

## A Technical Lemmas

### Lemma A.1

$$|e^z - 1 - z| \leq z^2, \text{ for each } |z| \leq \frac{1}{2}.$$

PROOF First, note that  $g(z) := e^z - 1 - z \geq 0$ , for each  $z \in \mathbb{R}$ . Indeed,  $g'(z) = e^z - 1$  and  $g''(z) = e^z > 0$ . So,  $g$  is strictly convex. Next, note that  $g'(z) = 0 \iff z = 0$ . So,  $g$  achieves its unique minimum at  $z = 0$ . Since  $g(0) = 0$ , we see that  $g(z) \geq 0$ , for each  $z \in \mathbb{R}$ . Thus,  $|e^z - 1 - z| = e^z - 1 - z$ , for each  $z \in \mathbb{R}$ . Hence, to prove the Lemma, it suffices to prove that  $h(z) := z^2 - (e^z - 1 - z) = z^2 + z + 1 - e^z \geq 0$ , for each  $z \in [-1/2, 1/2]$ . Now,  $h'(z) = 2z + 1 - e^z$  and  $h''(z) = 2 - e^z$ . So,  $h''(z) = 0 \iff z = \log(2) \approx 0.6931$ , and  $h''(z) > 0$  for each  $z \in (-\infty, \log(2)) \supset [-1/2, 1/2]$ . So,  $h$  is strictly convex on the interval  $[-1/2, 1/2]$ . But  $h(0) = h'(0) = 0$ . Hence,  $z = 0$  is the unique minimum of  $h$  on  $[-1/2, 1/2]$ , and we may now conclude that  $h(z) \geq 0$ , for each  $z \in [-1/2, 1/2]$ , as required.  $\square$

**Lemma A.2 (p.358, [1])** Let  $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m \in \mathbb{C}$ . Then,

$$|a_i|, |b_i| \leq 1, \text{ for each } i = 1, 2, \dots, m \implies |a_1 a_2 \cdots a_m - b_1 b_2 \cdots b_m| \leq \sum_{i=1}^m |a_i - b_i|$$

PROOF Equality holds trivially for  $m = 1$ . We first prove the inequality for  $m = 2$ .

$$\begin{aligned} |a_1 a_2 - b_1 b_2| &= |a_1 a_2 - b_1 a_2 + b_1 a_2 - b_1 b_2| \leq |a_1 a_2 - b_1 a_2| + |b_1 a_2 - b_1 b_2| \\ &\leq |a_1 a_2 - b_1 a_2| + |b_1 a_2 - b_1 b_2| \leq |a_1 - b_1| |a_2| + |b_1| |a_2 - b_2| \\ &\leq |a_1 - b_1| + |a_2 - b_2|, \text{ since } |a_2|, |b_1| \leq 1, \text{ by hypothesis.} \end{aligned}$$

The general case now follows by induction: Assume the Lemma is valid for  $1, 2, \dots, m$ , and we prove that it is also valid for  $m + 1$ .

$$\begin{aligned} |a_1 \cdots a_m a_{m+1} - b_1 \cdots b_m b_{m+1}| &\leq |a_1 \cdots a_m - b_1 \cdots b_m| + |a_{m+1} - b_{m+1}| \\ &\leq \sum_{i=1}^m |a_i - b_i| + |a_{m+1} - b_{m+1}| = \sum_{i=1}^{m+1} |a_i - b_i| \end{aligned}$$

The proof of the Lemma is complete.  $\square$

### Lemma A.3 (p.343, [1])

$$\left| e^{ix} - \sum_{k=0}^n \frac{(ix)^k}{k!} \right| \leq \min \left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right\}, \text{ for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

PROOF We first establish a number of Claims, which will easily imply the Lemma.

#### Claim 1:

$$\int_0^x (x-s)^n e^{is} ds = \frac{x^{n+1}}{n+1} + \frac{i}{n+1} \int_0^x (x-s)^{n+1} e^{is} ds, \text{ for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof of Claim 1: We proceed by integration by parts. Let  $u = e^{is}$  and  $dv = (x-s)^n ds$ . Then,  $du = ie^{is}$  and  $v = -(x-s)^{n+1}/(n+1)$ . Hence,

$$\begin{aligned} \int_0^x (x-s)^n e^{is} ds &= \int u dv = uv - \int v du \\ &= \left[ e^{is} \cdot \frac{(-1)(x-s)^{n+1}}{n+1} \right]_{s=0}^{s=x} - \int_0^x \frac{(-1)(x-s)^{n+1}}{n+1} \cdot ie^{is} ds, \\ &= \frac{x^{n+1}}{n+1} + \frac{i}{n+1} \int_0^x (x-s)^{n+1} e^{is} ds. \end{aligned}$$

This proves Claim 1.

**Claim 2:**

$$e^{ix} = \sum_{k=0}^n \frac{(ix)^k}{k!} + \frac{i^{n+1}}{n!} \int_0^x (x-s)^n e^{is} ds, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof of Claim 2: We proceed by induction. For  $n = 0$ , we have:

$$\begin{aligned} \text{RHS}(n=0) &= \sum_{k=0}^0 \frac{(ix)^k}{k!} + \frac{i^{0+1}}{0!} \int_0^x (x-s)^0 e^{is} ds = 1 + i \int_0^x e^{is} ds = 1 + i \left[ \frac{e^{is}}{i} \right]_{s=0}^{s=x} \\ &= 1 + (e^{ix} - 1) = e^{ix}. \end{aligned}$$

Thus, Claim 2 is indeed true for  $n = 0$ . Next, by induction hypothesis, assume Claim 2 is true for  $n$ , and we verify that Claim 2 is also true for  $n + 1$ .

$$\begin{aligned} \text{RHS}(n+1) &= \sum_{k=0}^{n+1} \frac{(ix)^k}{k!} + \frac{i^{n+2}}{(n+1)!} \int_0^x (x-s)^{n+1} e^{is} ds \\ &= \sum_{k=0}^n \frac{(ix)^k}{k!} + \frac{(ix)^{n+1}}{(n+1)!} + \frac{i^{n+2}}{(n+1)!} \cdot \frac{n+1}{i} \left[ \int_0^x (x-s)^n e^{is} ds - \frac{x^{n+1}}{n+1} \right] \\ &= \sum_{k=0}^n \frac{(ix)^k}{k!} + \frac{i^{n+1}}{n!} \int_0^x (x-s)^n e^{is} ds + \frac{(ix)^{n+1}}{(n+1)!} - \frac{i^{n+1}}{n!} \cdot \frac{x^{n+1}}{n+1} = e^{ix}, \end{aligned}$$

where the second equality follows from Claim 1 and the last equality follows from the induction hypothesis (that Claim 2 holds for  $n$ ). This proves Claim 2.

**Claim 3:**

$$e^{ix} = \sum_{k=0}^n \frac{(ix)^k}{k!} + \frac{i^n}{(n-1)!} \int_0^x (x-s)^{n-1} (e^{is} - 1) ds, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 1.$$

Proof of Claim 3: By Claim 1, we have (replacing  $n$  with  $n - 1$ ):

$$\int_0^x (x-s)^{n-1} e^{is} ds = \frac{x^n}{n} + \frac{i}{n} \int_0^x (x-s)^n e^{is} ds, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 1.$$

Isolating the integral on the right-hand-side, we have:

$$\int_0^x (x-s)^n e^{is} ds = \frac{n}{i} \left[ \int_0^x (x-s)^{n-1} e^{is} ds - \frac{x^n}{n} \right], \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 1.$$

Next, note that, for any  $x \in \mathbb{R}$  and any  $n \geq 1$ ,

$$\int_0^x (x-s)^{n-1} ds = - \left[ \frac{(x-s)^n}{n} \right]_{s=0}^{s=x} = - \left[ 0 - \frac{x^n}{n} \right] = \frac{x^n}{n}$$

Hence, we have:

$$\begin{aligned} \int_0^x (x-s)^n e^{is} ds &= \frac{n}{i} \left[ \int_0^x (x-s)^{n-1} e^{is} ds - \frac{x^n}{n} \right], \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 1. \\ &= \frac{n}{i} \left[ \int_0^x (x-s)^{n-1} e^{is} ds - \int_0^x (x-s)^{n-1} ds \right], \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 1. \\ &= \frac{n}{i} \left[ \int_0^x (x-s)^{n-1} (e^{is} - 1) ds \right], \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 1. \end{aligned}$$

Substituting the above into the right-hand-side of Claim 2, we have:

$$\begin{aligned}
 e^{ix} &= \sum_{k=0}^n \frac{(ix)^k}{k!} + \frac{i^{n+1}}{n!} \int_0^x (x-s)^n e^{is} ds, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0 \\
 &= \sum_{k=0}^n \frac{(ix)^k}{k!} + \frac{i^{n+1}}{n!} \cdot \frac{n}{i} \cdot \left[ \int_0^x (x-s)^{n-1} (e^{is} - 1) ds \right], \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 1 \\
 &= \sum_{k=0}^n \frac{(ix)^k}{k!} + \frac{i^n}{(n-1)!} \cdot \left[ \int_0^x (x-s)^{n-1} (e^{is} - 1) ds \right], \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 1
 \end{aligned}$$

This proves Claim 3.

**Claim 4:**

$$\left| \int_0^x (x-s)^n e^{is} ds \right| \leq \frac{|x|^{n+1}}{n+1}, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 0.$$

Proof of Claim 4: First, consider  $x \geq 0$ , in which case, we have, for any  $n \geq 0$ ,

$$\left| \int_0^x (x-s)^n e^{is} ds \right| \leq \int_0^x |x-s|^n ds \leq \int_0^x (x-s)^n ds = \dots = \frac{x^{n+1}}{n+1} = \frac{|x|^{n+1}}{n+1}$$

Next, for  $x < 0$ , let  $y := -x > 0$ . Then,

$$\begin{aligned}
 \left| \int_0^x (x-s)^n e^{is} ds \right| &= \left| \int_0^{-y} (-y-s)^n e^{is} ds \right| = \left| \int_0^y (-y+t)^n e^{-it} dt \right| \\
 &\leq \int_0^y |y-t|^n dt = \int_0^y (y-t)^n dt = \dots = \frac{y^{n+1}}{n+1} = \frac{|x|^{n+1}}{n+1}
 \end{aligned}$$

This completes the proof Claim 4.

**Claim 5:**

$$\left| \int_0^x (x-s)^{n-1} (e^{is} - 1) ds \right| \leq \frac{2|x|^n}{n}, \quad \text{for any } x \in \mathbb{R} \text{ and any } n \geq 1.$$

Proof of Claim 5: First, consider  $x \geq 0$ , in which case, we have, for any  $n \geq 1$ ,

$$\left| \int_0^x (x-s)^{n-1} (e^{is} - 1) ds \right| \leq \int_0^x |(x-s)^{n-1} (e^{is} - 1)| ds \leq 2 \int_0^x (x-s)^{n-1} ds = \frac{2x^n}{n} = \frac{2|x|^n}{n},$$

where the second last equality follows from the simple calculation:

$$\int_0^x (x-s)^{n-1} ds = - \left[ \frac{(x-s)^n}{n} \right]_{s=0}^{s=x} = - \left[ 0 - \frac{x^n}{n} \right] = \frac{x^n}{n}.$$

Next, for  $x < 0$ , let  $y := -x > 0$ . Then,

$$\begin{aligned}
 \left| \int_0^x (x-s)^{n-1} (e^{is} - 1) ds \right| &= \left| \int_0^{-y} (-y-s)^{n-1} (e^{is} - 1) ds \right| = \left| - \int_0^y (-y+t)^{n-1} (e^{-it} - 1) dt \right| \\
 &\leq 2 \int_0^y |t-y|^{n-1} dt = 2 \int_0^y (y-t)^{n-1} dt = \frac{2y^n}{n} = \frac{2|x|^n}{n}.
 \end{aligned}$$

This completes the proof of Claim 5.

The proof of the Lemma now follows readily from the preceding Claims.

$$\begin{aligned}
 & \left| e^{ix} - \sum_{k=0}^n \frac{(ix)^k}{k!} \right| \\
 & \leq \min \left\{ \left| \frac{i^{n+1}}{n!} \int_0^x (x-s)^n e^{is} ds \right|, \left| \frac{i^n}{(n-1)!} \int_0^x (x-s)^{n-1} (e^{is} - 1) ds \right| \right\}, \text{ by Claims 2 and 3} \\
 & \leq \min \left\{ \frac{1}{n!} \cdot \frac{|x|^{n+1}}{n+1}, \frac{1}{(n-1)!} \cdot \frac{2|x|^n}{n} \right\}, \text{ by Claims 4 and 5} \\
 & \leq \min \left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right\}
 \end{aligned}$$

This completes the proof of the Lemma. □

**Lemma A.4 (§7.1, [2])**

Let  $\{\theta_{nj} \in \mathbb{C} \mid 1 \leq j \leq k_n, n \in \mathbb{N}\}$  be doubly indexed array of complex numbers. If all the following three conditions are true:

(a) there exists  $M > 0$  such that

$$\sum_{j=1}^{k_n} |\theta_{nj}| \leq M, \quad \text{for each } n \in \mathbb{N},$$

(b)  $\lim_{n \rightarrow \infty} \max_{1 \leq j \leq k_n} |\theta_{nj}| = 0$ , and

(c) there exists  $\theta \in \mathbb{C}$  such that

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \theta_{nj} = \theta,$$

then

$$\lim_{n \rightarrow \infty} \prod_{j=1}^{k_n} (1 + \theta_{nj}) = e^\theta.$$

PROOF First, note that hypothesis (b) immediately implies that there exists some  $n_0 \in \mathbb{N}$  such that

$$|\theta_{nj}| \leq \frac{1}{2}, \quad \text{for each } n \geq n_0, \text{ for each } 1 \leq j \leq k_n.$$

Thus, without loss of generality, we may assume that:

$$|\theta_{nj}| \leq \frac{1}{2}, \quad \text{for each } n \in \mathbb{N}, \text{ for each } 1 \leq j \leq k_n.$$

We denote by  $\log(1 + \theta_{nj})$  the (unique) complex logarithm<sup>1</sup> of  $1 + \theta_{nj}$  with argument in  $(-\pi, \pi]$ . Next, recall the MacLaurin Series for  $\log(1 + x)$ :

$$\log(1 + x) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{x^m}{m}, \quad \text{for any } x \in \mathbb{C} \text{ with } |x| < 1.$$

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<sup>1</sup>Recall that the complex exponential function is defined by  $\exp : \mathbb{C} \rightarrow \mathbb{C} : x + iy \mapsto e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$ . Clearly,  $\exp$  is not injective. More precisely, for  $x_1 + iy_1, x_2 + iy_2 \in \mathbb{C} \setminus \{0\}$ , we have  $e^{x_1 + iy_1} = e^{x_2 + iy_2}$  if and only if  $x_1 = x_2 \in \mathbb{R} \setminus \{0\}$  and  $y_1 - y_2 \in 2\pi\mathbb{Z}$ . For  $z = re^{i\theta} \in \mathbb{C} \setminus \{0\}$ , a complex logarithm of  $z$  is any  $w = x + iy \in \mathbb{C} \setminus \{0\}$  such that  $e^{x+iy} = e^w = z = re^{i\theta}$ , i.e.  $x = \log r$  and  $y = \theta + 2\pi\mathbb{Z}$ . In particular, let  $\mathcal{D} := \{x + iy \in \mathbb{C} \mid x \in \mathbb{R}, y \in (-\pi, \pi]\}$ . Then, the restriction  $\exp : \mathcal{D} \rightarrow \mathbb{C} \setminus \{0\}$  is bijective.

Hence, we have the following inequality: for each  $n \in \mathbb{N}$  and for each  $1 \leq j \leq k_n$ ,

$$\begin{aligned} |\log(1 + \theta_{nj}) - \theta_{nj}| &= \left| \sum_{m=2}^{\infty} (-1)^{n+1} \frac{(\theta_{nj})^m}{m} \right| \leq \sum_{m=2}^{\infty} \frac{|\theta_{nj}|^m}{m} \leq \frac{|\theta_{nj}|^2}{2} \sum_{m=2}^{\infty} |\theta_{nj}|^{m-2} \\ &\leq \frac{|\theta_{nj}|^2}{2} \sum_{m=2}^{\infty} \left(\frac{1}{2}\right)^{m-2} = \frac{|\theta_{nj}|^2}{2} \sum_{i=2}^{\infty} \left(\frac{1}{2}\right)^{i-2} = \frac{|\theta_{nj}|^2}{2} \cdot 2 = |\theta_{nj}|^2. \end{aligned}$$

This in turn implies: for each  $n \in \mathbb{N}$ ,

$$\left| \sum_{j=1}^{k_n} \log(1 + \theta_{nj}) - \sum_{j=1}^{k_n} \theta_{nj} \right| = \left| \sum_{j=1}^{k_n} (\log(1 + \theta_{nj}) - \theta_{nj}) \right| \leq \sum_{j=1}^{k_n} |\log(1 + \theta_{nj}) - \theta_{nj}| \leq \sum_{j=1}^{k_n} |\theta_{nj}|^2.$$

Thus, for each  $n \in \mathbb{N}$ , there exists  $\Lambda_n \in \mathbb{C}$  with  $|\Lambda_n| \leq 1$  such that

$$\sum_{j=1}^{k_n} \log(1 + \theta_{nj}) = \sum_{j=1}^{k_n} \theta_{nj} + \Lambda_n \cdot \sum_{j=1}^{k_n} |\theta_{nj}|^2.$$

(Since for any  $z \in \mathbb{C}$ ,  $|z| \leq A \implies z = A \cdot w$ , for some  $w \in \mathbb{C}$  with  $|w| \leq 1$ .) Next note that, hypotheses (a) and (b) together imply:

$$\sum_{j=1}^{k_n} |\theta_{nj}|^2 \leq \left( \max_{1 \leq j \leq k_n} |\theta_{nj}| \right) \left( \sum_{j=1}^{k_n} |\theta_{nj}| \right) \leq M \cdot \left( \max_{1 \leq j \leq k_n} |\theta_{nj}| \right) \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Therefore, since  $|\Lambda_n| \leq 1$  for each  $n \in \mathbb{N}$ , we now see that

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \log(1 + \theta_{nj}) = \lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \theta_{nj} + \lim_{n \rightarrow \infty} \left( \Lambda_n \cdot \sum_{j=1}^{k_n} |\theta_{nj}|^2 \right) = \theta + 0 = \theta.$$

We may now conclude, by continuity of the exponential function  $\exp(\cdot)$ :

$$\begin{aligned} \lim_{n \rightarrow \infty} \prod_{j=1}^{k_n} (1 + \theta_{nj}) &= \lim_{n \rightarrow \infty} \exp \left( \log \prod_{j=1}^{k_n} (1 + \theta_{nj}) \right) = \lim_{n \rightarrow \infty} \exp \left( \sum_{j=1}^{k_n} \log(1 + \theta_{nj}) \right) \\ &= \exp \left( \lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \log(1 + \theta_{nj}) \right) = \exp(\theta) \end{aligned}$$

This completes the proof of the Lemma. □

## B The Central Limit Theorems

**Theorem B.1 (Lindeberg's Central Limit Theorem, Theorem 1.15, [3])**

*Suppose:*

- $\{k_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$  is a sequence of natural numbers such that  $k_n \rightarrow \infty$  as  $n \rightarrow \infty$ , and
- for each  $n \in \mathbb{N}$ ,  $X_1^{(n)}, X_2^{(n)}, \dots, X_{k_n}^{(n)} : \Omega_n \rightarrow \mathbb{R}$  are independent (but not necessarily identically distributed)  $\mathbb{R}$ -valued random variables defined on a common probability space  $(\Omega_n, \mathcal{A}_n, \mu_n)$  such that

$$\mu_j^{(n)} := E[X_j^{(n)}] \in \mathbb{R} \text{ exists, for each } 1 \leq j \leq k_n, \text{ and } 0 < \sigma_n^2 := \text{Var} \left[ \sum_{j=1}^{k_n} X_j^{(n)} \right] < \infty.$$

Then, Lindeberg's condition implies

$$\frac{1}{\sigma_n} \sum_{j=1}^{k_n} \left( X_j^{(n)} - \mu_j^{(n)} \right) \xrightarrow{\mathcal{L}} N(0, 1),$$

where  $N(0, 1)$  denotes the standard Gaussian distribution on  $\mathbb{R}$ , and **Lindeberg's condition** is the following condition:

$$\lim_{n \rightarrow \infty} \frac{1}{\sigma_n^2} \sum_{j=1}^{k_n} E \left[ \left( X_j^{(n)} - \mu_j^{(n)} \right)^2 \cdot I_{\{|X_j^{(n)} - \mu_j^{(n)}| \geq \varepsilon \sigma_n\}} \right] = 0, \text{ for each } \varepsilon > 0.$$

PROOF Considering  $\left( X_j^{(n)} - \mu_j^{(n)} \right) / \sigma_n$ , we may assume, without loss of generality, that

$$E \left[ X_j^{(n)} \right] = 0, \quad \text{and} \quad \sigma_n^2 := \text{Var} \left[ \sum_{j=1}^{k_n} X_j^{(n)} \right] = 1.$$

Let  $t \in \mathbb{R}$  be fixed. Let  $\sigma_{nj}^2 := \text{Var} \left[ X_j^{(n)} \right]$ . Note that  $\sum_{j=1}^{k_n} \sigma_{nj}^2 = \sigma_n^2 = 1$ .

**Claim 1:**

$$\lim_{n \rightarrow \infty} \max_{1 \leq j \leq k_n} \{ \sigma_{nj}^2 \} = 0.$$

Proof of Claim 1: First, note that, for an arbitrary  $\varepsilon > 0$ , we have

$$\begin{aligned} 0 \leq \sigma_{nj}^2 &= \text{Var} \left[ X_j^{(n)} \right] = \int x^2 d\mu_{X_j^{(n)}}(x) \leq \int_{\{|X_j^{(n)}| < \varepsilon\}} x^2 d\mu_{X_j^{(n)}}(x) + \int_{\{|X_j^{(n)}| \geq \varepsilon\}} x^2 d\mu_{X_j^{(n)}}(x) \\ &\leq \varepsilon^2 + E \left[ \left( X_j^{(n)} \right)^2 \cdot I_{\{|X_j^{(n)}| \geq \varepsilon\}} \right] \leq \varepsilon^2 + \sum_{i=1}^{k_n} E \left[ \left( X_i^{(n)} \right)^2 \cdot I_{\{|X_i^{(n)}| \geq \varepsilon\}} \right]. \end{aligned}$$

It follows that

$$0 \leq \max_{1 \leq j \leq k_n} \{ \sigma_{nj}^2 \} \leq \varepsilon^2 + \sum_{i=1}^{k_n} E \left[ \left( X_i^{(n)} \right)^2 \cdot I_{\{|X_i^{(n)}| \geq \varepsilon\}} \right].$$

Hence, Lindeberg's condition implies:

$$0 \leq \limsup_{n \rightarrow \infty} \max_{1 \leq j \leq k_n} \{ \sigma_{nj}^2 \} \leq \varepsilon^2 + \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} E \left[ \left( X_i^{(n)} \right)^2 \cdot I_{\{|X_i^{(n)}| \geq \varepsilon\}} \right] = \varepsilon^2.$$

Since  $\varepsilon > 0$  is arbitrary, we see that:

$$0 \leq \limsup_{n \rightarrow \infty} \max_{1 \leq j \leq k_n} \{ \sigma_{nj}^2 \} = 0,$$

which implies

$$\lim_{n \rightarrow \infty} \max_{1 \leq j \leq k_n} \{ \sigma_{nj}^2 \} = 0$$

This proves Claim 1.

**Claim 2:** For each sufficiently large  $n \in \mathbb{N}$ , we have:

$$0 \leq 1 - \frac{t^2 \sigma_{nj}^2}{2} \leq 1, \text{ for each } 1 \leq j \leq k_n.$$

Proof of Claim 2: Recall that  $t \in \mathbb{R}$  is fixed in this argument. Hence, Claim 2 follows immediately from Claim 1.

**Claim 3:** For each sufficiently large  $n \in \mathbb{N}$ , we have:

$$\left| \prod_{j=1}^{k_n} \varphi_{X_j^{(n)}}(t) - \prod_{j=1}^{k_n} \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| \leq \sum_{j=1}^{k_n} \left| \varphi_{X_j^{(n)}}(t) - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right|.$$

Proof of Claim 3: This follows immediately from Lemma A.2, Claim 2, and the fact that characteristic functions of  $\mathbb{R}$ -valued random variables (or probability measures defined on  $\mathbb{R}$ ) always map into the closed unit disk in the complex plane.

**Claim 4:** For each sufficiently large  $n \in \mathbb{N}$ , we have:

$$\left| \prod_{j=1}^{k_n} e^{-t^2 \sigma_{nj}^2 / 2} - \prod_{j=1}^{k_n} \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| \leq \sum_{j=1}^{k_n} \left| e^{-t^2 \sigma_{nj}^2 / 2} - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right|.$$

Proof of Claim 4: This follows immediately from Lemma A.2, Claim 2, and the fact that  $\exp\{(-\infty, 0]\} \subset [0, 1]$ .

**Claim 5:**

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \left| \varphi_{X_j^{(n)}}(t) - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| = 0.$$

Proof of Claim 5: By Lemma A.3, we have

$$\left| e^{itx} - \left( 1 + itx - \frac{t^2 x^2}{2} \right) \right| \leq \min\{|tx|^2, |tx|^3\}.$$

Next, note that

$$\begin{aligned} \int e^{itx} - \left( 1 + itx - \frac{t^2 x^2}{2} \right) d\mu_{X_j^{(n)}}(x) &= \int e^{itx} d\mu_{X_j^{(n)}}(x) - \int \left( 1 + itx - \frac{t^2 x^2}{2} \right) d\mu_{X_j^{(n)}}(x) \\ &= \varphi_{X_j^{(n)}}(t) - \left( 1 + it \cdot E[X_j^{(n)}] - \frac{t^2}{2} \cdot E[(X_j^{(n)})^2] \right) \\ &= \varphi_{X_j^{(n)}}(t) - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \end{aligned}$$

Hence, for an arbitrary  $\epsilon > 0$ ,

$$\begin{aligned} \left| \varphi_{X_j^{(n)}}(t) - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| &= \left| \int e^{itx} - \left( 1 + itx - \frac{t^2 x^2}{2} \right) d\mu_{X_j^{(n)}}(x) \right| \leq \int \left| e^{itx} - \left( 1 + itx - \frac{t^2 x^2}{2} \right) \right| d\mu_{X_j^{(n)}}(x) \\ &\leq \int \min\{|tx|^2, |tx|^3\} d\mu_{X_j^{(n)}}(x) \\ &= \int_{\{|X_j^{(n)}| < \epsilon\}} \min\{|tx|^2, |tx|^3\} d\mu_{X_j^{(n)}}(x) + \int_{\{|X_j^{(n)}| \geq \epsilon\}} \min\{|tx|^2, |tx|^3\} d\mu_{X_j^{(n)}}(x) \\ &\leq \int_{\{|X_j^{(n)}| < \epsilon\}} |tx|^3 d\mu_{X_j^{(n)}}(x) + \int_{\{|X_j^{(n)}| \geq \epsilon\}} |tx|^2 d\mu_{X_j^{(n)}}(x) \\ &\leq \epsilon |t|^3 \int_{\{|X_j^{(n)}| < \epsilon\}} |x|^2 d\mu_{X_j^{(n)}}(x) + |t|^2 \int_{\{|X_j^{(n)}| \geq \epsilon\}} |x|^2 d\mu_{X_j^{(n)}}(x) \\ &\leq \epsilon |t|^3 \cdot \sigma_{nj}^2 + |t|^2 \cdot E\left[ (X_j^{(n)})^2 \cdot I_{\{|X_j^{(n)}| \geq \epsilon\}} \right] \end{aligned}$$

Thus, for an arbitrary  $\epsilon > 0$ ,

$$\sum_{j=1}^{k_n} \left| \varphi_{X_j^{(n)}}(t) - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| \leq \epsilon |t|^3 \cdot \sum_{j=1}^{k_n} \sigma_{nj}^2 + |t|^2 \cdot \sum_{j=1}^{k_n} E\left[ (X_j^{(n)})^2 \cdot I_{\{|X_j^{(n)}| \geq \epsilon\}} \right].$$

Recall that  $t \in \mathbb{R}$  is fixed and  $\sum_{j=1}^{k_n} \sigma_{nj}^2 = 1$ . Lindeberg's condition therefore implies:

$$0 \leq \limsup_{n \rightarrow \infty} \sum_{j=1}^{k_n} \left| \varphi_{X_j^{(n)}}(t) - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| \leq \varepsilon |t|^3 + |t|^2 \cdot \lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} E \left[ \left( X_j^{(n)} \right)^2 \cdot I_{\{|X_j^{(n)}| \geq \varepsilon\}} \right] = \varepsilon |t|^3.$$

Since  $\varepsilon > 0$  is arbitrary, we have

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \left| \varphi_{X_j^{(n)}}(t) - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| = 0.$$

This proves Claim 5.

**Claim 6:**

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \left| e^{-t^2 \sigma_{nj}^2 / 2} - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| = 0$$

Proof of Claim 6: Since  $t \in \mathbb{R}$  is fixed, by Claim 1, we have that, for each sufficiently large  $n \in \mathbb{N}$ ,

$$\left| \frac{t^2 \sigma_{nj}^2}{2} \right| \leq \frac{1}{2}, \quad \text{for each } 1 \leq j \leq k_n.$$

Thus, Lemma A.1 implies that, for each sufficiently large  $n \in \mathbb{N}$ ,

$$\left| e^{-t^2 \sigma_{nj}^2 / 2} - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| = \left| e^{-t^2 \sigma_{nj}^2 / 2} - 1 - \left( -\frac{t^2 \sigma_{nj}^2}{2} \right) \right| \leq \left| \frac{t^2 \sigma_{nj}^2}{2} \right|^2 \leq t^4 \sigma_{nj}^4$$

Summing over  $j$ , we have: for each sufficiently large  $n$ ,

$$\begin{aligned} 0 \leq \sum_{j=1}^{k_n} \left| e^{-t^2 \sigma_{nj}^2 / 2} - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| &\leq t^4 \cdot \sum_{j=1}^{k_n} \sigma_{nj}^4 \\ &\leq t^4 \cdot \sum_{j=1}^{k_n} \left( \sigma_{nj}^2 \cdot \max_{1 \leq i \leq k_n} \{ \sigma_{ni}^2 \} \right) \\ &= t^4 \cdot \left( \sum_{j=1}^{k_n} \sigma_{nj}^2 \right) \left( \max_{1 \leq i \leq k_n} \{ \sigma_{ni}^2 \} \right) \\ &= t^4 \cdot \left( \max_{1 \leq i \leq k_n} \{ \sigma_{ni}^2 \} \right). \end{aligned}$$

Claim 1 now implies

$$0 \leq \limsup_{n \rightarrow \infty} \sum_{j=1}^{k_n} \left| e^{-t^2 \sigma_{nj}^2 / 2} - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| = t^4 \cdot \lim_{n \rightarrow \infty} \left( \max_{1 \leq i \leq k_n} \{ \sigma_{ni}^2 \} \right) = 0,$$

which in turn implies

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{k_n} \left| e^{-t^2 \sigma_{nj}^2 / 2} - \left( 1 - \frac{t^2 \sigma_{nj}^2}{2} \right) \right| = 0.$$

This proves Claim 6. □

## References

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