

1 Chapter 1

Exercise 1.1(a)

Let X be the sum of the two number obtained.

Let X_1 be the number obtained on Die 1.

Let X_2 be the number obtained on Die 2.

Thus, $X = X_1 + X_2$, and

$$E_x = \{X = x\} = \{X_1 + X_2 = x\} = \{X_1 = x_1, X_2 = x - x_1 \mid 1 \leq x_1, x - x_1 \leq 6\}$$

Now,

$$1 \leq x - x_1 \leq 6 \iff -1 \geq x_1 - x \geq -6 \iff x - 1 \geq x_1 \geq x - 6 \iff x - 6 \leq x_1 \leq x - 1$$

Hence,

$$E_x = \{X = x\} = \{X_1 + X_2 = x\} = \{X_1 = x_1, X_2 = x - x_1 \mid \max\{1, x - 6\} \leq x_1 \leq \min\{6, x - 1\}\}$$

$$\begin{aligned} P(E_x) &= \sum_{x_1=\max\{1, x-6\}}^{\min\{6, x-1\}} P(X_1 = x_1, X_2 = x - x_1) = \sum_{x_1=\max\{1, x-6\}}^{\min\{6, x-1\}} \frac{1}{6^2} \\ &= \frac{1}{6^2} (\min\{6, x - 1\} - \max\{1, x - 6\} + 1) \end{aligned}$$

Next, note that

$$\min\{6, x - 1\} = \begin{cases} x - 1, & \text{if } x = 2, 3, \dots, 6 \\ 6, & \text{if } x = 7, 8, \dots, 12 \end{cases} \quad \text{and} \quad \max\{1, x - 6\} = \begin{cases} 1, & \text{if } x = 2, 3, \dots, 6 \\ x - 6, & \text{if } x = 7, 8, \dots, 12 \end{cases}$$

Hence,

$$\begin{aligned} P(E_x) &= \frac{1}{6^2} (\min\{6, x - 1\} - \max\{1, x - 6\} + 1) = \frac{1}{36} \begin{cases} (x - 1) - 1 + 1, & \text{if } x = 2, 3, \dots, 6 \\ 6 - (x - 6) + 1, & \text{if } x = 7, 8, \dots, 12 \end{cases} \\ &= \frac{1}{36} \begin{cases} x - 1, & \text{if } x = 2, 3, \dots, 6 \\ 13 - x, & \text{if } x = 7, 8, \dots, 12 \end{cases} \end{aligned}$$

□

Exercise 1.18

Recapitulation of the rules of craps: Let x be the number obtained on the first roll. If $x \in \{7, 11\}$, then the player wins. If $x \in \{2, 3, 12\}$, then the player loses. If $x \in \{4, 5, 6, 8, 9, 10\}$, then the player keeps rolling, until either 7 is rolled or x is rolled. If x is rolled first (before 7 is rolled), then the player wins. If 7 is rolled first (before x is rolled), then the player loses.

Let W be the $\{0, 1\}$ -valued random variable such that $W = 1$ if the player wins, and $W = 0$ if the player loses. We thus seek to compute $P(W = 1)$. Let X be (the random variable of) the sum of the two numbers obtained on the first roll. Note that $\text{Range}(X) = \{2, 3, 4, \dots, 12\}$. Then,

$$\begin{aligned} P(W = 1) &= \sum_{x=2}^{12} P(W = 1|X = x) \cdot P(X = x) \\ &= P(W = 1|X = 7) P(X = 7) + P(W = 1|X = 11) P(X = 11) + \sum_{x \in \{4, 5, 6, 8, 9, 10\}} P(W = 1|X = x) \cdot P(X = x) \end{aligned}$$

Now, note that $P(W = 1|X = 7) = P(W = 1|X = 11) = 1$, $P(X = 7) = \frac{6}{36} = \frac{1}{6}$, and $P(X = 11) = \frac{2}{36} = \frac{1}{18}$.

From Exercise 1.1(a), we have:

$$\begin{aligned} P(X = x) &= \frac{1}{6^2} (\min\{6, x-1\} - \max\{1, x-6\} + 1) = \frac{1}{36} \begin{cases} (x-1) - 1 + 1, & \text{if } x = 2, 3, \dots, 6 \\ 6 - (x-6) + 1, & \text{if } x = 7, 8, \dots, 12 \end{cases} \\ &= \frac{1}{36} \begin{cases} x-1, & \text{if } x = 2, 3, \dots, 6 \\ 13-x, & \text{if } x = 7, 8, \dots, 12 \end{cases} \end{aligned}$$

Next, let Y_n be the random variable of the sum of the two numbers obtained on the $(n+1)$ st roll. Then,

$$\begin{aligned} P(W = 1|X = x) &= \sum_{n=1}^{\infty} [1 - P(Y_n = 7) - P(Y_n = x)]^{n-1} \cdot P(X = x) \\ &= P(X = x) \cdot \sum_{n=1}^{\infty} [1 - P(Y_n = 7) - P(Y_n = x)]^{n-1} \\ &= P(X = x) \cdot \frac{1}{1 - [1 - P(Y = 7) - P(Y = x)]} \\ &= \frac{P(X = x)}{P(Y = 7) + P(Y = x)} \\ &= \frac{P(X = x)}{\frac{1}{6} + P(Y = x)} \end{aligned}$$

Exercises and Solutions in Biostatistical Theory

Kenneth Chu

Kupper-Neelon-O'Brien, Chapman & Hall/CRC Press, 2011

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Hence,

$$\begin{aligned}
 P(W = 1) &= \sum_{x=2}^{12} P(W = 1|X = x) \cdot P(X = x) \\
 &= P(W = 1|X = 7) P(X = 7) + P(W = 1|X = 11) P(X = 11) + \sum_{x \in \{4,5,6,8,9,10\}} P(W = 1|X = x) \cdot P(X = x) \\
 &= \frac{6}{36} + \frac{2}{36} + \sum_{x \in \{4,5,6,8,9,10\}} \frac{P(X = x)^2}{\frac{1}{6} + P(X = x)} \\
 &= \frac{6}{36} + \frac{2}{36} + \frac{(\frac{4-1}{36})^2}{\frac{1}{6} + \frac{4-1}{36}} + \frac{(\frac{5-1}{36})^2}{\frac{1}{6} + \frac{5-1}{36}} + \frac{(\frac{6-1}{36})^2}{\frac{1}{6} + \frac{6-1}{36}} + \frac{(\frac{13-8}{36})^2}{\frac{1}{6} + \frac{13-8}{36}} + \frac{(\frac{13-9}{36})^2}{\frac{1}{6} + \frac{13-9}{36}} + \frac{(\frac{13-10}{36})^2}{\frac{1}{6} + \frac{13-10}{36}} \\
 &= \frac{6}{36} + \frac{2}{36} + \frac{(1/36)^2}{1/36} \left(\frac{3^2}{6+3} + \frac{4^2}{6+4} + \frac{5^2}{6+5} + \frac{5^2}{6+5} + \frac{4^2}{6+4} + \frac{3^2}{6+3} \right) \\
 &= \frac{6}{36} + \frac{2}{36} + \frac{2}{36} \left(\frac{3^2}{6+3} + \frac{4^2}{6+4} + \frac{5^2}{6+5} \right) = \frac{1}{36} \left[6 + 2 + 2 \left(\frac{9}{9} + \frac{16}{10} + \frac{25}{11} \right) \right] \\
 &= \frac{1}{36} \left[8 + 2 \left(\frac{536}{110} \right) \right] = \frac{1}{36} \left[\frac{1952}{110} \right] = \frac{1}{2^2 \cdot 3^2} \left[\frac{2^5 \cdot 61}{2 \cdot 5 \cdot 11} \right] \\
 &= \frac{2^2 \cdot 61}{3^2 \cdot 5 \cdot 11} \approx 0.4929293
 \end{aligned}$$

□

Exercise 1.19(a)

Let n be the number of workers in the sample. Let X_i , $i = 1, 2, \dots, n$, be $\{0, 1\}$ -valued random variables defined by:

$$X_i = \begin{cases} 1, & \text{if the } i\text{th subject is highly exposed,} \\ 0, & \text{if the } i\text{th subject is NOT highly exposed} \end{cases}$$

Define

$$S_n := \sum_{i=1}^n X_i, \quad \text{and} \quad S_{n-1} := \sum_{i=1}^{n-1} X_i.$$

First, note that

$$\theta_n = P(S_n \text{ is even}), \quad \text{and} \quad \theta_{n-1} = P(S_{n-1} \text{ is even}).$$

Note also that

$$\begin{aligned} \theta_n &= P(S_n \text{ is even}) = P(X_n = 1)P(S_{n-1} \text{ is odd}) + P(X_n = 0)P(S_{n-1} \text{ is even}) \\ &= \pi_h(1 - \theta_{n-1}) + (1 - \pi_h)\theta_{n-1} = \pi_h + (1 - 2\pi_h)\theta_{n-1} \end{aligned}$$

Thus, the desired difference equation is:

$$\theta_n = \pi_h + (1 - 2\pi_h)\theta_{n-1} \tag{1.1}$$

Exercise 1.19(b)

To solve the difference equation (1.1) obtained in Exercise 1.19(a), we assume that θ_n has the following form:

$$\theta_n = \alpha + \beta\gamma^n \tag{1.2}$$

where α , β , and γ are unknown constants to be determined. We first make the following:

Observation: $\beta \neq 0$ and $\gamma \notin \{0, 1\}$.

Indeed, if $\beta = 0$ or $\gamma \in \{0, 1\}$, then θ_n would be constant in n . In that case, define $\theta := \theta_n = \theta_{n-1} = \dots$. By the difference equation (1.1), we would then have

$$\theta = \pi_h + (1 - 2\pi_h)\theta \implies 0 = \pi_h(1 - 2\theta) \implies \theta = \frac{1}{2} \quad (\text{since } \pi_h > 0)$$

However, this contradicts the initial condition that $\theta_0 = 1$. Thus, this proves the assertion that $\beta \neq 0$ and $\gamma \notin \{0, 1\}$. (Note that if the sample size is 0, then the number of highly exposed subjects must be 0; hence $\theta_0 = P(S_0 \text{ is even}) = 1$, since we have here adopted the convention that 0 is “even.”)

Now, substituting (1.2) into (1.1) yields:

$$\begin{aligned} \alpha + \beta\gamma^n &= \theta_n = \pi_h + (1 - 2\pi_h)\theta_{n-1} \\ &= \pi_h + (1 - 2\pi_h)(\alpha + \beta\gamma^{n-1}) \\ &= \alpha + \pi_h(1 - 2\alpha) + \beta\gamma^{n-1}(1 - 2\pi_h) \end{aligned}$$

Collecting terms involving γ on the right-hand side yields:

$$\pi_h(2\alpha - 1) = \beta\gamma^{n-1}(1 - 2\pi_h - \gamma)$$

Now, note that the left-hand side of the preceding equation is independent of γ , while the right-hand side is a scalar multiple of the $(n - 1)$ th power of γ ; in other words, the right-hand side is a scalar multiple of a power of γ which is constant in n .

Exercises and Solutions in Biostatistical Theory

Kenneth Chu

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May 26, 2013

This happens if and only if either $\gamma \in \{0, 1\}$, or if the coefficient $\beta(1 - 2\pi_h - \gamma) = 0$. The preceding Observation (i.e. $\beta \neq 0$ and $\gamma \notin \{0, 1\}$) thus implies:

$$\gamma = 1 - 2\pi_h$$

Since $\pi_h > 0$, we furthermore conclude that

$$\alpha = \frac{1}{2}$$

We therefore have:

$$\theta_n = \frac{1}{2} + \beta(1 - 2\pi_h)^n$$

The initial condition $\theta_0 = 1$ now implies:

$$1 = \theta_0 = \frac{1}{2} + \beta(1 - 2\pi_h)^0 = \frac{1}{2} + \beta \implies \beta = \frac{1}{2}$$

We may now conclude:

$$\theta_n = \frac{1}{2} + \frac{1}{2}(1 - 2\pi_h)^n$$

Lastly, if $\pi_h = 0.05$, then

$$\theta_{50} = \frac{1}{2} + \frac{1}{2}(1 - 2 \times 0.05)^{50} \approx 0.5025769$$

□

Comment: For $0 < \pi_h < \frac{1}{2}$, the formula $\theta_n = \frac{1}{2} + \frac{1}{2}(1 - 2\pi_h)^n$ implies that $\theta_n > \frac{1}{2}$, for any $n = 1, 2, 3, \dots$; in other words, there is a higher than 50 : 50 chance that the number of highly exposed subjects in the sample is “even”, whenever $0 < \pi_h < \frac{1}{2}$. This apparent asymmetry between odd and even is NOT surprising given the fact that 0 is regarded as “even” here, and that the probability that there are no highly exposed workers in the sample is high if π_h is “small” (e.g. $0 < \pi_h < \frac{1}{2}$).

Exercise 1.20(a)

$$p(D|S, x) = \frac{p(D, S, x)}{p(S, x)} = \frac{p(D, S, x)}{p(D, x)} \frac{p(D, x)}{p(S, x)} = p(S|D, x) \frac{p(D, x)/p(x)}{p(S, x)/p(x)} = p(S|D, x) \frac{p(D|x)}{p(S|x)}$$

Now, we are given that

$$p(S|D, x) = \pi_1, \quad \text{and} \quad p(D|x) = \frac{\exp(\beta_0 + \beta^T x)}{1 + \exp(\beta_0 + \beta^T x)}$$

So, we now proceed to compute $p(S|x)$. To this end,

$$\begin{aligned} p(S|x) &= \frac{p(S, x)}{p(x)} = \frac{1}{p(x)} (p(S, D, x) + p(S, \bar{D}, x)) = \frac{1}{p(x)} \left(\frac{p(S, D, x)}{p(D, x)} p(D, x) + \frac{p(S, \bar{D}, x)}{p(\bar{D}, x)} p(\bar{D}, x) \right) \\ &= p(S|D, x)p(D|x) + p(S|\bar{D}, x)p(\bar{D}|x) \end{aligned}$$

Hence,

$$\begin{aligned} p(D|S, x) &= p(S|D, x) \frac{p(D|x)}{p(S|x)} = \frac{p(S|D, x) p(D|x)}{p(S|D, x) p(D|x) + p(S|\bar{D}, x) p(\bar{D}|x)} = \frac{\pi_1 \cdot p(D|x)}{\pi_1 \cdot p(D|x) + \pi_0 \cdot p(\bar{D}|x)} \\ &= \frac{\pi_1 \cdot \frac{\exp(\beta_0 + \beta^T x)}{1 + \exp(\beta_0 + \beta^T x)}}{\pi_1 \cdot \frac{\exp(\beta_0 + \beta^T x)}{1 + \exp(\beta_0 + \beta^T x)} + \pi_0 \cdot \frac{1}{1 + \exp(\beta_0 + \beta^T x)}} = \frac{\pi_1 \cdot \exp(\beta_0 + \beta^T x)}{\pi_1 \cdot \exp(\beta_0 + \beta^T x) + \pi_0} \\ &= \frac{\frac{\pi_1}{\pi_0} \cdot \exp(\beta_0 + \beta^T x)}{1 + \frac{\pi_1}{\pi_0} \cdot \exp(\beta_0 + \beta^T x)} = \frac{\exp[\log(\pi_1/\pi_0) + \beta_0 + \beta^T x]}{1 + \exp[\log(\pi_1/\pi_0) + \beta_0 + \beta^T x]}, \end{aligned}$$

as required.

Comment: The above derivations show that, in a case-control study, if one has knowledge (or good estimate) of the ratio π_1/π_0 , one can obtain an estimate for $p(D|x)$, the disease risk associated to covariate value x , from the quantity $p(D|S, x)$, which can be estimated from case-control study data as follows:

$$p(D|S, x) \approx \frac{\#(\text{subjects in sample with disease and covariate value } x)}{\#(\text{subjects in sample with covariate value } x)}$$

However, in practice, the ratio π_1/π_0 is rarely, if ever, known. And, without knowledge or estimate of π_1/π_0 , the disease risk $p(D|x)$ associated to covariate value x can NOT be estimated based on data from a case-control study.

Exercise 1.20(b)

First, note that

$$\frac{p(D|x^*)}{p(\bar{D}|x^*)} = \frac{\exp(\beta_0 + \beta^T x^*) / (1 + \exp(\beta_0 + \beta^T x^*))}{1 / (1 + \exp(\beta_0 + \beta^T x^*))} = \exp(\beta_0 + \beta^T x^*)$$

Similarly,

$$\frac{p(D|x)}{p(\bar{D}|x)} = \exp(\beta_0 + \beta^T x)$$

Hence,

$$\theta_r = \theta_r(x^*, x) = \frac{p(D|x^*)/p(\bar{D}|x^*)}{p(D|x)/p(\bar{D}|x)} = \frac{\exp(\beta_0 + \beta^T x^*)}{\exp(\beta_0 + \beta^T x)} = \exp[\beta^T(x^* - x)],$$

Exercises and Solutions in Biostatistical Theory

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as required. Next,

$$\theta_c = \theta_c(x^*, x) = \frac{p(D|S, x^*)/p(\bar{D}|S, x^*)}{p(D|S, x)/p(\bar{D}|S, x)} = \frac{\exp[\log(\pi_1/\pi_0) + \beta_0 + \beta^T x^*]}{\exp[\log(\pi_1/\pi_0) + \beta_0 + \beta^T x]} = \exp[\beta^T(x^* - x)] ,$$

as required.

Comment: Exercise 1.20(a) showed that, without knowledge or estimate of the ratio π_1/π_0 , case-control study data can NOT be used to estimate the disease $p(D|x)$ associated to covariate value x . On the other hand, case-control study data can be readily used to estimate the odds ratio

$$\theta_c = \theta_c(x^*, x) := \frac{p(D|S, x^*)/p(\bar{D}|S, x^*)}{p(D|S, x)/p(\bar{D}|S, x)}$$

Exercise 1.20(b) shows that θ_c is equal to

$$\theta_r = \theta_r(x^*, x) := \frac{p(D|x^*)/p(\bar{D}|x^*)}{p(D|x)/p(\bar{D}|x)}$$

Thus, Exercise 1.20(a) and Exercise 1.20(b) together show that, while case-control study data can NOT be used to estimate disease risk $p(D|x)$ associated to covariate value x , they can be used to estimate the disease odds ratio

$$\theta_r = \theta_r(x^*, x) := \frac{p(D|x^*)/p(\bar{D}|x^*)}{p(D|x)/p(\bar{D}|x)}$$

associated to the covariate value x^* against x .

References