# Survival Analysis

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## 1 Outline

Suppose:

- $(\Omega, \mathcal{A}, \mu)$ is a probability space.
- $n \in \mathbb{N}$ is an natural number (positive integer).
- $T_1, T_2, \ldots, T_n : \Omega \longrightarrow [0, \infty]$ are independent identically distributed extended $\mathbb{R}$-valued random variables.
- $U_1, U_2, \ldots, U_n : \Omega \longrightarrow [0, \infty]$ are independent identically distributed extended $\mathbb{R}$-valued random variables.
- For each $i = 1, 2, \ldots, n$, let $X_i := \min\{T_i, U_i\}$, and $C_i := I_{\{T_i \,\leq\, U_i\}}$.

For each subject $i = 1, 2, \ldots, n$, the random variable $T_i$ is interpreted to be the "survival time" of subject $i$, while $U_i$ is interpreted to be the "censoring time" of subject $i$.

We wish to make inference about the (common) *survival function*

$$S(t) \;\; := \;\; P(\, T > t \,) \;\; = \;\; \mu\Big(\Big\{\, \omega \in \Omega \;\Big|\; T(\omega) > t \,\Big\}\Big)$$

of $T_1, T_2, \ldots, T_n$. However, in survival analysis, the inference about $S(t)$ is made based on the *right-censored survival time data* $\{\, X_i,\, C_i \,\}$, $i = 1, 2, \ldots, n$ (rather than on the $T_i$'s directly).

The *hazard function*:

$$\lambda(t) \;\; := \;\; \lim_{h \to 0^+} \frac{1}{h} \cdot P\Big(\, t \leq T < t + h \;\Big|\; t \leq T \,\Big)$$

The *cumulative hazard function*:

$$\Lambda(t) \;\; := \;\; \int_0^t \lambda(t)\,\mathrm{d}t$$

The *Nelson-Aalen estimator* for the cumulative hazard function $\Lambda(t)$:

$$\widehat{\Lambda}(\omega, t) \;\; := \;\; \sum_{\substack{C_i(\omega) \,=\, 1 \\ T_i(\omega) \,\leq\, t}} \frac{1}{Y(\omega, T_i(\omega))}\,,$$

where

$$Y_i(\omega, t) \;\; := \;\; \left\{\begin{array}{ll} 1, & t - h < X_i(\omega),\; \text{for each } h > 0 \\ 0, & \text{otherwise} \end{array}\right.$$

and

$$Y(\omega, t) \;\; := \;\; \sum_{i\,=\,1}^{n} Y_i(\omega, t)$$

The aggregated counting process for subject $i$:

$$N_i(\omega, t) \;\; := \;\; I_{\{\, X_i(\omega) \,\leq\, t \,\}}$$

The aggregated counting process:

$$N(\omega, t) \;\; := \;\; \sum_{i\,=\,1}^{n} N_i(\omega, t) \;\; = \;\; \sum_{i\,=\,1}^{n} I_{\{\, X_i(\omega) \,\leq\, t \,\}}$$

The aggregated intensity process:

$$\alpha(\omega,t) \;\; := \;\; \lim_{h\to 0^+} \frac{1}{h}\cdot P\left( N(\omega,t+h) - N(\omega,t) = 1 \;\middle|\; \mathcal{F}_t \right) \;\; = \;\; \lim_{h\to 0^+} \frac{1}{h}\cdot E\left[ N(\omega,t+h) - N(\omega,t) \;\middle|\; \mathcal{F}_t \right]$$

The aggregated cumulative intensity process:

$$A(\omega,t) \;\; := \;\; \int_0^t \alpha(\omega,t)\,\mathrm{d}t$$

Then, the process

$$M(\omega,t) \;\; := \;\; N(\omega,t) - A(\omega,t) \;\; = \;\; N(\omega,t) - \int_0^t \alpha(\omega,t)\,\mathrm{d}t$$

is a martingale process. In particular, $M(\,\cdot\,,t)$ satisfies

$$E\left[\, M(\,\cdot\,,t+h) - M(\,\cdot\,,t) \;\middle|\; \mathcal{F}_t \right](\omega) \;\; = \;\; M(\omega,t)$$

# A Integration on product measure spaces

**Definition A.1 (Product $\sigma$-algebra)**
*Suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces. The product $\sigma$-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ of $\mathcal{A}_1$ and $\mathcal{A}_2$ is, by definition, the following:*

$$\mathcal{A}_1 \otimes \mathcal{A}_2 \;:=\; \sigma\left(\left\{\; A_1 \times A_2 \in \Omega_1 \times \Omega_2 \;\middle|\; A_1 \in \mathcal{A}_1,\; A_2 \in \mathcal{A}_2 \;\right\}\right).$$

*In other words, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the $\sigma$-algebra of subsets of $\Omega_1 \times \Omega_2$ containing all Cartesian products $A_1 \times A_2$, where $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.*

**Definition A.2 (Horizontal and vertical sections in a set-theoretic Cartesian product)**
*Suppose $X$ and $Y$ are two non-empty sets. For each $x \in X$, $y \in Y$, and $V \subset X \times Y$, we define:*

$$\begin{aligned} V_{(x,\,\cdot\,)} \;&:=\; \left\{\; y \in Y \;\middle|\; (x,y) \in V \;\right\} \\ V_{(\,\cdot\,,y)} \;&:=\; \left\{\; x \in X \;\middle|\; (x,y) \in V \;\right\} \end{aligned}$$

**Theorem A.3 (Sections of measurable subsets in a product measurable space are themselves measurable.)**
*Suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces. Then,*

(i) *$V_{(x,\,\cdot\,)} \in \mathcal{A}_2$, for each $x \in \Omega_1$ and each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$, and*

(ii) *$V_{(\,\cdot\,,y)} \in \mathcal{A}_1$, for each $y \in \Omega_2$ and each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$.*

Proof We give only the proof of (i); that of (ii) is similar. Define $\mathcal{F} \subset \mathcal{P}(\Omega_1 \times \Omega_2)$ as follows:

$$\mathcal{F} \;:=\; \left\{\; V \in \Omega_1 \times \Omega_2 \;\middle|\; V_{(x,\,\cdot\,)} \in \mathcal{A}_2,\; \text{for each } x \in \Omega_1 \;\right\}.$$

**Claim 1:** $\left\{\; A_1 \times A_2 \in \Omega_1 \times \Omega_2 \;\middle|\; A_1 \in \mathcal{A}_1,\; A_2 \in \mathcal{A}_2 \;\right\} \;\subset\; \mathcal{F}$

**Claim 2:** $\mathcal{F}$ is a $\sigma$-algebra of subsets of $\Omega_1 \times \Omega_2$.

Proof of Claim 1: Suppose $x \in \Omega_1$ and $V = A_1 \times A_2$, where $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. Then,

$$V_{(x,\,\cdot\,)} \;=\; \begin{cases} A_2, & \text{if } x \in A_1 \\ \varnothing, & \text{otherwise} \end{cases}$$

This proves that $V_{(x,\,\cdot\,)} = (A_1 \times A_2)_{(x,\,\cdot\,)} \subset \mathcal{F}$. Since $x \in \Omega_1$, $A_1 \in \mathcal{A}_1$, and $A_2 \in \mathcal{A}_2$ are arbitrary, Claim 1 follows.

Proof of Claim 2: First, note that, for each $x \in \Omega_1$, we have $(\Omega_1 \times \Omega_2)_{(x,\,\cdot\,)} := \left\{\; y \in \Omega \;\middle|\; (x,y) \in \Omega_1 \times \Omega_2 \;\right\} = \Omega_2 \in \mathcal{A}_2$. Hence, $\Omega_1 \times \Omega_2 \in \mathcal{F}$. Next, suppose $V \in \mathcal{F}$ and $V^c := (\Omega_1 \times \Omega_2) \setminus V$. Then, for each $x \in \Omega_1$,

$$\begin{aligned} (\,V^c\,)_{(x,\,\cdot\,)} \;&=\; \left\{\; y \in \Omega_2 \;\middle|\; (x,y) \in V^c \;\right\} \;=\; \left\{\; y \in \Omega_2 \;\middle|\; (x,y) \notin V \;\right\} \\ &=\; \Omega_2 \Big\backslash \left\{\; y \in \Omega_2 \;\middle|\; (x,y) \in V \;\right\} \;=\; \left(\,V_{(x,\,\cdot\,)}\,\right)^c \;\in\; \mathcal{A}_2, \end{aligned}$$

where the last containment follows from the fact that $\mathcal{A}_2$ is a $\sigma$-algebra (hence closed under complementation) and that $V \in \mathcal{F}$ (hence $V_{(x,\,\cdot\,)} \in \mathcal{A}_2$). This proves that $\mathcal{F}$ is closed under complementation. Lastly, suppose $V_1, V_2, \ldots, \in \mathcal{F}$. Then,

$$\left(\bigcup_{i=1}^{\infty} V_i\right)_{(x,\,\cdot\,)} \;=\; \left\{\; y \in \Omega_2 \;\middle|\; (x,y) \in \bigcup_{i=1}^{\infty} V_i \;\right\} \;=\; \bigcup_{i=1}^{\infty} \left\{\; y \in \Omega_2 \;\middle|\; (x,y) \,\in\, V_i \;\right\} \;=\; \bigcup_{i=1}^{\infty} (\,V_i\,)_{(x,\,\cdot\,)} \;\in\; \mathcal{A}_2,$$

where the last containment follows from the fact that $\mathcal{A}_2$ is a $\sigma$-algebra (hence closed under countable union) and that each $V_i \in \mathcal{F}$ (hence $(V_i)_{(x,\,\cdot\,)} \in \mathcal{A}_2$). This proves that $\mathcal{F}$ is closed under countable union. This completes the proof of Claim 2.

Claim 1 and Claim 2 together immediately imply that

$$\mathcal{A}_1 \otimes \mathcal{A}_2 \;:=\; \sigma\left(\left\{\; A_1 \times A_2 \in \Omega_1 \times \Omega_2 \;\middle|\; A_1 \in \mathcal{A}_1,\; A_2 \in \mathcal{A}_2 \;\right\}\right) \;\subset\; \mathcal{F} \;:=\; \left\{\; V \in \Omega_1 \times \Omega_2 \;\middle|\; \begin{array}{c} V_{(x,\,\cdot\,)} \in \mathcal{A}_2\,, \\ \text{for each } x \in \Omega_1 \end{array} \right\}.$$

This completes the proof of statement (i) in the present Theorem. □

**Theorem A.4 (Sections of measurable maps are themselves measurable.)**
*Suppose* $(\Omega_1, \mathcal{A}_1)$, $(\Omega_2, \mathcal{A}_2)$, $(S, \mathcal{S})$ *are measurable spaces, and* $f : (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \longrightarrow (S, \mathcal{S})$ *is an* $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{S})$*-measurable map. Then,*

(i) $f(x,\,\cdot\,) : \Omega_2 \longrightarrow S : y \longmapsto f(x,y)$ *is an* $(\mathcal{A}_2, \mathcal{S})$*-measurable map for each* $x \in \Omega_1$.

(ii) $f(\,\cdot\,,y) : \Omega_1 \longrightarrow S : x \longmapsto f(x,y)$ *is an* $(\mathcal{A}_1, \mathcal{S})$*-measurable map for each* $y \in \Omega_2$.

Proof

(i) We need to show that $f(x,\,\cdot\,)^{-1}(V) \in \mathcal{A}_2$, for each $x \in \Omega_1$, and each $V \in \mathcal{S}$. To this end, note that

$$f(x,\,\cdot\,)^{-1}(V) \;=\; \left\{\; y \in \Omega_2 \;\middle|\; f(x,y) \in V \;\right\} \;=\; \left\{\; y \in \Omega_2 \;\middle|\; (x,y) \in f^{-1}(V) \;\right\} \;=\; f^{-1}(V)_{(x,\,\cdot\,)} \;\in\; \mathcal{A}_2\,,$$

where the last containment follows, by Theorem A.3, from the fact that $f^{-1}(V) \in \mathcal{A}_1 \otimes \mathcal{A}_2$ (since $V \in \mathcal{S}$ and $f$ is $(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{S})$-measurable).

(ii) The proof here is similar to that of (i). □

**Definition A.5 (Elementary subsets of the set-theoretic Cartesian product of two measurable spaces)**
*Suppose* $(\Omega_1, \mathcal{A}_1)$ *and* $(\Omega_2, \mathcal{A}_2)$ *are two measurable spaces. The collection of* <u>*elementary subsets*</u> *of* $\Omega_1 \times \Omega_2$ *with respect to their respective* $\sigma$*-algebras* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *is, by definition, the following:*

$$\mathcal{E}(\mathcal{A}_1, \mathcal{A}_2) \;:=\; \left\{\; \bigsqcup_{i\,=\,1}^{n} A_1^{(i)} \times A_2^{(i)} \;\in\; \Omega_1 \times \Omega_2 \;\middle|\; \begin{array}{l} A_k^{(i)} \in \mathcal{A}_k,\;\; \text{for } k = 1, 2, \\ \text{for each } i = 1, 2, \ldots, n, \\ \text{for each } n \in \mathbb{N} \end{array} \right\}$$

**Definition A.6 (Monotone class)**
*Suppose* $X$ *is a non-empty set. Then, a collection* $\mathcal{M}$ *of subsets of* $X$ *is called a* <u>*monotone class*</u> *if* $\mathcal{M}$ *satisfies both of the following two conditions:*

(i)
$$A \;:=\; \bigcup_{i\,=\,1}^{\infty} A_i \;\in\; \mathcal{M}, \quad \text{whenever } \{\, A_i \,\}_{i\in\mathbb{N}} \text{ satistfies } A_i \in \mathcal{M} \text{ and } A_i \subset A_{i+1}, \text{ for each } i \in \mathbb{N}.$$

(ii)
$$B \;:=\; \bigcap_{i\,=\,1}^{\infty} B_i \;\in\; \mathcal{M}, \quad \text{whenever } \{\, B_i \,\}_{i\in\mathbb{N}} \text{ satistfies } B_i \in \mathcal{M} \text{ and } B_i \supset B_{i+1}, \text{ for each } i \in \mathbb{N}.$$

**Theorem A.7 (An arbitrary intersection of monotone classes is itself a monotone class)**
*Suppose $X$ is a non-empty set and $\{\,\mathcal{M}_t\,\}_{t\in T}$ is a family of monotone classes of subsets of $X$ indexed by the non-empty set $T$. Then,*

$$\mathcal{M} \;:=\; \bigcap_{t\,\in\,T} \mathcal{M}_t \;\subset\; \mathcal{P}(\,X\,)$$

*is itself a monotone class of subsets of $X$.*

PROOF Suppose $\{\,A_i\,\}_{i\in\mathbb{N}}$ satisfies $A_i \subset A_{i+1}$, for each $i \in \mathbb{N}$. Then, note the following implications:

$$\begin{aligned}
& A_i \,\in\, \mathcal{M} \,=\, \bigcap_{t\in T} M_t, \text{ for each } i \in \mathbb{N} \\
\Longleftrightarrow\quad & A_i \in \mathcal{M}_t, \;\; \text{for each } i \in \mathbb{N}, \text{ each } t \in T \\
\Longrightarrow\quad & A \,:=\, \bigcup_{i\,=\,1}^{\infty} A_i \,\in\, \mathcal{M}_t, \;\; \text{for each } t \in T \quad (\text{since each } \mathcal{M}_t \text{ is a monotone class}) \\
\Longrightarrow\quad & A \,:=\, \bigcup_{i\,=\,1}^{\infty} A_i \;\;\in\;\; \mathcal{M} := \bigcap_{t\in T} M_t
\end{aligned}$$

Similarly, suppose $\{\,B_i\,\}_{i\in\mathbb{N}}$ satisfies $B_i \supset B_{i+1}$, for each $i \in \mathbb{N}$. Then, note the following implications:

$$\begin{aligned}
& B_i \,\in\, \mathcal{M} \,=\, \bigcap_{t\in T} M_t, \text{ for each } i \in \mathbb{N} \\
\Longleftrightarrow\quad & B_i \in \mathcal{M}_t, \;\; \text{for each } i \in \mathbb{N}, \text{ each } t \in T \\
\Longrightarrow\quad & B \,:=\, \bigcap_{i\,=\,1}^{\infty} B_i \,\in\, \mathcal{M}_t, \;\; \text{for each } t \in T \quad (\text{since each } \mathcal{M}_t \text{ is a monotone class}) \\
\Longrightarrow\quad & B \,:=\, \bigcap_{i\,=\,1}^{\infty} B_i \;\;\in\;\; \mathcal{M} := \bigcap_{t\in T} M_t
\end{aligned}$$

This shows that $\mathcal{M} \,:=\, \bigcap_{t\,\in\,T} \mathcal{M}_t$ is indeed a monotone class, and completes the proof of the Theorem. □

**Theorem A.8**
*Suppose $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ are two measurable spaces. Then, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the smallest monotone class which satistifes $\mathcal{E}(\mathcal{A}_1, \mathcal{A}_2) \subset \mathcal{A}_1 \otimes \mathcal{A}_2$.*

PROOF First note that, since $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a $\sigma$-algebra, it is closed under countable intersections and countable unions. Hence, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is in particular a monotone class. It is also immediate that $\mathcal{E}(\mathcal{A}_1, \mathcal{A}_2) \subset \mathcal{A}_1 \otimes \mathcal{A}_2$, since $\mathcal{A}_1 \otimes \mathcal{A}_2$ is closed under finite disjoint unions (being closed under countable unions) and it contains all subsets of $\Omega_1 \times \Omega_2$ of the form $A_1 \times A_2$ with $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. So, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a monotone class of subsets of $\Omega_1 \times \Omega_2$ which contains $\mathcal{E}(\mathcal{A}_1, \mathcal{A}_2)$.

Let $\mathcal{M}$ be the smallest monotone class containing $\mathcal{E}(\mathcal{A}_1, \mathcal{A}_2)$. By Theorem A.7, $\mathcal{M}$ exists and equals the intersection of all monotone classes of subsets of $\Omega_1 \times \Omega_2$ which contain $\mathcal{E}(\mathcal{A}_1, \mathcal{A}_2)$. Thus, we have $\mathcal{M} \subset \mathcal{A}_1 \otimes \mathcal{A}_2$. Letting $\mathcal{A}_1 \times \mathcal{A}_2 \;:=\; \left\{\; A_1 \times A_2 \in \Omega_1 \times \Omega_2 \;\middle|\; A_1 \in \mathcal{A}_1,\; A_2 \in \mathcal{A}_2 \;\right\}$, we have the following series of containment:

$$\mathcal{A}_1 \times \mathcal{A}_2 \;\;\subset\;\; \mathcal{E}(\mathcal{A}_1, \mathcal{A}_2) \;\;\subset\;\; \mathcal{M} \;\;\subset\;\; \mathcal{A}_1 \otimes \mathcal{A}_2 \;\;:=\;\; \sigma(\,\mathcal{A}_1 \times \mathcal{A}_2\,) \;\;\subset\;\; \mathcal{P}(\,\Omega_1 \times \Omega_2\,)\,.$$

Thus, the present Theorem is equivalent to the equality $\mathcal{M} \;=\; \mathcal{A}_1 \otimes \mathcal{A}_2$, which will follow immediately from the following:

**Claim:** $\mathcal{M}$ is a $\sigma$-algebra of subsets of $\Omega_1 \times \Omega_2$.

Proof of Claim:

$\square$

**Theorem A.9 (Well-definition of the product measure of two $\sigma$-finite measures)**
*Suppose $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ are two $\sigma$-finite measure spaces. Let $(\mathbb{R}, \mathcal{B})$ be $\mathbb{R}$ equipped with its Borel $\sigma$-algebra $\mathcal{B} = \sigma(\mathcal{O}(\mathbb{R}))$. Then, for each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$, the following statements hold:*

(i) *the map* $\Omega_1 \longrightarrow \mathbb{R} : x \longmapsto \mu_2\big(V_{(x,\,\cdot\,)}\big) = \int_{\Omega_2} 1_V(x,y)\,\mathrm{d}\mu_2(y)$ *is $(\mathcal{A}_1, \mathcal{B})$-measurable,*

(ii) *the map* $\Omega_2 \longrightarrow \mathbb{R} : y \longmapsto \mu_1\big(V_{(\,\cdot\,,y)}\big) = \int_{\Omega_1} 1_V(x,y)\,\mathrm{d}\mu_1(x)$ *is $(\mathcal{A}_2, \mathcal{B})$-measurable, and*

(iii) *the following equality of Lebesgue integrals (of measurable $\mathbb{R}$-valued functions) holds:*

$$\int_{\Omega_1} \mu_2\big(V_{(x,\,\cdot\,)}\big)\;\mathrm{d}\mu_1(x) \;\; = \;\; \int_{\Omega_2} \mu_1\big(V_{(\,\cdot\,,y)}\big)\;\mathrm{d}\mu_2(y),$$

*or equivalently,*

$$\int_{\Omega_1}\left(\int_{\Omega_2} 1_V(x,y)\,\mathrm{d}\mu_2(y)\right)\mathrm{d}\mu_1(x) \;\; = \;\; \int_{\Omega_2}\left(\int_{\Omega_1} 1_V(x,y)\,\mathrm{d}\mu_1(x)\right)\mathrm{d}\mu_2(y).$$

Proof Define $\mathcal{C} \subset \mathcal{A}_1 \otimes \mathcal{A}_2$ as follows:

$$\mathcal{C} \;\; := \;\; \left\{\, V \in \mathcal{A}_1 \otimes \mathcal{A}_2 \;\;\middle|\;\; \int_{\Omega_1} \mu_2\big(V_{(x,\,\cdot\,)}\big)\;\mathrm{d}\mu_1(x) \, = \, \int_{\Omega_2} \mu_1\big(V_{(\,\cdot\,,y)}\big)\;\mathrm{d}\mu_2(y) \,\right\}.$$

**Claim 1:** $A_1 \times A_2 \in \mathcal{C}$, for each $A_1 \in \mathcal{A}_1$ and each $A_2 \in \mathcal{A}_2$.

**Claim 2:** $V \, := \, \bigcup_{i=1}^{\infty} V_i \, \in \, \mathcal{C}$, whenever $\{\, V_i \,\}_{i\in\mathbb{N}} \subset \mathcal{C}$ and $V_i \subset V_{i+1}$, for each $i \in \mathbb{N}$.

**Claim 3:** $V \, := \, \bigsqcup_{i=1}^{\infty} V_i \, \in \, \mathcal{C}$, whenever $\{\, V_i \,\}_{i\in\mathbb{N}} \subset \mathcal{C}$ is a disjoint countable collection of members in $\mathcal{C}$.

**Claim 4:** Suppose $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$, with $\mu_1(A_1), \mu_2(A_2) < \infty$. Suppose also that $\{\, V_i \,\}_{i\in\mathbb{N}} \subset \mathcal{C}$ satisfies $A_1 \times A_2 \supset V_1 \supset V_2 \supset V_3 \supset \cdots$. Then, $V \, := \, \bigcap_{i=1}^{\infty} V_i \, \in \, \mathcal{C}$.

Proof of Claim 1:

Proof of Claim 2:

Proof of Claim 3:

Proof of Claim 4:

Next, note that, since $(\Omega_1, \mathcal{A}_1, \mu_1)$ is a $\sigma$-finite measure space, there exist mutually disjoint $\Omega_1^{(1)}, \Omega_1^{(2)}, \ldots \in \mathcal{A}_1$ such that

$$\Omega_1 \;=\; \bigsqcup_{n=1}^{\infty} \Omega_1^{(n)}, \quad \text{and} \quad \mu_1\!\left(\Omega_1^{(n)}\right) \;<\; \infty, \;\; \text{for each } n \in \mathbb{N}.$$

Similarly, there exist mutually disjoint $\Omega_2^{(1)}, \Omega_2^{(2)}, \ldots \in \mathcal{A}_2$ such that

$$\Omega_2 \;=\; \bigsqcup_{n=1}^{\infty} \Omega_2^{(n)}, \quad \text{and} \quad \mu_2\!\left(\Omega_2^{(n)}\right) \;<\; \infty, \;\; \text{for each } n \in \mathbb{N}.$$

We now define

$$\mathcal{M} \;:=\; \left\{\, V \in \mathcal{A}_1 \otimes \mathcal{A}_2 \;\middle|\; V \cap \left(\Omega_1^{(m)} \times \Omega_2^{(n)}\right) \,\in\, \mathcal{C}, \text{ for each } m, n \in \mathbb{N} \,\right\}.$$

**Claim 5:** $\mathcal{M}$ is a monotone class.

**Claim 6:**

$$\mathcal{E} \;\subset\; \mathcal{M}$$

Proof of Claim 5: Suppose $V_1, V_2, \ldots \in \mathcal{M}$, with $V_1 \subset V_2 \subset V_3 \subset \cdots$. We need to show $V := \bigcup_{i=1}^{\infty} V_i \in \mathcal{M}$. To this end, note that, for each $m, n \in \mathbb{N}$, we have

$$V \cap \left(\Omega_1^{(m)} \times \Omega_2^{(n)}\right) \;=\; \left(\bigcup_{i=1}^{\infty} V_i\right) \bigcap \left(\Omega_1^{(m)} \times \Omega_2^{(n)}\right) \;=\; \bigcup_{i=1}^{\infty} \underbrace{\left(V_i \cap (\Omega_1^{(m)} \times \Omega_2^{(n)})\right)}_{\in\, \mathcal{C}} \;\in\; \mathcal{C},$$

where the last containment follows from Claim 3. Thus, we see that we indeed have $V \in \mathcal{M}$. Next, suppose that $W_1, W_2, \ldots \in \mathcal{M}$, with $W_1 \supset W_2 \supset W_3 \supset \cdots$. We need to show $W := \bigcap_{i=1}^{\infty} W_i \in \mathcal{M}$. Now, for each $m, n \in \mathbb{N}$, we have:

$$W \cap \left(\Omega_1^{(m)} \times \Omega_2^{(n)}\right) \;=\; \left(\bigcap_{i=1}^{\infty} W_i\right) \bigcap \left(\Omega_1^{(m)} \times \Omega_2^{(n)}\right) \;=\; \bigcap_{i=1}^{\infty} \underbrace{\left(W_i \cap (\Omega_1^{(m)} \times \Omega_2^{(n)})\right)}_{\in\, \mathcal{C}} \;\in\; \mathcal{C}.$$

where the last containment follows from Claim 4. This proves that $\mathcal{M}$ is indeed a monotone class and completes the proof of Claim 5.

It follows from Claim 5, Claim 6 and Theorem **??** that $\mathcal{M} = \mathcal{A}_1 \otimes \mathcal{A}_2$, which in turn implies that $V \cap \left(\Omega_1^{(m)} \times \Omega_2^{(n)}\right) \in \mathcal{C}$, for each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$ and each $m, n \in \mathbb{N}$. Hence, for each $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$, we have

$$V \;=\; V \,\cap\, (\,\Omega_1 \times \Omega_2) \;=\; V \bigcap \left(\bigsqcup_{m,n \in \mathbb{N}} \Omega_1^{(m)} \times \Omega_2^{(n)}\right) \;=\; \bigsqcup_{m,n \in \mathbb{N}} \underbrace{V \cap \left(\Omega_1^{(m)} \times \Omega_2^{(n)}\right)}_{\in\, \mathcal{C}} \;\in\; \mathcal{C},$$

where the last containment follows from Claim 3. Lastly, recall that $V \in \mathcal{C}$ is equivalent to

$$\int_{\Omega_1} \mu_2\big(V_{(x,\,\cdot\,)}\big)\,\mathrm{d}\mu_1(x) \;=\; \int_{\Omega_2} \mu_1\big(V_{(\,\cdot\,,y)}\big)\,\mathrm{d}\mu_2(y).$$

This completes the proof of the present Theorem. $\square$

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