A The Central Limit Theorems

Lemma A.1 (§7.1, [1])

Let $\{\, \theta_{nj} \in \mathbb{C} \mid 1 \le j \le k_n, n \in \mathbb{N} \,\}$ *be doubly indexed array of complex numbers. If all the following three conditions are true:*

(a) *there exists* $M > 0$ *such that*

$$\sum_{j=1}^{k_n} |\, \theta_{nj} \,| \;\le\; M, \quad \text{for each } n \in \mathbb{N},$$

(b) $\displaystyle\lim_{n\to\infty} \max_{1\le j\le k_n} |\, \theta_{nj} \,| \;=\; 0$, *and*

(c) *there exists* $\theta \in \mathbb{C}$ *such that*

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \theta_{nj} \;=\; \theta,$$

then

$$\lim_{n\to\infty} \prod_{j=1}^{k_n} (1 + \theta_{nj}) \;=\; e^{\theta}.$$

Proof First, note that hypothesis (b) immediately implies that there exists some $n_0 \in \mathbb{N}$ such that

$$|\, \theta_{nj} \,| \;\le\; \frac{1}{2}, \quad \text{for each } n \ge n_0, \text{ for each } 1 \le j \le k_n.$$

Thus, without loss of generality, we may assume that:

$$|\, \theta_{nj} \,| \;\le\; \frac{1}{2}, \quad \text{for each } n \in \mathbb{N}, \text{ for each } 1 \le j \le k_n.$$

We denote by $\log(1+\theta_{nj})$ the (unique) complex logarithm[1] of $1+\theta_{nj}$ with argument in $(-\pi, \pi]$. Next, recall the MacLaurin Series for $\log(1+x)$:

$$\log(1+x) \;=\; \sum_{m=1}^{\infty} (-1)^{n+1} \frac{x^m}{m}, \quad \text{for any } x \in \mathbb{C} \text{ with } |\, x \,| < 1.$$

Hence, we have the following inequality: for each $n \in \mathbb{N}$ and for each $1 \le j \le k_n$,

$$\begin{aligned} |\, \log(1+\theta_{nj}) - \theta_{nj} \,| \;&=\; \left| \sum_{m=2}^{\infty} (-1)^{n+1} \frac{(\theta_{nj})^m}{m} \right| \;\le\; \sum_{m=2}^{\infty} \frac{|\, \theta_{nj} \,|^m}{m} \;\le\; \frac{|\, \theta_{nj} \,|^2}{2} \sum_{m=2}^{\infty} |\, \theta_{nj} \,|^{m-2} \\ &\le\; \frac{|\, \theta_{nj} \,|^2}{2} \sum_{m=2}^{\infty} \left(\frac{1}{2}\right)^{m-2} \;=\; \frac{|\, \theta_{nj} \,|^2}{2} \sum_{i=2}^{\infty} \left(\frac{1}{2}\right)^{i-2} \;=\; \frac{|\, \theta_{nj} \,|^2}{2} \cdot 2 \;=\; |\, \theta_{nj} \,|^2 . \end{aligned}$$

[1] Recall that the complex exponential function is defined by $\exp : \mathbb{C} \to \mathbb{C} : x + \mathbf{i}\, y \mapsto e^x \cdot e^{\mathbf{i}\, y} = e^x (\cos y + \mathbf{i} \sin y)$. Clearly, exp is not injective. More precisely, for $x_1 + \mathbf{i}\, y_1,\, x_2 + \mathbf{i}\, y_2 \in \mathbb{C}\backslash\{\, 0 \,\}$, we have $e^{x_1 + \mathbf{i} y_1} = e^{x_2 + \mathbf{i} y_2}$ if and only if $x_1 = x_2 \in \mathbb{R}\backslash\{\, 0 \,\}$ and $y_1 - y_2 \in 2\pi\mathbb{Z}$. For $z = re^{\mathbf{i}\theta} \in \mathbb{C}\backslash\{\, 0 \,\}$, a complex logarithm of z is any $w = x + \mathbf{i}\, y \in \mathbb{C}\backslash\{\, 0 \,\}$ such that $e^{x + \mathbf{i} y} = e^w = z = re^{\mathbf{i}\theta}$, i.e. $x = \log r$ and $y = \theta + 2\pi\mathbb{Z}$. In particular, let $\mathcal{D} := \{\, x + \mathbf{i}\, y \in \mathbb{C} \mid x \in \mathbb{R}, y \in (-\pi, \pi] \,\}$. Then, the restriction $\exp : \mathcal{D} \to \mathbb{C}\backslash\{\, 0 \,\}$ is bijective.

This in turn implies: for each $n \in \mathbb{N}$,

$$\left|\,\sum_{j=1}^{k_n}\log(1+\theta_{nj}) \;-\; \sum_{j=1}^{k_n}\theta_{nj}\,\right| \;=\; \left|\,\sum_{j=1}^{k_n}\left(\log(1+\theta_{nj})-\theta_{nj}\right)\right| \;\leq\; \sum_{j=1}^{k_n}\left|\,\log(1+\theta_{nj}) \;-\; \theta_{nj}\,\right| \;\leq\; \sum_{j=1}^{k_n}\left|\,\theta_{nj}\,\right|^2.$$

Thus, for each $n \in \mathbb{N}$, there exists $\Lambda_n \in \mathbb{C}$ with $|\,\Lambda_n\,| \leq 1$ such that

$$\sum_{j=1}^{k_n}\log(1+\theta_{nj}) \;=\; \sum_{j=1}^{k_n}\theta_{nj} \;+\; \Lambda_n\cdot\sum_{j=1}^{k_n}\left|\,\theta_{nj}\,\right|^2.$$

(Since for any $z \in \mathbb{C}$, $|\,z\,| \leq A \implies z = A\cdot w$, for some $w \in \mathbb{C}$ with $|\,w\,| \leq 1$.) Next note that, hypotheses (a) and (b) together imply:

$$\sum_{j=1}^{k_n}\left|\,\theta_{nj}\,\right|^2 \;\leq\; \left(\max_{1\leq j\leq k_n}\left|\,\theta_{nj}\,\right|\right)\left(\sum_{j=1}^{k_n}\left|\,\theta_{nj}\,\right|\right) \;\leq\; M\cdot\left(\max_{1\leq j\leq k_n}\left|\,\theta_{nj}\,\right|\right) \quad\longrightarrow\quad 0, \quad \text{as } n \longrightarrow \infty.$$

Therefore, since $|\,\Lambda_n\,| \leq 1$ for each $n \in \mathbb{N}$, we now see that

$$\lim_{n\to\infty}\sum_{j=1}^{k_n}\log(1+\theta_{nj}) \;=\; \lim_{n\to\infty}\sum_{j=1}^{k_n}\theta_{nj} \;+\; \lim_{n\to\infty}\left(\Lambda_n\cdot\sum_{j=1}^{k_n}\left|\,\theta_{nj}\,\right|^2\right) \;=\; \theta+0 \;=\; \theta.$$

We may now conclude, by continuity of the exponential function $\exp(\,\cdot\,)$:

$$\begin{aligned}\lim_{n\to\infty}\prod_{j=1}^{k_n}(\,1+\theta_{nj}\,) \;&=\; \lim_{n\to\infty}\exp\left(\log\prod_{j=1}^{k_n}(\,1+\theta_{nj}\,)\right) \;=\; \lim_{n\to\infty}\exp\left(\sum_{j=1}^{k_n}\log(\,1+\theta_{nj}\,)\right)\\ &=\; \exp\left(\lim_{n\to\infty}\sum_{j=1}^{k_n}\log(\,1+\theta_{nj}\,)\right) \;=\; \exp(\,\theta\,)\end{aligned}$$

This completes the proof of the Lemma. □

References

[1] Chung, K. L. *A Course in Probability Theory*, third ed. Academic Press, 2001.