The Wiener Measure on $\big(\, C[0,1]\,,\|\,\cdot\,\|_\infty \,\big)$

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In these notes, we give the definition of the well-known **Wiener measure** defined on the separable Banach space $\big(\, C[0,1]\,,\|\,\cdot\,\|_\infty \,\big)$, and give the proof that it indeed exists. Here is the outline:

- The Wiener measure will be defined as any Borel probability measure on $\big(C[0,1],\|\,\cdot\,\|_\infty\big)$ with certain prescribed finite-dimensional distributions. Since the collection of finite-dimensional subsets of $C[0,1]$ is a separating class for its Borel σ-algebra (see Example 1.3, [3]), we see immediately that if a Wiener measure exists, it is unique.
- However, the above definition is axiomatic, rather than constructive; the existence of a probability measure on $\big(C[0,1],\|\,\cdot\,\|_\infty\big)$ satisfying the defining properties of a Wiener measure must therefore be proved.
- To present this proof, we will start by showing that the "scaling limits" of the finite-dimensional distributions of the sequence of probability measures induced on $\big(C[0,1],\|\,\cdot\,\|_\infty\big)$ by linearly interpolated random walks with I.I.D. steps possess exactly the defining properties (of the finite-dimensional distributions) of the Wiener measure.
- We then show that the sequence of probability measures induced on $\big(C[0,1],\|\,\cdot\,\|_\infty\big)$ by linearly interpolated random walks with I.I.D. steps is tight, and thus admits a weakly convergent subsequence.
- The Wiener measure is, by definition, the limit of this weakly convergent subsequence of probability measures on $\big(C[0,1],\|\,\cdot\,\|_\infty\big)$

1 Equivalence of $(C[0,1]\,,\|\cdot\|_\infty)$-valued random variables and $\mathbb{R}$-valued stochastic processes indexed by $[0,1]$ with continuous sample paths

Proposition 1.1 (The "one-dimensional subsets" of $C[0,1]$ generate its Borel σ-algebra)
Let $(C[0,1]\,,\|\cdot\|_\infty)$ be the metric space of continuous $\mathbb{R}$-valued functions defined on the closed unit interval equipped with the supremum norm. For each $t\in[0,1]$, let $\mathrm{ev}_t : C[0,1] \longrightarrow \mathbb{R} : x \longmapsto x(t)$. Define:

$$\mathcal{S} \;:=\; \left\{\; \mathrm{ev}_t^{-1}(H) \,\subset\, C[0,1] \;\middle|\; \begin{array}{c} t\in[0,1] \\ H\in\mathcal{O} \end{array} \right\} \;\subset\; \mathcal{P}(\,C[0,1]\,)\,.$$

Then, $\mathcal{S}$ generates the Borel σ-algebra $\mathcal{B} := \mathcal{B}(C[0,1],\|\cdot\|_\infty)$ of the metric space $(C[0,1],\|\cdot\|_\infty)$; in other words,

$$\sigma(\,\mathcal{S}\,) \;=\; \mathcal{B}.$$

Proof First, note that $\sigma(\,\mathcal{S}\,)\subset\mathcal{B}$. Indeed, recall that, for each $t\in[0,1]$, $\mathrm{ev}_t : C[0,1]\longrightarrow\mathbb{R}$ is continuous, hence $(\mathcal{B},\mathcal{O})$-measurable, by Corollary B.4. In particular, $\mathrm{ev}_t^{-1}(H)\in\mathcal{B}$, for each $t\in[0,1]$ and $H\in\mathcal{O}$. Thus, $\mathcal{S}\subset\mathcal{B}$; hence, $\sigma(\,\mathcal{S}\,)\subset\mathcal{B}$.

It remains to establish the reverse inclusion. To this end, first observe that, for each $x\in C[0,1]$ and each $\varepsilon>0$, we have

$$\overline{B(x,\varepsilon)} \;=\; \bigcap_{r\,\in\,\mathbb{Q}\,\cap\,[0,1]} \left\{\; y\in C[0,1] \;\middle|\; |\,y(r)-x(r)\,|\le\varepsilon \;\right\} \;=\; \bigcap_{r\,\in\,\mathbb{Q}\,\cap\,[0,1]} \mathrm{ev}_r^{-1}\Big(\,[x(r)-\varepsilon,x(r)+\varepsilon]\,\Big),$$

which shows that $\sigma(\,\mathcal{S}\,)$ contains all the closed balls in $C[0,1]$. On the other hand, recall that, in any metric space, every open ball can be expressed as a countable union of closed balls; indeed, for any y in the given metric space, and any $\delta>0$, we have:

$$B(y,\delta) \;=\; \bigcup_{n\in\mathbb{N}} \overline{B\bigg(y,\delta-\frac{1}{n}\bigg)}.$$

We thus see that $\sigma(\,\mathcal{S}\,)$ contains all the open balls in $C[0,1]$. By the separability of $C[0,1]$ and Theorem C.1, we see that every open subset of $C[0,1]$ can be expressed as a countable union of open balls. Hence, $\sigma(\,\mathcal{S}\,)$ in fact contains all the open subsets of $C[0,1]$, which immediately yields $\mathcal{B}\,\subset\,\sigma(\,\mathcal{S}\,)$. This proves $\sigma(\,\mathcal{S}\,)\,=\,\mathcal{B}$. $\square$

Theorem 1.2

Suppose:

- *$(\Omega, \mathcal{A})$ is a measurable space, and $\mathcal{O}$ is the Borel σ-algebra of $\mathbb{R}$ (equipped with usual Euclidean metric).*
- *$(C[0,1]\,, \|\cdot\|_\infty)$ is the metric space of continuous $\mathbb{R}$-valued functions defined on the compact unit interval equipped with the supremum norm, and $\mathcal{B} := \mathcal{B}(C[0,1]\,, \|\cdot\|_\infty)$ is its Borel σ-algebra.*
- *$X : \Omega \longrightarrow C[0,1]$ is a function with domain Ω and codomain $C[0,1]$, but otherwise arbitrary.*
- *For each $t \in [0,1]$, let $\mathrm{ev}_t : C[0,1] \longrightarrow \mathbb{R} : x \longmapsto x(t)$.*
- *For each $t \in [0,1]$, let $X_t := \mathrm{ev}_t \circ X$. In other words, $X_t : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto \mathrm{ev}_t(X(\omega)) = X(\omega)(t)$.*

Then, X is $(\mathcal{A}, \mathcal{B})$-measurable if and only if, for each $t \in [0,1]$, X_t is $(\mathcal{A}, \mathcal{O})$-measurable.

PROOF

<u>$(\Longrightarrow)$</u>

It is trivial to see that, for each $t \in [0,1]$, $\mathrm{ev}_t : (\,C[0,1]\,, \|\cdot\|_\infty\,) \longrightarrow (\,\mathbb{R}\,, |\cdot|\,) : x \longmapsto x(t)$ is continuous. Recall that continuous maps are necessarily Borel measurable; see Corollary B.4. Hence, $\mathrm{ev}_t : (\,C[0,1]\,, \|\cdot\|_\infty\,) \longrightarrow (\,\mathbb{R}\,, |\cdot|\,)$ is $(\mathcal{B}\,, \mathcal{O})$-measurable, for each $t \in [0,1]$. Now, suppose $X : \Omega \longrightarrow C[0,1]$ is $(\mathcal{A}, \mathcal{B})$-measurable. Then, for each $t \in [0,1]$, the composition $X_t := \mathrm{ev}_t \circ X$ is $(\mathcal{A}\,, \mathcal{O})$-measurable, as required.

<u>$(\Longleftarrow)$</u>

Suppose that, for each $t \in [0,1]$, $X_t := \mathrm{ev}_t \circ X$ is $(\mathcal{A}\,, \mathcal{O})$-measurable. We seek to establish that $X : (\Omega, \mathcal{A}) \longrightarrow (C[0,1], \mathcal{B})$ is $(\mathcal{A}, \mathcal{B})$-measurable. To this end, let

$$\mathcal{S} \;:=\; \left\{ \mathrm{ev}_t^{-1}(H) \,\subset\, C[0,1] \;\middle|\; \begin{array}{c} t \in [0,1] \\ H \in \mathcal{O} \end{array} \right\} \;\subset\; \mathcal{P}(\,C[0,1]\,)\,.$$

Then, note that the $(\mathcal{A}\,, \mathcal{B})$-measurable of X follows immediately from Theorem B.3, Proposition 1.1, and the following

Claim: $X^{-1}(\,\mathcal{S}\,) \,\subset\, \mathcal{A}$.

Proof of Claim: Every set in $\mathcal{S}$ has the form $\mathrm{ev}_t^{-1}(H)$, for some $t \in [0,1]$ and some $H \in \mathcal{O}$. Note that

$$X^{-1}\big(\mathrm{ev}_t^{-1}(H)\big) \;=\; (\mathrm{ev}_t \circ X)^{-1}\Big(\,H\,\Big) \;=\; X_t^{-1}\Big(\,H\,\Big) \;\in\; \mathcal{A},$$

where the last containment follows immediately from the $(\mathcal{A}\,, \mathcal{O})$-measurability hypothesis on X_t, for each $t \in [0,1]$. This shows that $X^{-1}(\,\mathcal{S}\,) \subset \mathcal{A}$ and completes the proof of the Claim.

The proof of the Theorem is now complete. □

Theorem 1.3

Suppose:

- *$(\Omega, \mathcal{A})$ is a measurable space, and $\mathcal{O}$ is the Borel σ-algebra of $\mathbb{R}$ (equipped with usual Euclidean metric).*
- *$(C[0,1]\,, \|\cdot\|_\infty)$ is the metric space of continuous $\mathbb{R}$-valued functions defined on the closed unit interval equipped with the supremum norm, and $\mathcal{B} := \mathcal{B}(C[0,1]\,, \|\cdot\|_\infty)$ is its Borel σ-algebra.*
- *$X : \Omega \longrightarrow C[0,1]$ is a function with domain Ω and codomain $C[0,1]$, but otherwise arbitrary.*

- *For each* $t \in [0,1]$, *let* $\mathrm{ev}_t : C[0,1] \longrightarrow \mathbb{R} : x \longmapsto x(t)$.
- *For each* $t \in [0,1]$, *let* $X_t := \mathrm{ev}_t \circ X$. *In other words,* $X_t : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto \mathrm{ev}_t(X(\omega)) = X(\omega)(t)$.

Then, the following are equivalent:

(i) X *is a* $(C[0,1]\,,\|\cdot\|_\infty)$*-valued random variable (in other words,* X *is* $(\mathcal{A}\,,\mathcal{B})$*-measurable).*

(ii) *For each* $t \in [0,1]$, X_t *is an* $\mathbb{R}$*-valued random variable (in other words, each* X_t *is* $(\mathcal{A}\,,\mathcal{O})$*-measurable).*

(iii) $\{\, X_t : \Omega \longrightarrow \mathbb{R} \,\}_{t\in[0,1]}$ *is a stochastic process indexed by the closed unit interval defined on the probability space* $(\Omega, \mathcal{A}, \mu)$ *with state space* $\mathbb{R}$ *and continuous sample paths.*

Proof The equivalence of (i) and (ii) is immediate by the preceding Theorem. The equivalence of (ii) and (iii) is immediate by the definition of stochastic processes. □

2 Scaling limits of finite-dimensional distributions of linearly interpolated random walks are multivariate Gaussian

Definition 2.1

- *A **random walk on** $\mathbb{R}$ is a sequence* $\{\, S_n : \Omega \longrightarrow \mathbb{R} \,\}_{n\in\mathbb{N}}$ *of* $\mathbb{R}$*-valued random variables defined on a probability space* Ω *of the following form: There exists a sequence* $\xi_1, \xi_2, \ldots : \Omega \longrightarrow \mathbb{R}$ *of* <u>*independent and identically distributed*</u> $\mathbb{R}$*-valued random variables such that*

$$S_n \;=\; \sum_{i\,=\,1}^{n} \xi_i, \quad \text{for each } n \in \mathbb{N}.$$

- *Let* $\left\{\, S_n \,=\, \sum\limits_{i\,=\,1}^{n} \xi_i \,:\, \Omega \,\longrightarrow\, \mathbb{R} \,\right\}_{n\in\mathbb{N}}$ *be a random walk on* $\mathbb{R}$. *Let* $S_0 : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto 0$ *be the (constant) random variable defined on* Ω *with constant value* $0 \in \mathbb{R}$.

  For each $n \in \mathbb{N}$, *the **piecewise linear equi-spaced lattice interpolation of*** $\{\, S_0, S_1, S_2, \ldots, S_n \,\}$ ***over the unit interval*** $[0,1]$ *is the function* $\overline{S}^{(n)} : \Omega \longrightarrow C[0,1]$ *defined as follows:*

$$\overline{S}^{(n)}(\omega)(t) \;:=\; \frac{1}{\sigma\cdot\sqrt{n}}\left\{\, S_{i-1}(\omega) \;+\; n\left(t - \frac{i-1}{n}\right)\xi_i(\omega) \,\right\}, \;\text{ for each } \omega\in\Omega,\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],\; i = 1,2,3,\ldots,n.$$

- *For each* $n \in \mathbb{N}$ *and each* $t \in [0,1]$, *define* $\overline{S}_t^{(n)} : \Omega \longrightarrow \mathbb{R}$ *as follows:*

$$\overline{S}_t^{(n)}(\omega) \;:=\; \overline{S}^{(n)}(\omega)(t), \quad \text{for each } \omega \in \Omega.$$

Theorem 2.2

Let $\left\{\, S_n \,=\, \sum\limits_{i\,=\,1}^{n} \xi_i \,:\, \Omega \,\longrightarrow\, \mathbb{R} \,\right\}_{n\in\mathbb{N}}$ *be a random walk on* $\mathbb{R}$, *and let* $\overline{S}^{(n)} : \Omega \longrightarrow C[0,1]$ *and* $\overline{S}_t^{(n)} : \Omega \longrightarrow \mathbb{R}$ *be the associated functions as defined in Definition 2.1. Let* $S_0 : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto 0$ *be the (constant)* $\mathbb{R}$*-valued random variable defined on* Ω *with constant value* $0 \in \mathbb{R}$. *Then, the following statements are true:*

(i) *For each* $\omega \in \Omega$ *and each* $n \in \mathbb{N}$,

$$\overline{S}^{(n)}(\omega)\left(\frac{i}{n}\right) \;=\; \frac{1}{\sigma\cdot\sqrt{n}}\cdot S_i(\omega), \quad \text{for } i = 0,1,2,\ldots,n.$$

(ii) *For each $\omega \in \Omega$ and each $n \in \mathbb{N}$,*

$$\overline{S}^{(n)}(\omega)(t) \;\; \textit{is the linear interpolation from} \;\; \frac{1}{\sigma\cdot\sqrt{n}}\,S_{i-1}(\omega) \;\; \textit{to} \;\; \frac{1}{\sigma\cdot\sqrt{n}}\,S_{i}(\omega) \;\; \textit{over} \;\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],$$

where $i = 1, 2, \ldots, n$. In particular, for each $\omega \in \Omega$, $\overline{S}^{(n)}(\omega)$ is a continuous $\mathbb{R}$-valued function defined on $[0,1]$.

(iii) *For each $n \in \mathbb{N}$ and each $t \in [0,1]$, the function $\overline{S}^{(n)}_t : \Omega \longrightarrow \mathbb{R}$ is an $\mathbb{R}$-valued random variable defined on the probability space Ω. Hence, for each $n \in \mathbb{N}$, $\left\{ \overline{S}^{(n)}_t \right\}_{t\in[0,1]}$ is a stochastic process indexed by $[0,1]$ with state space $\mathbb{R}$ and continuous sample paths.*

(iv) *For each $n \in \mathbb{N}$, the function $\overline{S}^{(n)} : \Omega \longrightarrow C[0,1]$ is a $C[0,1]$-valued random variable defined on the probability space Ω.*

Proof

(i) Obvious.

(ii) Obvious.

(iii) For each $n \in \mathbb{N}$ and each $t \in [0,1]$, the function $\overline{S}^{(n)}_t$ is an $\mathbb{R}$-valued random variable defined on Ω because it is a linear combination, with coefficients in $\mathbb{R}$, of the $\mathbb{R}$-valued random variables $\xi_1, \xi_2, \ldots, \xi_n$. Thus, $\left\{ \overline{S}^{(n)}_t \right\}_{t\in[0,1]}$ is a stochastic process indexed by $[0,1]$ with state space $\mathbb{R}$. By (ii), this stochastic process has continuous sample paths.

(iv) Immediate by (iii) and Theorem 1.3.

□

Theorem 2.3

- *Let $\xi_1, \xi_2, \ldots : \Omega \longrightarrow \mathbb{R}$ be a sequence of independent and identically distributed $\mathbb{R}$-valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mu)$, with expectation value zero and common finite variance $\sigma^2 > 0$.*
- *Define the random variables:*

$$\begin{cases} S_0 \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto 0, & \text{and} \\ S_n \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto \displaystyle\sum_{i=1}^{n} \xi_i(\omega), & \text{for each } n \in \mathbb{N}. \end{cases}$$

- *For each $n \in \mathbb{N}$, define $X^{(n)} : \Omega \longrightarrow C[0,1]$ as follows:*

$$X^{(n)}(\omega)(t) \;:=\; \frac{1}{\sigma\cdot\sqrt{n}}\left\{ S_{i-1}(\omega) \,+\, n\left(t - \frac{i-1}{n}\right)\xi_i(\omega) \right\}, \;\; \text{for each } \omega \in \Omega,\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],\; i = 1, 2, 3, \ldots, n.$$

- *For each $n \in \mathbb{N}$ and each $t \in [0,1]$, define $X^{(n)}_t : \Omega \longrightarrow \mathbb{R}$ as follows:*

$$X^{(n)}_t(\omega) \;:=\; X^{(n)}(\omega)(t), \quad \text{for each } \omega \in \Omega.$$

Then, the following statements are true:

(i) *For any* $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$,

$$\left(X^{(n)}_{t_1} - X^{(n)}_{t_0}, \;\ldots\;, X^{(n)}_{t_k} - X^{(n)}_{t_{k-1}} \right) \;\overset{d}{\longrightarrow}\; N\left(\mu = \mathbf{0}\,,\, \Sigma = \mathrm{diag}(\, t_1 - t_0, \;\ldots\;, t_k - t_{k-1}\,) \right), \;\text{ as }\; n \longrightarrow \infty.$$

(ii) *For any pairwise distinct* $0 \leq t_1, t_2, \ldots, t_k \leq 1$,

$$\left(X^{(n)}_{t_1}, X^{(n)}_{t_2}, \;\ldots\;, X^{(n)}_{t_k} \right) \;\overset{d}{\longrightarrow}\; N\left(\mu = \mathbf{0}\,,\, \Sigma = \Big[\min\{t_i, t_j\} \Big]_{1 \leq i,j \leq k} \right), \;\text{ as }\; n \longrightarrow \infty.$$

Proof First, note that, by Theorem 2.2, for each $n \in \mathbb{N}$, $X^{(n)} : \Omega \longrightarrow C[0,1]$ is a $C[0,1]$-valued random variable defined on Ω.

(i) For each $\omega \in \Omega$, $n \in \mathbb{N}$, and $t \in [0,1]$, we have

$$X^{(n)}_t(\omega) \;=\; \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{\lfloor nt \rfloor}(\omega) \;+\; \Big(nt - \lfloor nt \rfloor \Big) \cdot \xi_{\lfloor nt \rfloor + 1}(\omega) \right\},$$

where $\lfloor \cdot \rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$, defined by

$$\lfloor x \rfloor \;:=\; \max \left\{ k \in \mathbb{Z} \;\middle|\; k \leq x \right\}, \quad \text{for each } x \in \mathbb{R},$$

is the round-down function. We next state three Claims, whose proofs will be given below. We note that the desired conclusion follows readily from Claim 3 and the Cramér-Wold Theorem (Theorem 1.9(iii), p.56, [5]); hence the present proof is complete once we establish the three Claims below.

Claim 1: If $\{ a_n \}_{n \in \mathbb{N}}$ is a sequence of non-negative integers and $\{ b_n \}_{n\in\mathbb{N}} \subset \mathbb{N}$ a sequence of positive integers satisfying:

$$a_n \;<\; b_n, \text{ for sufficiently large } n \in \mathbb{N}, \qquad \text{and} \qquad \lim_{n \to \infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$

then

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i = 1 + a_n}^{b_n} \xi_i \;\overset{d}{\longrightarrow}\; \sqrt{c} \cdot Z, \quad \text{where } Z \sim N(0,1).$$

Claim 2: For each fixed $t \in [0,1]$,

$$W(t)_n \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot \Big(nt - \lfloor nt \rfloor \Big) \cdot \xi_{\lfloor nt \rfloor + 1} \;\overset{d}{\longrightarrow}\; 0.$$

Claim 3: For $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$, and arbitrary $c_1, c_2, \ldots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^{k} c_i \left(X^{(n)}_{t_i} - X^{(n)}_{t_{i-1}} \right) \;\overset{d}{\longrightarrow}\; N\left(0\,,\, \sum_{i=1}^{k} c_i^2\,(t_i - t_{i-1}) \right), \quad \text{as } n \longrightarrow \infty.$$

<u>Proof of Claim 1:</u> Note that, for sufficiently large $n \in \mathbb{N}$, we may write

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \;=\; \frac{\sqrt{b_n - a_n}}{\sqrt{n}} \cdot \left(\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i=1+a_n}^{b_n} \xi_i \right).$$

Since, by hypothesis, that

$$\lim_{n\to\infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$

Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [4]), once we establish the following:

$$\frac{1}{\sigma\cdot\sqrt{b_n - a_n}} \cdot \sum_{i\,=\,1+a_n}^{b_n} \xi_i \;\overset{d}{\longrightarrow}\; N(0,1), \quad \text{as } n \longrightarrow \infty.$$

We establish the above convergence by invoking the Lindeberg Central Limit Theorem (Theorem 1.15, §1.5.5, p.67, [5]). In the present context, the Lindeberg Condition is the following:

$$\lim_{n\to\infty} \frac{1}{B_n^2} \cdot E\left[\, \sum_{i\,=\,1+a_n}^{b_n} \xi_i^2 \cdot I_{\left\{\,|\,\xi_i\,|\,\geq\,\varepsilon\, S_n\right\}} \,\right] \;=\; 0, \quad \text{for each } \varepsilon > 0,$$

where

$$B_n^2 \;:=\; \mathrm{Var}\left[\, \sum_{i\,=\,1+a_n}^{b_n} \xi_i \,\right] \;=\; (b_n - a_n)\,\sigma^2 \;\;>\;\; 0.$$

The last equality used the hypothesis that $\xi_1,\ \xi_2,\ \ldots$ are independent and identically distributed with common finite variance $0 < \sigma^2 < \infty$. Hence, for each $\varepsilon > 0$,

$$\begin{aligned}
\frac{1}{B_n^2} \cdot E\left[\, \sum_{i\,=\,1+a_n}^{b_n} \xi_i^2 \cdot I_{\left\{\,|\,\xi_i\,|\,\geq\,\varepsilon\, B_n\right\}} \,\right] \;&=\; \frac{1}{(b_n - a_n)\,\sigma^2} \cdot (b_n - a_n) \cdot E\left[\, \xi_1^2 \cdot I_{\left\{\,|\,\xi_1\,|\,\geq\,\varepsilon\sigma\,\sqrt{b_n - a_n}\right\}} \,\right] \\
&=\; \frac{1}{\sigma^2} \cdot E\left[\, \xi_1^2 \cdot I_{\left\{\,|\,\xi_1\,|/\varepsilon\sigma\,\geq\,\sqrt{b_n - a_n}\right\}} \,\right] \;\longrightarrow\; 0, \quad \text{as } \; n \longrightarrow \infty,
\end{aligned}$$

since $\lim\limits_{n\to\infty} \sqrt{b_n - a_n} \;=\; \infty$ and $\sigma^2 \;=\; E\left[\, \xi_1^2 \,\right]$ is finite. This verifies that the Lindeberg Condition indeed holds in the present context, and completes the proof of Claim 1.

<u>Proof of Claim 2:</u> First, note that $E[\, W(t)_n \,] = 0$. We now argue that $W(t)_n \overset{p}{\longrightarrow} 0$. To this end, let $\varepsilon > 0$ be given. Then,

$$\begin{aligned}
\varepsilon^2 \cdot P(\,|\,W(t)_n\,|\;\geq\;\varepsilon\,) \;&\leq\; E\left[\, W(t)_n^2 \cdot I_{\{\,|\,W(t)_n\,|\,\geq\,\varepsilon\,\}} \,\right] \\
&\leq\; E\left[\, W(t)_n^2 \,\right] \;=\; \mathrm{Var}(\, W(t)_n \,) \;=\; \mathrm{Var}\left[\, \frac{1}{\sigma\cdot\sqrt{n}} \cdot \left(nt - \lfloor nt \rfloor \right) \cdot \xi_{\lfloor nt\rfloor+1} \,\right] \\
&=\; \frac{1}{n\cdot\sigma^2} \cdot \left(nt - \lfloor\, nt\,\rfloor \right)^2 \cdot \mathrm{Var}\left(\, \xi_{\lfloor nt\rfloor+1} \,\right) \;=\; \frac{1}{n\cdot\sigma^2} \cdot \left(nt - \lfloor\, nt\,\rfloor \right)^2 \cdot \sigma^2 \\
&\leq\; \frac{1}{n},
\end{aligned}$$

which implies

$$\lim_{n\to\infty} P(\,|\,W(t)_n\,|\;\geq\;\varepsilon\,) \;=\; 0, \;\; \text{for each } \varepsilon > 0,$$

i.e. $W(t)_n \overset{p}{\longrightarrow} 0$, as $n \longrightarrow \infty$ (Definition 2, Chapter 1, [4]), which is equivalent to $W(t)_n \overset{d}{\longrightarrow} 0$, as $n \longrightarrow \infty$ (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [4]). This proves Claim 2.

Proof of Claim 3: Let $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$, and $c_1, c_2, \ldots, c_k \in \mathbb{R}$ be arbitrary. Observe that:

$$
\begin{aligned}
& \sum_{i=1}^{k} c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \\
= & \sum_{i=1}^{k} \frac{c_i}{\sigma\cdot\sqrt{n}} \left\{ S_{\lfloor nt_i \rfloor} - S_{\lfloor nt_{i-1} \rfloor} \right\} + \sum_{i=1}^{k} \frac{c_i}{\sigma\cdot\sqrt{n}} \left\{ \Big(nt_i - \lfloor nt_i \rfloor \Big) \cdot \xi_{\lfloor nt_i \rfloor + 1} - \Big(nt_{i-1} - \lfloor nt_{i-1} \rfloor \Big) \cdot \xi_{\lfloor nt_{i-1} \rfloor + 1} \right\} \\
= & \sum_{i=1}^{k} \frac{c_i}{\sigma\cdot\sqrt{n}} \left\{ \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \right\} + \sum_{i=1}^{k} c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \\
= & \sum_{i=1}^{k} c_i\, Y_i^{(n)} + \sum_{i=1}^{k} c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\}
\end{aligned}
$$

By Claim 2 and Slutsky's Theorem (Corollary, p.40, [4]),

$$
\sum_{i=1}^{k} c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \overset{d}{\longrightarrow} 0, \text{ as } n \longrightarrow \infty. \tag{2.1}
$$

Next, note that since $\xi_1, \xi_2, \xi_3, \ldots$ are independent, we see that, for each fixed $n \in \mathbb{N}$,

$$
Y_i^{(n)} := \frac{1}{\sigma\cdot\sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j, \quad i = 1, 2, 3, \ldots, k,
$$

are independent. Now, since $0 \leq t_{i-1} < t_i \leq 1$, it follows that $\lfloor nt_{i-1} \rfloor < \lfloor nt_i \rfloor$ for sufficiently large $n \in \mathbb{N}$. In addition,

$$
\begin{aligned}
\frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} &= \frac{\lfloor nt_i \rfloor}{n} - \frac{\lfloor nt_{i-1} \rfloor}{n} = \left(\frac{nt_i}{n} + \frac{\lfloor nt_i \rfloor - nt_i}{n} \right) - \left(\frac{nt_{i-1}}{n} + \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right) \\
&= t_i - t_{i-1} + \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n},
\end{aligned}
$$

which implies

$$
\left| \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} - (t_i - t_{i-1}) \right| = \left| \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right| \leq \frac{2}{n} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
$$

Thus,

$$
\lim_{n\to\infty} \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} = t_i - t_{i-1} > 0.
$$

Thus, by Claim 1, we see that, for each $i = 1, 2, \ldots, k$,

$$
Y_i^{(n)} := \frac{1}{\sigma\cdot\sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \overset{d}{\longrightarrow} \sqrt{t_i - t_{i-1}} \cdot N(0,1) = N\Big(0\,,\, t_i - t_{i-1}\Big), \quad \text{as } n \longrightarrow \infty. \tag{2.2}
$$

By (2.1), (2.2), Proposition A.1, and Slutsky's Theorem (Corollary, p.40, [4]), we now see that

$$
\sum_{i=1}^{k} c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) = \sum_{i=1}^{k} c_i\, Y_i^{(n)} + \sum_{i=1}^{k} c_i \left\{ W(t_i)_n - W(t_{i-1})_n \right\} \overset{d}{\longrightarrow} N\left(0\,,\, \sum_{i=1}^{k} c_i^2 (t_i - t_{i-1}) \right).
$$

This completes the proof of Claim 3.

(ii) Let $t_0 := 0$, hence, $X^{(n)}_{t_0} \equiv 0$ for each $n \in \mathbb{N}$. Without loss of generality, we may re-label $t_1, t_2, \ldots, t_k$ if necessary such that $t_1 < t_2 < \cdots < t_k$. We thus have, for each $n \in \mathbb{N}$,

$$\left(\begin{array}{c} X^{(n)}_{t_1} \\ \\ X^{(n)}_{t_2} \\ \\ \vdots \\ \\ X^{(n)}_{t_k} \end{array}\right) \;=\; \underbrace{\left[\begin{array}{ccccccc} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{array}\right]}_{T} \cdot \left(\begin{array}{c} X^{(n)}_{t_1} \\ \\ X^{(n)}_{t_2} - X^{(n)}_{t_1} \\ \\ \vdots \\ \\ X^{(n)}_{t_k} - X^{(n)}_{t_{k-1}} \end{array}\right).$$

By (i), we know that

$$\left(\begin{array}{c} X^{(n)}_{t_1} \\ \\ X^{(n)}_{t_2} - X^{(n)}_{t_1} \\ \\ \vdots \\ \\ X^{(n)}_{t_k} - X^{(n)}_{t_{k-1}} \end{array}\right) \;\overset{d}{\longrightarrow}\; \left(\begin{array}{c} Z_{t_1} \\ \\ Z_{t_2-t_1} \\ \\ \vdots \\ \\ Z_{t_k-t_{k-1}} \end{array}\right) \;\sim\; N\Big(\,\mu = \mathbf{0}\,,\;\Sigma = \mathrm{diag}(t_1, t_2 - t_1, \ldots, t_k - t_{k-1})\,\Big),\;\;\text{as}\;\; n \longrightarrow \infty.$$

Since the map $\mathbb{R}^k \longrightarrow \mathbb{R}^k : x \longmapsto T \cdot x$ is continuous, we see immediately by Slutsky's Theorem (Theorem 6(a), p.39, [4]) that

$$\left(\begin{array}{c} X^{(n)}_{t_1} \\ \\ X^{(n)}_{t_2} \\ \\ \vdots \\ \\ X^{(n)}_{t_k} \end{array}\right) \;\overset{d}{\longrightarrow}\; T \cdot \left(\begin{array}{c} Z_{t_1} \\ \\ Z_{t_2-t_1} \\ \\ \vdots \\ \\ Z_{t_k-t_{k-1}} \end{array}\right),\quad \text{as}\;\; n \longrightarrow \infty.$$

Since the map $\mathbb{R}^k \longrightarrow \mathbb{R}^k : x \longmapsto T \cdot x$ is an invertible linear automorphism on $\mathbb{R}^k$, we see that

$$L \;=\; \left(\begin{array}{c} L_{t_1} \\ \\ L_{t_2} \\ \\ \vdots \\ \\ L_{t_k} \end{array}\right) \;:=\; T \cdot \left(\begin{array}{c} Z_{t_1} \\ \\ Z_{t_2-t_1} \\ \\ \vdots \\ \\ Z_{t_k-t_{k-1}} \end{array}\right)$$

is still an $\mathbb{R}^k$-valued Gaussian random variable, and it clearly has expectation value $\mathbf{0} \in \mathbb{R}^k$, since each of Z_{t_1}, $Z_{t_2-t_1}, \ldots, Z_{t_k-t_{k-1}}$ has expectation value $0 \in \mathbb{R}$. It remains only to compute the covariance matrix of the $\mathbb{R}^k$-valued Gaussian random variable L. To this end, consider $1 \leq i \leq j \leq k$, i.e. $t_i \leq t_j$. Then, using the alternative notation $Z_{t_1 - t_0} := Z_{t_1}$, we have

$$\begin{aligned}
\mathrm{Cov}\big(L_{t_i}\,,\,L_{t_j} \big) &= \mathrm{Cov}\big(Z_{t_1} + Z_{t_2 - t_1} + \cdots + Z_{t_i - t_{i-1}}\,,\, Z_{t_1} + Z_{t_2 - t_1} + \cdots + Z_{t_j - t_{j-1}} \big) \\
&= \mathrm{Cov}\left(\sum_{a=1}^{i} Z_{t_a - t_{a-1}}\,,\, \sum_{b=1}^{j} Z_{t_b - t_{b-1}} \right) \;=\; \sum_{a=1}^{i}\sum_{b=1}^{j} \mathrm{Cov}\big(Z_{t_a - t_{a-1}}\,,\, Z_{t_b - t_{b-1}} \big) \\
&= \sum_{a=1}^{i} \mathrm{Cov}\big(Z_{t_a - t_{a-1}}\,,\, Z_{t_a - t_{a-1}} \big) \;=\; \sum_{a=1}^{i} \mathrm{Var}\big(Z_{t_a - t_{a-1}} \big) \;=\; \sum_{a=1}^{i} (\, t_a - t_{a-1}\,) \\
&= (\, t_1 - t_0\,) + (\, t_2 - t_1\,) + \cdots + (\, t_{i-1} - t_{i-2}\,) + (\, t_i - t_{i-1}\,) \\
&= t_i \;=\; \min\{\, t_i\,, t_j\,\}\,,
\end{aligned}$$

as required.

□

3 A sufficient condition for the tightness of a sequence of linearly interpolated random walks

Lemma 3.1 (Lemma, p.88, [3])

- *Let $\xi_1, \xi_2, \ldots : \Omega \longrightarrow \mathbb{R}$ be a sequence of independent and identically distributed $\mathbb{R}$-valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mu)$, with expectation value zero and common finite variance $\sigma^2 > 0$.*
- *Define the random variables:*

$$\begin{cases} S_0 \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto 0, & \text{and} \\[2ex] S_n \;:\; \Omega \longrightarrow \mathbb{R} \;:\; \omega \longmapsto \displaystyle\sum_{i=1}^{n} \xi_i(\omega), & \text{for each } n \in \mathbb{N}. \end{cases}$$

- *For each $n \in \mathbb{N}$, define $X^{(n)} : \Omega \longrightarrow C[0,1]$ as follows:*

$$X^{(n)}(\omega)(t) \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) \;+\; n\left(t - \frac{i-1}{n} \right) \xi_i(\omega) \right\}, \;\text{ for each } \omega \in \Omega,\; t \in \left[\frac{i-1}{n}, \frac{i}{n} \right],\; i = 1, 2, 3, \ldots, n.$$

If

(i) *the sequence $\{\, \xi_n \,\}_{n \in \mathbb{N}}$ is stationary*

(i.e. for each fixed $j = 0, 1, 2, \ldots$, the distribution of $(\, \xi_k, \xi_{k+1}, \ldots, \xi_{k+j}\,)$ is the same of all $k \in \mathbb{N}$), and

(ii)

$$\lim_{\lambda \to \infty}\, \limsup_{n \to \infty}\; \lambda^2 \cdot P\left(\max_{1 \le k \le n} |\, S_k \,| \;\ge\; \lambda\, \sigma \sqrt{n} \right) \;=\; 0,$$

then $\left\{\, X^{(n)} \,\right\}_{n \in \mathbb{N}}$ is tight.

Proof We apply the necessary and sufficient condition for tightness in Theorem F.1(iv). Thus, it suffices to prove the following two claims:

Claim 1: For each $\eta > 0$, there exist $a > 0$ and $n_0 \in \mathbb{N}$ such that

$$P_{X^{(n)}}\left(\left\{\, f \in C[0,1] \;\middle|\; \vert\, f(0)\,\vert \geq a \,\right\}\right) \;=\; P\left(\,\left\vert\, X_0^{(n)} \,\right\vert \geq a\,\right) \;\leq\; \eta, \;\; \text{for each } n \geq n_0,$$

Claim 2: For each $\varepsilon > 0$,

$$\lim_{\delta\to 0^+}\limsup_{n\to\infty} P_{X^{(n)}}\left(\left\{\, f \in C[0,1] \;\middle|\; w(f,\delta) \geq \varepsilon \,\right\}\right) \;=\; \lim_{\delta\to 0^+}\limsup_{n\to\infty} P\left(\, w(X^{(n)},\delta) \,\geq\, \varepsilon\,\right) \;=\; 0.$$

Proof of Claim 1: Since, for each $n \in \mathbb{N}$, $X_0^{(n)}$ is identically zero, we may choose $a = 1$ and $n_0 = 1$. We then have, for any $\eta > 0$,

$$P_{X^{(n)}}\left(\left\{\, f \in C[0,1] \;\middle|\; \vert\, f(0)\,\vert \geq a \,\right\}\right) \;=\; P\left(\,\left\vert\, X_0^{(n)} \,\right\vert \geq a\,\right) \;=\; P\left(\,\left\vert\, X_0^{(n)} \,\right\vert \geq 1\,\right) \;=\; 0 \;\leq\; \eta, \;\; \text{for each } n \geq n_0 := 1.$$

This proves Claim 1.

In order to simplify the proof of Claim 2, before presenting it, we first prove instead the following:

Claim 3: For each $\varepsilon > 0$ and $\delta \in (0,1)$,

$$P\left(\, w(X^{(n)},\delta) \,\geq\, 3\,\varepsilon\,\right) \;\leq\; \frac{2}{\delta}\cdot P\left(\,\max_{1\leq k\leq\lceil n\delta\rceil} \vert\, S_k\,\vert \;\geq\; \frac{\varepsilon}{\sqrt{2\delta}}\,\sigma\sqrt{\lceil\, n\delta\,\rceil}\,\right), \quad \text{for all sufficiently large } n \in \mathbb{N}.$$

Proof of Claim 3: For each $\delta \in (0,1)$ and $n \in \mathbb{N}$, let $m := \lceil\, n\,\delta\,\rceil$ be the round-up ("smallest integer greater than or equal to") of $n\,\delta$ and $v = \lceil n/m \rceil$ be that of n/m. Let

$$t_0 = 0, \quad t_1 = \frac{m}{n}, \quad t_2 = \frac{2\cdot m}{n}, \quad \ldots, \quad t_{v-1} = \frac{(v-1)\cdot m}{n}, \quad t_v = 1.$$

Then,

$$t_i - t_{i-1} \;=\; \frac{i\cdot m}{n} - \frac{(i-1)\cdot m}{n} \;=\; \frac{m}{n} \;=\; \frac{\lceil n\,\delta\rceil}{n} \;\geq\; \frac{n\,\delta}{n} \;=\; \delta, \quad \text{for } i = 1,2,\ldots,v-1,$$

which implies, by Proposition D.5(i), that for any $\varepsilon > 0$, any Borel probability measure P on $(C[0,1], \Vert\cdot\Vert_\infty)$, and any $f \in C[0,1]$, we have

$$w(f,\delta) \;\leq\; 3\cdot\max_{1\,\leq\, i\,\leq\, v}\left\{\,\sup_{s\,\in\,[t_{i-1},t_i]}\left\vert\, f(s) - f(t_{i-1})\,\right\vert\,\right\}.$$

Thus, in particular, we have

$$w(X^{(n)}(\omega),\delta) \;\leq\; 3\cdot\max_{1\,\leq\, i\,\leq\, v}\left\{\,\sup_{s\,\in\,[t_{i-1},t_i]}\left\vert\, X^{(n)}(\omega)(s) - X^{(n)}(\omega)(t_{i-1})\,\right\vert\,\right\}, \quad \text{for each } \omega \in \Omega.$$

Next, recall that $X^{(n)}$ is continuous and piecewise linear with its set of discontinuity points contained in $\{\, 1/n,\, 2/n,\, \ldots,\, (n-1)/n,\, 1\,\}$, and

$$X^{(n)}(\omega)\left(\frac{i}{n}\right) \;=\; \frac{1}{\sigma\cdot\sqrt{n}}\cdot S_i(\omega), \quad \text{for } \; i = 1,2,\ldots,n.$$

Hence, writing $m_i := i\cdot m$, $i = 0,1,2,\ldots,v-1$, and $m_v := n$ (hence $t_i = m_i/n$, $i = 0,1,2,\ldots,v$), the preceding inequality becomes:

$$w(X^{(n)}(\omega),\delta) \;\leq\; 3\cdot\max_{1\,\leq\, i\,\leq\, v}\left\{\,\max_{m_{i-1}\,\leq\, k\,\leq\, m_i}\frac{\left\vert\, S_k(\omega) - S_{m_{i-1}}(\omega)\,\right\vert}{\sigma\sqrt{n}}\,\right\}, \quad \text{for each } \omega \in \Omega,$$

which implies

$$\begin{aligned}
P\Big(\Big\{\,\omega\in\Omega\;\Big|\; w(X^{(n)}(\omega),\delta)\;\geq\;3\,\varepsilon\,\Big\}\Big) \;&\leq\; P\left(\left\{\,\omega\in\Omega\;\left|\;\max_{1\leq i\leq v}\left\{\max_{m_{i-1}\,\leq\,k\,\leq\,m_i}\frac{\left|\,S_k(\omega)-S_{m_{i-1}}(\omega)\,\right|}{\sigma\sqrt{n}}\right\}\;\geq\;\varepsilon\,\right.\right\}\right)\\
&=\; P\left(\bigcup_{i=1}^{v}\left\{\,\omega\in\Omega\;\left|\;\max_{m_{i-1}\,\leq\,k\,\leq\,m_i}\frac{\left|\,S_k(\omega)-S_{m_{i-1}}(\omega)\,\right|}{\sigma\sqrt{n}}\;\geq\;\varepsilon\,\right.\right\}\right)\\
&\leq\; \sum_{i\,=\,1}^{v}P\Big(\Big\{\,\omega\in\Omega\;\Big|\;\max_{m_{i-1}\,\leq\,k\,\leq\,m_i}\Big|\,S_k(\omega)-S_{m_{i-1}}(\omega)\,\Big|\;\geq\;\varepsilon\,\sigma\sqrt{n}\,\Big\}\Big)\\
&=\; \sum_{i\,=\,1}^{v}P\Big(\Big\{\,\omega\in\Omega\;\Big|\;\max_{1\,\leq\,k\,\leq\,m_i-m_{i-1}}\Big|\,S_k(\omega)-S_0(\omega)\,\Big|\;\geq\;\varepsilon\,\sigma\sqrt{n}\,\Big\}\Big)
\end{aligned}$$

Next, note that

$$\begin{aligned}
n\,\delta\;\leq\;m\;:=\;\lceil\,n\,\delta\,\rceil\;\leq\;n\cdot\delta+1 \quad&\Longrightarrow\quad \frac{1}{n\,\delta}\;\geq\;\frac{1}{m}\;\geq\;\frac{1}{n\cdot\delta+1}\\
&\Longrightarrow\quad \frac{1}{\delta}\;=\;\frac{n}{n\,\delta}\;\geq\;\frac{n}{m}\;\geq\;\frac{n}{n\cdot\delta+1}\;=\;\frac{1}{\delta+1/n},
\end{aligned}$$

which implies $\lim\limits_{n\to\infty}\dfrac{n}{m}=\dfrac{1}{\delta}$. In particular,

$$\frac{n}{m}\;>\;\frac{1}{2\delta}\,,\quad\text{for sufficiently large }\;n.$$

Consequently,

$$\varepsilon\,\sigma\sqrt{n}\;=\;\varepsilon\,\sigma\sqrt{\frac{n}{m}}\,\sqrt{m}\;\geq\;\frac{\varepsilon\,\sigma}{\sqrt{2\,\delta}}\,\sqrt{m}\,,\quad\text{for sufficiently large }\;n.$$

Furthermore, we have $m_i - m_{i-1} \;=\; im-(i-1)m \;=\; m$, for $i = 1,2,\ldots,v-1$, and

$$v\;:=\;\lceil\,n/m\,\rceil\quad\Longrightarrow\quad(v-1)m\;<\;n\;\leq\;vm\quad\Longrightarrow\quad\left\{\begin{array}{l} m_v-m_{v-1}\;=\;n-(v-1)m\;\leq\;vm-(v-1)m\;\leq\;m\\[2ex] v\;<\;\dfrac{n}{m}+1\;<\;\dfrac{n}{m}+\dfrac{1}{\delta}\;\leq\;\dfrac{1}{\delta}+\dfrac{1}{\delta}\;=\;\dfrac{2}{\delta}\end{array}\right.$$

Thus, we see that, for sufficiently large n,

$$\begin{aligned}
P\Big(\Big\{\,\omega\in\Omega\;\Big|\; w(X^{(n)}(\omega),\delta)\;\geq\;3\,\varepsilon\,\Big\}\Big) \;&\leq\; \sum_{i\,=\,1}^{v}P\Big(\Big\{\,\omega\in\Omega\;\Big|\;\max_{1\,\leq\,k\,\leq\,m_i-m_{i-1}}\Big|\,S_k(\omega)-S_0(\omega)\,\Big|\;\geq\;\varepsilon\,\sigma\sqrt{n}\,\Big\}\Big)\\
&\leq\; \sum_{i\,=\,1}^{v}P\Big(\Big\{\,\omega\in\Omega\;\Big|\;\max_{1\,\leq\,k\,\leq\,m}\Big|\,S_k(\omega)-S_0(\omega)\,\Big|\;\geq\;\frac{\varepsilon\,\sigma}{\sqrt{2\,\delta}}\,\sqrt{m}\,\Big\}\Big)\\
&\leq\; v\cdot P\Big(\Big\{\,\omega\in\Omega\;\Big|\;\max_{1\,\leq\,k\,\leq\,m}\Big|\,S_k(\omega)-S_0(\omega)\,\Big|\;\geq\;\frac{\varepsilon}{\sqrt{2\,\delta}}\,\sigma\,\sqrt{m}\,\Big\}\Big)\\
&\leq\; \frac{2}{\delta}\cdot P\Big(\Big\{\,\omega\in\Omega\;\Big|\;\max_{1\,\leq\,k\,\leq\,m}|\,S_k(\omega)\,|\;\geq\;\frac{\varepsilon}{\sqrt{2\,\delta}}\,\sigma\,\sqrt{m}\,\Big\}\Big).
\end{aligned}$$

This completes the proof of Claim 3.

Proof of Claim 2: By Claim 3, we have, for each $\varepsilon > 0$, each $\delta\in(0,1)$ and each sufficiently large $n\in\mathbb{N}$,

$$P\Big(\Big\{\,\omega\in\Omega\;\Big|\; w(X^{(n)}(\omega),\delta)\;\geq\;3\,\varepsilon\,\Big\}\Big)\;\leq\;\frac{2}{\delta}\cdot P\Big(\Big\{\,\omega\in\Omega\;\Big|\;\max_{1\,\leq\,k\,\leq\,\lceil n\delta\rceil}|\,S_k(\omega)\,|\;\geq\;\frac{\varepsilon}{\sqrt{2\,\delta}}\,\sigma\,\sqrt{\lceil n\delta\rceil}\,\Big\}\Big).$$

This implies:

$$\begin{aligned}
& \limsup_{n\to\infty}\; P\Big(\Big\{\;\omega\in\Omega\;\Big|\;w(X^{(n)}(\omega),\delta)\;\geq\;3\,\varepsilon\;\Big\}\Big) \\
\leq\;& \limsup_{n\to\infty}\;\frac{2}{\delta}\cdot P\left(\left\{\;\omega\in\Omega\;\middle|\;\max_{1\,\leq\,k\,\leq\,\lceil n\delta\rceil}|\,S_k(\omega)\,|\;\geq\;\frac{\varepsilon}{\sqrt{2\,\delta}}\,\sigma\,\sqrt{\lceil n\delta\rceil}\;\right\}\right) \\
\leq\;& \limsup_{n\to\infty}\;\frac{2}{\delta}\cdot P\left(\left\{\;\omega\in\Omega\;\middle|\;\max_{1\,\leq\,k\,\leq\,n}|\,S_k(\omega)\,|\;\geq\;\frac{\varepsilon}{\sqrt{2\,\delta}}\,\sigma\,\sqrt{n}\;\right\}\right) \\
=\;& \frac{4}{\varepsilon^2}\cdot\limsup_{n\to\infty}\left(\frac{\varepsilon}{\sqrt{2\delta}}\right)^2\cdot P\left(\left\{\;\omega\in\Omega\;\middle|\;\max_{1\,\leq\,k\,\leq\,n}|\,S_k(\omega)\,|\;\geq\;\frac{\varepsilon}{\sqrt{2\,\delta}}\,\sigma\,\sqrt{n}\;\right\}\right),
\end{aligned}$$

where the second inequality follows from the fact that, for each fixed $\delta\in(0,1)$, $\{\,\lceil\,n\delta\,\rceil\,\}_{n\in\mathbb{N}}$ is a sequence of non-decreasing natural numbers but which may contain repeated entries. The preceding inequality and hypothesis (ii) together imply:

$$\begin{aligned}
& \lim_{\delta\to0^+}\;\limsup_{n\to\infty}\; P\Big(\Big\{\;\omega\in\Omega\;\Big|\;w(X^{(n)}(\omega),\delta)\;\geq\;3\,\varepsilon\;\Big\}\Big) \\
\leq\;& \frac{4}{\varepsilon^2}\cdot\lim_{\delta\to0^+}\;\limsup_{n\to\infty}\left(\frac{\varepsilon}{\sqrt{2\delta}}\right)^2\cdot P\left(\left\{\;\omega\in\Omega\;\middle|\;\max_{1\,\leq\,k\,\leq\,n}|\,S_k(\omega)\,|\;\geq\;\frac{\varepsilon}{\sqrt{2\,\delta}}\,\sigma\,\sqrt{n}\;\right\}\right) \\
=\;& \frac{4}{\varepsilon^2}\cdot\lim_{\lambda\to\infty}\;\limsup_{n\to\infty}\Big(\lambda\Big)^2\cdot P\left(\left\{\;\omega\in\Omega\;\middle|\;\max_{1\,\leq\,k\,\leq\,n}|\,S_k(\omega)\,|\;\geq\;\lambda\,\sigma\,\sqrt{n}\;\right\}\right) \\
=\;& \frac{4}{\varepsilon^2}\cdot 0\;=\;0.
\end{aligned}$$

This completes the proof of Claim 2, as well as that of the present Lemma. □

4 The Wiener measure on $(C[0,1],\|\cdot\|_\infty)$

Definition 4.1 (Wiener measure on $(C[0,1],\|\cdot\|_\infty)$)

A Borel probability measure W on $(C[0,1],\|\cdot\|_\infty)$ is called a ***Wiener measure*** *if it satisfies the following two conditions:*

(i) *Its induced measure $W\circ\mathrm{ev}_0^{-1}$ on $\mathbb{R}$ via the evaluation map $\mathrm{ev}_0 : C[0,1]\longrightarrow\mathbb{R} : f\longmapsto f(0)$ is the point-mass measure on $\mathbb{R}$ concentrated at $0\in\mathbb{R}$, i.e.*

$$W\big(\,\mathrm{ev}_0^{-1}(\,\{\,0\,\}\,)\,\big)\;=\;W\Big(\Big\{\;f\in C[0,1]\;\Big|\;f(0)=0\;\Big\}\Big)\;=\;1\,.$$

(ii) *For any $0\leq t_0<t_1<t_2<\cdots<t_k\leq1$,*

$$W\circ\Pi^{-1}_{t_0t_1\ldots t_k}\;=\;N\Big(\;\mu=\mathbf{0}\,,\,\Sigma=\mathrm{diag}(\,t_1-t_0,\;\ldots\;,\;t_k-t_{k-1}\,)\;\Big),$$

where the map $\Pi_{t_0t_1\ldots t_k}:C[0,1]\longrightarrow\mathbb{R}^k$ is defined as follows:

$$\Pi_{t_0t_1\ldots t_k}\;:\;C[0,1]\longrightarrow\mathbb{R}^k\;:\;f\;\longmapsto\;\Big(\,f(t_1)-f(t_0),\,f(t_2)-f(t_1),\,\ldots,\,f(t_k)-f(t_{k-1})\,\Big).$$

Theorem 4.2 (Finite-dimensional distributions of the Wiener measure)

Let W be a Wiener measure defined on $(C[0,1],\|\cdot\|_\infty)$. Then, for any pairwise distinct $0\;\leq\;t_1,\,t_2,\;\ldots\;,\,t_k\leq1$,

$$W\circ\pi^{-1}_{t_1\ldots t_k}\;=\;N\left(\;\mu=\mathbf{0}\,,\,\Sigma=\Big[\;\min\{t_i,t_j\}\;\Big]_{1\leq i,j\leq k}\;\right),$$

where the map $\pi_{t_1 \ldots t_k} : C[0,1] \longrightarrow \mathbb{R}^k$ *is defined as follows:*

$$\pi_{t_1 \ldots t_k} \;:\; C[0,1] \longrightarrow \mathbb{R}^k \;:\; f \;\longmapsto\; \Big(f(t_1),\, f(t_2),\, \ldots,\, f(t_k) \Big).$$

PROOF Let $t_0 := 0$. Re-labeling the t_i's if necessary, without loss of generality, we may assume that $t_1 < t_2 < \cdots < t_k$. Let $\mathcal{B}$ denote the Borel σ-algebra of $(C[0,1], \|\cdot\|_\infty)$. Then, $(C[0,1], \mathcal{B}, W)$ is the probability space obtained by equipping the measurable space $(C[0,1], \mathcal{B})$ with the Wiener measure W.

Define the following $\mathbb{R}$-valued random variables on the probability space $(C[0,1], \mathcal{B}, W)$:

$$\begin{array}{lclccll}
Z_{t_1 - t_0} & : & C[0,1] \longrightarrow \mathbb{R}, & Z_{t_1-t_0}(f) & := & f(t_1) & \\
Z_{t_i - t_{i-1}} & : & C[0,1] \longrightarrow \mathbb{R}, & Z_{t_i-t_{i-1}}(f) & := & f(t_i) - f(t_{i-1}), & \text{for } \; i = 2, \ldots, k \\
L_{t_i} & : & C[0,1] \longrightarrow \mathbb{R}, & L_{t_i}(f) & := & f(t_i), & \text{for } \; i = 1, \ldots, k
\end{array}$$

Then, we have

$$\left(\begin{array}{c} L_{t_1}(f) \\ \\ L_{t_2}(f) \\ \\ \vdots \\ \\ L_{t_k}(f) \end{array}\right) = \left(\begin{array}{c} f(t_1) \\ \\ f(t_2) \\ \\ \vdots \\ \\ f(t_k) \end{array}\right) = \underbrace{\left[\begin{array}{ccccccc} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{array}\right]}_{T} \cdot \left(\begin{array}{c} f(t_1) \\ \\ f(t_2) - f(t_1) \\ \\ \vdots \\ \\ f(t_k) - f(t_{k-1}) \end{array}\right) = T \cdot \left(\begin{array}{c} Z_{t_1-t_0}(f) \\ \\ Z_{t_2-t_1}(f) \\ \\ \vdots \\ \\ Z_{t_k-t_{k-1}}(f) \end{array}\right)$$

The present Theorem will be proved once we establish that the $\mathbb{R}^k$-valued random variable $\pi_{t_1 \ldots t_k} = (\, L_{t_1},\, L_{t_2},\, \ldots,\, L_{t_k} \,) \;:\; C[0,1] \longrightarrow \mathbb{R}^k$ defined on $(C[0,1], \mathcal{B}, W)$ has the following multivariate Gaussian distribution:

$$\pi_{t_1 \ldots t_k} \;=\; \left(\begin{array}{c} L_{t_1} \\ \\ L_{t_2} \\ \\ \vdots \\ \\ L_{t_k} \end{array}\right) \;\sim\; N\!\left(\mu = \mathbf{0}\,,\, \Sigma = \Big[\, \min\{t_i, t_j\} \,\Big]_{1 \le i,j \le k} \right).$$

To this end, note that since $t_0 := 0$, by the definition of a Wiener measure, the $\mathbb{R}^k$-valued random variable $\big(Z_{t_1-t_0},\, Z_{t_2-t_1},\, \ldots,\, Z_{t_k-t_{k-1}} \big) : C[0,1] \longrightarrow \mathbb{R}^k$ equals $\Pi_{t_0 t_1 \ldots t_k}$ W-almost-surely (since $W\Big(f(0) = 0\Big) = 1$). These two $\mathbb{R}^k$-valued random variables therefore have the same probability distribution. By the definition of a Wiener measure again, this common distribution is the following multivariate Gaussian distribution:

$$\left(\begin{array}{c} Z_{t_1-t_0} \\ \\ Z_{t_2-t_1} \\ \\ \vdots \\ \\ Z_{t_k-t_{k-1}} \end{array}\right) \;\sim\; N\!\left(\mu = \mathbf{0}\,,\, \Sigma = \mathrm{diag}(\, t_1 - t_0,\, \ldots\,,\, t_k - t_{k-1} \,) \right).$$

Since the $\mathbb{R}^k$-valued random variable $(L_{t_1}, L_{t_2}, \ldots, L_{t_k}) : C[0,1] \longrightarrow \mathbb{R}^k$ can be obtained from $\left(Z_{t_1-t_0}, Z_{t_2-t_1}, \ldots, Z_{t_k-t_{k-1}} \right) : C[0,1] \longrightarrow \mathbb{R}^k$ via the non-singular linear transformation $T : \mathbb{R}^k \longrightarrow \mathbb{R}^k$, it follows that $(L_{t_1}, L_{t_2}, \ldots, L_{t_k})$ also has a multivariate Gaussian distribution. Since $\left(Z_{t_1-t_0}, Z_{t_2-t_1}, \ldots, Z_{t_k-t_{k-1}} \right)$ has mean vector zero, so does $(L_{t_1}, L_{t_2}, \ldots, L_{t_k})$. It remains only to compute the covariance matrix of $(L_{t_1}, L_{t_2}, \ldots, L_{t_k})$, which we now do: For $t_i < t_j$, we have

$$\begin{aligned}
\mathrm{Cov}\left(L_{t_i}, L_{t_j} \right) &= \mathrm{Cov}\left(Z_{t_1-t_0} + Z_{t_2-t_1} + \cdots + Z_{t_i-t_{i-1}}, Z_{t_1-t_0} + Z_{t_2-t_1} + \cdots + Z_{t_j-t_{j-1}} \right) \\
&= \mathrm{Cov}\left(\sum_{a=1}^{i} Z_{t_a-t_{a-1}}, \sum_{b=1}^{j} Z_{t_b-t_{b-1}} \right) = \sum_{a=1}^{i}\sum_{b=1}^{j} \mathrm{Cov}\left(Z_{t_a-t_{a-1}}, Z_{t_b-t_{b-1}} \right) \\
&= \sum_{a=1}^{i} \mathrm{Cov}\left(Z_{t_a-t_{a-1}}, Z_{t_a-t_{a-1}} \right) = \sum_{a=1}^{i} \mathrm{Var}\left(Z_{t_a-t_{a-1}} \right) = \sum_{a=1}^{i} (t_a - t_{a-1}) \\
&= (t_1 - t_0) + (t_2 - t_1) + \cdots + (t_{i-1} - t_{i-2}) + (t_i - t_{i-1}) \\
&= t_i = \min\{ t_i, t_j \},
\end{aligned}$$

as required. This completes the proof of the Theorem. □

Theorem 4.3 (Uniqueness of the Wiener measure)

Any two Wiener measures on $(C[0,1], \|\cdot\|_\infty)$ must in fact be equal.

PROOF By Theorem 4.2, the collection of finite-dimensional distributions of a Wiener measure on $(C[0,1], \|\cdot\|_\infty)$ is completely determined. Thus, any two Wiener measures W_1 and W_2 must the exactly the same finite-dimensional distributions, which in turn implies that W_1 and W_2 must agree on the entire collection of finite-dimensional subsets of $C[0,1]$. Lastly, recall that the finite-dimensional subsets of $C[0,1]$ form a separating class of the Borel σ-algebra of $(C[0,1], w\|\cdot\|_\infty)$ (Example 1.3, p.11, [3]). We may now conclude that $W_1 = W_2$, as Borel probability measures on $(C[0,1], w\|\cdot\|_\infty)$. □

Theorem 4.4 (Existence of the Wiener measure)

There exists a Wiener measure on $(C[0,1], \|\cdot\|_\infty)$.

PROOF Let $\xi_1, \xi_2, \ldots : \Omega \longrightarrow \mathbb{R}$ be a sequence of independent and identically distributed standard Gaussian $\mathbb{R}$-valued random variables defined on the probability space $(\Omega, \mathcal{A}, \mu)$ (i.e. with expectation value zero and common finite variance $\sigma^2 = 1$). Define the random variables:

$$\begin{cases}
S_0 : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto 0, & \text{and} \\
S_n : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto \displaystyle\sum_{i=1}^{n} \xi_i(\omega), & \text{for each } n \in \mathbb{N}.
\end{cases}$$

For each $n \in \mathbb{N}$, define $X^{(n)} : \Omega \longrightarrow C[0,1]$ as follows:

$$X^{(n)}(\omega)(t) := \frac{1}{\sqrt{n}} \left\{ S_{i-1}(\omega) + n\left(t - \frac{i-1}{n} \right) \xi_i(\omega) \right\}, \text{ for each } \omega \in \Omega,\ t \in \left[\frac{i-1}{n}, \frac{i}{n} \right],\ i = 1, 2, 3, \ldots, n.$$

For each $n \in \mathbb{N}$ and each $t \in [0,1]$, define $X_t^{(n)} : \Omega \longrightarrow \mathbb{R}$ as follows:

$$X_t^{(n)}(\omega) := X^{(n)}(\omega)(t), \quad \text{for each } \omega \in \Omega.$$

Claim 1:

The sequence $\{ P_{X^{(n)}} \}_{n\in\mathbb{N}}$ of Borel probability measures on $(C[0,1], \|\cdot\|_\infty)$ induced by the $X^{(n)}$'s is tight.

□

A Technical Lemmas

Note that $X_n \overset{d}{\longrightarrow} X$ and $Y_n \overset{d}{\longrightarrow} Y$ does NOT in general imply $X_n + Y_n \overset{d}{\longrightarrow} X + Y$. But the implication does hold if X_n and Y_n are independent for each $n \in \mathbb{N}$, and both X and Y are Gaussian random variables, as the following Proposition shows.

Proposition A.1 *Let $k \in \mathbb{N}$ be fixed. Suppose:*

- *For each $n \in \mathbb{N}$,*
$$Y_1^{(n)},\, Y_2^{(n)},\, \ldots,\, Y_k^{(n)} \,:\, \Omega^{(n)} \longrightarrow \mathbb{R}$$
are independent $\mathbb{R}$-valued random variables defined on the probability space $\Omega^{(n)}$.
- *For each $i = 1, 2, \ldots, k$,*
$$Y_i^{(n)} \overset{d}{\longrightarrow} N\big(\,\mu_i,\, \sigma_i^2\,\big)\,, \quad \text{as } \, n \longrightarrow \infty.$$

Then, for any $c_1,\, c_2,\, \ldots,\, c_k \,\in \mathbb{R}$,
$$\sum_{i=1}^{k} c_i\, Y_i^{(n)} \overset{d}{\longrightarrow} N\left(\sum_{i=1}^{k} c_i\,\mu_i\,,\ \sum_{i=1}^{k} c_i^2\,\sigma_i^2 \right), \quad \text{as } \, n \longrightarrow \infty.$$

Proof Let $Y^{(n)} \,:=\, \sum_{i=1}^{k} c_i\, Y_i^{(n)}$. Let φ_X denote the characteristic function of a $\mathbb{R}$-valued random variable X. Then,

$$\begin{aligned}
\varphi_{Y^{(n)}}(t) &= \varphi_{\sum_i^k c_i Y_i^{(n)}}(t) \\
&= \prod_{i=1}^{k} \varphi_{c_i Y_i^{(n)}}(t), \quad \text{since } Y_1^{(n)},\, \ldots,\, Y_k^{(n)} \text{ are independent} \\
&= \prod_{i=1}^{k} \varphi_{Y_i^{(n)}}(\,c_i t\,) \\
&\longrightarrow \prod_{i=1}^{k} \exp\!\left\{ \sqrt{-1}\,\mu_i\,(c_i\,t) \,-\, \frac{1}{2}\,\sigma_i^2\,(\,c_i t\,)^2 \right\} \\
&= \exp\!\left\{ \sqrt{-1}\left(\sum_{i=1}^{k} c_i\mu_i \right) t \,-\, \frac{1}{2}\left(\sum_{i=1}^{k} c_i^2\sigma_i^2 \right) t^2 \right\}, \quad \text{as } \, n \longrightarrow \infty,
\end{aligned}$$

where the second and third equalities follow from the properties of characteristic functions of random variables (see p.21, [4]), while the expression of the limit follows from the fact that the characteristic function φ_Z of a random variable Z with distribution $N\big(\,\mu,\, \sigma^2\,\big)$ is

$$\varphi_Z \;=\; \exp\!\left\{ \sqrt{-1}\,\mu t \,-\, \frac{1}{2}\,\sigma^2\,t^2 \right\}.$$

The Proposition now follows immediately from the Lévy-Cramér Continuity Theorem (Theorem 1.9(ii), p.56, [5]). □

Proposition A.2

Suppose:

- *$k \in \mathbb{N}$ and $Q_1,\, Q_2,\, \ldots\, Q_k$ are Borel probability measures on $(C[0,1]\|\,\cdot\,\|_\infty)$.*
- *$\{\, P_n \,\}_{n\in\mathbb{N}}$ is a sequence of Borel probability measures on $(C[0,1]\|\,\cdot\,\|_\infty)$.*

Then, the following are equivalent:

(i) $\{ P_n \}_{n\in\mathbb{N}}$ *is tight.*

(ii) $\{ Q_1, \ldots, Q_k \} \cup \{ P_n \}_{n\in\mathbb{N}}$ *is tight.*

Proof Note that once we prove the Proposition for $k = 1$, the cases $k \geq 2$ will follow immediately by finite induction. We now proceed to prove the Proposition for $k = 1$, and we write Q for Q_1.

<u>(ii) $\Longrightarrow$ (i)</u> This is trivial. Indeed,

$$\begin{array}{cl} & \{ Q \} \cup \{ P_n \}_{n\in\mathbb{N}} \text{ is tight} \\ \Longleftrightarrow & \text{for each } \varepsilon > 0,\ \exists \text{ compact } K \subset C[0,1] \text{ such that } 1-\varepsilon < Q(K) \leq 1 \text{ and } 1-\varepsilon < P_n(K) \leq 1, \text{ for all } n \in \mathbb{N} \\ \Longrightarrow & \text{for each } \varepsilon > 0,\ \exists \text{ compact } K \subset C[0,1] \text{ such that } 1-\varepsilon < P_n(K) \leq 1, \text{ for all } n \in \mathbb{N} \\ \Longleftrightarrow & \{ P_n \}_{n\in\mathbb{N}} \text{ is tight} \end{array}$$

This proves that (ii) $\Longrightarrow$ (i).

<u>(i) $\Longrightarrow$ (ii)</u>

Let $\varepsilon > 0$ be given. Since $(C[0,1], \|\cdot\|_\infty)$ is separable and complete, the single Borel probability measure Q on $(C[0,1], \|\cdot\|_\infty)$ is tight (Theorem 1.3, [3]). Thus, there exists a compact subset $K_1 \subset C[0,1]$ such that $1-\varepsilon < Q(K_1) \leq 1$. On the other hand, the tightness hypothesis on $\{ P_n \}_{n\in\mathbb{N}}$ implies there exists a compact subset $K_2 \subset C[0,1]$ such that $1-\varepsilon < P_n(K_2) \leq 1$, for each $n \in \mathbb{N}$. Let $K := K_1 \cup K_2$. Then, K is itself a compact subset of $C[0,1]$, and

$$Q(\,K\,) \;=\; Q(\,K_1 \cup K_2\,) \;\geq\; Q(\,K_1\,) \;>\; 1-\varepsilon, \quad \text{and} \quad P_n(\,K\,) \;=\; P_n(\,K_1 \cup K_2\,) \;\geq\; P_n(\,K_2\,) \;>\; 1-\varepsilon.$$

This proves the tightness of $\{ Q \} \cup \{ P_n \}_{n\in\mathbb{N}}$, as required. $\square$

B Continuous maps are Borel measurable

Lemma B.1 (The pre-image of a σ-algebra is itself a σ-algebra.)
Suppose Ω is a non-empty set, $(X, \mathcal{X})$ is a measurable space, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then,

$$f^{-1}(\mathcal{X}) \;:=\; \left\{\, f^{-1}(V) \subset \Omega \;\middle|\; V \in \mathcal{X} \,\right\}$$

is a σ-algebra of subsets of Ω.

Proof

<u>$\Omega \in f^{-1}(\mathcal{X})$</u> $f(\Omega) \subset X \;\Longrightarrow\; \Omega = f^{-1}(X) \in f^{-1}(\mathcal{X})$.

<u>$f^{-1}(\mathcal{X})$ is closed under complementations</u> Let $V \in \mathcal{X}$. Then, $X \setminus V \in \mathcal{X}$, and

$$\Omega \setminus f^{-1}(V) \;=\; \{\, \omega \in \Omega \mid f(\omega) \notin V \,\} \;=\; \{\, \omega \in \Omega \mid f(\omega) \in X \setminus V \,\} \;=\; f^{-1}(X \setminus V) \;\in\; f^{-1}(\mathcal{X})\,,$$

which shows that $f^{-1}(\mathcal{X})$ is indeed closed under complementations.

<u>$f^{-1}(\mathcal{X})$ is closed countable unions</u> Let $V_1, V_2, \ldots \in \mathcal{X}$. Then, $\displaystyle\bigcup_{i=1}^{\infty} V_i \in \mathcal{X}$, and

$$\bigcup_{i=1}^{\infty} f^{-1}(V_i) \;=\; \left\{\, \omega \in \Omega \;\middle|\; \begin{array}{c} f(\omega) \in V_i \\ \text{for some } i \in \mathbb{N} \end{array} \,\right\} \;=\; f^{-1}\!\left(\bigcup_{i=1}^{\infty} V_i \right) \;\in\; f^{-1}(\mathcal{X})\,,$$

which proves that $f^{-1}(\mathcal{X})$ is indeed closed under countable unions.

This concludes the proof that that $f^{-1}(\mathcal{X})$ is a σ-algebra of subsets of Ω. $\square$

Lemma B.2 (The push-forward of a σ-algebra is itself a σ-algebra.)
Suppose $(\Omega, \mathcal{A})$ is a measurable space, X is a non-empty set, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then,

$$\mathcal{F} \;:=\; \left\{\, V \subset X \;\middle|\; f^{-1}(V) \in \mathcal{A} \,\right\}$$

is a σ-algebra of subsets of X.

Proof
<u>$X \in \mathcal{F}$</u> $f^{-1}(X) = \Omega \in \mathcal{A} \implies X \in \mathcal{F}$.

<u>$\mathcal{F}$ is closed under complementations</u> $V \in \mathcal{F} \implies f^{-1}(V) \in \mathcal{A} \implies f^{-1}(X \setminus V) = \Omega \setminus f^{-1}(V) \in \mathcal{A} \implies X \setminus V \in \mathcal{F}$, which proves that $\mathcal{F}$ is indeed closed under complementations.

<u>$\mathcal{F}$ is closed under countable unions</u>

$$\begin{aligned} V_1, V_2, \ldots \;\in\; \mathcal{F} \;&\implies\; f^{-1}(V_1), f^{-1}(V_2), \ldots \;\in\; \mathcal{A} \\ &\implies\; f^{-1}\left(\bigcup_{i=1}^{\infty} V_i \right) \;=\; \bigcup_{i=1}^{\infty} f^{-1}(V_i) \;\in\; \mathcal{A} \\ &\implies\; \bigcup_{i=1}^{\infty} V_i \;\in\; \mathcal{F}, \end{aligned}$$

which proves that $\mathcal{F}$ is indeed closed under countable unions. $\square$

Theorem B.3
Suppose $(\Omega, \mathcal{A})$ and $(X, \mathcal{X})$ are measurable spaces, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then, f is $(\mathcal{A}, \mathcal{X})$-measurable if there exists $\mathcal{S} \subset \mathcal{X}$ satisfying the following conditions:

- *$\mathcal{S}$ generates $\mathcal{X}$, i.e. $\sigma(\mathcal{S}) = \mathcal{X}$, and*
- *$f^{-1}(\mathcal{S}) \subset \mathcal{A}$.*

Proof By Lemma B.2,

$$\mathcal{F} \;:=\; \left\{\, V \subset X \;\middle|\; f^{-1}(V) \in \mathcal{A} \,\right\}$$

is a σ-algebra of subsets of X. By hypothesis, $\mathcal{S} \subset \mathcal{F}$; hence, $\mathcal{X} = \sigma(\mathcal{S}) \subset \mathcal{F}$. Thus, $f^{-1}(\mathcal{X}) \subset \mathcal{A}$; equivalently, f is $(\mathcal{A}, \mathcal{X})$-measurable. $\square$

Corollary B.4 (Continuous maps are Borel measurable.)
Suppose X_1, X_2 are topological spaces, and $\mathcal{B}_1$, $\mathcal{B}_2$ are their respective Borel σ-algebras. Then, every continuous map $f : X_1 \longrightarrow X_2$ is $(\mathcal{B}_1, \mathcal{B}_2)$-measurable.

C Topology

Theorem C.1 (Appendix M3, [3])
Suppose S is a metric space. Then, the following conditions are equivalent:

(i) *S is separable.*

(ii) *The topology of S has a countable basis.*

(iii) *Every open cover of each subset of S has a countable subcover.*

D Modulus of continuity

Definition D.1 (Modulus of continuity)

*Let $\mathbb{R}^{[0,1]}$ denote the set of all arbitrary $\mathbb{R}$-valued functions defined on the closed unit interval $[0,1]$. The **modulus of continuity** is, by definition, the following function:*

$$w\;:\;\mathbb{R}^{[0,1]}\times(0,1]\;\longrightarrow\;[0,\infty]\;:\;(f,\delta)\;\longmapsto\;\sup\left\{\,|\,f(s)-f(t)\,|\;\middle|\;\begin{array}{c}(s,t)\in[0,1]\times[0,1]\\|s-t|\;\leq\;\delta\end{array}\right\}.$$

Proposition D.2

(i) *The restriction of w to $C[0,1]\times(0,1]$ takes values in $[0,\infty)$.*

(ii) *For each $\delta\in(0,1]$ and each $f,g\in C[0,1]$, we have*

$$\Big|\,w(f,\delta)\;-\;w(g,\delta)\,\Big|\;\;\leq\;\;2\;\|\;f\;-\;g\;\|_\infty\,.$$

(iii) *For each $\delta\in(0,1]$, the map $w(\,\cdot\,,\delta):(C[0,1],\|\cdot\|_\infty)\longrightarrow\mathbb{R}:f\longmapsto w(f,\delta)$ is continuous.*

PROOF

For each $\delta\in(0,1]$, the set

$$D(\delta)\;:=\;\left\{\,(s,t)\in[0,1]\times[0,1]\;\middle|\;|\,s-t\,|\leq\delta\,\right\}$$

is a subset of the compact set $[0,1]\times[0,1]$.

(i) For each $f\in C[0,1]$, the map

$$[0,1]\times[0,1]\;\longrightarrow\;\mathbb{R}\;:\;(s,t)\;\longmapsto\;|\,f(s)-f(t)\,|$$

is continuous on the compact set $[0,1]\times[0,1]$; hence, it is bounded and attains its supremum on $[0,1]\times[0,1]$; in particular, its supremum on $[0,1]\times[0,1]$ is a (finite non-negative) real number. Hence,

$$\begin{aligned}w(f,\delta)\;\;&:=\;\;\sup\left\{\,|\,f(s)-f(t)\,|\;\middle|\;\begin{array}{c}(s,t)\in[0,1]\times[0,1]\\|s-t|\;\leq\;\delta\end{array}\right\}\;\;=\;\;\sup_{(s,t)\,\in\,D(\delta)}\left\{\,\Big|\,f(s)-f(t)\,\Big|\,\right\}\\&\leq\;\;\sup_{(s,t)\,\in\,[0,1]\times[0,1]}\left\{\,\Big|\,f(s)-f(t)\,\Big|\,\right\}\;\;<\;\;\infty.\end{aligned}$$

This proves (i).

(ii) Recall that for any $a,b\in\mathbb{R}$, we have

$$|\,a\,|\;\;=\;\;|\,a-b+b\,|\;\;\leq\;\;|\,a-b\,|\;+\;|\,b\,|,$$

which in turn implies:

$$|\,a\,|\;-\;|\,b\,|\;\;\leq\;\;|\,a-b\,|.$$

Consequently, for each $f,g\in C[0,1]$ and each $s,t\in[0,1]$, we have:

$$\begin{aligned}\Big|\,f(s)-f(t)\,\Big|\;-\;\Big|\,g(s)-g(t)\,\Big|\;\;&\leq\;\;\Big|\,f(s)-f(t)-g(s)+g(t)\,\Big|\\&\leq\;\;\Big|\,f(s)-g(s)\,\Big|\;+\;\Big|\,f(t)-g(t)\,\Big|\;\;\leq\;\;2\,\Big\|\,f-g\,\Big\|_\infty.\end{aligned}$$

On the other hand, note that for any $g\in C[0,1]$ and any $(s,t)\in[0,1]\times[0,1]$ with $|\,s-t\,|\;\leq\;\delta$, i.e. $(s,t)\in D(\delta)$, we have

$$\Big|\,g(s)-g(t)\,\Big|\;\;\leq\;\;\sup_{(\xi,\zeta)\,\in\,D(\delta)}\left\{\,|\,g(\xi)-g(\zeta)\,|\,\right\}\;\;=:\;\;w(g,\delta),$$

hence,

$$-\,w(g,\delta) \;\;\leq\;\; -\,\left|\,g(s)-g(t)\,\right|, \quad \text{for any } g\in C[0,1] \text{ and any } (s,t)\in D(\delta).$$

Thus, we see that, for any $f,g\in C[0,1]$ and any $(s,t)\in D(\delta)$, we have

$$\left|\,f(s)-f(t)\,\right| \;-\; w(g,\delta) \;\;\leq\;\; \left|\,f(s)-f(t)\,\right| \;-\; \left|\,g(s)-g(t)\,\right| \;\;\leq\;\; 2\,\left\|\,f-g\,\right\|_\infty .$$

Taking supremum of the left-hand side of the preceding inequality over $(s,t)\in D(\delta)$ now yields:

$$\begin{aligned} w(f,\delta) \;-\; w(g,\delta) \;\;&=\;\; \sup_{(\xi,\zeta)\,\in\,D(\delta)} \left\{\,\left|\,f(\xi)-f(\zeta)\,\right|\,\right\} \;-\; w(g,\delta) \\ &=\;\; \sup_{(\xi,\zeta)\,\in\,D(\delta)} \left\{\,\left|\,f(\xi)-f(\zeta)\,\right| \;-\; w(g,\delta)\,\right\} \\ &\leq\;\; 2\,\left\|\,f-g\,\right\|_\infty . \end{aligned}$$

Interchanging f and g in the preceding inequality now yields:

$$\left|\,w(f,\delta) \;-\; w(g,\delta)\,\right| \;\;\leq\;\; 2\,\left\|\,f-g\,\right\|_\infty .$$

This completes the proof of (ii).

(iii) This is an immediate consequence of (ii).

□

Corollary D.3

For any $\delta\in(0,1]$, the map $w(\,\cdot\,,\delta) : C[0,1] \longrightarrow \mathbb{R}$ is an $\mathbb{R}$-valued random variable (i.e. an $\mathbb{R}$-valued Borel measurable function).

Proof $w(\,\cdot\,,\delta)$ is continuous by the preceding Theorem, and hence, Borel measurable, by Corollary B.4. □

Proposition D.4

For each fixed $f\in C[0,1]$, the map $w(f,\,\cdot\,) : (0,1] \longrightarrow \mathbb{R}$ is a non-decreasing function on $(0,1]$.

Proof For $\delta_1,\delta_2\in(0,1]$ with $\delta_1\leq\delta_2$, we have

$$D(\delta_1) \;:=\; \left\{\,(s,t)\in[0,1]\times[0,1]\;\middle|\;|\,s-t\,|\leq\delta_1\,\right\} \;\;\subset\;\; \left\{\,(s,t)\in[0,1]\times[0,1]\;\middle|\;|\,s-t\,|\leq\delta_2\,\right\} \;=:\; D(\delta_2).$$

Hence, for each $f\in C[0,1]$ and each $\delta_1,\delta_2\in(0,1)$ with $\delta_1\leq\delta_2$, we have

$$w(f,\delta_1) \;\;:=\;\; \sup_{(s,t)\,\in\,D(\delta_1)} \left\{\,|\,f(s)-f(t)\,|\,\right\} \;\;\leq\;\; \sup_{(s,t)\,\in\,D(\delta_2)} \left\{\,|\,f(s)-f(t)\,|\,\right\} \;\;=:\;\; w(f,\delta_2).$$

□

Proposition D.5 (Theorem 7.4, [3])

Suppose:

- $\delta>0$ *and* $0 \,=\, t_0 \,<\, t_1 \,<\, \cdots \,<\, t_n = 1$ *satisfy:*

$$\min_{1\,<\,i\,<\,n} \left\{\,t_i - t_{i-1}\,\right\} \;\;\geq\;\; \delta.$$

- *$C[0,1]$ is the Banach space of continuous $\mathbb{R}$-valued functions defined on $[0,1]$ equipped with the supremum norm.*

Then, the following statements are true:

(i) *For each $f \in C[0,1]$, we have:*

$$w(f,\delta) \;\leq\; 3 \cdot \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1},t_i]} \left\vert\, f(s) - f(t_{i-1}) \,\right\vert \right\}.$$

(ii) *For each $\varepsilon > 0$ and each Borel probability measure P on the (separable) Banach space $(C[0,1], \Vert\cdot\Vert_\infty)$, we have:*

$$P\Big(\Big\{\, f \in C[0,1] \;\Big\vert\; w(f,\delta) \,\geq\, 3\,\varepsilon \,\Big\}\Big) \;\leq\; \sum_{i=1}^{n} P\left(\left\{\, f \in C[0,1] \;\middle\vert\; \sup_{s \in [t_{i-1},t_i]} \left\vert\, f(s) - f(t_{i-1}) \,\right\vert \,\geq\, \varepsilon \,\right\}\right).$$

Proof

(i) First, note that

$$\left.\begin{array}{lcc} \vert\, s - t \,\vert & \leq & \delta \\ \displaystyle\min_{1 < i < n} \left\{ t_i - t_{i-1} \right\} & \geq & \delta \end{array}\right\} \;\Longrightarrow\; \left\{\begin{array}{cll} \text{either} & s,t \,\in\, [t_{i-1},t_i], & \text{for some } i \in \{1,2,\ldots,n\} \\ \text{or} & s,t \,\in\, [t_{i-2},t_{i-1}] \cup [t_{i-1},t_i], & \text{for some } i \in \{2,\ldots,n\} \end{array}\right.$$

For the case in which both s and t lie in the same subinterval $[t_{i-1},t_i]$, for some $i \in \{1,2,\ldots,n\}$, we have

$$\left\vert\, f(s) - f(t) \,\right\vert \;\leq\; \left\vert\, f(s) - f(t_{i-1}) \,\right\vert + \left\vert\, f(t_{i-1}) - f(t) \,\right\vert \;\leq\; 2 \cdot \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1},t_i]} \left\vert\, f(s) - f(t_{i-1}) \,\right\vert \right\}.$$

For the case in which s and t lie in adjacent subintervals, say $s \in [t_{i-2},t_{i-1}]$ and $t \in [t_{i-1},t_i]$, for some $i \in \{2,\ldots,n\}$, we have

$$\begin{aligned} \left\vert\, f(s) - f(t) \,\right\vert &\leq \left\vert\, f(s) - f(t_{i-2}) \,\right\vert + \left\vert\, f(t_{i-2}) - f(t_{i-1}) \,\right\vert + \left\vert\, f(t_{i-1}) - f(t) \,\right\vert \\ &\leq 3 \cdot \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1},t_i]} \left\vert\, f(s) - f(t_{i-1}) \,\right\vert \right\}. \end{aligned}$$

Thus we see that, for any $f \in C[0,1]$ and any $(s,t) \in [0,1] \times [0,1]$ with $\vert\, s - t \,\vert \leq \delta$, we have

$$\left\vert\, f(s) - f(t) \,\right\vert \;\leq\; 3 \cdot \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1},t_i]} \left\vert\, f(s) - f(t_{i-1}) \,\right\vert \right\},$$

which implies, for each $f \in C[0,1]$,

$$w(f,\delta) \;:=\; \sup\left\{ \vert\, f(s) - f(t) \,\vert \;\middle\vert\; \begin{array}{c} (s,t) \in [0,1] \times [0,1] \\ \vert s - t \vert \,\leq\, \delta \end{array} \right\} \;\leq\; 3 \cdot \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1},t_i]} \left\vert\, f(s) - f(t_{i-1}) \,\right\vert \right\}.$$

This proves (i).

(ii) By (i), we see that, for any $\varepsilon, \delta > 0$ and any $f \in C[0,1]$, we have:

$$3\,\varepsilon \;\leq\; w(f,\delta) \quad\Longrightarrow\quad \varepsilon \;\leq\; \max_{1 \leq i \leq n} \left\{ \sup_{s \in [t_{i-1},t_i]} \left\vert\, f(s) - f(t_{i-1}) \,\right\vert \right\}.$$

For any $\varepsilon, \delta > 0$ and any $C[0,1]$-valued random variable $X : \left(\Omega, \mathcal{A}, \mu\right) \longrightarrow (C[0,1], \|\cdot\|_\infty)$, we have:

$$\left\{\, f \in C[0,1] \;\middle|\; 3\,\varepsilon \,\leq\, w(f,\delta) \,\right\} \;\subset\; \left\{\, f \in C[0,1] \;\middle|\; \varepsilon \,\leq\, \max_{1\leq i\leq n}\left\{ \sup_{s\,\in\,[t_{i-1},t_i]} \left| f(s) - f(t_{i-1}) \right| \right\} \right\}$$
$$= \bigcup_{i=1}^{n} \left\{\, f \in C[0,1] \;\middle|\; \varepsilon \,\leq\, \sup_{s\,\in\,[t_{i-1},t_i]} \left| f(s) - f(t_{i-1}) \right| \,\right\},$$

and (ii) now follows by sub-additivity of measures.

□

Corollary D.6 (Corollary of Theorem 7.4, [3])

Suppose $\{\,P_n\,\}_{n\in\mathbb{N}}$ is a sequence of Borel probability measures on $(C[0,1], \|\cdot\|_\infty)$.
Then, (i) implies (ii):

(i) *For each $\varepsilon, \eta > 0$, there exist $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that*

$$\frac{1}{\delta}\cdot P_n\left(\left\{\, f \in C[0,1] \;\middle|\; \varepsilon \,\leq \sup_{s\,\in\,\left[\,t\,,\;\min\{1,t+\delta\}\right]} \left| f(s) - f(t) \right| \,\right\}\right) \;\leq\; \eta, \quad \text{for each } t \in [0,1] \text{ and each } n \geq n_0.$$

(ii) *For each $\varepsilon, \eta > 0$, there exist $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that*

$$P_n\left(\left\{\, f \in C[0,1] \;\middle|\; \varepsilon \,\leq\, w(f,\delta) \,\right\}\right) \;\leq\; \eta, \quad \text{for each } n \geq n_0.$$

PROOF Suppose (i) holds and let $\varepsilon, \eta > 0$ be given. By (i), there exists $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\delta}\cdot P_n\left(\left\{\, f \in C[0,1] \;\middle|\; \frac{\varepsilon}{3} \,\leq \sup_{s\,\in\,\left[\,t\,,\;\min\{1,t+\delta\}\right]} \left| f(s) - f(t) \right| \,\right\}\right) \;\leq\; \eta, \quad \text{for each } t \in [0,1] \text{ and each } n \geq n_0.$$

Now, let $t_0 = 0$, and $t_i = i\delta$, for $i = 1, 2, 3, \ldots, k := \lfloor\, 1/\delta \,\rfloor$. Then, the preceding inequality and the preceding Theorem together imply:

$$P_n\left(\left\{\, f \in C[0,1] \;\middle|\; \varepsilon \,\leq\, w(f,\delta) \,\right\}\right) \;\leq\; \sum_{i\,=\,1}^{k} P_n\left(\left\{\, f \in C[0,1] \;\middle|\; \frac{\varepsilon}{3} \leq \sup_{s\,\in\,[t_{i-1},t_i]} \left| f(s) - f(t_{i-1}) \right| \,\right\}\right)$$
$$\leq\; k\cdot\delta\cdot\eta \;=\; \lfloor\, 1/\delta \,\rfloor\cdot\delta\cdot\eta \;\leq\; 1\cdot\eta \;=\; \eta, \quad \text{for each } n \geq n_0.$$

This completes the proof of the Corollary. □

E The Arzelà-Ascoli Theorem: compactness of subsets of $C(X)$

Recall that the space $C(X)$ of continuous $\mathbb{R}$-valued functions defined on a compact topological space X equipped with the supremum norm is a complete metric space (see Theorem 9.3, [1]). The Arzelà-Ascoli Theorem characterizes compactness of subsets of $C(X)$.

Definition E.1 (Equicontinuity)

Let X be a topological space and (Y,d) a metric space. Let Y^X denote the set of arbitrary functions from X into Y.

- *A subset $S \subset Y^X$ is said to be **equicontinuous at** $x_0 \in X$ if, for each $\varepsilon > 0$, there exists an open subset $V \subset X$ satisfying:*

$$x_0 \in V\,, \quad \text{and} \quad \sup_{(x,f)\,\in\,V\times S}\left\{\, d\Big(f(x), f(x_0)\Big) \,\right\} \;\leq\; \varepsilon.$$

- *A subset $S \subset Y^X$ is said to be **equicontinuous** if it is equicontinuous at each $x_0 \in X$.*

Proposition E.2

Let X be a topological space and (Y,d) a metric space. Let Y^X denote the set of arbitrary functions from X into Y. Suppose $x_0 \in X$ and $S_1, S_2 \subset Y^X$. Then,

$$\left.\begin{array}{c} S_1 \subset S_2, \;\; \text{and} \\ S_2 \text{ is equicontinuous at } x_0 \end{array}\right\} \quad\Longrightarrow\quad S_1 \text{ is equicontinuous at } x_0$$

Proof Let $\varepsilon > 0$ be given. By the equicontinuity of S_2 at x_0, there exists open $V \subset X$ such that

$$x_0 \in V \quad \text{and} \quad \sup_{(x,f)\,\in\,V\times S_2}\left\{\, d\Big(f(x), f(x_0)\Big) \,\right\} \;\leq\; \varepsilon.$$

However, the hypothesis $S_1 \subset S_2$ implies that

$$\sup_{(x,f)\,\in\,V\times S_1}\left\{\, d\Big(f(x), f(x_0)\Big) \,\right\} \;\leq\; \sup_{(x,f)\,\in\,V\times S_2}\left\{\, d\Big(f(x), f(x_0)\Big) \,\right\}$$

which immediately implies the following conditions hold:

$$x_0 \in V \quad \text{and} \quad \sup_{(x,f)\,\in\,V\times S_1}\left\{\, d\Big(f(x), f(x_0)\Big) \,\right\} \;\leq\; \varepsilon.$$

This proves the equicontinuity of S_1 at x_0, as required. □

Definition E.3 (Uniform equicontinuity)

*Let (X,ρ) and (Y,d) two metric spaces. Let Y^X denote the set of arbitrary functions from X into Y. A subset $S \subset Y^X$ is said to be **uniformly equicontinuous** if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that:*

$$\sup\left\{\, d\Big(f(x_1), f(x_2)\Big) \;\middle|\; \begin{array}{c} f \in S,\; x_1, x_2 \in X, \\ \rho(x_1,x_2) < \delta \end{array} \right\} \;\leq\; \varepsilon.$$

Proposition E.4

Let (X,ρ) and (Y,d) two metric spaces. Let Y^X denote the set of arbitrary functions from X into Y. Then, the following are true:

(i) *Uniform equicontinuity of a subset $S \subset Y^X$ implies equicontinuity of S.*

(ii) *Suppose furthermore that (X,ρ) is compact. Then, equicontinuity of a subset $S \subset Y^X$ implies uniform equicontinuity of S.*

Proof

(i) Suppose $S \subset Y^X$ is uniformly equicontinuous; we seek to prove that S is also equicontinuous. Let $x_0 \in X$ and $\varepsilon > 0$. By uniform equicontinuity of S, there exists $\delta > 0$ such that:

$$\sup\left\{\, d\Big(f(x_1), f(x_2)\Big) \;\middle|\; \begin{array}{c} f \in S,\; x_1, x_2 \in X, \\ \rho(x_1,x_2) < \delta \end{array} \right\} \;\leq\; \varepsilon.$$

Let $V(x_0) := \left\{\, x \in X \;\middle|\; \rho(x,x_0) < \delta \,\right\}$. Then, $V(x_0)$ is an open subset of X, with $x_0 \in V(x_0)$, and

$$\sup_{(x,f)\,\in\,V(x_0)\times S}\left\{\, d\Big(f(x), f(x_0)\Big) \,\right\} \;\leq\; \sup\left\{\, d\Big(f(x_1), f(x_2)\Big) \;\middle|\; \begin{array}{c} f \in S,\; x_1, x_2 \in X, \\ \rho(x_1,x_2) < \delta \end{array} \right\} \;\leq\; \varepsilon.$$

This proves the equicontinuity of S.

(ii) Suppose (X,ρ) is compact and $S \subset Y^X$ is equicontinuous. Let $\varepsilon > 0$ be given. By equicontinuity of S, for each $x \in X$, there exists an open ball $B(x,\delta_x) \subset X$ such that

$$\sup_{(\xi,f)\,\in\,B(x,\delta_x)\times S} \left\{\, d\Big(f(\xi), f(x)\Big) \,\right\} \;\leq\; \frac{\varepsilon}{2}.$$

Thus, $X \;=\; \bigcup_{x\,\in\,X} B\Big(x,\delta_x/\,2\Big)$ is an open cover of X. By compactness of X, this open cover admits a finite subcover:

$$X \;=\; \bigcup_{i\,=\,1}^{n} B\Big(x_i,\delta_{x_i}/\,2\Big).$$

Define $\delta \;:=\; \min_{1\leq i\leq n}\left\{\,\delta_{x_i}/\,2\,\right\} \;>\; 0$. Now note the uniform equicontinuity of S will be established once we prove the validity of the following:

<u>Claim:</u> For any $\xi_1,\xi_2 \in X$, and any $f \in S$, we have:

$$\rho(\xi_1,\xi_2) \;<\; \delta \quad\Longrightarrow\quad d\Big(f(\xi_1), f(\xi_2)\Big) \;\leq\; \varepsilon.$$

Proof of Claim: Suppose $\rho(\xi_1,\xi_2) < \delta$. Note that $\xi_1 \in B\Big(x_i,\delta_{x_i}/\,2\Big)$, for some $i = 1,2,\ldots,n$. Next, observe that

$$\rho(x_i,\xi_2) \;\leq\; \rho(x_i,\xi_1) \,+\, \rho(\xi_1,\xi_2) \;\leq\; \frac{\delta_{x_i}}{2} \,+\, \delta \;\leq\; \frac{\delta_{x_i}}{2} \,+\, \frac{\delta_{x_i}}{2} \;=\; \delta_{x_i}.$$

This shows that both $\xi_1,\xi_2 \in B(x_i,\delta_{x_i})$, which implies

$$d\Big(f(\xi_1), f(x_i)\Big) \;\leq\; \frac{\varepsilon}{2}\,, \quad\text{and}\quad d\Big(f(\xi_2), f(x_i)\Big) \;\leq\; \frac{\varepsilon}{2}\,, \quad\text{for each } f \in S,$$

which in turn implies:

$$d\Big(f(\xi_1), f(\xi_2)\Big) \;\leq\; d\Big(f(\xi_1), f(x_i)\Big) \,+\, d\Big(f(f(x_i),\xi_2)\Big) \;\leq\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon, \quad\text{for each } f \in S.$$

This completes the proof of the Claim and the uniform equicontinuity of S. □

Theorem E.5 (Arzelà-Ascoli, Theorem 9.10, [1])

Suppose X is a compact topological space and $C(X)$ is the space of continuous $\mathbb{R}$-valued functions defined on X equipped with the supremum norm. Then, for each $S \subset C(X)$, the following conditions are equivalent:

(i) *S is a compact subset of $C(X)$.*

(ii) *S is closed, bounded, and equicontinuous subset of $C(X)$.*

PROOF

<u>(i) $\Longrightarrow$ (ii)</u>

Recall that every compact subset in a metric space is closed and bounded. Thus, it remains only to show that $S \subset C(X)$ is equicontinuous. To this end, let $\varepsilon > 0$ be given. Recall that a metric space is compact if and only if it is complete and totally bounded (Theorem 7.8, [1]). Thus, the compactness hypothesis on S implies S is totally bounded; in particular, there exist $f_1,\ldots,f_n \in S$ such that

$$S \;\subset\; \bigcup_{i\,=\,1}^{n} B\Big(f_i\,,\,\frac{\varepsilon}{3}\Big).$$

Hence, for each $x_0 \in X$, we may define

$$V(x_0) \;:=\; \bigcap_{i\,=\,1}^{n} f_i^{-1}\Big(\Big(f_i(x_0) - \frac{\varepsilon}{3}\,, f_i(x_0) + \frac{\varepsilon}{3}\Big)\Big).$$

Note that $V(x_0)$ is open and $x_0 \in V(x_0)$. Now, let $f \in S$ and $x \in V(x_0)$ be given. We may choose $i \in \{\,1,2,\ldots,n\,\}$ such that $f \in B\Big(f_i\,,\,\dfrac{\varepsilon}{3}\Big)$, i.e. $\|\,f - f_i\,\|_\infty \,\leq\, \dfrac{\varepsilon}{3}$. Hence,

$$\Big|\,f(x) \,-\, f(x_0)\,\Big| \;\leq\; \Big|\,f(x) \,-\, f_i(x)\,\Big| \,+\, \Big|\,f_i(x) \,-\, f_i(x_0)\,\Big| \,+\, \Big|\,f_i(x_0) \,-\, f(x_0)\,\Big| \;\leq\; \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \;=\; \varepsilon.$$

Thus,

$$\sup_{(x,f)\,\in\,V(x_0)\times S}\left\{\,\Big|\,f(x) \,-\, f(x_0)\,\Big|\,\right\} \;\leq\; \varepsilon.$$

This shows equicontinuity of S at $x_0 \in X$. Since $x_0 \in X$ is arbitrary, we may conclude that S is equicontinuous.

<u>(ii) $\Longrightarrow$ (i)</u>

Suppose $S \in C(X)$ is closed, bounded, and equicontinuous. We need to show that S is a compact subset of $C(X)$. Recall that every subset of a metric space is compact if and only if it is sequentially compact (Theorem 7.3, [1]). Thus, it suffices to show that every sequence $\{\,f_n\,\}_{n\in\mathbb{N}} \subset S$ has a convergent subsequence with limit in S. We start by stating and proving the following:

> <u>Claim 1:</u>
> For each $k \in \mathbb{N}$, there exists a finite subset $F_k \subset X$ and open neighbourhoods $\{\,V_y\,\}_{y\in F_k}$ such that
>
> $$X \;=\; \bigcup_{y\,\in\,F_k} V_y,, \quad \text{and} \quad \sup\left\{\,\Big|\,f(x) - f(y)\,\Big|\;\middle|\; \begin{array}{c} x \in V_y,\; y \in F_k \\ f \in S \end{array}\right\} \;\leq\; \frac{1}{3k}.$$

Proof of Claim 1: By equicontinuity of S, for each $y \in X$, there exists an open neighbourhood $V_y \subset X$ of $y \in X$ such that

$$\sup\left\{\,\Big|\,f(x) - f(y)\,\Big|\;\middle|\; \begin{array}{c} x \in V_y \\ f \in S \end{array}\right\} \;\leq\; \frac{1}{3k}.$$

Thus, $X = \bigcup_{y\,\in\,X} V_y$ is an open cover of X. Compactness of X now implies that this open cover of X admits a finite subcover, i.e.

$$X \;=\; \bigcup_{y\,\in\,F_k} V_y, \quad \text{for some finite subset } F_k \subset X.$$

Lastly, note that

$$\sup\left\{\,\Big|\,f(x) - f(y)\,\Big|\;\middle|\; \begin{array}{c} x \in V_y,\; y \in F_k \\ f \in S \end{array}\right\} \;=\; \sup_{y\,\in\,F_k}\left\{\,\sup\left\{\,\Big|\,f(x) - f(y)\,\Big|\;\middle|\; \begin{array}{c} x \in V_y \\ f \in S \end{array}\right\}\right\} \;\leq\; \frac{1}{3k}.$$

This completes the proof of Claim 1.

Next, let $F := \bigcup_{k\,=\,1}^{\infty} F_k$. Note that F is a countably infinite set. Let $F = \{\,x_1, x_2, \ldots\,\}$ be an enumeration of F. Recall that we wish to prove that every sequence $\{\,f_n\,\}_{n\in\mathbb{N}} \subset S$ contains a convergent subsequence with limit in S. Now, consider the array of real numbers:

$$\begin{array}{cccc} f_1(x_1) & f_2(x_1) & f_3(x_1) & \cdots \\ f_1(x_2) & f_2(x_2) & f_3(x_2) & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

Since $S \subset C(X)$ is bounded (with respect to the $\|\cdot\|_\infty$ norm on $C(X)$), there exists $M > 0$ such that $\sup_{f\,\in\, S} \left\| f \right\|_\infty \leq M$. In particular, every row in the above array is bounded. By Theorem A.14, p.538, [2], there exists an increasing sequence of positive integers $n(1), n(2), n(3), \ldots$ such that the limit

$$\lim_{i\,\to\,\infty} f_{n(i)}(x_k) \;\text{ exists, for each } k = 1, 2, \ldots.$$

<u>Claim 2:</u> $\left\{ f_{n(i)} \right\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $(\,C(X)\,,\|\cdot\|_\infty\,)$.

Proof of Claim 2: For each $k \in \mathbb{N}$, the convergence of $\left\{ f_{n(i)}(x_k) \right\}_{i\in\mathbb{N}}$ in $\mathbb{R}$ implies that each $\left\{ f_{n(i)}(x_k) \right\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Since the set F_k is finite, we see that there exists $m_k \in \mathbb{N}$ such that

$$\left| f_{n(i)}(y) - f_{n(j)}(y) \right| \;<\; \frac{1}{3k}\,, \quad \text{for any } i, j \geq m_k, \text{ and each } y \in F_k.$$

Now, for each $x \in X$, there exists $y \in F_k$ such that $x \in V_y$. Hence, for any $i, j \geq m_k$ and any $x \in X$, we have

$$\begin{aligned}
\left| f_{n(i)}(x) - f_{n(j)}(x) \right| &\leq \left| f_{n(i)}(x) - f_{n(i)}(y) \right| + \left| f_{n(i)}(y) - f_{n(j)}(y) \right| + \left| f_{n(j)}(y) - f_{n(j)}(x) \right| \\
&\leq \frac{1}{3k} + \frac{1}{3k} + \frac{1}{3k} = \frac{1}{k}.
\end{aligned}$$

In other words,

$$\left\| f_{n(i)} - f_{n(j)} \right\|_\infty \;\leq\; \frac{1}{k}, \quad \text{for any } i, j \geq m_k.$$

This shows that $\left\{ f_{n(i)} \right\}_{i\in\mathbb{N}}$ is indeed a Cauchy sequence in $(\,C(X)\,,\|\cdot\|_\infty\,)$ and completes the proof of Claim 2.

Lastly, by Theorem 9.3, [1], $(\,C(X)\,,\|\cdot\|_\infty\,)$ is a complete metric space. Thus, the Cauchy sequence $\left\{ f_{n(i)} \right\}_{i\in\mathbb{N}} \subset C(X)$ converges to some element $f_0 \in C(X)$. Since $S \subset C(X)$ is, by hypothesis, a closed subset of $C(X)$, we see furthermore that $f_0 \in S$. This proves the sequential compactness of S and completes the proof of the Arzelà-Ascoli Theorem. □

Proposition E.6

Suppose X is a compact topological space and $C(X)$ is the space of continuous $\mathbb{R}$-valued functions defined on X equipped with the supremum norm. Let $S \subset C(X)$.

(i) *S is equicontinuous at $x_0 \in X$ if and only if its closure $\overline{S}$ in $C(X)$ is equicontinuous at x_0.*

(ii) *S is equicontinuous if and only if its closure $\overline{S}$ in $C(X)$ is equicontinuous.*

Proof It is obvious that (ii) is an immediate consequence of (i). Thus, it suffices to establish (i). First, by Proposition E.2, we immediately see that the equicontinuity of $\overline{S}$ at x_0 implies the equicontinuity of S at x_0. It remains to prove the converse. So, suppose that $S \subset C(X)$ is equicontinuous at $x_0 \in X$. Thus, for each $\varepsilon > 0$, there exists an open subset $V \subset X$ satisfying:

$$x_0 \in V\,, \quad \text{and} \quad \sup_{(x,f)\,\in\, V\times S} \left\{ \left| f(x) - f(x_0) \right| \right\} \leq \varepsilon.$$

Observe that, in order to show the equicontinuity of $\overline{S}$ at x_0, it suffices to show that the following inequality is also valid:

$$\sup_{(x,g)\,\in\, V\times\overline{S}} \left\{ \left| g(x) - g(x_0) \right| \right\} \leq \varepsilon.$$

To this end, let $g \in \overline{S} \subset C(X)$. Then, there exist a sequence $f_1, f_2, \ldots \in S$ such that

$$\lim_{n\,\to\,\infty} \| f_n - g \|_\infty \;=\; \lim_{n\,\to\,\infty} \sup_{x\,\in\, X} \left\{ \left| f_n(x) - g(x) \right| \right\} \;=\; 0.$$

Consequently, for any $x \in V$ and $n \in \mathbb{N}$, we have:

$$\begin{aligned}\left| g(x) - g(x_0) \right| \;&\leq\; \left| g(x) - f_n(x) \right| + \left| f_n(x) - f_n(x_0) \right| + \left| f_n(x_0) - g(x_0) \right| \\ &\leq\; \|\, g - f_n \,\|_\infty \;+\; \varepsilon \;+\; \|\, f_n - g \,\|_\infty \;\longrightarrow\; 0 \,+\, \varepsilon \,+\, 0 \;=\; \varepsilon.\end{aligned}$$

This implies:

$$\sup_{(x,g)\,\in\, V\times\overline{S}} \left\{ \left| g(x) \,-\, g(x_0) \right| \right\} \;\leq\; \varepsilon,$$

as desired. This completes the proof of the Proposition. □

Theorem E.7 (Theorem 7.2, p.81, [3])

Let $C[0,1]$ denote the space of continuous $\mathbb{R}$-valued functions defined on the closed unit interval $[0,1]$ equipped with the supremum norm. Then, for each subset $S \subset C[0,1]$, the following are equivalent:

(i) *S is a relatively compact subset of $C[0,1]$, i.e. the closure of S is a compact subset of $C[0,1]$.*

(ii) $\displaystyle\sup_{f\,\in\,S}\left\{ \|\, f \,\|_\infty \right\} < \infty$, *and S is uniformly equicontinuous.*

(iii) $\displaystyle\sup_{f\,\in\,S}\left\{ |\, f(0) \,| \right\} < \infty$, *and for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_{f\,\in\,S}\left\{\, w(f,\delta) \,\right\} \;=\; \sup\left\{ \left| f(t_1) - f(t_2) \right| \;\middle|\; \begin{array}{c} f \in S,\; t_1, t_2 \in [0,1], \\ |\, t_1 - t_2 \,| < \delta \end{array} \right\} \;\leq\; \varepsilon.$$

(iv) $\displaystyle\sup_{f\,\in\,S}\left\{ |\, f(0) \,| \right\} < \infty$, *and* $\displaystyle\lim_{\delta\to 0^+}\,\sup_{f\,\in\,S}\left\{\, w(f,\delta) \,\right\} \;=\; 0.$

PROOF

(i) $\Longleftrightarrow$ (ii)

$$\begin{array}{rcll}
\text{(i)} & \Longleftrightarrow & \overline{S} \text{ is compact,} & \text{(by definition of relative compactness)} \\
& \Longleftrightarrow & \overline{S} \text{ is bounded and equicontinuous,} & \text{(by the Arzelà-Ascoli Theorem)} \\
& \Longleftrightarrow & S \text{ is bounded and equicontinuous,} & \text{(by Proposition E.6)} \\
& \Longleftrightarrow & S \text{ is bounded and uniformly equicontinuous,} & \text{(by Proposition E.4)} \\
& \Longleftrightarrow & \text{(ii).} &
\end{array}$$

(ii) $\Longleftrightarrow$ (iii)

Noting that the second condition in (iii) is precisely uniform equicontinuity of S, we see immediately that (ii) $\Longrightarrow$ (iii). Conversely, we may conclude that (iii) $\Longrightarrow$ (ii) once we prove that (iii) implies $\displaystyle\sup_{f\,\in\,S}\left\{ \|\, f \,\|_\infty \right\} < \infty$. To this end, take $\varepsilon = 1$. Then, by the second condition in (iii) (uniform equicontinuity of S), there exists $\delta > 0$ such that

$$\sup\left\{ \left| f(t_1) - f(t_2) \right| \;\middle|\; \begin{array}{c} f \in S,\; t_1, t_2 \in [0,1], \\ |\, t_1 - t_2 \,| < \delta \end{array} \right\} \;\leq\; \varepsilon \;:=\; 1.$$

Next, choose $k \in \mathbb{N}$ sufficiently large such that $\dfrac{1}{k} < \delta$. Hence, for any $f \in S$ and any $t \in [0,1]$, we have

$$\begin{aligned}
\left| f(t) \right| &= \left| f(t) - f\left(\frac{k-1}{k}\cdot t\right) + f\left(\frac{k-1}{k}\cdot t\right) - \;\cdots\; - f\left(\frac{1}{k}\cdot t\right) + f\left(\frac{1}{k}\cdot t\right) - f(0) + f(0) \right| \\
&\leq \left| f(0) \right| \;+\; \sum_{i=1}^{k} \left| f\left(\frac{i}{k}\cdot t\right) + f\left(\frac{i-1}{k}\cdot t\right) \right| \;\leq\; \left| f(0) \right| \;+\; k\cdot 1 \\
&\leq \sup_{f\,\in\,S}\left\{ \,|\,f(0)\,|\, \right\} \;+\; k,
\end{aligned}$$

where the last inequality follows from the first condition in (iii). Consequently, we see that, for each $f \in S$,

$$\|\,f\,\|_\infty \;:=\; \sup_{t\,\in\,[0,1]}\left\{ \,|\,f(t)\,|\, \right\} \;\leq\; \sup_{f\,\in\,S}\left\{ \,|\,f(0)\,|\, \right\} \;+\; k \;<\; \infty,$$

which in turn implies

$$\sup_{f\,\in\,S}\left\{ \,\|\,f\,\|_\infty \right\} \;\leq\; \sup_{f\,\in\,S}\left\{ \,|\,f(0)\,|\, \right\} \;+\; k \;<\; \infty.$$

This completes the proof that (ii) $\Longleftrightarrow$ (iii).

(iii) $\Longleftrightarrow$ (iv)

This follows trivially from the definition of the right-limit at zero of a $\mathbb{R}$-valued function defined on an interval $[0,\delta_0)$, for some $\delta_0 > 0$.

□

F Tightness of sequences of Borel measures on $(C[0,1], \|\cdot\|_\infty)$

Theorem F.1 (Theorem 7.3, [3])

Suppose $\{\,P_n\,\}_{n\in\mathbb{N}}$ is a sequence of Borel probability measures on $(C[0,1], \|\cdot\|_\infty)$.
Then, the following are equivalent:

(i) $\{\,P_n\,\}_{n\in\mathbb{N}}$ *is tight.*

(ii) *For each ε, $\eta > 0$, there exist $a > 0$ and $\delta \in (0,1)$ such that*

$$P_n\Big(\Big\{\; f \in C[0,1] \;\Big|\; |\,f(0)\,| \geq a \;\Big\}\Big) \;\leq\; \eta, \;\text{ and }\; P_n\Big(\Big\{\; f \in C[0,1] \;\Big|\; w(f,\delta) \geq \varepsilon \;\Big\}\Big) \;\leq\; \eta, \;\text{ for each } n \in \mathbb{N}.$$

(iii) *For each ε, $\eta > 0$, there exist $a > 0$, $\delta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that*

$$P_n\Big(\Big\{\; f \in C[0,1] \;\Big|\; |\,f(0)\,| \geq a \;\Big\}\Big) \;\leq\; \eta, \;\text{ and }\; P_n\Big(\Big\{\; f \in C[0,1] \;\Big|\; w(f,\delta) \geq \varepsilon \;\Big\}\Big) \;\leq\; \eta, \;\text{ for each } n \geq n_0.$$

(iv) *For each $\eta > 0$, there exist $a > 0$ and $n_0 \in \mathbb{N}$ such that*

$$P_n\Big(\Big\{\; f \in C[0,1] \;\Big|\; |\,f(0)\,| \geq a \;\Big\}\Big) \;\leq\; \eta, \;\text{ for each } n \geq n_0,$$

and, for each $\varepsilon > 0$,

$$\lim_{\delta\to 0^+}\limsup_{n\to\infty} P_n\Big(\Big\{\; f \in C[0,1] \;\Big|\; w(f,\delta) \geq \varepsilon \;\Big\}\Big) \;=\; 0.$$

Proof

(i) $\Longrightarrow$ (ii)

Suppose $\{P_n\}_{n\in\mathbb{N}}$ is tight and let $\varepsilon, \eta > 0$ be given. By tightness of $\{P_n\}_{n\in\mathbb{N}}$, there exists a compact subset $K \subset C[0,1]$ such that $1-\eta \leq P_n(K) \leq 1$, for every $n \in \mathbb{N}$. By Theorem E.7(iii),

$$\begin{aligned}
& \text{compactness of } K \subset C[0,1] \\
\Longrightarrow\quad & \exists\, a>0 \text{ such that } \sup_{f\in K}\left\{\,|\,f(0)\,|\,\right\} < a < \infty, \quad \text{and} \quad \exists\,\delta>0 \text{ such that } \sup_{f\in K}\left\{\,w(f,\delta)\,\right\} < \varepsilon \\
\Longrightarrow\quad & \exists\, a>0 \text{ such that } K \subset \left\{\, f\in C[0,1] \;\middle|\; |\,f(0)\,| < a \,\right\}, \text{ and } \exists\,\delta>0 \text{ such that } K \subset \left\{\, f\in C[0,1] \;\middle|\; w(f,\delta) < \varepsilon \,\right\}
\end{aligned}$$

Hence, with $a>0$ and $\delta\in(0,1)$ as chosen above, we see that, for each $n\in\mathbb{N}$, we have:

$$P_n\!\left(\left\{\, f\in C[0,1] \;\middle|\; |\,f(0)\,| \geq a \,\right\}\right) \;\leq\; P_n(\,C[0,1]\setminus K\,) \;=\; 1-P_n(\,K\,) \;\leq\; \eta,$$

and

$$P_n\!\left(\left\{\, f\in C[0,1] \;\middle|\; w(f,\delta) \geq \varepsilon \,\right\}\right) \;\leq\; P_n(\,C[0,1]\setminus K\,) \;=\; 1-P_n(\,K\,) \;\leq\; \eta.$$

This completes the proof that (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i)

Let $\eta>0$ be given. We need to show that $\{P_n\}_{n\in\mathbb{N}}$ is tight; equivalently, we need to find a compact subset $K\subset C[0,1]$ such that $1-\eta < P_n(\,K\,) \leq 1$, for each $n\in\mathbb{N}$.

By (ii), we may choose $a>0$ such that

$$P_n\!\left(\left\{\, f\in C[0,1] \;\middle|\; |\,f(0)\,| \geq a \,\right\}\right) \;\leq\; \frac{\eta}{2}, \quad \text{for each } n\in\mathbb{N}.$$

Let $B_0 := \left\{\, f\in C[0,1] \;\middle|\; |\,f(0)\,| < a \,\right\}$. Note that $P(B_0^c) \leq \dfrac{\eta}{2}$.

Next, applying (ii) again, for each $k\in\mathbb{N}$, we may choose $\delta_k>0$ such that

$$P_n\!\left(\left\{\, f\in C[0,1] \;\middle|\; w(f,\delta_k) \geq \frac{1}{k} \,\right\}\right) \;\leq\; \frac{\eta}{2^{k+1}}, \quad \text{for each } n\in\mathbb{N}.$$

Let $B_k := \left\{\, f\in C[0,1] \;\middle|\; w(f,\delta_k) < \dfrac{1}{k} \,\right\}$, for $k\in\mathbb{N}$. Note that $P(B_k^c) \leq \dfrac{\eta}{2^{k+1}}$, for each $k\in\mathbb{N}$.

Now, let $A := \bigcap\limits_{k=0}^{\infty} B_k \subset C[0,1]$, and $K := \overline{A} \subset C[0,1]$. Note that the desired implication (ii) $\Longrightarrow$ (i) follows from the following two Claims:

Claim 1: $1-\eta \leq P_n(\,K\,) \leq 1$, for every $n\in\mathbb{N}$. In particular, K is non-empty.

Claim 2: A is a relatively compact subset of $C[0,1]$, hence its closure K is a compact subset of $C[0,1]$.

Proof of Claim 1: Note that

$$A^c \;=\; \left(\bigcap_{k=0}^{\infty} B_k\right)^{c} \;=\; \bigcup_{k=0}^{\infty} B_k^c,$$

which implies, for each $n\in\mathbb{N}$,

$$\begin{aligned}
P_n(\,A^c\,) \;&=\; P_n\!\left(\bigcup_{k=0}^{\infty} B_k^c\right) \;\leq\; \sum_{k=0}^{\infty} P_n(\,B_k^c\,) \;=\; P_n(\,B_0^c\,) + \sum_{k=1}^{\infty} P_n(\,B_k^c\,) \\
&\leq\; \frac{\eta}{2} + \sum_{k=1}^{\infty}\frac{\eta}{2^{k+1}} \;=\; \cdots \;=\; \eta,
\end{aligned}$$

which in turn implies

$$1-\eta \;\leq\; 1-P_n(\,A^c\,) \;=\; P_n(\,A\,) \;\leq\; P_n(\,\overline{A}\,) \;=\; P_n(\,K\,)\,, \quad \text{for each } n\in\mathbb{N}.$$

This proves Claim 1.

Proof of Claim 2: Note that $f\in A \Longrightarrow f\in B_0 \Longrightarrow |\,f(0)\,|<a<\infty$. Hence, $\sup\limits_{f\,\in\,A}\left\{\,|\,f(0)\,|\,\right\}\leq a<\infty$. Secondly, let $\nu>0$ be given. Choose $k\in\mathbb{N}$ such that $\dfrac{1}{k}<\nu$. Then, $f\in A\Longrightarrow f\in B_k\Longrightarrow w(f,\delta_k)<\dfrac{1}{k}$, which in turn implies:

$$\sup_{f\,\in\,A}\left\{\,w(f,\delta_k)\,\right\}\;\leq\;\frac{1}{k}\;<\;\nu.$$

By Theorem E.7(iii) (or, essentially, the Arzelà-Ascoli Theorem), A is a relatively compact subset of $C[0,1]$. Its closure $K:=\overline{A}$ is therefore a compact subset of $C[0,1]$. This proves Claim 2, and completes of the proof of (ii) $\Longrightarrow$ (i).

(ii) $\Longleftrightarrow$ (iii)

It is obvious that (ii) $\Longrightarrow$ (iii). It remains to prove the reverse implication (iii) $\Longrightarrow$ (ii). So, suppose (iii) holds. Let $\varepsilon,\eta>0$ be given. By (iii), there exists $a'>0$, $\delta'\in(0,1)$, and $n_0\in\mathbb{N}$ such that

$$P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;|\,f(0)\,|\geq a'\,\Big\}\Big)\;\leq\;\eta,\;\;\text{and}\;\;P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;w(f,\delta')\geq\varepsilon\,\Big\}\Big)\;\leq\;\eta,\;\;\text{for each } n\geq n_0.$$

On the other hand, since $(C[0,1],\|\cdot\|_\infty)$ is separable and complete, every Borel probability measure on $(C[0,1],\|\cdot\|_\infty)$ is tight, by Theorem 1.3, p.8, [3]. Thus, by the tightness of each $P_1,P_2,\ldots,P_{n_0-1}$, and the equivalence (i) $\Longleftrightarrow$ (ii), which we have already established, there exist $a_1,a_2,\ldots,a_{n_0-1}>0$ and $\delta_1,\delta_2,\ldots,\delta_{n_0-1}\in(0,1)$ such that

$$P_i\Big(\Big\{\,f\in C[0,1]\;\Big|\;|\,f(0)\,|\geq a_i\,\Big\}\Big)\;\leq\;\eta,\;\;\text{and}\;\;P_i\Big(\Big\{\,f\in C[0,1]\;\Big|\;w(f,\delta_i)\geq\varepsilon\,\Big\}\Big)\;\leq\;\eta,\;\;\text{for } i=1,2,\ldots,n_0-1.$$

Now, let $a:=\max\{\,a',a_1,a_2,\ldots,a_{n_0-1}\,\}$ and $\delta:=\min\{\,\delta',\delta_1,\delta_2,\ldots,\delta_{n_0-1}\,\}$. Then,

$$\begin{aligned}
P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;|\,f(0)\,|\geq a\,\Big\}\Big) &\leq P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;|\,f(0)\,|\geq a'\,\Big\}\Big)\leq\eta, \quad \text{for } n\,\geq\,n_0, \quad \text{and}\\
P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;|\,f(0)\,|\geq a\,\Big\}\Big) &\leq P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;|\,f(0)\,|\geq a_n\,\Big\}\Big)\leq\eta, \quad \text{for } n=1,2,\ldots,n_0-1,
\end{aligned}$$

which together imply

$$P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;|\,f(0)\,|\geq a\,\Big\}\Big)\;\leq\;\eta,\quad\text{for each } n\in\mathbb{N}.$$

Next, recall that for each fixed $f\in C[0,1]$, the map $\delta\longmapsto w(f,\delta)$ is non-decreasing in δ, by Proposition D.4. We thus see that:

$$\begin{aligned}
P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;w(f,\delta)\geq\varepsilon\,\Big\}\Big) &\leq P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;w(f,\delta')\geq\varepsilon\,\Big\}\Big)\leq\eta, \quad \text{for } n\,\geq\,n_0, \quad \text{and}\\
P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;w(f,\delta)\geq\varepsilon\,\Big\}\Big) &\leq P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;w(f,\delta_n)\geq\varepsilon\,\Big\}\Big)\leq\eta, \quad \text{for } n=1,2,\ldots,n_0-1,
\end{aligned}$$

which together imply

$$P_n\Big(\Big\{\,f\in C[0,1]\;\Big|\;w(f,\delta)\geq\varepsilon\,\Big\}\Big)\;\leq\;\eta,\quad\text{for each } n\in\mathbb{N}.$$

This proves (iii) $\Longrightarrow$ (ii), and completes the proof of the equivalence (ii) $\Longleftrightarrow$ (iii).

(iii) $\Longrightarrow$ (iv)

(iii) $\Longleftrightarrow$ for each $\varepsilon,\,\eta>0$, there exist $a>0$, $\delta\in(0,1)$ and $n_0\in\mathbb{N}$ such that
$P_n\left(\left\{\,f\in C[0,1]\;\middle|\;|\,f(0)\,|\geq a\,\right\}\right)\;\leq\;\eta$, and $P_n\left(\left\{\,f\in C[0,1]\;\middle|\;w(f,\delta)\geq\varepsilon\,\right\}\right)\;\leq\;\eta$, for each $n\geq n_0$

$\Longleftrightarrow$ for each $\eta>0$, $\exists\;a>0$ and $n_0\in\mathbb{N}$ such that $P_n\left(\left\{\,f\in C[0,1]\;\middle|\;|\,f(0)\,|\geq a\,\right\}\right)\leq\eta$, for each $n\geq n_0$, and for each $\varepsilon,\,\eta>0$, $\exists\;\delta\in(0,1)$ such that $\displaystyle\limsup_{n\to\infty}\,P_n\left(\left\{\,f\in C[0,1]\;\middle|\;w(f,\delta)\geq\varepsilon\,\right\}\right)\;\leq\;\eta$

$\Longleftrightarrow$ for each $\eta>0$, $\exists\;a>0$ and $n_0\in\mathbb{N}$ such that $P_n\left(\left\{\,f\in C[0,1]\;\middle|\;|\,f(0)\,|\geq a\,\right\}\right)\leq\eta$, for each $n\geq n_0$, and for each $\varepsilon>0$, $\displaystyle\lim_{\delta\to0^+}\;\limsup_{n\to\infty}\,P_n\left(\left\{\,f\in C[0,1]\;\middle|\;w(f,\delta)\geq\varepsilon\,\right\}\right)\;=\;0$

$\Longleftrightarrow$ (iv)

The proof of the Theorem is now complete. □

G Weak convergence of Borel probability measures on $(C[0,1],\|\cdot\|_\infty)$

Theorem G.1
Suppose $P,P_1,P_2,\ldots$ are Borel probability measures on $(C[0,1],\|\cdot\|_\infty)$. If

$$P_n\circ\pi^{-1}_{t_1t_2\cdots t_k}\;\overset{d}{\longrightarrow}\;P\circ\pi^{-1}_{t_1t_2\cdots t_k},\quad\text{as }\;n\longrightarrow\infty,\quad\text{for any }\;t_1,t_2,\ldots,t_k\in[0,1]\,,$$

then the limit of every weakly convergent subsequence of $\{\,P_n\,\}$ must be P.

Proof Suppose $P_{n(i)}\overset{d}{\longrightarrow}Q$, as $i\longrightarrow\infty$, where Q is some Borel probability measure on $(C[0,1],\|\cdot\|_\infty)$. We need to show that Q in fact must be P. Since, for any $t_1,t_2,\ldots,t_k\in[0,1]$, the map $\pi_{t_1t_2\cdots t_k}:C[0,1]\longrightarrow\mathbb{R}^k:f\longmapsto\big(f(t_1),\ldots,f(t_k)\big)$ is continuous, it follows that $P_{n(i)}\circ\pi^{-1}_{t_1t_2\cdots t_k}\overset{d}{\longrightarrow}Q\circ\pi^{-1}_{t_1t_2\cdots t_k}$, as $i\longrightarrow\infty$, for any $t_1,t_2,\ldots,t_k\in[0,1]$, by the Continuous Mapping Theorem (Theorem 2.7, [3], or simply see remark on p.20, [3]). Then, for each bounded continuous function $\varphi:\mathbb{R}^k\longrightarrow\mathbb{R}$, we have

$$\int_{\mathbb{R}^k}\varphi\,\mathrm{d}(Q\circ\pi^{-1}_{t_1\cdots t_k})\;=\;\lim_{i\to\infty}\int_{\mathbb{R}^k}\varphi\,\mathrm{d}(P_{n(i)}\circ\pi^{-1}_{t_1\cdots t_k})\;=\;\lim_{n\to\infty}\int_{\mathbb{R}^k}\varphi\,\mathrm{d}(P_n\circ\pi^{-1}_{t_1\cdots t_k})\;=\;\int_{\mathbb{R}^k}\varphi\,\mathrm{d}(P\circ\pi^{-1}_{t_1\cdots t_k}),$$

where the first and third equalities follow directly from the definition of weak convergence of probability measures, while the second equality follows from the elementary fact that every subsequence of a convergent sequence of real numbers converges to the same limit as the full sequence. By Theorem 1.2, p.8, [3], we see that $Q\circ\pi^{-1}_{t_1\cdots t_k}=P\circ\pi^{-1}_{t_1\cdots t_k}$, as Borel measures on $\mathbb{R}^k$, for any $t_1,t_2,\ldots,t_k\in[0,1]$. This in turn implies that $Q\circ\pi^{-1}_{t_1\cdots t_k}(B)=P\circ\pi^{-1}_{t_1\cdots t_k}(B)$, for every Borel subset $B\subset\mathbb{R}^k$. In other words, Q and P agree on the collection of finite-dimensional subsets of $C[0,1]$. Since the finite-dimensional subsets of $C[0,1]$ form a separating class (Example 1.3, p.11, [3]), we may now conclude that $Q=P$. This completes the proof of the Theorem. □

Theorem G.2 (Example 5.1, p.57, [3])
Suppose $P,P_1,P_2,\ldots$ are Borel probability measures on $(C[0,1],\|\cdot\|_\infty)$. If

(i) *the sequence $\{\,P_n\,\}_{n\in\mathbb{N}}$ of Borel probability measures on $(C[0,1],\|\cdot\|_\infty)$ is relatively compact, and*

(ii) *for any $t_1,t_2,\ldots,t_k\in[0,1]$, we have*

$$P_n\circ\pi^{-1}_{t_1t_2\cdots t_k}\;\overset{d}{\longrightarrow}\;P\circ\pi^{-1}_{t_1t_2\cdots t_k},\quad\text{as }\;n\longrightarrow\infty\,,$$

then $P_n \overset{d}{\longrightarrow} P$, *as* $n \longrightarrow \infty$.

PROOF Recall that, by Theorem 2.6, p.20, [3], $P_n \overset{d}{\longrightarrow} P$ if and only if every subsequence of $\{ P_n \}$ contains a further subsequence that weakly converges to P. So, let $\left\{ P_{n(i)} \right\}_{i\in\mathbb{N}}$ be a subsequence of $\{ P_n \}_{n\in\mathbb{N}}$. By hypothesis (i) (i.e. relative compactness of $\{ P_n \}_{n\in\mathbb{N}}$), the subsequence $\left\{ P_{n(i)} \right\}_{i\in\mathbb{N}}$ contains a weakly convergent further subsequence $\left\{ P_{n(i_m)} \right\}_{m\in\mathbb{N}}$, say $P_{n(i_m)} \overset{d}{\longrightarrow} Q$, as $m \longrightarrow \infty$, where Q is some Borel probability measure on $(C[0,1], \|\cdot\|_\infty)$. By hypothesis (ii) and Theorem G.1, we see that in fact $Q = P$. Thus, we have shown that every subsequence $\left\{ P_{n(i)} \right\}_{i\in\mathbb{N}}$ of $\{ P_n \}_{n\in\mathbb{N}}$ contains a further subsequence $P_{n(i_m)}$ which weakly converges to P. By Theorem 2.6, p.20, [3], we may now conclude that the full original sequence P_n converges weakly to P. This completes the proof of the Theorem. □

Theorem G.3 (Theorem 7.5, p.84, [3])
Suppose:

- $(\Omega, \mathcal{A}, \mu)$, $\left(\Omega^{(1)}, \mathcal{A}^{(1)}, \mu^{(1)}\right)$, $\left(\Omega^{(2)}, \mathcal{A}^{(2)}, \mu^{(2)}\right)$, ... *are probability spaces.*
- $X : \Omega \longrightarrow C[0,1]$ *is a* $C[0,1]$*-valued random variable, and* $\left\{ X^{(n)} : \Omega^{(n)} \longrightarrow C[0,1] \right\}_{n\in\mathbb{N}}$ *is a sequence of* $C[0,1]$*-valued random variables.*

If

(i) *for any* $t_1, t_2, \ldots, t_k \in [0,1]$,

$$\left(X^{(n)}_{t_1}, X^{(n)}_{t_2}, \ldots, X^{(n)}_{t_k} \right) \;\overset{d}{\longrightarrow}\; \left(X_{t_1}, X_{t_2}, \ldots, X_{t_k} \right), \quad \text{as } n \longrightarrow \infty, \text{ and}$$

(ii) *for each* $\varepsilon > 0$,

$$\lim_{\delta\to 0^+} \limsup_{n\to\infty} P\Big(w(X^{(n)}, \delta) \,\geq\, \varepsilon \Big) \;=\; \lim_{\delta\to 0^+} \limsup_{n\to\infty} \mu^{(n)}\left(\left\{ \omega \in \Omega^{(n)} \,\middle|\, w(X^{(n)}(\omega), \delta) \,\geq\, \varepsilon \right\} \right) \;=\; 0,$$

then

$$X^{(n)} \overset{d}{\longrightarrow} X, \text{ as } n \longrightarrow \infty.$$

PROOF Note that, in order to prove the Theorem, it suffices to establish tightness of $\left\{ X^{(n)} \right\}_{n\in\mathbb{N}}$. This is because tightness implies relative compactness (Prokhorov's Theorem, Theorem 5.1, p.59, [3]), which, together with hypothesis (i) and Theorem G.2, will imply the desired weak convergence $X^{(n)} \overset{d}{\longrightarrow} X$. We use Theorem F.1(iv) to establish the tightness of $\left\{ X^{(n)} \right\}_{n\in\mathbb{N}}$. Thus, the proof the present Theorem will be complete once we prove the following two claims:

> **Claim 1:** $\left\{ X^{(n)} \right\}_{n\in\mathbb{N}}$ satisfies the first condition in Theorem F.1(iv).
> In other words, for each $\eta > 0$, there exist $a > 0$ and $n_0 \in \mathbb{N}$ such that
>
> $$P_{X^{(n)}}\left(\left\{ f \in C[0,1] \,\middle|\, |\, f(0) \,| \geq a \right\} \right) \;\leq\; \eta, \quad \text{for each } n \,\geq\, n_0.$$
>
> **Claim 2:** $\left\{ X^{(n)} \right\}_{n\in\mathbb{N}}$ satisfies the second condition in Theorem F.1(iv).
> In other words, for each $\varepsilon > 0$,
>
> $$\lim_{\delta\to 0^+} \limsup_{n\to\infty} P_{X^{(n)}}\left(\left\{ f \in C[0,1] \,\middle|\, w(f,\delta) \geq \varepsilon \right\} \right) \;=\; 0.$$

Proof of Claim 1: By hypothesis (i), $X_0^{(n)} \overset{d}{\longrightarrow} X_0$, which is equivalent to the statement that every subsequence of $\left\{ X_0^{(n)} \right\}_{n\in\mathbb{N}}$ admits a further subsequence which weakly converges to X_0 (Theorem 2.6, p.20, [3]). In particular, weak convergence of $\left\{ X_0^{(n)} \right\}_{n\in\mathbb{N}}$ implies relative compactness of $\left\{ X_0^{(n)} \right\}_{n\in\mathbb{N}}$. Since each $X_0^{(n)}$ induces a Borel probability measure on $\mathbb{R}$, and $\mathbb{R}$ is separable and complete, Prokhorov's Theorem (Theorem 5.2, p.60, [3]) implies that $\left\{ X_0^{(n)} \right\}_{n\in\mathbb{N}}$ is tight (as a sequence of Borel probability measures on $\mathbb{R}$). Thus, for any $\eta > 0$, there exists a compact set $K \subset \mathbb{R}$ such that

$$1-\eta \;<\; P_{X_0^{(n)}}(\,K\,) \;=\; P\Big(X_0^{(n)} \in K \Big) \;\leq\; 1, \quad \text{for each } n\in\mathbb{N}.$$

Since $K\subset\mathbb{R}$ is compact, it is bounded, and thus there exists $a>0$ such that $K\subset(-a,a)$. Thus, we see that:

$$1-\eta \;\;\leq\;\; P\Big(X_0^{(n)} \in K \Big) \;\;\leq\;\; P\Big(\Big| X_0^{(n)} \Big| < a \Big) \;\;=\;\; 1 - P\Big(\Big| X_0^{(n)} \Big| \geq a \Big), \quad \text{for each } n\in\mathbb{N},$$

which implies that

$$P_{X^{(n)}}\Big(\Big\{\, f\in C[0,1] \;\Big|\; |\,f(0)\,| \geq a \,\Big\} \Big) \;\;=\;\; P\Big(\Big| X_0^{(n)} \Big| \geq a \Big) \;\;\leq\;\; \eta, \quad \text{for each } n\in\mathbb{N}.$$

This completes the proof of Claim 1.

Proof of Claim 2: Claim 2 holds precisely by hypothesis (ii), since

$$P_{X^{(n)}}\Big(\Big\{\, f\in C[0,1] \;\Big|\; w(f,\delta)\geq\varepsilon \,\Big\} \Big) \;\;=\;\; P\Big(w(X^{(n)},\delta) \;\geq\; \varepsilon \Big).$$

This completes the proof of Claim 2, as well as that of the present Theorem. □

H Etemadi's inequality

Theorem H.1 (Appendix M19, [3])

Suppose $X_1,\ldots,X_n:\Omega\longrightarrow\mathbb{R}$ are independent $\mathbb{R}$-valued random variables defined on the common probability space Ω. Let $S_k := \sum_{i=1}^{k} X_i$, for $k=1,2,\ldots,n$. Then, for any $x>0$,

$$P\left(\left\{\, \omega\in\Omega \;\middle|\; \max_{1\leq k\leq n} |\,S_k(\omega)\,| \;\geq\; 3x \,\right\} \right) \;\;\leq\;\; 3\cdot\max_{1\leq k\leq n} P\left(\left\{\, \omega\in\Omega \;\middle|\; |\,S_k(\omega)\,| \;\geq\; x \,\right\} \right).$$

PROOF Define:

$$\begin{aligned}
A \;&:=\; \left\{\, \omega\in\Omega \;\middle|\; \max_{1\leq k\leq n} |\,S_k(\omega)\,| \;\geq\; 3x \,\right\}, \\
A_1 \;&:=\; \left\{\, \omega\in\Omega \;\middle|\; |\,S_1(\omega)\,| \geq 3x \,\right\}, \\
A_k \;&:=\; \left\{\, \omega\in\Omega \;\middle|\; \max_{1\leq i\leq k-1} |\,S_i(\omega)\,| < 3x \;\text{ and }\; |\,S_k(\omega)\,| \;\geq\; 3x \,\right\}, \quad \text{for } k=2,3,\ldots,n.
\end{aligned}$$

Note that A can be expressed as the following disjoint union: $A \;=\; \bigsqcup_{k=1}^{n} A_k$.

Claim 1: $A_k \cap \Big\{ |\,S_n\,| < x \Big\} \;\subset\; A_k \cap \Big\{ |\,S_n - S_k\,| > 2x \Big\}$, for each $k=1,2,\ldots,n$.

Claim 2: $\Big\{ |\,S_n - S_k\,| > 2x \Big\} \;\subset\; \Big\{ |\,S_n\,| > x \Big\} \bigcup \Big\{ |\,S_k\,| > x \Big\}$, for each $k=1,2,\ldots,n$.

Proof of Claim 1:

$$
\begin{aligned}
& \omega \in A_k \cap \left\{\, |\, S_n \,| < x \,\right\} \\
\Longleftrightarrow\quad & \max_{1 \le i \le k-1} \left| S_i(\omega) \right| < 3x \;\text{ and }\; \left| S_k(\omega) \right| \,\ge\, 3x \;\text{ and }\; \left| S_n(\omega) \right| \,<\, x \\
\Longrightarrow\quad & \max_{1 \le i \le k-1} \left| S_i(\omega) \right| < 3x \;\text{ and }\; \left| S_k(\omega) \right| \,\ge\, 3x \;\text{ and }\; \left| S_k(\omega) - S_n(\omega) \right| \,\ge\, \left| S_k(\omega) \right| - \left| S_n(\omega) \right| \,>\, 3x - x \,=\, 2x \\
\Longrightarrow\quad & \omega \in A_k \cap \left\{\, |\, S_n - S_k \,| > 2x \,\right\}
\end{aligned}
$$

This proves Claim 1.

Proof of Claim 2: Simply note that for any two real numbers $a, b \in \mathbb{R}$, we have:

$$
|\,a\,| \,\le\, x \;\text{ and }\; |\,b\,| \,\le\, x \quad\Longrightarrow\quad |\,a - b\,| \,\le\, |\,a\,| \,+\, |\,b\,| \,\le\, x \,+\, x \,=\, 2x,
$$

whose contrapositive is:

$$
|\,a - b\,| \,>\, 2x, \quad\Longrightarrow\quad |\,a\,| \,>\, x \;\text{ or }\; |\,b\,| \,>\, x.
$$

This proves Claim 2.

Using the above two Claims, we now see that:

$$
\begin{aligned}
P(A) \;&=\; P\Big(A \cap \{\, |\, S_n \,| \ge x \,\} \Big) \,+\, P\Big(A \cap \{\, |\, S_n \,| < x \,\} \Big) \\
&\le\; P\Big(|\, S_n \,| \ge x \Big) \,+\, P\Big(A \cap \{\, |\, S_n \,| < x \,\} \Big) \\
&=\; P\Big(|\, S_n \,| \ge x \Big) \,+\, \sum_{k=1}^{n} P\Big(A_k \cap \{\, |\, S_n \,| < x \,\} \Big) \\
&\le\; P\Big(|\, S_n \,| \ge x \Big) \,+\, \sum_{k=1}^{n} P\Big(A_k \cap \{\, |\, S_n - S_k \,| > 2x \,\} \Big), \qquad \text{by Claim 1} \\
&=\; P\Big(|\, S_n \,| \ge x \Big) \,+\, \sum_{k=1}^{n} P(\, A_k \,) \cdot P\Big(\,|\, S_n - S_k \,| > 2x \,\Big), \qquad \text{by independence of summands} \\
&\le\; P\Big(|\, S_n \,| \ge x \Big) \,+\, \left(\max_{1 \le k \le n} P\Big(\,|\, S_n - S_k \,| > 2x \Big) \right) \cdot \sum_{k=1}^{n} P(\, A_k \,) \\
&=\; P\Big(|\, S_n \,| \ge x \Big) \,+\, P(\, A \,) \cdot \max_{1 \le k \le n} P\Big(\,|\, S_n - S_k \,| > 2x \Big) \\
&=\; P\Big(|\, S_n \,| \ge x \Big) \,+\, \max_{1 \le k \le n} P\Big(\,|\, S_n - S_k \,| > 2x \Big) \\
&\le\; P\Big(|\, S_n \,| \ge x \Big) \,+\, \max_{1 \le k \le n} \left\{ P\Big(|\, S_n \,| > x \Big) \,+\, P\Big(|\, S_k \,| > x \Big) \right\}, \qquad \text{by Claim 2} \\
&\le\; 3 \cdot \max_{1 \le k \le n} \left\{ P\Big(\,|\, S_k \,| \ge x \,\Big) \right\}.
\end{aligned}
$$

This completes the proof of Etemadi's inequality. □

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