## 1 The Portmanteau Theorem

**Theorem 1.1 (The Portmanteau Theorem, Theorem 2.1, [1])**

*Suppose:*

- $(S, \rho)$ *is a metric space,* $\mathcal{B}(S)$ *its the Borel* $\sigma$*-algebra,* $(S, \mathcal{B}(S))$ *is the corresponding measurable space.*
- $P, P_1, P_2, \ldots \in \mathcal{M}_1(S, \mathcal{B}(S))$ *are probability measures on* $(S, \mathcal{B}(S))$*.*

*Then, the following are equivalent:*

(i) $P_n$ *converges weakly to* $P$*, i.e. for each bounded continuous* $\mathbb{R}$*-valued function* $f : S \longrightarrow \mathbb{R}$*, we have*

$$\lim_{n\to\infty} \int_S f(s)\,\mathrm{d}P_n(s) \;=\; \int_S f(s)\,\mathrm{d}P(s).$$

(ii) *For each closed set* $F \subset S$*, we have*

$$\limsup_{n\to\infty} P_n(F) \;\leq\; P(F).$$

(iii) *For each open set* $G \subset S$*, we have*

$$\liminf_{n\to\infty} P_n(G) \;\geq\; P(G).$$

(iv) *For each* $P$*-continuity set* $A \in \mathcal{B}(S)$*, i.e.* $P(\partial A) = 0$*, we have*

$$\lim_{n\to\infty} P_n(A) \;=\; P(A).$$

Proof

<u>(i) $\Longrightarrow$ (ii)</u>

For each $\varepsilon > 0$, by Lemma A.2(ii), choose a bounded continuous functions $f_\varepsilon : S \longrightarrow [0, 1]$ such that

$$I_F \;\leq\; f_\varepsilon \;\leq\; I_{F^\varepsilon}.$$

This implies, for each $\varepsilon > 0$, we have

$$P_n(F) \;=\; \int_S I_F(x)\,\mathrm{d}P_n(x) \;\leq\; \int_S f_\varepsilon(x)\,\mathrm{d}P_n(x).$$

By (i), we thus have

$$\limsup_{n\to\infty} P_n(F) \;\leq\; \lim_{n\to\infty} \int_S f_\varepsilon(x)\,\mathrm{d}P_n(x) \;=\; \int_S f_\varepsilon(x)\,\mathrm{d}P(x) \;\leq\; \int_S I_{F^\varepsilon}(x)\,\mathrm{d}P(x) \;=\; P(F^\varepsilon)\,.$$

By Lemma A.2(i), we have $F^\varepsilon \downarrow F$ as $\varepsilon \downarrow 0$. Hence, $P(F^\varepsilon) \downarrow P(F)$ as $\varepsilon \downarrow 0$. We may now conclude:

$$\limsup_{n\to\infty} P_n(F) \;\leq\; \lim_{\varepsilon\to 0^+} P(F^\varepsilon) \;=\; P(F)\,.$$

<u>(ii) $\Longrightarrow$ (iii)</u>

Assume (ii) holds. Let $G \subset S$ be a open subset. Then, $F := S \setminus G$ is closed. By (ii), we have:

$$\begin{aligned} 1 - \liminf_{n\to\infty} P_n(G) &= \limsup_{n\to\infty} \{\, 1 - P_n(G) \,\} = \limsup_{n\to\infty} P_n(\, S \setminus G \,) = \limsup_{n\to\infty} P_n(F) \\ &\leq P(F) = P(\, S \setminus G \,) = 1 - P(G)\,, \end{aligned}$$

which yields

$$\liminf_{n\to\infty} P_n(G) \geq P(G)\,. \tag{1.1}$$

<u>(ii) $\Longrightarrow$ (iii)</u>

Assume (iii) holds. Let $F \subset S$ be an closed subset. Then, $G := S \setminus F$ is open. By (iii), we have:

$$\begin{aligned} 1 - \limsup_{n\to\infty} P_n(F) &= \liminf_{n\to\infty} \{\, 1 - P_n(F) \,\} = \liminf_{n\to\infty} P_n(\, S \setminus F \,) = \liminf_{n\to\infty} P_n(G) \\ &\geq P(G) = P(\, S \setminus F \,) = 1 - P(F)\,, \end{aligned}$$

which yields

$$\limsup_{n\to\infty} P_n(F) \leq P(F)\,. \tag{1.2}$$

<u>(ii) and (iii) $\Longrightarrow$ (iv)</u>

Let $A \in \mathcal{B}(S)$. Then, by (ii) and (iii), we have:

$$P(A^\circ) \leq \liminf_{n\to\infty} P_n(A^\circ) \leq \liminf_{n\to\infty} P_n(A) \leq \limsup_{n\to\infty} P_n(A) \leq \limsup_{n\to\infty} P_n\big(\overline{A}\big) \leq P\big(\overline{A}\big)\,.$$

Hence, if $\partial A := \overline{A} \setminus A^\circ$ is a $P$-continuity set, i.e. $P(\partial A) = 0$, hence $P(A^\circ) = P(\,A\,) = P\big(\overline{A}\big)$, then (iv) follows.

<u>(iv) $\Longrightarrow$ (ii)</u>

<u>(iii) $\Longrightarrow$ (i)</u>

Let $g : S \longrightarrow [0,\infty)$ be continuous $\mathbb{R}$-valued function on $S$. Then, for each $t \in (0,\infty)$, the set $g^{-1}(\,(t,\infty)\,) = \{\, s \in S \mid g(s) > t \,\}$ is an open subset of $S$. Hence, by (iii), Lemma **??**, and Fatou's Lemma, we have

$$\begin{aligned} \int_S g(s)\,\mathrm{d}P(s) &= \int_0^\infty P(g > t)\,\mathrm{d}t \leq \int_0^\infty \liminf_{n\to\infty} P_n(g > t)\,\mathrm{d}t \\ &\leq \liminf_{n\to\infty} \int_0^\infty P_n(g > t)\,\mathrm{d}t \leq \liminf_{n\to\infty} \int_S g(s)\,\mathrm{d}P_n(s). \end{aligned}$$

Now, let $f : S \longrightarrow \mathbb{R}$ be continuous and bounded with $|\,f\,| \leq c < \infty$. Then, $c \pm f : S \longrightarrow [0,\infty)$ are continuous and non-negative $\mathbb{R}$-valued functions on $S$. Applying the preceding inequality to each yields:

$$\begin{aligned} \int_S c + f(s)\,\mathrm{d}P(s) &\leq \liminf_{n\to\infty} \int_S c + f(s)\,\mathrm{d}P_n(s) \\ \int_S c - f(s)\,\mathrm{d}P(s) &\leq \liminf_{n\to\infty} \int_S c - f(s)\,\mathrm{d}P_n(s). \end{aligned}$$

These respectively imply:

$$\begin{aligned} \int_S f(s)\,\mathrm{d}P(s) &\leq \liminf_{n\to\infty} \int_S f(s)\,\mathrm{d}P_n(s) \\ \limsup_{n\to\infty} \int_S f(s)\,\mathrm{d}P_n(s) &\leq \int_S f(s)\,\mathrm{d}P(s), \end{aligned}$$

which proves (i). $\square$

## A Technical Lemmas

**Lemma A.1** *Suppose $(S, \rho)$ is a metric space, and $A \subset S$ is an arbitrary non-empty subset. Define*

$$\rho(\,\cdot\,, A) \;:\; S \;\longrightarrow\; \mathbb{R} \;:\; x \;\longmapsto\; \inf_{y \in A} \{\, \rho(x, y) \,\}$$

*Then,*

(i) *$\rho(\,\cdot\,, A)$ is a continuous $\mathbb{R}$-valued function on $S$.*

(ii) *For each $x \in S$, $\rho(x, A) = 0$ if and only if $x \in \overline{A}$.*

PROOF

(i) Suppose $x_n \longrightarrow x$. We need to prove $\rho(x_n, A) \longrightarrow \rho(x, A)$, which follows immediately from the following two Claims:

**Claim 1:** $\rho(x, A) \;\leq\; \liminf_{n \to \infty} \rho(x_n, A)$.

**Claim 2:** $\limsup_{n \to \infty} \rho(x_n, A) \;\leq\; \rho(x, A)$.

<u>Proof of Claim 1:</u> For each $y \in S$, we have:

$$\rho(x, y) \;\leq\; \rho(x, x_n) \;+\; \rho(x_n, y).$$

Hence,

$$\rho(x, A) \;=\; \inf_{y \in A} \rho(x, y) \;\leq\; \rho(x, x_n) \;+\; \inf_{y \in A} \rho(x_n, y) \;=\; \rho(x, x_n) \;+\; \rho(x_n, A).$$

Since $\rho(x, x_n) \longrightarrow 0$, the preceding inequality implies

$$\rho(x, A) \;\leq\; \liminf_{n \to \infty} \rho(x_n, A).$$

This proves Claim 1.

<u>Proof of Claim 2:</u> For each $y \in S$, we have:

$$\rho(x_n, y) \;\leq\; \rho(x_n, x) \;+\; \rho(x, y).$$

Hence,

$$\rho(x_n, A) \;=\; \inf_{y \in A} \rho(x_n, y) \;\leq\; \rho(x_n, x) \;+\; \inf_{y \in A} \rho(x, y) \;=\; \rho(x_n, x) \;+\; \rho(x, A).$$

Since $\rho(x, x_n) \longrightarrow 0$, the preceding inequality implies

$$\limsup_{n \to \infty} \rho(x_n, A) \;\leq\; \rho(x, A).$$

This proves Claim 2.

(ii)

$$\begin{aligned}
\rho(x, A) = 0 \quad &\Longleftrightarrow \quad \inf_{y \in A} \rho(x, y) = 0 \\
&\Longleftrightarrow \quad \text{For each } \varepsilon > 0 \text{, there exists } y \in A \text{ such that } \rho(x, y) < \varepsilon \\
&\Longleftrightarrow \quad y \in \overline{A}
\end{aligned}$$

$\square$

**Lemma A.2** *Suppose $(S, \rho)$ is a metric space, and $A \subset S$ is an arbitrary non-empty subset. For each $\varepsilon > 0$, define*

$$A^{\varepsilon} \;:=\; \{\; s \in S \;\mid\; \rho(s, A) < \varepsilon \;\}\,.$$

*Then the following are true:*

(i) *$A^{\varepsilon}$ is an open subset of $S$. In particular, $A^{\varepsilon}$ is a $\mathcal{B}(S)$-measurable subset of $S$.*

(ii) *$A^{\varepsilon} \downarrow \overline{A}$, as $\varepsilon \downarrow 0$.*

(iii) *There exists a bounded continuous $\mathbb{R}$-valued function $f : S \longrightarrow \mathbb{R}$ such that*

$$I_{\bar{A}}(x) \;\leq\; f(x) \;\leq\; I_{A^{\varepsilon}}(x)\,, \quad \text{for each } x \in S.$$

Proof

(i)

(ii)

(iii) Define $f : S \longrightarrow \mathbb{R}$ as follows:

$$f(x) \;:=\; \max\left\{\; 0\,,\, 1 - \frac{\rho(x, A)}{\varepsilon} \;\right\}.$$

Then, by Lemma A.1(i), $f$ is continuous $\mathbb{R}$-valued function on $S$. Clear, $0 \leq f(x) \leq 1$, for each $x \in S$. By Lemma A.1(ii), we have

$$x \;\in\; \overline{A} \quad\Longleftrightarrow\quad \rho(x, F) \;=\; 0 \quad\Longleftrightarrow\quad f(x) \;=\; 1.$$

This proves $I_{\bar{A}}(x) \leq 1 = f(x)$, for each $x \in \overline{A}$, and hence for each $x \in S$ (since $I_{\bar{A}}(x) = 0$ for $x \in S \setminus \overline{A}$, and the inequality holds trivially). On the other hand,

$$x \;\in\; S \setminus A^{\varepsilon} \quad\Longleftrightarrow\quad \varepsilon \;\leq\; \rho(x, A) \quad\Longleftrightarrow\quad 1 - \frac{\rho(x, A)}{\varepsilon} \;\leq\; 0 \quad\Longrightarrow\quad f(x) \;=\; 0.$$

This proves $f(x) = 0 \leq I_{A^{\varepsilon}}(x)$, for each $x \in S \setminus A^{\varepsilon}$, and hence for each $x \in S$ (since $I_{A^{\varepsilon}}(x) = 1$ for each $x \in A^{\varepsilon}$ and the inequality holds trivially). This completes the proof of (ii).

□

## References

[1] Billingsley, P. *Convergence of Probability Measures*, second ed. John Wiley & Sons, 1999.