1 The metric space structure on $\mathbb{R}^\infty$

Definition 1.1 (The metric space $\mathbb{R}^\infty$, Example 1.2, [1])
Let $\mathbb{R}^\infty$ denotes the set of all infinite sequences of real numbers, i.e.

$$\mathbb{R}^\infty \;:=\; \{\, (x_1, x_2, \dots) \mid x_i \in \mathbb{R}, \text{ for each } i \in \mathbb{N} \,\}.$$

Define $\rho : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow [\,0,1\,]$ as follows:

$$\rho(x,y) \;:=\; \sum_{n=1}^{\infty} \frac{\min\{\,1\,, |\,x_n - y_n\,|\,\}}{2^n}.$$

Remark 1.2 *Recall that*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \;=\; \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^{n-1}} \;=\; \frac{1}{2}\cdot\left(\frac{1}{1-\frac{1}{2}}\right) \;=\; 1,$$

which proves indeed that $0 \leq \rho(x,y) \leq 1$, for any $x, y \in \mathbb{R}^\infty$.

Theorem 1.3

(i) $(\,\mathbb{R}^\infty, \rho\,)$ *is a metric space. Let $\mathbb{R}^\infty$ denote also this metric space in the remainder of this Theorem.*

(ii) *For $x, x^{(1)}, x^{(2)}, x^{(3)}, \dots, \in \mathbb{R}^\infty$, we have:*

$$\rho\Big(\,x^{(n)},\,x\,\Big) \;\longrightarrow\; 0 \quad\Longleftrightarrow\quad \text{for each } i \in \mathbb{N},\; \lim_{n\to\infty}\left|\,x_i^{(n)} - x_i\,\right| \;=\; 0$$

(iii) *For each $n \in \mathbb{N}$, the "natural projection to the initial segment of length n"*

$$\pi_n \;:\; \mathbb{R}^\infty \;\longrightarrow\; \mathbb{R}^n \;:\; x \;\longmapsto\; (x_1, x_2, \dots, x_n)$$

is continuous, where $\mathbb{R}^n$ has the usual Euclidean topology.

(iv) *For each $x \in \mathbb{R}^\infty$, $n \in \mathbb{N}$, and $\varepsilon > 0$, let $C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)$ denote the open hypercube in $\mathbb{R}^n$ of side length 2ε centred at $\pi_n(x) \in \mathbb{R}^n$, i.e.*

$$C_{\mathbb{R}^n}(\pi_n(x),\varepsilon) \;\;:=\;\; \left\{\; y \in \mathbb{R}^n \;\middle|\; \begin{array}{c} |\,y_i - x_i\,| \;<\; \varepsilon, \\ i = 1, 2, \dots, n \end{array} \;\right\}$$

Then, its pre-image in $\mathbb{R}^\infty$ under π_n

$$\pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \;\;=\;\; \left\{\; y \in \mathbb{R}^\infty \;\middle|\; \begin{array}{c} |\,y_i - x_i\,| \;<\; \varepsilon, \\ i = 1, 2, \dots, n \end{array} \;\right\}$$

is an open subset of $\mathbb{R}^\infty$.

(v) *The collection*

$$\left\{\; \pi_n^{-1}(\,C_{\mathbb{R}^n}(\pi_n(x),\varepsilon)\,) \subset \mathbb{R}^\infty \;\middle|\; n \in \mathbb{N},\, x \in \mathbb{R}^\infty,\, \varepsilon > 0 \;\right\}$$

of all pre-images under π^n of open hypercubes in $\mathbb{R}^n$, for all $n \in \mathbb{N}$, forms a basis for the topology of $\mathbb{R}^\infty$.

(vi) $\mathbb{R}^\infty$ *is a separable and complete metric space. Hence, every probability measure on $\mathbb{R}^\infty$ is tight.*

PROOF

(i) Clearly, ρ is non-negative and symmetric. We now show that, for any $x, y \in \mathbb{R}^\infty$, we have $\rho(x,y) = 0$ implies $x = y$. Indeed,

$$\begin{aligned}\rho(x,y) = 0 \quad &\Longleftrightarrow \quad \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - y_i\,|\,\}}{2^i} \;=\; 0 \\ &\Longleftrightarrow \quad \min\{\,1\,,|\,x_i - y_i\,|\,\} \;=\; 0, \;\; \text{for each } i \in \mathbb{N} \\ &\Longleftrightarrow \quad |\,x_i - y_i\,| \;=\; 0, \;\; \text{for each } i \in \mathbb{N} \\ &\Longleftrightarrow \quad x \;=\; y.\end{aligned}$$

In order to show that ρ is a metric, it remains only to establish the Triangle Inequality. By Lemma A.2, for any $x, y, z \in \mathbb{R}^\infty$, we have

$$\begin{aligned}\rho(x,y) \;&=\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - y_i\,|\,\}}{2^i} \\ &\leq\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - z_i\,|\,\} \,+\, \min\{\,1\,,|\,z_i - y_i\,|\,\}}{2^i} \\ &=\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i - z_i\,|\,\}\,\}}{2^i} \;+\; \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,z_i - y_i\,|\,\}}{2^i} \\ &=\; \rho(x,z) + \rho(z,y),\end{aligned}$$

where we have used the fact that $0 \leq \rho \leq 1$ to split the infinite sum into two terms in second-to-last equality. This proves that ρ satisfies the Triangle Inequality, and it is thus a metric on $\mathbb{R}^\infty$.

(ii) $\underline{\lim_{n\to\infty} \rho\big(x^{(n)}, x\big) = 0 \;\;\Longrightarrow\;\; \lim_{n\to\infty} \left| x_i^{(n)} - x_i \right| = 0, \text{ for each } i \in \mathbb{N}}$

$$\begin{aligned}\lim_{n\to\infty} \rho\Big(x^{(n)}, x\Big) = 0 \quad &\Longrightarrow \quad \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i^{(n)} - x_i\,|\,\}}{2^i} = 0 \\ &\Longrightarrow \quad \lim_{n\to\infty} \min\{\,1\,,|\,x_i^{(n)} - x_i\,|\,\} = 0, \;\; \text{for each } i \in \mathbb{N} \\ &\Longrightarrow \quad \lim_{n\to\infty} \left| x_i^{(n)} - x_i \right| = 0, \;\; \text{for each } i \in \mathbb{N}\end{aligned}$$

$\underline{\lim_{n\to\infty} \left| x_i^{(n)} - x_i \right| = 0, \text{ for each } i \in \mathbb{N} \;\;\Longrightarrow\;\; \lim_{n\to\infty} \rho\big(x^{(n)}, x\big) = 0}$

This follows from the Weierstrass M-test. Suppose $\lim_{n\to\infty} \left| x_i^{(n)} - x_i \right| = 0$, for each $i \in \mathbb{N}$. Then,

$$\lim_{n\to\infty} \frac{\min\{\,1\,,|\,x_i^{(n)} - x_i\,|\,\}}{2^i} \;=\; 0 \;=:\; y_i, \;\; \text{for each } i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$, let $M_i \;:=\; \dfrac{1}{2^i}$. Then,

$$\frac{\min\{1\,,|\,x_i^{(n)} - x_i\,|\}}{2^i} \;\leq\; M_i \quad \text{and} \quad \sum_{i=1}^{\infty} M_i \;<\; \infty.$$

Hence, by the Weierstrass M-test (Lemma A.3), we have

$$\lim_{n\to\infty} \rho\Big(x^{(n)}, x\Big) \;=\; \lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{\min\{\,1\,,|\,x_i^{(n)} - x_i\,|\,\}}{2^i} \;=\; \sum_{i=1}^{\infty} y_i \;=\; 0.$$

(iii) Immediate by (ii).

(iv) Since $C_{\mathbb{R}^n}(\pi_n(x),\varepsilon) \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$, its pre-image under the continuous (by (ii)) map $\pi_n : \mathbb{R}^\infty \longrightarrow \mathbb{R}^n$ is an open subset of $\mathbb{R}^\infty$.

(v) It suffices to show that every open ball in $B(x,\varepsilon) \subset \mathbb{R}^\infty$ contains the pre-image of an open hypercube in $\mathbb{R}^n$ under π_n.

□

A Technical Lemmas

Lemma A.1 *Define*

$$\phi : [\,0,\infty\,) \longrightarrow [\,0,1\,] : t \longmapsto \min\{\,1\,,t\,\}.$$

Then, ϕ satisfies:

$$\phi(s+t) \;\leq\; \phi(s) \;+\; \phi(t), \;\;\text{for each } s,t \in [0,\infty).$$

Proof For any $s,t \in [0,\infty)$, either $s+t \geq 1$ or $s+t < 1$. If $s+t \geq 1$, then

$$\phi(s+t) \;=\; \min\{1,s+t\} \;=\; 1 \;<\; 2 \;=\; 1+1 \;\leq\; \min\{1,s\}+\min\{1,t\} \;=\; \phi(s)+\phi(t),$$

hence, the required inequality holds. On the other hand, if $s+t<1$, then we must also have $s<1$ and $t<1$ (since $s,t \geq 0$). Hence,

$$\phi(s+t) \;=\; \min\{1,s+t\} \;=\; s+t \;=\; \min\{1,s\}+\min\{1,t\} \;=\; \phi(s)+\phi(t),$$

thus, the required inequality also holds. □

Lemma A.2 *For any $x,y,z \in \mathbb{R}$, we have:*

$$\min\{1,|\,x-y\,|\} \;\leq\; \min\{1,|\,x-z\,|\} \;+\; \min\{1,|\,z-y\,|\}.$$

Proof Observe that $|\,x-y\,| \;\leq\; |\,x-z\,|+|\,z-y\,|$ implies

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; |\,x-z\,|+|\,z-y\,|.$$

The above inequality, together with $\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; 1$, thus in turn imply:

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; \min\{\,1\,,|\,x-z\,|+|\,z-y\,|\,\}.$$

By Lemma A.1, we therefore have:

$$\min\{\,1\,,|\,x-y\,|\,\} \;\leq\; \min\{\,1\,,|\,x-z\,|+|\,z-y\,|\,\}. \;\leq\; \min\{\,1\,,|\,x-z\,|\,\} \;+\; \min\{\,1\,,|\,z-y\,|\,\},$$

which proves the present Lemma. □

Lemma A.3 (The Weierstrass M-test, Theorem A.28, [2])

Suppose that $\lim\limits_{n\to\infty} x_i^{(n)} \;=\; x_i$, for each $i \in \mathbb{N}$, and that $\left|\,x_i^{(n)}\,\right| \;\leq\; M_i$, where $\sum\limits_{i=1}^{\infty} M_i < \infty$. Then,

(i) $\sum\limits_{i=1}^{\infty} x_i$ *exists, and* $\sum\limits_{i=1}^{\infty} x_i^{(n)}$ *exists for each $n \in \mathbb{N}$.*

(ii) *Furthermore,*

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} x_i^{(n)} \;=\; \sum_{i=1}^{\infty} x_i.$$

Proof

(i) $\sum_{i=1}^{\infty} M_i < \infty$ and $\left| x_i^{(n)} \right| \leq M_i \implies$ the series $\sum_{i=1}^{\infty} x_i$ and $\sum_{i=1}^{\infty} x_i^{(n)}$, $n \in \mathbb{N}$, converge absolutely.

(ii) Let $\varepsilon > 0$ be given. Choose $K \in \mathbb{N}$ sufficiently large such that $\sum_{j=K+1}^{\infty} M_i < \dfrac{\varepsilon}{3}$. Next, choose $N \in \mathbb{N}$ sufficiently large such that

$$\left| x_i^{(n)} - x_i \right| < \frac{\varepsilon}{3K}, \quad \text{for any } n > N \text{ and } i = 1, 2, \ldots, K.$$

Then, we have, for each $n > N$,

$$\begin{aligned}
\left| \sum_{i=1}^{\infty} x_i^{(n)} - \sum_{i=1}^{\infty} x_i \right| &= \left| \sum_{i=1}^{K} \left(x_i^{(n)} - x_i \right) + \sum_{i=K+1}^{\infty} x_i^{(n)} - \sum_{i=K+1}^{\infty} x_i \right| \\
&\leq \sum_{i=1}^{K} \left| x_i^{(n)} - x_i \right| + \sum_{i=K+1}^{\infty} \left| x_i^{(n)} \right| + \sum_{i=K+1}^{\infty} | x_i | \\
&\leq K \cdot \frac{\varepsilon}{3K} + \sum_{i=K+1}^{\infty} M_i + \sum_{i=K+1}^{\infty} M_i \leq \frac{\varepsilon}{3} + 2 \cdot \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since ε is arbitrary, this proves:

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} x_i^{(n)} = \sum_{i=1}^{\infty} x_i.$$

□

References

[1] Billingsley, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.

[2] Billingsley, P. *Probability and Measure*, anniversary ed. John Wiley & Sons, 2012.