

## 1 Donsker's Theorem for $(C[0, 1], \|\cdot\|_\infty)$

### Proposition 1.1

- Let  $\xi_1, \xi_2, \dots : \Omega \rightarrow \mathbb{R}$  be a sequence of independent and identically distributed  $\mathbb{R}$ -valued random variables defined on the probability space  $(\Omega, \mathcal{A}, \mu)$ , with expectation value zero and common finite variance  $\sigma^2 > 0$ .
- Define the random variables:

$$\begin{cases} S_0 & : \Omega \rightarrow \mathbb{R} : \omega \mapsto 0, & \text{and} \\ S_n & : \Omega \rightarrow \mathbb{R} : \omega \mapsto \sum_{i=1}^n \xi_i(\omega), & \text{for each } n \in \mathbb{N}. \end{cases}$$

- For each  $n \in \mathbb{N}$ , define  $X^{(n)} : \Omega \rightarrow C[0, 1]$  as follows:

$$X^{(n)}(\omega)(t) := \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) + n \left( t - \frac{i-1}{n} \right) \xi_i(\omega) \right\}, \text{ for each } \omega \in \Omega, t \in \left[ \frac{i-1}{n}, \frac{i}{n} \right], i = 1, 2, 3, \dots, n.$$

- For each  $n \in \mathbb{N}$  and each  $t \in [0, 1]$ , define  $X_t^{(n)} : \Omega \rightarrow \mathbb{R}$  as follows:

$$X_t^{(n)}(\omega) := X^{(n)}(\omega)(t), \text{ for each } \omega \in \Omega.$$

Then, the following statements are true:

- (i) For each  $\omega \in \Omega$  and each  $n \in \mathbb{N}$ ,

$$X^{(n)}(\omega) \left( \frac{i}{n} \right) = \frac{1}{\sigma \cdot \sqrt{n}} \cdot S_i(\omega), \text{ for } i = 0, 1, 2, \dots, n.$$

- (ii) For each  $\omega \in \Omega$  and each  $n \in \mathbb{N}$ ,

$$X^{(n)}(\omega)(t) \text{ is the linear interpolation from } \frac{1}{\sigma \cdot \sqrt{n}} S_{i-1}(\omega) \text{ to } \frac{1}{\sigma \cdot \sqrt{n}} S_i(\omega) \text{ over } t \in \left[ \frac{i-1}{n}, \frac{i}{n} \right],$$

where  $i = 1, 2, \dots, n$ .

- (iii) For each  $t \in [0, 1]$ ,

$$X_t^{(n)} \xrightarrow{d} \sqrt{t} \cdot N(0, 1), \text{ as } n \rightarrow \infty.$$

- (iv) For any  $0 \leq t_0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq 1$ ,

$$\left( X_{t_1}^{(n)} - X_{t_0}^{(n)}, \dots, X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \right) \xrightarrow{d} N \left( \mu = \mathbf{0}, \Sigma = \text{diag}(t_1 - t_0, \dots, t_k - t_{k-1}) \right), \text{ as } n \rightarrow \infty.$$

- (v) For any  $0 \leq t_1, t_2, \dots, t_k \leq 1$ ,

$$\left( X_{t_1}^{(n)}, X_{t_2}^{(n)}, \dots, X_{t_k}^{(n)} \right) \xrightarrow{d} N \left( \mu = \mathbf{0}, \Sigma = \left[ \min\{t_i, t_j\} \right]_{1 \leq i, j \leq k} \right), \text{ as } n \rightarrow \infty.$$

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PROOF

$$\begin{aligned} E[X_t^{(n)}] &= E\left[ \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1} + n \left( t - \frac{i-1}{n} \right) \xi_i \right\} \right] \\ &= \frac{1}{\sigma \cdot \sqrt{n}} \left\{ E[S_{i-1}] + n \left( t - \frac{i-1}{n} \right) \cdot E[\xi_i] \right\} = 0. \end{aligned}$$

And, for each  $n \in \mathbb{N}$ , and each  $t \in \left[ \frac{i-1}{n}, \frac{i}{n} \right]$ ,  $i = 1, 2, \dots, n$ ,

$$\begin{aligned} \text{Var}[X_t^{(n)}] &= \text{Var}\left[ \frac{1}{\sigma^2 \cdot \sqrt{n}} \left\{ S_{i-1} + n \left( t - \frac{i-1}{n} \right) \xi_i \right\} \right] \\ &= \frac{1}{n \sigma^2} \left\{ \text{Var}[S_{i-1}] + n^2 \left( t - \frac{i-1}{n} \right)^2 \cdot \text{Var}[\xi_i] \right\} \\ &= \frac{1}{n \sigma^2} \left\{ (i-1) \cdot \sigma^2 + n^2 \left( t - \frac{i-1}{n} \right)^2 \cdot \sigma^2 \right\} \\ &= \frac{1}{n} \cdot \left\{ (i-1) + n^2 \left( t - \frac{i-1}{n} \right)^2 \right\}. \end{aligned}$$

□

## Remark 1.2

By the Central Limit Theorem,

$$X_t^{(n)}$$

## A Technical Lemmas

### Definition A.1

Let  $\Omega$  be a non-empty set and  $\mathcal{P}(\Omega)$  denote the power set of  $\Omega$ . An **outer measure** on  $\Omega$  is a function  $\varphi : \mathcal{P}(\Omega) \rightarrow [0, \infty]$  satisfying the following conditions:

- $\varphi(\emptyset) = 0$ .
- *monotonicity*:  $\varphi(A) \leq \varphi(B)$ , for every  $A, B \in \mathcal{P}(\Omega)$  with  $A \subset B$ .
- *countable sub-additivity*:

$$\varphi\left(\bigcap_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \varphi(A_i), \quad \text{for any } A_1, A_2, \dots \in \mathcal{P}(\Omega).$$

### Definition A.2

Let  $\Omega$  be a non-empty set and  $\mathcal{P}(\Omega)$  denote the power set of  $\Omega$ . Let  $\varphi : \mathcal{P}(\Omega) \rightarrow [0, \infty]$  be an outer measure on  $\Omega$ . A subset  $A \subset \Omega$  is said to be  $\varphi$ -measurable if

$$\varphi(E) = \varphi(A \cap E) + \varphi(A^c \cap E), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

### Theorem A.3

Let  $\Omega$  be a non-empty set and  $\mathcal{P}(\Omega)$  denote the power set of  $\Omega$ . Let  $\varphi : \mathcal{P}(\Omega) \rightarrow [0, \infty]$  be an outer measure on  $\Omega$ .

- (i) A subset  $A \subset \Omega$  is  $\varphi$ -measurable if and only if

$$\varphi(E) \geq \varphi(A \cap E) + \varphi(A^c \cap E), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

- (ii) The collection  $\mathcal{A}(\varphi)$  of  $\varphi$ -measurable subsets of  $\Omega$  forms a  $\sigma$ -algebra of subsets of  $\Omega$ .
- (iii) The restriction  $\varphi|_{\mathcal{A}(\varphi)}$  of the outer measure  $\varphi$  to the  $\sigma$ -algebra  $\mathcal{A}(\varphi)$  is a (countably additive) complete measure on the measurable space  $(\Omega, \mathcal{A}(\varphi))$ .

### Lemma A.4

Every compact subset of a metric space is also a separable subset of that metric space.

PROOF Let  $(X, \rho)$  be a metric space and  $K \subset X$  be a compact subset of  $X$ . For each  $x \in X$  and positive  $r > 0$ , let

$$B(x, r) := \{y \in X \mid \rho(x, y) < r\} \subset X,$$

i.e.  $B(x, r)$  is the open ball in  $X$  centred at  $x$  with radius  $r > 0$ . For each  $n \in \mathbb{N}$ , the following forms an open cover of  $K$ :

$$\mathcal{C}_n := \left\{ B\left(x, \frac{1}{n}\right) \subset X \mid x \in K \right\}.$$

Since  $K$  is compact, each  $\mathcal{C}_n$  admits a finite subcover:

$$\mathcal{F}_n := \left\{ B\left(x_i^{(n)}, \frac{1}{n}\right) \subset X \mid x_i^{(n)} \in K, \ i = 1, 2, \dots, J_n \right\}.$$

Let

$$\mathcal{D}_n := \left\{ x_i^{(n)} \in K \mid i = 1, 2, \dots, J_n \right\} \subset K,$$

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and let  $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset K$ . We claim that  $\mathcal{D}$  is dense in  $K$ . Indeed, let  $y \in K$ . Since each  $\mathcal{F}_n$  is a (finite) open cover of  $K$ , we have:

$$y \in K \subset \bigcup_{i=1}^{J_n} B\left(x_i^{(n)}, \frac{1}{n}\right), \quad \text{for each } n \in \mathbb{N}.$$

Since  $x_i^{(n)} \in \mathcal{D}$ , for each  $i = 1, 2, \dots, J_n$  and for each  $n \in \mathbb{N}$ , the above inclusion shows that, for each  $n \in \mathbb{N}$ , there exists some  $x \in \mathcal{D}$  such that  $\rho(y, x) < \frac{1}{n}$ . In particular,  $\mathcal{D}$  contains a sequence that converges to  $y \in K$ . Since  $y \in K$  is an arbitrary element of  $K$ , we see that  $\overline{\mathcal{D}} \supset K$ . Since  $\mathcal{D} \subset K$  and  $K$  is compact, hence closed, we trivially have  $\overline{\mathcal{D}} \subset K$ . We may now conclude that  $\overline{\mathcal{D}} = K$ . This completes the proof of the Lemma.  $\square$

## Lemma A.5

*Every countable union of separable subsets of a metric space is itself a separable subset of that metric space.*

PROOF Let  $S := \bigcup_{i=1}^{\infty} S_i \subset X$  be a countable union of separable subsets  $S_i$  of a metric space  $X$ . For each fixed  $i \in \mathbb{N}$ , since  $S_i$  is separable, there exists countable  $D_i \subset S_i$  which is dense in  $S_i$ . Let  $D := \bigcup_{i=1}^{\infty} D_i$ . Then,  $D$  is a countable subset of  $S$ . The Lemma is proved once we establish that  $D$  is dense in  $S$ . To this end, let  $x \in S = \bigcup_{i=1}^{\infty} S_i$ . Then,  $x \in S_i$  for some  $i \in \mathbb{N}$ . Since  $D_i$  is dense in  $S_i$ , there exists a sequence  $\{y_k\} \subset D_i \subset D$  such that  $y_k \rightarrow x$ , as  $k \rightarrow \infty$ . This proves that  $D$  is indeed dense in  $S$ , and completes the proof of the Lemma.  $\square$

## Lemma A.6 (second theorem in Appendix M3, [?])

*Let  $(S, \rho)$  be a metric space and  $\Sigma \subset S$  a separable subset of  $S$ . Then, there exists a countable collection  $\mathcal{A}$  of open subsets of  $S$  satisfying the following property: For each  $x \in S$  and each open subset  $G$  of  $S$ ,*

$$x \in G \cap \Sigma \implies x \in A \subset \overline{A} \subset G, \quad \text{for some } A \in \mathcal{A}.$$

PROOF Let  $D \subset \Sigma$  be a countable dense subset of  $\Sigma$ . Let

$$\mathcal{A} := \left\{ B(d, r) \subset S \mid \begin{array}{l} d \in D, \\ r \in \mathbb{Q}, r > 0 \end{array} \right\}.$$

Then,  $\mathcal{A}$  is a countable collection of open balls in  $S$ . Now, let  $G \subset S$  be an arbitrary open subset of  $S$  and  $x \in G \cap \Sigma$ . First, choose  $\varepsilon > 0$  such that  $B(x, \varepsilon) \subset G$ . Next, since  $x \in \Sigma$  and  $D$  is dense in  $\Sigma$ , we may choose  $d \in D$  such that  $d \in B(x, \varepsilon/2)$ , or equivalently  $\rho(x, d) < \varepsilon/2$ . Finally choose positive rational  $r > 0$  such that  $\rho(x, d) < r < \varepsilon/2$ .

Now, note that  $\overline{B(d, r)} \subset B(x, \varepsilon)$ ; indeed,

$$y \in \overline{B(d, r)} \iff \rho(y, d) \leq r \implies \rho(x, y) \leq \rho(x, d) + \rho(d, y) < \varepsilon/2 + r < \varepsilon/2 + \varepsilon/2 \implies y \in B(x, \varepsilon).$$

Thus, we have

$$x \in B(d, r) \subset \overline{B(d, r)} \subset B(x, \varepsilon) \subset G.$$

This completes the proof of the Lemma.  $\square$

## Theorem A.7 (The Diagonal Method, Appendix A.14, [?])

# Donsker's Theorems (Functional Central Limit Theorems)

Suppose that each row of the array

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

is a bounded sequence of real numbers. Then, there exists an increasing sequence

$$n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$$

of positive integers such that the limit

$$\lim_{k \rightarrow \infty} x_{r,n_k} \text{ exists, for each } r = 1, 2, 3, \dots$$

PROOF From the first row, select a convergent subsequence

$$x_{1,n(1,1)}, x_{1,n(1,2)}, x_{1,n(1,3)}, \dots$$

Here, we have  $n(1,1) < n(1,2) < n(1,3) < \cdots \in \mathbb{N}$ , and  $\lim_{k \rightarrow \infty} x_{1,n(1,k)} \in \mathbb{R}$  exists. Next, note that the following subsequence of the second row:

$$x_{2,n(1,1)}, x_{2,n(1,2)}, x_{2,n(1,3)}, \dots$$

is still a bounded sequence of real numbers, and we may thus select a convergent subsequence:

$$x_{2,n(2,1)}, x_{2,n(2,2)}, x_{2,n(2,3)}, \dots$$

Here, we have  $n(2,1) < n(2,2) < n(2,3) < \cdots \in \{n(1,k)\}_{k \in \mathbb{N}}$ , and  $\lim_{k \rightarrow \infty} x_{2,n(2,k)} \in \mathbb{R}$  exists. Continuing inductively, we obtain an array of positive integers

$$\begin{array}{cccc} n(1,1) & n(1,2) & n(1,3) & \cdots \\ n(2,1) & n(2,2) & n(2,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

which satisfies: For each  $r \in \mathbb{N}$ , we have

- each row is an increasing sequence of positive integers, i.e.  $n(r,1) < n(r,2) < n(r,3) < \cdots$ ,
- the  $(r+1)^{\text{th}}$  row is a subsequence of the  $r^{\text{th}}$  row, i.e.  $\{n(r+1,k)\}_{k \in \mathbb{N}} \subset \{n(r,k)\}_{k \in \mathbb{N}}$ , and
- $\lim_{k \rightarrow \infty} x_{r,n(r,k)} \in \mathbb{R}$  exists.

Note that the first two properties together imply:

$$n(k,k) < n(k,k+1) \leq n(k+1,k+1), \text{ for each } k \in \mathbb{N}.$$

Now, define  $n_k := n(k,k)$ , for  $k \in \mathbb{N}$ . We then see that

$$n_k := n(k,k) < n(k+1,k+1) =: n_{k+1},$$

i.e.,  $\{n_k\}_{k \in \mathbb{N}}$  is a strictly increasing sequence of positive integers. Lastly, for each  $r \in \mathbb{N}$ , consider the sequence

$$x_{r,n_1}, x_{r,n_2}, x_{r,n_3}, \dots$$

Note that, for each  $r \in \mathbb{N}$ ,

$$x_{r,n_r}, x_{r,n_{r+1}}, x_{r,n_{r+2}}, \dots$$

is a subsequence of  $\{x_{r,n(r,k)}\}_{k \in \mathbb{N}}$ . We saw above that  $\lim_{k \rightarrow \infty} x_{r,n(r,k)}$  exists, which in turn implies that  $\lim_{k \rightarrow \infty} x_{r,n_k}$  exists. Since  $r \in \mathbb{N}$  is arbitrary, the proof of the Theorem is now complete.  $\square$