1 Examples of separating and convergence-determining classes of $\mathbb{R}^\infty$

Definition 1.1 (The metric on $\mathbb{R}^\infty$, Example 1.2, [1])
Let $\mathbb{R}^\infty$ denotes the set of all infinite sequences of real numbers, i.e.

$$\mathbb{R}^\infty := \{\, (x_1, x_2, \dots) \mid x_i \in \mathbb{R}, \text{ for each } i \in \mathbb{N} \,\}.$$

Define $\rho : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow [\,0,1\,]$ as follows:

$$\rho(x,y) := \sum_{n=1}^{\infty} \frac{\min\{\,1\,, |\,x_n - y_n\,|\,\}}{2^n}.$$

Remark 1.2 *Recall that*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^{n-1}} = \frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}\right) = 1,$$

which proves indeed that $0 \leq \rho(x,y) \leq 1$, for any $x, y \in \mathbb{R}^\infty$.

Theorem 1.3 (The metric space properties of $\mathbb{R}^\infty$)

(i) *$(\,\mathbb{R}^\infty, \rho\,)$ is a metric space. Let $\mathbb{R}^\infty$ denote also this metric space in the remainder of this Theorem.*

(ii) *For $x, x^{(1)}, x^{(2)}, x^{(3)}, \dots, \in \mathbb{R}^\infty$, we have:*

$$\rho\Big(\,x^{(n)},\,x\,\Big) \longrightarrow 0 \quad\Longleftrightarrow\quad \text{for each } i \in \mathbb{N},\ \lim_{n\to\infty}\left|\,x_i^{(n)} - x_i\,\right| = 0$$

(iii) *For each $n \in \mathbb{N}$, the "natural projection to the initial segment of length n"*

$$\pi_n \,:\, \mathbb{R}^\infty \longrightarrow \mathbb{R}^n \,:\, x \longmapsto (x_1, x_2, \dots, x_n)$$

is continuous, where $\mathbb{R}^n$ has the usual Euclidean topology.

(iv) *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, let $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)$ denote the open hypercube in $\mathbb{R}^n$ of side length 2ε centred at $\pi_n(x) \in \mathbb{R}^n$, i.e.*

$$C_{\mathbb{R}^n}(\pi_n(x),\varepsilon) := \left\{\, y \in \mathbb{R}^n \;\middle|\; \begin{array}{c} |\,y_i - x_i\,| < \varepsilon, \\ i = 1, 2, \dots, n \end{array} \right\}$$

Then, its pre-image in $\mathbb{R}^\infty$ under π_n

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) = \left\{\, y \in \mathbb{R}^\infty \;\middle|\; \begin{array}{c} |\,y_i - x_i\,| < \varepsilon, \\ i = 1, 2, \dots, n \end{array} \right\}$$

is an open subset of $\mathbb{R}^\infty$.

(v) *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, we have:*

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \subset B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon + \frac{1}{2^n}\right),$$

where $B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon + \frac{1}{2^n}\right)$ is the open ball in $\mathbb{R}^\infty$ centred at x of radius $\varepsilon + \frac{1}{2^n}$, i.e.

$$B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon + \frac{1}{2^n}\right) := \left\{\, y \in \mathbb{R}^\infty \;\middle|\; \rho(y,x) < \varepsilon + \frac{1}{2^n} \,\right\}$$

(vi) *The collection*

$$\left\{\ \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \subset \mathbb{R}^\infty \ \middle|\ n\in\mathbb{N},\, x\in\mathbb{R}^\infty,\, \varepsilon>0\ \right\}$$

of all pre-images under π_n of open hypercubes in $\mathbb{R}^n$, for all $n\in\mathbb{N}$, forms a basis for the topology of $\mathbb{R}^\infty$.

(vii) $\mathbb{R}^\infty$ *is a separable metric space.*

(viii) $\mathbb{R}^\infty$ *is a complete metric space.*

PROOF

(i) Clearly, ρ is non-negative and symmetric. We now show that, for any $x,y\in\mathbb{R}^\infty$, we have $\rho(x,y)=0$ implies $x=y$. Indeed,

$$\begin{aligned}
\rho(x,y) \;=\; 0 \quad &\Longleftrightarrow\quad \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-y_i\,|\,\}}{2^i} \;=\; 0 \\
&\Longleftrightarrow\quad \min\{\,1\,,|\,x_i-y_i\,|\,\} \;=\; 0,\ \ \text{for each } i\in\mathbb{N} \\
&\Longleftrightarrow\quad |\,x_i-y_i\,| \;=\; 0,\ \ \text{for each } i\in\mathbb{N} \\
&\Longleftrightarrow\quad x \;=\; y.
\end{aligned}$$

In order to show that ρ is a metric, it remains only to establish the Triangle Inequality. By Lemma A.2, for any $x,y,z\in\mathbb{R}^\infty$, we have

$$\begin{aligned}
\rho(x,y) \;&=\; \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-y_i\,|\,\}}{2^i} \\
&\le\; \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-z_i\,|\,\}\,+\,\min\{\,1\,,|\,z_i-y_i\,|\,\}}{2^i} \\
&=\; \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i-z_i\,|\,\}\,\}}{2^i} \;+\; \sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,z_i-y_i\,|\,\}}{2^i} \\
&=\; \rho(x,z)+\rho(z,y),
\end{aligned}$$

where we have used the fact that $0\le\rho\le1$ to split the infinite sum into two terms in second-to-last equality. This proves that ρ satisfies the Triangle Inequality, and it is thus a metric on $\mathbb{R}^\infty$.

(ii) $\underline{\lim\limits_{n\to\infty}\rho\big(x^{(n)},x\big)=0 \;\Longrightarrow\; \lim\limits_{n\to\infty}\left|\,x_i^{(n)}-x_i\,\right|=0, \text{ for each } i\in\mathbb{N}}$

$$\begin{aligned}
\lim_{n\to\infty}\rho\Big(x^{(n)},x\Big)=0 \quad &\Longrightarrow\quad \lim_{n\to\infty}\sum_{i=1}^{\infty}\frac{\min\{\,1\,,|\,x_i^{(n)}-x_i\,|\,\}}{2^i}=0 \\
&\Longrightarrow\quad \lim_{n\to\infty}\min\{\,1\,,|\,x_i^{(n)}-x_i\,|\,\}=0,\ \ \text{for each } i\in\mathbb{N} \\
&\Longrightarrow\quad \lim_{n\to\infty}\left|\,x_i^{(n)}-x_i\,\right|=0,\ \ \text{for each } i\in\mathbb{N}
\end{aligned}$$

$\underline{\lim\limits_{n\to\infty}\left|\,x_i^{(n)}-x_i\,\right|=0, \text{ for each } i\in\mathbb{N} \;\Longrightarrow\; \lim\limits_{n\to\infty}\rho\big(x^{(n)},x\big)=0}$

This follows from the Weierstrass M-test. Suppose $\lim\limits_{n\to\infty}\left|\,x_i^{(n)}-x_i\,\right|=0$, for each $i\in\mathbb{N}$. Then,

$$\lim_{n\to\infty}\frac{\min\{\,1\,,|\,x_i^{(n)}-x_i\,|\,\}}{2^i} \;=\; 0 \;=:\; y_i,\ \ \text{for each } i\in\mathbb{N}.$$

For each $i \in \mathbb{N}$, let $M_i := \dfrac{1}{2^i}$. Then,

$$\frac{\min\{1\,,|\,x_i^{(n)} - x_i\,|\}}{2^i} \;\leq\; M_i \quad \text{and} \quad \sum_{i=1}^\infty M_i \;<\; \infty.$$

Hence, by the Weierstrass M-test (Lemma A.3), we have

$$\lim_{n\to\infty} \rho\Big(x^{(n)}, x\Big) \;=\; \lim_{n\to\infty} \sum_{i=1}^\infty \frac{\min\{\,1\,,|\,x_i^{(n)} - x_i\,|\,\}}{2^i} \;=\; \sum_{i=1}^\infty y_i \;=\; 0.$$

(iii) Immediate by (ii).

(iv) Since $C_{\mathbb{R}^n}(\pi_n(x),\varepsilon) \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, its pre-image under the continuous (by (iii)) map $\pi_n : \mathbb{R}^\infty \longrightarrow \mathbb{R}^n$ is an open subset of $\mathbb{R}^\infty$.

(v) For $y \in \mathbb{R}^\infty$, we have

$$\begin{aligned} y \in \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;&\Longrightarrow\; |\,y_i - x_i\,| < \varepsilon, \text{ for each } i = 1,2,\ldots,n \\ &\Longrightarrow\; \rho(x,y) \;:=\; \sum_{i=1}^\infty \frac{\min\{\,1\,,|\,x_i - y_i\,|\,\}}{2^i} \;\leq\; \sum_{i=1}^n \frac{\varepsilon}{2^i} + \sum_{i=n+1}^\infty \frac{1}{2^i} \;=\; \varepsilon + \frac{1}{2^n}. \end{aligned}$$

This proves:

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\subset\; B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon + \frac{1}{2^n}\right).$$

(vi) It suffices to show that every open ball in $B_{\mathbb{R}^\infty}(x,r) \subset \mathbb{R}^\infty$, $r > 0$, contains the pre-image of an open hypercube centred at $\pi_n(x) \in \mathbb{R}^n$ under π_n. To this end, for $r > 0$, choose $\varepsilon > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large such that $\varepsilon + \dfrac{1}{2^n} < r$. Then, for any $x \in \mathbb{R}^\infty$, by (v), we have:

$$x \;\in\; \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\subset\; B_{\mathbb{R}^\infty}\left(x\,,\,\varepsilon + \frac{1}{2^n}\right) \;\subset\; B_{\mathbb{R}^\infty}(\,x\,,r\,),$$

as required.

(vii) It suffices to exhibit a countable subset of $\mathbb{R}^\infty$ that intersects every open ball in $\mathbb{R}^\infty$. To this end, let

$$D \;:=\; \bigcup_{n=1}^\infty \left\{\, x = (x_1, x_2, \ldots\,) \in \mathbb{R}^\infty \;\middle|\; \begin{array}{l} x_i \in \mathbb{Q}, \text{ for each } i \in \mathbb{N} \\ x_i = 0, \text{ for all } i \geq n \end{array} \right\}.$$

Clearly, D is a countable subset of $\mathbb{R}^\infty$. Now let $B_{\mathbb{R}^\infty}(x,\varepsilon)$ be an arbitrary open ball in $\mathbb{R}^\infty$. Choose $\delta > 0$ small enough and $n \in \mathbb{N}$ large enough such that $\delta + \dfrac{1}{2^n} < \varepsilon$. Then,

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\delta)\,) \;\subset\; B_{\mathbb{R}^\infty}\left(x\,,\,\delta + \frac{1}{2^n}\right) \;\subset\; B_{\mathbb{R}^\infty}(\,x\,,\varepsilon\,),$$

Now, for each $i = 1,2,\ldots,n$, choose $z_i \in \mathbb{Q} \cap (x_i - \delta, x_i + \delta)$. Let $z = (z_1, z_2, \ldots, z_n,\,0,\,0,\,\ldots) \in \mathbb{R}^\infty$. Then, we have

$$z \;\in\; D \bigcap \left\{\, y \in \mathbb{R}^\infty \;\middle|\; \begin{array}{c} y_i \in (x_i - \delta, x_i + \delta), \\ \text{for each } i = 1,2,\ldots,n \end{array} \right\} \;=\; D \bigcap \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\delta)\,) \;\subset\; D \bigcap B_{\mathbb{R}^\infty}(\,x\,,\varepsilon\,).$$

This proves the the countable subset $D \subset \mathbb{R}^\infty$ has non-empty intersection with every open ball in $\mathbb{R}^\infty$, i.e. D is dense in $\mathbb{R}^\infty$. Hence, $\mathbb{R}^\infty$ is separable.

(viii) We need to show that every Cauchy sequence in $\mathbb{R}^\infty$ converges to any element in $\mathbb{R}^\infty$.

$$\begin{aligned}
& \left\{ x^{(n)} \right\}_{n\in\mathbb{N}} \subset \mathbb{R}^\infty \text{ is a Cauchy sequence in } \mathbb{R}^\infty \\
\Longleftrightarrow\quad & \text{for each } \varepsilon > 0, \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that } \rho\Big(x^{(m)}, x^{(n)}\Big) \;<\; \varepsilon, \text{ for any } m,n > N_\varepsilon \\
\Longrightarrow\quad & \text{for each } i \in \mathbb{N}, \text{ we have:} \\
& \text{for each } \varepsilon > 0, \text{ there exists } N_\varepsilon \in \mathbb{N} \text{ such that } \left| x_i^{(m)} - x_i^{(n)} \right| \;<\; \varepsilon, \text{ for any } m,n > N_\varepsilon \\
\Longrightarrow\quad & \text{for each } i \in \mathbb{N}, \left\{ x_i^{(n)} \right\}_{n\in\mathbb{N}} \subset \mathbb{R} \text{ is a Cauchy sequence in } \mathbb{R};\ \text{hence } x_i := \lim_{n\to\infty} x_i^{(n)} \in \mathbb{R} \text{ exists} \\
\Longrightarrow\quad & \lim_{n\to\infty} \rho\Big(x^{(n)}, x\Big) \;=\; 0, \ \text{ where } x := (x_1, x_2, \ldots) \in \mathbb{R}^\infty \quad \text{(by (ii))}
\end{aligned}$$

This proves that $\mathbb{R}^\infty$ indeed is a complete metric space.

□

A Technical Lemmas

Lemma A.1 *Define*

$$\phi \,:\, [\,0,\infty\,) \longrightarrow [\,0,1\,] \,:\, t \longmapsto \min\{\,1\,,t\,\}.$$

Then, ϕ satisfies:

$$\phi(s+t) \;\leq\; \phi(s) \;+\; \phi(t), \ \text{ for each } s,t \in [0,\infty).$$

Proof For any $s,t \in [0,\infty)$, either $s+t \geq 1$ or $s+t < 1$. If $s+t \geq 1$, then

$$\phi(s+t) \;=\; \min\{1, s+t\} \;=\; 1 \;<\; 2 \;=\; 1+1 \;\leq\; \min\{1,s\} + \min\{1,t\} \;=\; \phi(s)+\phi(t),$$

hence, the required inequality holds. On the other hand, if $s+t < 1$, then we must also have $s < 1$ and $t < 1$ (since $s,t \geq 0$). Hence,

$$\phi(s+t) \;=\; \min\{1, s+t\} \;=\; s+t \;=\; \min\{1,s\} + \min\{1,t\} \;=\; \phi(s)+\phi(t),$$

thus, the required inequality also holds. □

Lemma A.2 *For any $x,y,z \in \mathbb{R}$, we have:*

$$\min\{1, |\,x-y\,|\} \;\leq\; \min\{1, |\,x-z\,|\} \;+\; \min\{1, |\,z-y\,|\}.$$

Proof Observe that $|\,x-y\,| \;\leq\; |\,x-z\,| + |\,z-y\,|$ implies

$$\min\{\,1\,, |\,x-y\,|\,\} \;\leq\; |\,x-z\,| + |\,z-y\,|.$$

The above inequality, together with $\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; 1$, thus in turn imply:

$$\min\{\,1\,, |\,x-y\,|\,\} \;\leq\; \min\{\,1\,, |\,x-z\,| + |\,z-y\,|\,\}.$$

By Lemma A.1, we therefore have:

$$\min\{\,1\,, |\,x-y\,|\,\} \;\leq\; \min\{\,1\,, |\,x-z\,| + |\,z-y\,|\,\}. \;\leq\; \min\{\,1\,, |\,x-z\,|\,\} \;+\; \min\{\,1\,, |\,z-y\,|\,\},$$

which proves the present Lemma. □

Lemma A.3 (The Weierstrass M-test, Theorem A.28, [2])
Suppose that $\lim_{n\to\infty} x_i^{(n)} = x_i$, *for each* $i \in \mathbb{N}$, *and that* $\left| x_i^{(n)} \right| \leq M_i$, *where* $\sum_{i=1}^{\infty} M_i < \infty$. *Then,*

(i) $\sum_{i=1}^{\infty} x_i$ *exists, and* $\sum_{i=1}^{\infty} x_i^{(n)}$ *exists for each* $n \in \mathbb{N}$.

(ii) *Furthermore,*

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

Proof

(i) $\sum_{i=1}^{\infty} M_i < \infty$ and $\left| x_i^{(n)} \right| \leq M_i \implies$ the series $\sum_{i=1}^{\infty} x_i$ and $\sum_{i=1}^{\infty} x_i^{(n)}$, $n \in \mathbb{N}$, converge absolutely.

(ii) Let $\varepsilon > 0$ be given. Choose $K \in \mathbb{N}$ sufficiently large such that $\sum_{j=K+1}^{\infty} M_i < \frac{\varepsilon}{3}$. Next, choose $N \in \mathbb{N}$ sufficiently large such that

$$\left| x_i^{(n)} - x_i \right| < \frac{\varepsilon}{3K}, \quad \text{for any } n > N \text{ and } i = 1, 2, \ldots, K.$$

Then, we have, for each $n > N$,

$$\begin{aligned}
\left| \sum_{i=1}^{\infty} x_i^{(n)} - \sum_{i=1}^{\infty} x_i \right| &= \left| \sum_{i=1}^{K} \left(x_i^{(n)} - x_i \right) + \sum_{i=K+1}^{\infty} x_i^{(n)} - \sum_{i=K+1}^{\infty} x_i \right| \\
&\leq \sum_{i=1}^{K} \left| x_i^{(n)} - x_i \right| + \sum_{i=K+1}^{\infty} \left| x_i^{(n)} \right| + \sum_{i=K+1}^{\infty} |\, x_i \,| \\
&\leq K \cdot \frac{\varepsilon}{3K} + \sum_{i=K+1}^{\infty} M_i + \sum_{i=K+1}^{\infty} M_i \leq \frac{\varepsilon}{3} + 2 \cdot \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since ε is arbitrary, this proves:

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

□

References

[1] Billingsley, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.

[2] Billingsley, P. *Probability and Measure*, anniversary ed. John Wiley & Sons, 2012.