Some Asymptotic Theory for the Bootstrap

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This set of notes aims to provide complete proofs of a number of asymptotic results regarding the Bootstrap [2] contained in Bickel and Freedman [1].

1 Bootstrap asymptotics for sample mean

Theorem 1.1 (Bootstrap Central Limit Theorem for I.I.D. sample mean, Theorem 2.1 [1])
Let $(\Omega, \mathcal{A}, \nu)$ be a probability space. Let $X, X_1, X_2, \ldots : \Omega \longrightarrow \mathbb{R}$ be a sequence of independent and identically distributed $\mathbb{R}$-valued random variables defined on Ω with finite expectation value $\mu_X \in \mathbb{R}$ and variance $\sigma_X^2 < \infty$. For each $n \in \mathbb{N}$ be fixed, define:

$$\overline{X}_n \;:\; \Omega \;\longrightarrow\; \mathbb{R} \;:\; \omega \;\longmapsto\; \frac{1}{n}\sum_{i=1}^{n} X_i(\omega).$$

For $n, m \in \mathbb{N}$, define $\mathcal{S}_m^{(n)}$ to be the set of all functions from $\{1,2,\ldots,m\} \longrightarrow \{1,2,\ldots,n\}$. Thus, each

$$s \;=\; (s(1), s(2), \ldots, s(m)) \;\in\; \mathcal{S}_m^{(n)}$$

can be regarded as a length-m finite (ordered) sequence of positive integers between 1 and n, inclusive. Note that $\mathcal{S}_m^{(n)}$ is a finite set with $\mid \mathcal{S}_m^{(n)} \mid = n^m$. Endow $\mathcal{S}_m^{(n)}$ with the probability space structure induced by the uniform probability function:

$$P_{\mathcal{S}_m^{(n)}}(s) \;:=\; \frac{1}{n^m}, \quad \text{for each } s \in \mathcal{S}_m^{(n)}.$$

Let $\Omega \times \mathcal{S}_m^{(n)}$ be the product probability space of Ω and $\mathcal{S}_m^{(n)}$. Define:

$$\overline{X}_m^{(n)} \;:\; \Omega \times \mathcal{S}_m^{(n)} \;\longrightarrow\; \mathbb{R} \;:\; (\omega, s) \;\longmapsto\; \frac{1}{m}\sum_{j=1}^{n} X_{s(j)}(\omega).$$

For each $\omega \in \Omega$, define:

$$\Phi_{m,\omega}^{(n)} \;:\; \mathcal{S}_m^{(n)} \;\longrightarrow\; \mathbb{R} \;:\; s \;\longmapsto\; \sqrt{m}\left(\overline{X}_m^{(n)}(\omega, s) - \overline{X}_n(\omega)\right)$$

Then,

$$P\left(\,\Phi_{m,\omega}^{(n)} \xrightarrow{d} N(0,\sigma_X^2), \text{ as } n, m \to \infty\,\right) \;=\; \nu\left(\left\{\,\omega \in \Omega \;\middle|\; \Phi_{m,\omega}^{(n)} \xrightarrow{d} N(0,\sigma_X^2), \text{ as } n, m \to \infty\,\right\}\right) \;=\; 1.$$

Remark 1.2
*For each fixed $\omega \in \Omega$, $\left\{\,\Phi_{m,\omega}^{(n)} : \mathcal{S}_m^{(n)} \longrightarrow \mathbb{R}\,\right\}_{n,m\in\mathbb{N}}$ is a doubly indexed sequence of $\mathbb{R}$-valued random variables. Note that their respective domains $\mathcal{S}_m^{(n)}$ are pairwise distinct probability spaces. The **Bootstrap Central Limit Theorem for I.I.D. sample mean** asserts that for almost every $\omega \in \Omega$, the doubly indexed sequence $\left\{\,\Phi_{m,\omega}^{(n)}\,\right\}$ of $\mathbb{R}$-valued random variables converges in distribution to $N(0,\sigma_X^2)$ as $n, m \longrightarrow \infty$.*

Remark 1.3 *The following results are well known from classical asymptotic theory:*
*By the **Weak Law of Large Numbers**, $\overline{X}_n$ converges in probability to μ_X, as $n \longrightarrow \infty$; in other words,*

$$\lim_{n\to\infty} P\left(\,\left|\overline{X}_n - \mu_X\right| > \varepsilon\,\right) \;=\; \lim_{n\to\infty} \nu\left(\left\{\,\omega \in \Omega \;:\; \left|\overline{X}_n(\omega) - \mu_X\right| > \varepsilon\,\right\}\right) \;=\; 0, \quad \text{for each } \varepsilon > 0.$$

*By the **Strong Law of Large Numbers**, $\overline{X}_n$ converges almost surely to μ_X, as $n \longrightarrow \infty$; in other words,*

$$P\left(\,\lim_{n\to\infty} \overline{X}_n = \mu_X\,\right) \;=\; \nu\left(\left\{\,\omega \in \Omega \;\middle|\; \lim_{n\to\infty} \overline{X}_n(\omega) = \mu_X\,\right\}\right) \;=\; 1.$$

*By the **Central Limit Theorem**, $\sqrt{n}\left(\overline{X}_n - \mu_X\right)$ converges in distribution to $N(0,\sigma_X^2)$.*

References

[1] Bickel, P. J., and Freedman, D. A. Some asymptotic theory for the bootsrap. *The Annals of Statistics 9*, 6 (1981), 1196–1217.

[2] Efron, B. Bootsrap methods: another look at the jackknife. *The Annals of Statistics 7*, 1 (1979), 1–26.