The metric space structures on $\mathbb{R}^n$, $\mathbb{R}^\infty$ and $C([0,1],\mathbb{R})$

1 The metric space structure on $\mathbb{R}^\infty$

Definition 1.1 (The metric space $\mathbb{R}^\infty$, Example 1.2, [1])
Let $\mathbb{R}^\infty$ denotes the set of all infinite sequences of real numbers, i.e.

$$\mathbb{R}^\infty \ := \ \{\ (x_1, x_2, \ldots)\ |\ x_i \in \mathbb{R},\ \text{for each } i \in \mathbb{N}\ \}.$$

Define $\rho : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow [\,0,1\,]$ as follows:

$$\rho(x,y) \ := \ \sum_{n=1}^{\infty} \frac{\min\{\,1\,,|\,x_n - y_n\,|\,\}}{2^n}.$$

Remark 1.2 *Recall that*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \ = \ \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \ = \ \frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}\right) \ = \ 1,$$

which proves indeed that $0 \le \rho(x,y) \le 1$, for any $x, y \in \mathbb{R}^\infty$.

Theorem 1.3

(i) $(\,\mathbb{R}^\infty, \rho\,)$ *is a metric space. Let $\mathbb{R}^\infty$ denote also this metric space in the remainder of this Theorem.*

(ii) *For $x, x^{(1)}, x^{(2)}, x^{(3)}, \ldots, \in \mathbb{R}^\infty$, we have:*

$$\rho\Big(\,x^{(n)},\,x\,\Big) \ \longrightarrow \ 0 \quad \Longleftrightarrow \quad \text{for each } i \in \mathbb{N},\ \lim_{n\to\infty}\left|\,x_i^{(n)} - x_i\,\right| \ = \ 0$$

(iii) *For each $n \in \mathbb{N}$, the "natural projection to the initial segment of length n"*

$$\pi_n \ :\ \mathbb{R}^\infty \ \longrightarrow \ \mathbb{R}^n \ :\ x \ \longmapsto \ (x_1, x_2, \ldots, x_n)$$

is continuous, where $\mathbb{R}^n$ has the usual Euclidean topology.

(iv) *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, let $C_{\mathbb{R}^n}(\pi_n(x), \varepsilon)$ denote the open hypercube in $\mathbb{R}^n$ of side length 2ε centred at $\pi_n(x) \in \mathbb{R}^n$, i.e.*

$$C_{\mathbb{R}^n}(\pi_n(x),\varepsilon) \ \ := \ \left\{\ y \in \mathbb{R}^n \ \middle|\ \begin{array}{c} |\,y_i - x_i\,| \ < \ \varepsilon, \\ i = 1,2,\ldots,n \end{array}\ \right\}$$

Then, its pre-image in $\mathbb{R}^\infty$ under π_n

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \ \ = \ \left\{\ y \in \mathbb{R}^\infty \ \middle|\ \begin{array}{c} |\,y_i - x_i\,| \ < \ \varepsilon, \\ i = 1,2,\ldots,n \end{array}\ \right\}$$

is an open subset of $\mathbb{R}^\infty$.

(v) *The collection*

$$\left\{\ \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \subset \mathbb{R}^\infty \ \middle|\ n \in \mathbb{N}, x \in \mathbb{R}^\infty, \varepsilon > 0\ \right\}$$

of all pre-images under π^n of open hypercubes in $\mathbb{R}^n$, for all $n \in \mathbb{N}$, forms a basis for the topology of $\mathbb{R}^\infty$.

(vi) $\mathbb{R}^\infty$ *is a separable and complete metric space. Hence, every probability measure on $\mathbb{R}^\infty$ is tight.*

Proof

(i) Clearly, ρ is non-negative and symmetric. We now show that, for any $x, y \in \mathbb{R}^\infty$, we have $\rho(x,y) = 0$ implies $x = y$. Indeed,

$$\begin{aligned}\rho(x,y) = 0 \quad &\Longleftrightarrow \quad \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - y_i\,|\,\}}{2^i} \;=\; 0 \\ &\Longleftrightarrow \quad \min\{\,1\,,|\,x_i - y_i\,|\,\} \;=\; 0, \;\; \text{for each } i \in \mathbb{N} \\ &\Longleftrightarrow \quad |\,x_i - y_i\,| \;=\; 0, \;\; \text{for each } i \in \mathbb{N} \\ &\Longleftrightarrow \quad x \;=\; y.\end{aligned}$$

In order to show that ρ is a metric, it remains only to establish the Triangle Inequality. By Lemma A.2, for any $x, y, z \in \mathbb{R}^\infty$, we have

$$\begin{aligned}\rho(x,y) \;&=\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - y_i\,|\,\}}{2^i} \\ &\leq\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - z_i\,|\,\} \;+\; \min\{\,1\,,|\,z_i - y_i\,|\,\}}{2^i} \\ &=\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - z_i\,|\,\}\,\}}{2^i} \;+\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,z_i - y_i\,|\,\}}{2^i} \\ &=\; \rho(x,z) + \rho(z,y),\end{aligned}$$

where we have used the fact that $0 \leq \rho \leq 1$ to split the infinite sum into two terms in second-to-last equality. This proves that ρ satisfies the Triangle Inequality, and it is thus a metric on $\mathbb{R}^\infty$.

$\square$

A Technical Lemmas

Lemma A.1 *Define*

$$\phi \,:\, [\,0,\infty\,) \longrightarrow [\,0,1\,] \,:\, t \longmapsto \min\{\,1\,,t\,\}.$$

Then, ϕ satisfies:

$$\phi(s+t) \;\leq\; \phi(s) \;+\; \phi(t), \;\; \text{for each } s,t \in [0,\infty).$$

Proof For any $s, t \in [0,\infty)$, either $s + t \geq 1$ or $s + t < 1$. If $s + t \geq 1$, then

$$\phi(s+t) \;=\; \min\{1, s+t\} \;=\; 1 \;<\; 2 \;=\; 1+1 \;\leq\; \min\{1,s\} + \min\{1,t\} \;=\; \phi(s) + \phi(t),$$

hence, the required inequality holds. On the other hand, if $s + t < 1$, then we must also have $s < 1$ and $t < 1$ (since $s, t \geq 0$). Hence,

$$\phi(s+t) \;=\; \min\{1, s+t\} \;=\; s+t \;=\; \min\{1,s\} + \min\{1,t\} \;=\; \phi(s) + \phi(t),$$

thus, the required inequality also holds. $\square$

Lemma A.2 *For any $x, y, z \in \mathbb{R}$, we have:*

$$\min\{1, |\,x - y\,|\} \;\leq\; \min\{1, |\,x - z\,|\} \;+\; \min\{1, |\,z - y\,|\}.$$

PROOF Observe that $|\,x-y\,| \;\leq\; |\,x-z\,|+|\,z-y\,|$ implies

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; |\,x-z\,|+|\,z-y\,|.$$

The above inequality, together with $\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; 1$, thus in turn imply:

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; \min\{\,1\,,|\,x-z\,|+|\,z-y\,|\,\}.$$

By Lemma A.1, we therefore have:

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; \min\{\,1\,,|\,x-z\,|+|\,z-y\,|\,\}. \;\leq\; \min\{\,1\,,|\,x-z\,|\,\} \;+\; \min\{\,1\,,|\,z-y\,|\,\},$$

which proves the present Lemma. □

References

[1] BILLINGSLEY, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.