1 The Prokhorov Theorem

Definition 1.1 (Tightness and weak sequential compactness)

Suppose:

- (S, ρ) *is a metric space,* $\mathcal{B}(S)$ *its the Borel* σ*-algebra,* $(S, \mathcal{B}(S))$ *is the corresponding measurable space.*
- $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ *is a family of probability measures on* $(S, \mathcal{B}(S))$.

The family Π *is said to be:*

(i) ***tight*** *if, for each* $\varepsilon > 0$*, there exists a compact subset* $K_\varepsilon \subset S$ *such that*

$$1 - \epsilon \ < \ P(\, K_\varepsilon \,) \ \leq \ 1, \quad \text{for each } P \in \Pi.$$

(ii) ***weakly sequentially compact*** *if, for every sequence* $\{\, P_n \,\}_{n\in\mathbb{N}} \ \subset \ \Pi$*, there exists a probability measure* $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ *and subsequence* $\{\, P_{n(i)} \,\}_{i\in\mathbb{N}}$ *such that*

$$P_{n(i)} \ \overset{w}{\longrightarrow} \ P, \quad \text{as } i \longrightarrow \infty.$$

Theorem 1.2 (The Prokhorov Theorem, Theorems 5.1 & 5.2, [2])

Suppose:

- (S, ρ) *is a metric space,* $\mathcal{B}(S)$ *its the Borel* σ*-algebra,* $(S, \mathcal{B}(S))$ *is the corresponding measurable space.*
- $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ *is a collection of probability measures on* $(S, \mathcal{B}(S))$.

Then, the following statements hold:

(i) *Tightness of* Π *implies weak sequential compactness of* Π.

(ii) *Suppose further that* (S, ρ) *is complete and separable.*
Then, weak sequential compactness of Π *implies tightness of* Π.

Proof We first prove statement (ii), then statement (i).

Proof of (ii)

Suppose S is complete and separable. Let $\varepsilon > 0$ be fixed. We need to find a compact subset $K \subset S$ such that

$$1 - \varepsilon \quad < \quad P(\, K \,) \quad \leq \quad 1, \quad \text{for each } P \in \Pi.$$

Now, separability of S implies that every open cover of every subset of S admits a countable subcover (Appendix M3, [2]). Denote by $B(x, r) \subset S$ the open ball in S centred at $x \in S$ of radius $r > 0$. For each $k \in \mathbb{N}$, the open cover

$$\left\{\, B\!\left(x, \frac{1}{k}\right) \,\right\}_{x \in S}$$

of S admits a countable subcover, say,

$$\{\, A_{ki} \,\}_{i\in\mathbb{N}} \quad \subset \quad \left\{\, B\!\left(x, \frac{1}{k}\right) \,\right\}_{x \in S}.$$

Let $G_{kn} := \bigcup_{i=1}^{n} A_{ki}$. Then, each G_{kn} is an open subset of S and $G_{kn} \uparrow S$, as $n \longrightarrow \infty$. Hence, by the Claim below, there exists $n_k \in \mathbb{N}$ such that

$$1 - \frac{\varepsilon}{2^k} \;<\; P\left(\bigcup_{i=1}^{n_k} A_{ki} \right) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Now, let

$$K \;:=\; \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}}$$

Note that K, being a closed subset of the complete metric space S, is itself complete. Note also that the set $\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}$ is totally bounded; hence so is its closure K. Being complete and totally bounded, K is therefore compact (Appendix M5, [2]). It now remains only to show that $1 - \varepsilon < P(K) \leq 1$, for each $P \in \Pi$; or equivalently, that $P(K^c) \leq \varepsilon$, for each $P \in \Pi$. To this end, write $B_k := \bigcup_{i=1}^{n_k} A_{ki}$. Then,

$$1 - \frac{\varepsilon}{2^k} \;<\; P(\,B_k\,) \;\leq\; 1\,; \quad \text{equivalently,} \quad P(\,B_k^c\,) \;\leq\; \frac{\varepsilon}{2^k}.$$

Also,

$$K \;:=\; \overline{\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} A_{ki}} \;:=\; \overline{\bigcap_{k=1}^{\infty} B_k} \;\supset\; \bigcap_{k=1}^{\infty} B_k.$$

Hence,

$$K^c \;\subset\; \left(\bigcap_{k=1}^{\infty} B_k \right)^c \;=\; \bigcup_{k=1}^{\infty} B_k^c,$$

which implies:

$$P(\,K^c\,) \;\leq\; \sum_{k=1}^{\infty} P(B_k^c) \;\leq\; \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} \;=\; \varepsilon.$$

Thus, the proof of (ii) will be complete once we prove the following:

> **Claim:** Let $\{\,G_n\,\}_{n \in \mathbb{N}}$ be a sequence of open subsets of S with $G_n \uparrow S$. Then, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that
>
> $$1 - \varepsilon \;<\; P(\,G_{n_\varepsilon}\,) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Proof of Claim: Suppose the Claim is false, and we derive a contradiction. The failure of the Claim implies that there exists some $0 < \varepsilon < 1$ such that for each $n \in \mathbb{N}$, there exists $P_n \in \Pi$ such that

$$P_n(\,G_n\,) \;\leq\; 1 - \varepsilon.$$

By the hypothesis of weak sequential compactness of Π, there exists some probability measure $Q \in \mathcal{M}_1(S, \mathcal{B}(S))$ and the subsequence $\{\,P_{n(i)}\,\}$ of $\{\,P_n\,\}$ such that $P_{n(i)} \xrightarrow{w} Q$, as $i \longrightarrow \infty$. Now, for each fixed $n \in \mathbb{N}$, we have:

$$\begin{aligned} Q(\,G_n\,) \;&\leq\; \liminf_{i \to \infty} P_{n(i)}(\,G_n\,)\,, \quad \text{by the Portmanteau Theorem} \\ &\leq\; \liminf_{i \to \infty} P_{n(i)}\big(\,G_{n(i)}\,\big)\,, \quad \text{since } \{\,G_n\,\} \text{ is increasing} \\ &\leq\; 1 - \varepsilon, \quad \text{by choice of } P_n \end{aligned}$$

But, by hypothesis, we also have $G_n \uparrow S$. Hence, we therefore have:

$$1 \;=\; Q(S) \;=\; \lim_{n \to \infty} Q(G_n) \;\leq\; 1 - \varepsilon,$$

which is the desired contradiction. This completes the proof of the Claim, hence that of (ii).

Proof of (i)

Suppose $\Pi \subset \mathcal{M}_1(S, \mathcal{B}(S))$ is tight. We need to establish that Π is weakly sequentially compact. In other words, if $\{ P_n \} \subset \Pi$ is a sequence of probability measures contained in Π, we need to establish that there exists a Borel probability measure $P \in \mathcal{M}_1(S, \mathcal{B}(S))$ and a subsequence $\{ P_{n(i)} \} \subset \{ P_n \}$ such that $P_{n(i)} \overset{w}{\longrightarrow} P$, as $i \longrightarrow \infty$.

So, let $\{ P_n \} \subset \Pi$. We prove the Theorem by establishing the following series of Claims. Note that the proof of the Theorem is complete once we establish Claim 8.

Claim 1: There exists an increasing sequence of compact subsets $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ such that

$$1 - \frac{1}{m} \;<\; P_n(K_m) \;\leq\; 1, \quad \text{for every } \; m,\, n \,\in\, \mathbb{N}.$$

Claim 2: Let $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$ be one such sequence of compact subsets of S as in Claim 1. Then, the following statements are true:

(a) $\Sigma := \bigcup_{m=1}^{\infty} K_m$ is a separable subset of S.

(b) There exists a countable collection $\mathcal{A}$ of open subsets of S satisfying the following property: For each $x \in S$ and for each open subset G of S,

$$x \,\in\, G \bigcap \left(\bigcup_{m=1}^{\infty} K_m \right) \quad\Longrightarrow\quad x \,\in\, A \,\subset\, \overline{A} \,\subset\, G, \;\text{ for some } A \in \mathcal{A}.$$

(c) The collection $\mathcal{A}$ is an open cover of Σ.

Claim 3: Define:

$$\mathcal{H} \;:=\; \{ \varnothing \} \bigcup \left\{ \begin{array}{c} \text{all finite unions of sets of the form} \\ \overline{A} \cap K_m, \;\text{ where } A \in \mathcal{A} \text{ and } m \in \mathbb{N} \end{array} \right\}.$$

Then, the following statements are true:

(a) $K_m \,\in\, \mathcal{H}$, for each $m \in \mathbb{N}$.

(b) There exists a subsequence $\{ P_{n(i)} \} \subset \{ P_n \}$ such that the limit

$$\lim_{i \to \infty} P_{n(i)}(H) \;\text{ exists, }\; \text{for each } H \in \mathcal{H}.$$

We may therefore define the following function:

$$\alpha \,:\, \mathcal{H} \,\longrightarrow\, [\,0,1\,] \,:\, H \,\longmapsto\, \lim_{i \to \infty} P_{n(i)}(H).$$

Claim 4: The function $\alpha : \mathcal{H} \longrightarrow [\,0,1\,]$ satisfies the following properties:

(a) $\alpha(\,\varnothing\,) = 0$.

(b) monotonicity: $\alpha(H_1) \;\leq\; \alpha(H_2)$, for any $H_1, H_2 \in \mathcal{H}$ with $H_1 \subset H_2$.

(c) finite additivity for disjoint sets:

$\alpha(\,H_1 \cup H_2\,) = \alpha(H_1) + \alpha(H_2)$, for any $H_1, H_2 \in \mathcal{H}$ with $H_1 \cap H_2 = \varnothing$.

(d) finite sub-additivity: $\alpha(H_1 \cup H_2) \leq \alpha(H_1) + \alpha(H_2)$, for any $H_1, H_2 \in \mathcal{H}$.

Claim 5: Let $\mathcal{O}(S)$ denote the collection of all open subsets of S. Define the following function:

$$\beta \;:\; \mathcal{O}(S) \;\longrightarrow\; [\,0,1\,] \;:\; G \;\longmapsto\; \sup\left\{\,\alpha(H) \,\in\, [\,0,1\,] \;\middle|\; \begin{array}{l} H \in \mathcal{H},\ \text{and} \\ H \subset G \end{array}\right\}.$$

(Note that the supremum above is always taken over a non-empty set: For each open $G \subset S$, the set $\{\,H \in \mathcal{H} \mid H \subset G\,\}$ is non-empty, since $\varnothing \in \mathcal{H}$.)

Next, let $\mathcal{P}(S)$ be the power set of S, i.e. the collection of all subsets of S. Define the following function:

$$\gamma \;:\; \mathcal{P}(S) \;\longrightarrow\; [\,0,1\,] \;:\; W \;\longmapsto\; \inf\left\{\,\beta(G) \,\in\, [\,0,1\,] \;\middle|\; \begin{array}{l} G \in \mathcal{O}(S),\ \text{and} \\ W \subset G \end{array}\right\}.$$

Then, the function $\gamma : \mathcal{P}(S) \longrightarrow [\,0,1\,]$ is an outer measure defined on S.

Claim 6: The σ-algebra $\mathcal{A}(\gamma)$ of γ-measurable subsets of S contains the Borel σ-algebra $\mathcal{B}(S)$ of S.

Claim 7: The restriction $P \;:=\; \gamma\,|_{\mathcal{B}(S)}$ of γ to $\mathcal{B}(S)$ is a Borel probability measure which statisfies:

$$P(G) \;=\; \gamma(G) \;=\; \beta(G) \;:=\; \sup\left\{\,\alpha(H) \,\in\, [0,1] \;\middle|\; \begin{array}{l} H \in \mathcal{H}, \\ H \subset G \end{array}\right\},\ \ \text{for each open subset } G \subset S.$$

Claim 8: $P_{n(i)} \overset{w}{\longrightarrow} P$, as $i \longrightarrow \infty$.

Proof of Claim 1: By tightness hypothesis on Π, for each $m \in \mathbb{N}$, there exists a compact subset $L_m \subset S$ such that

$$1 - \frac{1}{m} \;<\; P(L_m) \;\leq\; 1, \quad \text{for each } P \in \Pi.$$

Define, for each $m \in \mathbb{N}$, $K_m \;:=\; \bigcup_{i=1}^{m} L_i$. Then, each K_m is compact (since finite unions of compact subsets are themselves compact). Next, we trivially have $K_1 \subset K_2 \subset K_3 \subset \cdots \subset S$. Also,

$$P(K_m) \;=\; P\left(\bigcup_{i=1}^{m} L_i\right) \;\geq\; L_m \;>\; 1 - \frac{1}{m}, \quad \text{for each } P \in \Pi.$$

In particular, the above inequality holds for each P_n. This proves Claim 1.

Proof of Claim 2: Separability of $\Sigma := \bigcup_{m=1}^{\infty} K_m$ is an immediate consequence of Lemma A.4 and Lemma A.5. Then, the existence of $\mathcal{A}$ follows immediately from the separability of Σ and Lemma A.6.

It remains to show that $\mathcal{A}$ is an open cover of Σ. To this end, first note that the collection $\mathcal{O}(S)$ of open subsets of S forms an open cover of S, hence $\mathcal{O}(S)$ is also an open cover of Σ. Therefore, for each $x \in \Sigma$, we may choose $G_x \in \mathcal{O}(S)$ such that $x \in G_x$. Thus, $x \in \Sigma \cap G_x$, and by properties of $\mathcal{A}$, we may furthermore choose $A_x \in \mathcal{A}$ such that $x \in A_x \subset \overline{A_x} \subset G_x$. We thus see that the collection

$$\left\{ A_x \in \mathcal{A} \;\middle\vert\; x \in \Sigma \right\} \;\subset\; \mathcal{A}$$

is an open cover of Σ consisting of subsets in $\mathcal{A}$. This completes the proof of Claim 2.

Proof of Claim 3:

(a) By Claim 2, $\mathcal{A}$ is an open cover of $\Sigma := \bigcup_{m=1}^{\infty} K_m$. In particular, $\mathcal{A}$ is an open cover of K_m for each $m \in \mathbb{N}$. Compactness of K_m implies that $\mathcal{A}$ admits a finite subcover of K_m. Thus we have

$$K_m \;\subset\; \bigcup_{i=1}^{J_m} A_i^{(m)} \;\subset\; \bigcup_{i=1}^{J_m} \overline{A_i^{(m)}}, \quad \text{for some } A_1^{(m)},\, A_2^{(m)},\, \ldots,\, A_{J_m}^{(m)} \in \mathcal{A},$$

which implies

$$K_m \;=\; K_m \bigcap \left(\bigcup_{i=1}^{J_m} \overline{A_i^{(m)}} \right) \;=\; \bigcup_{i=1}^{J_m} \left(K_m \bigcap \overline{A_i^{(m)}} \right) \;\in\; \mathcal{H}.$$

(b) Note that $\mathcal{H}$ is a countable collection of subsets of S. Let $\mathcal{H} = \{ H_1, H_2, H_3, \ldots \}$ be an enumeration of $\mathcal{H}$. Consider the following array of real numbers:

$$\begin{array}{cccc} P_1(H_1) & P_2(H_1) & P_3(H_1) & \cdots \\ P_1(H_2) & P_2(H_2) & P_3(H_2) & \cdots \\ P_1(H_3) & P_2(H_3) & P_3(H_3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Note that each row of the above array is bounded between 0 and 1. Hence, by Theorem A.7, there exists an increasing sequence

$$n(1) \;<\; n(2) \;<\; n(3) \;<\; \cdots \in \mathbb{N}$$

of natural numbers such that the limit

$$\lim_{k\to\infty} P_{n(k)}(H_r), \;\; \text{exists for each } r \in \mathbb{N}.$$

This completes the proof of Claim 3.

Proof of Claim 4:

(a) Obviously, $P_{n(i)}(\varnothing) = 0$, for each $i \in \mathbb{N}$. Hence,

$$\alpha(\varnothing) \;=\; \lim_{i\to\infty} P_{n(i)}(\varnothing) \;=\; \lim_{i\to\infty} (0) \;=\; 0.$$

(b) For $H_1, H_2 \in \mathcal{H}$ with $H_1 \subset H_2$, we have $P_{n(i)}(H_1) \;\leq\; P_{n(i)}(H_2)$, for each $i \in \mathbb{N}$. Hence,

$$\alpha(H_1) \;:=\; \lim_{i\to\infty} P_{n(i)}(H_1) \;\leq\; \lim_{i\to\infty} P_{n(i)}(H_2) \;=:\; \alpha(H_2).$$

(c) For $H_1, H_2 \in \mathcal{H}$ with $H_1 \cap H_2 = \varnothing$, we have $P_{n(i)}(\,H_1 \cup H_2\,) = P_{n(i)}(H_1) + P_{n(i)}(H_2)$, for each $i \in \mathbb{N}$. Hence,

$$\begin{aligned}
\alpha(\,H_1 \cup H_2\,) &= \lim_{i\to\infty} P_{n(i)}(\,H_1 \cup H_2\,) \\
&= \lim_{i\to\infty} \left[\,P_{n(i)}(H_1) + P_{n(i)}(H_2)\,\right] \\
&= \lim_{i\to\infty} P_{n(i)}(H_1) + \lim_{i\to\infty} P_{n(i)}(H_2) \;=\; \alpha(H_1) + \alpha(H_2).
\end{aligned}$$

(d) For $H_1, H_2 \in \mathcal{H}$, we have $P_{n(i)}(\,H_1 \cup H_2\,) \leq P_{n(i)}(H_1) + P_{n(i)}(H_2)$, for each $i \in \mathbb{N}$. Hence,

$$\begin{aligned}
\alpha(\,H_1 \cup H_2\,) &= \lim_{i\to\infty} P_{n(i)}(\,H_1 \cup H_2\,) \\
&\leq \lim_{i\to\infty} \left[\,P_{n(i)}(H_1) + P_{n(i)}(H_2)\,\right] \\
&= \lim_{i\to\infty} P_{n(i)}(H_1) + \lim_{i\to\infty} P_{n(i)}(H_2) \;=\; \alpha(H_1) + \alpha(H_2).
\end{aligned}$$

Proof of Claim 5:

Claim 5A: For any subsets $F,\, G \subset S$, where F is closed and G is open, we have

$$\left.\begin{array}{c} F \subset G, \text{ and} \\ \text{there exists } H \in \mathcal{H} \text{ such that } F \subset H \end{array}\right\} \;\Longrightarrow\; \text{there exists } H_0 \in \mathcal{H} \text{ such that } F \subset H_0 \subset G$$

Proof of Claim 5A: For each $x \in F$, choose $A_x \in \mathcal{A}$ such that $x \in A_x \subset \overline{A_x} \subset G$. Recall that $H \in \mathcal{H}$ is compact, hence F is also compact (being a closed subset of the compact set H). Therefore, the open cover $\{\,A_x\,\}_{x\in F}$ of F admits a finite sub-cover, say $\{\,A_{x_1},\, A_{x_2},\, \ldots\,,A_{x_k}\,\}$. Secondly, note that $F \subset K_m$, for some $m \in \mathbb{N}$. Let $H_0 := \bigcup_{i=1}^{k} (\,\overline{A}_{x_i} \cap K_m\,)$. It is clear that $H_0 \in \mathcal{H}$. Furthermore,

$$F \;\subset\; \left(\bigcup_{i=1}^{k} \overline{A}_{x_i}\right) \bigcap K_m \;=\; \underbrace{\bigcup_{i=1}^{k} (\,\overline{A}_{x_i} \cap K_m\,)}_{H_0} \;\subset\; \left(\bigcup_{i=1}^{k} \overline{A}_{x_i}\right) \;\subset\; G$$

This proves Claim 5A.

Claim 5B: $\beta : \mathcal{O}(S) \longrightarrow [\,0,1\,]$ is finitely subadditive.

Proof of Claim 5B: Suppose $G_1,\, G_2 \subset S$ are two arbitrary open subsets of S. We need to show:

$$\beta(\,G_1 \cup G_2\,) \;\leq\; \beta(G_1) + \beta(G_2)\,.$$

Recall that

$$\beta(\,G_1 \cup G_2\,) \;:=\; \sup\left\{\,\alpha(H) \in [\,0,1\,] \;\middle|\; \begin{array}{l} H \in \mathcal{H}, \text{ and} \\ H \subset G_1 \cup G_2 \end{array}\right\}.$$

Now, for each $H \in \mathcal{H}$ with $H \subset G_1 \cup G_2$, define

$$F_1 \;:=\; \left\{\,x \in H \;\middle|\; \rho(x, G_1^c) \geq \rho(x, G_2^c)\,\right\}, \quad \text{and} \quad F_2 \;:=\; \left\{\,x \in H \;\middle|\; \rho(x, G_1^c) \leq \rho(x, G_2^c)\,\right\}.$$

We immediately have $H = F_1 \cup F_2$. Note that $F_1 \subset G_1$. Indeed,

$$x \in F_1 \cap G_1^c \;\Longrightarrow\; \left\{\begin{array}{l} x \in H \cap G_1^c \subset G_2, \text{ and} \\ \rho(x, G_2^c) \leq \rho(x, G_1^c) = 0 \end{array}\right. \;\Longrightarrow\; 0 < \rho(x, G_2^c) \leq \rho(x, G_1^c) = 0, \text{ a contradiction.}$$

This proves indeed that $F_1 \subset G_1$. Similarly, we have $F_2 \subset G_2$. Now,

$$\left.\begin{array}{c} F_1 \subset H,\ H \in \mathcal{H} \\ \text{Claim 5A} \end{array}\right\} \quad \Longrightarrow \quad F_1 \;\subset\; H_1 \;\subset\; G_1, \;\; \text{for some } H_1 \in \mathcal{H}.$$

Similarly, $F_2 \subset H_2 \subset G_2$, for some $H_2 \in \mathcal{H}$. We thus have $H = F_1 \cup F_2 \subset H_1 \cup H_2$. By Claim 4, definition of $\beta : \mathcal{O}(S) \longrightarrow [0,1]$, we therefore have

$$\alpha(H) \;\leq\; \alpha(\,H_1 \cup H_2\,) \;\leq\; \alpha(\,H_1\,) + \alpha(\,H_2\,) \;\leq\; \beta(\,G_1\,) + \beta(\,G_2\,).$$

Recall that $H \subset G_1 \cup G_2$ and $H \in \mathcal{H}$, but otherwise H is arbitrary; hence, we have

$$\beta(\,G_1 \cup G_2\,) \;:=\; \sup\left\{ \alpha(H) \in [\,0,1\,] \;\middle|\; \begin{array}{c} H \in \mathcal{H}, \text{ and} \\ H \subset G_1 \cup G_2 \end{array} \right\} \;\leq\; \beta(\,G_1\,) + \beta(\,G_2\,).$$

This proves Claim 5B.

Claim 5C: $\beta : \mathcal{O}(S) \longrightarrow [\,0,1\,]$ is countably subadditive.

Proof of Claim 5C: Let $G_1, G_2, \ldots \subset S$ be open subsets of S. We need to show:

$$\beta\left(\bigcup_{i=1}^{\infty} G_i\right) \;\leq\; \sum_{i=1}^{\infty} \beta(G_i).$$

Recall that

$$\beta\left(\bigcup_{i=1}^{\infty} G_i\right) \;:=\; \sup\left\{ \alpha(H) \in [\,0,1\,] \;\middle|\; \begin{array}{c} H \in \mathcal{H}, \text{ and} \\ H \subset \bigcup_{i=1}^{\infty} G_i \end{array} \right\}.$$

Now, for any $H \in \mathcal{H}$ with $H \subset \bigcup_{i=1}^{\infty} G_i$, compactness of H implies $H \subset \bigcup_{i=1}^{k} G_i$, for some $k \in \mathbb{N}$. Therefore,

$$\alpha(H) \;\leq\; \sup\left\{ \alpha(H') \in [\,0,1\,] \;\middle|\; \begin{array}{c} H' \in \mathcal{H}, \text{ and} \\ H' \subset \bigcup_{i=1}^{k} G_i \end{array} \right\} \;=:\; \beta\left(\bigcup_{i=1}^{k} G_i\right) \;\leq\; \sum_{i=1}^{k} \beta(\,G_i\,) \;\leq\; \sum_{i=1}^{\infty} \beta(\,G_i\,),$$

which in turn immediately implies

$$\beta\left(\bigcup_{i=1}^{\infty} G_i\right) \;:=\; \sup\left\{ \alpha(H) \in [\,0,1\,] \;\middle|\; \begin{array}{c} H \in \mathcal{H}, \text{ and} \\ H \subset \bigcup_{i=1}^{\infty} G_i \end{array} \right\} \;\leq\; \sum_{i=1}^{\infty} \beta(\,G_i\,).$$

This proves Claim 5C.

Claim 5D: $\gamma : \mathcal{P}(S) \longrightarrow [\,0,1\,]$ is outer measure on S.

<u>Proof of Claim 6:</u> First, note that

$$\gamma(\,F \cap G\,) + \gamma(\,F^c \cap G\,) \;\leq\; \beta(G), \quad \text{for each closed } F \subset S \text{ and each open } G \subset S. \tag{1.1}$$

Indeed, let $\varepsilon > 0$ be given. Since $F^c \cap G \in \mathcal{O}(S)$, by definition of β as a supremum, we may choose $H_1 \in \mathcal{H}$ such that

$$H_1 \;\subset\; F^c \cap G \quad \text{and} \quad \beta(\,F^c \cap G\,) - \varepsilon \;<\; \alpha(H_1) \;\leq\; \beta(\,F^c \cap G\,).$$

The first inclusion immediately implies that $F \cap G \subset H_1^c \cap G$. Now, recall that $H_1 \subset S$ is a closed subset; hence, $H_1^c \cap G$ is open. Thus, we may choose $H_0 \in \mathcal{H}$ such that

$$H_0 \;\subset\; H_1^c \cap G \quad \text{and} \quad \beta(\,H_1^c \cap G\,) - \varepsilon \;<\; \alpha(H_0) \;\leq\; \beta(\,H_1^c \cap G\,).$$

Since $H_0 \cap H_1 = \varnothing$, $H_0 \cup H_1 \subset G$, and $F \cap G \subset H_1^c \cap G$, we have

$$\begin{aligned} \beta(G) &\geq \alpha(\,H_0 \cup H_1\,) \;=\; \alpha(H_0) + \alpha(H_1) \\ &> \beta(\,H_1^c \cap G\,) - \varepsilon + \beta(\,F^c \cap G\,) - \varepsilon \\ &\geq \gamma(\,F \cap G\,) + \gamma(\,F^c \cap G\,) - 2\,\varepsilon, \end{aligned}$$

which implies $\beta(G) \geq \gamma(\,F \cap G\,) + \gamma(\,F^c \cap G\,)$, since $\varepsilon > 0$ is arbitrary.

Now, let $F \subset S$ be an arbitrary closed subset of S, and $A \subset S$ be an arbitrary subset of S. Then, by (1.1), we have

$$\gamma(\,F \cap A\,) + \gamma(\,F^c \cap A\,) \;\leq\; \gamma(\,F \cap G\,) + \gamma(\,F^c \cap G\,) \;\leq\; \beta(G), \quad \text{for each open } G \subset S \text{ with } A \subset G.$$

Hence,

$$\gamma(\,F \cap A\,) + \gamma(\,F^c \cap A\,) \quad\leq\quad \inf\left\{ \beta(G) \;\middle|\; \begin{array}{c} G \in \mathcal{O}(S) \\ A \subset G \end{array} \right\} \;=:\; \gamma(A).$$

This proves the γ-measurability of each closed subset $F \subset S$, and hence the γ-measurability of each open subset of S, i.e. $\mathcal{O}(S) \subset \mathcal{A}(\gamma)$. It now follows immediately that

$$\mathcal{B}(S) \;=\; \sigma(\,\mathcal{O}(S)\,) \;\subset\; \mathcal{A}(\gamma).$$

This completes the proof of Claim 6.

<u>Proof of Claim 7:</u> Immediate by definition of $P : \mathcal{B}(S) \longrightarrow [0,1]$.

<u>Proof of Claim 8:</u> Let $G \subset S$ be an arbitrary open subset of S. Then, we have

$$\alpha(H) \;:=\; \lim_{i\to\infty} P_{n(i)}(H) \;\leq\; \liminf_{i\to\infty} P_{n(i)}(G), \quad \text{for each } H \in \mathcal{H} \text{ with } H \subset G.$$

The preceding inequality and Claim 7 together imply:

$$P(G) \;=\; \sup\left\{ \alpha(H) \in [0,1] \;\middle|\; \begin{array}{c} H \in \mathcal{H} \\ H \subset G \end{array} \right\} \;\leq\; \liminf_{i\to\infty} P_{n(i)}(G), \;\; \text{for each open subset } G \subset S,$$

which is equivalent to the weak convergence $P_{n(i)} \overset{w}{\longrightarrow} P$, as $i \longrightarrow \infty$, by the Portmanteau Theorem (Theorem 2.1, [2]). This completes the proof of Claim 8. $\square$

A Technical Lemmas

Definition A.1

*Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω. An **outer measure** on Ω is a function $\varphi : \mathcal{P}(\Omega) \longrightarrow [0,\infty]$ satisfying the following conditions:*

- $\varphi(\varnothing) = 0$.
- *monotonicity:* $\varphi(A) \leq \varphi(B)$, *for every* $A, B \in \mathcal{P}(\Omega)$ *with* $A \subset B$.
- *countable sub-additivity:*

$$\varphi\left(\bigcap_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \varphi(A_i), \quad \text{for any } A_1, A_2, \ldots \in \mathcal{P}(\Omega).$$

Definition A.2

Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω. Let $\varphi : \mathcal{P}(\Omega) \longrightarrow [0,\infty]$ be an outer measure on Ω. A subset $A \subset \Omega$ is said to be φ-measurable if

$$\varphi(E) = \varphi(A \cap E) + \varphi(A^c \cap E), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

Theorem A.3

Let Ω be a non-empty set and $\mathcal{P}(\Omega)$ denote the power set of Ω. Let $\varphi : \mathcal{P}(\Omega) \longrightarrow [0,\infty]$ be an outer measure on Ω.

(i) *A subset $A \subset \Omega$ is φ-measurable if and only if*

$$\varphi(E) \geq \varphi(A \cap E) + \varphi(A^c \cap E), \quad \text{for every } E \in \mathcal{P}(\Omega).$$

(ii) *The collection $\mathcal{A}(\varphi)$ of φ-measurable subsets of Ω forms a σ-algebra of subsets of Ω.*

(iii) *The restriction $\varphi|_{\mathcal{A}(\varphi)}$ of the outer measure φ to the σ-algebra $\mathcal{A}(\varphi)$ is a (countably additive) complete measure on the measurable space $(\Omega, \mathcal{A}(\varphi))$.*

Lemma A.4

Every compact subset of a metric space is also a separable subset of that metric space.

Proof Let (X,ρ) be a metric space and $K \subset X$ be a compact subset of X. For each $x \in X$ and positive $r > 0$, let

$$B(x,r) := \{\, y \in X \mid \rho(x,y) < r \,\} \subset X,$$

i.e. $B(x,r)$ is the open ball in X centred at x with radius $r > 0$. For each $n \in \mathbb{N}$, the following forms an open cover of K:

$$\mathcal{C}_n := \left\{ B\left(x, \frac{1}{n}\right) \subset X \;\middle|\; x \in K \right\}.$$

Since K is compact, each $\mathcal{C}_n$ admits a finite subcover:

$$\mathcal{F}_n := \left\{ B\left(x_i^{(n)}, \frac{1}{n}\right) \subset X \;\middle|\; x_i^{(n)} \in K,\ i = 1, 2, \ldots, J_n \right\}.$$

Let

$$\mathcal{D}_n := \left\{ x_i^{(n)} \in K \;\middle|\; i = 1, 2, \ldots, J_n \right\} \subset K,$$

and let $\mathcal{D} := \bigcup_{n=1}^{\infty} \mathcal{D}_n \subset K$. We claim that $\mathcal{D}$ is dense in K. Indeed, let $y \in K$. Since each $\mathcal{F}_n$ is a (finite) open cover of K, we have:

$$y \;\in\; K \;\subset\; \bigcup_{i=1}^{J_n} B\left(x_i^{(n)}, \frac{1}{n} \right), \quad \text{for each } n \in \mathbb{N}.$$

Since $x_i^{(n)} \in \mathcal{D}$, for each $i = 1, 2, \ldots, J_n$ and for each $n \in \mathbb{N}$, the above inclusion shows that, for each $n \in \mathbb{N}$, there exists some $x \in \mathcal{D}$ such that $\rho(y, x) < \dfrac{1}{n}$. In particular, $\mathcal{D}$ contains a sequence that converges to $y \in K$. Since $y \in K$ is an arbitrary element of K, we see that $\overline{D} \supset K$. Since $\mathcal{D} \subset K$ and K is compact, hence closed, we trivially have $\overline{D} \subset K$. We may now conclude that $\overline{D} = K$. This completes the proof of the Lemma. □

Lemma A.5

Every countable union of separable subsets of a metric space is itself a separable subset of that metric space.

Proof Let $S := \bigcup_{i=1}^{\infty} S_i \subset X$ be a countable union of separable subsets S_i of a metric space X. For each fixed $i \in \mathbb{N}$, since S_i is separable, there exists countable $D_i \subset S_i$ which is dense in S_i. Let $D := \bigcup_{i=1}^{\infty} D_i$. Then, D is a countable subset of S. The Lemma is proved once we establish that D is dense in S. To this end, let $x \in S = \bigcup_{i=1}^{\infty} S_i$. Then, $x \in S_i$ for some $i \in \mathbb{N}$. Since D_i is dense in S_i, there exists a sequence $\{ y_k \} \subset D_i \subset D$ such that $y_k \longrightarrow x$, as $k \longrightarrow \infty$. This proves that D is indeed dense in S, and completes the proof of the Lemma. □

Lemma A.6 (second theorem in Appendix M3, [2])

Let (S, ρ) be a metric space and $\Sigma \subset S$ a separable subset of S. Then, there exists a countable collection $\mathcal{A}$ of open subsets of S satisfying the following property: For each $x \in S$ and each open subset G of S,

$$x \;\in\; G \bigcap \Sigma \quad \Longrightarrow \quad x \;\in\; A \;\subset\; \overline{A} \;\subset\; G, \;\; \text{for some } A \in \mathcal{A}.$$

Proof Let $D \subset \Sigma$ be a countable dense subset of Σ. Let

$$\mathcal{A} \;:=\; \left\{ \; B(d, r) \;\subset\; S \;\middle|\; \begin{array}{c} d \in D, \\ r \in \mathbb{Q}, \; r > 0 \end{array} \right\}.$$

Then, $\mathcal{A}$ is a countable collection of open balls in S. Now, let $G \subset S$ be an arbitrary open subset of S and $x \in G \bigcap \Sigma$. First, choose $\varepsilon > 0$ such that $B(x, \varepsilon) \subset G$. Next, since $x \in \Sigma$ and D is dense in Σ, we may choose $d \in D$ such that $d \in B(x, \varepsilon/2)$, or equivalently $\rho(x, d) < \varepsilon/2$. Finally choose positive rational $r > 0$ such that $\rho(x, d) < r < \varepsilon/2$.

Now, note that $\overline{B(d, r)} \subset B(x, \varepsilon)$; indeed,

$$y \in \overline{B(d, r)} \quad \Longleftrightarrow \quad \rho(y, d) \leq r \quad \Longrightarrow \quad \rho(x, y) \;\leq\; \rho(x, d) + \rho(d, y) \;<\; \varepsilon/2 + r \;<\; \varepsilon/2 + \varepsilon/2 \quad \Longrightarrow \quad y \in B(x, \varepsilon).$$

Thus, we have

$$x \;\in\; B(d, r) \;\subset\; \overline{B(d, r)} \;\subset\; B(x, \varepsilon) \;\subset\; G.$$

This completes the proof of the Lemma. □

Theorem A.7 (The Diagonal Method, Appendix A.14, [1])

Suppose that each row of the array

$$\begin{array}{cccc} x_{1,1} & x_{1,2} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

is a bounded sequence of real numbers. Then, there exists an increasing sequence

$$n_1 \; < \; n_2 \; < \; n_3 \; < \cdots \; \in \; \mathbb{N}$$

of positive integers such that the limit

$$\lim_{k\to\infty} x_{r,n_k} \; \text{exists}, \quad \text{for each } r = 1,2,3,\ldots$$

Proof From the first row, select a convergent subsequence

$$x_{1,n(1,1)}, \;\; x_{1,n(1,2)}, \;\; x_{1,n(1,3)}, \;\; \cdots\cdots$$

Here, we have $n(1,1) < n(1,2) < n(1,3) < \cdots \in \mathbb{N}$, and $\lim\limits_{k\to\infty} x_{1,n(1,k)} \in \mathbb{R}$ exists. Next, note that the following subsequence of the second row:

$$x_{2,n(1,1)}, \;\; x_{2,n(1,2)}, \;\; x_{2,n(1,3)}, \;\; \cdots\cdots$$

is still a bounded sequence of real numbers, and we may thus select a convergent subsequence:

$$x_{2,n(2,1)}, \;\; x_{2,n(2,2)}, \;\; x_{2,n(2,3)}, \;\; \cdots\cdots$$

Here, we have $n(2,1) < n(2,2) < n(2,3) < \cdots \in \{\, n(1,k) \,\}_{k\in\mathbb{N}}$, and $\lim\limits_{k\to\infty} x_{2,n(2,k)} \in \mathbb{R}$ exists. Continuing inductively, we obtain an array of positive integers

$$\begin{array}{cccc} n(1,1) & n(1,2) & n(1,3) & \cdots \\ n(2,1) & n(2,2) & n(2,3) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

which satisfies: For each $r \in \mathbb{N}$, we have

- each row is an increasing sequence of positive integers, i.e. $n(r,1) < n(r,2) < n(r,3) < \cdots$,
- the $(r+1)^{\text{th}}$ row is a subsequence of the r^{th} row, i.e. $\{\, n(r+1,k) \,\}_{k\in\mathbb{N}} \subset \{\, n(r,k) \,\}_{k\in\mathbb{N}}$, and
- $\lim\limits_{k\to\infty} x_{r,n(r,k)} \in \mathbb{R}$ exists.

Note that the first two properties together imply:

$$n(k,k) \;\; < \;\; n(k,k+1) \;\; \leq \;\; n(k+1,k+1), \quad \text{for each } k \in \mathbb{N}.$$

Now, define $n_k := n(k,k)$, for $k \in \mathbb{N}$. We then see that

$$n_k \;\; := \;\; n(k,k) \;\; < n(k+1,k+1) \;\; =: \;\; n_{k+1},$$

i.e., $\{\, n_k \,\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive integers. Lastly, for each $r \in \mathbb{N}$, consider the sequence

$$x_{r,n_1}\,, \;\; x_{r,n_2}\,, \;\; x_{r,n_3}\,, \;\; \cdots$$

Note that, for each $r \in \mathbb{N}$,

$$x_{r,n_r}\,, \;\; x_{r,n_{r+1}}\,, \;\; x_{r,n_{r+2}}\,, \;\; \cdots$$

is a subsequence of $\{\, x_{r,n(r,k)} \,\}_{k\in\mathbb{N}}$. We saw above that $\lim\limits_{k\to\infty} x_{r,n(r,k)}$ exists, which in turn implies that $\lim\limits_{k\to\infty} x_{r,n_k}$ exists. Since $r \in \mathbb{N}$ is arbitrary, the proof of the Theorem is now complete. $\square$

References

[1] Billingsley, P. *Probability and Measure*, third ed. John Wiley & Sons, 1995.

[2] Billingsley, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.