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$$e_t = e_{t-1} + x_t$$

$$x_t = \phi x_{t-1} + \varepsilon_t. \qquad Var(\varepsilon_t) = \sigma_\varepsilon^2 = 1$$

$$x_t = \underline{\underline{\phi x_{t-1}}} + \varepsilon_t = e_t - e_{t-1}$$

$$x_t = \varepsilon_t = e_t - \underline{e_{t-1}}$$

$$\varepsilon_{t-1} = \underline{e_{t-1}} - \underline{e_{t-2}}$$

$$\varepsilon_{t-2} = \underline{e_{t-2}} - \underline{e_{t-3}}$$

$$\varepsilon_{t-3} = \underline{e_{t-3}} - e_{t-4}.$$

$$y_t. = \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \varepsilon_{t-3} = e_t - e_{t-4}$$

$$E[y_t] = E[\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \varepsilon_{t-3}] = 0$$

$Cov(y_t, y_{t-j}), \quad j = 0,$

$$Cov(y_t, y_t) = Var(y_t) = \varepsilon_t^2 + \varepsilon_{t-1}^2 + \varepsilon_{t-2}^2 + \varepsilon_{t-3}^2$$

$$= 4.$$

$j = 1$

$$Cov(y_t, y_{t-1}) = E[y_t - \bar{y}][y_{t-i} - \bar{y}] = E[y_t, y_{t-i}] - E[y_t]E[y_{t-i}]$$

$$= E[y_t \cdot y_{t-1}] = E[(\varepsilon_t + \cdots + \varepsilon_{t-3})(\varepsilon_{t-1} + \cdots + \varepsilon_{t-4})]$$

$$= E[\varepsilon_{t-1}^2 + \varepsilon_{t-2}^2 + \varepsilon_{t-3}^2]$$

$$= 3$$

$j=2 \quad Cov(y_t, y_{t-2}) = E[\varepsilon_{t-2}^2 + \varepsilon_{t-3}^2] = 2$

$j=3 \quad Cov(y_t, y_{t-3}) = E[\varepsilon_{t-3}^2] = 1$

$j=4. \quad Cov(y_t, y_{t-4}). = 0 \quad j=5, \; Cov(y_t, y_{t-5}) = 0.$

2. Auto Covariance Drops to 0 after 3 lag
Not Correlated with $y_{t-i}$ term.

Hence it is a MA(3) Model with AR = 0.

## Problem 2

1. $Var(R^e_{t+1}) = \beta^2 Var(x_t) + Var(\varepsilon_{t+1})$

$= 1 \cdot 0.05^2 + 0.15^2 = 0.025$

$\sigma(R^e_{t+1}) = \sqrt{0.025} = 0.1581$

2. $R^2 = \rho^2 = \left[\frac{Cov(R^e_{t+1}, x_t)}{\sqrt{Var(R^e_{t+1}) Var(x_t)}}\right]^2$,

$= \left[\frac{0.0025}{\sqrt{0.025} \times 0.05}\right]^2$

$= 0.1$

$Cov(R^e_{t+1}, x_t)$

$= E[R^e_{t+1} \cdot x_t] - E[R^e_{t+1}]E[x_t]$

$= E[(a+\beta x_t + \varepsilon_{t+1})x_t] - E[a+\beta x_t + \varepsilon_{t+1}] \cdot E[x_t]$

$= E[\beta x_t^2 + x_t \varepsilon_{t+1}] - E[x_t] \cdot 0.05$

$= E[x_t^2] + E[x_t \varepsilon_{t+1}] - 0.05^2$

$= 0.005 + 0 - 0.05^2 = 0.0025$

$Var[x_t] = E[x_t^2] - E^2[x_t]$

$E[x_t^2] = 0.05^2 + 0.05^2$

$= 0.005$

3. sharpe ratio $= \frac{\text{excess return}}{\sigma_{mkt}}$

$$= \frac{0.05}{\sqrt{0.025}}$$

$$= 0.316 = 31.6\%$$

4. $\gamma = 40/9$. $\sigma_t(\varepsilon_{t+1}) = 0.15$

$a = 0,\ \beta = 1.$

$$a_t = \frac{E[a+\beta x_t+\varepsilon_{t+1}]}{\gamma\ \sigma_t^2[a+\beta x_t+\varepsilon_{t+1}]}$$

$$= \frac{\beta E[x_t]}{\gamma(\beta^2\sigma_t^2(x_t)+\sigma_t^2(\varepsilon_{t+1}))}$$

$$\sigma_t^2[a+\beta x_t+\varepsilon_{t+1}] = \beta^2\sigma_t^2[x_t]+\sigma_t^2[\varepsilon_{t+1}]$$

if $x_t = 0\%$, $E[x_t] = 0$

$a_t = 0.$ $\Rightarrow$ sharpe ratio $= \frac{E[R^e_{t+1}]}{\sigma(R^e_{t+1})} = 0.$

if $x_t = 10\%$

$$a_t = \frac{0.1}{40/9 \times 0.15^2} = 1$$

100% in risky asset.

sharpe ratio $= \frac{E[R^e_{t+1}]}{\sigma(R^e_{t+1})} = \frac{0.1}{0.15} = 0.667.$

5(a) $E[a_t R_{t+1}] = E\left[\frac{E_t[R^e_{t+1}] R_{t+1}}{\gamma \sigma_t^2[R_{t+1}]}\right] = \frac{E_t^2[R_{t+1}]}{\frac{40}{9} E[\sigma_t^2[R_{t+1}]]} = \frac{0.05^2}{\frac{40}{9} \cdot 0.025}$

$\sigma_t^2[R_{t+1}] = Var(X_t) + Var(\varepsilon_{t+1})$

$= 0.0025 + 0.15$

$= 0.025$

$= 0.0225.$

(b) $Var(\alpha_t R_{t+1}) = E[\alpha_t^2 (X_t^2 + \sigma_t^2(\varepsilon_{t+1}))] - E[\alpha_t x_t]^2$

$= (E[X_t^2 + \sigma_t^2(\varepsilon_{t+1})] - E[X_t]^2) \times \alpha_t^2$

$= (0.0025 + 0.05^2 + 0.15^2 - 0.05^2) \times 0.45^2$

$= 0.0050625$

$\alpha_t^2 = \left(\frac{E_t[R_{t+1}]}{\gamma \sigma_t^2[R_{t+1}]}\right)^2$

$= \left(\frac{0.05}{\frac{40}{9} \times 0.025}\right)^2$

$= 0.45^2$

standard deviation $= \sqrt{0.0050625}$

$= 0.07115125.$

$Var(X_t) = E[X_t^2] - E[X_t]^2$

$0.0025 = E[X_t^2] - 0.05^2$

$E[X_t^2] = 0.0025 + 0.05^2$

(c) sharpe ratio $= \frac{0.0225}{0.07115125} = 0.316$

$= 31.6\%$

(d) (i) $X_t = -0.05$ or $0.15$ $\quad E[X_t] = 0.05.$

$Var(R_{t+1}) = Var(X_t) + Var(\varepsilon_{t+1})$

$= 0.01 + 0.15^2 = 0.0325$

$E[R_{t+1}] = E[X_t] = 0.05.$

$Var(X_t) = 0.01$

$Var(X_t) = E[X_t^2] - E[X_t]^2$

$E[X_t^2] = 0.01 + 0.05^2$

$= 0.0125$

$$R^2=\rho^2=\left[\frac{Cov(R^e_{t+1}, x_t)}{\sqrt{Var(R^e_{t+1})Var(x_t)}}\right]^2,$$

$$=\left[\frac{0.01}{\sqrt{0.0325\times 0.01}}\right]^2$$

$$=0.30769$$

$$Cov(R^e_{t+1}, x_t)$$
$$=E[R^e_{t+1}\cdot x_t]-E[R^e_{t+1}]E[x_t]$$
$$=E[(a+\beta x_t+\varepsilon_{t+1})x_t]-E[a+\beta x_t+\varepsilon_{t+1}]\cdot E[x_t]$$
$$=E[\beta x_t^2+x_t\varepsilon_{t+1}]-E[x_t]\cdot 0.05$$
$$=E[x_t^2]-0.05^2$$
$$=0.0125-0.05^2$$
$$=0.01$$

(ii) $x_t=-5\%$ or $15\%$.

$$a_t=\frac{E_t[R_{t+1}]}{\gamma\sigma_t^2[R_{t+1}]}=\frac{0.05}{\frac{40}{9}\cdot 0.0325}$$

$$=0.3461538$$

$$E[a_tR_{t+1}]=0.3461538\,E[R_{t+1}]$$
$$=0.3461538\times 0.05$$
$$=0.0173076\text{9}$$

$$Var(a_tR_{t+1})=E[a_t^2(x_t^2+\sigma_t^2(\varepsilon_{t+1}))]-E[a_tx_t]^2$$
$$=a_t^2(E[x_t^2]+E[\sigma_t^2(\varepsilon_{t+1})]-E[x_t]^2)$$
$$=0.346^2\times(0.0125+0.15^2-0.05^2)$$
$$=0.0038907$$

$$\text{Sharpe ratio}=\frac{0.01730769}{\sqrt{0.0038907}}=0.277$$