The Prokhorov Theorem

Study Notes   September 26, 2015   Kenneth Chu

1 The Prokhorov Theorem

Definition 1.1 (Tightness and weak sequential compactness)

Suppose:

- *(S, ρ) is a metric space, $\mathcal{B}(S)$ its the Borel σ-algebra, $(S, \mathcal{B}(S))$ is the corresponding measurable space.*

- *$\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is a family of probability measures on $(S, \mathcal{B}(S))$.*

The family Π is said to be:

(i)   ***tight*** *if, for each $\varepsilon > 0$, there exists a compact subset $K_\varepsilon \subset S$ such that*

$$1 - \epsilon \ < \ P(\, K_\varepsilon \,) \ \leq \ 1, \quad \text{for each } P \in \Pi.$$

(ii)   ***weakly sequentially compact*** *if, for every sequence $\{\, P_n \,\}_{n \in \mathbb{N}} \subset \Pi$, there exists a probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ and subsequence $\{\, P_{n(i)} \,\}_{i \in \mathbb{N}}$ such that*

$$P_{n(i)} \ \xrightarrow{\ w\ } \ P, \quad \text{as } i \longrightarrow \infty.$$

Theorem 1.2 (The Prokhorov Theorem, Theorems 5.1 & 5.2, [1])

Suppose:

- *(S, ρ) is a metric space, $\mathcal{B}(S)$ its the Borel σ-algebra, $(S, \mathcal{B}(S))$ is the corresponding measurable space.*

- *$\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is a collection of probability measures on $(S, \mathcal{B}(S))$.*

Then, the following statements hold:

(i)   *Tightness of Π implies weak sequential compactness of Π.*

(ii)   *Suppose further that (S, ρ) is complete and separable.*
      Then, weak sequential compactness of Π implies tightness of Π.

PROOF   We first prove statement (ii), then statement (i).

Proof of (ii)

Suppose S is complete and separable. Let $\varepsilon > 0$ be fixed. We need to find a compact subset $K \subset S$ such that

$$1 - \varepsilon \ < \ P(\, K \,) \ \leq \ 1, \quad \text{for each } P \in \Pi.$$

Now, separability of S implies that every open cover of every subset of S admits a countable subcover (Appendix M3, [1]). Denote by $B(x, r) \subset S$ the open ball in S centred at $x \in S$ of radius $r > 0$. For each $k \in \mathbb{N}$, the open cover

$$\left\{ \, B\left(x, \frac{1}{k} \right) \, \right\}_{x \in S}$$

of S admits a countable subcover, say,

$$\{\, A_{ki} \,\}_{i \in \mathbb{N}} \ \subset \ \left\{ \, B\left(x, \frac{1}{k} \right) \, \right\}_{x \in S}.$$

Let $G_{kn} := \bigcup_{i=1}^{n} A_{ki}$. Then, each G_{kn} is an open subset of S and $G_{kn} \uparrow S$, as $n \longrightarrow \infty$. Hence, by the Claim below, there exists $n_k \in \mathbb{N}$ such that

$$1 - \frac{\varepsilon}{2^k} \;<\; P\left(\bigcup_{i=1}^{n_k} A_{ki} \right) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Now, let

$$K \;:=\; \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}}$$

Note that K, being a closed subset of the complete metric space S, is itself complete. Note also that the set $\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}$ is totally bounded; hence so is its closure K. Being complete and totally bounded, K is therefore compact (Appendix M5, [1]). It now remains only to show that $1 - \varepsilon < P(K) \leq 1$, for each $P \in \Pi$; or equivalently, that $P(K^c) \leq \varepsilon$, for each $P \in \Pi$. To this end, write $B_k := \bigcup_{i=1}^{n_k} A_{ki}$. Then,

$$1 - \frac{\varepsilon}{2^k} \;<\; P(\,B_k\,) \;\leq\; 1; \quad \text{equivalently,} \;\; P(\,B_k^c\,) \;\leq\; \frac{\varepsilon}{2^k}.$$

Also,

$$K \;:=\; \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}} \;:=\; \overline{\bigcap_{k=1}^{\infty} B_k} \;\supset\; \bigcap_{k=1}^{\infty} B_k.$$

Hence,

$$K^c \;\subset\; \left(\bigcap_{k=1}^{\infty} B_k \right)^c \;=\; \bigcup_{k=1}^{\infty} B_k^c,$$

which implies:

$$P(\,K^c\,) \;\leq\; \sum_{k=1}^{\infty} P(B_k^c) \;\leq\; \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} \;=\; \varepsilon.$$

Thus, the proof of (ii) will be complete once we prove the following:

> **Claim:**   Let $\{\, G_n \,\}_{n \in \mathbb{N}}$ be a sequence of open subsets of S with $G_n \uparrow S$. Then, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that
>
> $$1 - \varepsilon \;<\; P(\, G_{n_\varepsilon} \,) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Proof of Claim: Suppose the Claim is false, and we derive a contradiction. The failure of the Claim implies that there exists some $0 < \varepsilon < 1$ such that for each $n \in \mathbb{N}$, there exists $P_n \in \Pi$ such that

$$P_n(\, G_n \,) \;\leq\; 1 - \varepsilon.$$

By the hypothesis of weak sequential compactness of Π, there exists some probability measure $Q \in \mathcal{M}_1(S, \mathcal{B}(S))$ and the subsequence $\{\, P_{n(i)} \,\}$ of $\{\, P_n \,\}$ such that $P_{n(i)} \xrightarrow{w} Q$, as $i \longrightarrow \infty$. Now, for each fixed $n \in \mathbb{N}$, we have:

$$
\begin{aligned}
Q(\, G_n \,) \;&\leq\; \liminf_{i \to \infty} P_{n(i)}(\, G_n \,), \quad \text{by the Portmanteau Theorem} \\
&\leq\; \liminf_{i \to \infty} P_{n(i)}\big(\, G_{n(i)} \,\big), \quad \text{since } \{\, G_n \,\} \text{ is increasing} \\
&\leq\; 1 - \varepsilon, \quad \text{by choice of } P_n
\end{aligned}
$$

But, by hypothesis, we also have $G_n \uparrow S$. Hence, we therefore have:

$$1 \;=\; Q(S) \;=\; \lim_{n \to \infty} Q(G_n) \;\leq\; 1 - \varepsilon,$$

which is the desired contradiction. This completes the proof of the Claim, hence that of (ii).

<u>Proof of (i)</u>

Suppose $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is tight. We need to establish that Π is weakly sequentially compact. In other words, if $\{P_n\} \subset \Pi$ is a sequence of probability measures contained in Π, we need to establish that there exists a Borel probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ and a subsequence $\{P_{n(i)}\} \subset \{P_n\}$ such that $P_{n(i)} \xrightarrow{w} P$, as $i \longrightarrow \infty$.

So, let $\{P_n\} \subset \Pi$. We prove the Theorem by establishing the following series of Claims. Note that the proof of the Theorem is complete once we establish Claim 5.

Claim 1: There exists an increasing sequence of compact subsets $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ such that

$$1 - \frac{1}{m} < P_n(K_m) \leq 1, \quad \text{for every } m, n \in \mathbb{N}.$$

Claim 2: Let $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ be one such sequence of compact subsets of S as in Claim 1. Then, $\bigcup_{m=1}^{\infty} K_m$ is a separable subset of S, and there exists a countable collection $\mathcal{A}$ of open subsets of S satisfying the following property: For each $x \in S$ and for each open subset G of S,

$$x \in G \bigcap \left(\bigcup_{m=1}^{\infty} K_m \right) \implies x \in A \subset \overline{A} \subset G, \text{ for some } A \in \mathcal{A}.$$

Claim 3: Define:

$$\mathcal{H} := \{\varnothing\} \bigcup \left\{ \begin{array}{c} \text{all finite unions of sets of the form} \\ \overline{A} \cap K_m, \text{ where } A \in \mathcal{A} \text{ and } m \in \mathbb{N} \end{array} \right\}.$$

Then, there exists a subsequence $\{P_{n(i)}\} \subset \{P_n\}$ such that the limit

$$\alpha(H) := \lim_{i \to \infty} P_{n(i)}(H) \text{ exists, for each } H \in \mathcal{H}.$$

Claim 4: There exists a Borel probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ such that

$$P(G) := \sup_{H \subset G} \alpha(H), \quad \text{for each open subset } G \subset S.$$

Claim 5: $P_{n(i)} \xrightarrow{w} P$, as $i \longrightarrow \infty$.

<u>Proof of Claim 1:</u> By tightness hypothesis on Π, for each $m \in \mathbb{N}$, there exists a compact subset $L_m \subset S$ such that

$$1 - \frac{1}{m} < P(L_m) \leq 1, \quad \text{for each } P \in \Pi.$$

Define, for each $m \in \mathbb{N}$, $K_m := \bigcup_{i=1}^{m} L_i$. Then, each K_m is compact (since finite unions of compact subsets are themselves compact). Next, we trivially have $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$. Also,

$$P(K_m) = P\left(\bigcup_{i=1}^{m} L_i \right) \geq L_m > 1 - \frac{1}{m}, \quad \text{for each } P \in \Pi.$$

In particular, the above inequality holds for each P_n. This proves Claim 1.

<u>Proof of Claim 2:</u>   Separability of $\bigcup_{m=1}^{\infty} K_m$ is an immediate consequence of Lemma A.1 and Lemma A.2. Then, the existence of $\mathcal{A}$ follows immediately from the separability of $\bigcup_{m=1}^{\infty} K_m$ and Lemma A.3. This proves Claim 2.

<u>Proof of Claim 3:</u>

<u>Proof of Claim 4:</u>

<u>Proof of Claim 5:</u>

$\square$

A   Technical Lemmas

Lemma A.1

Every compact subset of a metric space is also a separable subset of that metric space.

PROOF   Let (X, ρ) be a metric space and $K \subset X$ be a compact subset of X. For each $x \in X$ and positive $r > 0$, let

$$B(x, r) \ := \ \{ \, y \in X \mid \rho(x, y) < r \, \} \ \subset \ X,$$

i.e. $B(x, r)$ is the open ball in X centred at x with radius $r > 0$. For each $n \in \mathbb{N}$, the following forms an open cover of K:

$$\mathcal{C}_n \ := \ \left\{ \, B\left(x, \frac{1}{n}\right) \subset X \ \bigg| \ x \in K \, \right\}.$$

Since K is compact, each $\mathcal{C}_n$ admits a finite subcover:

$$\mathcal{F}_n \ := \ \left\{ \, B\left(x_i^{(n)}, \frac{1}{n}\right) \subset X \ \bigg| \ x_i^{(n)} \in K, \ \ i = 1, 2, \ldots, J_n \, \right\}.$$

Let

$$\mathcal{D}_n \ := \ \left\{ \, x_i^{(n)} \in K \ \bigg| \ i = 1, 2, \ldots, J_n \, \right\} \ \subset \ K,$$

and let $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset K$. We claim that $\mathcal{D}$ is dense in K. Indeed, let $y \in K$. Since each $\mathcal{F}_n$ is a (finite) open cover of K, we have:

$$y \ \in \ K \ \subset \ \bigcup_{i=1}^{J_n} B\left(x_i^{(n)}, \frac{1}{n}\right), \quad \text{for each } n \in \mathbb{N}.$$

Since $x_i^{(n)} \in \mathcal{D}$, for each $i = 1, 2, \ldots, J_n$ and for each $n \in \mathbb{N}$, the above inclusion shows that, for each $n \in \mathbb{N}$, there exists some $x \in \mathcal{D}$ such that $\rho(y, x) < \frac{1}{n}$. In particular, $\mathcal{D}$ contains a sequence that converges to $y \in K$. Since $y \in K$ is an arbitrary element of K, we see that $\overline{\mathcal{D}} \supset K$. Since $\mathcal{D} \subset K$ and K is compact, hence closed, we trivially have $\overline{\mathcal{D}} \subset K$. We may now conclude that $\overline{\mathcal{D}} = K$. This completes the proof of the Lemma. $\square$

Lemma A.2

Every countable union of separable subsets of a metric space is itself a separable subset of that metric space.

PROOF   Let $S := \bigcup_{i=1}^{\infty} S_i \subset X$ be a countable union of separable subsets S_i of a metric space X. For each fixed $i \in \mathbb{N}$, since S_i is separable, there exists countable $D_i \subset S_i$ which is dense in S_i. Let $D := \bigcup_{i=1}^{\infty} D_i$. Then, D is a countable subset of S. The Lemma is proved once we establish that D is dense in S. To this end, let $x \in S = \bigcup_{i=1}^{\infty} S_i$. Then, $x \in S_i$ for some $i \in \mathbb{N}$. Since D_i is dense in S_i, there exists a sequence $\{ y_k \} \subset D_i \subset D$ such that $y_k \longrightarrow x$, as $k \longrightarrow \infty$. This proves that D is indeed dense in S, and completes the proof of the Lemma. $\square$

Lemma A.3 (second theorem in Appendix M3, [1])

Let (S, ρ) be a metric space and $\Sigma \subset S$ a separable subset of S. Then, there exists a countable collection $\mathcal{A}$ of open subsets of S satisfying the following property: For each $x \in S$ and each open subset G of S,

$$x \in G \bigcap \Sigma \ \implies \ x \in A \subset \overline{A} \subset G, \ \text{for some } A \in \mathcal{A}.$$

PROOF   Let $D \subset \Sigma$ be a countable dense subset of Σ. Let

$$\mathcal{A} \ := \ \left\{ \ B(d,r) \ \subset \ S \ \middle| \ \begin{array}{l} d \in D, \\ r \in \mathbb{Q}, \ r > 0 \end{array} \right\}.$$

Then, $\mathcal{A}$ is a countable collection of open balls in S. Now, let $G \subset S$ be an arbitrary open subset of S and $x \in G \bigcap \Sigma$. First, choose $\varepsilon > 0$ such that $B(x,\varepsilon) \subset G$. Next, since $x \in \Sigma$ and D is dense in Σ, we may choose $d \in D$ such that $d \in B(x,\varepsilon/2)$, or equivalently $\rho(x,d) < \varepsilon/2$. Finally choose positive rational $r > 0$ such that $\rho(x,d) < r < \varepsilon/2$.

Now, note that $\overline{B(d,r)} \subset B(x,\varepsilon)$; indeed,

$$y \in \overline{B(d,r)} \quad \Longleftrightarrow \quad \rho(y,d) \leq r \quad \Longrightarrow \quad \rho(x,y) \ \leq \ \rho(x,d) + \rho(d,y) \ < \ \varepsilon/2 + r \ < \ \varepsilon/2 + \varepsilon/2 \quad \Longrightarrow \quad y \in B(x,\varepsilon).$$

Thus, we have

$$x \ \in \ B(d,r) \ \subset \ \overline{B(d,r)} \ \subset \ B(x,\varepsilon) \ \subset \ G.$$

This completes the proof of the Lemma. $\square$

References

[1] BILLINGSLEY, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.