

## 1 Outline

Suppose:

- $(\Omega, \mathcal{A}, \mu)$  is a probability space.
- $n \in \mathbb{N}$  is an natural number (positive integer).
- $T_1, T_2, \dots, T_n : \Omega \longrightarrow [0, \infty]$  are independent identically distributed extended  $\mathbb{R}$ -valued random variables.
- $U_1, U_2, \dots, U_n : \Omega \longrightarrow [0, \infty]$  are independent identically distributed extended  $\mathbb{R}$ -valued random variables.
- For each  $i = 1, 2, \dots, n$ , let  $X_i := \min\{T_i, U_i\}$ , and  $C_i := I_{\{T_i \leq U_i\}}$ .

For each subject  $i = 1, 2, \dots, n$ , the random variable  $T_i$  is interpreted to be the “survival time” of subject  $i$ , while  $U_i$  is interpreted to be the “censoring time” of subject  $i$ .

We wish to make inference about the (common) *survival function*

$$S(t) := P(T > t) = \mu\left(\left\{\omega \in \Omega \mid T(\omega) > t\right\}\right)$$

of  $T_1, T_2, \dots, T_n$ . However, in survival analysis, the inference about  $S(t)$  is made based on the *right-censored survival time data*  $\{X_i, C_i\}$ ,  $i = 1, 2, \dots, n$  (rather than on the  $T_i$ ’s directly).

The *hazard function*:

$$\lambda(t) := \lim_{h \rightarrow 0^+} \frac{1}{h} \cdot P\left(t \leq T < t + h \mid t \leq T\right)$$

The *cumulative hazard function*:

$$\Lambda(t) := \int_0^t \lambda(t) dt$$

The *Nelson-Aalen estimator* for the cumulative hazard function  $\Lambda(t)$ :

$$\hat{\Lambda}(\omega, t) := \sum_{\substack{C_i(\omega)=1 \\ T_i(\omega) \leq t}} \frac{1}{Y(\omega, T_i(\omega))},$$

where

$$Y_i(\omega, t) := \begin{cases} 1, & t - h < X_i(\omega), \text{ for each } h > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$Y(\omega, t) := \sum_{i=1}^n Y_i(\omega, t)$$

The aggregated counting process for subject  $i$ :

$$N_i(\omega, t) := I_{\{X_i(\omega) \leq t\}}$$

The aggregated counting process:

$$N(\omega, t) := \sum_{i=1}^n N_i(\omega, t) = \sum_{i=1}^n I_{\{X_i(\omega) \leq t\}}$$

The aggregated intensity process:

$$\alpha(\omega, t) := \lim_{h \rightarrow 0^+} \frac{1}{h} \cdot P\left(N(\omega, t+h) - N(\omega, t) = 1 \mid \mathcal{F}_t\right) = \lim_{h \rightarrow 0^+} \frac{1}{h} \cdot E\left[N(\omega, t+h) - N(\omega, t) \mid \mathcal{F}_t\right]$$

The aggregated cumulative intensity process:

$$A(\omega, t) := \int_0^t \alpha(\omega, t) dt$$

Then, the process

$$M(\omega, t) := N(\omega, t) - A(\omega, t) = N(\omega, t) - \int_0^t \alpha(\omega, t) dt$$

is a martingale process. In particular,  $M(\cdot, t)$  satisfies

$$E\left[M(\cdot, t+h) - M(\cdot, t) \mid \mathcal{F}_t\right](\omega) = 0$$

## A Integration on product measure spaces

### Definition A.1 (Product $\sigma$ -algebra)

Suppose  $(\Omega_1, \mathcal{A}_1)$  and  $(\Omega_2, \mathcal{A}_2)$  are two measurable spaces. The product  $\sigma$ -algebra  $\mathcal{A}_1 \otimes \mathcal{A}_2$  of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  is, by definition, the following:

$$\mathcal{A}_1 \otimes \mathcal{A}_2 := \sigma \left( \left\{ A_1 \times A_2 \in \Omega_1 \times \Omega_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2 \right\} \right).$$

In other words,  $\mathcal{A}_1 \otimes \mathcal{A}_2$  is the  $\sigma$ -algebra of subsets of  $\Omega_1 \times \Omega_2$  containing all Cartesian products  $A_1 \times A_2$ , where  $A_1 \in \mathcal{A}_1$  and  $A_2 \in \mathcal{A}_2$ .

### Definition A.2 (Horizontal and vertical sections in a set-theoretic Cartesian product)

Suppose  $X$  and  $Y$  are two non-empty sets. For each  $x \in X$ ,  $y \in Y$ , and  $V \subset X \times Y$ , we define:

$$\begin{aligned} V_{(x, \cdot)} &:= \left\{ y \in Y \mid (x, y) \in V \right\} \\ V_{(\cdot, y)} &:= \left\{ x \in X \mid (x, y) \in V \right\} \end{aligned}$$

### Theorem A.3 (Sections of measurable subsets in a product measurable space are themselves measurable.)

Suppose  $(\Omega_1, \mathcal{A}_1)$  and  $(\Omega_2, \mathcal{A}_2)$  are two measurable spaces. Then,

- (i)  $V_{(x, \cdot)} \in \mathcal{A}_2$ , for each  $x \in \Omega_1$  and each  $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$ , and
- (ii)  $V_{(\cdot, y)} \in \mathcal{A}_1$ , for each  $y \in \Omega_2$  and each  $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$ .

PROOF We give only the proof of (i); that of (ii) is similar. Define  $\mathcal{F} \subset \mathcal{P}(\Omega_1 \times \Omega_2)$  as follows:

$$\mathcal{F} := \left\{ V \in \Omega_1 \times \Omega_2 \mid V_{(x, \cdot)} \in \mathcal{A}_2, \text{ for each } x \in \Omega_1 \right\}.$$

**Claim 1:**  $\left\{ A_1 \times A_2 \in \Omega_1 \times \Omega_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2 \right\} \subset \mathcal{F}$

**Claim 2:**  $\mathcal{F}$  is a  $\sigma$ -algebra of subsets of  $\Omega_1 \times \Omega_2$ .

Proof of Claim 1: Suppose  $x \in \Omega_1$  and  $V = A_1 \times A_2$ , where  $A_1 \in \mathcal{A}_1$  and  $A_2 \in \mathcal{A}_2$ . Then,

$$V_{(x, \cdot)} = \begin{cases} A_2, & \text{if } x \in A_1 \\ \emptyset, & \text{otherwise} \end{cases}$$

This proves that  $V_{(x, \cdot)} = (A_1 \times A_2)_{(x, \cdot)} \subset \mathcal{F}$ . Since  $x \in \Omega_1$ ,  $A_1 \in \mathcal{A}_1$ , and  $A_2 \in \mathcal{A}_2$  are arbitrary, Claim 1 follows.

Proof of Claim 2: First, note that, for each  $x \in \Omega_1$ , we have  $(\Omega_1 \times \Omega_2)_{(x, \cdot)} := \left\{ y \in \Omega_2 \mid (x, y) \in \Omega_1 \times \Omega_2 \right\} = \Omega_2 \in \mathcal{A}_2$ . Hence,  $\Omega_1 \times \Omega_2 \in \mathcal{F}$ . Next, suppose  $V \in \mathcal{F}$  and  $V^c := (\Omega_1 \times \Omega_2) \setminus V$ . Then, for each  $x \in \Omega_1$ ,

$$\begin{aligned} (V^c)_{(x, \cdot)} &= \left\{ y \in \Omega_2 \mid (x, y) \in V^c \right\} = \left\{ y \in \Omega_2 \mid (x, y) \notin V \right\} \\ &= \Omega_2 \setminus \left\{ y \in \Omega_2 \mid (x, y) \in V \right\} = (V_{(x, \cdot)})^c \in \mathcal{A}_2, \end{aligned}$$

where the last containment follows from the fact that  $\mathcal{A}_2$  is a  $\sigma$ -algebra (hence closed under complementation) and that  $V \in \mathcal{F}$  (hence  $V_{(x, \cdot)} \in \mathcal{A}_2$ ). This proves that  $\mathcal{F}$  is closed under complementation. Lastly, suppose  $V_1, V_2, \dots \in \mathcal{F}$ . Then,

$$\left( \bigcup_{i=1}^{\infty} V_i \right)_{(x, \cdot)} = \left\{ y \in \Omega_2 \mid (x, y) \in \bigcup_{i=1}^{\infty} V_i \right\} = \bigcup_{i=1}^{\infty} \left\{ y \in \Omega_2 \mid (x, y) \in V_i \right\} = \bigcup_{i=1}^{\infty} (V_i)_{(x, \cdot)} \in \mathcal{A}_2,$$

where the last containment follows from the fact that  $\mathcal{A}_2$  is a  $\sigma$ -algebra (hence closed under countable union) and that each  $V_i \in \mathcal{F}$  (hence  $(V_i)_{(x, \cdot)} \in \mathcal{A}_2$ ). This proves that  $\mathcal{F}$  is closed under countable union. This completes the proof of Claim 2.

Claim 1 and Claim 2 together immediately imply that

$$\mathcal{A}_1 \otimes \mathcal{A}_2 := \sigma \left( \left\{ A_1 \times A_2 \in \Omega_1 \times \Omega_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2 \right\} \right) \subset \mathcal{F} := \left\{ V \in \Omega_1 \times \Omega_2 \mid \begin{array}{l} V_{(x, \cdot)} \in \mathcal{A}_2, \\ \text{for each } x \in \Omega_1 \end{array} \right\}.$$

This completes the proof of statement (i) in the present Theorem.  $\square$

**Theorem A.4 (Sections of measurable maps are themselves measurable.)**

Suppose  $(\Omega_1, \mathcal{A}_1)$ ,  $(\Omega_2, \mathcal{A}_2)$ ,  $(S, \mathcal{S})$  are measurable spaces, and  $f : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \longrightarrow (S, \mathcal{S})$  is an  $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{S})$ -measurable map. Then,

- (i)  $f(x, \cdot) : \Omega_2 \longrightarrow S : y \longmapsto f(x, y)$  is an  $(\mathcal{A}_2, \mathcal{S})$ -measurable map for each  $x \in \Omega_1$ .
- (ii)  $f(\cdot, y) : \Omega_1 \longrightarrow S : x \longmapsto f(x, y)$  is an  $(\mathcal{A}_1, \mathcal{S})$ -measurable map for each  $y \in \Omega_2$ .

PROOF

- (i) We need to show that  $f(x, \cdot)^{-1}(V) \in \mathcal{A}_2$ , for each  $x \in \Omega_1$ , and each  $V \in \mathcal{S}$ . To this end, note that

$$f(x, \cdot)^{-1}(V) = \left\{ y \in \Omega_2 \mid f(x, y) \in V \right\} = \left\{ y \in \Omega_2 \mid (x, y) \in f^{-1}(V) \right\} = f^{-1}(V)_{(x, \cdot)} \in \mathcal{A}_2,$$

where the last containment follows, by Theorem A.3, from the fact that  $f^{-1}(V) \in \mathcal{A}_1 \otimes \mathcal{A}_2$  (since  $V \in \mathcal{S}$  and  $f$  is  $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{S})$ -measurable).

- (ii) The proof here is similar to that of (i).  $\square$

**Theorem A.5 (Well-definition of the product measure of two  $\sigma$ -finite measures)**

Suppose  $(\Omega_1, \mathcal{A}_1, \mu_1)$  and  $(\Omega_2, \mathcal{A}_2, \mu_2)$  are two  $\sigma$ -finite measure spaces. Let  $(\mathbb{R}, \mathcal{B})$  be  $\mathbb{R}$  equipped with its Borel  $\sigma$ -algebra  $\mathcal{B} = \sigma(\mathcal{O}(\mathbb{R}))$ . Then,

- (i) the map  $\Omega_1 \longrightarrow \mathbb{R} : x \longmapsto \mu_2(V_{(x, \cdot)}) = \int_{\Omega_2} 1_V(x, y) d\mu_2(y)$  is  $(\mathcal{A}_1, \mathcal{B})$ -measurable.
- (ii) the map  $\Omega_2 \longrightarrow \mathbb{R} : y \longmapsto \mu_1(V_{(\cdot, y)}) = \int_{\Omega_1} 1_V(x, y) d\mu_1(x)$  is  $(\mathcal{A}_2, \mathcal{B})$ -measurable.
- (iii) the following equality of Lebesgue integrals (of measurable  $\mathbb{R}$ -valued functions) holds:

$$\int_{\Omega_1} \mu_2(V_{(x, \cdot)}) d\mu_1(x) = \int_{\Omega_2} \mu_1(V_{(\cdot, y)}) d\mu_2(y), \quad \text{for each } V \in \mathcal{A}_1 \otimes \mathcal{A}_2,$$

or equivalently,

$$\int_{\Omega_1} \left( \int_{\Omega_2} 1_V(x, y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_2} \left( \int_{\Omega_1} 1_V(x, y) d\mu_1(x) \right) d\mu_2(y), \quad \text{for each } V \in \mathcal{A}_1 \otimes \mathcal{A}_2.$$

## References

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