

1 Donsker's Theorem for $(C[0, 1], \|\cdot\|_\infty)$

Proposition 1.1

- Let $\xi_1, \xi_2, \dots : \Omega \rightarrow \mathbb{R}$ be a sequence of independent and identically distributed \mathbb{R} -valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mu)$, with expectation value zero and common finite variance $\sigma^2 > 0$.
- Define the random variables:

$$\begin{cases} S_0 & : \Omega \rightarrow \mathbb{R} : \omega \mapsto 0, & \text{and} \\ S_n & : \Omega \rightarrow \mathbb{R} : \omega \mapsto \sum_{i=1}^n \xi_i(\omega), & \text{for each } n \in \mathbb{N}. \end{cases}$$

- For each $n \in \mathbb{N}$, define $X^{(n)} : \Omega \rightarrow C[0, 1]$ as follows:

$$X^{(n)}(\omega)(t) := \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) + n \left(t - \frac{i-1}{n} \right) \xi_i(\omega) \right\}, \text{ for each } \omega \in \Omega, t \in \left[\frac{i-1}{n}, \frac{i}{n} \right], i = 1, 2, 3, \dots, n.$$

- For each $n \in \mathbb{N}$ and each $t \in [0, 1]$, define $X_t^{(n)} : \Omega \rightarrow \mathbb{R}$ as follows:

$$X_t^{(n)}(\omega) := X^{(n)}(\omega)(t), \text{ for each } \omega \in \Omega.$$

Then, the following statements are true:

- (i) For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,

$$X^{(n)}(\omega) \left(\frac{i}{n} \right) = \frac{1}{\sigma \cdot \sqrt{n}} \cdot S_i(\omega), \text{ for } i = 0, 1, 2, \dots, n.$$

- (ii) For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,

$$X^{(n)}(\omega)(t) \text{ is the linear interpolation from } \frac{1}{\sigma \cdot \sqrt{n}} S_{i-1}(\omega) \text{ to } \frac{1}{\sigma \cdot \sqrt{n}} S_i(\omega) \text{ over } t \in \left[\frac{i-1}{n}, \frac{i}{n} \right],$$

where $i = 1, 2, \dots, n$.

- (iii) For each $t \in [0, 1]$,

$$X_t^{(n)} \xrightarrow{d} \sqrt{t} \cdot N(0, 1), \text{ as } n \rightarrow \infty.$$

- (iv) For any $0 \leq t_0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq 1$,

$$\left(X_{t_1}^{(n)} - X_{t_0}^{(n)}, \dots, X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \right) \xrightarrow{d} N \left(\mu = \mathbf{0}, \Sigma = \text{diag}(t_1 - t_0, \dots, t_k - t_{k-1}) \right), \text{ as } n \rightarrow \infty.$$

- (v) For any $0 \leq t_1, t_2, \dots, t_k \leq 1$,

$$\left(X_{t_1}^{(n)}, X_{t_2}^{(n)}, \dots, X_{t_k}^{(n)} \right) \xrightarrow{d} N \left(\mu = \mathbf{0}, \Sigma = \left[\min\{t_i, t_j\} \right]_{1 \leq i, j \leq k} \right), \text{ as } n \rightarrow \infty.$$

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PROOF

- (i) Obvious.
- (ii) Obvious.
- (iii) The statement holds trivially for $t = 0$. We prove the statement for $t \in (0, 1]$. Now, for each $t \in (0, 1]$, note that

$$X_t^{(n)}(\omega) = \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{[nt]}(\omega) + \left(nt - [nt] \right) \cdot \xi_{[nt]+1}(\omega) \right\},$$

where $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$, defined by

$$\lfloor x \rfloor := \max \left\{ k \in \mathbb{Z} \mid k \leq x \right\}, \quad \text{for each } x \in \mathbb{R},$$

is the round-down function.

Claim 1: For each fixed $t \in (0, 1]$,

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot S_{[nt]} \xrightarrow{d} \sqrt{t} \cdot Z, \quad \text{where } Z \sim N(0, 1).$$

Claim 2: For each fixed $t \in (0, 1]$,

$$B_n := \frac{1}{\sigma \cdot \sqrt{n}} \cdot \left(nt - [nt] \right) \cdot \xi_{[nt]+1} \xrightarrow{d} 0.$$

The desired statement now follows by Slutsky's Theorem (Corollary, p.40, [3]).

Proof of Claim 1: Note that

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot S_{[nt]} = \frac{\sqrt{[nt]}}{\sqrt{n}} \left(\frac{1}{\sigma \cdot \sqrt{[nt]}} \cdot S_{[nt]} \right),$$

and

$$\frac{\sqrt{[nt]}}{\sqrt{n}} \rightarrow \sqrt{t}, \quad \text{as } n \rightarrow \infty.$$

Hence, Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [3]), once we establish the following:

$$\frac{1}{\sigma \cdot \sqrt{[nt]}} \cdot S_{[nt]} \xrightarrow{d} N(0, 1), \quad \text{as } n \rightarrow \infty.$$

By Theorem 2.6, p.20, [2], it suffices to show that:

$$\text{Every subsequence } \{A_{n_i}\}_{i \in \mathbb{N}} \text{ of } \left\{ A_n := \frac{1}{\sigma \cdot \sqrt{[nt]}} \cdot S_{[nt]} \right\}_{n \in \mathbb{N}} \text{ contains a further} \quad (1.1)$$

subsequence that converges in distribution to $N(0, 1)$.

To this end, first recall that by the Central Limit Theorem,

$$\frac{1}{\sigma \cdot \sqrt{m}} \cdot S_m \xrightarrow{d} N(0, 1), \quad \text{as } m \rightarrow \infty.$$

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By Theorem 2.6, p.20, [2], this is equivalent to:

$$\text{Every subsequence of } \left\{ \frac{1}{\sigma \cdot \sqrt{m}} \cdot S_m \right\}_{m \in \mathbb{N}} \text{ contains a further subsequence which converges} \quad (1.2)$$

in distribution to $N(0, 1)$.

Next, note that, for each fixed $t \in (0, 1]$, $\{ \lfloor nt \rfloor \}_{n \in \mathbb{N}}$ is a sequence of positive integers non-decreasing in $n \in \mathbb{N}$ and satisfying $\lim_{n \rightarrow \infty} \lfloor nt \rfloor = \infty$. Thus, $\{ \lfloor nt \rfloor \}_{n \in \mathbb{N}}$ is a subsequence of $\mathbb{N} = \{1, 2, 3, \dots\}$. Hence, for every subsequence $\{n_i\}_{i \in \mathbb{N}}$ of $\mathbb{N} = \{1, 2, 3, \dots\}$, $\{ \lfloor n_i \cdot t \rfloor \}_{i \in \mathbb{N}}$ is itself a subsequence of $\mathbb{N} = \{1, 2, 3, \dots\}$. Therefore, by (1.2), $\left\{ A_{n_i} := \frac{1}{\sigma \cdot \sqrt{\lfloor n_i \cdot t \rfloor}} \cdot S_{\lfloor n_i \cdot t \rfloor} \right\}_{i \in \mathbb{N}}$ contains a further subsequence which converges in distribution to $N(0, 1)$; in other words, (1.1) holds. This proves Claim 1.

Proof of Claim 2: First, note that $E[B_n] = 0$. We now argue that $B_n \xrightarrow{p} 0$. To this end, let $\varepsilon > 0$ be given. Then,

$$\begin{aligned} \varepsilon^2 \cdot P(|B_n| \geq \varepsilon) &\leq E[B_n^2 \cdot I_{\{|B_n| \geq \varepsilon\}}] \\ &\leq E[B_n^2] = \text{Var}(B_n) = \text{Var}\left[\frac{1}{\sigma \cdot \sqrt{n}} \cdot (nt - \lfloor nt \rfloor) \cdot \xi_{\lfloor nt \rfloor + 1}\right] \\ &= \frac{1}{n \cdot \sigma^2} \cdot (nt - \lfloor nt \rfloor)^2 \cdot \text{Var}(\xi_{\lfloor nt \rfloor + 1}) = \frac{1}{n \cdot \sigma^2} \cdot (nt - \lfloor nt \rfloor)^2 \cdot \sigma^2 \\ &\leq \frac{1}{n}, \end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} P(|B_n| \geq \varepsilon) = 0, \text{ for each } \varepsilon > 0,$$

i.e. $B_n \xrightarrow{p} 0$, as $n \rightarrow \infty$ (Definition 2, Chapter 1, [3]), which is equivalent to $B_n \xrightarrow{d} 0$, as $n \rightarrow \infty$ (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [3]). This proves Claim 2.

First, recall that, by the Central Limit Theorem, we have:

$$\frac{1}{\sigma \sqrt{n}} \cdot S_n \xrightarrow{d} N(0, 1), \text{ as } n \rightarrow \infty.$$

By Theorem 2.6, [2], the above convergence is equivalent to:

For each subsequence $\{n_i\}_{i \in \mathbb{N}}$ of $\{1, 2, \dots\}$, there exists a further subsequence $\{n_{i(k)}\}_{k \in \mathbb{N}}$ such that

$$\frac{1}{\sigma \sqrt{n_{i(k)}}} \cdot S_{n_{i(k)}} \xrightarrow{d} N(0, 1), \text{ as } k \rightarrow \infty.$$

Note that $\lfloor \cdot \rfloor$ is non-decreasing over all of \mathbb{R} . Hence, for each fixed $t \in (0, 1]$, $\{ \lfloor nt \rfloor \}_{n \in \mathbb{N}}$ is a non-decreasing sequence of non-negative integers satisfying $\lfloor nt \rfloor \rightarrow \infty$ as $n \rightarrow \infty$. □

Remark 1.2

By the Central Limit Theorem,

$$X_t^{(n)}$$

A Technical Lemmas

Definition A.1

Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω . An **outer measure** on Ω is a function $\varphi : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ satisfying the following conditions:

- $\varphi(\emptyset) = 0$.
- *monotonicity*: $\varphi(A) \leq \varphi(B)$, for every $A, B \in \mathcal{P}(\Omega)$ with $A \subset B$.
- *countable sub-additivity*:

$$\varphi\left(\bigcap_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \varphi(A_i), \quad \text{for any } A_1, A_2, \dots \in \mathcal{P}(\Omega).$$

Definition A.2

Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω . Let $\varphi : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ be an outer measure on Ω . A subset $A \subset \Omega$ is said to be φ -measurable if

$$\varphi(E) = \varphi(A \cap E) + \varphi(A^c \cap E), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

Theorem A.3

Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω . Let $\varphi : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ be an outer measure on Ω .

- (i) A subset $A \subset \Omega$ is φ -measurable if and only if

$$\varphi(E) \geq \varphi(A \cap E) + \varphi(A^c \cap E), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

- (ii) The collection $\mathcal{A}(\varphi)$ of φ -measurable subsets of Ω forms a σ -algebra of subsets of Ω .
- (iii) The restriction $\varphi|_{\mathcal{A}(\varphi)}$ of the outer measure φ to the σ -algebra $\mathcal{A}(\varphi)$ is a (countably additive) complete measure on the measurable space $(\Omega, \mathcal{A}(\varphi))$.

Lemma A.4

Every compact subset of a metric space is also a separable subset of that metric space.

PROOF Let (X, ρ) be a metric space and $K \subset X$ be a compact subset of X . For each $x \in X$ and positive $r > 0$, let

$$B(x, r) := \{y \in X \mid \rho(x, y) < r\} \subset X,$$

i.e. $B(x, r)$ is the open ball in X centred at x with radius $r > 0$. For each $n \in \mathbb{N}$, the following forms an open cover of K :

$$\mathcal{C}_n := \left\{ B\left(x, \frac{1}{n}\right) \subset X \mid x \in K \right\}.$$

Since K is compact, each \mathcal{C}_n admits a finite subcover:

$$\mathcal{F}_n := \left\{ B\left(x_i^{(n)}, \frac{1}{n}\right) \subset X \mid x_i^{(n)} \in K, \ i = 1, 2, \dots, J_n \right\}.$$

Let

$$\mathcal{D}_n := \left\{ x_i^{(n)} \in K \mid i = 1, 2, \dots, J_n \right\} \subset K,$$

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and let $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset K$. We claim that \mathcal{D} is dense in K . Indeed, let $y \in K$. Since each \mathcal{F}_n is a (finite) open cover of K , we have:

$$y \in K \subset \bigcup_{i=1}^{J_n} B\left(x_i^{(n)}, \frac{1}{n}\right), \quad \text{for each } n \in \mathbb{N}.$$

Since $x_i^{(n)} \in \mathcal{D}$, for each $i = 1, 2, \dots, J_n$ and for each $n \in \mathbb{N}$, the above inclusion shows that, for each $n \in \mathbb{N}$, there exists some $x \in \mathcal{D}$ such that $\rho(y, x) < \frac{1}{n}$. In particular, \mathcal{D} contains a sequence that converges to $y \in K$. Since $y \in K$ is an arbitrary element of K , we see that $\overline{\mathcal{D}} \supset K$. Since $\mathcal{D} \subset K$ and K is compact, hence closed, we trivially have $\overline{\mathcal{D}} \subset K$. We may now conclude that $\overline{\mathcal{D}} = K$. This completes the proof of the Lemma. \square

Lemma A.5

Every countable union of separable subsets of a metric space is itself a separable subset of that metric space.

PROOF Let $S := \bigcup_{i=1}^{\infty} S_i \subset X$ be a countable union of separable subsets S_i of a metric space X . For each fixed $i \in \mathbb{N}$, since S_i is separable, there exists countable $D_i \subset S_i$ which is dense in S_i . Let $D := \bigcup_{i=1}^{\infty} D_i$. Then, D is a countable subset of S . The Lemma is proved once we establish that D is dense in S . To this end, let $x \in S = \bigcup_{i=1}^{\infty} S_i$. Then, $x \in S_i$ for some $i \in \mathbb{N}$. Since D_i is dense in S_i , there exists a sequence $\{y_k\} \subset D_i \subset D$ such that $y_k \rightarrow x$, as $k \rightarrow \infty$. This proves that D is indeed dense in S , and completes the proof of the Lemma. \square

Lemma A.6 (second theorem in Appendix M3, [2])

Let (S, ρ) be a metric space and $\Sigma \subset S$ a separable subset of S . Then, there exists a countable collection \mathcal{A} of open subsets of S satisfying the following property: For each $x \in S$ and each open subset G of S ,

$$x \in G \cap \Sigma \implies x \in A \subset \overline{A} \subset G, \quad \text{for some } A \in \mathcal{A}.$$

PROOF Let $D \subset \Sigma$ be a countable dense subset of Σ . Let

$$\mathcal{A} := \left\{ B(d, r) \subset S \mid \begin{array}{l} d \in D, \\ r \in \mathbb{Q}, r > 0 \end{array} \right\}.$$

Then, \mathcal{A} is a countable collection of open balls in S . Now, let $G \subset S$ be an arbitrary open subset of S and $x \in G \cap \Sigma$. First, choose $\varepsilon > 0$ such that $B(x, \varepsilon) \subset G$. Next, since $x \in \Sigma$ and D is dense in Σ , we may choose $d \in D$ such that $d \in B(x, \varepsilon/2)$, or equivalently $\rho(x, d) < \varepsilon/2$. Finally choose positive rational $r > 0$ such that $\rho(x, d) < r < \varepsilon/2$.

Now, note that $\overline{B(d, r)} \subset B(x, \varepsilon)$; indeed,

$$y \in \overline{B(d, r)} \iff \rho(y, d) \leq r \implies \rho(x, y) \leq \rho(x, d) + \rho(d, y) < \varepsilon/2 + r < \varepsilon/2 + \varepsilon/2 \implies y \in B(x, \varepsilon).$$

Thus, we have

$$x \in B(d, r) \subset \overline{B(d, r)} \subset B(x, \varepsilon) \subset G.$$

This completes the proof of the Lemma. \square

Theorem A.7 (The Diagonal Method, Appendix A.14, [1])

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Suppose that each row of the array

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

is a bounded sequence of real numbers. Then, there exists an increasing sequence

$$n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$$

of positive integers such that the limit

$$\lim_{k \rightarrow \infty} x_{r,n_k} \text{ exists, for each } r = 1, 2, 3, \dots$$

PROOF From the first row, select a convergent subsequence

$$x_{1,n(1,1)}, x_{1,n(1,2)}, x_{1,n(1,3)}, \dots$$

Here, we have $n(1,1) < n(1,2) < n(1,3) < \cdots \in \mathbb{N}$, and $\lim_{k \rightarrow \infty} x_{1,n(1,k)} \in \mathbb{R}$ exists. Next, note that the following subsequence of the second row:

$$x_{2,n(1,1)}, x_{2,n(1,2)}, x_{2,n(1,3)}, \dots$$

is still a bounded sequence of real numbers, and we may thus select a convergent subsequence:

$$x_{2,n(2,1)}, x_{2,n(2,2)}, x_{2,n(2,3)}, \dots$$

Here, we have $n(2,1) < n(2,2) < n(2,3) < \cdots \in \{n(1,k)\}_{k \in \mathbb{N}}$, and $\lim_{k \rightarrow \infty} x_{2,n(2,k)} \in \mathbb{R}$ exists. Continuing inductively, we obtain an array of positive integers

$$\begin{array}{cccc} n(1,1) & n(1,2) & n(1,3) & \cdots \\ n(2,1) & n(2,2) & n(2,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

which satisfies: For each $r \in \mathbb{N}$, we have

- each row is an increasing sequence of positive integers, i.e. $n(r,1) < n(r,2) < n(r,3) < \cdots$,
- the $(r+1)^{\text{th}}$ row is a subsequence of the r^{th} row, i.e. $\{n(r+1,k)\}_{k \in \mathbb{N}} \subset \{n(r,k)\}_{k \in \mathbb{N}}$, and
- $\lim_{k \rightarrow \infty} x_{r,n(r,k)} \in \mathbb{R}$ exists.

Note that the first two properties together imply:

$$n(k,k) < n(k,k+1) \leq n(k+1,k+1), \text{ for each } k \in \mathbb{N}.$$

Now, define $n_k := n(k,k)$, for $k \in \mathbb{N}$. We then see that

$$n_k := n(k,k) < n(k+1,k+1) =: n_{k+1},$$

i.e., $\{n_k\}_{k \in \mathbb{N}}$ is a strictly increasing sequence of positive integers. Lastly, for each $r \in \mathbb{N}$, consider the sequence

$$x_{r,n_1}, x_{r,n_2}, x_{r,n_3}, \dots$$

Note that, for each $r \in \mathbb{N}$,

$$x_{r,n_r}, x_{r,n_{r+1}}, x_{r,n_{r+2}}, \dots$$

is a subsequence of $\{x_{r,n(r,k)}\}_{k \in \mathbb{N}}$. We saw above that $\lim_{k \rightarrow \infty} x_{r,n(r,k)}$ exists, which in turn implies that $\lim_{k \rightarrow \infty} x_{r,n_k}$ exists. Since $r \in \mathbb{N}$ is arbitrary, the proof of the Theorem is now complete. \square

References

- [1] BILLINGSLEY, P. *Probability and Measure*, third ed. John Wiley & Sons, 1995.
- [2] BILLINGSLEY, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.
- [3] FERGUSON, T. S. *A Course in Large Sample Theory*, first ed. Texts in Statistical Science. CRC Press, 1996.