1 Sample space of two-stage sampling & its probability function

Let $U^{(1)}$ be a finite set of size $N^{(1)}$. Let $U_1^{(2)}, U_2^{(2)}, \ldots, U_{N^{(1)}}^{(2)}$ be finite sets of sizes $N_1^{(2)}, N_2^{(2)}, \ldots, N_{N^{(1)}}^{(2)}$, respectively. For each $i = 1, 2, \ldots, N^{(1)}$, we enumerate the elements of $U_i^{(2)}$ as follows:

$$U_i^{(2)} \;=\; \left\{\, u_{i1}, u_{i2}, \ldots, u_{iN_i^{(2)}} \,\right\} \;=\; \left\{\, u_{ik} \;\middle|\; k = 1, 2, \ldots, N_i^{(2)} \,\right\}$$

Let

$$U \;:=\; \bigsqcup_{i \in U^{(1)}} U_i^{(2)} \;=\; \left\{\, u_{ik} \;\middle|\; i = 1, 2, \ldots, N^{(1)},\ k = 1, 2, \ldots, N_i^{(2)} \,\right\}$$

Let $p^{(1)} : \mathcal{S}^{(1)} \longrightarrow (0, 1]$ be our chosen first-stage sampling design, where $\mathcal{S}^{(1)} \subseteq \mathcal{P}\big(U^{(1)}\big)$ is the set of all possible first-stage samples in the design, and $\mathcal{P}\big(U^{(1)}\big)$ is the power set of $U^{(1)}$.

For each $i \in U^{(1)}$, let $p_i^{(2)} : \mathcal{S}_i^{(2)} \longrightarrow (0, 1]$ be our chosen second-stage sampling design, where $\mathcal{S}_i^{(2)} \subseteq \mathcal{P}\big(U_i^{(2)}\big)$ is the set of all possible second-stage samples in the design, and $\mathcal{P}\big(U_i^{(2)}\big)$ is the power set of $U_i^{(2)}$.

The sample space $\mathcal{S}$ of the two-stage sampling design is:

$$\mathcal{S} \;:=\; \left\{\, \left(s^{(1)}, \left\{ s_i^{(2)} \right\}_{i \in U^{(1)}} \right) \in \mathcal{S}^{(1)} \times \prod_{i \in U^{(1)}} \mathcal{S}_i^{(2)} \;\middle|\; \begin{array}{ll} s_i^{(2)} \in \mathcal{S}_i^{(2)}, & \text{if } i \in s^{(1)} \\ s_i^{(2)} = \varnothing, & \text{if } i \notin s^{(1)} \end{array} \right\}$$

We will use the following abbreviation for an element in $\mathcal{S}$:

$$s \;=\; \left(s^{(1)}, \left\{ s_i^{(2)} \right\}_{i \in s^{(i)}} \right)$$

We now define the probability function $p : \mathcal{S} \longrightarrow (0, 1]$ as follows: For each $s \in \mathcal{S}$,

$$p(s) \;:=\; p\left(\left(s^{(1)}, \left\{ s_i^{(2)} \right\}_{i \in s^{(1)}} \right) \right) \;=\; p^{(1)}\left(s^{(1)} \right) \cdot \prod_{i \in s^{(1)}} p_i^{(2)}\left(s_i^{(2)} \right)$$

We will prove in Proposition 1.2 below that $p : \mathcal{S} \longrightarrow (0, 1]$ defined above is indeed a probability function on $\mathcal{S}$.

Lemma 1.1 *For each first-stage sample $s^{(1)} \in \mathcal{S}^{(1)}$, let $\Omega\big(s^{(1)}\big) := \left\{ s_i^{(2)} \in \mathcal{S}_i^{(2)} \;\middle|\; i \in s^{(1)} \right\}$, i.e. $\Omega\big(s^{(1)}\big)$ is the collection of all second-stage samples compatible with the first-stage sample $s^{(1)} \in \mathcal{S}^{(1)}$. Then, we have:*

$$\sum_{\xi \in \Omega(s^{(1)})} p\left(\left(s^{(1)}, \xi \right) \right) \;=\; p^{(1)}\left(s^{(1)} \right).$$

PROOF   Let n be the number of elements in $s^{(1)}$, we write $s^{(1)} = \{ i_1, i_2, \ldots, i_n \}$. Then,

$$p\left(\left(s^{(1)}, \left\{ s_{i_1}^{(2)}, s_{i_2}^{(2)}, \ldots, s_{i_n}^{(2)} \right\} \right) \right) \;=\; p^{(1)}\left(s^{(1)} \right) \cdot p_{i_1}^{(2)}\left(s_{i_1}^{(2)} \right) \cdot p_{i_2}^{(2)}\left(s_{i_2}^{(2)} \right) \cdots p_{i_n}^{(2)}\left(s_{i_n}^{(2)} \right)$$

Hence,

$$
\begin{aligned}
\sum_{\xi \in \Omega(s^{(1)})} p\Big(\big(\,s^{(1)}, \xi\,\big)\Big)
&= \sum_{\zeta_1 \in \mathcal{S}_{i_1}^{(2)}} \sum_{\zeta_2 \in \mathcal{S}_{i_2}^{(2)}} \cdots \sum_{\zeta_n \in \mathcal{S}_{i_n}^{(2)}} p\Big(\big(\,s^{(1)}, \{\,\zeta_1, \zeta_2, \ldots, \zeta_n\,\}\,\big)\Big) \\
&= \sum_{\zeta_1 \in \mathcal{S}_{i_1}^{(2)}} \sum_{\zeta_2 \in \mathcal{S}_{i_2}^{(2)}} \cdots \sum_{\zeta_n \in \mathcal{S}_{i_n}^{(2)}} p^{(1)}\Big(s^{(1)}\Big) \cdot p_{i_1}^{(2)}(\zeta_1) \cdot p_{i_2}^{(2)}(\zeta_2) \cdots p_{i_n}^{(2)}(\zeta_n) \\
&= p^{(1)}\Big(s^{(1)}\Big) \cdot \left(\sum_{\zeta_1 \in \mathcal{S}_{i_1}^{(2)}} p_{i_1}^{(2)}(\zeta_1)\right) \cdot \left(\sum_{\zeta_2 \in \mathcal{S}_{i_2}^{(2)}} p_{i_2}^{(2)}(\zeta_2)\right) \cdots \left(\sum_{\zeta_n \in \mathcal{S}_{i_n}^{(2)}} p_{i_n}^{(2)}(\zeta_n)\right) \\
&= p^{(1)}\Big(s^{(1)}\Big) \cdot (\,1\,) \cdot (\,1\,) \cdots (\,1\,) \\
&= p^{(1)}\Big(s^{(1)}\Big)
\end{aligned}
$$

$\square$

Proposition 1.2

$$
\sum_{s \in \mathcal{S}} p(s) = 1
$$

PROOF

$$
\sum_{s \in \mathcal{S}} p(s) = \sum_{(s^{(1)}, \xi) \in \mathcal{S}} p(s^{(1)}, \xi) = \sum_{s^{(1)} \in \mathcal{S}^{(1)}} \sum_{\xi \in \Omega(s^{(1)})} p(s^{(1)}, \xi) = \sum_{s^{(1)} \in \mathcal{S}^{(1)}} p^{(1)}(s^{(1)}) = 1,
$$

where the second-last equality follows from the preceding Lemma. $\square$

2 Estimation in two-stage sampling

Let $\mathbf{y} : U \longrightarrow \mathbb{R}^m$ be an $\mathbb{R}^m$-valued function defined on U (such a $\mathbf{y}$ commonly called a "population parameter"). We will use the common notation $\mathbf{y}_{kl}$ for $\mathbf{y}(u_{kl})$, for $k = 1, 2, \ldots, N^{(1)}$ and $l = 1, 2, \ldots, N_k^{(2)}$. We wish to estimate

$$
\mathbf{T_y} := \sum_{u \in U} \mathbf{y}(u) = \sum_{k \in U^{(1)}} \sum_{l \in U_k^{(2)}} \mathbf{y}_{kl} = \sum_{k=1}^{N^{(1)}} \sum_{l=1}^{N_k^{(2)}} \mathbf{y}_{kl} \in \mathbb{R}^m
$$

via two-stage sampling. We consider estimators for $\widehat{T}_{\mathbf{y}}$ of the following form:

$$
\begin{aligned}
\widehat{\mathbf{T}}_{\mathbf{y}} : \quad \mathcal{S} &\longrightarrow \mathbb{R}^m \\
\Big(s^{(1)}, \{s_k^{(2)}\}_{k \in s^{(1)}}\Big) &\longmapsto \sum_{k \in s^{(1)}} w_k^{(1)}(s^{(1)}) \, \widehat{\mathbf{T}}_{\mathbf{y}|k}(s_k^{(2)}) = \sum_{k \in U^{(1)}} I_k(s^{(1)}) \, w_k^{(1)}(s^{(1)}) \, \widehat{\mathbf{T}}_{\mathbf{y}|k}(s_k^{(2)}),
\end{aligned}
$$

where, for each $k \in U^{(1)}$, $w_k^{(1)} : \mathcal{S}^{(1)} \longrightarrow \mathbb{R}$ and $\widehat{\mathbf{T}}_{\mathbf{y}|k} : \mathcal{S}_k^{(2)} \longrightarrow \mathbb{R}^m$ are random variables.

Proposition 2.1   *Suppose:*

- *The first-stage weights $w_k^{(1)} : \mathcal{S}^{(1)} \longrightarrow (0, 1]$ satisfy the following:*

$$
E^{(1)}\Big[\,\widehat{T}_z\,\Big] = T_z := \sum_{k \in U^{(1)}} z_k, \quad \text{for any function } z : U^{(1)} \longrightarrow \mathbb{R},
$$

where $\widehat{T}_z : \mathcal{S}^{(1)} \longrightarrow \mathbb{R}$ is a random variable defined by $\widehat{T}_z(s^{(1)}) := \sum_{k \in s^{(1)}} w_k^{(1)}(s^{(1)}) \, z_k$.

- *For each $k \in U^{(1)}$, the random variable $\widehat{\mathbf{T}}_{\mathbf{y}|k} : \mathcal{S}_k^{(2)} : \longrightarrow \mathbb{R}$ is a design-unbiased estimator for $T_{\mathbf{y}|k}$, i.e.*

$$E_k^{(2)}\left[\, \widehat{\mathbf{T}}_{\mathbf{y}\,|\,k} \,\right] \;\; = \;\; \mathbf{T}_{\mathbf{y}\,|\,k} \;\; := \;\; \sum_{l \in U_k^{(2)}} \mathbf{y}_{kl}$$

Then, the random variable $\widehat{\mathbf{T}}_{\mathbf{y}}$ is an unbiased estimator for $\mathbf{T}_{\mathbf{y}}$.

Definition 2.2

A random variable $\widehat{\mathbf{T}}_{\mathbf{y}} : \mathcal{S} \longrightarrow \mathbb{R}^m$ is said to be <u>*linear in the population parameter*</u> *$\mathbf{y} : U \longrightarrow \mathbb{R}$ if it has the following form:*

$$\begin{aligned} \widehat{\mathbf{T}}_{\mathbf{y}} : \quad \mathcal{S} \quad &\longrightarrow \quad \mathbb{R}^m \\ s \quad &\longmapsto \quad \sum_{k \in s} w_k(s)\,\mathbf{y}_k \;\; = \;\; \sum_{k \in U} I_k(s)\,w_k(s)\,\mathbf{y}_k, \end{aligned}$$

where, for each $k \in U$, $w_k : \mathcal{S} \longrightarrow \mathbb{R}$ is itself an $\mathbb{R}$-valued random variable, and $I_k : \mathcal{S} \longrightarrow \{0,1\}$ is the indicator random variable defined by:

$$I_k(s) \;\; = \;\; \left\{ \begin{array}{ll} 1, & \text{if } k \in s, \\ 0, & \text{otherwise} \end{array} \right.$$

We call the w_k's the <u>*weights*</u> *of $\widehat{\mathbf{T}}_{\mathbf{y}}$, and we use the notation $\widehat{\mathbf{T}}_{\mathbf{y};w}$ to indicate that the random variable depends on the weights w_k.*

Nomenclature In the context of finite-population probability sampling, under a design $p : \mathcal{S} \longrightarrow (0,1]$, an "estimator" is precisely just a random variable defined on the space $\mathcal{S}$ of all admissible samples in the design.

Proposition 2.3

Let $\widehat{\mathbf{T}}_{\mathbf{y};w} : \mathcal{S} \longrightarrow \mathbb{R}^m$, with $\widehat{\mathbf{T}}_{\mathbf{y};w}(s) = \sum_{k \in U} I_k(s)\,w_k(s)\,\mathbf{y}_k = \sum_{k \in s} w_k(s)\,\mathbf{y}_k$, be a random variable linear in the population parameter $\mathbf{y} : U \longrightarrow \mathbb{R}$. Then,

$$E\left[\, \widehat{\mathbf{T}}_{\mathbf{y};w} \,\right] \;\; = \;\; \mathbf{T}_{\mathbf{y}}, \text{ for arbitrary } \mathbf{y} \quad \Longleftrightarrow \quad E[\, I_k\,w_k \,] \;\; = \;\; 1, \text{ for each } k \in U.$$

PROOF   Note:

$$E\left[\, \widehat{\mathbf{T}}_{\mathbf{y};w} \,\right] \;\; = \;\; E\left[\, \sum_{k \in s} w_k\,\mathbf{y}_k \,\right] \;\; = \;\; E\left[\, \sum_{k \in U} I_k\,w_k\,\mathbf{y}_k \,\right] \;\; = \;\; \sum_{k \in U} E[\, I_k\,w_k \,]\,\mathbf{y}_k$$

Hence, since $\mathbf{y} : U \longrightarrow \mathbb{R}$ is arbitrary,

$$E\left[\, \widehat{\mathbf{T}}_{\mathbf{y};w} \,\right] \;\; = \;\; \mathbf{T}_{\mathbf{y}} \;\; := \;\; \sum_{k \in U} \mathbf{y}_k \quad \Longleftrightarrow \quad \sum_{k \in U} (E[\, I_k\,w_k \,] - 1)\cdot\mathbf{y}_k \;\; = \;\; \mathbf{0} \quad \Longleftrightarrow \quad E[\, I_k\,w_k \,] \;\; = \;\; 1, \text{ for each } k \in U.$$

The proof of the Proposition is now complete.   □

Corollary 2.4

Let $U = \{1, 2, \ldots, N\}$ be a finite population. For any fixed but arbitrary population parameter $\mathbf{y} : U \longrightarrow \mathbb{R}^m$ and for any sampling design $p : \mathcal{S} \longrightarrow (0,1]$ such that each of its first-order inclusion probabilities is strictly positive, the Horvitz-Thompson estimator $\widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}}$ is well-defined and it is the unique unbiased estimator for $\mathbf{T}_{\mathbf{y}}$, which is linear in $\mathbf{y}$ and whose weights are constant in s.

PROOF   Recall that the Horvitz-Thompson estimator is defined as:

$$\widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}}(s) \;\; := \;\; \sum_{k \in s} \frac{1}{\pi_k}\,\mathbf{y}_k \;\; := \;\; \sum_{k \in U} I_k(s)\,\frac{1}{\pi_k}\,\mathbf{y}_k,$$

where $\pi_k := E[\,I_k\,] = \sum\limits_{k \in U} p(s)\, I_k(s) = \sum\limits_{s \ni k} p(s)$ is the inclusion probability of $k \in U$ under the sampling design $p : \mathcal{S} \longrightarrow (0,1]$. Clearly, $\widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}}$ is linear in $\mathbf{y}$ with weights constant in s. Next, note that:

$$
E\left[\,\widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}}\,\right] \;=\; E\left[\sum_{k \in s} \frac{1}{\pi_k}\,\mathbf{y}_k\right] \;=\; E\left[\sum_{k \in U} I_k\,\frac{\mathbf{y}_k}{\pi_k}\right] \;=\; \sum_{k \in U} E[\,I_k\,]\,\frac{\mathbf{y}_k}{\pi_k} \;=\; \sum_{k \in U} \pi_k\,\frac{\mathbf{y}_k}{\pi_k} \;=\; \sum_{k \in U} \mathbf{y}_k \;=\; \mathbf{T}_y
$$

Hence, $\widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}}$ is an unbiased estimator for $\mathbf{T}_{\mathbf{y}}$. Conversely, let

$$
\widehat{\mathbf{T}}_{y;w}(s) \;=\; \sum_{k \in s} w_k\,\mathbf{y}_k
$$

be any unbiased estimator for $\mathbf{T}_{\mathbf{y}}$ which linear in $\mathbf{y}$ with weights w_k constant in s. Thus,

$$
\sum_{k \in U} \mathbf{y}_k \;=\; \mathbf{T}_{\mathbf{y}} \;=\; E\left[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\right] \;=\; E\left[\sum_{k \in s} w_k\,\mathbf{y}_k\right] \;=\; E\left[\sum_{k \in U} I_k\,w_k\,\mathbf{y}_k\right] \;=\; \sum_{k \in U} E[\,I_k\,]\,w_k\,\mathbf{y}_k \;=\; \sum_{k \in U} \pi_k\,w_k\,\mathbf{y}_k.
$$

Since $\mathbf{y}$ is arbitrary, the above equation immediately implies that

$$
\pi_k w_k - 1 = 0,
$$

or equivalently, $w_k = \dfrac{1}{\pi_k}$; in other words, $\widehat{\mathbf{T}}_{\mathbf{y};w}$ is in fact equal to the Horvitz-Thompson estimator. The proof of the Corollary is now complete. $\qquad\square$

Lemma 2.5

Let $(\Omega, \mathcal{A}, p)$ be a probability space, $X, Y : \Omega \longrightarrow \mathbb{R}$ be two $\mathbb{R}$-valued random variables defined on Ω, and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ be two fixed vectors in $\mathbb{R}^m$. Then,

$$
\mathrm{Cov}(\,X \cdot \mathbf{u}\,,\, Y \cdot \mathbf{v}\,) \;=\; \mathrm{Cov}(X,Y) \cdot \mathbf{u} \cdot \mathbf{v}^T \;\in\; \mathbb{R}^{m \times m}
$$

PROOF   Note:

$$
\begin{aligned}
\mathrm{Cov}(\,X \cdot \mathbf{u}\,,\, Y \cdot \mathbf{v}\,) \;&:=\; E\left[\,(X\,\mathbf{u} - \mu_X \mathbf{u}) \cdot (Y\,\mathbf{v} - \mu_Y \mathbf{v})^T\,\right] \;=\; E\left[\,(X - \mu_X)\,\mathbf{u} \cdot (Y - \mu_Y)\,\mathbf{v}^T\,\right] \\
&=\; E\left[\,(X - \mu_X)\,(Y - \mu_Y) \cdot \mathbf{u} \cdot \mathbf{v}^T\,\right] \;=\; E\left[\,(X - \mu_X)\,(Y - \mu_Y)\,\right] \cdot \mathbf{u} \cdot \mathbf{v}^T \\
&=\; \mathrm{Cov}(X,Y) \cdot \mathbf{u} \cdot \mathbf{v}^T,
\end{aligned}
$$

as required. $\qquad\square$

Proposition 2.6

Let $\widehat{\mathbf{T}}_{\mathbf{y};w} : \mathcal{S} \longrightarrow \mathbb{R}$, with $\widehat{\mathbf{T}}_{\mathbf{y};w}(s) = \sum\limits_{k \in s} w_k(s)\,\mathbf{y}_k = \sum\limits_{k \in U} I_k(s)\,w_k(s)\,\mathbf{y}_k$, be a random variable linear in the population parameter $\mathbf{y} : U \longrightarrow \mathbb{R}$. Then, the covariance matrix of $\widehat{\mathbf{T}}_{\mathbf{y};w}$ is given by:

$$
\mathrm{Var}\left[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\right] \;=\; \sum_{i \in U} \sum_{k \in U} \mathrm{Cov}[\,I_i\,w_i\,,\, I_k\,w_k\,]\,\mathbf{y}_i \cdot \mathbf{y}_k^T \;\in \mathbb{R}^{m \times m}
$$

Furthermore, if the first-order and second-order inclusion probabilities of the sampling design $p : \mathcal{S} \longrightarrow (0,1]$ are all strictly positive, i.e. $\pi_k = \pi_{kk} := \sum\limits_{s \ni k} p(s) > 0$, for each $k \in U$, and $\pi_{ik} := \sum\limits_{s \ni i,k} p(s) > 0$, for any distinct $i, k \in U$, then an unbiased estimator for $\mathrm{Var}\left[\,\widehat{\mathbf{T}}_{y;w}\,\right]$ is given by:

$$
\widehat{\mathrm{Var}}\left[\,\widehat{\mathbf{T}}_{y;w}\,\right](s) \;:=\; \sum_{i,k \in s} \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}}\,\mathbf{y}_i \cdot \mathbf{y}_k^T \;=\; \sum_{k \in s} \frac{\mathrm{Var}(I_k w_k)}{\pi_k}\,\mathbf{y}_k \cdot \mathbf{y}_k^T + \sum_{\substack{i,k \in s \\ i \neq k}} \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}}\,\mathbf{y}_i \cdot \mathbf{y}_k^T, \;\; \text{for each } s \in \mathcal{S}.
$$

PROOF First, note that Lemma 2.5 implies:

$$\mathrm{Var}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\Big] \;=\; \mathrm{Cov}\Bigg[\sum_{i \in U} I_i\, w_i\, \mathbf{y}_i \,,\, \sum_{k \in U} I_k\, w_k\, \mathbf{y}_k\Bigg] \;=\; \sum_{i \in U}\sum_{k \in U} \mathrm{Cov}[\, I_i\, w_i \,,\, I_k\, w_k\,] \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T \;\in\; \mathbb{R}^{m \times m}$$

Next,

$$
\begin{aligned}
E\Big(\,\widehat{\mathrm{Var}}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\Big]\,\Big) \;&=\; \sum_{s \in \mathcal{S}} p(s) \cdot \widehat{\mathrm{Var}}\Big[\,\widehat{\mathbf{T}}_{\mathbf{y};w}\,\Big](s) \;=\; \sum_{s \in \mathcal{S}} p(s) \cdot \Bigg(\sum_{i,k \in s} \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}} \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T\Bigg) \\[2mm]
&=\; \sum_{s \in \mathcal{S}} p(s) \cdot \Bigg(\sum_{i,k \in U} I_i(s) I_k(s) \cdot \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}} \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T\Bigg) \\[2mm]
&=\; \sum_{i,k \in U} \Bigg(\sum_{s \in \mathcal{S}} p(s) I_i(s) I_k(s)\Bigg) \cdot \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}} \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T \\[2mm]
&=\; \sum_{i,k \in U} \Bigg(\sum_{s \ni i,k} p(s)\Bigg) \cdot \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}} \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T \\[2mm]
&=\; \sum_{i,k \in U} \pi_{ik} \cdot \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}} \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T \;=\; \sum_{i,k \in U} \mathrm{Cov}(I_i w_i, I_k w_k) \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T \\[2mm]
&=\; \mathrm{Var}\Big[\,\widehat{\mathbf{T}}_{y;w}\,\Big]
\end{aligned}
$$

Lastly, recall that $\pi_{kk} = \pi_k$ and $\mathrm{Cov}(I_k w_k, I_k w_k) = \mathrm{Var}[\, I_k w_k\,]$, and the validity of the following identity is thus trivial:

$$\sum_{i,k \in s} \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}} \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T \;=\; \sum_{k \in s} \frac{\mathrm{Var}(I_k w_k)}{\pi_k} \cdot \mathbf{y}_k \cdot \mathbf{y}_k^T + \sum_{\substack{i,k \in s \\ i \neq k}} \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}} \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T$$

The proof of the Proposition is complete.   □

3   Calibrated linear estimators for (multivariate) population totals

Definition 3.1

Let $\widehat{\mathbf{T}}_{\mathbf{y};w} : \mathcal{S} \longrightarrow \mathbb{R}^m$ be an $\mathbb{R}^m$-valued random variable which is linear in the $\mathbb{R}^m$-valued population parameter $\mathbf{y} : U \longrightarrow \mathbb{R}^m$, i.e.

$$
\begin{aligned}
\widehat{\mathbf{T}}_{\mathbf{y};w} : \quad \mathcal{S} \quad &\longrightarrow \quad \mathbb{R}^m \\
s \quad &\longmapsto \quad \sum_{k \in s} w_k(s) \cdot \mathbf{y}_k \;=\; \sum_{k \in U} I_k(s)\, w_k(s) \cdot \mathbf{y}_k,
\end{aligned}
$$

where, for each $k \in U$, $w_k : \mathcal{S} \longrightarrow \mathbb{R}$ is itself an $\mathbb{R}$-valued random variable, and $I_k : \mathcal{S} \longrightarrow \{0,1\}$ is the indicator random variable defined by:

$$I_k(s) \;=\; \begin{cases} 1, & \text{if } k \in s, \\ 0, & \text{otherwise} \end{cases}$$

Let $x : U \longrightarrow \mathbb{R}$ be an $\mathbb{R}$-valued population parameter and $T_x := \sum_{k \in U} x_k$.

Then, $\widehat{\mathbf{T}}_{\mathbf{y};w}$ is said to be <u>calibrated with respect to x</u> *if*

$$\sum_{k \in s} w_k(s)\, x_k \;=\; T_x, \;\; \text{for } \textcolor{red}{\text{each }} s \in \mathcal{S}.$$

Example 3.2

If the sampling design has fixed sample size and each of its first-order inclusion probabilities is strictly positive, then Horvitz-Thompson estimator is calibrated with respect to the first-order inclusion probabilities.

To see this, let $U = \{1, 2, \ldots, N\}$ be a finite population, $\mathbf{y} : U \longrightarrow \mathbb{R}^m$ a population parameter, and $p : \mathcal{S} \subset \mathcal{P}(U) \longrightarrow (0, 1]$ a sampling design such that $\pi_k := \sum_{s \ni k} p(s) > 0$, for each $k \in U$. The Horvitz-Thompson estimator $\widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}} : \mathcal{S} \longrightarrow \mathbb{R}$ is then well-defined and is given by:

$$\widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}}(s) \ := \ \sum_{k \in s} \frac{\mathbf{y}_k}{\pi_k}$$

Let $x : U \longrightarrow \mathbb{R}$ be defined by

$$x_k \ = \ \pi_k, \ \text{ for each } k \in U,$$

i.e. x_k is simply the inclusion probability of $k \in U$ under the sampling design $p : \mathcal{S} \longrightarrow (0, 1]$.

Now, suppose that the sampling design has a fixed sample size n, and we shall show that $\widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}}$ is consequently calibrated with respect to $x : U \longrightarrow \mathbb{R}$. Indeed, recall that the weights of the Horvitz-Thompson estimator are simply $w_k(s) = 1/\pi_k$, for each $k \in U$ and each $s \in \mathcal{S}$. Hence,

$$\sum_{k \in s} w_k(s) \, x_k \ = \ \sum_{k \in s} \frac{1}{\pi_k} \pi_k \ = \ \sum_{k \in s} 1 \ = \ \left(\begin{array}{c} \text{sample} \\ \text{size of } s \end{array} \right) \ = \ n,$$

since the sampling design has fixed size n. On the other hand,

$$T_x \ = \ \sum_{k \in U} x_k \ = \ \sum_{k \in U} \pi_k \ = \ \sum_{k \in U} E[\, I_k \,] \ = \ E\left[\sum_{k \in U} I_k \right] \ = \ E\left[\begin{array}{c} \text{sample} \\ \text{size} \end{array} \right] \ = \ n,$$

again since the sample size is fixed and equals n. Therefore, we have, for any $s \in \mathcal{S}$,

$$\sum_{k \in s} w_k(s) \, x_k \ = \ n \ = \ T_x$$

Therefore, the Horvitz-Thompson estimator, under the assumption of fixed sample size, is indeed calibrated with respect to the inclusion probabilities $x : U \longrightarrow \mathbb{R}$, $x_k = \pi_k := \sum_{s \ni k} p(s)$, for each $k \in U$. $\square$

Proposition 3.3

Let $\widehat{\mathbf{T}}_{\mathbf{y};w,x} : \mathcal{S} \longrightarrow \mathbb{R}^m$ be an $\mathbb{R}^m$-valued random variable which is linear in the $\mathbb{R}^m$-valued population parameter $\mathbf{y} : U \longrightarrow \mathbb{R}^m$ and calibrated with respect to the population parameter $x : U \longrightarrow \mathbb{R}$, with $x_k \neq 0$ for each $k \in U$.
Then, the mean squared error matrix of $\widehat{\mathbf{T}}_{\mathbf{y};w,x}$ as an estimator of $\mathbf{T}_{\mathbf{y}}$ is given by:

$$\mathrm{MSE}\left[\widehat{\mathbf{T}}_{\mathbf{y};w,x} \right] \ = \ -\frac{1}{2} \sum_{\substack{i,k \in U \\ i \neq k}} a_{ik} \left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k} \right) \cdot \left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k} \right)^T x_i \, x_k \ \in \ \mathbb{R}^{m \times m}, \ \text{ where } \ a_{ik} \ := \ E[\, (I_i \, w_i - 1) \, (I_k \, w_k - 1) \,].$$

PROOF

$$\begin{aligned}
\mathrm{MSE}\left[\widehat{\mathbf{T}}_{\mathbf{y};w,x} \right] \ &= \ E\left[\left(\widehat{\mathbf{T}}_{\mathbf{y};w,x} - \mathbf{T}_{\mathbf{y}} \right) \cdot \left(\widehat{\mathbf{T}}_{\mathbf{y};w,x} - \mathbf{T}_{\mathbf{y}} \right)^T \right] \ = \ E\left[\left(\sum_{i \in U} (I_i w_i - 1) \, \mathbf{y}_i \right) \cdot \left(\sum_{k \in U} (I_k w_k - 1) \, \mathbf{y}_k \right)^T \right] \\[2mm]
&= \ \sum_{i \in U} \sum_{k \in U} E[\, (I_i w_i - 1) \, (I_k w_k - 1) \,] \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T \ = \ \sum_{k \in U} a_{kk} \cdot \mathbf{y}_k \cdot \mathbf{y}_k^T + \sum_{\substack{i,k \in U \\ i \neq k}} a_{ik} \cdot \mathbf{y}_i \cdot \mathbf{y}_k^T \\[2mm]
&= \ \sum_{k \in U} a_{kk} \left(\frac{\mathbf{y}_k \cdot \mathbf{y}_k^T}{x_k^2} \right) x_k^2 + \sum_{\substack{i,k \in U \\ i \neq k}} a_{ik} \left(\frac{\mathbf{y}_i}{x_i} \right) \cdot \left(\frac{\mathbf{y}_k}{x_k} \right)^T x_i \, x_k
\end{aligned}$$

On the other hand,

$$-\frac{1}{2} \sum_{\substack{i,k \in U \\ i \neq k}} a_{ik} \left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k} \right) \cdot \left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k} \right)^T x_i\, x_k$$

$$= -\frac{1}{2} \sum_{\substack{i,k \in U \\ i \neq k}} a_{ik} \left[\left(\frac{\mathbf{y}_i}{x_i} \right) \left(\frac{\mathbf{y}_i}{x_i} \right)^T - \left(\frac{\mathbf{y}_i}{x_i} \right) \left(\frac{\mathbf{y}_k}{x_k} \right)^T - \left(\frac{\mathbf{y}_k}{x_k} \right) \left(\frac{\mathbf{y}_i}{x_i} \right)^T + \left(\frac{\mathbf{y}_k}{x_k} \right) \left(\frac{\mathbf{y}_k}{x_k} \right)^T \right] x_i\, x_k$$

$$= -\frac{1}{2} \sum_{\substack{i,k \in U \\ i \neq k}} a_{ik} \left[\left(\frac{\mathbf{y}_i}{x_i} \right) \left(\frac{\mathbf{y}_i}{x_i} \right)^T + \left(\frac{\mathbf{y}_k}{x_k} \right) \left(\frac{\mathbf{y}_k}{x_k} \right)^T \right] x_i\, x_k + \sum_{\substack{i,k \in U \\ i \neq k}} a_{ik} \left(\frac{\mathbf{y}_i}{x_i} \right) \left(\frac{\mathbf{y}_k}{x_k} \right)^T x_i\, x_k$$

Thus, the proof of the present Proposition will be complete once we show:

$$\underbrace{\sum_{k \in U} a_{kk} \left(\frac{\mathbf{y}_k}{x_k} \right) \left(\frac{\mathbf{y}_k}{x_k} \right)^T x_k^2}_{\frac{1}{2} \sum_{\substack{i,k \in U \\ i=k}} a_{ik} \left[\left(\frac{\mathbf{y}_i}{x_i} \right) \left(\frac{\mathbf{y}_i}{x_i} \right)^T + \left(\frac{\mathbf{y}_k}{x_k} \right) \left(\frac{\mathbf{y}_k}{x_k} \right)^T \right] x_i\, x_k} = -\frac{1}{2} \sum_{\substack{i,k \in U \\ i \neq k}} a_{ik} \left[\left(\frac{\mathbf{y}_i}{x_i} \right) \left(\frac{\mathbf{y}_i}{x_i} \right)^T + \left(\frac{\mathbf{y}_k}{x_k} \right) \left(\frac{\mathbf{y}_k}{x_k} \right)^T \right] x_i\, x_k,$$

which is equivalent to:

$$\sum_{i \in U} \sum_{k \in U} a_{ik} \left[\left(\frac{\mathbf{y}_i}{x_i} \right) \left(\frac{\mathbf{y}_i}{x_i} \right)^T + \left(\frac{\mathbf{y}_k}{x_k} \right) \left(\frac{\mathbf{y}_k}{x_k} \right)^T \right] x_i\, x_k = 0. \tag{3.1}$$

Observe that

$$\text{LHS(3.1)} = \sum_{i \in U} \sum_{k \in U} a_{ik} \left(\frac{\mathbf{y}_i}{x_i} \right) \left(\frac{\mathbf{y}_i}{x_i} \right)^T x_i\, x_k + \sum_{i \in U} \sum_{k \in U} a_{ik} \left(\frac{\mathbf{y}_k}{x_k} \right) \left(\frac{\mathbf{y}_k}{x_k} \right)^T x_i\, x_k$$

$$= 2 \sum_{i \in U} \sum_{k \in U} a_{ik} \left(\frac{\mathbf{y}_i}{x_i} \right) \left(\frac{\mathbf{y}_i}{x_i} \right)^T x_i\, x_k = 2 \sum_{i \in U} x_i \left(\frac{\mathbf{y}_i}{x_i} \right) \left(\frac{\mathbf{y}_i}{x_i} \right)^T \left(\sum_{k \in U} a_{ik} x_k \right).$$

Hence, (3.1) follows once we show

$$\sum_{k \in U} a_{ik} x_k = 0, \quad \text{for each } i \in U. \tag{3.2}$$

Lastly, we now claim that (3.2) follows from the hypothesis that $\widehat{T}_{y;w;x}$ is calibrated with respect to x. Indeed,

$$\sum_{k \in U} a_{ik} x_k = \sum_{k \in U} E[\, (I_i w_i - 1)(I_k w_k - 1)\,] x_k = \sum_{k \in U} \left[\sum_{s \in \mathcal{S}} p(s)(I_i(s)w_i(s) - 1)(I_k(s)w_k(s) - 1) \right] x_k$$

$$= \sum_{s \in \mathcal{S}} p(s) \cdot (I_i(s)w_i(s) - 1) \cdot \left[\sum_{k \in U} (I_k(s)w_k(s) - 1) \cdot x_k \right]$$

$$= \sum_{s \in \mathcal{S}} p(s) \cdot (I_i(s)w_i(s) - 1) \cdot \left[\underbrace{\left(\sum_{k \in s} w_k(s)\, x_k \right) - T_x}_{0} \right]$$

$$= 0$$

The proof of the present Proposition is now complete. $\qquad\square$

Proposition 3.4 (The Yates-Grundy-Sen Variance Estimator for calibrated linear population total estimators)

Let $p : \mathcal{S} \longrightarrow (0,1]$ be a sampling design each of whose first-order and second-order inclusion probabilities is strictly positively. Let $\widehat{\mathbf{T}}_{\mathbf{y};w,x} : \mathcal{S} \longrightarrow \mathbb{R}^m$ be a random variable which is linear in the population parameter $\mathbf{y} : U \longrightarrow \mathbb{R}^m$ and calibrated with respect to the population parameter $x : U \longrightarrow \mathbb{R}$, with $x_k \neq 0$ for each $k \in U$. Suppose that $\widehat{\mathbf{T}}_{\mathbf{y};w,x}$ is an unbiased estimator for $\mathbf{T_y} := \sum_{k \in U} \mathbf{y}_k$, for arbitrary $\mathbf{y}$. Then, the following is an unbiased estimator of the variance $\mathrm{Var}\left[\ \widehat{\mathbf{T}}_{\mathbf{y};w,x}\ \right]$ *of $\widehat{\mathbf{T}}_{\mathbf{y};w,x}$: For each $s \in \mathcal{S}$ admissible in the sampling design $p : \mathcal{S} \longrightarrow (0,1]$,*

$$\widehat{\mathrm{Var}}\left[\ \widehat{\mathbf{T}}_{\mathbf{y};w,x}\ \right](s) \ := \ -\frac{1}{2}\sum_{\substack{i,k \in s \\ i \neq k}} \left(w_i(s)w_k(s) - \frac{1}{\pi_{ik}}\right)\left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k}\right)\cdot\left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k}\right)^T x_i\, x_k$$

Terminology: $\widehat{\mathrm{Var}}\left[\ \widehat{\mathbf{T}}_{\mathbf{y};w,x}\ \right]$ is called the Yates-Grundy-Sen Variance Estimator.

PROOF   Since $\widehat{\mathbf{T}}_{\mathbf{y};w,x}$ is an unbiased estimator for $\mathbf{T_y}$ by hypothesis, we have $\mathrm{Var}\left[\ \widehat{\mathbf{T}}_{\mathbf{y};w,x}\ \right] = \mathrm{MSE}\left[\ \widehat{\mathbf{T}}_{\mathbf{y};w,x}\ \right]$. By Proposition 3.3, we thus have:

$$\mathrm{Var}\left[\ \widehat{\mathbf{T}}_{\mathbf{y};w,x}\ \right] \ = \ -\frac{1}{2}\sum_{\substack{i,k \in U \\ i \neq k}} a_{ik}\left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k}\right)^2 x_i\, x_k, \quad \text{where} \ \ a_{ik} \ := \ E[\,(I_i\, w_i - 1)\,(I_k\, w_k - 1)\,].$$

On the other hand,

$$E\left(\ \widehat{\mathrm{Var}}\left[\ \widehat{\mathbf{T}}_{\mathbf{y};w,x}\ \right]\ \right) \ = \ -\frac{1}{2}\sum_{\substack{i,k \in U \\ i \neq k}} E\left[\ I_iI_k\left(w_iw_k - \frac{1}{\pi_{ik}}\right)\right]\left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k}\right)\cdot\left(\frac{\mathbf{y}_i}{x_i} - \frac{\mathbf{y}_k}{x_k}\right)^T x_i\, x_k$$

Thus, it remains only to show:

$$a_{ik} \ = \ E\left[\ I_iI_k\left(w_iw_k - \frac{1}{\pi_{ik}}\right)\right].$$

Now,

$$E\left[\ I_iI_k\left(w_iw_k - \frac{1}{\pi_{ik}}\right)\right] \ = \ E[\,I_iI_kw_iw_k\,] - \frac{1}{\pi_{ik}}E[\,I_iI_k\,] \ = \ E[\,I_iI_kw_iw_k\,] - \frac{1}{\pi_{ik}}\pi_{ik} \ = \ E[\,I_iI_kw_iw_k\,] - 1,$$

and

$$\begin{aligned} a_{ik} \ &= \ E[\,(I_i\, w_i - 1)\,(I_k\, w_k - 1)\,] \ = \ E[\,I_i\, I_k\, w_i\, w_k\,] - E[\,I_i\, w_i\,] - E[\,I_k\, w_k\,] + 1 \\ &= \ E[\,I_i\, I_k\, w_i\, w_k\,] - 1 - 1 + 1 \ = \ E[\,I_i\, I_k\, w_i\, w_k\,] - 1 \\ &= \ E\left[\ I_iI_k\left(w_iw_k - \frac{1}{\pi_{ik}}\right)\right], \end{aligned}$$

where third last equality follows from Proposition 2.3 and the unbiasedness hypothesis on $\widehat{\mathbf{T}}_{\mathbf{y};w,x}$ as an estimator for $\mathbf{T_y}$. The proof of the present Proposition is now complete. $\square$

4   Unbiased variance estimators for the Horvitz-Thompson Estimator

Let $U = \{1, 2, \ldots, N\}$ be a finite population. Let $\mathbf{y} = (y_1, y_2, \ldots, y_m) : U \longrightarrow \mathbb{R}^m$ be an $\mathbb{R}^m$-valued function defined on U (commonly called a "population parameter"). We will use the common notation $\mathbf{y}_k$ for $\mathbf{y}(k)$. We wish to estimate $\mathbf{T_y} := \sum_{k \in U} \mathbf{y}_k \in \mathbb{R}^m$ via survey sampling. Let $p : \mathcal{S} \longrightarrow (0,1]$ be our chosen sampling design, where $\mathcal{S} \subseteq \mathcal{P}(U)$ is the set of all possible samples in the design, and $\mathcal{P}(U)$ is the power set of U.

Proposition 4.1

Suppose the first-order and second-order inclusion probabilities of $p : \mathcal{S} \longrightarrow (0,1]$ are all strictly positive, i.e.

$$\pi_k := \sum_{s \ni k} p(s) = \sum_{k \in U} I_k(s) p(s) > 0 \quad \text{and} \quad \pi_{ik} := \sum_{s \ni i,k} p(s) = \sum_{i,k \in U} I_i(s) I_k(s) p(s) > 0,$$

for any $i, k \in U$. Then, the Horvitz-Thompson estimator for $\mathbf{T_y}$ is:

$$\widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}}(s) \ := \ \sum_{k \in s} \frac{1}{\pi_k} \mathbf{y}_k,$$

and the covariance matrix of the Horvitz-Thompson estimator can be given by:

$$\mathrm{Var}\Big[\, \widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}} \,\Big] \ = \ \sum_{i,k \in U} (\pi_{ik} - \pi_i \pi_k) \cdot \left(\frac{\mathbf{y}_i}{\pi_i}\right) \cdot \left(\frac{\mathbf{y}_k}{\pi_k}\right)^T$$

An unbiased estimator for the covariance matrix of the Horvitz-Thompson estimator is given by:

$$\widehat{\mathrm{Var}}\Big[\, \widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}} \,\Big](s) \ = \ \sum_{i,k \in s} \left(\frac{\pi_{ik} - \pi_i \pi_k}{\pi_{ik}}\right) \cdot \left(\frac{\mathbf{y}_i}{\pi_i}\right) \cdot \left(\frac{\mathbf{y}_k}{\pi_k}\right)^T, \quad \text{for each } s \in \mathcal{S}.$$

Furthermore, if the sampling design has fixed sample size, then an alternative expression of the covariance matrix of the Horvitz-Thompson estimator is:

$$\mathrm{Var}\Big[\, \widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}} \,\Big] \ = \ -\frac{1}{2} \sum_{i,k \in U} (\pi_{ik} - \pi_i \pi_k) \cdot \left(\frac{\mathbf{y}_i}{\pi_i} - \frac{\mathbf{y}_k}{\pi_k}\right) \cdot \left(\frac{\mathbf{y}_i}{\pi_i} - \frac{\mathbf{y}_k}{\pi_k}\right)^T$$

and the corresponding Yates-Grundy-Sen variance estimator is:

$$\widehat{\mathrm{Var}}^{\mathrm{YGS}}\Big[\, \widehat{\mathbf{T}}_{\mathbf{y}}^{\mathrm{HT}} \,\Big](s) \ := \ -\frac{1}{2} \sum_{i,k \in s} \left(\frac{\pi_{ik} - \pi_i \pi_k}{\pi_{ik}}\right) \cdot \left(\frac{\mathbf{y}_i}{\pi_i} - \frac{\mathbf{y}_k}{\pi_k}\right) \cdot \left(\frac{\mathbf{y}_i}{\pi_i} - \frac{\mathbf{y}_k}{\pi_k}\right)^T$$

PROOF   By Proposition 2.6, for any random variable (a.k.a. estimator) $\widehat{\mathbf{T}}_{\mathbf{y};w}$ linear in the population parameter $\mathbf{y} : \mathcal{S} \longrightarrow \mathbb{R}^m$ with weights $w_k : \mathcal{S} \longrightarrow \mathbb{R}$, $k \in U$, the following

$$\widehat{\mathrm{Var}}\Big[\, \widehat{\mathbf{T}}_{y;w} \,\Big](s) \ := \ \sum_{i,k \in s} \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}} \mathbf{y}_i \cdot \mathbf{y}_k^T \tag{4.3}$$

always gives an unbiased estimator for the covariance matrix of $\widehat{\mathbf{T}}_{y;w}$. For the Horvitz-Thompson estimator, the weights are $w_k = 1/\pi_k$, for each $k \in U$, and the weights are independent of the sample $s \in \mathcal{S}$. Thus, for the Horvitz-Thompson estimator, the right-hand side of equation (4.3) becomes:

$$\sum_{i,k \in s} \frac{\mathrm{Cov}(I_i w_i, I_k w_k)}{\pi_{ik}} \mathbf{y}_i \cdot \mathbf{y}_k^T \ = \ \sum_{i,k \in s} \frac{\mathrm{Cov}(I_i, I_k)}{\pi_{ik}} \left(\frac{\mathbf{y}_i}{\pi_i}\right) \cdot \left(\frac{\mathbf{y}_k}{\pi_k}\right)^T$$

$$= \ \sum_{i,k \in s} \frac{E(I_i I_k) - E(I_i) E(I_k)}{\pi_{ik}} \left(\frac{\mathbf{y}_i}{\pi_i}\right) \cdot \left(\frac{\mathbf{y}_k}{\pi_k}\right)^T$$

$$= \ \sum_{i,k \in s} \frac{\pi_{ik} - \pi_i \pi_k}{\pi_{ik}} \left(\frac{\mathbf{y}_i}{\pi_i}\right) \cdot \left(\frac{\mathbf{y}_k}{\pi_k}\right)^T,$$

which coincides with the right-hand side of the equation of the conclusion of the present Proposition. Thus this present Proposition is but a special case of Proposition 2.6, specialized to the Horvitz-Thompson estimator, and the proof is now complete. □

5   Estimation of Domain Totals

6   Conditional inference in finite-population sampling

In this section, we give a justification for making inference conditional on the observed sample size for sampling designs with random sample size.

Observation ("mixture" of experiments) [see [?], p.15.]

Consider a population $\mathcal{U}$ of 1000 units. We wish to estimate the total T_y of a certain population characteristic $\mathbf{y} = (y_1, y_2, \ldots, y_{1000})$. Suppose we use the following two-step sampling scheme:

- Step 1: We first flip a fair coin.
  Define the random variable X by letting $X = 1$ if the coin lands heads, and $X = 0$ if it lands tails.

- Step 2: If $X = 1$, we select an SRS from $\mathcal{U}$ of size 100. If $X = 0$, we take a census on all of $\mathcal{U}$.

Let $\mathcal{S} \subset \mathcal{P}(\mathcal{U})$ denote the probability space of all possible samples induced by the (two-step) sampling design above. Note that $\mathcal{S} = \mathcal{S}_0 \sqcup \mathcal{S}_1$, where $\mathcal{S}_0 = \{\mathcal{U}\}$ and $\mathcal{S}_1$ is the set of all subsets of $\mathcal{U}$ of size 100. The sampling design is determined by the following probability distribution on $\mathcal{S}$:

$$P(\mathcal{U}) = \frac{1}{2} \quad \text{and} \quad P(s) = \frac{1}{2 \begin{pmatrix} 1000 \\ 100 \end{pmatrix}}, \quad \text{for each } s \in \mathcal{S}_1.$$

Let $\widehat{T}_y : \mathcal{S} \longrightarrow \mathbb{R}$ denote our chosen estimator for T_y. Then the (unconditional) probability distribution of $\widehat{T}_y$ can be "decomposed" as follows:

$$
\begin{aligned}
P\left(\widehat{T}_y = t \mid \mathbf{y}\right) &= P\left(\widehat{T}_y = t,\, X = 0 \mid \mathbf{y}\right) + P\left(\widehat{T}_y = t,\, X = 1 \mid \mathbf{y}\right) \\
&= P\left(\widehat{T}_y = t \mid X = 0,\, \mathbf{y}\right) \cdot P(X = 0 \mid \mathbf{y}) + P\left(\widehat{T}_y = t \mid X = 1,\, \mathbf{y}\right) \cdot P(X = 1 \mid \mathbf{y}) \\
&= P\left(\widehat{T}_y = t \mid X = 0,\, \mathbf{y}\right) \cdot P(X = 0) + P\left(\widehat{T}_y = t \mid X = 1,\, \mathbf{y}\right) \cdot P(X = 1),
\end{aligned}
$$

where the last equality follows because the distribution of X is independent of $\mathbf{y}$. Suppose the observation we make consists of $\left(\widehat{T}_y,\, X\right)$. The unconditional probability distribution of $\widehat{T}_y$, given by $P\left(\widehat{T}_y = t \mid \mathbf{y}\right)$ above, describes of course the randomness of the estimator $\widehat{T}_y$ as induced by both the randomness of the sample $s \in \mathcal{S} = \mathcal{S}_0 \sqcup \mathcal{S}_1$ as well as that of X (the outcome of the coin flip in Step 1). Now, suppose we have indeed carried out the sampling procedure and have obtained an observation of $\left(\widehat{T}_y,\, X\right)$. Suppose it happened that $X = 1$. Hence, we know that the estimate $\widehat{T}_y(s)$ we actually obtained was generated from an SRS of size 100 (rather than a census). Note also that the probability distribution of X is independent of $\mathbf{y}$ and the observation of X gives no information about $\mathbf{y}$. One school of thought therefore argues that downstream inferences about $\mathbf{y}$ should be carried out using the conditional probability $P\left(\widehat{T}_y = t \mid X = 1,\, \mathbf{y}\right)$, rather than the unconditional probability $P\left(\widehat{T}_y = t \mid \mathbf{y}\right)$. In other words, in the present example, as far as making inferences about $\mathbf{y}$ is concerned, only the randomness in Step 2 is relevant, and the randomness in Step 1 (i.e. the randomness of X, the outcome of the coin flip) is irrelevant to any inference about $\mathbf{y}$. Consequently randomness of X "should" be removed in any inference procedure for $\mathbf{y}$, and this is achieved by conditioning on the observed value of X. □

Conditioning on obtained sample size for sample designs with random sample size

Suppose $\mathcal{U}$ is a finite population. We wish to estimate the total $T_y = \sum_{i \in \mathcal{U}} y_i$ of a population characteristic $\mathbf{y} : \mathcal{U} \longrightarrow \mathbb{R}$, using a sample design $p : \mathcal{S} \longrightarrow [0, 1]$ and a estimator $\widehat{T} : \mathcal{S} \longrightarrow \mathbb{R}$. We make the assumption that the sampling design p is independent of $\mathbf{y}$. Let $N : \mathcal{S} \longrightarrow \mathbb{N} \cup \{0\}$ be the random variable of sample size, i.e. $N(s) = $ number of elements

in s, for each possible sample $s \in \mathcal{S}$. Then,

$$
\begin{aligned}
P\left(\widehat{T} = t \,\middle|\, \mathbf{y} \right)
&= \sum_{n} P\left(\widehat{T} = t,\, N = n \,\middle|\, \mathbf{y} \right) \\
&= \sum_{n} P\left(\widehat{T} = t \,\middle|\, N = n,\, \mathbf{y} \right) \cdot P\left(N = n \,\middle|\, \mathbf{y} \right) \\
&= \sum_{n} P\left(\widehat{T} = t \,\middle|\, N = n,\, \mathbf{y} \right) \cdot P\left(N = n \right),
\end{aligned}
$$

where the last equality follows from the assumed independence of the probability distribution $p : \mathcal{S} \longrightarrow [\,0\,,\,1\,]$ (hence that of N) from $\mathbf{y}$. The key observation to make now is that: Although the actual sampling procedure operationally may or may not have been a two-step procedure, the independence of p from $\mathbf{y}$ makes it probabilistically equivalent to a two-step procedure, as shown by the above decomposition of $P\left(\widehat{T} = t \,\middle|\, \mathbf{y} \right)$ — Step (1): randomly select a sample size $N = n$ according to the distribution $P(N = n)$, and then Step (2): randomly select a sample s of size n chosen in Step (1) according to the distribution $P(\, s \mid N = n\,)$. By the statistical reasoning explained in the preceding observation, it follows that post-sampling inference about $\mathbf{y}$ should be made based on the conditional distribution $P\left(\widehat{T} = t \,\middle|\, N = n\,,\, \mathbf{y} \right)$, rather than the unconditional distribution $P\left(\widehat{T} = t \,\middle|\, \mathbf{y} \right)$. This is because the sampling scheme is probabilistically equivalent to a two-step procedure, with the probability distribution of the first step (choosing a sample size) independent of the parameters of interest (T_y), and thus only the probability distribution of the second step (choosing a sample of the size chosen in first step) should be used to make inference about T_y. $\square$

Caution

In more formal parlance, the random variable $N : \mathcal{S} \longrightarrow \mathbb{N} \cup \{\,0\,\}$ is <u>ancillary</u> to the parameter $\mathbf{y}$. Thus, conditioning on sample size, for finite-population sampling schemes with random sample size, *partially* conforms to the **Conditionality Principle**, which states that statistical inference about a parameter should be made conditioned on observed values of statistics ancillary to that parameter. The conformance is only partial due to the (obvious) fact that it is the sample s itself which is ancillary to the parameter of interest $\mathbf{y}$, not just its sample size $N(s)$. Thus, full conformance to the Conditionality Principle would require inference about $\mathbf{y}$ be made conditioned on the observed sample s itself (rather than its size $N(s)$). However, if we did condition on the obtained sample s itself, the domain of the estimator $\widehat{T}$ would be restricted to the singleton $\{\,s\,\}$, and $\widehat{T}$ could then attain only one value under conditioning on s, and no randomization-based (i.e. design-based) inference — apart from the observed value of $\widehat{T}(s)$ — could be made any longer.