

In these notes, we give the definition of the well-known **Wiener measure** defined on the separable Banach space $(C[0, 1], \|\cdot\|_\infty)$, and give the proof that it indeed exists. Here is the outline:

- The Wiener measure will be defined as any Borel probability measure on $(C[0, 1], \|\cdot\|_\infty)$ with certain prescribed finite-dimensional distributions. Since the collection of finite-dimensional subsets of $C[0, 1]$ is a separating class for its Borel σ -algebra (see Example 1.3, [3]), we see immediately that if a Wiener measure exists, it is unique.
- However, the above definition is NOT constructive; the existence of a probability measure on $(C[0, 1], \|\cdot\|_\infty)$ satisfying the defining properties of a Wiener measure must therefore be proved.
- To present this proof, we will start by showing that the “scaling limits” of the finite-dimensional distributions of the sequence of probability measures induced on $(C[0, 1], \|\cdot\|_\infty)$ by linearly interpolated random walks with I.I.D. steps possess exactly the defining properties (of the finite-dimensional distributions) of the Wiener measure.
- We then show that the sequence of probability measures induced on $(C[0, 1], \|\cdot\|_\infty)$ by linearly interpolated random walks with I.I.D. steps is tight, and thus admits a weakly convergent subsequence.
- The Wiener measure is, by definition, the limit of this weakly convergent subsequence of probability measures on $(C[0, 1], \|\cdot\|_\infty)$

1 Equivalence of $(C[0, 1], \|\cdot\|_\infty)$ -valued random variables and \mathbb{R} -valued stochastic processes indexed by $[0, 1]$ with continuous sample paths

Proposition 1.1 (The “one-dimensional subsets” of $C[0, 1]$ generate its Borel σ -algebra)

Let $(C[0, 1], \|\cdot\|_\infty)$ be the metric space of continuous \mathbb{R} -valued functions defined on the closed unit interval equipped with the supremum norm. For each $t \in [0, 1]$, let $\text{ev}_t : C[0, 1] \rightarrow \mathbb{R} : x \mapsto x(t)$. Define:

$$\mathcal{S} := \left\{ \text{ev}_t^{-1}(H) \subset C[0, 1] \mid \begin{array}{l} t \in [0, 1] \\ H \in \mathcal{O} \end{array} \right\} \subset \mathcal{P}(C[0, 1]).$$

Then, \mathcal{S} generates the Borel σ -algebra $\mathcal{B} := \mathcal{B}(C[0, 1], \|\cdot\|_\infty)$ of the metric space $(C[0, 1], \|\cdot\|_\infty)$; in other words,

$$\sigma(\mathcal{S}) = \mathcal{B}.$$

PROOF First, note that $\sigma(\mathcal{S}) \subset \mathcal{B}$. Indeed, recall that, for each $t \in [0, 1]$, $\text{ev}_t : C[0, 1] \rightarrow \mathbb{R}$ is continuous, hence $(\mathcal{B}, \mathcal{O})$ -measurable, by Corollary B.4. In particular, $\text{ev}_t^{-1}(H) \in \mathcal{B}$, for each $t \in [0, 1]$ and $H \in \mathcal{O}$. Thus, $\mathcal{S} \subset \mathcal{B}$; hence, $\sigma(\mathcal{S}) \subset \mathcal{B}$.

It remains to establish the reverse inclusion. To this end, first observe that, for each $x \in C[0, 1]$ and each $\varepsilon > 0$, we have

$$\overline{B(x, \varepsilon)} = \bigcap_{r \in \mathbb{Q} \cap [0, 1]} \left\{ y \in C[0, 1] \mid |y(r) - x(r)| \leq \varepsilon \right\} = \bigcap_{r \in \mathbb{Q} \cap [0, 1]} \text{ev}_r^{-1}([x(r) - \varepsilon, x(r) + \varepsilon]),$$

which shows that $\sigma(\mathcal{S})$ contains all the closed balls in $C[0, 1]$. On the other hand, recall that, in any metric space, every open ball can be expressed as a countable union of closed balls; indeed, for any y in the given metric space, and any $\delta > 0$, we have:

$$B(y, \delta) = \bigcup_{n \in \mathbb{N}} \overline{B\left(y, \delta - \frac{1}{n}\right)}.$$

We thus see that $\sigma(\mathcal{S})$ contains all the open balls in $C[0, 1]$. By the separability of $C[0, 1]$ and Theorem C.1, we see that every open subset of $C[0, 1]$ can be expressed as a countable union of open balls. Hence, $\sigma(\mathcal{S})$ in fact contains all the open subsets of $C[0, 1]$, which immediately yields $\mathcal{B} \subset \sigma(\mathcal{S})$. This proves $\sigma(\mathcal{S}) = \mathcal{B}$. \square

Theorem 1.2

Suppose:

- (Ω, \mathcal{A}) is a measurable space, and \mathcal{O} is the Borel σ -algebra of \mathbb{R} (equipped with usual Euclidean metric).
- $(C[0, 1], \|\cdot\|_\infty)$ is the metric space of continuous \mathbb{R} -valued functions defined on the compact unit interval equipped with the supremum norm, and $\mathcal{B} := \mathcal{B}(C[0, 1], \|\cdot\|_\infty)$ is its Borel σ -algebra.
- $X : \Omega \longrightarrow C[0, 1]$ is a function with domain Ω and codomain $C[0, 1]$, but otherwise arbitrary.
- For each $t \in [0, 1]$, let $\text{ev}_t : C[0, 1] \longrightarrow \mathbb{R} : x \longmapsto x(t)$.
- For each $t \in [0, 1]$, let $X_t := \text{ev}_t \circ X$. In other words, $X_t : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto \text{ev}_t(X(\omega)) = X(\omega)(t)$.

Then, X is $(\mathcal{A}, \mathcal{B})$ -measurable if and only if, for each $t \in [0, 1]$, X_t is $(\mathcal{A}, \mathcal{O})$ -measurable.

PROOF

(\implies)

It is trivial to see that, for each $t \in [0, 1]$, $\text{ev}_t : (C[0, 1], \|\cdot\|_\infty) \longrightarrow (\mathbb{R}, |\cdot|) : x \longmapsto x(t)$ is continuous. Recall that continuous maps are necessarily Borel measurable; see Corollary B.4. Hence, $\text{ev}_t : (C[0, 1], \|\cdot\|_\infty) \longrightarrow (\mathbb{R}, |\cdot|)$ is $(\mathcal{B}, \mathcal{O})$ -measurable, for each $t \in [0, 1]$. Now, suppose $X : \Omega \longrightarrow C[0, 1]$ is $(\mathcal{A}, \mathcal{B})$ -measurable. Then, for each $t \in [0, 1]$, the composition $X_t := \text{ev}_t \circ X$ is $(\mathcal{A}, \mathcal{O})$ -measurable, as required.

(\impliedby)

Suppose that, for each $t \in [0, 1]$, $X_t := \text{ev}_t \circ X$ is $(\mathcal{A}, \mathcal{O})$ -measurable. We seek to establish that $X : (\Omega, \mathcal{A}) \longrightarrow (C[0, 1], \mathcal{B})$ is $(\mathcal{A}, \mathcal{B})$ -measurable. To this end, let

$$\mathcal{S} := \left\{ \text{ev}_t^{-1}(H) \subset C[0, 1] \mid \begin{array}{l} t \in [0, 1] \\ H \in \mathcal{O} \end{array} \right\} \subset \mathcal{P}(C[0, 1]).$$

Then, note that the $(\mathcal{A}, \mathcal{B})$ -measurability of X follows immediately from Theorem B.3, Proposition 1.1, and the following

Claim: $X^{-1}(\mathcal{S}) \subset \mathcal{A}$.

Proof of Claim: Every set in \mathcal{S} has the form $\text{ev}_t^{-1}(H)$, for some $t \in [0, 1]$ and some $H \in \mathcal{O}$. Note that

$$X^{-1}(\text{ev}_t^{-1}(H)) = (\text{ev}_t \circ X)^{-1}(H) = X_t^{-1}(H) \in \mathcal{A},$$

where the last containment follows immediately from the $(\mathcal{A}, \mathcal{O})$ -measurability hypothesis on X_t , for each $t \in [0, 1]$. This shows that $X^{-1}(\mathcal{S}) \subset \mathcal{A}$ and completes the proof of the Claim.

The proof of the Theorem is now complete. □

Theorem 1.3

Suppose:

- (Ω, \mathcal{A}) is a measurable space, and \mathcal{O} is the Borel σ -algebra of \mathbb{R} (equipped with usual Euclidean metric).
- $(C[0, 1], \|\cdot\|_\infty)$ is the metric space of continuous \mathbb{R} -valued functions defined on the closed unit interval equipped with the supremum norm, and $\mathcal{B} := \mathcal{B}(C[0, 1], \|\cdot\|_\infty)$ is its Borel σ -algebra.
- $X : \Omega \longrightarrow C[0, 1]$ is a function with domain Ω and codomain $C[0, 1]$, but otherwise arbitrary.

- For each $t \in [0, 1]$, let $\text{ev}_t : C[0, 1] \rightarrow \mathbb{R} : x \mapsto x(t)$.
- For each $t \in [0, 1]$, let $X_t := \text{ev}_t \circ X$. In other words, $X_t : \Omega \rightarrow \mathbb{R} : \omega \mapsto \text{ev}_t(X(\omega)) = X(\omega)(t)$.

Then, the following are equivalent:

- (i) X is a $(C[0, 1], \|\cdot\|_\infty)$ -valued random variable (in other words, X is $(\mathcal{A}, \mathcal{B})$ -measurable).
- (ii) For each $t \in [0, 1]$, X_t is an \mathbb{R} -valued random variable (in other words, each X_t is $(\mathcal{A}, \mathcal{O})$ -measurable).
- (iii) $\{X_t : \Omega \rightarrow \mathbb{R}\}_{t \in [0, 1]}$ is a stochastic process indexed by the closed unit interval defined on the probability space $(\Omega, \mathcal{A}, \mu)$ with state space \mathbb{R} and continuous sample paths.

PROOF The equivalence of (i) and (ii) is immediate by the preceding Theorem. The equivalence of (ii) and (iii) is immediate by the definition of stochastic processes. \square

2 Scaling limits of finite-dimensional distributions of linearly interpolated random walks are multivariate Gaussian

Definition 2.1

- A **random walk on \mathbb{R}** is a sequence $\{S_n : \Omega \rightarrow \mathbb{R}\}_{n \in \mathbb{N}}$ of \mathbb{R} -valued random variables defined on a probability space Ω of the following form: There exists a sequence $\xi_1, \xi_2, \dots : \Omega \rightarrow \mathbb{R}$ of independent and identically distributed \mathbb{R} -valued random variables such that

$$S_n = \sum_{i=1}^n \xi_i, \quad \text{for each } n \in \mathbb{N}.$$

- Let $\left\{ S_n = \sum_{i=1}^n \xi_i : \Omega \rightarrow \mathbb{R} \right\}_{n \in \mathbb{N}}$ be a random walk on \mathbb{R} . Let $S_0 : \Omega \rightarrow \mathbb{R} : \omega \mapsto 0$ be the (constant) random variable defined on Ω with constant value $0 \in \mathbb{R}$.

For each $n \in \mathbb{N}$, the **piecewise linear equi-spaced lattice interpolation of $\{S_0, S_1, S_2, \dots, S_n\}$ over the unit interval $[0, 1]$** is the function $\bar{S}^{(n)} : \Omega \rightarrow C[0, 1]$ defined as follows:

$$\bar{S}^{(n)}(\omega)(t) := \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) + n \left(t - \frac{i-1}{n} \right) \xi_i(\omega) \right\}, \quad \text{for each } \omega \in \Omega, \quad t \in \left[\frac{i-1}{n}, \frac{i}{n} \right], \quad i = 1, 2, 3, \dots, n.$$

- For each $n \in \mathbb{N}$ and each $t \in [0, 1]$, define $\bar{S}_t^{(n)} : \Omega \rightarrow \mathbb{R}$ as follows:

$$\bar{S}_t^{(n)}(\omega) := \bar{S}^{(n)}(\omega)(t), \quad \text{for each } \omega \in \Omega.$$

Theorem 2.2

Let $\left\{ S_n = \sum_{i=1}^n \xi_i : \Omega \rightarrow \mathbb{R} \right\}_{n \in \mathbb{N}}$ be a random walk on \mathbb{R} , and let $\bar{S}^{(n)} : \Omega \rightarrow C[0, 1]$ and $\bar{S}_t^{(n)} : \Omega \rightarrow \mathbb{R}$ be the associated functions as defined in Definition 2.1. Let $S_0 : \Omega \rightarrow \mathbb{R} : \omega \mapsto 0$ be the (constant) \mathbb{R} -valued random variable defined on Ω with constant value $0 \in \mathbb{R}$. Then, the following statements are true:

- (i) For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,

$$\bar{S}^{(n)}(\omega) \left(\frac{i}{n} \right) = \frac{1}{\sigma \cdot \sqrt{n}} \cdot S_i(\omega), \quad \text{for } i = 0, 1, 2, \dots, n.$$

(ii) For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,

$$\bar{S}^{(n)}(\omega)(t) \text{ is the linear interpolation from } \frac{1}{\sigma \cdot \sqrt{n}} S_{i-1}(\omega) \text{ to } \frac{1}{\sigma \cdot \sqrt{n}} S_i(\omega) \text{ over } t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],$$

where $i = 1, 2, \dots, n$. In particular, for each $\omega \in \Omega$, $\bar{S}^{(n)}(\omega)$ is a continuous \mathbb{R} -valued function defined on $[0, 1]$.

(iii) For each $n \in \mathbb{N}$ and each $t \in [0, 1]$, the function $\bar{S}_t^{(n)} : \Omega \rightarrow \mathbb{R}$ is an \mathbb{R} -valued random variable defined on the probability space Ω . Hence, for each $n \in \mathbb{N}$, $\left\{ \bar{S}_t^{(n)} \right\}_{t \in [0, 1]}$ is a stochastic process indexed by $[0, 1]$ with state space \mathbb{R} and continuous sample paths.

(iv) For each $n \in \mathbb{N}$, the function $\bar{S}^{(n)} : \Omega \rightarrow C[0, 1]$ is a $C[0, 1]$ -valued random variable defined on the probability space Ω .

PROOF

(i) Obvious.

(ii) Obvious.

(iii) For each $n \in \mathbb{N}$ and each $t \in [0, 1]$, the function $\bar{S}_t^{(n)}$ is an \mathbb{R} -valued random variable defined on Ω because it is a linear combination, with coefficients in \mathbb{R} , of the \mathbb{R} -valued random variables $\xi_1, \xi_2, \dots, \xi_n$. Thus, $\left\{ \bar{S}_t^{(n)} \right\}_{t \in [0, 1]}$ is a stochastic process indexed by $[0, 1]$ with state space \mathbb{R} . By (ii), this stochastic process has continuous sample paths.

(iv) Immediate by (iii) and Theorem 1.3.

□

Theorem 2.3

- Let $\xi_1, \xi_2, \dots : \Omega \rightarrow \mathbb{R}$ be a sequence of independent and identically distributed \mathbb{R} -valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mu)$, with expectation value zero and common finite variance $\sigma^2 > 0$.
- Define the random variables:

$$\begin{cases} S_0 & : \Omega \rightarrow \mathbb{R} : \omega \mapsto 0, & \text{and} \\ S_n & : \Omega \rightarrow \mathbb{R} : \omega \mapsto \sum_{i=1}^n \xi_i(\omega), & \text{for each } n \in \mathbb{N}. \end{cases}$$

- For each $n \in \mathbb{N}$, define $X^{(n)} : \Omega \rightarrow C[0, 1]$ as follows:

$$X^{(n)}(\omega)(t) := \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) + n \left(t - \frac{i-1}{n} \right) \xi_i(\omega) \right\}, \text{ for each } \omega \in \Omega, t \in \left[\frac{i-1}{n}, \frac{i}{n} \right], i = 1, 2, 3, \dots, n.$$

- For each $n \in \mathbb{N}$ and each $t \in [0, 1]$, define $X_t^{(n)} : \Omega \rightarrow \mathbb{R}$ as follows:

$$X_t^{(n)}(\omega) := X^{(n)}(\omega)(t), \text{ for each } \omega \in \Omega.$$

Then, the following statements are true:

(i) For any $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$,

$$\left(X_{t_1}^{(n)} - X_{t_0}^{(n)}, \dots, X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \right) \xrightarrow{d} N\left(\mu = \mathbf{0}, \Sigma = \text{diag}(t_1 - t_0, \dots, t_k - t_{k-1})\right), \text{ as } n \rightarrow \infty.$$

(ii) For any $0 \leq t_1, t_2, \dots, t_k \leq 1$,

$$\left(X_{t_1}^{(n)}, X_{t_2}^{(n)}, \dots, X_{t_k}^{(n)} \right) \xrightarrow{d} N\left(\mu = \mathbf{0}, \Sigma = \left[\min\{t_i, t_j\} \right]_{1 \leq i, j \leq k}\right), \text{ as } n \rightarrow \infty.$$

PROOF First, note that, by Theorem 2.2, for each $n \in \mathbb{N}$, $X^{(n)} : \Omega \rightarrow C[0, 1]$ is a $C[0, 1]$ -valued random variable defined on Ω .

(i) For each $\omega \in \Omega$, $n \in \mathbb{N}$, and $t \in [0, 1]$, we have

$$X_t^{(n)}(\omega) = \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{[nt]}(\omega) + (nt - [nt]) \cdot \xi_{[nt]+1}(\omega) \right\},$$

where $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$, defined by

$$\lfloor x \rfloor := \max \left\{ k \in \mathbb{Z} \mid k \leq x \right\}, \text{ for each } x \in \mathbb{R},$$

is the round-down function. We next state three Claims, whose proofs will be given below. We note that the desired conclusion follows readily from Claim 3 and the Cramér-Wold Theorem (Theorem 1.9(iii), p.56, [5]); hence the present proof is complete once we establish the three Claims below.

Claim 1: If $\{a_n\}_{n \in \mathbb{N}}$ is a sequence of non-negative integers and $\{b_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ a sequence of positive integers satisfying:

$$a_n < b_n, \text{ for sufficiently large } n \in \mathbb{N}, \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{b_n - a_n}{n} = c > 0,$$

then

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \xrightarrow{d} \sqrt{c} \cdot Z, \text{ where } Z \sim N(0, 1).$$

Claim 2: For each fixed $t \in [0, 1]$,

$$W(t)_n := \frac{1}{\sigma \cdot \sqrt{n}} \cdot (nt - [nt]) \cdot \xi_{[nt]+1} \xrightarrow{d} 0.$$

Claim 3: For $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$, and arbitrary $c_1, c_2, \dots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^k c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \xrightarrow{d} N\left(0, \sum_{i=1}^k c_i^2 (t_i - t_{i-1})\right), \text{ as } n \rightarrow \infty.$$

Proof of Claim 1: Note that, for sufficiently large $n \in \mathbb{N}$, we may write

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i = \frac{\sqrt{b_n - a_n}}{\sqrt{n}} \cdot \left(\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \right).$$

Since, by hypothesis, that

$$\lim_{n \rightarrow \infty} \frac{b_n - a_n}{n} = c > 0,$$

Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [4]), once we establish the following:

$$\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \xrightarrow{d} N(0, 1), \quad \text{as } n \rightarrow \infty.$$

We establish the above convergence by invoking the Lindeberg Central Limit Theorem (Theorem 1.15, §1.5.5, p.67, [5]). In the present context, the Lindeberg Condition is the following:

$$\lim_{n \rightarrow \infty} \frac{1}{B_n^2} \cdot E \left[\sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\{|\xi_i| \geq \varepsilon S_n\}} \right] = 0, \quad \text{for each } \varepsilon > 0,$$

where

$$B_n^2 := \text{Var} \left[\sum_{i=1+a_n}^{b_n} \xi_i \right] = (b_n - a_n) \sigma^2 > 0.$$

The last equality used the hypothesis that ξ_1, ξ_2, \dots are independent and identically distributed with common finite variance $0 < \sigma^2 < \infty$. Hence, for each $\varepsilon > 0$,

$$\begin{aligned} \frac{1}{B_n^2} \cdot E \left[\sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\{|\xi_i| \geq \varepsilon B_n\}} \right] &= \frac{1}{(b_n - a_n) \sigma^2} \cdot (b_n - a_n) \cdot E \left[\xi_1^2 \cdot I_{\{|\xi_1| \geq \varepsilon \sigma \sqrt{b_n - a_n}\}} \right] \\ &= \frac{1}{\sigma^2} \cdot E \left[\xi_1^2 \cdot I_{\{|\xi_1| / \varepsilon \sigma \geq \sqrt{b_n - a_n}\}} \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

since $\lim_{n \rightarrow \infty} \sqrt{b_n - a_n} = \infty$ and $\sigma^2 = E[\xi_1^2]$ is finite. This verifies that the Lindeberg Condition indeed holds in the present context, and completes the proof of Claim 1.

Proof of Claim 2: First, note that $E[W(t)_n] = 0$. We now argue that $W(t)_n \xrightarrow{p} 0$. To this end, let $\varepsilon > 0$ be given. Then,

$$\begin{aligned} \varepsilon^2 \cdot P(|W(t)_n| \geq \varepsilon) &\leq E[W(t)_n^2 \cdot I_{\{|W(t)_n| \geq \varepsilon\}}] \\ &\leq E[W(t)_n^2] = \text{Var}(W(t)_n) = \text{Var} \left[\frac{1}{\sigma \cdot \sqrt{n}} \cdot (nt - \lfloor nt \rfloor) \cdot \xi_{\lfloor nt \rfloor + 1} \right] \\ &= \frac{1}{n \cdot \sigma^2} \cdot (nt - \lfloor nt \rfloor)^2 \cdot \text{Var}(\xi_{\lfloor nt \rfloor + 1}) = \frac{1}{n \cdot \sigma^2} \cdot (nt - \lfloor nt \rfloor)^2 \cdot \sigma^2 \\ &\leq \frac{1}{n}, \end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} P(|W(t)_n| \geq \varepsilon) = 0, \quad \text{for each } \varepsilon > 0,$$

i.e. $W(t)_n \xrightarrow{p} 0$, as $n \rightarrow \infty$ (Definition 2, Chapter 1, [4]), which is equivalent to $W(t)_n \xrightarrow{d} 0$, as $n \rightarrow \infty$ (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [4]). This proves Claim 2.

Proof of Claim 3: Let $0 \leq t_0 < t_1 < t_2 < \dots < t_k \leq 1$, and $c_1, c_2, \dots, c_k \in \mathbb{R}$ be arbitrary. Observe that:

$$\begin{aligned}
 & \sum_{i=1}^k c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \\
 &= \sum_{i=1}^k \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ S_{\lfloor nt_i \rfloor} - S_{\lfloor nt_{i-1} \rfloor} \right\} + \sum_{i=1}^k \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \left(nt_i - \lfloor nt_i \rfloor \right) \cdot \xi_{\lfloor nt_i \rfloor + 1} - \left(nt_{i-1} - \lfloor nt_{i-1} \rfloor \right) \cdot \xi_{\lfloor nt_{i-1} \rfloor + 1} \right\} \\
 &= \sum_{i=1}^k \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \right\} + \sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \\
 &= \sum_{i=1}^k c_i Y_i^{(n)} + \sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\}
 \end{aligned}$$

By Claim 2 and Slutsky's Theorem (Corollary, p.40, [4]),

$$\sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \xrightarrow{d} 0, \text{ as } n \rightarrow \infty. \quad (2.1)$$

Next, note that since $\xi_1, \xi_2, \xi_3, \dots$ are independent, we see that, for each fixed $n \in \mathbb{N}$,

$$Y_i^{(n)} := \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j, \quad i = 1, 2, 3, \dots, k,$$

are independent. Now, since $0 \leq t_{i-1} < t_i \leq 1$, it follows that $\lfloor nt_{i-1} \rfloor < \lfloor nt_i \rfloor$ for sufficiently large $n \in \mathbb{N}$. In addition,

$$\begin{aligned}
 \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} &= \frac{\lfloor nt_i \rfloor}{n} - \frac{\lfloor nt_{i-1} \rfloor}{n} = \left(\frac{nt_i}{n} + \frac{\lfloor nt_i \rfloor - nt_i}{n} \right) - \left(\frac{nt_{i-1}}{n} + \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right) \\
 &= t_i - t_{i-1} + \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n},
 \end{aligned}$$

which implies

$$\left| \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} - (t_i - t_{i-1}) \right| = \left| \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right| \leq \frac{2}{n} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus,

$$\lim_{n \rightarrow \infty} \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} = t_i - t_{i-1} > 0.$$

Thus, by Claim 1, we see that, for each $i = 1, 2, \dots, k$,

$$Y_i^{(n)} := \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \xrightarrow{d} \sqrt{t_i - t_{i-1}} \cdot N(0, 1) = N\left(0, t_i - t_{i-1}\right), \text{ as } n \rightarrow \infty. \quad (2.2)$$

By (2.1), (2.2), Proposition A.1, and Slutsky's Theorem (Corollary, p.40, [4]), we now see that

$$\sum_{i=1}^k c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) = \sum_{i=1}^k c_i Y_i^{(n)} + \sum_{i=1}^k c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \xrightarrow{d} N\left(0, \sum_{i=1}^k c_i^2 (t_i - t_{i-1})\right).$$

This completes the proof of Claim 3.

(ii) Let $t_0 := 0$, hence, $X_{t_0}^{(n)} \equiv 0$ for each $n \in \mathbb{N}$. We thus have, for each $n \in \mathbb{N}$,

$$\begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}}_T \cdot \begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} - X_{t_1}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \end{pmatrix}.$$

By (i), we know that

$$\begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} - X_{t_1}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \end{pmatrix} \xrightarrow{d} \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix} \sim N\left(\mu = \mathbf{0}, \Sigma = \text{diag}(t_1, t_2 - t_1, \dots, t_k - t_{k-1})\right), \text{ as } n \rightarrow \infty.$$

Since the map $\mathbb{R}^k \rightarrow \mathbb{R}^k : x \mapsto T \cdot x$ is continuous, we see immediately by Slutsky's Theorem (Theorem 6(a), p.39, [4]) that

$$\begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} \end{pmatrix} \xrightarrow{d} T \cdot \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix}, \text{ as } n \rightarrow \infty.$$

Since the map $\mathbb{R}^k \rightarrow \mathbb{R}^k : x \mapsto T \cdot x$ is an invertible linear automorphism on \mathbb{R}^k , we see that

$$L = \begin{pmatrix} L_{t_1} \\ L_{t_2} \\ \vdots \\ L_{t_k} \end{pmatrix} := T \cdot \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix}$$

is still an \mathbb{R}^k -valued Gaussian random variable, and it clearly has expectation value $\mathbf{0} \in \mathbb{R}^k$, since each of $Z_{t_1}, Z_{t_2-t_1}, \dots, Z_{t_k-t_{k-1}}$ has expectation value $0 \in \mathbb{R}$. It remains only to compute the covariance matrix of the \mathbb{R}^k -valued Gaussian random variable L . To this end, consider $1 \leq i \leq j \leq k$, i.e. $t_i \leq t_j$. Then, using the

alternative notation $Z_{t_1-t_0} := Z_{t_1}$, we have

$$\begin{aligned}
 \text{Cov}(L_{t_i}, L_{t_j}) &= \text{Cov}(Z_{t_1} + Z_{t_2-t_1} + \cdots + Z_{t_i-t_{i-1}}, Z_{t_1} + Z_{t_2-t_1} + \cdots + Z_{t_j-t_{j-1}}) \\
 &= \text{Cov}\left(\sum_{a=1}^i Z_{t_a-t_{a-1}}, \sum_{b=1}^j Z_{t_b-t_{b-1}}\right) = \sum_{a=1}^i \sum_{b=1}^j \text{Cov}(Z_{t_a-t_{a-1}}, Z_{t_b-t_{b-1}}) \\
 &= \sum_{a=1}^i \text{Cov}(Z_{t_a-t_{a-1}}, Z_{t_a-t_{a-1}}) = \sum_{a=1}^i \text{Var}(Z_{t_a-t_{a-1}}) = \sum_{a=1}^i (t_a - t_{a-1}) \\
 &= (t_1 - t_0) + (t_2 - t_1) + \cdots + (t_{i-1} - t_{i-2}) + (t_i - t_{i-1}) \\
 &= t_i = \min\{t_i, t_j\},
 \end{aligned}$$

as required.

□

A Technical Lemmas

Note that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ does NOT in general imply $X_n + Y_n \xrightarrow{d} X + Y$. But the implication does hold if X_n and Y_n are independent for each $n \in \mathbb{N}$, and both X and Y are Gaussian random variables, as the following Proposition shows.

Proposition A.1 *Let $k \in \mathbb{N}$ be fixed. Suppose:*

- For each $n \in \mathbb{N}$,

$$Y_1^{(n)}, Y_2^{(n)}, \dots, Y_k^{(n)} : \Omega^{(n)} \longrightarrow \mathbb{R}$$

are independent \mathbb{R} -valued random variables defined on the probability space $\Omega^{(n)}$.

- For each $i = 1, 2, \dots, k$,

$$Y_i^{(n)} \xrightarrow{d} N(\mu_i, \sigma_i^2), \quad \text{as } n \longrightarrow \infty.$$

Then, for any $c_1, c_2, \dots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^k c_i Y_i^{(n)} \xrightarrow{d} N\left(\sum_{i=1}^k c_i \mu_i, \sum_{i=1}^k c_i^2 \sigma_i^2\right), \quad \text{as } n \longrightarrow \infty.$$

PROOF Let $Y^{(n)} := \sum_{i=1}^k c_i Y_i^{(n)}$. Let φ_X denote the characteristic function of a \mathbb{R} -valued random variable X . Then,

$$\begin{aligned} \varphi_{Y^{(n)}}(t) &= \varphi_{\sum_{i=1}^k c_i Y_i^{(n)}}(t) \\ &= \prod_{i=1}^k \varphi_{c_i Y_i^{(n)}}(t), \quad \text{since } Y_1^{(n)}, \dots, Y_k^{(n)} \text{ are independent} \\ &= \prod_{i=1}^k \varphi_{Y_i^{(n)}}(c_i t) \\ &\longrightarrow \prod_{i=1}^k \exp\left\{\sqrt{-1} \mu_i (c_i t) - \frac{1}{2} \sigma_i^2 (c_i t)^2\right\} \\ &= \exp\left\{\sqrt{-1} \left(\sum_{i=1}^k c_i \mu_i\right) t - \frac{1}{2} \left(\sum_{i=1}^k c_i^2 \sigma_i^2\right) t^2\right\}, \quad \text{as } n \longrightarrow \infty, \end{aligned}$$

where the second and third equalities follow from the properties of characteristic functions of random variables (see p.21, [4]), while the expression of the limit follows from the fact that the characteristic function φ_Z of a random variable Z with distribution $N(\mu, \sigma^2)$ is

$$\varphi_Z = \exp\left\{\sqrt{-1} \mu t - \frac{1}{2} \sigma^2 t^2\right\}.$$

The Proposition now follows immediately from the Lévy-Cramér Continuity Theorem (Theorem 1.9(ii), p.56, [5]). \square

Proposition A.2

Suppose:

- $k \in \mathbb{N}$ and Q_1, Q_2, \dots, Q_k are Borel probability measures on $(C[0, 1], \|\cdot\|_\infty)$.
- $\{P_n\}_{n \in \mathbb{N}}$ is a sequence of Borel probability measures on $(C[0, 1], \|\cdot\|_\infty)$.

Then, the following are equivalent:

- (i) $\{P_n\}_{n \in \mathbb{N}}$ is tight.
- (ii) $\{Q_1, \dots, Q_k\} \cup \{P_n\}_{n \in \mathbb{N}}$ is tight.

PROOF Note that once we prove the Proposition for $k = 1$, the cases $k \geq 2$ will follow immediately by finite induction. We now proceed to prove the Proposition for $k = 1$, and we write Q for Q_1 .

(ii) \implies (i) This is trivial. Indeed,

$$\begin{aligned} & \{Q\} \cup \{P_n\}_{n \in \mathbb{N}} \text{ is tight} \\ \iff & \text{for each } \varepsilon > 0, \exists \text{ compact } K \subset C[0, 1] \text{ such that } 1 - \varepsilon < Q(K) \leq 1 \text{ and } 1 - \varepsilon < P_n(K) \leq 1, \text{ for all } n \in \mathbb{N} \\ \implies & \text{for each } \varepsilon > 0, \exists \text{ compact } K \subset C[0, 1] \text{ such that } 1 - \varepsilon < P_n(K) \leq 1, \text{ for all } n \in \mathbb{N} \\ \iff & \{P_n\}_{n \in \mathbb{N}} \text{ is tight} \end{aligned}$$

This proves that (ii) \implies (i).

(i) \implies (ii)

Let $\varepsilon > 0$ be given. Since $(C[0, 1], \|\cdot\|_\infty)$ is separable and complete, the single Borel probability measure Q on $(C[0, 1], \|\cdot\|_\infty)$ is tight (Theorem 1.3, [3]). Thus, there exists a compact subset $K_1 \subset C[0, 1]$ such that $1 - \varepsilon < Q(K_1) \leq 1$. On the other hand, the tightness hypothesis on $\{P_n\}_{n \in \mathbb{N}}$ implies there exists a compact subset $K_2 \subset C[0, 1]$ such that $1 - \varepsilon < P_n(K_2) \leq 1$, for each $n \in \mathbb{N}$. Let $K := K_1 \cup K_2$. Then, K is itself a compact subset of $C[0, 1]$, and

$$Q(K) = Q(K_1 \cup K_2) \geq Q(K_1) > 1 - \varepsilon, \quad \text{and} \quad P_n(K) = P_n(K_1 \cup K_2) \geq P_n(K_2) > 1 - \varepsilon.$$

This proves the tightness of $\{Q\} \cup \{P_n\}_{n \in \mathbb{N}}$, as required. \square

B Continuous maps are Borel measurable

Lemma B.1 (The pre-image of a σ -algebra is itself a σ -algebra.)

Suppose Ω is a non-empty set, (X, \mathcal{X}) is a measurable space, and $f : \Omega \rightarrow X$ is a map from Ω into X . Then,

$$f^{-1}(\mathcal{X}) := \{f^{-1}(V) \subset \Omega \mid V \in \mathcal{X}\}$$

is a σ -algebra of subsets of Ω .

PROOF

$$\underline{f^{-1}(\mathcal{X})} \quad f(\Omega) \subset X \implies \Omega = f^{-1}(X) \in f^{-1}(\mathcal{X}).$$

$f^{-1}(\mathcal{X})$ is closed under complementations Let $V \in \mathcal{X}$. Then, $X \setminus V \in \mathcal{X}$, and

$$\Omega \setminus f^{-1}(V) = \{\omega \in \Omega \mid f(\omega) \notin V\} = \{\omega \in \Omega \mid f(\omega) \in X \setminus V\} = f^{-1}(X \setminus V) \in f^{-1}(\mathcal{X}),$$

which shows that $f^{-1}(\mathcal{X})$ is indeed closed under complementations.

$f^{-1}(\mathcal{X})$ is closed countable unions Let $V_1, V_2, \dots \in \mathcal{X}$. Then, $\bigcup_{i=1}^{\infty} V_i \in \mathcal{X}$, and

$$\bigcup_{i=1}^{\infty} f^{-1}(V_i) = \left\{ \omega \in \Omega \mid \begin{array}{c} f(\omega) \in V_i \\ \text{for some } i \in \mathbb{N} \end{array} \right\} = f^{-1}\left(\bigcup_{i=1}^{\infty} V_i\right) \in f^{-1}(\mathcal{X}),$$

which proves that $f^{-1}(\mathcal{X})$ is indeed closed under countable unions.

This concludes the proof that $f^{-1}(\mathcal{X})$ is a σ -algebra of subsets of Ω . \square

Lemma B.2 (The push-forward of a σ -algebra is itself a σ -algebra.)

Suppose (Ω, \mathcal{A}) is a measurable space, X is a non-empty set, and $f : \Omega \rightarrow X$ is a map from Ω into X . Then,

$$\mathcal{F} := \{ V \subset X \mid f^{-1}(V) \in \mathcal{A} \}$$

is a σ -algebra of subsets of X .

PROOF

$$\underline{X \in \mathcal{F}} \quad f^{-1}(X) = \Omega \in \mathcal{A} \implies X \in \mathcal{F}.$$

\mathcal{F} is closed under complementations $V \in \mathcal{F} \implies f^{-1}(V) \in \mathcal{A} \implies f^{-1}(X \setminus V) = \Omega \setminus f^{-1}(V) \in \mathcal{A} \implies X \setminus V \in \mathcal{F}$, which proves that \mathcal{F} is indeed closed under complementations.

\mathcal{F} is closed under countable unions

$$\begin{aligned} V_1, V_2, \dots \in \mathcal{F} &\implies f^{-1}(V_1), f^{-1}(V_2), \dots \in \mathcal{A} \\ &\implies f^{-1}\left(\bigcup_{i=1}^{\infty} V_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(V_i) \in \mathcal{A} \\ &\implies \bigcup_{i=1}^{\infty} V_i \in \mathcal{F}, \end{aligned}$$

which proves that \mathcal{F} is indeed closed under countable unions. □

Theorem B.3

Suppose (Ω, \mathcal{A}) and (X, \mathcal{X}) are measurable spaces, and $f : \Omega \rightarrow X$ is a map from Ω into X . Then, f is $(\mathcal{A}, \mathcal{X})$ -measurable if there exists $\mathcal{S} \subset \mathcal{X}$ satisfying the following conditions:

- \mathcal{S} generates \mathcal{X} , i.e. $\sigma(\mathcal{S}) = \mathcal{X}$, and
- $f^{-1}(\mathcal{S}) \subset \mathcal{A}$.

PROOF By Lemma B.2,

$$\mathcal{F} := \{ V \subset X \mid f^{-1}(V) \in \mathcal{A} \}$$

is a σ -algebra of subsets of X . By hypothesis, $\mathcal{S} \subset \mathcal{F}$; hence, $\mathcal{X} = \sigma(\mathcal{S}) \subset \mathcal{F}$. Thus, $f^{-1}(\mathcal{X}) \subset \mathcal{A}$; equivalently, f is $(\mathcal{A}, \mathcal{X})$ -measurable. □

Corollary B.4 (Continuous maps are Borel measurable.)

Suppose X_1, X_2 are topological spaces, and $\mathcal{B}_1, \mathcal{B}_2$ are their respective Borel σ -algebras. Then, every continuous map $f : X_1 \rightarrow X_2$ is $(\mathcal{B}_1, \mathcal{B}_2)$ -measurable.

C Topology

Theorem C.1 (Appendix M3, [3])

Suppose S is a metric space. Then, the following conditions are equivalent:

- (i) S is separable.
- (ii) The topology of S has a countable basis.
- (iii) Every open cover of *each subset* of S has a countable subcover.

D Modulus of continuity

Definition D.1 (Modulus of continuity)

Let $\mathbb{R}^{[0,1]}$ denote the set of all arbitrary \mathbb{R} -valued functions defined on the closed unit interval $[0, 1]$. The **modulus of continuity** is, by definition, the following function:

$$w : \mathbb{R}^{[0,1]} \times (0, 1] \longrightarrow [0, \infty] : (f, \delta) \longmapsto \sup \left\{ |f(s) - f(t)| \mid \begin{array}{l} (s, t) \in [0, 1] \times [0, 1] \\ |s - t| \leq \delta \end{array} \right\}.$$

Proposition D.2

- (i) The restriction of w to $C[0, 1] \times (0, 1]$ takes values in $[0, \infty)$.
- (ii) For each $\delta \in (0, 1]$ and each $f, g \in C[0, 1]$, we have

$$\left| w(f, \delta) - w(g, \delta) \right| \leq 2 \|f - g\|_\infty.$$

- (iii) For each $\delta \in (0, 1]$, the map $w(\cdot, \delta) : (C[0, 1], \|\cdot\|_\infty) \longrightarrow \mathbb{R} : f \longmapsto w(f, \delta)$ is continuous.

PROOF

For each $\delta \in (0, 1]$, the set

$$D(\delta) := \left\{ (s, t) \in [0, 1] \times [0, 1] \mid |s - t| \leq \delta \right\}$$

is a subset of the compact set $[0, 1] \times [0, 1]$.

- (i) For each $f \in C[0, 1]$, the map

$$[0, 1] \times [0, 1] \longrightarrow \mathbb{R} : (s, t) \longmapsto |f(s) - f(t)|$$

is continuous on the compact set $[0, 1] \times [0, 1]$; hence, it is bounded and attains its supremum on $[0, 1] \times [0, 1]$; in particular, its supremum on $[0, 1] \times [0, 1]$ is a (finite non-negative) real number. Hence,

$$\begin{aligned} w(f, \delta) &:= \sup \left\{ |f(s) - f(t)| \mid \begin{array}{l} (s, t) \in [0, 1] \times [0, 1] \\ |s - t| \leq \delta \end{array} \right\} = \sup_{(s, t) \in D(\delta)} \left\{ |f(s) - f(t)| \right\} \\ &\leq \sup_{(s, t) \in [0, 1] \times [0, 1]} \left\{ |f(s) - f(t)| \right\} < \infty. \end{aligned}$$

This proves (i).

- (ii) Recall that for any $a, b \in \mathbb{R}$, we have

$$|a| = |a - b + b| \leq |a - b| + |b|,$$

which in turn implies:

$$|a| - |b| \leq |a - b|.$$

Consequently, for each $f, g \in C[0, 1]$ and each $s, t \in [0, 1]$, we have:

$$\begin{aligned} \left| |f(s) - f(t)| - |g(s) - g(t)| \right| &\leq \left| |f(s) - f(t) - g(s) + g(t)| \right| \\ &\leq \left| |f(s) - g(s)| + |f(t) - g(t)| \right| \leq 2 \|f - g\|_\infty. \end{aligned}$$

On the other hand, note that for any $g \in C[0, 1]$ and any $(s, t) \in [0, 1] \times [0, 1]$ with $|s - t| \leq \delta$, i.e. $(s, t) \in D(\delta)$, we have

$$\left| g(s) - g(t) \right| \leq \sup_{(\xi, \zeta) \in D(\delta)} \left\{ |g(\xi) - g(\zeta)| \right\} =: w(g, \delta),$$

hence,

$$-w(g, \delta) \leq -\left|g(s) - g(t)\right|, \quad \text{for any } g \in C[0, 1] \text{ and any } (s, t) \in D(\delta).$$

Thus, we see that, for any $f, g \in C[0, 1]$ and any $(s, t) \in D(\delta)$, we have

$$\left|f(s) - f(t)\right| - w(g, \delta) \leq \left|f(s) - f(t)\right| - \left|g(s) - g(t)\right| \leq 2\|f - g\|_\infty.$$

Taking supremum of the left-hand side of the preceding inequality over $(s, t) \in D(\delta)$ now yields:

$$\begin{aligned} w(f, \delta) - w(g, \delta) &= \sup_{(\xi, \zeta) \in D(\delta)} \left\{ \left|f(\xi) - f(\zeta)\right| \right\} - w(g, \delta) \\ &= \sup_{(\xi, \zeta) \in D(\delta)} \left\{ \left|f(\xi) - f(\zeta)\right| - w(g, \delta) \right\} \\ &\leq 2\|f - g\|_\infty. \end{aligned}$$

Interchanging f and g in the preceding inequality now yields:

$$\left|w(f, \delta) - w(g, \delta)\right| \leq 2\|f - g\|_\infty.$$

This completes the proof of (ii).

(iii) This is an immediate consequence of (ii). □

Corollary D.3

For any $\delta \in (0, 1]$, the map $w(\cdot, \delta) : C[0, 1] \rightarrow \mathbb{R}$ is an \mathbb{R} -valued random variable (i.e. an \mathbb{R} -valued Borel measurable function).

PROOF $w(\cdot, \delta)$ is continuous by the preceding Theorem, and hence, Borel measurable, by Corollary B.4. □

Proposition D.4

For each fixed $f \in C[0, 1]$, the map $w(f, \cdot) : (0, 1] \rightarrow \mathbb{R}$ is a non-decreasing function on $(0, 1]$.

PROOF For $\delta_1, \delta_2 \in (0, 1]$ with $\delta_1 \leq \delta_2$, we have

$$D(\delta_1) := \left\{ (s, t) \in [0, 1] \times [0, 1] \mid |s - t| \leq \delta_1 \right\} \subset \left\{ (s, t) \in [0, 1] \times [0, 1] \mid |s - t| \leq \delta_2 \right\} =: D(\delta_2).$$

Hence, for each $f \in C[0, 1]$ and each $\delta_1, \delta_2 \in (0, 1]$ with $\delta_1 \leq \delta_2$, we have

$$w(f, \delta_1) := \sup_{(s, t) \in D(\delta_1)} \left\{ |f(s) - f(t)| \right\} \leq \sup_{(s, t) \in D(\delta_2)} \left\{ |f(s) - f(t)| \right\} =: w(f, \delta_2).$$

□

Proposition D.5 (Theorem 7.4, [3])

Suppose:

- $\delta > 0$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ satisfy:

$$\min_{1 \leq i \leq n} \left\{ t_i - t_{i-1} \right\} \geq \delta.$$

- $C[0, 1]$ is the Banach space of continuous \mathbb{R} -valued functions defined on $[0, 1]$ equipped with the supremum norm.

Then, the following statements are true:

- (i) For each $f \in C[0, 1]$, we have:

$$w(f, \delta) \leq 3 \cdot \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1}, t_i]} |f(s) - f(t_{i-1})| \right\}.$$

- (ii) For each $\varepsilon > 0$ and each Borel probability measure P on the (separable) Banach space $(C[0, 1], \|\cdot\|_\infty)$, we have:

$$P\left(\left\{f \in C[0, 1] \mid w(f, \delta) \geq 3\varepsilon\right\}\right) \leq \sum_{i=1}^n P\left(\left\{f \in C[0, 1] \mid \sup_{s \in [t_{i-1}, t_i]} |f(s) - f(t_{i-1})| \geq \varepsilon\right\}\right).$$

PROOF

- (i) First, note that

$$\min_{1 \leq i \leq n} \left\{ \begin{array}{l} |s - t| \leq \delta \\ |t_i - t_{i-1}| \geq \delta \end{array} \right\} \implies \left\{ \begin{array}{ll} \text{either } s, t \in [t_{i-1}, t_i], & \text{for some } i \in \{1, 2, \dots, n\} \\ \text{or } s, t \in [t_{i-1}, t_i] \cup [t_i - t_{i-2}], & \text{for some } i \in \{2, \dots, n\} \end{array} \right\}$$

For the case in which both s and t lie in the same subinterval $[t_{i-1}, t_i]$, for some $i \in \{1, 2, \dots, n\}$, we have

$$|f(s) - f(t)| \leq |f(s) - f(t_{i-1})| + |f(t_{i-1}) - f(t)| \leq 2 \cdot \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1}, t_i]} |f(s) - f(t_{i-1})| \right\}.$$

For the case in which s and t lie in adjacent subintervals, say $s \in [t_{i-2}, t_{i-1}]$ and $t \in [t_{i-1}, t_i]$, for some $i \in \{2, \dots, n\}$, we have

$$\begin{aligned} |f(s) - f(t)| &\leq |f(s) - f(t_{i-2})| + |f(t_{i-2}) - f(t_{i-1})| + |f(t_{i-1}) - f(t)| \\ &\leq 3 \cdot \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1}, t_i]} |f(s) - f(t_{i-1})| \right\}. \end{aligned}$$

Thus we see that, for any $f \in C[0, 1]$ and any $(s, t) \in [0, 1] \times [0, 1]$ with $|s - t| \leq \delta$, we have

$$|f(s) - f(t)| \leq 3 \cdot \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1}, t_i]} |f(s) - f(t_{i-1})| \right\},$$

which implies, for each $f \in C[0, 1]$,

$$w(f, \delta) := \sup \left\{ |f(s) - f(t)| \mid \begin{array}{l} (s, t) \in [0, 1] \times [0, 1] \\ |s - t| \leq \delta \end{array} \right\} \leq 3 \cdot \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1}, t_i]} |f(s) - f(t_{i-1})| \right\}.$$

This proves (i).

- (ii) By (i), we see that, for any $\varepsilon, \delta > 0$ and any $f \in C[0, 1]$, we have:

$$3\varepsilon \leq w(f, \delta) \implies \varepsilon \leq \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1}, t_i]} |f(s) - f(t_{i-1})| \right\}.$$

For any $\varepsilon, \delta > 0$ and any $C[0, 1]$ -valued random variable $X : (\Omega, \mathcal{A}, \mu) \rightarrow (C[0, 1], \|\cdot\|_\infty)$, we have:

$$\begin{aligned} \left\{ f \in C[0, 1] \mid 3\varepsilon \leq w(f, \delta) \right\} &\subset \left\{ f \in C[0, 1] \mid \varepsilon \leq \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1}, t_i]} |f(s) - f(t_{i-1})| \right\} \right\} \\ &= \bigcup_{i=1}^n \left\{ f \in C[0, 1] \mid \varepsilon \leq \sup_{s \in [t_{i-1}, t_i]} |f(s) - f(t_{i-1})| \right\}, \end{aligned}$$

and (ii) now follows by sub-additivity of measures. □

Corollary D.6 (Corollary of Theorem 7.4, [3])

Suppose $\{P_n\}_{n \in \mathbb{N}}$ is a sequence of Borel probability measures on $(C[0, 1], \|\cdot\|_\infty)$.

Then, (i) implies (ii):

(i) For each $\varepsilon, \eta > 0$, there exist $\delta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\delta} \cdot P_n \left(\left\{ f \in C[0, 1] \mid \varepsilon \leq \sup_{s \in [t, \min\{1, t+\delta\}]} |f(s) - f(t)| \right\} \right) \leq \eta, \quad \text{for each } t \in [0, 1] \text{ and each } n \geq n_0.$$

(ii) For each $\varepsilon, \eta > 0$, there exist $\delta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that

$$P_n \left(\left\{ f \in C[0, 1] \mid \varepsilon \leq w(f, \delta) \right\} \right) \leq \eta, \quad \text{for each } n \geq n_0.$$

PROOF Suppose (i) holds and let $\varepsilon, \eta > 0$ be given. By (i), there exists $\delta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\delta} \cdot P_n \left(\left\{ f \in C[0, 1] \mid \frac{\varepsilon}{3} \leq \sup_{s \in [t, \min\{1, t+\delta\}]} |f(s) - f(t)| \right\} \right) \leq \eta, \quad \text{for each } t \in [0, 1] \text{ and each } n \geq n_0.$$

Now, let $t_0 = 0$, and $t_i = i\delta$, for $i = 1, 2, 3, \dots, k := \lfloor 1/\delta \rfloor$. Then, the preceding inequality and the preceding Theorem together imply:

$$\begin{aligned} P_n \left(\left\{ f \in C[0, 1] \mid \varepsilon \leq w(f, \delta) \right\} \right) &\leq \sum_{i=1}^k P_n \left(\left\{ f \in C[0, 1] \mid \frac{\varepsilon}{3} \leq \sup_{s \in [t_{i-1}, t_i]} |f(s) - f(t_{i-1})| \right\} \right) \\ &\leq k \cdot \delta \cdot \eta = \lfloor 1/\delta \rfloor \cdot \delta \cdot \eta \leq 1 \cdot \eta = \eta, \quad \text{for each } n \geq n_0. \end{aligned}$$

This completes the proof of the Corollary. □

E The Arzelà-Ascoli Theorem: compactness of subsets of $C(X)$

Recall that the space $C(X)$ of continuous \mathbb{R} -valued functions defined on a compact topological space X equipped with the supremum norm is a complete metric space (see Theorem 9.3, [1]). The Arzelà-Ascoli Theorem characterizes compactness of subsets of $C(X)$.

Definition E.1 (Equicontinuity)

Let X be a topological space and (Y, d) a metric space. Let Y^X denote the set of arbitrary functions from X into Y .

- A subset $S \subset Y^X$ is said to be **equicontinuous at** $x_0 \in X$ if, for each $\varepsilon > 0$, there exists an open subset $V \subset X$ satisfying:

$$x_0 \in V, \quad \text{and} \quad \sup_{(x,f) \in V \times S} \left\{ d(f(x), f(x_0)) \right\} \leq \varepsilon.$$

- A subset $S \subset Y^X$ is said to be **equicontinuous** if it is equicontinuous at each $x_0 \in X$.

Proposition E.2

Let X be a topological space and (Y, d) a metric space. Let Y^X denote the set of arbitrary functions from X into Y . Suppose $x_0 \in X$ and $S_1, S_2 \subset Y^X$. Then,

$$\left. \begin{array}{l} S_1 \subset S_2, \text{ and} \\ S_2 \text{ is equicontinuous at } x_0 \end{array} \right\} \implies S_1 \text{ is equicontinuous at } x_0$$

PROOF Let $\varepsilon > 0$ be given. By the equicontinuity of S_2 at x_0 , there exists open $V \subset X$ such that

$$x_0 \in V \quad \text{and} \quad \sup_{(x,f) \in V \times S_2} \left\{ d(f(x), f(x_0)) \right\} \leq \varepsilon.$$

However, the hypothesis $S_1 \subset S_2$ implies that

$$\sup_{(x,f) \in V \times S_1} \left\{ d(f(x), f(x_0)) \right\} \leq \sup_{(x,f) \in V \times S_2} \left\{ d(f(x), f(x_0)) \right\}$$

which immediately implies the following conditions hold:

$$x_0 \in V \quad \text{and} \quad \sup_{(x,f) \in V \times S_1} \left\{ d(f(x), f(x_0)) \right\} \leq \varepsilon.$$

This proves the equicontinuity of S_1 at x_0 , as required. □

Definition E.3 (Uniform equicontinuity)

Let (X, ρ) and (Y, d) two metric spaces. Let Y^X denote the set of arbitrary functions from X into Y . A subset $S \subset Y^X$ is said to be **uniformly equicontinuous** if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\sup \left\{ d(f(x_1), f(x_2)) \mid \begin{array}{l} f \in S, x_1, x_2 \in X, \\ \rho(x_1, x_2) < \delta \end{array} \right\} \leq \varepsilon.$$

Proposition E.4

Let (X, ρ) and (Y, d) two metric spaces. Let Y^X denote the set of arbitrary functions from X into Y . Then, the following are true:

- (i) Uniform equicontinuity of a subset $S \subset Y^X$ implies equicontinuity of S .
- (ii) Suppose furthermore that (X, ρ) is compact. Then, equicontinuity of a subset $S \subset Y^X$ implies uniform equicontinuity of S .

PROOF

- (i) Suppose $S \subset Y^X$ is uniformly equicontinuous; we seek to prove that S is also equicontinuous. Let $x_0 \in X$ and $\varepsilon > 0$. By uniform equicontinuity of S , there exists $\delta > 0$ such that:

$$\sup \left\{ d(f(x_1), f(x_2)) \mid \begin{array}{l} f \in S, x_1, x_2 \in X, \\ \rho(x_1, x_2) < \delta \end{array} \right\} \leq \varepsilon.$$

Let $V(x_0) := \{x \in X \mid \rho(x, x_0) < \delta\}$. Then, $V(x_0)$ is an open subset of X , with $x_0 \in V(x_0)$, and

$$\sup_{(x,f) \in V(x_0) \times S} \left\{ d(f(x), f(x_0)) \right\} \leq \sup \left\{ d(f(x_1), f(x_2)) \mid \begin{array}{l} f \in S, x_1, x_2 \in X, \\ \rho(x_1, x_2) < \delta \end{array} \right\} \leq \varepsilon.$$

This proves the equicontinuity of S .

- (ii) Suppose (X, ρ) is compact and $S \subset Y^X$ is equicontinuous. Let $\varepsilon > 0$ be given. By equicontinuity of S , for each $x \in X$, there exists an open ball $B(x, \delta_x) \subset X$ such that

$$\sup_{(\xi, f) \in B(x, \delta_x) \times S} \left\{ d(f(\xi), f(x)) \right\} \leq \frac{\varepsilon}{2}.$$

Thus, $X = \bigcup_{x \in X} B(x, \delta_x/2)$ is an open cover of X . By compactness of X , this open cover admits a finite subcover:

$$X = \bigcup_{i=1}^n B(x_i, \delta_{x_i}/2).$$

Define $\delta := \min_{1 \leq i \leq n} \left\{ \delta_{x_i}/2 \right\} > 0$. Now note the uniform equicontinuity of S will be established once we prove the validity of the following:

Claim: For any $\xi_1, \xi_2 \in X$, and any $f \in S$, we have:

$$\rho(\xi_1, \xi_2) < \delta \implies d(f(\xi_1), f(\xi_2)) \leq \varepsilon.$$

Proof of Claim: Suppose $\rho(\xi_1, \xi_2) < \delta$. Note that $\xi_1 \in B(x_i, \delta_{x_i}/2)$, for some $i = 1, 2, \dots, n$. Next, observe that

$$\rho(x_i, \xi_2) \leq \rho(x_i, \xi_1) + \rho(\xi_1, \xi_2) \leq \frac{\delta_{x_i}}{2} + \delta \leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i}.$$

This shows that both $\xi_1, \xi_2 \in B(x_i, \delta_{x_i})$, which implies

$$d(f(\xi_1), f(x_i)) \leq \frac{\varepsilon}{2}, \quad \text{and} \quad d(f(\xi_2), f(x_i)) \leq \frac{\varepsilon}{2}, \quad \text{for each } f \in S,$$

which in turn implies:

$$d(f(\xi_1), f(\xi_2)) \leq d(f(\xi_1), f(x_i)) + d(f(x_i), f(\xi_2)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \text{for each } f \in S.$$

This completes the proof of the Claim and the uniform equicontinuity of S . □

Theorem E.5 (Arzelà-Ascoli, Theorem 9.10, [1])

Suppose X is a compact topological space and $C(X)$ is the space of continuous \mathbb{R} -valued functions defined on X equipped with the supremum norm. Then, for each $S \subset C(X)$, the following conditions are equivalent:

- (i) S is a compact subset of $C(X)$.
- (ii) S is closed, bounded, and equicontinuous subset of $C(X)$.

PROOF

(i) \implies (ii)

Recall that every compact subset in a metric space is closed and bounded. Thus, it remains only to show that $S \subset C(X)$ is equicontinuous. To this end, let $\varepsilon > 0$ be given. Recall that a metric space is compact if and only if it is complete and totally bounded (Theorem 7.8, [1]). Thus, the compactness hypothesis on S implies S is totally bounded; in particular, there exist $f_1, \dots, f_n \in S$ such that

$$S \subset \bigcup_{i=1}^n B(f_i, \frac{\varepsilon}{3}).$$

Hence, for each $x_0 \in X$, we may define

$$V(x_0) := \bigcap_{i=1}^n f_i^{-1} \left(\left(f_i(x_0) - \frac{\varepsilon}{3}, f_i(x_0) + \frac{\varepsilon}{3} \right) \right).$$

Note that $V(x_0)$ is open and $x_0 \in V(x_0)$. Now, let $f \in S$ and $x \in V(x_0)$ be given. We may choose $i \in \{1, 2, \dots, n\}$ such that $f \in B\left(f_i, \frac{\varepsilon}{3}\right)$, i.e. $\|f - f_i\|_\infty \leq \frac{\varepsilon}{3}$. Hence,

$$\left| f(x) - f(x_0) \right| \leq \left| f(x) - f_i(x) \right| + \left| f_i(x) - f_i(x_0) \right| + \left| f_i(x_0) - f(x_0) \right| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus,

$$\sup_{(x,f) \in V(x_0) \times S} \left\{ \left| f(x) - f(x_0) \right| \right\} \leq \varepsilon.$$

This shows equicontinuity of S at $x_0 \in X$. Since $x_0 \in X$ is arbitrary, we may conclude that S is equicontinuous.

(ii) \implies (i)

Suppose $S \in C(X)$ is closed, bounded, and equicontinuous. We need to show that S is a compact subset of $C(X)$. Recall that every subset of a metric space is compact if and only if it is sequentially compact (Theorem 7.3, [1]). Thus, it suffices to show that every sequence $\{f_n\}_{n \in \mathbb{N}} \subset S$ has a convergent subsequence with limit in S . We start by stating and proving the following:

Claim 1:

For each $k \in \mathbb{N}$, there exists a finite subset $F_k \subset X$ and open neighbourhoods $\{V_y\}_{y \in F_k}$ such that

$$X = \bigcup_{y \in F_k} V_y, \text{ and } \sup \left\{ \left| f(x) - f(y) \right| \mid \begin{array}{l} x \in V_y, y \in F_k \\ f \in S \end{array} \right\} \leq \frac{1}{3k}.$$

Proof of Claim 1: By equicontinuity of S , for each $y \in X$, there exists an open neighbourhood $V_y \subset X$ of $y \in X$ such that

$$\sup \left\{ \left| f(x) - f(y) \right| \mid \begin{array}{l} x \in V_y \\ f \in S \end{array} \right\} \leq \frac{1}{3k}.$$

Thus, $X = \bigcup_{y \in X} V_y$ is an open cover of X . Compactness of X now implies that this open cover of X admits a finite subcover, i.e.

$$X = \bigcup_{y \in F_k} V_y, \text{ for some finite subset } F_k \subset X.$$

Lastly, note that

$$\sup \left\{ \left| f(x) - f(y) \right| \mid \begin{array}{l} x \in V_y, y \in F_k \\ f \in S \end{array} \right\} = \sup_{y \in F_k} \left\{ \sup \left\{ \left| f(x) - f(y) \right| \mid \begin{array}{l} x \in V_y \\ f \in S \end{array} \right\} \right\} \leq \frac{1}{3k}.$$

This completes the proof of Claim 1.

Next, let $F := \bigcup_{k=1}^{\infty} F_k$. Note that F is a countably infinite set. Let $F = \{x_1, x_2, \dots\}$ be an enumeration of F . Recall that we wish to prove that every sequence $\{f_n\}_{n \in \mathbb{N}} \subset S$ contains a convergent subsequence with limit in S . Now, consider the array of real numbers:

$$\begin{array}{cccc} f_1(x_1) & f_2(x_1) & f_3(x_1) & \cdots \\ f_1(x_2) & f_2(x_2) & f_3(x_2) & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

Since $S \subset C(X)$ is bounded (with respect to the $\|\cdot\|_\infty$ norm on $C(X)$), there exists $M > 0$ such that $\sup_{f \in S} \|f\|_\infty \leq M$.

In particular, every row in the above array is bounded. By Theorem A.14, p.538, [2], there exists an increasing sequence of positive integers $n(1), n(2), n(3), \dots$ such that the limit

$$\lim_{i \rightarrow \infty} f_{n(i)}(x_k) \text{ exists, for each } k = 1, 2, \dots$$

Claim 2: $\{f_{n(i)}\}_{i \in \mathbb{N}}$ is a Cauchy sequence in $(C(X), \|\cdot\|_\infty)$.

Proof of Claim 2: For each $k \in \mathbb{N}$, the convergence of $\{f_{n(i)}(x_k)\}_{i \in \mathbb{N}}$ in \mathbb{R} implies that each $\{f_{n(i)}(x_k)\}_{i \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} . Since the set F_k is finite, we see that there exists $m_k \in \mathbb{N}$ such that

$$|f_{n(i)}(y) - f_{n(j)}(y)| < \frac{1}{3k}, \text{ for any } i, j \geq m_k, \text{ and each } y \in F_k.$$

Now, for each $x \in X$, there exists $y \in F_k$ such that $x \in V_y$. Hence, for any $i, j \geq m_k$ and any $x \in X$, we have

$$\begin{aligned} |f_{n(i)}(x) - f_{n(j)}(x)| &\leq |f_{n(i)}(x) - f_{n(i)}(y)| + |f_{n(i)}(y) - f_{n(j)}(y)| + |f_{n(j)}(y) - f_{n(j)}(x)| \\ &\leq \frac{1}{3k} + \frac{1}{3k} + \frac{1}{3k} = \frac{1}{k}. \end{aligned}$$

In other words,

$$\|f_{n(i)} - f_{n(j)}\|_\infty \leq \frac{1}{k}, \text{ for any } i, j \geq m_k.$$

This shows that $\{f_{n(i)}\}_{i \in \mathbb{N}}$ is indeed a Cauchy sequence in $(C(X), \|\cdot\|_\infty)$ and completes the proof of Claim 2.

Lastly, by Theorem 9.3, [1], $(C(X), \|\cdot\|_\infty)$ is a complete metric space. Thus, the Cauchy sequence $\{f_{n(i)}\}_{i \in \mathbb{N}} \subset C(X)$ converges to some element $f_0 \in C(X)$. Since $S \subset C(X)$ is, by hypothesis, a closed subset of $C(X)$, we see furthermore that $f_0 \in S$. This proves the sequential compactness of S and completes the proof of the Arzelà-Ascoli Theorem. \square

Proposition E.6

Suppose X is a compact topological space and $C(X)$ is the space of continuous \mathbb{R} -valued functions defined on X equipped with the supremum norm. Let $S \subset C(X)$.

- (i) *S is equicontinuous at $x_0 \in X$ if and only if its closure \bar{S} in $C(X)$ is equicontinuous at x_0 .*
- (ii) *S is equicontinuous if and only if its closure \bar{S} in $C(X)$ is equicontinuous.*

PROOF It is obvious that (ii) is an immediate consequence of (i). Thus, it suffices to establish (i). First, by Proposition E.2, we immediately see that the equicontinuity of \bar{S} at x_0 implies the equicontinuity of S at x_0 . It remains to prove the converse. So, suppose that $S \subset C(X)$ is equicontinuous at $x_0 \in X$. Thus, for each $\varepsilon > 0$, there exists an open subset $V \subset X$ satisfying:

$$x_0 \in V, \text{ and } \sup_{(x,f) \in V \times S} \left\{ |f(x) - f(x_0)| \right\} \leq \varepsilon.$$

Observe that, in order to show the equicontinuity of \bar{S} at x_0 , it suffices to show that the following inequality is also valid:

$$\sup_{(x,g) \in V \times \bar{S}} \left\{ |g(x) - g(x_0)| \right\} \leq \varepsilon.$$

To this end, let $g \in \bar{S} \subset C(X)$. Then, there exist a sequence $f_1, f_2, \dots \in S$ such that

$$\lim_{n \rightarrow \infty} \|f_n - g\|_\infty = \lim_{n \rightarrow \infty} \sup_{x \in X} \left\{ |f_n(x) - g(x)| \right\} = 0.$$

Consequently, for any $x \in V$ and $n \in \mathbb{N}$, we have:

$$\begin{aligned} \left| g(x) - g(x_0) \right| &\leq \left| g(x) - f_n(x) \right| + \left| f_n(x) - f_n(x_0) \right| + \left| f_n(x_0) - g(x_0) \right| \\ &\leq \|g - f_n\|_\infty + \varepsilon + \|f_n - g\|_\infty \longrightarrow 0 + \varepsilon + 0 = \varepsilon. \end{aligned}$$

This implies:

$$\sup_{(x,g) \in V \times \overline{S}} \left\{ \left| g(x) - g(x_0) \right| \right\} \leq \varepsilon,$$

as desired. This completes the proof of the Proposition. □

Theorem E.7 (Theorem 7.2, p.81, [3])

Let $C[0, 1]$ denote the space of continuous \mathbb{R} -valued functions defined on the closed unit interval $[0, 1]$ equipped with the supremum norm. Then, for each subset $S \subset C[0, 1]$, the following are equivalent:

- (i) S is a relatively compact subset of $C[0, 1]$, i.e. the closure of S is a compact subset of $C[0, 1]$.
- (ii) $\sup_{f \in S} \left\{ \|f\|_\infty \right\} < \infty$, and S is uniformly equicontinuous.
- (iii) $\sup_{f \in S} \left\{ |f(0)| \right\} < \infty$, and for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{f \in S} \left\{ w(f, \delta) \right\} = \sup \left\{ \left| f(t_1) - f(t_2) \right| \mid f \in S, t_1, t_2 \in [0, 1], |t_1 - t_2| < \delta \right\} \leq \varepsilon.$$

- (iv) $\sup_{f \in S} \left\{ |f(0)| \right\} < \infty$, and $\lim_{\delta \rightarrow 0^+} \sup_{f \in S} \left\{ w(f, \delta) \right\} = 0$.

PROOF

(i) \iff (ii)

- | | | | |
|-----|--------|---|---|
| (i) | \iff | \overline{S} is compact, | (by definition of relative compactness) |
| | \iff | \overline{S} is bounded and equicontinuous, | (by the Arzelà-Ascoli Theorem) |
| | \iff | S is bounded and equicontinuous, | (by Proposition E.6) |
| | \iff | S is bounded and uniformly equicontinuous, | (by Proposition E.4) |
| | \iff | (ii). | |

(ii) \iff (iii)

Noting that the second condition in (iii) is precisely uniform equicontinuity of S , we see immediately that (ii) \implies (iii).

Conversely, we may conclude that (iii) \implies (ii) once we prove that (iii) implies $\sup_{f \in S} \left\{ \|f\|_\infty \right\} < \infty$. To this end, take $\varepsilon = 1$. Then, by the second condition in (iii) (uniform equicontinuity of S), there exists $\delta > 0$ such that

$$\sup \left\{ \left| f(t_1) - f(t_2) \right| \mid f \in S, t_1, t_2 \in [0, 1], |t_1 - t_2| < \delta \right\} \leq \varepsilon := 1.$$

Next, choose $k \in \mathbb{N}$ sufficiently large such that $\frac{1}{k} < \delta$. Hence, for any $f \in S$ and any $t \in [0, 1]$, we have

$$\begin{aligned} |f(t)| &= \left| f(t) - f\left(\frac{k-1}{k} \cdot t\right) + f\left(\frac{k-1}{k} \cdot t\right) - \cdots - f\left(\frac{1}{k} \cdot t\right) + f\left(\frac{1}{k} \cdot t\right) - f(0) + f(0) \right| \\ &\leq |f(0)| + \sum_{i=1}^k \left| f\left(\frac{i}{k} \cdot t\right) - f\left(\frac{i-1}{k} \cdot t\right) \right| \leq |f(0)| + k \cdot 1 \\ &\leq \sup_{f \in S} \{ |f(0)| \} + k, \end{aligned}$$

where the last inequality follows from the first condition in (iii). Consequently, we see that, for each $f \in S$,

$$\|f\|_\infty := \sup_{t \in [0, 1]} \{ |f(t)| \} \leq \sup_{f \in S} \{ |f(0)| \} + k < \infty,$$

which in turn implies

$$\sup_{f \in S} \{ \|f\|_\infty \} \leq \sup_{f \in S} \{ |f(0)| \} + k < \infty.$$

This completes the proof that (ii) \iff (iii).

(iii) \iff (iv)

This follows trivially from the definition of the right-limit at zero of a \mathbb{R} -valued function defined on an interval $[0, \delta_0)$, for some $\delta_0 > 0$. □

F Tightness of sequences of Borel measures on $(C[0, 1], \|\cdot\|_\infty)$

Theorem F.1 (Theorem 7.3, [3])

Suppose $\{P_n\}_{n \in \mathbb{N}}$ is a sequence of Borel probability measures on $(C[0, 1], \|\cdot\|_\infty)$.

Then, the following are equivalent:

- (i) $\{P_n\}_{n \in \mathbb{N}}$ is tight.
- (ii) For each $\varepsilon, \eta > 0$, there exist $a > 0$ and $\delta \in (0, 1)$ such that

$$P_n\left(\left\{ f \in C[0, 1] \mid |f(0)| \geq a \right\}\right) \leq \eta, \text{ and } P_n\left(\left\{ f \in C[0, 1] \mid w(f, \delta) \geq \varepsilon \right\}\right) \leq \eta, \text{ for each } n \in \mathbb{N}.$$

- (iii) For each $\varepsilon, \eta > 0$, there exist $a > 0$, $\delta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that

$$P_n\left(\left\{ f \in C[0, 1] \mid |f(0)| \geq a \right\}\right) \leq \eta, \text{ and } P_n\left(\left\{ f \in C[0, 1] \mid w(f, \delta) \geq \varepsilon \right\}\right) \leq \eta, \text{ for each } n \geq n_0.$$

- (iv) For each $\eta > 0$, there exist $a > 0$ and $n_0 \in \mathbb{N}$ such that

$$P_n\left(\left\{ f \in C[0, 1] \mid |f(0)| \geq a \right\}\right) \leq \eta, \text{ for each } n \geq n_0,$$

and, for each $\varepsilon > 0$,

$$\lim_{\delta \rightarrow 0^+} \limsup_{n \rightarrow \infty} P_n\left(\left\{ f \in C[0, 1] \mid w(f, \delta) \geq \varepsilon \right\}\right) = 0.$$

PROOF

(i) \implies (ii)

Suppose $\{P_n\}_{n \in \mathbb{N}}$ is tight and let $\varepsilon, \eta > 0$ be given. By tightness of $\{P_n\}_{n \in \mathbb{N}}$, there exists a compact subset $K \subset C[0, 1]$ such that $1 - \eta \leq P_n(K) \leq 1$, for every $n \in \mathbb{N}$. By Theorem E.7(iii),

compactness of $K \subset C[0, 1]$

$$\implies \exists a > 0 \text{ such that } \sup_{f \in K} \{ |f(0)| \} < a < \infty, \quad \text{and} \quad \exists \delta > 0 \text{ such that } \sup_{f \in K} \{ w(f, \delta) \} < \varepsilon$$

$$\implies \exists a > 0 \text{ such that } K \subset \left\{ f \in C[0, 1] \mid |f(0)| < a \right\}, \text{ and } \exists \delta > 0 \text{ such that } K \subset \left\{ f \in C[0, 1] \mid w(f, \delta) < \varepsilon \right\}$$

Hence, with $a > 0$ and $\delta \in (0, 1)$ as chosen above, we see that, for each $n \in \mathbb{N}$, we have:

$$P_n \left(\left\{ f \in C[0, 1] \mid |f(0)| \geq a \right\} \right) \leq P_n(C[0, 1] \setminus K) = 1 - P_n(K) \leq \eta,$$

and

$$P_n \left(\left\{ f \in C[0, 1] \mid w(f, \delta) \geq \varepsilon \right\} \right) \leq P_n(C[0, 1] \setminus K) = 1 - P_n(K) \leq \eta.$$

This completes the proof that (i) \implies (ii).

(ii) \implies (i)

Let $\eta > 0$ be given. We need to show that $\{P_n\}_{n \in \mathbb{N}}$ is tight; equivalently, we need to find a compact subset $K \subset C[0, 1]$ such that $1 - \eta < P_n(K) \leq 1$, for each $n \in \mathbb{N}$.

By (ii), we may choose $a > 0$ such that

$$P_n \left(\left\{ f \in C[0, 1] \mid |f(0)| \geq a \right\} \right) \leq \frac{\eta}{2}, \quad \text{for each } n \in \mathbb{N}.$$

Let $B_0 := \left\{ f \in C[0, 1] \mid |f(0)| < a \right\}$. Note that $P(B_0^c) \leq \frac{\eta}{2}$.

Next, applying (ii) again, for each $k \in \mathbb{N}$, we may choose $\delta_k > 0$ such that

$$P_n \left(\left\{ f \in C[0, 1] \mid w(f, \delta_k) \geq \frac{1}{k} \right\} \right) \leq \frac{\eta}{2^{k+1}}, \quad \text{for each } n \in \mathbb{N}.$$

Let $B_k := \left\{ f \in C[0, 1] \mid w(f, \delta_k) < \frac{1}{k} \right\}$, for $k \in \mathbb{N}$. Note that $P(B_k^c) \leq \frac{\eta}{2^{k+1}}$, for each $k \in \mathbb{N}$.

Now, let $A := \bigcap_{k=0}^{\infty} B_k \subset C[0, 1]$, and $K := \overline{A} \subset C[0, 1]$. Note that the desired implication (ii) \implies (i) follows from the following two Claims:

Claim 1: $1 - \eta \leq P_n(K) \leq 1$, for every $n \in \mathbb{N}$. In particular, K is non-empty.

Claim 2: A is a relatively compact subset of $C[0, 1]$, hence its closure K is a compact subset of $C[0, 1]$.

Proof of Claim 1: Note that

$$A^c = \left(\bigcap_{k=0}^{\infty} B_k \right)^c = \bigcup_{k=0}^{\infty} B_k^c,$$

which implies, for each $n \in \mathbb{N}$,

$$\begin{aligned} P_n(A^c) &= P_n \left(\bigcup_{k=0}^{\infty} B_k^c \right) \leq \sum_{k=0}^{\infty} P_n(B_k^c) = P_n(B_0^c) + \sum_{k=1}^{\infty} P_n(B_k^c) \\ &\leq \frac{\eta}{2} + \sum_{k=1}^{\infty} \frac{\eta}{2^{k+1}} = \dots = \eta, \end{aligned}$$

which in turn implies

$$1 - \eta \leq 1 - P_n(A^c) = P_n(A) \leq P_n(\overline{A}) = P_n(K), \quad \text{for each } n \in \mathbb{N}.$$

This proves Claim 1.

Proof of Claim 2: Note that $f \in A \implies f \in B_0 \implies |f(0)| < a < \infty$. Hence, $\sup_{f \in A} \{|f(0)|\} \leq a < \infty$. Secondly, let $\nu > 0$ be given. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \nu$. Then, $f \in A \implies f \in B_k \implies w(f, \delta_k) < \frac{1}{k}$, which in turn implies:

$$\sup_{f \in A} \{w(f, \delta_k)\} \leq \frac{1}{k} < \nu.$$

By Theorem E.7(iii) (or, essentially, the Arzelà-Ascoli Theorem), A is a relatively compact subset of $C[0, 1]$. Its closure $K := \overline{A}$ is therefore a compact subset of $C[0, 1]$. This proves Claim 2, and completes the proof of (ii) \implies (i).

(ii) \iff (iii)

It is obvious that (ii) \implies (iii). It remains to prove the reverse implication (iii) \implies (ii). So, suppose (iii) holds. Let $\varepsilon, \eta > 0$ be given. By (iii), there exists $a' > 0$, $\delta' \in (0, 1)$, and $n_0 \in \mathbb{N}$ such that

$$P_n\left(\left\{f \in C[0, 1] \mid |f(0)| \geq a'\right\}\right) \leq \eta, \quad \text{and} \quad P_n\left(\left\{f \in C[0, 1] \mid w(f, \delta') \geq \varepsilon\right\}\right) \leq \eta, \quad \text{for each } n \geq n_0.$$

On the other hand, since $(C[0, 1], \|\cdot\|_\infty)$ is separable and complete, every Borel probability measure on $(C[0, 1], \|\cdot\|_\infty)$ is tight, by Theorem 1.3, p.8, [3]. Thus, by the tightness of each $P_1, P_2, \dots, P_{n_0-1}$, and the equivalence (i) \iff (ii), which we have already established, there exist $a_1, a_2, \dots, a_{n_0-1} > 0$ and $\delta_1, \delta_2, \dots, \delta_{n_0-1} \in (0, 1)$ such that

$$P_i\left(\left\{f \in C[0, 1] \mid |f(0)| \geq a_i\right\}\right) \leq \eta, \quad \text{and} \quad P_i\left(\left\{f \in C[0, 1] \mid w(f, \delta_i) \geq \varepsilon\right\}\right) \leq \eta, \quad \text{for } i = 1, 2, \dots, n_0 - 1.$$

Now, let $a := \max\{a', a_1, a_2, \dots, a_{n_0-1}\}$ and $\delta := \min\{\delta', \delta_1, \delta_2, \dots, \delta_{n_0-1}\}$. Then,

$$\begin{aligned} P_n\left(\left\{f \in C[0, 1] \mid |f(0)| \geq a\right\}\right) &\leq P_n\left(\left\{f \in C[0, 1] \mid |f(0)| \geq a'\right\}\right) \leq \eta, \quad \text{for } n \geq n_0, \quad \text{and} \\ P_n\left(\left\{f \in C[0, 1] \mid |f(0)| \geq a\right\}\right) &\leq P_n\left(\left\{f \in C[0, 1] \mid |f(0)| \geq a_n\right\}\right) \leq \eta, \quad \text{for } n = 1, 2, \dots, n_0 - 1, \end{aligned}$$

which together imply

$$P_n\left(\left\{f \in C[0, 1] \mid |f(0)| \geq a\right\}\right) \leq \eta, \quad \text{for each } n \in \mathbb{N}.$$

Next, recall that for each fixed $f \in C[0, 1]$, the map $\delta \mapsto w(f, \delta)$ is non-decreasing in δ , by Proposition D.4. We thus see that:

$$\begin{aligned} P_n\left(\left\{f \in C[0, 1] \mid w(f, \delta) \geq \varepsilon\right\}\right) &\leq P_n\left(\left\{f \in C[0, 1] \mid w(f, \delta') \geq \varepsilon\right\}\right) \leq \eta, \quad \text{for } n \geq n_0, \quad \text{and} \\ P_n\left(\left\{f \in C[0, 1] \mid w(f, \delta) \geq \varepsilon\right\}\right) &\leq P_n\left(\left\{f \in C[0, 1] \mid w(f, \delta_n) \geq \varepsilon\right\}\right) \leq \eta, \quad \text{for } n = 1, 2, \dots, n_0 - 1, \end{aligned}$$

which together imply

$$P_n\left(\left\{f \in C[0, 1] \mid w(f, \delta) \geq \varepsilon\right\}\right) \leq \eta, \quad \text{for each } n \in \mathbb{N}.$$

This proves (iii) \implies (ii), and completes the proof of the equivalence (ii) \iff (iii).

(iii) \implies (iv)

- (iii) \iff for each $\varepsilon, \eta > 0$, there exist $a > 0, \delta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that
 $P_n\left(\left\{f \in C[0, 1] \mid |f(0)| \geq a\right\}\right) \leq \eta$, and $P_n\left(\left\{f \in C[0, 1] \mid w(f, \delta) \geq \varepsilon\right\}\right) \leq \eta$, for each $n \geq n_0$
- \iff for each $\eta > 0, \exists a > 0$ and $n_0 \in \mathbb{N}$ such that $P_n\left(\left\{f \in C[0, 1] \mid |f(0)| \geq a\right\}\right) \leq \eta$, for each $n \geq n_0$, and
 for each $\varepsilon, \eta > 0, \exists \delta \in (0, 1)$ such that $\limsup_{n \rightarrow \infty} P_n\left(\left\{f \in C[0, 1] \mid w(f, \delta) \geq \varepsilon\right\}\right) \leq \eta$
- \iff for each $\eta > 0, \exists a > 0$ and $n_0 \in \mathbb{N}$ such that $P_n\left(\left\{f \in C[0, 1] \mid |f(0)| \geq a\right\}\right) \leq \eta$, for each $n \geq n_0$, and
 for each $\varepsilon > 0, \lim_{\delta \rightarrow 0^+} \limsup_{n \rightarrow \infty} P_n\left(\left\{f \in C[0, 1] \mid w(f, \delta) \geq \varepsilon\right\}\right) = 0$
- \iff (iv)

The proof of the Theorem is now complete. □

G Weak convergence of Borel probability measures on $(C[0, 1], \|\cdot\|_\infty)$

Theorem G.1

Suppose P, P_1, P_2, \dots are Borel probability measures on $(C[0, 1], \|\cdot\|_\infty)$. If

$$P_n \circ \pi_{t_1 t_2 \dots t_k}^{-1} \xrightarrow{d} P \circ \pi_{t_1 t_2 \dots t_k}^{-1}, \quad \text{as } n \rightarrow \infty, \quad \text{for any } t_1, t_2, \dots, t_k \in [0, 1],$$

then the limit of every weakly convergent subsequence of $\{P_n\}$ must be P .

PROOF Suppose $P_{n(i)} \xrightarrow{d} Q$, as $i \rightarrow \infty$, where Q is some Borel probability measure on $(C[0, 1], \|\cdot\|_\infty)$. We need to show that Q in fact must be P . Since, for any $t_1, t_2, \dots, t_k \in [0, 1]$, the map $\pi_{t_1 t_2 \dots t_k} : C[0, 1] \rightarrow \mathbb{R}^k : f \mapsto (f(t_1), \dots, f(t_k))$ is continuous, it follows that $P_{n(i)} \circ \pi_{t_1 t_2 \dots t_k}^{-1} \xrightarrow{d} Q \circ \pi_{t_1 t_2 \dots t_k}^{-1}$, as $i \rightarrow \infty$, for any $t_1, t_2, \dots, t_k \in [0, 1]$, by the Continuous Mapping Theorem (Theorem 2.7, [3], or simply see remark on p.20, [3]). Then, for each bounded continuous function $\varphi : \mathbb{R}^k \rightarrow \mathbb{R}$, we have

$$\int_{\mathbb{R}^k} \varphi \, d(Q \circ \pi_{t_1 \dots t_k}^{-1}) = \lim_{i \rightarrow \infty} \int_{\mathbb{R}^k} \varphi \, d(P_{n(i)} \circ \pi_{t_1 \dots t_k}^{-1}) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^k} \varphi \, d(P_n \circ \pi_{t_1 \dots t_k}^{-1}) = \int_{\mathbb{R}^k} \varphi \, d(P \circ \pi_{t_1 \dots t_k}^{-1}),$$

where the first and third equalities follow directly from the definition of weak convergence of probability measures, while the second equality follows from the elementary fact that every subsequence of a convergent sequence of real numbers converges to the same limit as the full sequence. By Theorem 1.2, p.8, [3], we see that $Q \circ \pi_{t_1 \dots t_k}^{-1} = P \circ \pi_{t_1 \dots t_k}^{-1}$, as Borel measures on \mathbb{R}^k , for any $t_1, t_2, \dots, t_k \in [0, 1]$. This in turn implies that $Q \circ \pi_{t_1 \dots t_k}^{-1}(B) = P \circ \pi_{t_1 \dots t_k}^{-1}(B)$, for every Borel subset $B \subset \mathbb{R}^k$. In other words, Q and P agree on the collection of finite-dimensional subsets of $C[0, 1]$. Since the finite-dimensional subsets of $C[0, 1]$ form a separating class (Example 1.3, p.11, [3]), we may now conclude that $Q = P$. This completes the proof of the Theorem. □

Theorem G.2 (Example 5.1, p.57, [3])

Suppose P, P_1, P_2, \dots are Borel probability measures on $(C[0, 1], \|\cdot\|_\infty)$. If

- (i) the sequence $\{P_n\}_{n \in \mathbb{N}}$ of Borel probability measures on $(C[0, 1], \|\cdot\|_\infty)$ is relatively compact, and
- (ii) for any $t_1, t_2, \dots, t_k \in [0, 1]$, we have

$$P_n \circ \pi_{t_1 t_2 \dots t_k}^{-1} \xrightarrow{d} P \circ \pi_{t_1 t_2 \dots t_k}^{-1}, \quad \text{as } n \rightarrow \infty,$$

then $P_n \xrightarrow{d} P$, as $n \rightarrow \infty$.

PROOF Recall that, by Theorem 2.6, p.20, [3], $P_n \xrightarrow{d} P$ if and only if every subsequence of $\{P_n\}$ contains a further subsequence that weakly converges to P . So, let $\{P_{n(i)}\}_{i \in \mathbb{N}}$ be a subsequence of $\{P_n\}_{n \in \mathbb{N}}$. By the relative compactness hypothesis on $\{P_n\}_{n \in \mathbb{N}}$, the subsequence $\{P_{n(i)}\}_{i \in \mathbb{N}}$ contains a weakly convergent further subsequence $\{P_{n(i_m)}\}_{m \in \mathbb{N}}$, say $P_{n(i_m)} \xrightarrow{d} Q$, as $m \rightarrow \infty$, where Q is some Borel probability measure on $(C[0, 1], \|\cdot\|_\infty)$. By Theorem G.1, we see that in fact $Q = P$. Thus, we have shown that every subsequence $\{P_{n(i)}\}_{i \in \mathbb{N}}$ of $\{P_n\}_{n \in \mathbb{N}}$ contains a further subsequence $P_{n(i_m)}$ which weakly converges to P . By Theorem 2.6, p.20, [3], we may now conclude that the full original sequence P_n converges weakly to P . This completes the proof of the Theorem. \square

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