

1 Cumulative distribution functions

Definition 1.1 Let $X : (\Omega, \mathcal{A}, \mu) \rightarrow \mathbb{R}$ be a \mathbb{R} -valued random variable. The **cumulative distribution function** of X is, by definition, the function $F_X : \mathbb{R} \rightarrow [0, 1]$ defined as follows:

$$F_X(x) := P(X \leq x) = \mu(\{\omega \in \Omega \mid X(\omega) \leq x\}), \quad \text{for each } x \in \mathbb{R}.$$

Definition 1.2 A function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be

- **non-decreasing** if $f(x) \leq f(y)$, for any $x, y \in D$ with $x \leq y$.
- **non-increasing** if $f(x) \geq f(y)$, for any $x, y \in D$ with $x \leq y$.
- **monotone** if f is either non-decreasing or non-increasing.

Theorem 1.3 (Theorem 4.29, [1])

Let $f : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing function. Then,

$$f(x-) := \lim_{t \rightarrow x-} f(t) \quad \text{and} \quad f(x+) := \lim_{t \rightarrow x+} f(t)$$

exist for every $x \in (a, b)$. More precisely,

$$f(x-) = \sup_{a < t < x} f(t) \leq f(x) \leq \inf_{x < t < b} f(t) = f(x+).$$

Furthermore, if $a < x < y < b$, then

$$f(x+) \leq f(y-).$$

PROOF First note that, since f is non-decreasing, it immediately follows that:

$$\sup_{a < t < x} f(t) \leq f(x) \leq \inf_{x < t < b} f(t).$$

Next, we show that $f(x-) := \lim_{t \rightarrow x-} f(t)$ exists and equals $A := \sup_{a < t < x} f(t)$. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that

$$|f(t) - A| < \varepsilon, \quad \text{for every } t \in (x - \delta, x).$$

By definition of the supremum, there exists $\delta > 0$ with $x - \delta \in (a, x)$ such that

$$A - \varepsilon < f(x - \delta) \leq A.$$

Since f is non-decreasing, we have

$$f(x - \delta) \leq f(t) \leq A := \sup_{\xi \in (x - \delta, x)} f(\xi), \quad \text{for every } t \in (x - \delta, x).$$

We therefore see that

$$|f(t) - A| < \varepsilon,$$

as desired. This proves that $f(x-) := \lim_{t \rightarrow x-} f(t)$ indeed exists and equals $A := \sup_{t \in (a, x)} f(t)$. The proof that $f(x+)$ exists and equals $\inf_{t \in (x, b)} f(t)$ is analogous. Lastly, let $a < x < y < b$. Then, choose some $z \in (x, y) = (x, b) \cap (a, y)$. Hence, we have

$$f(x+) = \inf_{t \in (x, b)} f(t) \leq f(z) \leq \sup_{t \in (a, y)} f(t) = f(y-).$$

This proof the Theorem is complete. □

Remark 1.4 *The analogous results of the preceding Theorem for non-increasing functions hold, obviously.*

Definition 1.5 *Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$. A point $a \in \text{interior}(D)$ is a **jump discontinuity** of f if both*

$$\lim_{x \rightarrow a^-} f(x) \quad \text{and} \quad \lim_{x \rightarrow a^+} f(x)$$

exist but they are unequal.

Corollary 1.6 *A monotone \mathbb{R} -valued function defined on an interval of \mathbb{R} can have only jump discontinuities.*

Theorem 1.7 *A function $F : \mathbb{R} \rightarrow [0, 1]$ is a cumulative distribution function of some \mathbb{R} -valued random variable if and only if each of following four conditions holds:*

- F is non-decreasing.
- F is right-continuous.
- $\lim_{x \rightarrow -\infty} F(x) = 0$.
- $\lim_{x \rightarrow +\infty} F(x) = 1$.

PROOF If $F : \mathbb{R} \rightarrow [0, 1]$ is a cumulative distribution function of some \mathbb{R} -valued random variable $X : (\Omega, \mathcal{A}, \mu) \rightarrow \mathbb{R}$, then the four conditions follow immediately from the property of the probability measure μ . Conversely, suppose the four conditions hold. Let $\Omega := [0, 1]$ and $\mathcal{B}(\Omega)$ the Borel subsets of Ω . Let μ be the Lebesgue measure on $(\Omega, \mathcal{B}(\Omega))$, i.e. μ is determined by:

$$\mu([0, \omega]) := \omega, \quad \text{for each } \omega \in \Omega = [0, 1].$$

Define the random variable $X : (\Omega, \mathcal{B}(\Omega), \mu) \rightarrow \mathbb{R}$ by:

$$X(\omega) := \inf \{x \in \mathbb{R} \mid \omega \leq F(x)\}, \quad \text{for each } \omega \in \Omega = [0, 1].$$

Note that X is simply the quantile function of F .

Claim: Suppose $G : \mathbb{R} \rightarrow [0, 1]$ is non-decreasing and right-continuous. Then, for any $\omega \in [0, 1]$ and $x \in \mathbb{R}$,

$$\inf \{ \xi \in \mathbb{R} \mid \omega \leq G(\xi) \} \leq x \iff \omega \leq G(x).$$

Proof of Claim: Suppose $\omega \leq G(x)$. Then, $x \in \{ \xi \in \mathbb{R} \mid \omega \leq G(\xi) \}$. Hence, $\inf \{ \xi \in \mathbb{R} \mid \omega \leq G(\xi) \} \leq x$. Conversely, suppose $\inf \{ \xi \in \mathbb{R} \mid \omega \leq G(\xi) \} \leq x$. Since G is non-decreasing and right-continuous, we have:

$$\begin{aligned} \inf \{ \xi \in \mathbb{R} \mid \omega \leq G(\xi) \} \leq x &\implies \text{for any } \varepsilon > 0, \exists \xi \in \mathbb{R}, \text{ satisfying } \omega \leq G(\xi), \text{ such that } \xi \leq x + \varepsilon \\ &\implies \text{for any } \varepsilon > 0, \exists \xi \in \mathbb{R}, \text{ satisfying } \omega \leq G(\xi) \text{ and } G(\xi) \leq G(x + \varepsilon) \\ &\implies \text{for any } \varepsilon > 0, \omega \leq G(x + \varepsilon) \\ &\implies \omega \leq \lim_{\varepsilon \rightarrow 0^+} G(x + \varepsilon) = G(x). \end{aligned}$$

This completes the proof of the Claim.

Noting that, by hypothesis, F is non-decreasing right-continuous, and invoking the Claim above, we see that

$$\begin{aligned} P(X \leq x) &= P(\{ \omega \in \Omega \mid X(\omega) \leq x \}) = P(\{ \omega \in \Omega \mid \inf \{ \xi \in \mathbb{R} \mid \omega \leq F(\xi) \} \leq x \}) \\ &= P(\{ \omega \in \Omega \mid \omega \leq F(x) \}) = \mu(\{ \omega \in [0, 1] \mid \omega \leq F(x) \}) = \mu([0, F(x)]) \\ &= F(x) \end{aligned}$$

This shows that if F satisfies the four conditions, then F is the cumulative distribution function of the random variable X constructed above. The proof of the Theorem is now complete. \square

Theorem 1.8 (Darboux-Froda, Theorem 4.30, [1])

The set of discontinuities of a monotone \mathbb{R} -valued function defined on an interval of \mathbb{R} is at most countable.

PROOF We give the proof for non-decreasing functions; the proof for non-increasing functions is analogous. Let $f : (a, b) \rightarrow \mathbb{R}$ be non-decreasing, and let $\mathcal{D}(f) \subset (a, b)$ be the set of discontinuities of f . By Corollary 1.6, each $x \in \mathcal{D}(f)$ is a jump discontinuity of f , i.e. both one-sided limits $\lim_{t \rightarrow x^-} f(t)$ and $\lim_{t \rightarrow x^+} f(t)$ exist, and

$$\lim_{t \rightarrow x^-} f(t) < \lim_{t \rightarrow x^+} f(t)$$

Thus, for each $x \in \mathcal{D}(f)$, we may choose a rational number $r(x) \in \mathbb{Q}$ such that

$$\lim_{t \rightarrow x^-} f(t) < r(x) < \lim_{t \rightarrow x^+} f(t).$$

This defines a function $r : \mathcal{D}(f) \rightarrow \mathbb{Q}$. Note that this function is injective. Indeed, let $x, y \in \mathcal{D}(f)$ with $x < y$. Then, by Theorem 1.3,

$$r(x) < \lim_{t \rightarrow x^+} f(t) = f(x+) \leq f(y-) = \lim_{t \rightarrow y^-} f(t) < r(y)$$

This shows $r : \mathcal{D}(f) \rightarrow \mathbb{Q}$ is indeed injective. Since \mathbb{Q} is countable, we may now conclude that $\mathcal{D}(f)$ is at most countable. \square

Corollary 1.9 *The cumulative distribution function of an \mathbb{R} -valued random variable can have only jump discontinuities, and its set of (jump) discontinuities is at most countable.*

2 The O_P and o_P notations; convergence in distribution implies boundedness in probability

Definition 2.1 (The Big- O_P notation)

Let $\{X_n : (\Omega_n, \mathcal{A}_n, \mu_n) \rightarrow \mathbb{R}^k\}_{n \in \mathbb{N}}$ be a sequence of \mathbb{R}^k -valued random variables. Let $\{a_n\}_{n \in \mathbb{N}}$ be sequence of positive numbers. The notation $X_n = O_P(a_n)$ means:

For every $\varepsilon > 0$, there exist $C_\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that $P(|X_n| \leq C_\varepsilon \cdot a_n) > 1 - \varepsilon$, for every $n \geq n_\varepsilon$.

Proposition 2.2 *The following are equivalent:*

- (a) $X_n = O_P(a_n)$.
- (b) For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $P(|X_n| \leq C_\varepsilon \cdot a_n) > 1 - \varepsilon$, for each $n \in \mathbb{N}$.
- (c) For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $\limsup_{n \rightarrow \infty} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon$.
- (d) For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $\sup_{n \in \mathbb{N}} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon$.
- (e) $\lim_{C \rightarrow \infty} \limsup_{n \rightarrow \infty} P(|X_n| > C \cdot a_n) = 0$.
- (f) $\lim_{C \rightarrow \infty} \sup_{n \in \mathbb{N}} P(|X_n| > C \cdot a_n) = 0$.

PROOF

(a) \implies (b)

Let $\varepsilon > 0$ be given. By (a), there exist $B_\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that $P(|X_n| \leq B_\varepsilon \cdot a_n) > 1 - \varepsilon$, for each $n \geq n_\varepsilon$.

Claim: Let Y be an \mathbb{R}^k -valued random variable. Then, for each $\varepsilon > 0$, there exists $A_\varepsilon > 0$ such that $P(|Y| \leq A_\varepsilon) > 1 - \varepsilon$.

Proof of Claim: Suppose the Claim were false. Then, there exists some $\varepsilon > 0$ such that $P(|Y| \leq A) \leq 1 - \varepsilon$, for every $A > 0$; equivalently, $P(|Y| > A) > \varepsilon$, for every $A > 0$. This implies $\lim_{A \rightarrow \infty} P(|Y| > A) = \limsup_{A \rightarrow \infty} P(|Y| > A) \geq \varepsilon > 0$. But this is a contradiction since $\lim_{A \rightarrow \infty} P(|Y| > A) = 0$, for every \mathbb{R}^k -valued random variable Y . This proves the Claim.

By the Claim, for each $i = 1, 2, \dots, n_\varepsilon - 1$, there exists $B_\varepsilon^{(i)} > 0$ such that $P(|X_i| \leq B_\varepsilon^{(i)} \cdot a_i) > 1 - \varepsilon$. Now, let $C_\varepsilon := \max \{B_\varepsilon^{(1)}, B_\varepsilon^{(1)}, \dots, B_\varepsilon^{(n_\varepsilon-1)}, B_\varepsilon\}$. Then, $P(|X_n| \leq C_\varepsilon \cdot a_n) > 1 - \varepsilon$, for every $n \in \mathbb{N}$. This proves the implication (a) \implies (b).

(b) \implies (a) Trivial: Suppose (b) holds. Then (a) immediately follows with $n_\varepsilon = 1$.

(a) \iff (c) Let $\varepsilon > 0$ be given.

- (a) \iff There exist $C_\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that $P(|X_n| \leq C_\varepsilon \cdot a_n) > 1 - \varepsilon$, for every $n \geq n_\varepsilon$.
 \iff There exist $C_\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that $P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon$, for every $n \geq n_\varepsilon$.
 \iff There exist $C_\varepsilon > 0$ such that $\limsup_{n \rightarrow \infty} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon \iff$ (c)

(b) \iff (d) Let $\varepsilon > 0$ be given.

- (b) \iff There exists $C_\varepsilon > 0$ such that $P(|X_n| \leq C_\varepsilon \cdot a_n) > 1 - \varepsilon$, for every $n \in \mathbb{N}$.
 \iff There exists $C_\varepsilon > 0$ such that $P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon$, for every $n \in \mathbb{N}$.
 \iff There exist $C_\varepsilon > 0$ such that $\sup_{n \in \mathbb{N}} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon \iff$ (d)

(d) \iff (f) Let $\varepsilon > 0$ be given. We first establish that (f) \implies (d).

- (f) \iff There exists $C_\varepsilon > 0$ such that $\sup_{n \in \mathbb{N}} P(|X_n| > C \cdot a_n) \leq \varepsilon$, for each $C \geq C_\varepsilon$.
 \implies There exists $C_\varepsilon > 0$ such that $\sup_{n \in \mathbb{N}} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon \iff$ (d)

Conversely, suppose (d) holds and $C \geq C_\varepsilon$. Then, $|X_n| > C \cdot a_n \implies |X_n| > C_\varepsilon \cdot a_n$. Hence, $\{|X_n| > C \cdot a_n\} \subseteq \{|X_n| > C_\varepsilon \cdot a_n\}$, which in turn implies $P(|X_n| > C \cdot a_n) \leq P(|X_n| > C_\varepsilon \cdot a_n)$. Thus, we have

$$\sup_{n \in \mathbb{N}} P(|X_n| > C \cdot a_n) \leq \sup_{n \in \mathbb{N}} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon,$$

i.e. (f) holds.

(c) \iff (e) Let $\varepsilon > 0$ be given. We first establish that (e) \implies (c).

- (e) \iff There exists $C_\varepsilon > 0$ such that $\limsup_{n \rightarrow \infty} P(|X_n| > C \cdot a_n) \leq \varepsilon$, for each $C \geq C_\varepsilon$.
 \implies There exists $C_\varepsilon > 0$ such that $\limsup_{n \rightarrow \infty} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon \iff$ (c)

Conversely, suppose (c) holds and $C \geq C_\varepsilon$. Then, $|X_n| > C \cdot a_n \implies |X_n| > C_\varepsilon \cdot a_n$. Hence, $\{|X_n| > C \cdot a_n\} \subseteq \{|X_n| > C_\varepsilon \cdot a_n\}$, which in turn implies $P(|X_n| > C \cdot a_n) \leq P(|X_n| > C_\varepsilon \cdot a_n)$. Thus, we have

$$\limsup_{n \rightarrow \infty} P(|X_n| > C \cdot a_n) \leq \limsup_{n \rightarrow \infty} P(|X_n| > C_\varepsilon \cdot a_n) \leq \varepsilon,$$

i.e. (e) holds.

This completes the proof of the Proposition. □

Definition 2.3 (Bounded in probability)

A sequence $\{X_n : (\Omega_n, \mathcal{A}_n, \mu_n) \longrightarrow \mathbb{R}^k\}_{n \in \mathbb{N}}$ of \mathbb{R}^k -valued random variables is said to be **bounded in probability** if $X_n = O_P(1)$.

Theorem 2.4

If a sequence $\{X_n : (\Omega_n, \mathcal{A}_n, \mu_n) \longrightarrow \mathbb{R}\}_{n \in \mathbb{N}}$ of \mathbb{R} -valued random variables converges in distribution to some random variable $X : (\Omega, \mathcal{A}, \mu) \longrightarrow \mathbb{R}$, then the sequence $\{X_n\}$ is bounded in probability.

PROOF Let $\varepsilon > 0$ be given. We need to show that there exist $C_\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that

$$P(|X_n| > C_\varepsilon) \leq \varepsilon, \quad \text{for each } n \geq n_\varepsilon.$$

Denote by $F, F_n : \mathbb{R} \longrightarrow [0, 1]$ the cumulative distribution functions of X and X_n , respectively. By Theorem 1.7 and the Darboux-Froda Theorem (Theorem 1.8), the cumulative distribution function F satisfies: $\lim_{x \rightarrow +\infty} F(x) = 1$, $\lim_{x \rightarrow -\infty} F(x) = 0$, and that F can have at most countably many (jump) discontinuities. Thus for the given $\varepsilon > 0$, we may choose $C_\varepsilon > 0$ sufficiently large such that

$$0 \leq F(-C_\varepsilon) < \frac{\varepsilon}{4}, \quad |1 - F(C_\varepsilon)| < \frac{\varepsilon}{4}, \quad \text{and} \quad \{\pm C_\varepsilon\} \subset \mathcal{C}(F)$$

where $\mathcal{C}(F)$ denotes the continuity set of F . Now, since $\pm C_\varepsilon \in \mathcal{C}(F)$, the convergence in distribution $X_n \xrightarrow{\mathcal{L}} X$ implies that the convergences $F_n(-C_\varepsilon) \longrightarrow F(-C_\varepsilon)$ and $F_n(C_\varepsilon) \longrightarrow F(C_\varepsilon)$ (of sequences of real numbers). Thus, we may choose $n_\varepsilon \in \mathbb{N}$ sufficiently large such that

$$|F_n(-C_\varepsilon) - F(-C_\varepsilon)| < \frac{\varepsilon}{4}, \quad \text{and} \quad |F_n(C_\varepsilon) - F(C_\varepsilon)| < \frac{\varepsilon}{4}, \quad \text{for every } n \geq n_\varepsilon.$$

Therefore, for each $n \geq n_\varepsilon$, we have:

$$\begin{aligned} P(|X_n| > C_\varepsilon) &= P(X_n < -C_\varepsilon) + P(X_n > C_\varepsilon) = P(X_n < -C_\varepsilon) + 1 - P(X_n \leq C_\varepsilon) \\ &\leq P(X_n \leq -C_\varepsilon) + 1 - P(X_n \leq C_\varepsilon) = F_n(-C_\varepsilon) + 1 - F_n(C_\varepsilon) \\ &\leq |F_n(-C_\varepsilon) - F(-C_\varepsilon)| + |F(-C_\varepsilon)| + |1 - F(C_\varepsilon)| + |F(C_\varepsilon) - F_n(C_\varepsilon)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon \end{aligned}$$

This completes the proof that a sequence $\{X_n\}_{n \in \mathbb{N}}$ of \mathbb{R} -valued random variables is bounded in probability whenever it converges in distribution. □

References

- [1] RUDIN, W. *Principles of Mathematical Analysis*, third ed. McGraw-Hill, 1976.