Multi-stage Sampling

Study Notes January 11, 2015 Kenneth Chu

1 Variance estimation for multi-stage sampling

Let U be a finite population and $\mathcal{P}(U)$ the power set of U. Let $p : \mathcal{S} \longrightarrow (0,1]$ be a r-stage sampling design ($r \geq 2$), where $\mathcal{S} \subset \mathcal{P}(U)$ is the set of all admissible samples under the design p. We express the hierarchical structure of the population U, with respect to the r-stage design p, as follows:

$$U \;=\; \bigsqcup_{i \in U^{(1)}} U_i^{(2+)} \;=\; \bigsqcup_{i \in U^{(1)}} \bigsqcup_{a \in U_i^{(2)}} U_{ia}^{(3)} \;=\; \cdots \;=\; \bigsqcup_{i \in U^{(1)}} \cdots \bigsqcup_{b \in U_{i\cdots}^{(r-1)}} U_{i\cdots b}^{(r)} \tag{1.1}$$

where $U^{(1)}$ is the set of all primary sampling units (PSU), and for each PSU $i \in U^{(1)}$, $U_i^{(2)}$ denotes the set of all secondary sampling units (SSU) contained in $i \in U^{(1)}$, and for each SSU $a \in U_i^{(2)}$, $U_{ia}^{(3)}$ denotes the set of all tertiary sampling units (TSU) contained in $a \in U_i^{(2)}$, and so on. Similarly, we express the hierachical structure of every admissible sample $s \in \mathcal{S}$ as follows:

$$s \;=\; \bigsqcup_{i \in s^{(1)}} s_i^{(2+)} \;=\; \bigsqcup_{i \in s^{(1)}} \bigsqcup_{a \in s_i^{(2)}} s_{ia}^{(3)} \;=\; \cdots \tag{1.2}$$

Let $y : U \longrightarrow \mathbb{R}$ be a population characteristic. Let T be its population total, i.e.

$$T \;=\; \sum_{u \in U} y_u \;=\; \sum_{i \in U^{(1)}} T_i \;=\; \sum_{i \in U^{(1)}} \sum_{a \in U_{ia}^{(2)}} T_{ia} \;=\; \cdots \;=\; \sum_{i \in U^{(1)}} \cdots \sum_{u \in U_{i\cdots}^{(r)}} y_u \tag{1.3}$$

Theorem 1.1
If $\widehat{T}_i : \mathcal{S}_i^{(2+)} \longrightarrow \mathbb{R}$ is an unbiased estimator for T_i, for each PSU *$i \in U^{(1)}$, then the random variable $\widehat{T} : \mathcal{S} \longrightarrow \mathbb{R}$ defined as follows:*

$$\widehat{T}(s) \;:=\; \sum_{i \in s^{(1)}} \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \tag{1.4}$$

is a design-unbiased estimator for

$$T \;=\; \sum_{i \in U^{(1)}} T_i$$

If the r-stage sampling design has invariant and independent subsampling, then the design-variance of $\widehat{T}$ is given by:

$$\mathrm{Var}\Big[\,\widehat{T}\,\Big] \;=\; \underbrace{\mathrm{Var}^{(1)}\Big(E^{(2+)}\big(\,\widehat{T}\,\big|\,s^{(1)}\big)\Big)}_{\mathrm{V_{PSU}}} \;+\; \underbrace{E^{(1)}\Big(\mathrm{Var}^{(2+)}\big(\,\widehat{T}\,\big|\,s^{(1)}\big)\Big)}_{\mathrm{V_{subsampling}}}, \tag{1.5}$$

where

$$\begin{aligned}
\mathrm{Var}^{(1)}\Big(E^{(2+)}\big(\,\widehat{T}\,\big|\,s^{(1)}\big)\Big) \;&=\; \sum_{i \in U^{(1)}} \sum_{j \in U^{(1)}} \Delta_{ij}^{(1)} \frac{T_i}{\pi_i^{(1)}} \frac{T_j}{\pi_j^{(1)}}, \quad \text{and} \\
E^{(1)}\Big(\mathrm{Var}^{(2+)}\big(\,\widehat{T}\,\big|\,s^{(1)}\big)\Big) \;&=\; \sum_{i \in U^{(1)}} \frac{V_i}{\pi_i^{(1)}},
\end{aligned}$$

with

$$V_i \;:=\; \mathrm{Var}^{(2+)}\Big[\,\widehat{T}_i\,\Big] \quad \text{and} \quad \Delta_{ij}^{(1)} \;:=\; \begin{cases} \pi_i^{(1)}\left(1 - \pi_i^{(1)}\right), & \text{if } i = j \\[2ex] \pi_{ij}^{(1)} - \pi_i^{(1)}\pi_j^{(1)}, & \text{if } i \neq j \end{cases} \tag{1.6}$$

Furthermore, if $\widehat{V}_i : \mathcal{S}_i^{(2+)} \longrightarrow \mathbb{R}$ *is an unbiased estimator for* $V_i := \mathrm{Var}^{(2+)}\left[\,\widehat{T}_i\,\right]$*, and* $\pi_i^{(1)} > 0$*,* $\pi_{ij}^{(1)} > 0$ *for any* PSUs $i, j \in U^{(1)}$*, then*

$$\begin{aligned}
\widehat{\mathrm{Var}}\left[\,\widehat{T}\,\right](s) &:= \sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\cdot\frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} + \sum_{i\in s^{(1)}}\frac{\widehat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}}, \\
\widehat{\mathrm{Var}}^{(1)}\left[\,\widehat{T}\,\right](s) &:= \sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\cdot\frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}} + \sum_{i\in s^{(1)}}\left(1-\frac{1}{\pi_i^{(1)}}\right)\frac{\widehat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}}, \ \text{and} \\
\widehat{\mathrm{Var}}^{(2+)}\left[\,\widehat{T}\,\right](s) &:= \sum_{i\in s^{(1)}}\frac{\widehat{V}_i(s_i^{(2+)})}{\left(\pi_i^{(1)}\right)^2}
\end{aligned}$$

are unbiased estimators for $\mathrm{Var}\left[\,\widehat{T}\,\right]$, $\mathrm{V}_{\mathrm{PSU}} := \mathrm{Var}^{(1)}\left(E^{(2+)}\left(\,\widehat{T}\,\middle|\,s^{(1)}\right)\right)$*, and* $\mathrm{V}_{\mathrm{subsampling}} := E^{(1)}\left(\mathrm{Var}^{(2+)}\left(\,\widehat{T}\,\middle|\,s^{(1)}\right)\right)$*, respectively.*

Corollary 1.2

$$\widehat{\mathrm{Var}}^{(1)}\left[\,\widehat{T}\,\right](s) \;=\; \widehat{\mathrm{Var}}\left[\,\widehat{T}\,\right](s) \;-\; \widehat{\mathrm{Var}}^{(2+)}\left[\,\widehat{T}\,\right](s) \tag{1.7}$$

Proof

$$\begin{aligned}
\mathrm{Var}^{(1)}\left[E^{(2+)}\left[\,\widehat{T}\,\middle|\,s^{(1)}\,\right]\right] &= \mathrm{Var}^{(1)}\left[E^{(2+)}\left[\,\sum_{i\in s^{(1)}}\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\,\middle|\,s^{(1)}\,\right]\right] \\
&= \mathrm{Var}^{(1)}\left[\sum_{i\in s^{(1)}}\frac{E^{(2+)}\left[\,\widehat{T}_i(s_i^{(2+)})\,\right]}{\pi_i^{(1)}}\right] \\
&= \mathrm{Var}^{(1)}\left[\sum_{i\in s^{(1)}}\frac{T_i}{\pi_i^{(1)}}\right] \\
&= \sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}}\Delta_{ij}^{(1)}\frac{T_i}{\pi_i^{(1)}}\frac{T_j}{\pi_j^{(1)}}
\end{aligned}$$

$$\begin{aligned}
E^{(1)}\left[\mathrm{Var}^{(2+)}\left[\,\widehat{T}\,\middle|\,s^{(1)}\,\right]\right] &= E^{(1)}\left[\mathrm{Var}^{(2+)}\left[\,\sum_{i\in s^{(1)}}\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\,\middle|\,s^{(1)}\,\right]\right] \\
&= E^{(1)}\left[\sum_{i\in s^{(1)}}\mathrm{Var}^{(2+)}\left[\,\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\,\right]\right] \\
&= E^{(1)}\left[\sum_{i\in s^{(1)}}\frac{\mathrm{Var}^{(2+)}\left[\widehat{T}_i(s_i^{(2+)})\,\right]}{\left(\pi_i^{(1)}\right)^2}\right] \\
&= E^{(1)}\left[\sum_{i\in s^{(1)}}\frac{V_i/\pi_i^{(1)}}{\pi_i^{(1)}}\right] \\
&= \sum_{i\in U^{(1)}}\frac{V_i}{\pi_i^{(1)}}
\end{aligned}$$

$$
\begin{aligned}
E\left(\widehat{\mathrm{Var}}^{(2+)}\left(\widehat{T}\right)\right) &= E\left(\sum_{i\in s^{(1)}}\frac{\widehat{V}_i(s_i^{(2+)})}{\left(\pi_i^{(1)}\right)^2}\right) \;=\; E^{(1)}\left(E^{(2+)}\left(\sum_{i\in s^{(1)}}\frac{\widehat{V}_i(s_i^{(2+)})/\pi_i^{(1)}}{\pi_i^{(1)}}\,\middle|\,s^{(1)}\right)\right) \\
&= E^{(1)}\left(\sum_{i\in s^{(1)}}\frac{E^{(2+)}\left[\,\widehat{V}_i(s_i^{(2+)})\,\middle|\,s^{(1)}\right]\Big/\pi_i^{(1)}}{\pi_i^{(1)}}\right) \;=\; E^{(1)}\left(\sum_{i\in s^{(1)}}\frac{V_i/\pi_i^{(1)}}{\pi_i^{(1)}}\right) \;=\; \sum_{i\in U^{(1)}}\frac{V_i}{\pi_i^{(1)}} \\
&= E^{(1)}\left(\mathrm{Var}^{(2+)}\left(\widehat{T}\,\middle|\,s^{(1)}\right)\right) \;=\; \mathrm{V}_{\mathrm{PSU}}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E\left(\sum_{i\in s^{(1)}}\frac{\widehat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}}\right) &= E^{(1)}\left(E^{(2+)}\left(\sum_{i\in s^{(1)}}\frac{\widehat{V}_i(s_i^{(2+)})}{\pi_i^{(1)}}\,\middle|\,s^{(1)}\right)\right) \;=\; E^{(1)}\left(\sum_{i\in s^{(1)}}\frac{E^{(2+)}\left[\,\widehat{V}_i(s_i^{(2+)})\,\middle|\,s^{(1)}\right]}{\pi_i^{(1)}}\right) \\
&= E^{(1)}\left(\sum_{i\in s^{(1)}}\frac{V_i}{\pi_i^{(1)}}\right) \;=\; \sum_{i\in U^{(1)}} V_i
\end{aligned}
$$

Next, observe that

$$
\begin{aligned}
E\left[\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}}\frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\cdot\frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}}\right] &= E^{(1)}\left(E^{(2+)}\left(\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}}\frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\cdot\frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}}\,\middle|\,s^{(1)}\right)\right) \\
&= E^{(1)}\left(\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}}\frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{E^{(2+)}\left[\,\widehat{T}_i(s_i^{(2+)})\cdot\widehat{T}_j(s_j^{(2+)})\,\middle|\,s^{(1)}\right]}{\pi_i^{(1)}\,\pi_j^{(1)}}\right)
\end{aligned}
$$

Now, observe (the key technical observation) that

$$
E^{(2+)}\left[\,\widehat{T}_i(s_i^{(2+)})\cdot\widehat{T}_j(s_j^{(2+)})\,\middle|\,s^{(1)}\right] \;=\; E^{(2+)}\left[\,\widehat{T}_i(s_i^{(2+)})\cdot\widehat{T}_j(s_j^{(2+)})\right] \;=\; \begin{cases}\mathrm{Var}^{(2+)}\left(\widehat{T}_i\right)+E^{(2+)}\left(\widehat{T}_i\right)^2, & \text{if } i=j,\\ E^{(2+)}\left(\widehat{T}_i\right)\cdot E^{(2+)}\left(\widehat{T}_i\right), & \text{if } i\neq j\end{cases}
$$

Hence,

$$
\begin{aligned}
E\left[\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}}\frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}}\cdot\frac{\widehat{T}_j(s_j^{(2+)})}{\pi_j^{(1)}}\right] &= E^{(1)}\left(\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}}\frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{E^{(2+)}\left[\,\widehat{T}_i(s_i^{(2+)})\cdot\widehat{T}_j(s_j^{(2+)})\,\middle|\,s^{(1)}\right]}{\pi_i^{(1)}\,\pi_j^{(1)}}\right) \\
&= E^{(1)}\left(\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}}\frac{\Delta_{ij}^{(1)}}{\pi_{ij}^{(1)}}\cdot\frac{T_i}{\pi_i^{(1)}}\cdot\frac{Tj}{\pi_j^{(1)}}+\sum_{i\in s^{(1)}}\frac{(1-\pi_i^{(1)})\,V_i\Big/\pi_i^{(1)}}{\pi_i^{(1)}}\right) \\
&= \sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}}\Delta_{ij}^{(1)}\cdot\frac{T_i}{\pi_i^{(1)}}\cdot\frac{Tj}{\pi_j^{(1)}}+\sum_{i\in U^{(1)}}\frac{(1-\pi_i^{(1)})}{\pi_i^{(1)}}\cdot V_i \\
&= \sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}}\Delta_{ij}^{(1)}\cdot\frac{T_i}{\pi_i^{(1)}}\cdot\frac{Tj}{\pi_j^{(1)}}+\sum_{i\in U^{(1)}}\frac{V_i}{\pi_i^{(1)}}-\sum_{i\in U^{(1)}}V_i
\end{aligned}
$$

We may now establish that

$$\begin{aligned}
E\Big[\,\widehat{\mathrm{Var}}\Big(\widehat{T}\Big)\,\Big] &= E\left[\sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}}\frac{\Delta^{(1)}_{ij}}{\pi^{(1)}_{ij}}\cdot\frac{\widehat{T}_i(s^{(2+)}_i)}{\pi^{(1)}_i}\cdot\frac{\widehat{T}_j(s^{(2+)}_j)}{\pi^{(1)}_j} \;+\; \sum_{i\in s^{(1)}}\frac{\widehat{V}_i(s^{(2+)}_i)}{\pi^{(1)}_i}\right] \\
&= \left\{\sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}}\Delta^{(1)}_{ij}\cdot\frac{T_i}{\pi^{(1)}_i}\cdot\frac{Tj}{\pi^{(1)}_j} \;+\; \sum_{i\in U^{(1)}}\frac{V_i}{\pi^{(1)}_i} \;-\; \sum_{i\in U^{(1)}}V_i\right\} \;+\; \left\{\sum_{i\in U^{(1)}}V_i\right\} \\
&= \sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}}\Delta^{(1)}_{ij}\cdot\frac{T_i}{\pi^{(1)}_i}\cdot\frac{Tj}{\pi^{(1)}_j} \;+\; \sum_{i\in U^{(1)}}\frac{V_i}{\pi^{(1)}_i} \\
&= \mathrm{Var}\Big(\widehat{T}\Big)
\end{aligned}$$

Lastly, note that

$$\widehat{\mathrm{Var}}^{(1)}\Big[\widehat{T}\Big](s) \;=\; \widehat{\mathrm{Var}}\Big[\widehat{T}\Big](s) \;-\; \widehat{\mathrm{Var}}^{(2+)}\Big[\widehat{T}\Big](s)$$

Hence,

$$\begin{aligned}
E\Big[\,\widehat{\mathrm{Var}}^{(1)}\Big(\widehat{T}\Big)\Big] &= E\Big[\,\widehat{\mathrm{Var}}\Big(\widehat{T}\Big)\Big] \;-\; E\Big[\,\widehat{\mathrm{Var}}^{(2+)}\Big(\widehat{T}\Big)\Big] \\
&= \mathrm{Var}\Big[\,\widehat{T}\,\Big] \;-\; E^{(1)}\Big(\mathrm{Var}^{(2+)}(\,\widehat{T}\,\Big|\,s^{(1)})\Big) \\
&= \mathrm{Var}^{(1)}\Big(E^{(2+)}(\,\widehat{T}\,\Big|\,s^{(1)})\Big) \;+\; E^{(1)}\Big(\mathrm{Var}^{(2+)}(\,\widehat{T}\,\Big|\,s^{(1)})\Big) \;-\; E^{(1)}\Big(\mathrm{Var}^{(2+)}(\,\widehat{T}\,\Big|\,s^{(1)})\Big) \\
&= \mathrm{Var}^{(1)}\Big(E^{(2+)}(\,\widehat{T}\,\Big|\,s^{(1)})\Big) \;=\; \mathrm{V}_{\mathrm{subsampling}}
\end{aligned}$$

$\square$

2 Variance estimation for three-stage sampling

Let U be a finite population and $\mathcal{P}(U)$ the power set of U. Let $p:\mathcal{S}\longrightarrow(0,1]$ be a three-stage sampling design, where $\mathcal{S}\subset\mathcal{P}(U)$ is the set of all admissible samples under the design p. We express the three-stage structure of the population U, with respect to the three-stage design p, as follows:

$$U \;=\; \bigsqcup_{i\in U^{(1)}} U^{(2+)}_i \;=\; \bigsqcup_{i\in U^{(1)}}\bigsqcup_{a\in U^{(2)}_i} U^{(3)}_{ia}, \tag{2.1}$$

where $U^{(1)}$ is the set of all primary sampling units (PSU), and for each PSU $i\in U^{(1)}$, $U^{(2)}_i$ denotes the set of all secondary sampling units (SSU) contained in $i\in U^{(1)}$, and for each SSU $a\in U^{(2)}_i$, $U^{(3)}_{ia}$ denotes the set of all tertiary sampling units (TSU) contained in $a\in U^{(2)}_i$. Similarly, we express the three-stage structure of every admissible sample $s\in\mathcal{S}$ as follows:

$$s \;=\; \bigsqcup_{i\in s^{(1)}} s^{(2+)}_i \;=\; \bigsqcup_{i\in s^{(1)}}\bigsqcup_{a\in s^{(2)}_i} s^{(3)}_{ia} \tag{2.2}$$

Let $y:U\longrightarrow\mathbb{R}$ be a population characteristic. Let T be its population total, i.e.

$$T \;=\; \sum_{u\in U} y_u \;=\; \sum_{i\in U^{(1)}} T_i \;=\; \sum_{i\in U^{(1)}}\sum_{a\in U^{(2)}_{ia}} T_{ia} \;=\; \sum_{i\in U^{(1)}}\sum_{a\in U^{(2)}_i}\sum_{u\in U^{(3)}_{ia}} y_u \tag{2.3}$$

Theorem 2.1
For a three-stage sampling design with invariant and independent subsampling,

1. *The random variable* $\widehat{T} : \mathcal{S} \longrightarrow \mathbb{R}$*, defined as follows:*

$$\widehat{T}(s) \;:=\; \sum_{i\in s^{(1)}} \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \;=\; \sum_{i\in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a\in s_i^{(2)}} \frac{\widehat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \right) \;=\; \sum_{i\in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a\in s_i^{(2)}} \frac{1}{\pi_{i|a}^{(2)}} \left(\sum_{u\in s_{ia}^{(3)}} \frac{y_u}{\pi_{ia|u}^{(3)}} \right)\right)$$

is a design-unbiased estimator for T*, i.e.* $E\left[\,\widehat{T}\,\right] = T$.

2. *If the three-stage sampling design has invariant and independent subsampling, then the design-variance of* $\widehat{T}$ *can be given by:*

$$\begin{aligned}
&\mathrm{Var}\left[\,\widehat{T}\,\right] \\
=\;& \mathrm{Var}^{(1)}\Big(E^{(2+)}\Big(\,\widehat{T}\;\Big|\;s^{(1)}\Big)\Big) \;+\; E^{(1)}\Big(\mathrm{Var}^{(2+)}\Big(\,\widehat{T}\;\Big|\;s^{(1)}\Big)\Big) \\
=\;& \mathrm{Var}^{(1)}\Big(E^{(2+)}\Big(\,\widehat{T}\;\Big|\;s^{(1)}\Big)\Big) \;+\; E^{(1)}\Big\{\mathrm{Var}^{(2)}\Big(E^{(3)}\Big(\widehat{T}\;\Big|\;s^{(2)}\Big)\;\Big|\;s^{(1)}\Big) + E^{(2)}\Big(\mathrm{Var}^{(3)}\Big(\widehat{T}\;\Big|\;s^{(2)}\Big)\;\Big|\;s^{(1)}\Big)\Big\} \\
=\;& \underbrace{\mathrm{Var}^{(1)}\Big(E^{(2+)}\Big(\,\widehat{T}\;\Big|\;s^{(1)}\Big)\Big)}_{\mathrm{V_{PSU}}} \;+\; \underbrace{E^{(1)}\Big\{\mathrm{Var}^{(2)}\Big(E^{(3)}\Big(\widehat{T}\;\Big|\;s^{(2)}\Big)\;\Big|\;s^{(1)}\Big)\Big\}}_{\mathrm{V_{SSU}}} + \underbrace{E^{(1)}\Big\{E^{(2)}\Big(\mathrm{Var}^{(3)}\Big(\widehat{T}\;\Big|\;s^{(2)}\Big)\;\Big|\;s^{(1)}\Big)\Big\}}_{\mathrm{V_{TSU}}}
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{V_{PSU}} \;&:=\; \mathrm{Var}^{(1)}\Big(E^{(2+)}\Big(\,\widehat{T}\;\Big|\;s^{(1)}\,\Big)\Big) \;=\; \sum_{i\in U^{(1)}}\sum_{j\in U^{(1)}} \Delta_{ij}^{(1)} \frac{T_i}{\pi_i^{(1)}}\frac{T_j}{\pi_j^{(1)}}, \\
\mathrm{V_{SSU}} \;&:=\; E^{(1)}\Big\{\mathrm{Var}^{(2)}\Big(\,E^{(3)}(\,\widehat{T}\;\Big|\;s^{(2)}\,)\;\Big|\;s^{(1)}\Big)\Big\} \;=\; \sum_{i\in U^{(1)}} \frac{V_i^{(2)}}{\pi_i^{(1)}}, \quad \text{and} \\
\mathrm{V_{TSU}} \;&:=\; E^{(1)}\Big\{E^{(2)}\Big(\,\mathrm{Var}^{(3)}(\,\widehat{T}\;\Big|\;s^{(2)}\,)\;\Big|\;s^{(1)}\Big)\Big\} \;=\; \sum_{i\in U^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a\in U_i^{(2)}} \frac{V_{ia}^{(3)}}{\pi_{i|a}^{(2)}}\right),
\end{aligned}$$

with

$$\begin{aligned}
\Delta_{ij}^{(1)} \;&:=\; \begin{cases} \pi_i^{(1)}\left(1-\pi_i^{(1)}\right), & \text{if } i=j \\[2ex] \pi_{ij}^{(1)} - \pi_i^{(1)}\pi_j^{(1)}, & \text{if } i\neq j \end{cases} \\
V_i^{(2)} \;&:=\; \mathrm{Var}^{(2)}\left[\sum_{a\in s_i^{(2)}} \frac{T_{ia}}{\pi_{i|a}^{(2)}}\right] \;=\; \sum_{a\in U_i^{(2)}}\sum_{b\in U_i^{(2)}} \Delta_{i|ab}^{(2)} \cdot \frac{T_{ia}}{\pi_{i|a}^{(2)}} \cdot \frac{T_{ib}}{\pi_{i|b}^{(2)}} \\
\Delta_{i|ab}^{(2)} \;&:=\; \begin{cases} \pi_{i|a}^{(2)}\left(1-\pi_{i|a}^{(2)}\right), & \text{if } a=b \\[2ex] \pi_{i|ab}^{(2)} - \pi_{i|a}^{(2)}\pi_{i|b}^{(2)}, & \text{if } a\neq b \end{cases} \\
V_{ia}^{(3)} \;&:=\; \mathrm{Var}^{(3)}\Big(\,\widehat{T}_{ia}\;\Big|\;s^{(1)}, s^{(2)}\Big)
\end{aligned}$$

Theorem 2.2

For a three-stage sampling design $p : \mathcal{S} \subset \mathcal{P}(U) \longrightarrow \mathbb{R}$ *with invariant and independent subsampling, let* $\widehat{T} : \mathcal{S} \longrightarrow \mathbb{R}$ *be the random variable defined as follows:*

$$\widehat{T}(s) \;:=\; \sum_{i\in s^{(1)}} \frac{\widehat{T}_i(s_i^{(2+)})}{\pi_i^{(1)}} \;=\; \sum_{i\in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a\in s_i^{(2)}} \frac{\widehat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \right) \;=\; \sum_{i\in s^{(1)}} \frac{1}{\pi_i^{(1)}} \left(\sum_{a\in s_i^{(2)}} \frac{1}{\pi_{i|a}^{(2)}} \left(\sum_{u\in s_{ia}^{(3)}} \frac{y_u}{\pi_{ia|u}^{(3)}} \right)\right)$$

Recall that $\widehat{T}$ is an unbiased estimator of the population total

$$T \;:=\; \sum_{i\in U^{(1)}} T_i \;:=\; \sum_{i\in U^{(1)}}\sum_{a\in U^{(2)}_{ia}} T_{ia} \;:=\; \sum_{i\in U^{(1)}}\sum_{a\in U^{(2)}_{i}}\sum_{u\in U^{(3)}_{ia}} y_u$$

Let $\widehat{\mathrm{Var}}\Big[\,\widehat{T}\,\Big] : \mathcal{S} \longrightarrow \mathbb{R}$ be the random variable defined in a recursive manner as follows:

$$\begin{aligned}
\widehat{\mathrm{Var}}\Big[\,\widehat{T}\,\Big](s) &:= \sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta^{(1)}_{ij}}{\pi^{(1)}_{ij}}\cdot\frac{\widehat{T}_i(s^{(2+)}_i)}{\pi^{(1)}_i}\cdot\frac{\widehat{T}_j(s^{(2+)}_j)}{\pi^{(1)}_j} \;+\; \sum_{i\in s^{(1)}} \frac{\widehat{\mathrm{Var}}\Big[\,\widehat{T}_i\,\Big](s^{(2+)}_i)}{\pi^{(1)}_i}\\
\widehat{\mathrm{Var}}\Big[\,\widehat{T}_i\,\Big](s^{(2+)}_i) &:= \sum_{a\in s^{(2)}_i}\sum_{b\in s^{(2)}_i} \Delta^{(2)}_{i|ab}\cdot\frac{\widehat{T}_{ia}(s^{(3)}_{ia})}{\pi^{(2)}_{i|a}}\cdot\frac{\widehat{T}_{ib}(s^{(3)}_{ib})}{\pi^{(2)}_{i|b}} \;+\; \sum_{a\in s^{(2)}_i} \frac{\widehat{\mathrm{Var}}\Big[\,\widehat{T}_{ia}\,\Big](s^{(3)}_{ia})}{\pi^{(2)}_{i|a}}\\
\widehat{\mathrm{Var}}\Big[\,\widehat{T}_{ia}\,\Big](s^{(3)}_{ia}) &:= \sum_{u\in s^{(3)}_{ia}}\sum_{v\in s^{(3)}_{ia}} \Delta^{(3)}_{ia|uv}\cdot\frac{y_u}{\pi^{(3)}_{ia|u}}\cdot\frac{y_v}{\pi^{(3)}_{ia|v}}
\end{aligned}$$

where

$$\Delta^{(1)}_{ij} \;:=\; \begin{cases} \pi^{(1)}_i\left(1-\pi^{(1)}_i\right), & \text{if } i=j\\[1ex] \pi^{(1)}_{ij} - \pi^{(1)}_i\pi^{(1)}_j, & \text{if } i\neq j\end{cases}$$

$$\Delta^{(2)}_{i|ab} \;:=\; \begin{cases} \pi^{(2)}_{i|a}\left(1-\pi^{(2)}_{i|a}\right), & \text{if } a=b\\[1ex] \pi^{(2)}_{i|ab} - \pi^{(2)}_{i|a}\,\pi^{(2)}_{i|b}, & \text{if } a\neq b\end{cases}$$

$$\Delta^{(3)}_{ia|uv} \;:=\; \begin{cases} \pi^{(3)}_{ia|u}\left(1-\pi^{(3)}_{ia|u}\right), & \text{if } u=v\\[1ex] \pi^{(3)}_{ia|uv} - \pi^{(3)}_{ia|u}\,\pi^{(3)}_{ia|v}, & \text{if } u\neq v\end{cases}$$

Then, $\widehat{\mathrm{Var}}\Big[\,\widehat{T}\,\Big]$ is a design-unbiased estimator of the design variance $\mathrm{Var}\Big[\,\widehat{T}\,\Big]$ of the $\widehat{T}$.

Corollary 2.3
For a three-stage sampling design with invariant and independent subsampling, the fully expanded expression for $\widehat{\mathrm{Var}}\Big[\,\widehat{T}\,\Big]$ is as follows:

$$\begin{aligned}
\widehat{\mathrm{Var}}\Big[\,\widehat{T}\,\Big](s) \;&:=\; \sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta^{(1)}_{ij}}{\pi^{(1)}_{ij}}\cdot\frac{\widehat{T}_i(s^{(2+)}_i)}{\pi^{(1)}_i}\cdot\frac{\widehat{T}_j(s^{(2+)}_j)}{\pi^{(1)}_j} \;+\; \sum_{i\in s^{(1)}} \frac{\widehat{V}_i(s^{(2+)}_i)}{\pi^{(1)}_i}\\
&=\; \sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta^{(1)}_{ij}}{\pi^{(1)}_{ij}}\cdot\frac{\widehat{T}_i(s^{(2+)}_i)}{\pi^{(1)}_i}\cdot\frac{\widehat{T}_j(s^{(2+)}_j)}{\pi^{(1)}_j}\\
&\quad+\sum_{i\in s^{(1)}}\frac{1}{\pi^{(1)}_i}\left\{\sum_{a\in s^{(2)}_i}\sum_{b\in s^{(2)}_i}\Delta^{(2)}_{i|ab}\cdot\frac{\widehat{T}_{ia}(s^{(3)}_{ia})}{\pi^{(2)}_{i|a}}\cdot\frac{\widehat{T}_{ib}(s^{(3)}_{ib})}{\pi^{(2)}_{i|b}} + \sum_{a\in s^{(2)}_i}\frac{\widehat{V}_{ia}}{\pi^{(2)}_{i|a}}\right\}\\
&=\; \sum_{i\in s^{(1)}}\sum_{j\in s^{(1)}} \frac{\Delta^{(1)}_{ij}}{\pi^{(i)}_{ij}}\cdot\frac{\widehat{T}_i(s^{(2+)}_i)}{\pi^{(1)}_i}\cdot\frac{\widehat{T}_j(s^{(2+)}_j)}{\pi^{(1)}_j}\\
&\quad+\sum_{i\in s^{(1)}}\frac{1}{\pi^{(1)}_i}\left\{\sum_{a\in s^{(2)}_i}\sum_{b\in s^{(2)}_i}\Delta^{(2)}_{i|ab}\cdot\frac{\widehat{T}_{ia}(s^{(3)}_{ia})}{\pi^{(2)}_{i|a}}\cdot\frac{\widehat{T}_{ib}(s^{(3)}_{ib})}{\pi^{(2)}_{i|b}} + \sum_{a\in s^{(2)}_i}\frac{1}{\pi^{(2)}_{i|a}}\left(\sum_{u\in s^{(3)}_{ia}}\sum_{v\in s^{(3)}_{ia}}\Delta^{(3)}_{ia|uv}\frac{y_u}{\pi^{(3)}_{ia|u}}\frac{y_v}{\pi^{(3)}_{ia|v}}\right)\right\},
\end{aligned}$$

where

$$\widehat{T}_{ia}(s_{ia}^{(3)}) \;=\; \sum_{u\in s_{ia}^{(3)}} \frac{y_u}{\pi_{ia|u}^{(3)}} \quad \text{and} \quad \widehat{T}_i(s_i^{(2+)}) \;=\; \sum_{a\in s_i^{(2)}} \frac{\widehat{T}_{ia}(s_{ia}^{(3)})}{\pi_{i|a}^{(2)}} \;=\; \sum_{a\in s_i^{(2)}} \frac{1}{\pi_{i|a}^{(2)}} \left(\sum_{u\in s_{ia}^{(3)}} \frac{y_u}{\pi_{ia|u}^{(3)}} \right)$$

$$\Delta_{ia|uv}^{(3)} \;:=\; \begin{cases} \pi_{ia|u}^{(3)}\left(1-\pi_{ia|u}^{(3)}\right), & \text{if } u = v \\ \\ \pi_{ia|uv}^{(3)} - \pi_{ia|u}^{(3)}\,\pi_{ia|v}^{(3)}, & \text{if } u \neq v \end{cases}$$