Donsker's Theorems (Functional Central Limit Theorems)

1 Donsker's Theorem for $(\,C[0,1]\,,\,\|\cdot\|_\infty\,)$

Proposition 1.1

- *Let $\xi_1, \xi_2, \ldots : \Omega \longrightarrow \mathbb{R}$ be a sequence of independent and identically distributed $\mathbb{R}$-valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mu)$, with expectation value zero and common finite variance $\sigma^2 > 0$.*

- *Define the random variables:*

$$
\begin{cases}
S_0 & : \quad \Omega \longrightarrow \mathbb{R} \quad : \quad \omega \longmapsto 0, \qquad \qquad \text{and} \\[2mm]
S_n & : \quad \Omega \longrightarrow \mathbb{R} \quad : \quad \omega \longmapsto \sum_{i=1}^{n} \xi_i(\omega), \quad \text{for each } n \in \mathbb{N}.
\end{cases}
$$

- *For each $n \in \mathbb{N}$, define $X^{(n)} : \Omega \longrightarrow C[0,1]$ as follows:*

$$
X^{(n)}(\omega)(t) \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) \;+\; n\left(t - \frac{i-1}{n}\right)\xi_i(\omega) \right\}, \quad \text{for each } \omega \in \Omega,\ t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],\ i = 1,2,3,\ldots,n.
$$

- *For each $n \in \mathbb{N}$ and each $t \in [0,1]$, define $X_t^{(n)} : \Omega \longrightarrow \mathbb{R}$ as follows:*

$$
X_t^{(n)}(\omega) \;:=\; X^{(n)}(\omega)(t), \quad \text{for each } \omega \in \Omega.
$$

Then, the following statements are true:

(i) *For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,*

$$
X^{(n)}(\omega)\left(\frac{i}{n}\right) \;=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot S_i(\omega), \quad \text{for } i = 0, 1, 2, \ldots, n.
$$

(ii) *For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,*

$$
X^{(n)}(\omega)(t) \;\text{ is the linear interpolation from }\; \frac{1}{\sigma \cdot \sqrt{n}}\, S_{i-1}(\omega) \;\text{ to }\; \frac{1}{\sigma \cdot \sqrt{n}}\, S_i(\omega) \;\text{ over }\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],
$$

where $i = 1, 2, \ldots, n$.

(iii) *For any $0 \le t_0 < t_1 < t_2 < \cdots < t_k \le 1$,*

$$
\left(X_{t_1}^{(n)} - X_{t_0}^{(n)},\ \ldots,\ X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \right) \;\xrightarrow{d}\; N\left(\mu = \mathbf{0},\ \Sigma = \operatorname{diag}(\,t_1 - t_0,\ \ldots,\ t_k - t_{k-1}\,) \right), \;\text{ as }\; n \longrightarrow \infty.
$$

(iv) *For any $0 \le t_1, t_2, \cdots, t_k \le 1$,*

$$
\left(X_{t_1}^{(n)},\ X_{t_2}^{(n)},\ \ldots,\ X_{t_k}^{(n)} \right) \;\xrightarrow{d}\; N\left(\mu = \mathbf{0},\ \Sigma = \left[\, \min\{t_i, t_j\} \,\right]_{1 \le i,j \le k} \right), \;\text{ as }\; n \longrightarrow \infty.
$$

PROOF

(i)   Obvious.

(ii)   Obvious.

(iii)   First, note that, for each $\omega \in \Omega$, $n \in \mathbb{N}$, and $t \in [0,1]$, we have

$$X_t^{(n)}(\omega) \;=\; \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{\lfloor nt \rfloor}(\omega) \;+\; \left(nt - \lfloor nt \rfloor \right) \cdot \xi_{\lfloor nt \rfloor + 1}(\omega) \right\},$$

where $\lfloor \cdot \rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$, defined by

$$\lfloor x \rfloor \;:=\; \max \left\{ k \in \mathbb{Z} \;\middle|\; k \leq x \right\}, \quad \text{for each } x \in \mathbb{R},$$

is the round-down function. We next state three Claims, whose proofs will be given below. We note that the desired conclusion follows readily from Claim 3 and the Cramér-Wold Theorem (Theorem 1.9(iii), p.56, [4]); hence the present proof is complete once we establish the three Claims below.

> **Claim 1:**   If $\{\, a_n \,\}_{n \in \mathbb{N}}$ is a sequence of non-negative integers and $\{\, b_n \,\}_{n \in \mathbb{N}} \subset \mathbb{N}$ a sequence of positive integers satisfying:
>
> $$a_n \;<\; b_n, \text{ for sufficiently large } n \in \mathbb{N}, \qquad \text{and} \qquad \lim_{n \to \infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$
>
> then
>
> $$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i \,=\, 1 + a_n}^{b_n} \xi_i \;\xrightarrow{\;d\;}\; \sqrt{c} \cdot Z, \quad \text{where } Z \sim N(0,1).$$

> **Claim 2:**   For each fixed $t \in [0,1]$,
>
> $$W(t)_n \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot \left(nt - \lfloor nt \rfloor \right) \cdot \xi_{\lfloor nt \rfloor + 1} \;\xrightarrow{\;d\;}\; 0.$$

> **Claim 3:**   For $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$, and arbitrary $c_1, c_2, \ldots, c_k \in \mathbb{R}$,
>
> $$\sum_{i \,=\, 1}^{k} c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \;\xrightarrow{\;d\;}\; N\!\left(0 \,,\, \sum_{i \,=\, 1}^{k} c_i^2 \left(t_i - t_{i-1} \right) \right), \quad \text{as } n \longrightarrow \infty.$$

<u>Proof of Claim 1:</u>   Note that, for sufficiently large $n \in \mathbb{N}$, we may write

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i \,=\, 1 + a_n}^{b_n} \xi_i \;=\; \frac{\sqrt{b_n - a_n}}{\sqrt{n}} \cdot \left(\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i \,=\, 1 + a_n}^{b_n} \xi_i \right).$$

Since, by hypothesis, that

$$\lim_{n \to \infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$

Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [3]), once we establish the following:

$$\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i \,=\, 1 + a_n}^{b_n} \xi_i \;\xrightarrow{\;d\;}\; N(0,1), \quad \text{as } n \longrightarrow \infty.$$

We establish the above convergence by invoking the Lindeberg Central Limit Theorem (Theorem 1.15, §1.5.5, p.67, [4]). In the present context, the Lindeberg Condition is the following:

$$\lim_{n \to \infty} \frac{1}{B_n^2} \cdot E\left[\sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\left\{ |\xi_i| \geq \varepsilon S_n \right\}} \right] = 0, \quad \text{for each } \varepsilon > 0,$$

where

$$B_n^2 := \text{Var}\left[\sum_{i=1+a_n}^{b_n} \xi_i \right] = (b_n - a_n)\, \sigma^2 > 0.$$

The last equality used the hypothesis that $\xi_1, \xi_2, \ldots$ are independent and identically distributed with common finite variance $0 < \sigma^2 < \infty$. Hence, for each $\varepsilon > 0$,

$$\frac{1}{B_n^2} \cdot E\left[\sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\left\{ |\xi_i| \geq \varepsilon B_n \right\}} \right] = \frac{1}{(b_n - a_n)\, \sigma^2} \cdot (b_n - a_n) \cdot E\left[\xi_1^2 \cdot I_{\left\{ |\xi_1| \geq \varepsilon \sigma \sqrt{b_n - a_n} \right\}} \right]$$

$$= \frac{1}{\sigma^2} \cdot E\left[\xi_1^2 \cdot I_{\left\{ |\xi_1|/\varepsilon \sigma \geq \sqrt{b_n - a_n} \right\}} \right] \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,$$

since $\lim_{n \to \infty} \sqrt{b_n - a_n} = \infty$ and $\sigma^2 = E[\xi_1^2]$ is finite. This verifies that the Lindeberg Condition indeed holds in the present context, and completes the proof of Claim 1.

<u>Proof of Claim 2:</u>   First, note that $E[W(t)_n] = 0$. We now argue that $W(t)_n \xrightarrow{p} 0$. To this end, let $\varepsilon > 0$ be given. Then,

$$\varepsilon^2 \cdot P(|W(t)_n| \geq \varepsilon) \leq E\left[W(t)_n^2 \cdot I_{\{|W(t)_n| \geq \varepsilon\}} \right]$$

$$\leq E\left[W(t)_n^2 \right] = \text{Var}(W(t)_n) = \text{Var}\left[\frac{1}{\sigma \cdot \sqrt{n}} \cdot \left(nt - \lfloor nt \rfloor \right) \cdot \xi_{\lfloor nt \rfloor + 1} \right]$$

$$= \frac{1}{n \cdot \sigma^2} \cdot \left(nt - \lfloor nt \rfloor \right)^2 \cdot \text{Var}(\xi_{\lfloor nt \rfloor + 1}) = \frac{1}{n \cdot \sigma^2} \cdot \left(nt - \lfloor nt \rfloor \right)^2 \cdot \sigma^2$$

$$\leq \frac{1}{n},$$

which implies

$$\lim_{n \to \infty} P(|W(t)_n| \geq \varepsilon) = 0, \quad \text{for each } \varepsilon > 0,$$

i.e. $W(t)_n \xrightarrow{p} 0$, as $n \longrightarrow \infty$ (Definition 2, Chapter 1, [3]), which is equivalent to $W(t)_n \xrightarrow{d} 0$, as $n \longrightarrow \infty$ (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [3]). This proves Claim 2.

<u>Proof of Claim 3:</u>   Let $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$, and $c_1, c_2, \ldots, c_k \in \mathbb{R}$ be arbitrary. Observe that:

$$\sum_{i=1}^{k} c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right)$$

$$= \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ S_{\lfloor nt_i \rfloor} - S_{\lfloor nt_{i-1} \rfloor} \right\} + \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \left(nt_i - \lfloor nt_i \rfloor \right) \cdot \xi_{\lfloor nt_i \rfloor + 1} - \left(nt_{i-1} - \lfloor nt_{i-1} \rfloor \right) \cdot \xi_{\lfloor nt_{i-1} \rfloor + 1} \right\}$$

$$= \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \right\} + \sum_{i=1}^{k} c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\}$$

$$= \sum_{i=1}^{k} c_i Y_i^{(n)} + \sum_{i=1}^{k} c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\}$$

3

By Claim 2 and Slutsky's Theorem (Corollary, p.40, [3]),

$$\sum_{i=1}^{k} c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \xrightarrow{d} 0, \quad \text{as } n \longrightarrow \infty. \tag{1.1}$$

Next, note that since $\xi_1, \xi_2, \xi_3, \ldots$ are independent, we see that, for each fixed $n \in \mathbb{N}$,

$$Y_i^{(n)} := \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j, \quad i = 1, 2, 3, \ldots, k,$$

are independent. Now, since $0 \le t_{i-1} < t_i \le 1$, it follows that $\lfloor nt_{i-1} \rfloor < \lfloor nt_i \rfloor$ for sufficiently large $n \in \mathbb{N}$. In addition,

$$\frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} = \frac{\lfloor nt_i \rfloor}{n} - \frac{\lfloor nt_{i-1} \rfloor}{n} = \left(\frac{nt_i}{n} + \frac{\lfloor nt_i \rfloor - nt_i}{n} \right) - \left(\frac{nt_{i-1}}{n} + \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right)$$

$$= t_i - t_{i-1} + \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n},$$

which implies

$$\left| \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} - (t_i - t_{i-1}) \right| = \left| \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right| \le \frac{2}{n} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Thus,

$$\lim_{n \to \infty} \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} = t_i - t_{i-1} > 0.$$

Thus, by Claim 1, we see that, for each $i = 1, 2, \ldots, k$,

$$Y_i^{(n)} := \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \xrightarrow{d} \sqrt{t_i - t_{i-1}} \cdot N(0,1) = N\left(0, t_i - t_{i-1}\right), \quad \text{as } n \longrightarrow \infty. \tag{1.2}$$

By (1.1), (1.2), Proposition A.1, and Slutsky's Theorem (Corollary, p.40, [3]), we now see that

$$\sum_{i=1}^{k} c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) = \sum_{i=1}^{k} c_i Y_i^{(n)} + \sum_{i=1}^{k} c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \xrightarrow{d} N\left(0, \sum_{i=1}^{k} c_i^2 (t_i - t_{i-1}) \right).$$

This completes the proof of Claim 3.

(iv)   Let $t_0 := 0$, hence, $X_{t_0}^{(n)} \equiv 0$ for each $n \in \mathbb{N}$. We thus have, for each $n \in \mathbb{N}$,

$$\begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}}_{T} \cdot \begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} - X_{t_1}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \end{pmatrix}.$$

By (iii), we know that

$$
\begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} - X_{t_1}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \end{pmatrix} \xrightarrow{d} \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix} \sim N\Big(\mu = \mathbf{0} \, , \, \Sigma = \mathrm{diag}(t_1, t_2 - t_1, \ldots, t_k - t_{k-1}) \Big), \quad \text{as } n \longrightarrow \infty.
$$

Since the map $\mathbb{R}^k \longrightarrow \mathbb{R}^k : x \longmapsto T \cdot x$ is continuous, we see immediately by Slutsky's Theorem (Theorem 6(a), p.39, [3]) that

$$
\begin{pmatrix} X_{t_1}^{(n)} \\ X_{t_2}^{(n)} \\ \vdots \\ X_{t_k}^{(n)} \end{pmatrix} \xrightarrow{d} T \cdot \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix}, \quad \text{as } n \longrightarrow \infty.
$$

Since the map $\mathbb{R}^k \longrightarrow \mathbb{R}^k : x \longmapsto T \cdot x$ is an invertible linear automorphism on $\mathbb{R}^k$, we see that

$$
L = \begin{pmatrix} L_{t_1} \\ L_{t_2} \\ \vdots \\ L_{t_k} \end{pmatrix} := T \cdot \begin{pmatrix} Z_{t_1} \\ Z_{t_2-t_1} \\ \vdots \\ Z_{t_k-t_{k-1}} \end{pmatrix}
$$

is still an $\mathbb{R}^k$-valued Gaussian random variable, and it clearly has expectation value $\mathbf{0} \in \mathbb{R}^k$, since each of Z_{t_1}, $Z_{t_2-t_1}$, ..., $Z_{t_k-t_{k-1}}$ has expectation value $0 \in \mathbb{R}$. It remains only to compute the covariance matrix of the $\mathbb{R}^k$-valued Gaussian random variable L. To this end, consider $1 \leq i \leq j \leq k$, i.e. $t_i \leq t_j$. Then, using the alternative notation $Z_{t_1-t_0} := Z_{t_1}$, we have

$$
\begin{aligned}
\mathrm{Cov}\big(L_{t_i} \, , \, L_{t_j} \big) &= \mathrm{Cov}\big(Z_{t_1} + Z_{t_2-t_1} + \cdots + Z_{t_i-t_{i-1}} \, , \, Z_{t_1} + Z_{t_2-t_1} + \cdots + Z_{t_j-t_{j-1}} \big) \\
&= \mathrm{Cov}\bigg(\sum_{a=1}^{i} Z_{t_a-t_{a-1}} \, , \, \sum_{b=1}^{j} Z_{t_b-t_{b-1}} \bigg) = \sum_{a=1}^{i} \sum_{b=1}^{j} \mathrm{Cov}\big(Z_{t_a-t_{a-1}} \, , \, Z_{t_b-t_{b-1}} \big) \\
&= \sum_{a=1}^{i} \mathrm{Cov}\big(Z_{t_a-t_{a-1}} \, , \, Z_{t_a-t_{a-1}} \big) = \sum_{a=1}^{i} \mathrm{Var}\big(Z_{t_a-t_{a-1}} \big) = \sum_{a=1}^{i} (t_a - t_{a-1}) \\
&= (t_1 - t_0) + (t_2 - t_1) + \cdots + (t_{i-1} - t_{i-2}) + (t_i - t_{i-1}) \\
&= t_i = \min\{ t_i \, , \, t_j \},
\end{aligned}
$$

as required.

$\square$

Donsker's Theorems (Functional Central Limit Theorems)

Study Notes · November 1, 2015 · Kenneth Chu

A Technical Lemmas

Note that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ does NOT in general imply $X_n + Y_n \xrightarrow{d} X + Y$. But the implication does hold if X_n and Y_n are independent for each $n \in \mathbb{N}$, and both X and Y are Gaussian random variables, as the following Proposition shows.

Proposition A.1   *Let $k \in \mathbb{N}$ be fixed. Suppose:*

- *For each $n \in \mathbb{N}$,*

$$Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_k^{(n)} : \Omega^{(n)} \longrightarrow \mathbb{R}$$

  are independent $\mathbb{R}$-valued random variables defined on the probability space $\Omega^{(n)}$.

- *For each $i = 1, 2, \ldots, k$,*

$$Y_i^{(n)} \xrightarrow{d} N\left(\mu_i, \sigma_i^2 \right), \quad \text{as } n \longrightarrow \infty.$$

Then, for any $c_1, c_2, \ldots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^{k} c_i \, Y_i^{(n)} \xrightarrow{d} N\left(\sum_{i=1}^{k} c_i \, \mu_i \, , \, \sum_{i=1}^{k} c_i^2 \, \sigma_i^2 \right), \quad \text{as } n \longrightarrow \infty.$$

PROOF   Let $Y^{(n)} := \sum_{i=1}^{k} c_i \, Y_i^{(n)}$. Let φ_X denote the characteristic function of a $\mathbb{R}$-valued random variable X. Then,

$$
\begin{aligned}
\varphi_{Y^{(n)}}(t) \;&=\; \varphi_{\sum_i^k c_i Y_i^{(n)}}(t) \\
&=\; \prod_{i=1}^{k} \varphi_{c_i Y_i^{(n)}}(t), \quad \text{since } Y_1^{(n)}, \ldots, Y_k^{(n)} \text{ are independent} \\
&=\; \prod_{i=1}^{k} \varphi_{Y_i^{(n)}}(c_i t) \\
&\longrightarrow\; \prod_{i=1}^{k} \exp\left\{ \sqrt{-1}\, \mu_i\, (c_i\, t) \;-\; \frac{1}{2}\, \sigma_i^2\, (c_i\, t)^2 \right\} \\
&=\; \exp\left\{ \sqrt{-1}\left(\sum_{i=1}^{k} c_i \mu_i \right) t \;-\; \frac{1}{2} \left(\sum_{i=1}^{k} c_i^2 \sigma_i^2 \right) t^2 \right\}, \quad \text{as } n \longrightarrow \infty,
\end{aligned}
$$

where the second and third equalities follow from the properties of characteristic functions of random variables (see p.21, [3]), while the expression of the limit follows from the fact that the characteristic function φ_Z of a random variable Z with distribution $N\left(\mu, \sigma^2 \right)$ is

$$\varphi_Z \;=\; \exp\left\{ \sqrt{-1}\, \mu t \;-\; \frac{1}{2}\, \sigma^2\, t^2 \right\}.$$

The Proposition now follows immediately from the Lévy-Cramér Continuity Theorem (Theorem 1.9(ii), p.56, [4]). □

References

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