

## A Technical Lemmas

**Lemma A.1** (p.343, [1])

$$\left| e^{ix} - \sum_{k=0}^n \frac{(\mathbf{i}x)^k}{k!} \right| \leq \min \left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right\}, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

PROOF

**Claim 1:**

$$\int_0^x (x-s)^n e^{\mathbf{i}s} ds = \frac{x^{n+1}}{n+1} + \frac{\mathbf{i}}{n+1} \int_0^x (x-s)^{n+1} e^{\mathbf{i}s} ds, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof of Claim 1: We proceed by integration by parts. Let  $u = e^{\mathbf{i}s}$  and  $dv = (x-s)^n ds$ . Then,  $du = \mathbf{i}e^{\mathbf{i}s}$  and  $v = -(x-s)^{n+1}/(n+1)$ . Hence,

$$\begin{aligned} \int_0^x (x-s)^n e^{\mathbf{i}s} ds &= \int u dv = uv - \int v du \\ &= \left[ e^{\mathbf{i}s} \cdot \frac{(-1)(x-s)^{n+1}}{n+1} \right]_{s=0}^{s=x} - \int_0^x \frac{(-1)(x-s)^{n+1}}{n+1} \cdot \mathbf{i}e^{\mathbf{i}s} ds, \\ &= \frac{x^{n+1}}{n+1} + \frac{\mathbf{i}}{n+1} \int_0^x (x-s)^{n+1} e^{\mathbf{i}s} ds. \end{aligned}$$

This proves Claim 1.

**Claim 2:**

$$e^{ix} = \sum_{k=0}^n \frac{(\mathbf{i}x)^k}{k!} + \frac{\mathbf{i}^{n+1}}{n!} \int_0^x (x-s)^n e^{\mathbf{i}s} ds, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof of Claim 2: We proceed by induction. For  $n = 0$ , we have:

$$\begin{aligned} \text{RHS}(n=0) &= \sum_{k=0}^0 \frac{(\mathbf{i}x)^k}{k!} + \frac{\mathbf{i}^{0+1}}{0!} \int_0^x (x-s)^0 e^{\mathbf{i}s} ds = 1 + \mathbf{i} \int_0^x e^{\mathbf{i}s} ds = 1 + \mathbf{i} \left[ \frac{e^{\mathbf{i}s}}{\mathbf{i}} \right]_{s=0}^{s=x} \\ &= 1 + (e^{ix} - 1) = e^{ix}. \end{aligned}$$

Thus, Claim 2 is indeed true for  $n = 0$ . Next, by induction hypothesis, assume Claim 2 is true for  $n$ , and we verify that Claim 2 is also true for  $n+1$ .

$$\begin{aligned} \text{RHS}(n+1) &= \sum_{k=0}^{n+1} \frac{(\mathbf{i}x)^k}{k!} + \frac{\mathbf{i}^{n+2}}{(n+1)!} \int_0^x (x-s)^{n+1} e^{\mathbf{i}s} ds \\ &= \sum_{k=0}^n \frac{(\mathbf{i}x)^k}{k!} + \frac{(\mathbf{i}x)^{n+1}}{(n+1)!} + \frac{\mathbf{i}^{n+2}}{(n+1)!} \cdot \frac{n+1}{\mathbf{i}} \left[ \int_0^x (x-s)^n e^{\mathbf{i}s} ds - \frac{x^{n+1}}{n+1} \right] \\ &= \sum_{k=0}^n \frac{(\mathbf{i}x)^k}{k!} + \frac{\mathbf{i}^{n+1}}{n!} \int_0^x (x-s)^n e^{\mathbf{i}s} ds + \frac{(\mathbf{i}x)^{n+1}}{(n+1)!} - \frac{\mathbf{i}^{n+1}}{n!} \cdot \frac{x^{n+1}}{n+1} = e^{ix}, \end{aligned}$$

where the second equality follows from Claim 1 and the last equality follows from the induction hypothesis (that Claim 2 holds for  $n$ ). This proves Claim 2.

**Claim 3:**

$$e^{ix} = \sum_{k=0}^n \frac{(ix)^k}{k!} + \frac{i^n}{(n-1)!} \int_0^x (x-s)^{n-1} (e^{is} - 1) ds, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 1.$$

Proof of Claim 3:

□

**Lemma A.2 (§7.1, [2])**

Let  $\{\theta_{nj} \in \mathbb{C} \mid 1 \leq j \leq k_n, n \in \mathbb{N}\}$  be doubly indexed array of complex numbers. If all the following three conditions are true:

(a) there exists  $M > 0$  such that

$$\sum_{j=1}^{k_n} |\theta_{nj}| \leq M, \quad \text{for each } n \in \mathbb{N},$$

(b)  $\lim_{n \rightarrow \infty} \max_{1 \leq j \leq k_n} |\theta_{nj}| = 0$ , and

(c) there exists  $\theta \in \mathbb{C}$  such that

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \theta_{nj} = \theta,$$

then

$$\lim_{n \rightarrow \infty} \prod_{j=1}^{k_n} (1 + \theta_{nj}) = e^\theta.$$

PROOF First, note that hypothesis (b) immediately implies that there exists some  $n_0 \in \mathbb{N}$  such that

$$|\theta_{nj}| \leq \frac{1}{2}, \quad \text{for each } n \geq n_0, \text{ for each } 1 \leq j \leq k_n.$$

Thus, without loss of generality, we may assume that:

$$|\theta_{nj}| \leq \frac{1}{2}, \quad \text{for each } n \in \mathbb{N}, \text{ for each } 1 \leq j \leq k_n.$$

We denote by  $\log(1 + \theta_{nj})$  the (unique) complex logarithm<sup>1</sup> of  $1 + \theta_{nj}$  with argument in  $(-\pi, \pi]$ . Next, recall the MacLaurin Series for  $\log(1 + x)$ :

$$\log(1 + x) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{x^m}{m}, \quad \text{for any } x \in \mathbb{C} \text{ with } |x| < 1.$$

Hence, we have the following inequality: for each  $n \in \mathbb{N}$  and for each  $1 \leq j \leq k_n$ ,

$$\begin{aligned} |\log(1 + \theta_{nj}) - \theta_{nj}| &= \left| \sum_{m=2}^{\infty} (-1)^{m+1} \frac{(\theta_{nj})^m}{m} \right| \leq \sum_{m=2}^{\infty} \frac{|\theta_{nj}|^m}{m} \leq \frac{|\theta_{nj}|^2}{2} \sum_{m=2}^{\infty} |\theta_{nj}|^{m-2} \\ &\leq \frac{|\theta_{nj}|^2}{2} \sum_{m=2}^{\infty} \left(\frac{1}{2}\right)^{m-2} = \frac{|\theta_{nj}|^2}{2} \sum_{i=2}^{\infty} \left(\frac{1}{2}\right)^{i-2} = \frac{|\theta_{nj}|^2}{2} \cdot 2 = |\theta_{nj}|^2. \end{aligned}$$

<sup>1</sup>Recall that the complex exponential function is defined by  $\exp : \mathbb{C} \rightarrow \mathbb{C} : x + iy \mapsto e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$ . Clearly,  $\exp$  is not injective. More precisely, for  $x_1 + iy_1, x_2 + iy_2 \in \mathbb{C} \setminus \{0\}$ , we have  $e^{x_1 + iy_1} = e^{x_2 + iy_2}$  if and only if  $x_1 = x_2 \in \mathbb{R} \setminus \{0\}$  and  $y_1 - y_2 \in 2\pi\mathbb{Z}$ . For  $z = re^{i\theta} \in \mathbb{C} \setminus \{0\}$ , a complex logarithm of  $z$  is any  $w = x + iy \in \mathbb{C} \setminus \{0\}$  such that  $e^{x+iy} = e^w = z = re^{i\theta}$ , i.e.  $x = \log r$  and  $y = \theta + 2\pi\mathbb{Z}$ . In particular, let  $\mathcal{D} := \{x + iy \in \mathbb{C} \mid x \in \mathbb{R}, y \in (-\pi, \pi]\}$ . Then, the restriction  $\exp : \mathcal{D} \rightarrow \mathbb{C} \setminus \{0\}$  is bijective.

This in turn implies: for each  $n \in \mathbb{N}$ ,

$$\left| \sum_{j=1}^{k_n} \log(1 + \theta_{nj}) - \sum_{j=1}^{k_n} \theta_{nj} \right| = \left| \sum_{j=1}^{k_n} (\log(1 + \theta_{nj}) - \theta_{nj}) \right| \leq \sum_{j=1}^{k_n} |\log(1 + \theta_{nj}) - \theta_{nj}| \leq \sum_{j=1}^{k_n} |\theta_{nj}|^2.$$

Thus, for each  $n \in \mathbb{N}$ , there exists  $\Lambda_n \in \mathbb{C}$  with  $|\Lambda_n| \leq 1$  such that

$$\sum_{j=1}^{k_n} \log(1 + \theta_{nj}) = \sum_{j=1}^{k_n} \theta_{nj} + \Lambda_n \cdot \sum_{j=1}^{k_n} |\theta_{nj}|^2.$$

(Since for any  $z \in \mathbb{C}$ ,  $|z| \leq A \implies z = A \cdot w$ , for some  $w \in \mathbb{C}$  with  $|w| \leq 1$ .) Next note that, hypotheses (a) and (b) together imply:

$$\sum_{j=1}^{k_n} |\theta_{nj}|^2 \leq \left( \max_{1 \leq j \leq k_n} |\theta_{nj}| \right) \left( \sum_{j=1}^{k_n} |\theta_{nj}| \right) \leq M \cdot \left( \max_{1 \leq j \leq k_n} |\theta_{nj}| \right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Therefore, since  $|\Lambda_n| \leq 1$  for each  $n \in \mathbb{N}$ , we now see that

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \log(1 + \theta_{nj}) = \lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \theta_{nj} + \lim_{n \rightarrow \infty} \left( \Lambda_n \cdot \sum_{j=1}^{k_n} |\theta_{nj}|^2 \right) = \theta + 0 = \theta.$$

We may now conclude, by continuity of the exponential function  $\exp(\cdot)$ :

$$\begin{aligned} \lim_{n \rightarrow \infty} \prod_{j=1}^{k_n} (1 + \theta_{nj}) &= \lim_{n \rightarrow \infty} \exp \left( \log \prod_{j=1}^{k_n} (1 + \theta_{nj}) \right) = \lim_{n \rightarrow \infty} \exp \left( \sum_{j=1}^{k_n} \log(1 + \theta_{nj}) \right) \\ &= \exp \left( \lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \log(1 + \theta_{nj}) \right) = \exp(\theta) \end{aligned}$$

This completes the proof of the Lemma. □

## B The Central Limit Theorems

**Theorem B.1 (Lindeberg's Central Limit Theorem, Theorem 1.15, [3])**

*Suppose:*

- $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$  is a sequence of natural numbers such that  $k_n \rightarrow \infty$  as  $n \rightarrow \infty$ , and
- for each  $n \in \mathbb{N}$ ,  $X_1^{(n)}, X_2^{(n)}, \dots, X_{k_n}^{(n)} : \Omega_n \longrightarrow \mathbb{R}$  are independent (but not necessarily identically distributed)  $\mathbb{R}$ -valued random variables defined on a common probability space  $(\Omega_n, \mathcal{A}_n, \mu_n)$  such that

$$\mu_j^{(n)} := E[X_j^{(n)}] \in \mathbb{R} \text{ exists, for each } 1 \leq j \leq k_n, \quad \text{and} \quad 0 < \sigma_n^2 := \text{Var} \left[ \sum_{j=1}^{k_n} X_j^{(n)} \right] < \infty.$$

Let  $N(0, 1)$  denote the standard Gaussian distribution on  $\mathbb{R}$ . Then, the following implication holds: If

$$\lim_{n \rightarrow \infty} \frac{1}{\sigma_n^2} \sum_{j=1}^{k_n} E \left[ \left( X_j^{(n)} - \mu_j^{(n)} \right)^2 \cdot I_{\{|X_j^{(n)} - \mu_j^{(n)}| > \epsilon \sigma_n\}} \right] = 0, \quad \text{for each } \epsilon > 0,$$

then

$$\frac{1}{\sigma_n} \sum_{j=1}^{k_n} \left( X_j^{(n)} - \mu_j^{(n)} \right) \xrightarrow{\mathcal{L}} N(0, 1).$$

PROOF Considering  $\left( X_j^{(n)} - \mu_j^{(n)} \right) / \sigma_n$ , we may assume, without loss of generality, that

$$E \left[ X_j^{(n)} \right] = 0, \quad \text{and} \quad \sigma_n^2 := \text{Var} \left[ \sum_{j=1}^{k_n} X_j^{(n)} \right] = 1.$$

Lemma A.2. □

## References

- [1] BILLINGSLEY, P. *Probability and Measure*, third ed. John Wiley & Sons, 1995.
- [2] CHUNG, K. L. *A Course in Probability Theory*, third ed. Academic Press, 2001.
- [3] SHAO, J. *Mathematical Statistics*, second ed. Springer, 2003.