Donsker's Theorems (Functional Central Limit Theorems)

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1 Equivalence of $(C[0,1]\,,\|\cdot\|_\infty)$-valued random variables and stochastic processes indexed by $[0,1]$ with state space $\mathbb{R}$ and continuous sample paths

Proposition 1.1 (The "one-dimensional subsets" of $C[0,1]$ generate its Borel σ-algebra)
Let $(C[0,1]\,,\|\cdot\|_\infty)$ be the metric space of continuous $\mathbb{R}$-valued functions defined on the closed unit interval equipped with the supremum norm. For each $t\in[0,1]$, let $\mathrm{ev}_t : C[0,1]\longrightarrow\mathbb{R} : x\longmapsto x(t)$. Define:

$$\mathcal{S} \;:=\; \left\{\ \mathrm{ev}_t^{-1}(H)\,\subset\,C[0,1]\ \left|\ \begin{array}{c} t\in[0,1] \\ H\in\mathcal{O}\end{array}\right.\right\}\ \subset\ \mathcal{P}(\,C[0,1]\,)\,.$$

Then, $\mathcal{S}$ generates the Borel σ-algebra $\mathcal{B}:=\mathcal{B}(C[0,1],\|\cdot\|_\infty)$ of the metric space $(C[0,1],\|\cdot\|_\infty)$; in other words,

$$\sigma(\,\mathcal{S}\,)\;=\;\mathcal{B}.$$

Proof First, note that $\sigma(\,\mathcal{S}\,)\subset\mathcal{B}$. Indeed, recall that, for each $t\in[0,1]$, $\mathrm{ev}_t : C[0,1]\longrightarrow\mathbb{R}$ is continuous, hence $(\mathcal{B},\mathcal{O})$-measurable, by Corollary B.4. In particular, $\mathrm{ev}_t^{-1}(H)\in\mathcal{B}$, for each $t\in[0,1]$ and $H\in\mathcal{O}$. Thus, $\mathcal{S}\subset\mathcal{B}$; hence, $\sigma(\,\mathcal{S}\,)\subset\mathcal{B}$.

It remains to establish the reverse inclusion. To this end, first observe that, for each $x\in C[0,1]$ and each $\varepsilon>0$, we have

$$\overline{B(x,\varepsilon)}\;=\;\bigcap_{r\,\in\,\mathbb{Q}\,\cap\,[0,1]}\left\{\ y\in C[0,1]\ \left|\ \,|\,y(r)-x(r)\,|\leq\varepsilon\right.\right\}\;=\;\bigcap_{r\,\in\,\mathbb{Q}\,\cap\,[0,1]}\mathrm{ev}_r^{-1}\Big(\,[x(r)-\varepsilon,x(r)+\varepsilon]\,\Big),$$

which shows that $\sigma(\,\mathcal{S}\,)$ contains all the closed balls in $C[0,1]$. On the other hand, recall that, in any metric space, every open ball can be expressed as a countable union of closed balls; indeed, for any y in the given metric space, and any $\delta>0$, we have:

$$B(y,\delta)\;=\;\bigcup_{n\in\mathbb{N}}\overline{B\left(y,\delta-\frac{1}{n}\right)}.$$

We thus see that $\sigma(\,\mathcal{S}\,)$ contains all the open balls in $C[0,1]$. By the separability of $C[0,1]$ and Theorem C.1, we see that every open subset of $C[0,1]$ can be expressed as a countable union of open balls. Hence, $\sigma(\,\mathcal{S}\,)$ in fact contains all the open subsets of $C[0,1]$, which immediately yields $\mathcal{B}\,\subset\,\sigma(\,\mathcal{S}\,)$. This proves $\sigma(\,\mathcal{S}\,)\,=\,\mathcal{B}$. □

Theorem 1.2
Suppose:

- $(\Omega,\mathcal{A})$ *is a measurable space.*
- *Let $(C[0,1]\,,\|\cdot\|_\infty)$ denote the metric space of continuous $\mathbb{R}$-valued functions defined on the compact unit interval equipped with the supremum norm.*

  Let $\mathcal{B}:=\mathcal{B}(C[0,1],\|\cdot\|_\infty)$ denote the Borel σ-algebra of the metric space $(C[0,1],\|\cdot\|_\infty)$.
- *Let $\mathcal{O}$ denote the Borel σ-algebra of $\mathbb{R}$ (equipped with usual Euclidean metric).*
- *$X:\Omega\longrightarrow C[0,1]$ is a function with domain Ω and codomain $C[0,1]$, but otherwise arbitrary.*
- *For each $t\in[0,1]$, let $\mathrm{ev}_t : C[0,1]\longrightarrow\mathbb{R} : x\longmapsto x(t)$.*

- *For each $t \in [0,1]$, let $X_t := \mathrm{ev}_t \circ X$. In other words, $X_t : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto \mathrm{ev}_t(X(\omega)) = X(\omega)(t)$.*

Then, X is $(\mathcal{A},\mathcal{B})$-measurable if and only if, for each $t \in [0,1]$, X_t is $(\mathcal{A},\mathcal{O})$-measurable.

Proof

($\Longrightarrow$)

It is trivial to see that, for each $t \in [0,1]$, $\mathrm{ev}_t : (\,C[0,1]\,, \|\cdot\|_\infty\,) \longrightarrow (\,\mathbb{R}\,, |\cdot|\,) : x \longmapsto x(t)$ is continuous. Recall that continuous maps are necessarily Borel measurable; see Corollary B.4. Hence, $\mathrm{ev}_t : (\,C[0,1]\,, \|\cdot\|_\infty\,) \longrightarrow (\,\mathbb{R}\,, |\cdot|\,)$ is $(\mathcal{B},\mathcal{O})$-measurable, for each $t \in [0,1]$. Now, suppose $X : \Omega \longrightarrow C[0,1]$ is $(\mathcal{A},\mathcal{B})$-measurable. Then, for each $t \in [0,1]$, the composition $X_t := \mathrm{ev}_t \circ X$ is $(\mathcal{A}\,,\mathcal{O})$-measurable, as required.

($\Longleftarrow$)

Suppose that, for each $t \in [0,1]$, $X_t := \mathrm{ev}_t \circ X$ is $(\mathcal{A}\,,\mathcal{O})$-measurable. We seek to establish that $X : (\Omega,\mathcal{A}) \longrightarrow (C[0,1],\mathcal{B})$ is $(\mathcal{A},\mathcal{B})$-measurable. To this end, let

$$\mathcal{S} \;:=\; \left\{\; \mathrm{ev}_t^{-1}(H) \,\subset\, C[0,1] \;\middle|\; \begin{array}{c} t \in [0,1] \\ H \in \mathcal{O} \end{array} \right\} \;\subset\; \mathcal{P}(\,C[0,1]\,)\,.$$

Then, note that the $(\mathcal{A}\,,\mathcal{B})$-measurable of X follows immediately from Theorem B.3, Proposition 1.1, and the following

Claim: $X^{-1}(\,\mathcal{S}\,) \;\subset\; \mathcal{A}$.

Proof of Claim: Every set in $\mathcal{S}$ has the form $\mathrm{ev}_t^{-1}(H)$, for some $t \in [0,1]$ and some $H \in \mathcal{O}$. Note that

$$X^{-1}\big(\mathrm{ev}_t^{-1}(H)\big) \;=\; (\mathrm{ev}_t \circ X)^{-1}\Big(\,H\,\Big) \;=\; X_t^{-1}\Big(\,H\,\Big) \;\in\; \mathcal{A},$$

where the last containment follows immediately from the $(\mathcal{A}\,,\mathcal{O})$-measurability hypothesis on X_t, for each $t \in [0,1]$. This shows that $X^{-1}(\,\mathcal{S}\,) \subset \mathcal{A}$ and completes the proof of the Claim.

The proof of the Theorem is now complete. □

Theorem 1.3

Suppose:

- *$(\Omega,\mathcal{A},\mu)$ is a probability space.*
- *Let $(C[0,1]\,, \|\cdot\|_\infty)$ denote the metric space of continuous $\mathbb{R}$-valued functions defined on the closed unit interval equipped with the supremum norm.*
- *$X : \Omega \longrightarrow C[0,1]$ is a function with domain Ω and codomain $C[0,1]$, but otherwise arbitrary.*
- *For each $t \in [0,1]$, let $\mathrm{ev}_t : C[0,1] \longrightarrow \mathbb{R} : x \longmapsto x(t)$.*
- *For each $t \in [0,1]$, let $X_t := \mathrm{ev}_t \circ X$. In other words, $X_t : \Omega \longrightarrow \mathbb{R} : \omega \longmapsto \mathrm{ev}_t(X(\omega)) = X(\omega)(t)$.*

Then, the following are equivalent:

(i) *X is a $(C[0,1]\,, \|\cdot\|_\infty)$-valued random variable.*

(ii) *For each $t \in [0,1]$, X_t is an $\mathbb{R}$-valued random variable.*

(iii) *$\{\,X_t : \Omega \longrightarrow \mathbb{R}\,\}_{t\in[0,1]}$ is a stochastic process indexed by the closed unit interval defined on the probability space $(\Omega,\mathcal{A},\mu)$ with state space $\mathbb{R}$ and continuous sample paths.*

2 Donsker's Theorem for $(\,C[0,1]\,,\,\|\cdot\|_\infty\,)$

Proposition 2.1

- *Let* $\xi_1, \xi_2, \ldots : \Omega \longrightarrow \mathbb{R}$ *be a sequence of independent and identically distributed* $\mathbb{R}$*-valued random variables defined on the probability space* $(\Omega, \mathcal{A}, \mu)$*, with expectation value zero and common finite variance* $\sigma^2 > 0$*.*
- *Define the random variables:*

$$\left\{\begin{array}{lllllll} S_0 & : & \Omega \longrightarrow \mathbb{R} & : & \omega \longmapsto 0, & & \text{and} \\ S_n & : & \Omega \longrightarrow \mathbb{R} & : & \omega \longmapsto \displaystyle\sum_{i=1}^{n} \xi_i(\omega), & & \text{for each } n \in \mathbb{N}. \end{array}\right.$$

- *For each* $n \in \mathbb{N}$*, define* $X^{(n)} : \Omega \longrightarrow C[0,1]$ *as follows:*

$$X^{(n)}(\omega)(t) \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{i-1}(\omega) \;+\; n\left(t - \frac{i-1}{n}\right)\xi_i(\omega) \right\}, \;\; \text{for each } \omega \in \Omega,\;\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],\;\; i = 1, 2, 3, \ldots, n.$$

- *For each* $n \in \mathbb{N}$ *and each* $t \in [0,1]$*, define* $X_t^{(n)} : \Omega \longrightarrow \mathbb{R}$ *as follows:*

$$X_t^{(n)}(\omega) \;:=\; X^{(n)}(\omega)(t), \quad \text{for each } \omega \in \Omega.$$

Then, the following statements are true:

(i) *For each* $\omega \in \Omega$ *and each* $n \in \mathbb{N}$*,*

$$X^{(n)}(\omega)\left(\frac{i}{n}\right) \;=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot S_i(\omega), \quad \text{for } i = 0, 1, 2, \ldots, n.$$

(ii) *For each* $\omega \in \Omega$ *and each* $n \in \mathbb{N}$*,*

$$X^{(n)}(\omega)(t) \;\; \textit{is the linear interpolation from} \;\; \frac{1}{\sigma \cdot \sqrt{n}} S_{i-1}(\omega) \;\; \textit{to} \;\; \frac{1}{\sigma \cdot \sqrt{n}} S_i(\omega) \;\; \textit{over} \;\; t \in \left[\frac{i-1}{n}, \frac{i}{n}\right],$$

where $i = 1, 2, \ldots, n$*.*

(iii) *For any* $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$*,*

$$\left(X_{t_1}^{(n)} - X_{t_0}^{(n)},\; \ldots\;,\; X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \right) \;\overset{d}{\longrightarrow}\; N\Big(\mu = \mathbf{0}\,,\, \Sigma = \operatorname{diag}(\, t_1 - t_0,\; \ldots\;,\; t_k - t_{k-1}\,) \Big), \;\; \text{as } \; n \longrightarrow \infty.$$

(iv) *For any* $0 \leq t_1, t_2, \cdots, t_k \leq 1$*,*

$$\left(X_{t_1}^{(n)}, X_{t_2}^{(n)}, \ldots, X_{t_k}^{(n)} \right) \;\overset{d}{\longrightarrow}\; N\Big(\mu = \mathbf{0}\,,\, \Sigma = \Big[\, \min\{t_i, t_j\} \,\Big]_{1 \leq i,j \leq k} \Big), \;\; \text{as } \; n \longrightarrow \infty.$$

Proof

(i) Obvious.

(ii) Obvious.

(iii) First, note that, for each $\omega \in \Omega$, $n \in \mathbb{N}$, and $t \in [0,1]$, we have

$$X_t^{(n)}(\omega) \;=\; \frac{1}{\sigma \cdot \sqrt{n}} \left\{ S_{\lfloor nt \rfloor}(\omega) \;+\; \Big(nt - \lfloor nt \rfloor \Big) \cdot \xi_{\lfloor nt \rfloor + 1}(\omega) \right\},$$

where $\lfloor \cdot \rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$, defined by

$$\lfloor x \rfloor \;:=\; \max \left\{ k \in \mathbb{Z} \;\middle\vert\; k \leq x \right\}, \quad \text{for each } x \in \mathbb{R},$$

is the round-down function. We next state three Claims, whose proofs will be given below. We note that the desired conclusion follows readily from Claim 3 and the Cramér-Wold Theorem (Theorem 1.9(iii), p.56, [3]); hence the present proof is complete once we establish the three Claims below.

Claim 1: If $\{ a_n \}_{n \in \mathbb{N}}$ is a sequence of non-negative integers and $\{ b_n \}_{n \in \mathbb{N}} \subset \mathbb{N}$ a sequence of positive integers satisfying:

$$a_n \;<\; b_n, \text{ for sufficiently large } n \in \mathbb{N}, \qquad \text{and} \qquad \lim_{n \to \infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$

then

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i = 1 + a_n}^{b_n} \xi_i \;\overset{d}{\longrightarrow}\; \sqrt{c} \cdot Z, \quad \text{where } Z \sim N(0,1).$$

Claim 2: For each fixed $t \in [0,1]$,

$$W(t)_n \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot \Big(nt - \lfloor nt \rfloor \Big) \cdot \xi_{\lfloor nt \rfloor + 1} \;\overset{d}{\longrightarrow}\; 0.$$

Claim 3: For $0 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq 1$, and arbitrary $c_1, c_2, \ldots, c_k \in \mathbb{R}$,

$$\sum_{i=1}^{k} c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \;\overset{d}{\longrightarrow}\; N\left(0 \,,\, \sum_{i=1}^{k} c_i^2 \, (t_i - t_{i-1}) \right), \quad \text{as } n \longrightarrow \infty.$$

<u>Proof of Claim 1:</u> Note that, for sufficiently large $n \in \mathbb{N}$, we may write

$$\frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{i = 1 + a_n}^{b_n} \xi_i \;=\; \frac{\sqrt{b_n - a_n}}{\sqrt{n}} \cdot \left(\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i = 1 + a_n}^{b_n} \xi_i \right).$$

Since, by hypothesis, that

$$\lim_{n \to \infty} \frac{b_n - a_n}{n} \;=\; c \;>\; 0,$$

Claim 1 follows by Slutsky's Theorem (Example 6, p.40, [2]), once we establish the following:

$$\frac{1}{\sigma \cdot \sqrt{b_n - a_n}} \cdot \sum_{i = 1 + a_n}^{b_n} \xi_i \;\overset{d}{\longrightarrow}\; N(0,1), \quad \text{as } n \longrightarrow \infty.$$

We establish the above convergence by invoking the Lindeberg Central Limit Theorem (Theorem 1.15, §1.5.5, p.67, [3]). In the present context, the Lindeberg Condition is the following:

$$\lim_{n\to\infty} \frac{1}{B_n^2} \cdot E\left[\ \sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\left\{ |\,\xi_i\,| \geq \varepsilon\, S_n \right\}} \right] \;=\; 0, \quad \text{for each } \varepsilon > 0,$$

where

$$B_n^2 \;:=\; \mathrm{Var}\left[\ \sum_{i=1+a_n}^{b_n} \xi_i \right] \;=\; (b_n - a_n)\,\sigma^2 \;>\; 0.$$

The last equality used the hypothesis that $\xi_1,\ \xi_2,\ \ldots$ are independent and identically distributed with common finite variance $0 < \sigma^2 < \infty$. Hence, for each $\varepsilon > 0$,

$$\begin{aligned}
\frac{1}{B_n^2} \cdot E\left[\ \sum_{i=1+a_n}^{b_n} \xi_i^2 \cdot I_{\left\{ |\,\xi_i\,| \geq \varepsilon\, B_n \right\}} \right] &= \frac{1}{(b_n - a_n)\,\sigma^2} \cdot (b_n - a_n) \cdot E\left[\ \xi_1^2 \cdot I_{\left\{ |\,\xi_1\,| \geq \varepsilon\sigma\,\sqrt{b_n - a_n} \right\}} \right] \\
&= \frac{1}{\sigma^2} \cdot E\left[\ \xi_1^2 \cdot I_{\left\{ |\,\xi_1\,|/\varepsilon\sigma \,\geq\, \sqrt{b_n - a_n} \right\}} \right] \;\longrightarrow\; 0, \quad \text{as } n \longrightarrow \infty,
\end{aligned}$$

since $\lim\limits_{n\to\infty} \sqrt{b_n - a_n} \;=\; \infty$ and $\sigma^2 \;=\; E\left[\ \xi_1^2\ \right]$ is finite. This verifies that the Lindeberg Condition indeed holds in the present context, and completes the proof of Claim 1.

<u>Proof of Claim 2:</u> First, note that $E[\ W(t)_n\ ] = 0$. We now argue that $W(t)_n \xrightarrow{p} 0$. To this end, let $\varepsilon > 0$ be given. Then,

$$\begin{aligned}
\varepsilon^2 \cdot P(\,|\,W(t)_n\,| \,\geq\, \varepsilon\,) &\leq E\left[\ W(t)_n^2 \cdot I_{\{\,|\,W(t)_n\,| \geq \varepsilon\,\}}\ \right] \\
&\leq E\left[\ W(t)_n^2\ \right] \;=\; \mathrm{Var}(\ W(t)_n\ ) \;=\; \mathrm{Var}\left[\ \frac{1}{\sigma \cdot \sqrt{n}} \cdot \Big(nt - \lfloor nt \rfloor\Big) \cdot \xi_{\lfloor nt \rfloor + 1}\ \right] \\
&= \frac{1}{n \cdot \sigma^2} \cdot \Big(nt - \lfloor\, nt\,\rfloor\Big)^2 \cdot \mathrm{Var}\big(\ \xi_{\lfloor nt \rfloor + 1}\ \big) \;=\; \frac{1}{n \cdot \sigma^2} \cdot \Big(nt - \lfloor\, nt\,\rfloor\Big)^2 \cdot \sigma^2 \\
&\leq \frac{1}{n},
\end{aligned}$$

which implies

$$\lim_{n\to\infty} P(\ |\,W(t)_n\,| \,\geq\, \varepsilon\ ) \;=\; 0, \ \text{ for each } \varepsilon > 0,$$

i.e. $W(t)_n \xrightarrow{p} 0$, as $n \longrightarrow \infty$ (Definition 2, Chapter 1, [2]), which is equivalent to $W(t)_n \xrightarrow{d} 0$, as $n \longrightarrow \infty$ (by Theorem 1, Chapter 1 and Theorem 2, Chapter 2, [2]). This proves Claim 2.

<u>Proof of Claim 3:</u> Let $0 \,\leq\, t_0 \,<\, t_1 \,<\, t_2 \,<\, \cdots \,<\, t_k \,\leq\, 1$, and $c_1, c_2, \ldots, c_k \in \mathbb{R}$ be arbitrary. Observe that:

$$\begin{aligned}
&\sum_{i=1}^{k} c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \\
&= \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ S_{\lfloor nt_i \rfloor} \,-\, S_{\lfloor nt_{i-1} \rfloor} \right\} \,+\, \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \Big(nt_i - \lfloor nt_i \rfloor\Big) \cdot \xi_{\lfloor nt_i \rfloor + 1} \,-\, \Big(nt_{i-1} - \lfloor nt_{i-1} \rfloor\Big) \cdot \xi_{\lfloor nt_{i-1} \rfloor + 1} \right\} \\
&= \sum_{i=1}^{k} \frac{c_i}{\sigma \cdot \sqrt{n}} \left\{ \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \right\} \,+\, \sum_{i=1}^{k} c_i \left\{ W(t_i)_n \,-\, W(t_{i-1})_n \right\} \\
&= \sum_{i=1}^{k} c_i\, Y_i^{(n)} \,+\, \sum_{i=1}^{k} c_i \left\{ W(t_i)_n \,-\, W(t_{i-1})_n \right\}
\end{aligned}$$

By Claim 2 and Slutsky's Theorem (Corollary, p.40, [2]),

$$\sum_{i=1}^{k} c_i \left\{ W(t_i)_n \;-\; W(t_{i-1})_n \right\} \;\xrightarrow{d}\; 0, \;\text{ as }\; n \longrightarrow \infty. \tag{2.1}$$

Next, note that since $\xi_1, \xi_2, \xi_3, \ldots$ are independent, we see that, for each fixed $n \in \mathbb{N}$,

$$Y_i^{(n)} \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j, \quad i \;=\; 1, 2, 3, \ldots, k,$$

are independent. Now, since $0 \leq t_{i-1} < t_i \leq 1$, it follows that $\lfloor nt_{i-1} \rfloor \;<\; \lfloor nt_i \rfloor$ for sufficiently large $n \in \mathbb{N}$. In addition,

$$\begin{aligned}
\frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} \;&=\; \frac{\lfloor nt_i \rfloor}{n} - \frac{\lfloor nt_{i-1} \rfloor}{n} \;=\; \left(\frac{nt_i}{n} + \frac{\lfloor nt_i \rfloor - nt_i}{n} \right) - \left(\frac{nt_{i-1}}{n} + \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right) \\
&=\; t_i \;-\; t_{i-1} \;+\; \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n},
\end{aligned}$$

which implies

$$\left| \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} \;-\; (t_i - t_{i-1}) \right| \;=\; \left| \frac{\lfloor nt_i \rfloor - nt_i}{n} - \frac{\lfloor nt_{i-1} \rfloor - nt_{i-1}}{n} \right| \;\leq\; \frac{2}{n} \;\longrightarrow\; 0, \quad \text{as }\; n \longrightarrow \infty.$$

Thus,

$$\lim_{n \to \infty} \frac{\lfloor nt_i \rfloor - \lfloor nt_{i-1} \rfloor}{n} \;=\; t_i \;-\; t_{i-1} \;>\; 0.$$

Thus, by Claim 1, we see that, for each $i \;=\; 1, 2, \ldots, k$,

$$Y_i^{(n)} \;:=\; \frac{1}{\sigma \cdot \sqrt{n}} \cdot \sum_{j=1+\lfloor nt_{i-1} \rfloor}^{\lfloor nt_i \rfloor} \xi_j \;\xrightarrow{d}\; \sqrt{t_i - t_{i-1}} \cdot N(0,1) \;=\; N\Big(0\,,\, t_i - t_{i-1}\Big), \quad \text{as }\; n \longrightarrow \infty. \tag{2.2}$$

By (2.1), (2.2), Proposition A.1, and Slutsky's Theorem (Corollary, p.40, [2]), we now see that

$$\sum_{i=1}^{k} c_i \left(X_{t_i}^{(n)} - X_{t_{i-1}}^{(n)} \right) \;=\; \sum_{i=1}^{k} c_i \, Y_i^{(n)} \;+\; \sum_{i=1}^{k} c_i \left\{ W(t_i)_n \;-\; W(t_{i-1})_n \right\} \;\xrightarrow{d}\; N\left(0\,,\, \sum_{i=1}^{k} c_i^2 (t_i - t_{i-1}) \right).$$

This completes the proof of Claim 3.

(iv) Let $t_0 \;:=\; 0$, hence, $X_{t_0}^{(n)} \;\equiv\; 0$ for each $n \in \mathbb{N}$. We thus have, for each $n \in \mathbb{N}$,

$$\left(\begin{array}{c} X_{t_1}^{(n)} \\ \\ X_{t_2}^{(n)} \\ \\ \vdots \\ \\ X_{t_k}^{(n)} \end{array} \right) \;=\; \underbrace{\left[\begin{array}{ccccccc} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & \cdots & 1 & 1 \end{array} \right]}_{T} \cdot \left(\begin{array}{c} X_{t_1}^{(n)} \\ \\ X_{t_2}^{(n)} - X_{t_1}^{(n)} \\ \\ \vdots \\ \\ X_{t_k}^{(n)} - X_{t_{k-1}}^{(n)} \end{array} \right).$$

By (iii), we know that

$$\begin{pmatrix} X^{(n)}_{t_1} \\ \\ X^{(n)}_{t_2} - X^{(n)}_{t_1} \\ \\ \vdots \\ \\ X^{(n)}_{t_k} - X^{(n)}_{t_{k-1}} \end{pmatrix} \quad \overset{d}{\longrightarrow} \quad \begin{pmatrix} Z_{t_1} \\ \\ Z_{t_2-t_1} \\ \\ \vdots \\ \\ Z_{t_k-t_{k-1}} \end{pmatrix} \quad \sim \quad N\Big(\,\mu = \mathbf{0}\,,\ \Sigma = \operatorname{diag}(t_1, t_2 - t_1, \ldots, t_k - t_{k-1})\,\Big), \ \text{ as } \ n \longrightarrow \infty.$$

Since the map $\mathbb{R}^k \longrightarrow \mathbb{R}^k : x \longmapsto T \cdot x$ is continuous, we see immediately by Slutsky's Theorem (Theorem 6(a), p.39, [2]) that

$$\begin{pmatrix} X^{(n)}_{t_1} \\ \\ X^{(n)}_{t_2} \\ \\ \vdots \\ \\ X^{(n)}_{t_k} \end{pmatrix} \quad \overset{d}{\longrightarrow} \quad T \cdot \begin{pmatrix} Z_{t_1} \\ \\ Z_{t_2-t_1} \\ \\ \vdots \\ \\ Z_{t_k-t_{k-1}} \end{pmatrix}, \quad \text{as } \ n \longrightarrow \infty.$$

Since the map $\mathbb{R}^k \longrightarrow \mathbb{R}^k : x \longmapsto T \cdot x$ is an invertible linear automorphism on $\mathbb{R}^k$, we see that

$$L \quad = \quad \begin{pmatrix} L_{t_1} \\ \\ L_{t_2} \\ \\ \vdots \\ \\ L_{t_k} \end{pmatrix} \quad := \quad T \cdot \begin{pmatrix} Z_{t_1} \\ \\ Z_{t_2-t_1} \\ \\ \vdots \\ \\ Z_{t_k-t_{k-1}} \end{pmatrix}$$

is still an $\mathbb{R}^k$-valued Gaussian random variable, and it clearly has expectation value $\mathbf{0} \in \mathbb{R}^k$, since each of Z_{t_1}, $Z_{t_2-t_1}$, $\ldots$, $Z_{t_k-t_{k-1}}$ has expectation value $0 \in \mathbb{R}$. It remains only to compute the covariance matrix of the $\mathbb{R}^k$-valued Gaussian random variable L. To this end, consider $1 \leq i \leq j \leq k$, i.e. $t_i \leq t_j$. Then, using the alternative notation $Z_{t_1 - t_0} := Z_{t_1}$, we have

$$\begin{aligned}
\operatorname{Cov}\big(\,L_{t_i}\,,\,L_{t_j}\,\big) \;&=\; \operatorname{Cov}\big(\,Z_{t_1} + Z_{t_2-t_1} + \cdots + Z_{t_i-t_{i-1}}\,,\, Z_{t_1} + Z_{t_2-t_1} + \cdots + Z_{t_j-t_{j-1}}\,\big) \\
&=\; \operatorname{Cov}\left(\,\sum_{a=1}^{i} Z_{t_a-\,t_{a-1}}\,,\,\sum_{b=1}^{j} Z_{t_b-\,t_{b-1}}\right) \;=\; \sum_{a=1}^{i}\sum_{b=1}^{j}\operatorname{Cov}\big(\,Z_{t_a-\,t_{a-1}}\,,\,Z_{t_b-\,t_{b-1}}\,\big) \\
&=\; \sum_{a=1}^{i}\operatorname{Cov}\big(\,Z_{t_a-\,t_{a-1}}\,,\,Z_{t_a-\,t_{a-1}}\,\big) \;=\; \sum_{a=1}^{i}\operatorname{Var}\big(\,Z_{t_a-\,t_{a-1}}\,\big) \;=\; \sum_{a=1}^{i}\,(\,t_a - t_{a-1}\,) \\
&=\; (\,t_1 - t_0\,) + (\,t_2 - t_1\,) + \cdots + (\,t_{i-1} - t_{i-2}\,) + (\,t_i - t_{i-1}\,) \\
&=\; t_i \;=\; \min\{\,t_i\,,t_j\,\},
\end{aligned}$$

as required.

$\square$

A Technical Lemmas

Note that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ does NOT in general imply $X_n + Y_n \xrightarrow{d} X + Y$. But the implication does hold if X_n and Y_n are independent for each $n \in \mathbb{N}$, and both X and Y are Gaussian random variables, as the following Proposition shows.

Proposition A.1 *Let $k \in \mathbb{N}$ be fixed. Suppose:*

- *For each $n \in \mathbb{N}$,*
$$Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_k^{(n)} : \Omega^{(n)} \longrightarrow \mathbb{R}$$
are independent $\mathbb{R}$-valued random variables defined on the probability space $\Omega^{(n)}$.
- *For each $i = 1, 2, \ldots, k$,*
$$Y_i^{(n)} \xrightarrow{d} N\left(\mu_i, \sigma_i^2\right), \quad \text{as } n \longrightarrow \infty.$$

Then, for any $c_1, c_2, \ldots, c_k \in \mathbb{R}$,
$$\sum_{i=1}^{k} c_i\, Y_i^{(n)} \xrightarrow{d} N\left(\sum_{i=1}^{k} c_i\, \mu_i\,,\ \sum_{i=1}^{k} c_i^2\, \sigma_i^2\right), \quad \text{as } n \longrightarrow \infty.$$

Proof Let $Y^{(n)} := \sum_{i=1}^{k} c_i\, Y_i^{(n)}$. Let φ_X denote the characteristic function of a $\mathbb{R}$-valued random variable X. Then,

$$\begin{aligned}
\varphi_{Y^{(n)}}(t) &= \varphi_{\sum_i^k c_i Y_i^{(n)}}(t) \\
&= \prod_{i=1}^{k} \varphi_{c_i Y_i^{(n)}}(t), \quad \text{since } Y_1^{(n)}, \ldots, Y_k^{(n)} \text{ are independent} \\
&= \prod_{i=1}^{k} \varphi_{Y_i^{(n)}}(c_i t) \\
&\longrightarrow \prod_{i=1}^{k} \exp\left\{ \sqrt{-1}\,\mu_i\,(c_i\, t) - \frac{1}{2}\sigma_i^2\,(c_i t)^2 \right\} \\
&= \exp\left\{ \sqrt{-1}\left(\sum_{i=1}^{k} c_i \mu_i\right) t - \frac{1}{2}\left(\sum_{i=1}^{k} c_i^2 \sigma_i^2\right) t^2 \right\}, \quad \text{as } n \longrightarrow \infty,
\end{aligned}$$

where the second and third equalities follow from the properties of characteristic functions of random variables (see p.21, [2]), while the expression of the limit follows from the fact that the characteristic function φ_Z of a random variable Z with distribution $N\left(\mu, \sigma^2\right)$ is

$$\varphi_Z = \exp\left\{ \sqrt{-1}\,\mu t - \frac{1}{2}\sigma^2\, t^2 \right\}.$$

The Proposition now follows immediately from the Lévy-Cramér Continuity Theorem (Theorem 1.9(ii), p.56, [3]). $\square$

B Continuous maps are Borel measurable

Lemma B.1 (The pre-image of a σ-algebra is itself a σ-algebra.)
Suppose Ω is a non-empty set, $(X, \mathcal{X})$ is a measurable space, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then,
$$f^{-1}(\mathcal{X}) := \left\{ f^{-1}(V) \subset \Omega \mid V \in \mathcal{X} \right\}$$
is a σ-algebra of subsets of Ω.

Proof

<u>$\Omega \in f^{-1}(\mathcal{X})$</u> $f(\Omega) \subset X \implies \Omega = f^{-1}(X) \in f^{-1}(\mathcal{X})$.

<u>$f^{-1}(\mathcal{X})$ is closed under complementations</u> Let $V \in \mathcal{X}$. Then, $X \setminus V \in \mathcal{X}$, and

$$\Omega \setminus f^{-1}(V) \;=\; \{\,\omega \in \Omega \mid f(\omega) \notin V\,\} \;=\; \{\,\omega \in \Omega \mid f(\omega) \in X \setminus V\,\} \;=\; f^{-1}(X \setminus V) \;\in\; f^{-1}(\mathcal{X}),$$

which shows that $f^{-1}(\mathcal{X})$ is indeed closed under complementations.

<u>$f^{-1}(\mathcal{X})$ is closed countable unions</u> Let $V_1, V_2, \ldots \in \mathcal{X}$. Then, $\bigcup_{i=1}^{\infty} V_i \in \mathcal{X}$, and

$$\bigcup_{i=1}^{\infty} f^{-1}(V_i) \;=\; \left\{ \omega \in \Omega \;\middle|\; \begin{array}{c} f(\omega) \in V_i \\ \text{for some } i \in \mathbb{N} \end{array} \right\} \;=\; f^{-1}\left(\bigcup_{i=1}^{\infty} V_i\right) \;\in\; f^{-1}(\mathcal{X}),$$

which proves that $f^{-1}(\mathcal{X})$ is indeed closed under countable unions.

This concludes the proof that that $f^{-1}(\mathcal{X})$ is a σ-algebra of subsets of Ω. □

Lemma B.2 (The push-forward of a σ-algebra is itself a σ-algebra.)
Suppose $(\Omega, \mathcal{A})$ is a measurable space, X is a non-empty set, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then,

$$\mathcal{F} \;:=\; \left\{\, V \subset X \;\middle|\; f^{-1}(V) \in \mathcal{A} \,\right\}$$

is a σ-algebra of subsets of X.

Proof

<u>$X \in \mathcal{F}$</u> $f^{-1}(X) = \Omega \in \mathcal{A} \implies X \in \mathcal{F}$.

<u>$\mathcal{F}$ is closed under complementations</u> $V \in \mathcal{F} \implies f^{-1}(V) \in \mathcal{A} \implies f^{-1}(X \setminus V) = \Omega \setminus f^{-1}(V) \in \mathcal{A} \implies X \setminus V \in \mathcal{F}$, which proves that $\mathcal{F}$ is indeed closed under complementations.

<u>$\mathcal{F}$ is closed under countable unions</u>

$$\begin{aligned} V_1, V_2, \ldots \;\in\; \mathcal{F} \;&\implies\; f^{-1}(V_1), f^{-1}(V_2), \ldots \;\in\; \mathcal{A} \\ &\implies\; f^{-1}\left(\bigcup_{i=1}^{\infty} V_i\right) \;=\; \bigcup_{i=1}^{\infty} f^{-1}(V_i) \;\in\; \mathcal{A} \\ &\implies\; \bigcup_{i=1}^{\infty} V_i \;\in\; \mathcal{F}, \end{aligned}$$

which proves that $\mathcal{F}$ is indeed closed under countable unions. □

Theorem B.3
Suppose $(\Omega, \mathcal{A})$ and $(X, \mathcal{X})$ are measurable spaces, and $f : \Omega \longrightarrow X$ is a map from Ω into X. Then, f is $(\mathcal{A}, \mathcal{X})$-measurable if there exists $\mathcal{S} \subset \mathcal{X}$ satisfying the following conditions:

- *$\mathcal{S}$ generates $\mathcal{X}$, i.e. $\sigma(\mathcal{S}) = \mathcal{X}$, and*
- *$f^{-1}(\mathcal{S}) \subset \mathcal{A}$.*

Proof By Lemma B.2,

$$\mathcal{F} \;:=\; \left\{\, V \subset X \;\middle|\; f^{-1}(V) \in \mathcal{A} \,\right\}$$

is a σ-algebra of subsets of X. By hypothesis, $\mathcal{S} \subset \mathcal{F}$; hence, $\mathcal{X} = \sigma(\mathcal{S}) \subset \mathcal{F}$. Thus, $f^{-1}(\mathcal{X}) \subset \mathcal{A}$; equivalently, f is $(\mathcal{A}, \mathcal{X})$-measurable. □

Corollary B.4 (Continuous maps are Borel measurable.)
Suppose X_1, X_2 are topological spaces, and $\mathcal{B}_1$, $\mathcal{B}_2$ are their respective Borel σ-algebras. Then, every continuous map $f : X_1 \longrightarrow X_2$ is $(\mathcal{B}_1, \mathcal{B}_2)$-measurable.

C Topology

Theorem C.1 (Appendix M3, [1])
Suppose S is a metric space. Then, the following conditions are equivalent:

(i) *S is separable.*

(ii) *The topology of S has a countable basis.*

(iii) *Every open cover of each subset of S has a countable subcover.*

References

[1] Billingsley, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.

[2] Ferguson, T. S. *A Course in Large Sample Theory*, first ed. Texts in Statistical Science. CRC Press, 1996.

[3] Shao, J. *Mathematical Statistics*, second ed. Springer, 2003.