Scheffé's S-Method for Simultaneous Confidence Intervals

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1 Scheffé's S-Method: A Special Case

Let $X_1, \ldots, X_r$ be independent normal $\mathbb{R}$-valued random variables with common known variance $\sigma^2 = 1$, and $E(X_i) = \xi_i$. We seek to find simultaneous confidence intervals for the collection of linear functionals

$$f_{\mathbf{u}}(\xi) \;=\; \mathbf{u} \bullet \xi \;=\; \sum_{i=1}^{r} u_i\, \xi_i,$$

indexed by $\mathbf{u} = (u_1, u_2, \ldots, u_r)$ in the unit sphere S^{r-1} in $\mathbb{R}^r$, i.e.

$$\|\,\mathbf{u}\,\|^2 \;=\; \sum_{i=1}^{r} u_i^2 \;=\; 1.$$

In other words, we seek two functions $L, M : S^{r-1} \times \mathbb{R}^r \longrightarrow \mathbb{R}$ such that, for each $\mathbf{x} \in \mathbb{R}^r$, the set

$$S(\mathbf{x}; L, M) \;:=\; \left\{\; \zeta \in \mathbb{R}^r \;\middle|\; L(\mathbf{u}, \mathbf{x}) \leq \mathbf{u} \bullet \zeta \leq M(\mathbf{u}, \mathbf{x}),\ \text{for each } \mathbf{u} \in S^{r-1} \;\right\}$$

satisfies:

$$P_\zeta(\;\{\;\mathbf{x} \in \mathbb{R}^r \mid \zeta \in S(\mathbf{x}; L, M)\;\}\;) \;=\; \gamma, \quad \text{for each } \zeta \in \mathbb{R}^r.$$

Proposition 1.1 *Given any such $S(\mathbf{x}; L, M)$, there exists some constant $c > 0$ such that*

$$S(\mathbf{x}; L, M) \;=\; S(\mathbf{x}; c) \;:=\; \left\{\; \zeta \in \mathbb{R}^r \;\middle|\; \|\,\mathbf{u} \bullet (\mathbf{x} - \zeta)\,\| \leq c,\ \text{ for each } \mathbf{u} \in S^{r-1} \;\right\}$$

Proof We first make the following

CLAIM 1: Without loss of generality, we may assume that the functions L and M satisfy:

$$L(\mathbf{u}; \mathbf{x}) \;=\; -M(-\mathbf{u}; \mathbf{x}).$$

Indeed, since $\mathbf{u} \in S^{r-1} \Longleftrightarrow -\mathbf{u} \in S^{r-1}$, we have that, for any $\mathbf{u} \in S^{r-1}$,

$$L(\mathbf{u}, \mathbf{x}) \leq \mathbf{u} \bullet \zeta \leq M(\mathbf{u}, \mathbf{x}) \;\;\Longrightarrow\;\; L(-\mathbf{u}, \mathbf{x}) \leq -\mathbf{u} \bullet \zeta \leq M(-\mathbf{u}, \mathbf{x}) \;\;\Longrightarrow\;\; -M(-\mathbf{u}, \mathbf{x}) \leq \mathbf{u} \bullet \zeta \leq -L(-\mathbf{u}, \mathbf{x})$$

Hence, the inequalities $L(\mathbf{u}, \mathbf{x}) \leq \mathbf{u} \bullet \zeta \leq M(\mathbf{u}, \mathbf{x})$ in fact imply:

$$\widetilde{L}(\mathbf{u}; \mathbf{x}) \;:=\; \max\{L(\mathbf{u}; \mathbf{x}), -M(-\mathbf{u}; \mathbf{x})\} \;\leq\; \mathbf{u} \bullet \zeta \;\leq\; \min\{M(\mathbf{u}; \mathbf{x}), -L(-\mathbf{u}; \mathbf{x})\} \;=:\; \widetilde{M}(\mathbf{u}; \mathbf{x}).$$

Thus, we may replace the original bounding functions L and M with the tighter $\widetilde{L}$ and $\widetilde{M}$, respectively. Furthermore, note that

$$\widetilde{L}(\mathbf{u}; \mathbf{x}) \;:=\; \max\{L(\mathbf{u}; \mathbf{x}), -M(-\mathbf{u}; \mathbf{x})\} \;=\; -\min\{-L(\mathbf{u}; \mathbf{x}), M(-\mathbf{u}; \mathbf{x})\} \;=\; -\widetilde{M}(-\mathbf{u}; \mathbf{x})$$

This completes the proof of CLAIM 1.

CLAIM 2: The functions L and M are "invariant" with respect to the orthogonal group

$$O(r) \;:=\; \left\{\; Q \in \mathrm{GL}(\mathbb{R}^r) \;\middle|\; Q^{\mathrm{T}} \cdot Q = Q \cdot Q^{\mathrm{T}} = I_r \;\right\}$$

in the following sense:

$$L(Q\mathbf{u}; Q\mathbf{x}) = L(\mathbf{u}; \mathbf{x}), \quad \text{and} \quad M(Q\mathbf{u}; Q\mathbf{x}) = M(\mathbf{u}; \mathbf{x}), \quad \text{for each } \mathbf{u} \in S^{r-1}, \mathbf{x} \in \mathbb{R}^r, Q \in O(r).$$

$\square$

A Equivariance

Definition A.1 *Suppose:*

- $X : (\Omega, \mathcal{A}, \mu) \longrightarrow (\mathcal{X}, \mathcal{B})$ *is a random variable.*
- $G_{\mathcal{X}} \subset \mathrm{Aut}(\mathcal{X}, \mathcal{B})$ *is a subgroup of the automorphism group* $\mathrm{Aut}(\mathcal{X}, \mathcal{B})$ *of the codomain* $(\mathcal{X}, \mathcal{B})$ *of* X.
- $G_{\mathcal{T}} \in \mathrm{Aut}(\mathcal{T})$ *is a subgroup of the automorphism group* $\mathrm{Aut}(\mathcal{T})$ *of the decision space* $\mathcal{T}$.
- $h : G_{\mathcal{X}} \longrightarrow G_{\mathcal{T}}$ *is a homomorphism of groups. Thus,* $G_{\mathcal{X}}$ *induces an action of* $\mathcal{T}$ *via* h.
- $T : (\mathcal{X}, \mathcal{B}) \longrightarrow (\mathcal{T}, \mathcal{C})$ *be a measurable map.* T *will be called a* ***statistic****.*

The statistic T *is said to be* $(G_{\mathcal{X}}, h)$***-equivariant*** *if the following condition holds:*

$$T \circ g \;=\; h(g) \circ T\,; \quad \text{equivalently } T(g(x)) \;=\; h(g)[\,T(x)\,] \quad \text{for each } x \in \mathcal{X},\; g \in G_{\mathcal{X}}$$

References