1 Outline

Suppose:

- $(\Omega, \mathcal{A}, \mu)$ is a probability space.
- $n \in \mathbb{N}$ is an natural number (positive integer).
- $T_1, T_2, \ldots, T_n : \Omega \longrightarrow [0, \infty]$ are independent identically distributed extended $\mathbb{R}$-valued random variables.
- $U_1, U_2, \ldots, U_n : \Omega \longrightarrow [0, \infty]$ are independent identically distributed extended $\mathbb{R}$-valued random variables.
- For each $i = 1, 2, \ldots, n$, let $X_i := \min\{T_i, U_i\}$, and $C_i := I_{\{T_i \,\leq\, U_i\}}$.

For each subject $i = 1, 2, \ldots, n$, the random variable T_i is interpreted to be the "survival time" of subject i, while U_i is interpreted to be the "censoring time" of subject i.

We wish to make inference about the (common) *survival function*

$$S(t) \;\; := \;\; P(\, T > t \,) \;\; = \;\; \mu\Big(\Big\{\, \omega \in \Omega \;\Big|\; T(\omega) > t \,\Big\}\Big)$$

of $T_1, T_2, \ldots, T_n$. However, in survival analysis, the inference about $S(t)$ is made based on the *right-censored survival time data* $\{\, X_i,\, C_i \,\}$, $i = 1, 2, \ldots, n$ (rather than on the T_i's directly).

The *hazard function*:

$$\lambda(t) \;\; := \;\; \lim_{h \to 0^+} \frac{1}{h} \cdot P\Big(\, t \leq T < t + h \;\Big|\; t \leq T \,\Big)$$

The *cumulative hazard function*:

$$\Lambda(t) \;\; := \;\; \int_0^t \lambda(t)\, \mathrm{d}t$$

The *Nelson-Aalen estimator* for the cumulative hazard function $\Lambda(t)$:

$$\widehat{\Lambda}(\omega, t) \;\; := \;\; \sum_{\substack{C_i(\omega) \,=\, 1 \\ T_i(\omega) \,\leq\, t}} \frac{1}{Y(\omega, T_i(\omega))}\,,$$

where

$$Y_i(\omega, t) \;\; := \;\; \begin{cases} 1, & t - h < X_i(\omega), \text{ for each } h > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$Y(\omega, t) \;\; := \;\; \sum_{i \,=\, 1}^{n} Y_i(\omega, t)$$

The aggregated counting process for subject i:

$$N_i(\omega, t) \;\; := \;\; I_{\{\, X_i(\omega) \,\leq\, t \,\}}$$

The aggregated counting process:

$$N(\omega, t) \;\; := \;\; \sum_{i \,=\, 1}^{n} N_i(\omega, t) \;\; = \;\; \sum_{i \,=\, 1}^{n} I_{\{\, X_i(\omega) \,\leq\, t \,\}}$$

The aggregated intensity process:

$$\alpha(\omega,t) \;\;:=\;\; \lim_{h\to 0^+}\frac{1}{h}\cdot P\left(N(\omega,t+h)-N(\omega,t)=1 \;\middle|\; \mathcal{F}_t\right) \;\;=\;\; \lim_{h\to 0^+}\frac{1}{h}\cdot E\left[N(\omega,t+h)-N(\omega,t) \;\middle|\; \mathcal{F}_t\right]$$

The aggregated cumulative intensity process:

$$A(\omega,t) \;\;:=\;\; \int_0^t \alpha(\omega,t)\,\mathrm{d}t$$

Then, the process

$$M(\omega,t) \;\;:=\;\; N(\omega,t)-A(\omega,t) \;\;=\;\; N(\omega,t)-\int_0^t \alpha(\omega,t)\,\mathrm{d}t$$

is a martingale process. In particular, $M(\,\cdot\,,t)$ satisfies

$$E\left[\,M(\,\cdot\,,t+h)-M(\,\cdot\,,t) \;\middle|\; \mathcal{F}_t\right](\omega) \;\;=\;\; M(\omega,t)$$

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