1 The Prokhorov Theorem

Definition 1.1 (Tightness and weak sequential compactness)

Suppose:

- (S,ρ) *is a metric space,* $\mathcal{B}(S)$ *its the Borel* σ*-algebra,* $(S,\mathcal{B}(S))$ *is the corresponding measurable space.*
- $\Pi \subset \mathcal{M}_1(S,\mathcal{B}(S))$ *is a family of probability measures on* $(S,\mathcal{B}(S))$*.*

The family Π *is said to be:*

(i) ***tight*** *if, for each* $\varepsilon > 0$*, there exists a compact subset* $K_\varepsilon \subset S$ *such that*

$$1-\epsilon \;<\; P(\,K_\varepsilon\,) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

(ii) ***weakly sequentially compact*** *if, for every sequence* $\{\,P_n\,\}_{n\in\mathbb{N}} \subset \Pi$*, there exists a probability measure* $P \in \mathcal{M}_1(S,\mathcal{B}(S))$ *and subsequence* $\{\,P_{n(i)}\,\}_{i\in\mathbb{N}}$ *such that*

$$P_{n(i)} \;\overset{w}{\longrightarrow}\; P, \quad \text{as } i \longrightarrow \infty.$$

Theorem 1.2 (The Prokhorov Theorem, Theorems 5.1 & 5.2, [2])

Suppose:

- (S,ρ) *is a metric space,* $\mathcal{B}(S)$ *its the Borel* σ*-algebra,* $(S,\mathcal{B}(S))$ *is the corresponding measurable space.*
- $\Pi \subset \mathcal{M}_1(S,\mathcal{B}(S))$ *is a collection of probability measures on* $(S,\mathcal{B}(S))$*.*

Then, the following statements hold:

(i) *Tightness of* Π *implies weak sequential compactness of* Π*.*

(ii) *Suppose further that* (S,ρ) *is complete and separable.*
Then, weak sequential compactness of Π *implies tightness of* Π*.*

Proof We first prove statement (ii), then statement (i).

Proof of (ii)

Suppose S is complete and separable. Let $\varepsilon > 0$ be fixed. We need to find a compact subset $K \subset S$ such that

$$1-\varepsilon \;<\; P(\,K\,) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Now, separability of S implies that every open cover of every subset of S admits a countable subcover (Appendix M3, [2]). Denote by $B(x,r) \subset S$ the open ball in S centred at $x \in S$ of radius $r > 0$. For each $k \in \mathbb{N}$, the open cover

$$\left\{\, B\!\left(x, \frac{1}{k}\right) \right\}_{x\in S}$$

of S admits a countable subcover, say,

$$\{\,A_{ki}\,\}_{i\in\mathbb{N}} \;\subset\; \left\{\, B\!\left(x, \frac{1}{k}\right) \right\}_{x\in S}.$$

Let $G_{kn} := \bigcup_{i=1}^{n} A_{ki}$. Then, each G_{kn} is an open subset of S and $G_{kn} \uparrow S$, as $n \longrightarrow \infty$. Hence, by the Claim below, there exists $n_k \in \mathbb{N}$ such that

$$1 - \frac{\varepsilon}{2^k} \;<\; P\left(\bigcup_{i=1}^{n_k} A_{ki} \right) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Now, let

$$K \;:=\; \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}}$$

Note that K, being a closed subset of the complete metric space S, is itself complete. Note also that the set $\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}$ is totally bounded; hence so is its closure K. Being complete and totally bounded, K is therefore compact (Appendix M5, [2]). It now remains only to show that $1-\varepsilon < P(K) \leq 1$, for each $P \in \Pi$; or equivalently, that $P(K^c) \leq \varepsilon$, for each $P \in \Pi$. To this end, write $B_k := \bigcup_{i=1}^{n_k} A_{ki}$. Then,

$$1 - \frac{\varepsilon}{2^k} \;<\; P(\,B_k\,) \;\leq\; 1\,; \quad \text{equivalently,} \quad P(\,B_k^c\,) \;\leq\; \frac{\varepsilon}{2^k}.$$

Also,

$$K \;:=\; \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}} \;:=\; \overline{\bigcap_{k=1}^{\infty} B_k} \;\supset\; \bigcap_{k=1}^{\infty} B_k.$$

Hence,

$$K^c \;\subset\; \left(\bigcap_{k=1}^{\infty} B_k \right)^c \;=\; \bigcup_{k=1}^{\infty} B_k^c,$$

which implies:

$$P(\,K^c\,) \;\leq\; \sum_{k=1}^{\infty} P(B_k^c) \;\leq\; \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} \;=\; \varepsilon.$$

Thus, the proof of (ii) will be complete once we prove the following:

> **Claim:** Let $\{\,G_n\,\}_{n\in\mathbb{N}}$ be a sequence of open subsets of S with $G_n \uparrow S$. Then, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that
>
> $$1 - \varepsilon \;<\; P(\,G_{n_\varepsilon}\,) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Proof of Claim: Suppose the Claim is false, and we derive a contradiction. The failure of the Claim implies that there exists some $0 < \varepsilon < 1$ such that for each $n \in \mathbb{N}$, there exists $P_n \in \Pi$ such that

$$P_n(\,G_n\,) \;\leq\; 1 - \varepsilon.$$

By the hypothesis of weak sequential compactness of Π, there exists some probability measure $Q \in \mathcal{M}_1(S, \mathcal{B}(S))$ and the subsequence $\{\,P_{n(i)}\,\}$ of $\{\,P_n\,\}$ such that $P_{n(i)} \overset{w}{\longrightarrow} Q$, as $i \longrightarrow \infty$. Now, for each fixed $n \in \mathbb{N}$, we have:

$$\begin{aligned}
Q(\,G_n\,) \;&\leq\; \liminf_{i\to\infty} P_{n(i)}(\,G_n\,), \quad \text{by the Portmanteau Theorem} \\
&\leq\; \liminf_{i\to\infty} P_{n(i)}\big(\,G_{n(i)}\,\big), \quad \text{since } \{\,G_n\,\} \text{ is increasing} \\
&\leq\; 1 - \varepsilon, \quad \text{by choice of } P_n
\end{aligned}$$

But, by hypothesis, we also have $G_n \uparrow S$. Hence, we therefore have:

$$1 \;=\; Q(S) \;=\; \lim_{n\to\infty} Q(G_n) \;\leq\; 1 - \varepsilon,$$

which is the desired contradiction. This completes the proof of the Claim, hence that of (ii).

Proof of (i)

Suppose $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is tight. We need to establish that Π is weakly sequentially compact. In other words, if $\{ P_n \} \subset \Pi$ is a sequence of probability measures contained in Π, we need to establish that there exists a Borel probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ and a subsequence $\left\{ P_{n(i)} \right\} \subset \{ P_n \}$ such that $P_{n(i)} \overset{w}{\longrightarrow} P$, as $i \longrightarrow \infty$.

So, let $\{ P_n \} \subset \Pi$. We prove the Theorem by establishing the following series of Claims. Note that the proof of the Theorem is complete once we establish Claim 5.

Claim 1: There exists an increasing sequence of compact subsets $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ such that

$$1 - \frac{1}{m} \; < \; P_n(K_m) \; \leq \; 1, \quad \text{for every } \; m, n \, \in \, \mathbb{N}.$$

Claim 2: Let $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ be one such sequence of compact subsets of S as in Claim 1. Then, $\bigcup_{m=1}^{\infty} K_m$ is a separable subset of S, and there exists a countable collection $\mathcal{A}$ of open subsets of S satisfying the following property: For each $x \in S$ and for each open subset G of S,

$$x \; \in \; G \bigcap \left(\bigcup_{m=1}^{\infty} K_m \right) \quad \Longrightarrow \quad x \; \in \; A \; \subset \; \overline{A} \; \subset \; G, \;\; \text{for some } A \in \mathcal{A}.$$

Claim 3: Define:

$$\mathcal{H} \;\; := \;\; \{ \varnothing \} \bigcup \left\{ \begin{array}{c} \text{all finite unions of sets of the form} \\ \overline{A} \cap K_m, \;\text{where } A \in \mathcal{A} \text{ and } m \in \mathbb{N} \end{array} \right\}.$$

Then, there exists a subsequence $\{ P_{n(i)} \} \subset \{ P_n \}$ such that the limit

$$\alpha(H) \; := \; \lim_{i \to \infty} P_{n(i)}(H) \;\; \text{exists}, \;\; \text{for each } H \in \mathcal{H}.$$

Claim 4: There exists a Borel probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ such that

$$P(G) \;\; := \;\; \sup_{H \subset G} \alpha(H), \quad \text{for each open subset } G \subset S.$$

Claim 5: $P_{n(i)} \overset{w}{\longrightarrow} P$, as $i \longrightarrow \infty$.

Proof of Claim 1: By tightness hypothesis on Π, for each $m \in \mathbb{N}$, there exists a compact subset $L_m \subset S$ such that

$$1 - \frac{1}{m} \; < \; P(L_m) \; \leq \; 1, \quad \text{for each } P \in \Pi.$$

Define, for each $m \in \mathbb{N}$, $K_m \; := \; \bigcup_{i=1}^{m} L_i$. Then, each K_m is compact (since finite unions of compact subsets are themselves compact). Next, we trivially have $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$. Also,

$$P(K_m) \;\; = \;\; P\left(\bigcup_{i=1}^{m} L_i \right) \;\; \geq \;\; L_m \;\; > \;\; 1 - \frac{1}{m}, \quad \text{for each } P \in \Pi.$$

In particular, the above inequality holds for each P_n. This proves Claim 1.

Proof of Claim 2: Separability of $\bigcup_{m=1}^{\infty} K_m$ is an immediate consequence of Lemma A.1 and Lemma A.2. Then, the existence of $\mathcal{A}$ follows immediately from the separability of $\bigcup_{m=1}^{\infty} K_m$ and Lemma A.3. This proves Claim 2.

Proof of Claim 3:

Proof of Claim 4:

Proof of Claim 5:

$\square$

A Technical Lemmas

Lemma A.1

Every compact subset of a metric space is also a separable subset of that metric space.

PROOF Let (X, ρ) be a metric space and $K \subset X$ be a compact subset of X. For each $x \in X$ and positive $r > 0$, let

$$B(x,r) \;\; := \;\; \{\, y \in X \mid \rho(x,y) < r \,\} \;\; \subset \;\; X,$$

i.e. $B(x,r)$ is the open ball in X centred at x with radius $r > 0$. For each $n \in \mathbb{N}$, the following forms an open cover of K:

$$\mathcal{C}_n \;\; := \;\; \left\{ \; B\!\left(x, \frac{1}{n}\right) \; \subset \; X \;\; \middle| \;\; x \in K \; \right\}.$$

Since K is compact, each $\mathcal{C}_n$ admits a finite subcover:

$$\mathcal{F}_n \;\; := \;\; \left\{ \; B\!\left(x_i^{(n)}, \frac{1}{n}\right) \; \subset \; X \;\; \middle| \;\; x_i^{(n)} \in K, \;\; i = 1, 2, \ldots, J_n \; \right\}.$$

Let

$$\mathcal{D}_n \;\; := \;\; \left\{ \; x_i^{(n)} \in \; K \;\; \middle| \;\; i = 1, 2, \ldots, J_n \; \right\} \;\; \subset \;\; K,$$

and let $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset K$. We claim that $\mathcal{D}$ is dense in K. Indeed, let $y \in K$. Since each $\mathcal{F}_n$ is a (finite) open cover of K, we have:

$$y \;\; \in \;\; K \;\; \subset \;\; \bigcup_{i=1}^{J_n} B\!\left(x_i^{(n)}, \frac{1}{n}\right), \quad \text{for each } n \in \mathbb{N}.$$

Since $x_i^{(n)} \in \mathcal{D}$, for each $i = 1, 2, \ldots, J_n$ and for each $n \in \mathbb{N}$, the above inclusion shows that, for each $n \in \mathbb{N}$, there exists some $x \in \mathcal{D}$ such that $\rho(y, x) < \dfrac{1}{n}$. In particular, $\mathcal{D}$ contains a sequence that converges to $y \in K$. Since $y \in K$ is an arbitrary element of K, we see that $\overline{D} \supset K$. Since $\mathcal{D} \subset K$ and K is compact, hence closed, we trivially have $\overline{D} \subset K$. We may now conclude that $\overline{D} = K$. This completes the proof of the Lemma. □

Lemma A.2

Every countable union of separable subsets of a metric space is itself a separable subset of that metric space.

PROOF Let $S := \bigcup_{i=1}^{\infty} S_i \subset X$ be a countable union of separable subsets S_i of a metric space X. For each fixed $i \in \mathbb{N}$, since S_i is separable, there exists countable $D_i \subset S_i$ which is dense in S_i. Let $D := \bigcup_{i=1}^{\infty} D_i$. Then, D is a countable subset of S. The Lemma is proved once we establish that D is dense in S. To this end, let $x \in S = \bigcup_{i=1}^{\infty} S_i$. Then, $x \in S_i$ for some $i \in \mathbb{N}$. Since D_i is dense in S_i, there exists a sequence $\{\, y_k \,\} \subset D_i \subset D$ such that $y_k \longrightarrow x$, as $k \longrightarrow \infty$. This proves that D is indeed dense in S, and completes the proof of the Lemma. □

Lemma A.3 (second theorem in Appendix M3, [2])

Let (S, ρ) be a metric space and $\Sigma \subset S$ a separable subset of S. Then, there exists a countable collection $\mathcal{A}$ of open subsets of S satisfying the following property: For each $x \in S$ and each open subset G of S,

$$x \;\in\; G \bigcap \Sigma \quad \Longrightarrow \quad x \;\in\; A \;\subset\; \overline{A} \;\subset\; G, \;\; \text{for some } A \in \mathcal{A}.$$

Proof Let $D \subset \Sigma$ be a countable dense subset of Σ. Let

$$\mathcal{A} \;:=\; \left\{\; B(d,r) \;\subset\; S \;\middle|\; \begin{array}{c} d \in D, \\ r \in \mathbb{Q},\; r > 0 \end{array} \right\}.$$

Then, $\mathcal{A}$ is a countable collection of open balls in S. Now, let $G \subset S$ be an arbitrary open subset of S and $x \in G \bigcap \Sigma$. First, choose $\varepsilon > 0$ such that $B(x,\varepsilon) \subset G$. Next, since $x \in \Sigma$ and D is dense in Σ, we may choose $d \in D$ such that $d \in B(x,\varepsilon/2)$, or equivalently $\rho(x,d) < \varepsilon/2$. Finally choose positive rational $r > 0$ such that $\rho(x,d) < r < \varepsilon/2$.

Now, note that $\overline{B(d,r)} \subset B(x,\varepsilon)$; indeed,

$$y \in \overline{B(d,r)} \quad\Longleftrightarrow\quad \rho(y,d) \leq r \quad\Longrightarrow\quad \rho(x,y) \;\leq\; \rho(x,d) + \rho(d,y) \;<\; \varepsilon/2 + r \;<\; \varepsilon/2 + \varepsilon/2 \quad\Longrightarrow\quad y \in B(x,\varepsilon).$$

Thus, we have

$$x \;\in\; B(d,r) \;\subset\; \overline{B(d,r)} \;\subset\; B(x,\varepsilon) \;\subset\; G.$$

This completes the proof of the Lemma. □

Theorem A.4 (The Diagonal Method, Appendix A.14, [1])

Suppose that each row of the array

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

is a bounded sequence of real numbers. Then, there exists an increasing sequence

$$n_1 \;<\; n_2 \;<\; n_3 \;< \cdots \;\in\; \mathbb{N}$$

of positive integers such that the limit

$$\lim_{k\to\infty} x_{r,n_k} \;\; \text{exists,} \quad \text{for each } r = 1,2,3,\ldots$$

Proof From the first row, select a convergent subsequence

$$x_{1,n(1,1)},\;\; x_{1,n(1,2)},\;\; x_{1,n(1,3)},\;\; \cdots\cdots$$

Here, we have $n(1,1) < n(1,2) < n(1,3) < \cdots \;\in\; \mathbb{N}$, and $\lim_{k\to\infty} x_{1,n(1,k)} \in \mathbb{R}$ exists. Next, note that the following subsequence of the second row:

$$x_{2,n(1,1)},\;\; x_{2,n(1,2)},\;\; x_{2,n(1,3)},\;\; \cdots\cdots$$

is still a bounded sequence of real numbers, and we may thus select a convergent subsequence:

$$x_{2,n(2,1)},\;\; x_{2,n(2,2)},\;\; x_{2,n(2,3)},\;\; \cdots\cdots$$

Here, we have $n(2,1) < n(2,2) < n(2,3) < \cdots \;\in\; \{\, n(1,k)\,\}_{k\in\mathbb{N}}$, and $\lim_{k\to\infty} x_{2,n(2,k)} \in \mathbb{R}$ exists. Continuing inductively, we obtain an array of positive integers

$$\begin{array}{cccc} n(1,1) & n(1,2) & n(1,3) & \cdots \\ n(2,1) & n(2,2) & n(2,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

which satisfies: For each $r \in \mathbb{N}$, we have

- each row is an increasing sequence of positive integers, i.e. $n(r,1) \;<\; n(r,2) \;<\; n(r,3) \;<\; \cdots$,

- the $(r+1)^{\text{th}}$ row is a subsequence of the r^{th} row, i.e. $\{\,n(r+1,k)\,\}_{k\in\mathbb{N}} \subset \{\,n(r,k)\,\}_{k\in\mathbb{N}}$, and
- $\lim\limits_{k\to\infty} x_{r,n(r,k)} \in \mathbb{R}$ exists.

Note that the first two properties together imply:

$$n(k,k) \;<\; n(k,k+1) \;\leq\; n(k+1,k+1), \quad \text{for each } k \in \mathbb{N}.$$

Now, define $n_k := n(k,k)$, for $k \in \mathbb{N}$. We then see that

$$n_k \;:=\; n(k,k) \;< n(k+1,k+1) \;=:\; n_{k+1},$$

i.e., $\{\,n_k\,\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. Lastly, for each $r \in \mathbb{N}$, consider the sequence

$$x_{r,n_1}\,,\;\; x_{r,n_2}\,,\;\; x_{r,n_3}\,,\;\; \cdots$$

Note that, for each $r \in \mathbb{N}$,

$$x_{r,n_r}\,,\;\; x_{r,n_{r+1}}\,,\;\; x_{r,n_{r+2}}\,,\;\; \cdots$$

is a subsequence of $\{\,x_{r,n(r,k)}\,\}_{k\in\mathbb{N}}$. We saw above that $\lim\limits_{k\to\infty} x_{r,n(r,k)}$ exists, which in turn implies that $\lim\limits_{k\to\infty} x_{r,n_k}$ exists. Since $r \in \mathbb{N}$ is arbitrary, the proof of the Theorem is now complete. □

References

[1] Billingsley, P. *Probability and Measure*, third ed. John Wiley & Sons, 1995.

[2] Billingsley, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.