of Ω.*

(iii) *The restriction $\varphi\,|_{\mathcal{A}(\varphi)}$ of the outer measure φ to the σ-algebra $\mathcal{A}(\varphi)$ is a (countably additive) complete measure on the measurable space $(\,\Omega, \mathcal{A}(\varphi)\,)$.*

Lemma A.4

Every compact subset of a metric space is also a separable subset of that metric space.

Proof Let (X, ρ) be a metric space and $K \subset X$ be a compact subset of X. For each $x \in X$ and positive $r > 0$, let

$$B(x,r) \;\;:=\;\; \{\, y \in X \mid \rho(x,y) < r \,\} \;\;\subset\;\; X,$$

i.e. $B(x,r)$ is the open ball in X centred at x with radius $r > 0$. For each $n \in \mathbb{N}$, the following forms an open cover of K:

$$\mathcal{C}_n \;\;:=\;\; \left\{\; B\!\left(x, \frac{1}{n}\right) \;\subset\; X \;\middle|\; x \in K \;\right\}.$$

Since K is compact, each $\mathcal{C}_n$ admits a finite subcover:

$$\mathcal{F}_n \;\;:=\;\; \left\{\; B\!\left(x_i^{(n)}, \frac{1}{n}\right) \;\subset\; X \;\middle|\; x_i^{(n)} \in K,\;\; i = 1, 2, \ldots, J_n \;\right\}.$$

Let

$$\mathcal{D}_n \;\;:=\;\; \left\{\; x_i^{(n)} \in\; K \;\middle|\; i = 1, 2, \ldots, J_n \;\right\} \;\;\subset\;\; K,$$

and let $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset K$. We claim that $\mathcal{D}$ is dense in K. Indeed, let $y \in K$. Since each $\mathcal{F}_n$ is a (finite) open cover of K, we have:

$$y \;\in\; K \;\subset\; \bigcup_{i=1}^{J_n} B\left(x_i^{(n)}, \frac{1}{n}\right), \quad \text{for each } n \in \mathbb{N}.$$

Since $x_i^{(n)} \in \mathcal{D}$, for each $i = 1, 2, \ldots, J_n$ and for each $n \in \mathbb{N}$, the above inclusion shows that, for each $n \in \mathbb{N}$, there exists some $x \in \mathcal{D}$ such that $\rho(y, x) < \dfrac{1}{n}$. In particular, $\mathcal{D}$ contains a sequence that converges to $y \in K$. Since $y \in K$ is an arbitrary element of K, we see that $\overline{D} \supset K$. Since $\mathcal{D} \subset K$ and K is compact, hence closed, we trivially have $\overline{D} \subset K$. We may now conclude that $\overline{D} = K$. This completes the proof of the Lemma. □

Lemma A.5

Every countable union of separable subsets of a metric space is itself a separable subset of that metric space.

Proof Let $S := \bigcup_{i=1}^{\infty} S_i \subset X$ be a countable union of separable subsets S_i of a metric space X. For each fixed $i \in \mathbb{N}$, since S_i is separable, there exists countable $D_i \subset S_i$ which is dense in S_i. Let $D := \bigcup_{i=1}^{\infty} D_i$. Then, D is a countable subset of S. The Lemma is proved once we establish that D is dense in S. To this end, let $x \in S = \bigcup_{i=1}^{\infty} S_i$. Then, $x \in S_i$ for some $i \in \mathbb{N}$. Since D_i is dense in S_i, there exists a sequence $\{\, y_k \,\} \subset D_i \subset D$ such that $y_k \longrightarrow x$, as $k \longrightarrow \infty$. This proves that D is indeed dense in S, and completes the proof of the Lemma. □

Lemma A.6 (second theorem in Appendix M3, [2])

Let (S, ρ) be a metric space and $\Sigma \subset S$ a separable subset of S. Then, there exists a countable collection $\mathcal{A}$ of open subsets of S satisfying the following property: For each $x \in S$ and each open subset G of S,

$$x \;\in\; G \bigcap \Sigma \quad\Longrightarrow\quad x \;\in\; A \;\subset\; \overline{A} \;\subset\; G, \;\text{ for some } A \in \mathcal{A}.$$

Proof Let $D \subset \Sigma$ be a countable dense subset of Σ. Let

$$\mathcal{A} \;:=\; \left\{\; B(d, r) \;\subset\; S \;\;\middle|\;\; \begin{array}{c} d \in D, \\ r \in \mathbb{Q},\; r > 0 \end{array} \right\}.$$

Then, $\mathcal{A}$ is a countable collection of open balls in S. Now, let $G \subset S$ be an arbitrary open subset of S and $x \in G \bigcap \Sigma$. First, choose $\varepsilon > 0$ such that $B(x, \varepsilon) \subset G$. Next, since $x \in \Sigma$ and D is dense in Σ, we may choose $d \in D$ such that $d \in B(x, \varepsilon/2)$, or equivalently $\rho(x, d) < \varepsilon/2$. Finally choose positive rational $r > 0$ such that $\rho(x, d) < r < \varepsilon/2$.

Now, note that $\overline{B(d, r)} \subset B(x, \varepsilon)$; indeed,

$$y \in \overline{B(d,r)} \quad\Longleftrightarrow\quad \rho(y, d) \leq r \quad\Longrightarrow\quad \rho(x, y) \;\leq\; \rho(x, d) + \rho(d, y) \;<\; \varepsilon/2 + r \;<\; \varepsilon/2 + \varepsilon/2 \quad\Longrightarrow\quad y \in B(x, \varepsilon).$$

Thus, we have

$$x \;\in\; B(d, r) \;\subset\; \overline{B(d, r)} \;\subset\; B(x, \varepsilon) \;\subset\; G.$$

This completes the proof of the Lemma. □

Theorem A.7 (The Diagonal Method, Appendix A.14, [1])

Suppose that each row of the array

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

is a bounded sequence of real numbers. Then, there exists an increasing sequence

$$n_1 \,<\, n_2 \,<\, n_3 \,<\cdots \,\in\, \mathbb{N}$$

of positive integers such that the limit

$$\lim_{k\to\infty} x_{r,n_k} \;\text{exists,}\quad \text{for each } r = 1,2,3,\ldots$$

PROOF From the first row, select a convergent subsequence

$$x_{1,n(1,1)},\;\; x_{1,n(1,2)},\;\; x_{1,n(1,3)},\;\; \cdots\cdots$$

Here, we have $n(1,1) < n(1,2) < n(1,3) < \cdots \in \mathbb{N}$, and $\lim\limits_{k\to\infty} x_{1,n(1,k)} \in \mathbb{R}$ exists. Next, note that the following subsequence of the second row:

$$x_{2,n(1,1)},\;\; x_{2,n(1,2)},\;\; x_{2,n(1,3)},\;\; \cdots\cdots$$

is still a bounded sequence of real numbers, and we may thus select a convergent subsequence:

$$x_{2,n(2,1)},\;\; x_{2,n(2,2)},\;\; x_{2,n(2,3)},\;\; \cdots\cdots$$

Here, we have $n(2,1) < n(2,2) < n(2,3) < \cdots \in \{\, n(1,k)\,\}_{k\in\mathbb{N}}$, and $\lim\limits_{k\to\infty} x_{2,n(2,k)} \in \mathbb{R}$ exists. Continuing inductively, we obtain an array of positive integers

$$\begin{array}{cccc} n(1,1) & n(1,2) & n(1,3) & \cdots \\ n(2,1) & n(2,2) & n(2,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

which satisfies: For each $r \in \mathbb{N}$, we have

- each row is an increasing sequence of positive integers, i.e. $n(r,1) < n(r,2) < n(r,3) < \cdots$,
- the $(r+1)^{\text{th}}$ row is a subsequence of the r^{th} row, i.e. $\{\, n(r+1,k)\,\}_{k\in\mathbb{N}} \subset \{\, n(r,k)\,\}_{k\in\mathbb{N}}$, and
- $\lim\limits_{k\to\infty} x_{r,n(r,k)} \in \mathbb{R}$ exists.

Note that the first two properties together imply:

$$n(k,k) \;\;<\;\; n(k,k+1) \;\;\leq\;\; n(k+1,k+1), \quad \text{for each } k \in \mathbb{N}.$$

Now, define $n_k := n(k,k)$, for $k \in \mathbb{N}$. We then see that

$$n_k \;\;:=\;\; n(k,k) \;\;< n(k+1,k+1) \;\;=:\;\; n_{k+1},$$

i.e., $\{\, n_k \,\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. Lastly, for each $r \in \mathbb{N}$, consider the sequence

$$x_{r,n_1}\,,\;\; x_{r,n_2}\,,\;\; x_{r,n_3}\,,\;\; \cdots$$

Note that, for each $r \in \mathbb{N}$,

$$x_{r,n_r}\,,\;\; x_{r,n_{r+1}}\,,\;\; x_{r,n_{r+2}}\,,\;\; \cdots$$

is a subsequence of $\{\, x_{r,n(r,k)}\,\}_{k\in\mathbb{N}}$. We saw above that $\lim\limits_{k\to\infty} x_{r,n(r,k)}$ exists, which in turn implies that $\lim\limits_{k\to\infty} x_{r,n_k}$ exists. Since $r \in \mathbb{N}$ is arbitrary, the proof of the Theorem is now complete. □

References

[1] Billingsley, P. *Probability and Measure*, third ed. John Wiley & Sons, 1995.

[2] Billingsley, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.

[3] Ferguson, T. S. *A Course in Large Sample Theory*, first ed. Texts in Statistical Science. CRC Press, 1996.