Donsker's Theorems (Functional Central Limit Theorems)

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1 Donsker's Theorem for $(\,C[0,1]\,,\,\|\cdot\|_\infty\,)$

Proposition 1.1

- *Let $\xi_1, \xi_2, \ldots : \Omega \longrightarrow \mathbb{R}$ be a sequence of independent and identically distributed $\mathbb{R}$-valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mu)$, with expectation value zero and common finite variance $\sigma^2 > 0$.*
- *Define the random variables:*

$$\begin{cases} S_0 \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto 0, & \text{and} \\ S_n \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto \displaystyle\sum_{i=1}^{n} \xi_i(\omega), & \text{for each } n \in \mathbb{N}. \end{cases}$$

- *For each $n \in \mathbb{N}$, define $X^{(n)} : \Omega \longrightarrow C[0,1]$ as follows:*

$$X^{(n)}(\omega)(t) \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) \;+\; n\left(t - \frac{i-1}{n}\right)\xi_i(\omega) \right\}, \;\text{for each } \omega \in \Omega,\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],\; i = 1, 2, 3, \ldots, n.$$

- *For each $n \in \mathbb{N}$ and each $t \in [0,1]$, define $X_t^{(n)} : \Omega \longrightarrow \mathbb{R}$ as follows:*

$$X_t^{(n)}(\omega) \;:=\; X^{(n)}(\omega)(t), \quad \text{for each } \omega \in \Omega.$$

Then, the following statements are true:

(i) *For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,*

$$X^{(n)}(\omega)\left(\frac{i}{n}\right) \;=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot S_i(\omega), \quad \text{for } i = 0, 1, 2, \ldots, n.$$

(ii) *For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,*

$$X^{(n)}(\omega)(t) \;\; \textit{is the linear interpolation from} \;\; \frac{1}{\sigma \cdot \sqrt{n}} S_{i-1}(\omega) \;\; \textit{to} \;\; \frac{1}{\sigma \cdot \sqrt{n}} S_i(\omega) \;\; \textit{over} \;\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],$$

where $i = 1, 2, \ldots, n$.

(iii) *For each $t \in [0,1]$,*

$$X_t^{(n)} \;\xrightarrow{d}\; \sqrt{t} \cdot N(0,1), \;\text{ as }\; n \longrightarrow \infty.$$

(iv) *For any $0 \le t_0 < t_1 < t_2 < \cdots < t_k \le 1$,*

$$\left(X_{t_1}^{(n)} - X_{t_0}^{(n)}, \;\ldots\;, X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \right) \;\xrightarrow{d}\; N\left(\mu = \mathbf{0}\,,\, \Sigma = \mathrm{diag}(\, t_1 - t_0, \;\ldots\;, t_k - t_{k-1}\,) \right), \;\text{ as }\; n \longrightarrow \infty.$$

(v) *For any $0 \le t_1, t_2, \cdots, t_k \le 1$,*

$$\left(X_{t_1}^{(n)}, X_{t_2}^{(n)}, \;\ldots\;, X_{t_k}^{(n)} \right) \;\xrightarrow{d}\; N\left(\mu = \mathbf{0}\,,\, \Sigma = \left[\min\{t_i, t_j\} \right]_{1 \le i,j \le k} \right), \;\text{ as }\; n \longrightarrow \infty.$$

PROOF

(i) Obvious.

(ii) Obvious.

(iii) The statement holds trivially for $t = 0$. We prove the statement for $t \in (0,1]$. Now, for each $t \in (0,1]$, note that

$$X_t^{(n)}(\omega) \;=\; \frac{1}{\sigma\cdot\sqrt{n}}\left\{\, S_{\lfloor nt\rfloor}(\omega) \;+\; \left(nt - \lfloor nt\rfloor\right)\cdot \xi_{\lfloor nt\rfloor+1}(\omega)\,\right\},$$

where $\lfloor\cdot\rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$, defined by

$$\lfloor x\rfloor \;:=\; \max\left\{\, k\in\mathbb{Z} \;\middle|\; k\leq x \,\right\}, \quad \text{for each } x\in\mathbb{R},$$

is the round-down function.

Claim 1: For each fixed $t \in (0,1]$,

$$\frac{1}{\sigma\cdot\sqrt{n}}\cdot S_{\lfloor nt\rfloor} \;\xrightarrow{d}\; \sqrt{t}\cdot Z, \quad \text{where } Z \sim N(0,1).$$

Claim 2: For each fixed $t \in (0,1]$,

$$B_n \;:=\; \frac{1}{\sigma\cdot\sqrt{n}}\cdot\left(nt - \lfloor nt\rfloor\right)\cdot \xi_{\lfloor nt\rfloor+1} \;\xrightarrow{d}\; 0.$$

The desired statement now follows by Slutsky's Theorem (Corollary, p.40, [3]).

Proof of Claim 1: Note that

$$\frac{1}{\sigma\cdot\sqrt{n}}\cdot S_{\lfloor nt\rfloor} \;=\; \frac{\sqrt{\lfloor nt\rfloor}}{\sqrt{n}}\left(\frac{1}{\sigma\cdot\sqrt{\lfloor nt\rfloor}}\cdot S_{\lfloor nt\rfloor}\right),$$

and

$$\frac{\sqrt{\lfloor nt\rfloor}}{\sqrt{n}} \;\longrightarrow\; \sqrt{t}, \quad \text{as } n\longrightarrow\infty.$$

Hence, Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [3]), once we establish the following:

$$\frac{1}{\sigma\cdot\sqrt{\lfloor nt\rfloor}}\cdot S_{\lfloor nt\rfloor} \;\xrightarrow{d}\; N(0,1), \quad \text{as } n\longrightarrow\infty.$$

By Theorem 2.6, p.20, [2], it suffices to show that:

Every subsequence $\{A_{n_i}\}_{i\in\mathbb{N}}$ of $\left\{ A_n \;:=\; \dfrac{1}{\sigma\cdot\sqrt{\lfloor nt\rfloor}}\cdot S_{\lfloor nt\rfloor}\right\}_{n\in\mathbb{N}}$ contains a further subsequence that converges in distribution to $N(0,1)$. (1.1)

To this end, first recall that by the Central Limit Theorem,

$$\frac{1}{\sigma\cdot\sqrt{m}}\cdot S_m \;\xrightarrow{d}\; N(0,1), \quad \text{as } m\longrightarrow\infty.$$

By Theorem 2.6, p.20, [2], this is equivalent to:

$$\text{Every subsequence of } \left\{ \frac{1}{\sigma\cdot\sqrt{m}}\cdot S_m \right\}_{m\in\mathbb{N}} \text{ contains a further subsequence which converges in distribution to } N(0,1). \tag{1.2}$$

Next, note that, for each fixed $t \in (0,1]$, $\{ \lfloor nt \rfloor \}_{n\in\mathbb{N}}$ is a sequence of positive integers non-decreasing in $n \in \mathbb{N}$ and satisfying $\lim\limits_{n\to\infty} \lfloor nt \rfloor = \infty$. Thus, $\{ \lfloor nt \rfloor \}_{n\in\mathbb{N}}$ is a subsequence of $\mathbb{N} = \{1,2,3,\ldots\}$. Hence, for every subsequence $\{ n_i \}_{i\in\mathbb{N}}$ of $\mathbb{N} = \{1,2,3,\ldots\}$, $\{ \lfloor n_i \cdot t \rfloor \}_{i\in\mathbb{N}}$ is itself a subsequence of $\mathbb{N} = \{1,2,3,\ldots\}$. Therefore, by (1.2), $\left\{ A_{n_i} := \dfrac{1}{\sigma\cdot\sqrt{\lfloor n_i\cdot t\rfloor}}\cdot S_{\lfloor n_i\cdot t\rfloor} \right\}_{i\in\mathbb{N}}$ contains a further subsequence which converges in distribution to $N(0,1)$; in other words, (1.1) holds. This proves Claim 1.

<u>Proof of Claim 2:</u> First, note that $E[\, B_n \,] = 0$. We now argue that $B_n \xrightarrow{p} 0$. To this end, let $\varepsilon > 0$ be given. Then,

$$\begin{aligned}
\varepsilon^2 \cdot P(\,|\,B_n\,| \geq \varepsilon\,) &\leq E\left[\, B_n^2 \cdot I_{\{\,|\,B_n\,|\geq\varepsilon\,\}} \,\right] \\
&\leq E\left[\, B_n^2 \,\right] \;=\; \mathrm{Var}(\,B_n\,) \;=\; \mathrm{Var}\left[\frac{1}{\sigma\cdot\sqrt{n}} \cdot \Big(nt - \lfloor nt \rfloor \Big) \cdot \xi_{\lfloor nt\rfloor+1} \right] \\
&= \frac{1}{n\cdot\sigma^2} \cdot \Big(nt - \lfloor nt \rfloor \Big)^2 \cdot \mathrm{Var}\big(\, \xi_{\lfloor nt\rfloor+1} \,\big) \;=\; \frac{1}{n\cdot\sigma^2} \cdot \Big(nt - \lfloor nt \rfloor \Big)^2 \cdot \sigma^2 \\
&\leq \frac{1}{n},
\end{aligned}$$

which implies

$$\lim_{n\to\infty} P(\,|\,B_n\,| \geq \varepsilon\,) \;=\; 0, \quad \text{for each } \varepsilon > 0,$$

i.e. $B_n \xrightarrow{p} 0$, as $n \longrightarrow \infty$ (Definition 2, Chapter 1, [3]), which is equivalent to $B_n \xrightarrow{d} 0$, as $n \longrightarrow \infty$ (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [3]). This proves Claim 2.

(iv) First, note that, for each $\omega \in \Omega$, $n \in \mathbb{N}$, and $t \in [0,1]$, we have

$$X_t^{(n)}(\omega) \;=\; \frac{1}{\sigma\cdot\sqrt{n}} \left\{ S_{\lfloor nt\rfloor}(\omega) \;+\; \Big(nt - \lfloor nt \rfloor \Big) \cdot \xi_{\lfloor nt\rfloor+1}(\omega) \right\},$$

where $\lfloor\cdot\rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$, defined by

$$\lfloor x \rfloor \;:=\; \max\left\{ k \in \mathbb{Z} \;\middle|\; k \leq x \right\}, \quad \text{for each } x \in \mathbb{R},$$

is the round-down function.

Claim 1: If $\{ a_n \}_{n\in\mathbb{N}}$ is a sequence of non-negative integers and $\{ b_n \}_{n\in\mathbb{N}} \subset \mathbb{N}$ a sequence of positive integers satisfying:

$$a_n < b_n, \text{ for sufficiently large } n \in \mathbb{N}, \qquad \text{and} \qquad \lim_{n\to\infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$

then

$$\frac{1}{\sigma\cdot\sqrt{n}} \cdot \sum_{i\,=\,1+a_n}^{b_n} \xi_i \;\xrightarrow{d}\; \sqrt{c}\cdot Z, \quad \text{where } Z \sim N(0,1).$$

Claim 2: For fixed $0 \leq s < t \leq 1$,

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \left(S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor} \right) \overset{d}{\longrightarrow} \sqrt{t-s} \cdot Z, \quad \text{where } Z \sim N(0,1).$$

Claim 3: For each fixed $t \in [0,1]$,

$$B(t)_n \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot \left(nt - \lfloor nt \rfloor \right) \cdot \xi_{\lfloor nt \rfloor + 1} \overset{d}{\longrightarrow} 0.$$

Claim 4: For $0 \leq s < t \leq 1$,

$$X^{(n)}_t - X^{(n)}_s \overset{d}{\longrightarrow} \sqrt{t-s}\, N(0,1), \quad \text{as } n \longrightarrow \infty.$$

Claim 5: For $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$, and arbitrary $c_1, c_2, \ldots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^{k} c_i \left(X^{(n)}_{t_i} - X^{(n)}_{t_{i-1}} \right) \overset{d}{\longrightarrow} \sum_{i=1}^{k} c_i \cdot \sqrt{t_i - t_{i-1}} \cdot Z_i \;\sim\; N\left(0\,,\, \sum_{i=1}^{k} c_i^2 \,(t_i - t_{i-1}) \right), \text{ as } n \longrightarrow \infty,$$

where $Z_1, Z_2, \ldots, Z_k$ are independent standard Gaussian $\mathbb{R}$-valued random variables.

Proof of Claim 1: Note that, for sufficiently large $n \in \mathbb{N}$, we may write

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \;=\; \frac{\sqrt{b_n - a_n}}{\sqrt{n}} \cdot \left(\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \right).$$

Since, by hypothesis, that

$$\lim_{n \to \infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$

Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [3]), once we establish the following:

$$\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \overset{d}{\longrightarrow} N(0,1), \quad \text{as } n \longrightarrow \infty.$$

We establish the above convergence by invoking the Lindeberg Central Limit Theorem (Theorem 1.15, §1.5.5, p.67, [4]). In the present context, the Lindeberg Condition is the following:

$$\lim_{n \to \infty} \frac{1}{B_n^2} \cdot E\left[\sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\left\{ |\,\xi_i\,| \,\geq\, \varepsilon\, S_n \right\}} \right] \;=\; 0, \quad \text{for each } \varepsilon > 0,$$

where

$$B_n^2 \;:=\; \mathrm{Var}\left[\sum_{i=1+a_n}^{b_n} \xi_i \right] \;=\; (b_n - a_n)\,\sigma^2 \;>\; 0.$$

The last equality used the hypothesis that $\xi_1,\ \xi_2,\ \ldots$ are independent and identically distributed with common finite variance $0 < \sigma^2 < \infty$. Hence, for each $\varepsilon > 0$,

$$\begin{aligned}
\frac{1}{B_n^2} \cdot E\left[\sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\left\{ |\,\xi_i\,| \,\geq\, \varepsilon\, B_n \right\}} \right] &= \frac{1}{(b_n - a_n)\,\sigma^2} \cdot (b_n - a_n) \cdot E\left[\xi_1^2 \cdot I_{\left\{ |\,\xi_1\,| \,\geq\, \varepsilon\sigma\, \sqrt{b_n - a_n} \right\}} \right] \\
&= \frac{1}{\sigma^2} \cdot E\left[\xi_1^2 \cdot I_{\left\{ |\,\xi_1\,|/\varepsilon\sigma \,\geq\, \sqrt{b_n - a_n} \right\}} \right] \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,
\end{aligned}$$

since $\lim\limits_{n\to\infty} \sqrt{b_n - a_n} = \infty$ and $\sigma^2 = E\left[\,\xi_1^2\,\right]$ is finite. This verifies that the Lindeberg Condition indeed holds in the present context, and completes the proof of Claim 1.

<u>Proof of Claim 2:</u> Let $a_n := \lfloor ns \rfloor$ and $b_n := \lfloor nt \rfloor$. Since $0 \leq s < t \leq 1$, it follows that $a_n < b_n$ for sufficiently large $n \in \mathbb{N}$. In addition,

$$\begin{aligned}
\frac{b_n - a_n}{n} &= \frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n} = \frac{\lfloor nt \rfloor}{n} - \frac{\lfloor ns \rfloor}{n} = \left(\frac{nt}{n} + \frac{\lfloor nt \rfloor - nt}{n}\right) - \left(\frac{ns}{n} + \frac{\lfloor ns \rfloor - ns}{n}\right) \\
&= t - s + \frac{\lfloor nt \rfloor - nt}{n} - \frac{\lfloor ns \rfloor - ns}{n},
\end{aligned}$$

which implies

$$\left| \frac{b_n - a_n}{n} - (t-s) \right| = \left| \frac{\lfloor nt \rfloor - nt}{n} - \frac{\lfloor ns \rfloor - ns}{n} \right| \leq \frac{2}{n} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Thus,

$$\lim_{n\to\infty} \frac{b_n - a_n}{n} = t - s > 0.$$

Next,

$$\begin{aligned}
\frac{1}{\sigma\cdot\sqrt{n}} \cdot \left(S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor} \right) &= \frac{1}{\sigma\cdot\sqrt{n}} \cdot \left(\sum_{i=1}^{\lfloor nt \rfloor} \xi_i - \sum_{i=1}^{\lfloor ns \rfloor} \xi_i \right) = \frac{1}{\sigma\cdot\sqrt{n}} \cdot \left(\sum_{i=1+\lfloor ns \rfloor}^{\lfloor nt \rfloor} \xi_i \right) \\
&= \frac{1}{\sigma\cdot\sqrt{n}} \cdot \left(\sum_{i=1+a_n}^{b_n} \xi_i \right) \xrightarrow{d} \sqrt{t-s}\, N(0,1),
\end{aligned}$$

where the last convergence follows by Claim 1. This completes the proof of Claim 2.

<u>Proof of Claim 3:</u> First, note that $E[\,B(t)_n\,] = 0$. We now argue that $B(t)_n \xrightarrow{p} 0$. To this end, let $\varepsilon > 0$ be given. Then,

$$\begin{aligned}
\varepsilon^2 \cdot P(\,|\,B(t)_n\,| \geq \varepsilon\,) &\leq E\left[\,B(t)_n^2 \cdot I_{\{\,|\,B(t)_n\,| \geq \varepsilon\,\}}\,\right] \\
&\leq E\left[\,B(t)_n^2\,\right] = \mathrm{Var}(\,B(t)_n\,) = \mathrm{Var}\left[\frac{1}{\sigma\cdot\sqrt{n}} \cdot \left(nt - \lfloor nt \rfloor\right) \cdot \xi_{\lfloor nt \rfloor + 1}\right] \\
&= \frac{1}{n\cdot\sigma^2} \cdot \left(nt - \lfloor nt \rfloor\right)^2 \cdot \mathrm{Var}(\,\xi_{\lfloor nt \rfloor + 1}\,) = \frac{1}{n\cdot\sigma^2} \cdot \left(nt - \lfloor nt \rfloor\right)^2 \cdot \sigma^2 \\
&\leq \frac{1}{n},
\end{aligned}$$

which implies

$$\lim_{n\to\infty} P(\,|\,B(t)_n\,| \geq \varepsilon\,) = 0, \quad \text{for each } \varepsilon > 0,$$

i.e. $B(t)_n \xrightarrow{p} 0$, as $n \longrightarrow \infty$ (Definition 2, Chapter 1, [3]), which is equivalent to $B(t)_n \xrightarrow{d} 0$, as $n \longrightarrow \infty$ (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [3]). This proves Claim 3.

<u>Proof of Claim 4:</u> For $0 \leq s < t \leq 1$,

$$X_t^{(n)} - X_s^{(n)} = \frac{1}{\sigma\cdot\sqrt{n}} \left\{ S_{\lfloor nt \rfloor} - S_{\lfloor ns \rfloor} \right\} + \frac{1}{\sigma\cdot\sqrt{n}} \left\{ \left(nt - \lfloor nt \rfloor\right) \cdot \xi_{\lfloor nt \rfloor + 1} - \left(ns - \lfloor ns \rfloor\right) \cdot \xi_{\lfloor ns \rfloor + 1} \right\}.$$

Hence, by Slutsky's Theorem (Corollary, p.40, [3]), and Claim 2 and Claim 3, we have:

$$X_t^{(n)} - X_s^{(n)} \overset{d}{\longrightarrow} \sqrt{t-s}\,N(0,1), \quad \text{as } n \longrightarrow \infty.$$

This proves Claim 4.

Proof of Claim 5: For $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$, and arbitrary $c_1, c_2, \ldots, c_k \in \mathbb{R}$,

$$\begin{aligned}
& \sum_{i=1}^{k} c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \\
= \; & \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ S_{\lfloor nt_i \rfloor} - S_{\lfloor nt_{i-1} \rfloor} \right\} + \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \left(nt_i - \lfloor nt_i \rfloor \right) \cdot \xi_{\lfloor nt_i \rfloor + 1} - \left(nt_{i-1} - \lfloor nt_{i-1} \rfloor \right) \cdot \xi_{\lfloor nt_{i-1} \rfloor + 1} \right\} \\
= \; & \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \right\} + \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \left(nt_i - \lfloor nt_i \rfloor \right) \cdot \xi_{\lfloor nt_i \rfloor + 1} - \left(nt_{i-1} - \lfloor nt_{i-1} \rfloor \right) \cdot \xi_{\lfloor nt_{i-1} \rfloor + 1} \right\} \\
\overset{d}{\longrightarrow} \; & \sum_{i=1}^{k} c_i \cdot \sqrt{t_i - t_{i-1}} \cdot Z_i, \;\; \text{as } n \longrightarrow \infty,
\end{aligned}$$

where $Z_1, Z_2, \ldots, Z_k$ are independent standard Gaussian $\mathbb{R}$-valued random variables, and the convergence in distribution above follows by Slutsky's Theorem (Corollary, p.40, [3]), Claim 2 and Claim 3. This completes the proof of Claim 5.

$\square$

A Technical Lemmas

Definition A.1
*Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω. An **outer measure** on Ω is a function $\varphi : \mathcal{P}(\Omega) \longrightarrow [0,\infty]$ satisfying the following conditions:*

- $\varphi(\,\varnothing\,) = 0$.
- *monotonicity:* $\varphi(A) \;\leq\; \varphi(B)$, *for every* $A, B \in \mathcal{P}(\Omega)$ *with* $A \subset B$.
- *countable sub-additivity:*

$$\varphi\left(\,\bigcap_{i=1}^{\infty} A_i\right) \;\;\leq\;\; \sum_{i=1}^{\infty} \varphi(\,A_i\,)\,, \quad \text{for any } A_1, A_2, \ldots \in \mathcal{P}(\Omega).$$

Definition A.2
Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω. Let $\varphi : \mathcal{P}(\Omega) \longrightarrow [0,\infty]$ be an outer measure on Ω. A subset $A \subset \Omega$ is said to be φ-measurable if

$$\varphi(E) \;=\; \varphi(\,A \cap E\,) \;+\; \varphi(\,A^c \cap E\,), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

Theorem A.3
Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω. Let $\varphi : \mathcal{P}(\Omega) \longrightarrow [0,\infty]$ be an outer measure on Ω.

(i) *A subset $A \subset \Omega$ is φ-measurable if and only if*

$$\varphi(E) \;\;\geq\;\; \varphi(\,A \cap E\,) \;+\; \varphi(\,A^c \cap E\,), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

(ii) *The collection $\mathcal{A}(\varphi)$ of φ-measurable subsets of Ω forms a σ-algebra of subsets of Ω.*

(iii) *The restriction $\varphi|_{\mathcal{A}(\varphi)}$ of the outer measure φ to the σ-algebra $\mathcal{A}(\varphi)$ is a (countably additive) complete measure on the measurable space $(\,\Omega, \mathcal{A}(\varphi)\,)$.*

Lemma A.4

Every compact subset of a metric space is also a separable subset of that metric space.

Proof Let (X,ρ) be a metric space and $K \subset X$ be a compact subset of X. For each $x \in X$ and positive $r > 0$, let

$$B(x,r) \;\;:=\;\; \{\, y \in X \mid \rho(x,y) < r \,\} \;\;\subset\;\; X,$$

i.e. $B(x,r)$ is the open ball in X centred at x with radius $r > 0$. For each $n \in \mathbb{N}$, the following forms an open cover of K:

$$\mathcal{C}_n \;\;:=\;\; \left\{\; B\!\left(x, \frac{1}{n}\right) \;\subset\; X \;\middle|\; x \in K \;\right\}.$$

Since K is compact, each $\mathcal{C}_n$ admits a finite subcover:

$$\mathcal{F}_n \;\;:=\;\; \left\{\; B\!\left(x_i^{(n)}, \frac{1}{n}\right) \;\subset\; X \;\middle|\; x_i^{(n)} \in K,\;\; i = 1, 2, \ldots, J_n \;\right\}.$$

Let

$$\mathcal{D}_n \;\;:=\;\; \left\{\; x_i^{(n)} \in\; K \;\middle|\; i = 1, 2, \ldots, J_n \;\right\} \;\;\subset\;\; K,$$

and let $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset K$. We claim that $\mathcal{D}$ is dense in K. Indeed, let $y \in K$. Since each $\mathcal{F}_n$ is a (finite) open cover of K, we have:

$$y \;\in\; K \;\subset\; \bigcup_{i=1}^{J_n} B\left(x_i^{(n)}, \frac{1}{n} \right), \quad \text{for each } n \in \mathbb{N}.$$

Since $x_i^{(n)} \in \mathcal{D}$, for each $i = 1, 2, \ldots, J_n$ and for each $n \in \mathbb{N}$, the above inclusion shows that, for each $n \in \mathbb{N}$, there exists some $x \in \mathcal{D}$ such that $\rho(y, x) < \dfrac{1}{n}$. In particular, $\mathcal{D}$ contains a sequence that converges to $y \in K$. Since $y \in K$ is an arbitrary element of K, we see that $\overline{D} \supset K$. Since $\mathcal{D} \subset K$ and K is compact, hence closed, we trivially have $\overline{D} \subset K$. We may now conclude that $\overline{D} = K$. This completes the proof of the Lemma. $\square$

Lemma A.5

Every countable union of separable subsets of a metric space is itself a separable subset of that metric space.

Proof Let $S := \bigcup_{i=1}^{\infty} S_i \subset X$ be a countable union of separable subsets S_i of a metric space X. For each fixed $i \in \mathbb{N}$, since S_i is separable, there exists countable $D_i \subset S_i$ which is dense in S_i. Let $D := \bigcup_{i=1}^{\infty} D_i$. Then, D is a countable subset of S. The Lemma is proved once we establish that D is dense in S. To this end, let $x \in S = \bigcup_{i=1}^{\infty} S_i$. Then, $x \in S_i$ for some $i \in \mathbb{N}$. Since D_i is dense in S_i, there exists a sequence $\{ y_k \} \subset D_i \subset D$ such that $y_k \longrightarrow x$, as $k \longrightarrow \infty$. This proves that D is indeed dense in S, and completes the proof of the Lemma. $\square$

Lemma A.6 (second theorem in Appendix M3, [2])

Let (S, ρ) be a metric space and $\Sigma \subset S$ a separable subset of S. Then, there exists a countable collection $\mathcal{A}$ of open subsets of S satisfying the following property: For each $x \in S$ and each open subset G of S,

$$x \;\in\; G \bigcap \Sigma \quad\Longrightarrow\quad x \;\in\; A \subset \overline{A} \subset G, \;\text{ for some } A \in \mathcal{A}.$$

Proof Let $D \subset \Sigma$ be a countable dense subset of Σ. Let

$$\mathcal{A} \;:=\; \left\{ \; B(d, r) \;\subset\; S \;\middle|\; \begin{array}{c} d \in D, \\ r \in \mathbb{Q},\; r > 0 \end{array} \right\}.$$

Then, $\mathcal{A}$ is a countable collection of open balls in S. Now, let $G \subset S$ be an arbitrary open subset of S and $x \in G \bigcap \Sigma$. First, choose $\varepsilon > 0$ such that $B(x, \varepsilon) \subset G$. Next, since $x \in \Sigma$ and D is dense in Σ, we may choose $d \in D$ such that $d \in B(x, \varepsilon/2)$, or equivalently $\rho(x, d) < \varepsilon/2$. Finally choose positive rational $r > 0$ such that $\rho(x, d) < r < \varepsilon/2$.

Now, note that $\overline{B(d, r)} \subset B(x, \varepsilon)$; indeed,

$$y \in \overline{B(d,r)} \quad\Longleftrightarrow\quad \rho(y, d) \leq r \quad\Longrightarrow\quad \rho(x, y) \;\leq\; \rho(x, d) + \rho(d, y) \;<\; \varepsilon/2 + r \;<\; \varepsilon/2 + \varepsilon/2 \quad\Longrightarrow\quad y \in B(x, \varepsilon).$$

Thus, we have

$$x \;\in\; B(d, r) \;\subset\; \overline{B(d, r)} \;\subset\; B(x, \varepsilon) \;\subset\; G.$$

This completes the proof of the Lemma. $\square$

Theorem A.7 (The Diagonal Method, Appendix A.14, [1])

Suppose that each row of the array

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

is a bounded sequence of real numbers. Then, there exists an increasing sequence

$$n_1 \; < \; n_2 \; < \; n_3 \; < \cdots \; \in \; \mathbb{N}$$

of positive integers such that the limit

$$\lim_{k\to\infty} x_{r,n_k} \;\text{ exists,} \quad \text{for each } r = 1,2,3,\ldots$$

Proof From the first row, select a convergent subsequence

$$x_{1,n(1,1)},\;\; x_{1,n(1,2)},\;\; x_{1,n(1,3)},\;\; \cdots\cdots$$

Here, we have $n(1,1) < n(1,2) < n(1,3) < \cdots \;\in\; \mathbb{N}$, and $\lim\limits_{k\to\infty} x_{1,n(1,k)} \in \mathbb{R}$ exists. Next, note that the following subsequence of the second row:

$$x_{2,n(1,1)},\;\; x_{2,n(1,2)},\;\; x_{2,n(1,3)},\;\; \cdots\cdots$$

is still a bounded sequence of real numbers, and we may thus select a convergent subsequence:

$$x_{2,n(2,1)},\;\; x_{2,n(2,2)},\;\; x_{2,n(2,3)},\;\; \cdots\cdots$$

Here, we have $n(2,1) < n(2,2) < n(2,3) < \cdots \;\in\; \{\, n(1,k) \,\}_{k\in\mathbb{N}}$, and $\lim\limits_{k\to\infty} x_{2,n(2,k)} \in \mathbb{R}$ exists. Continuing inductively, we obtain an array of positive integers

$$\begin{array}{cccc} n(1,1) & n(1,2) & n(1,3) & \cdots \\ n(2,1) & n(2,2) & n(2,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

which satisfies: For each $r \in \mathbb{N}$, we have

- each row is an increasing sequence of positive integers, i.e. $n(r,1) \,<\, n(r,2) \,<\, n(r,3) \,<\, \cdots$,
- the $(r+1)^{\text{th}}$ row is a subsequence of the r^{th} row, i.e. $\{\, n(r+1,k) \,\}_{k\in\mathbb{N}} \;\subset\; \{\, n(r,k) \,\}_{k\in\mathbb{N}}$, and
- $\lim\limits_{k\to\infty} x_{r,n(r,k)} \;\in\; \mathbb{R}$ exists.

Note that the first two properties together imply:

$$n(k,k) \;\;<\;\; n(k,k+1) \;\;\leq\;\; n(k+1,k+1), \quad \text{for each } k \in \mathbb{N}.$$

Now, define $n_k \;:=\; n(k,k)$, for $k \in \mathbb{N}$. We then see that

$$n_k \;\;:=\;\; n(k,k) \;\; < n(k+1,k+1) \;\;=:\;\; n_{k+1},$$

i.e., $\{\, n_k \,\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. Lastly, for each $r \in \mathbb{N}$, consider the sequence

$$x_{r,n_1}\,,\;\; x_{r,n_2}\,,\;\; x_{r,n_3}\,,\;\; \cdots$$

Note that, for each $r \in \mathbb{N}$,

$$x_{r,n_r}\,,\;\; x_{r,n_{r+1}}\,,\;\; x_{r,n_{r+2}}\,,\;\; \cdots$$

is a subsequence of $\{\, x_{r,n(r,k)} \,\}_{k\in\mathbb{N}}$. We saw above that $\lim\limits_{k\to\infty} x_{r,n(r,k)}$ exists, which in turn implies that $\lim\limits_{k\to\infty} x_{r,n_k}$ exists. Since $r \in \mathbb{N}$ is arbitrary, the proof of the Theorem is now complete. □

References

[1] Billingsley, P. *Probability and Measure*, third ed. John Wiley & Sons, 1995.

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[4] Shao, J. *Mathematical Statistics*, second ed. Springer, 2003.