Covariance and Correlation

Study Notes March 19, 2011 Kenneth Chu

1 Covariance

Let X and Y be two random variables, with mean μ_X and μ_Y respectively. The *covariance* $\mathrm{Cov}(X,Y)$ of X and Y is defined as:

$$\mathrm{Cov}(X,Y) \;:=\; E[\,(X-\mu_X)(Y-\mu_Y)\,]\,.$$

Observations

- $\mathrm{Cov}(X,X) = E\,[\,(X-\mu_X)(X-\mu_X)\,] = \mathrm{Var}(X)$.
- $\mathrm{Cov}(X,Y) \;:=\; E[\,(X-\mu_X)(Y-\mu_Y)\,] \;=\; E[\,XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y\,] \;=\; E[\,XY\,] - \mu_X\mu_Y$.
- X, Y independent $\Longrightarrow$ $\mathrm{Cov}(X,Y) = 0$.

  PROOF X, Y independent $\Longrightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y) \Longrightarrow E[\,XY\,] = \displaystyle\int xy\cdot f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$

  $$= \int xy\cdot f_X(x)f_Y(y)\,\mathrm{d}x\,\mathrm{d}y = \left(\int x\cdot f_X(x)\,\mathrm{d}x\right)\left(\int y\cdot f_Y(y)\,\mathrm{d}y\right) = E[\,X\,]\cdot E[\,Y\,] = \mu_X\,\mu_Y\,. \qquad \square$$

Fact Let $X_1, X_2, \ldots, X_n$ be random variables, and $\alpha_1,\ldots,\alpha_n \in \mathbb{R}$. Then,

$$\mathrm{Var}(\alpha_1X_1 + \alpha_2X_2 + \cdots + \alpha_nX_n) \;=\; \sum_{i=1}^{n}\alpha_i^2\,\mathrm{Var}(X_i) \;+ \sum_{1\le i<j\le n}\alpha_i\alpha_j\,\mathrm{Cov}(X_i,X_j)$$

2 Correlation

Let X and Y be two random variables with nonzero variance (equivalently, X, Y are non-constant). Let μ_X and μ_Y be the means of X and Y respectively. Let σ_X^2 and σ_Y^2 be the variances of X and Y respectively. The *correlation* $\rho(X,Y)$ of X and Y is defined as:

$$\rho(X,Y) \;:=\; E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\cdot\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right]$$

Observations

$$\rho(X,Y) \;=\; \mathrm{Cov}\left[\frac{X-\mu_X}{\sigma_X}\,,\;\frac{Y-\mu_Y}{\sigma_Y}\right] \;=\; \frac{\mathrm{Cov}(\,X,Y\,)}{\sigma_X\,\sigma_Y}$$

Theorem 2.1 *For any two random variables X and Y,*

- $|\,\rho(X,Y)\,| \le 1$, *and*
- $|\,\rho(X,Y)\,| = 1$ *if and only if X and Y are affinely dependent, i.e. there exist $\alpha,\beta,\gamma\in\mathbb{R}$, not all zero, such that $\alpha\,X + \beta\,Y + \gamma = 0$ almost surely.*