# Donsker's Theorems (Functional Central Limit Theorems)

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## 1 Donsker's Theorem for $(\, C[0,1]\,,\, \|\cdot\|_\infty \,)$

**Proposition 1.1**

- *Let $\xi_1, \xi_2, \ldots : \Omega \longrightarrow \mathbb{R}$ be a sequence of independent and identically distributed $\mathbb{R}$-valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mu)$, with expectation value zero and common finite variance $\sigma^2 > 0$.*
- *Define the random variables:*

$$\begin{cases} S_0 \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto 0, & \text{and} \\ S_n \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto \displaystyle\sum_{i=1}^{n} \xi_i(\omega), & \text{for each } n \in \mathbb{N}. \end{cases}$$

- *For each $n \in \mathbb{N}$, define $X^{(n)} : \Omega \longrightarrow C[0,1]$ as follows:*

$$X^{(n)}(\omega)(t) \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) \;+\; n\left(t - \frac{i-1}{n}\right)\xi_i(\omega) \right\}, \;\text{ for each } \omega \in \Omega,\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],\; i = 1, 2, 3, \ldots, n.$$

- *For each $n \in \mathbb{N}$ and each $t \in [0,1]$, define $X_t^{(n)} : \Omega \longrightarrow \mathbb{R}$ as follows:*

$$X_t^{(n)}(\omega) \;:=\; X^{(n)}(\omega)(t), \quad \text{for each } \omega \in \Omega.$$

*Then, the following statements are true:*

(i) *For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,*

$$X^{(n)}(\omega)\left(\frac{i}{n}\right) \;=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot S_i(\omega), \quad \text{for } i = 0, 1, 2, \ldots, n.$$

(ii) *For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,*

$$X^{(n)}(\omega)(t) \;\textit{ is the linear interpolation from }\; \frac{1}{\sigma \cdot \sqrt{n}} S_{i-1}(\omega) \;\textit{ to }\; \frac{1}{\sigma \cdot \sqrt{n}} S_i(\omega) \;\textit{ over }\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],$$

*where $i = 1, 2, \ldots, n$.*

(iii) *For any $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$,*

$$\left( X_{t_1}^{(n)} - X_{t_0}^{(n)}, \;\ldots\;, X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \right) \;\xrightarrow{d}\; N\left( \mu = \mathbf{0}\,,\, \Sigma = \operatorname{diag}(\, t_1 - t_0, \;\ldots\;, t_k - t_{k-1}\,) \right), \;\text{ as } n \longrightarrow \infty.$$

(iv) *For any $0 \leq t_1, t_2, \cdots, t_k \leq 1$,*

$$\left( X_{t_1}^{(n)}, X_{t_2}^{(n)}, \;\ldots\;, X_{t_k}^{(n)} \right) \;\xrightarrow{d}\; N\left( \mu = \mathbf{0}\,,\, \Sigma = \Big[ \min\{t_i, t_j\} \Big]_{1 \leq i,j \leq k} \right), \;\text{ as } n \longrightarrow \infty.$$

PROOF

(i) Obvious.

(ii) Obvious.

(iii) First, note that, for each $\omega \in \Omega$, $n \in \mathbb{N}$, and $t \in [0,1]$, we have

$$X_t^{(n)}(\omega) \;=\; \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{\lfloor nt \rfloor}(\omega) \;+\; \Big( nt - \lfloor nt \rfloor \Big) \cdot \xi_{\lfloor nt \rfloor + 1}(\omega) \right\},$$

where $\lfloor \cdot \rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$, defined by

$$\lfloor x \rfloor \;:=\; \max \left\{ k \in \mathbb{Z} \;\middle\vert\; k \leq x \right\}, \quad \text{for each } x \in \mathbb{R},$$

is the round-down function. We next state three Claims, whose proofs will be given below. We note that the desired conclusion follows readily from Claim 3 and the Cramér-Wold Theorem (Theorem 1.9, p.56, [4]); hence the present proof is complete once we establish the three Claims below.

**Claim 1:** If $\{ a_n \}_{n \in \mathbb{N}}$ is a sequence of non-negative integers and $\{ b_n \}_{n \in \mathbb{N}} \subset \mathbb{N}$ a sequence of positive integers satisfying:

$$a_n \;<\; b_n, \text{ for sufficiently large } n \in \mathbb{N}, \qquad \text{and} \qquad \lim_{n \to \infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$

then

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i = 1 + a_n}^{b_n} \xi_i \;\xrightarrow{d}\; \sqrt{c} \cdot Z, \quad \text{where } Z \sim N(0,1).$$

**Claim 2:** For each fixed $t \in [0,1]$,

$$W(t)_n \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot \Big( nt - \lfloor nt \rfloor \Big) \cdot \xi_{\lfloor nt \rfloor + 1} \;\xrightarrow{d}\; 0.$$

**Claim 3:** For $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$, and arbitrary $c_1, c_2, \ldots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^{k} c_i \left( X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \;\xrightarrow{d}\; N\!\left( 0 \,,\, \sum_{i=1}^{k} c_i^2 \, (t_i - t_{i-1}) \right), \quad \text{as } n \longrightarrow \infty.$$

<u>Proof of Claim 1:</u> Note that, for sufficiently large $n \in \mathbb{N}$, we may write

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i = 1 + a_n}^{b_n} \xi_i \;=\; \frac{\sqrt{b_n - a_n}}{\sqrt{n}} \cdot \left( \frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i = 1 + a_n}^{b_n} \xi_i \right).$$

Since, by hypothesis, that

$$\lim_{n \to \infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$

Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [3]), once we establish the following:

$$\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i = 1 + a_n}^{b_n} \xi_i \;\xrightarrow{d}\; N(0,1), \quad \text{as } n \longrightarrow \infty.$$

We establish the above convergence by invoking the Lindeberg Central Limit Theorem (Theorem 1.15, §1.5.5, p.67, [4]). In the present context, the Lindeberg Condition is the following:

$$\lim_{n\to\infty} \frac{1}{B_n^2} \cdot E\left[ \sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\left\{ |\,\xi_i\,| \geq \varepsilon\, S_n \right\}} \right] \;=\; 0, \quad \text{for each } \varepsilon > 0,$$

where

$$B_n^2 \;:=\; \mathrm{Var}\left[ \sum_{i=1+a_n}^{b_n} \xi_i \right] \;=\; (b_n - a_n)\,\sigma^2 \;>\; 0.$$

The last equality used the hypothesis that $\xi_1,\ \xi_2,\ \ldots$ are independent and identically distributed with common finite variance $0 < \sigma^2 < \infty$. Hence, for each $\varepsilon > 0$,

$$\begin{aligned}
\frac{1}{B_n^2} \cdot E\left[ \sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\left\{ |\,\xi_i\,| \geq \varepsilon\, B_n \right\}} \right] &= \frac{1}{(b_n - a_n)\,\sigma^2} \cdot (b_n - a_n) \cdot E\left[ \xi_1^2 \cdot I_{\left\{ |\,\xi_1\,| \geq \varepsilon\sigma\,\sqrt{b_n - a_n} \right\}} \right] \\
&= \frac{1}{\sigma^2} \cdot E\left[ \xi_1^2 \cdot I_{\left\{ |\,\xi_1\,|/\varepsilon\sigma \,\geq\, \sqrt{b_n - a_n} \right\}} \right] \;\longrightarrow\; 0, \quad \text{as } n \longrightarrow \infty,
\end{aligned}$$

since $\lim\limits_{n\to\infty} \sqrt{b_n - a_n} = \infty$ and $\sigma^2 = E\left[\,\xi_1^2\,\right]$ is finite. This verifies that the Lindeberg Condition indeed holds in the present context, and completes the proof of Claim 1.

Proof of Claim 2: First, note that $E[\,W(t)_n\,] = 0$. We now argue that $W(t)_n \overset{p}{\longrightarrow} 0$. To this end, let $\varepsilon > 0$ be given. Then,

$$\begin{aligned}
\varepsilon^2 \cdot P(\,|\,W(t)_n\,| \,\geq\, \varepsilon\,) &\leq E\left[\,W(t)_n^2 \cdot I_{\{\,|\,W(t)_n\,| \geq \varepsilon\,\}}\,\right] \\
&\leq E\left[\,W(t)_n^2\,\right] \;=\; \mathrm{Var}(\,W(t)_n\,) \;=\; \mathrm{Var}\left[ \frac{1}{\sigma \cdot \sqrt{n}} \cdot \Big( nt - \lfloor nt \rfloor \Big) \cdot \xi_{\lfloor nt \rfloor + 1} \right] \\
&= \frac{1}{n \cdot \sigma^2} \cdot \Big( nt - \lfloor\, nt\,\rfloor \Big)^2 \cdot \mathrm{Var}\big(\,\xi_{\lfloor nt \rfloor + 1}\,\big) \;=\; \frac{1}{n \cdot \sigma^2} \cdot \Big( nt - \lfloor\, nt\,\rfloor \Big)^2 \cdot \sigma^2 \\
&\leq \frac{1}{n},
\end{aligned}$$

which implies

$$\lim_{n\to\infty} P(\,|\,W(t)_n\,| \,\geq\, \varepsilon\,) \;=\; 0, \;\; \text{for each } \varepsilon > 0,$$

i.e. $W(t)_n \overset{p}{\longrightarrow} 0$, as $n \longrightarrow \infty$ (Definition 2, Chapter 1, [3]), which is equivalent to $W(t)_n \overset{d}{\longrightarrow} 0$, as $n \longrightarrow \infty$ (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [3]). This proves Claim 2.

Proof of Claim 3: Let $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$, and $c_1, c_2, \ldots, c_k \in \mathbb{R}$ be arbitrary. Observe that:

$$\begin{aligned}
& \sum_{i=1}^{k} c_i \left( X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \\
=\; & \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ S_{\lfloor nt_i \rfloor} - S_{\lfloor nt_{i-1} \rfloor} \right\} + \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \Big( nt_i - \lfloor nt_i \rfloor \Big) \cdot \xi_{\lfloor nt_i \rfloor + 1} - \Big( nt_{i-1} - \lfloor nt_{i-1} \rfloor \Big) \cdot \xi_{\lfloor nt_{i-1} \rfloor + 1} \right\} \\
=\; & \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \right\} + \sum_{i=1}^{k} c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \\
=\; & \sum_{i=1}^{k} c_i\, Y_i^{(n)} + \sum_{i=1}^{k} c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\}
\end{aligned}$$

By Claim 2 and Slutsky's Theorem (Corollary, p.40, [3]),

$$\sum_{i=1}^{k} c_i \left\{ W(t_i)_n \; - \; W(t_{i-1})_n \right\} \;\overset{d}{\longrightarrow}\; 0, \;\text{ as }\; n \longrightarrow \infty. \tag{1.1}$$

Next, note that since $\xi_1, \xi_2, \xi_3, \ldots$ are independent, we see that, for each fixed $n \in \mathbb{N}$,

$$Y_i^{(n)} \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j\,=\,1+\lfloor nt_{i-1}\rfloor}^{\lfloor nt_i \rfloor} \xi_j, \quad i \,=\, 1, 2, 3, \ldots, k,$$

are independent. Now, since $0 \leq t_{i-1} < t_i \leq 1$, it follows that $\lfloor nt_{i-1} \rfloor < \lfloor nt_i \rfloor$ for sufficiently large $n \in \mathbb{N}$. In addition,

$$\begin{aligned}
\frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} \;&=\; \frac{\lfloor nt_i \rfloor}{n} - \frac{\lfloor nt_{i-1} \rfloor}{n} \;=\; \left( \frac{nt_i}{n} + \frac{\lfloor nt_i \rfloor - nt_i}{n} \right) - \left( \frac{nt_{i-1}}{n} + \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right) \\
&=\; t_i \;-\; t_{i-1} \;+\; \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n},
\end{aligned}$$

which implies

$$\left| \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} \;-\; (t_i - t_{i-1}) \right| \;=\; \left| \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right| \;\leq\; \frac{2}{n} \;\longrightarrow\; 0, \quad \text{as } \; n \longrightarrow \infty.$$

Thus,

$$\lim_{n\to\infty} \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} \;=\; t_i \;-\; t_{i-1} \;>\; 0.$$

Thus, by Claim 1, we see that, for each $i \,=\, 1, 2, \ldots, k$,

$$Y_i^{(n)} \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j\,=\,1+\lfloor nt_{i-1}\rfloor}^{\lfloor nt_i \rfloor} \xi_j \;\overset{d}{\longrightarrow}\; \sqrt{t_i - t_{i-1}} \cdot N(0,1) \;=\; N\Big(0\,,\, t_i - t_{i-1}\Big), \quad \text{as } \; n \longrightarrow \infty. \tag{1.2}$$

By (1.1), (1.2), Proposition A.1, and Slutsky's Theorem (Corollary, p.40, [3]), we now see that

$$\sum_{i=1}^{k} c_i \left( X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \;=\; \sum_{i=1}^{k} c_i \, Y_i^{(n)} \;+\; \sum_{i=1}^{k} c_i \left\{ W(t_i)_n \;-\; W(t_{i-1})_n \right\} \;\overset{d}{\longrightarrow}\; N\left( 0\,,\, \sum_{i=1}^{k} c_i^2 (t_i - t_{i-1}) \right).$$

This completes the proof of Claim 3.

$\square$

## A Technical Lemmas

Note that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ does NOT in general imply $X_n + Y_n \xrightarrow{d} X + Y$. But the implication does hold if $X_n$ and $Y_n$ are independent for each $n \in \mathbb{N}$, and both $X$ and $Y$ are Gaussian random variables, as the following Proposition shows.

**Proposition A.1** *Let $k \in \mathbb{N}$ be fixed. Suppose:*

- *For each $n \in \mathbb{N}$,*
$$Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_k^{(n)} : \Omega^{(n)} \longrightarrow \mathbb{R}$$
*are independent $\mathbb{R}$-valued random variables defined on the probability space $\Omega^{(n)}$.*
- *For each $i = 1, 2, \ldots, k$,*
$$Y_i^{(n)} \xrightarrow{d} N\big(\mu_i, \sigma_i^2\big), \quad \text{as } n \longrightarrow \infty.$$

*Then, for any $c_1, c_2, \ldots, c_k \in \mathbb{R}$,*
$$\sum_{i=1}^{k} c_i\, Y_i^{(n)} \xrightarrow{d} N\left(\sum_{i=1}^{k} c_i\,\mu_i\,,\ \sum_{i=1}^{k} c_i^2\,\sigma_i^2\right), \quad \text{as } n \longrightarrow \infty.$$

PROOF Let $Y^{(n)} := \sum_{i=1}^{k} c_i\, Y_i^{(n)}$. Let $\varphi_X$ denote the characteristic function of a $\mathbb{R}$-valued random variable $X$. Then,
$$\begin{aligned}
\varphi_{Y^{(n)}}(t) &= \varphi_{\sum_i^k c_i Y_i^{(n)}}(t) \\
&= \prod_{i=1}^{k} \varphi_{c_i Y_i^{(n)}}(t), \quad \text{since } Y_1^{(n)}, \ldots, Y_k^{(n)} \text{ are independent} \\
&= \prod_{i=1}^{k} \varphi_{Y_i^{(n)}}(\, c_i t\,) \\
&\longrightarrow \prod_{i=1}^{k} \exp\left\{ \sqrt{-1}\,\mu_i\,(c_i\,t) \;-\; \frac{1}{2}\,\sigma_i^2\,(\,c_i t\,)^2 \right\} \\
&= \exp\left\{ \sqrt{-1}\left(\sum_{i=1}^{k} c_i\mu_i\right) t \;-\; \frac{1}{2}\left(\sum_{i=1}^{k} c_i^2\sigma_i^2\right) t^2 \right\}, \quad \text{as } n \longrightarrow \infty,
\end{aligned}$$
where the second and third equalities follow from the properties of characteristic functions of random variables (see p.21, [3]), while the expression of the limit follows from the fact that the characteristic function $\varphi_Z$ of a random variable $Z$ with distribution $N\big(\mu, \sigma^2\big)$ is
$$\varphi_Z = \exp\left\{ \sqrt{-1}\,\mu t \;-\; \frac{1}{2}\,\sigma^2\, t^2 \right\}.$$
The Proposition now follows immediately from the Lévy-Cramér Continuity Theorem (Theorem 1.9(ii), p.56, [4]). □

## References


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[3] FERGUSON, T. S. *A Course in Large Sample Theory*, first ed. Texts in Statistical Science. CRC Press, 1996.

[4] SHAO, J. *Mathematical Statistics*, second ed. Springer, 2003.