An estimator for the rendering equation

- The rendering equation:

$$L_o(\mathbf{x}, \vec{\omega}) = L_e(\mathbf{x}, \vec{\omega}) + \int_{2\pi} f_r(\mathbf{x}, \vec{\omega}', \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}') \cos\theta \, \mathrm{d}\omega' \quad .$$

- The Monte Carlo estimator:

$$L_N(\mathbf{x}, \vec{\omega}) = L_e(\mathbf{x}, \vec{\omega}) + \frac{1}{N} \sum_{i=1}^{N} \frac{f_r(\mathbf{x}, \vec{\omega}_i', \vec{\omega}) L_i(\mathbf{x}, \vec{\omega}_i') \cos\theta}{\mathrm{pdf}(\vec{\omega}_i')} \quad .$$

- The Lambertian BRDF:

$$f_r(\mathbf{x}, \vec{\omega}', \vec{\omega}) = \rho_d / \pi \quad .$$

- A good choice of pdf would be:

$$\mathrm{pdf}(\vec{\omega}_i') = \cos\theta / \pi \quad .$$

Cosine-weighted hemisphere sampling

- Sampling directions according to the distribution:
  $\mathrm{pdf}(\vec{\omega}_i') = \cos\theta/\pi \quad , \quad \mathrm{pdf}(\theta,\phi) = \cos\theta\sin\theta/\pi \ .$
- Compute the marginal and conditional density functions:

$$\begin{aligned}
\mathrm{pdf}(\theta) &= \int_0^{2\pi} \frac{\cos\theta}{\pi}\sin\theta\,\mathrm{d}\phi = 2\cos\theta\sin\theta \ . \\
\mathrm{pdf}(\phi|\theta) &= \frac{\cos\theta\sin\theta/\pi}{2\cos\theta\sin\theta} = \frac{1}{2\pi} \ .
\end{aligned}$$

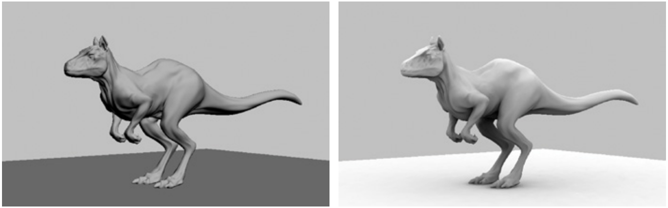
- The cdf for the marginal density function:

$$\begin{aligned}
P(\theta) &= 2\int_0^{\theta}\cos\theta'\sin\theta'\,\mathrm{d}\theta' = 2\int_1^{\cos\theta}(-\cos\theta')\,\mathrm{d}\cos\theta' = 1-\cos^2\theta \\
P(\phi|\theta) &= \phi/(2\pi) \ .
\end{aligned}$$

- Invert these to find the sampling strategy:

$$\vec{\omega}_i' = (\theta,\phi) = (\cos^{-1}\sqrt{\xi_1},\, 2\pi\xi_2) \ .$$

Ambient occlusion



- Using the Lambertian BRDF for materials, $f_r = \rho_d/\pi$; the cosine weighted hemisphere for sampling, $\mathrm{pdf}(\vec{\omega}_i') = \cos\theta/\pi$; and a visibility term V for incident illumination, the Monte Carlo estimator for ambient occlusion is simply:

$$L_N(\mathbf{x},\vec{\omega}) = \frac{1}{N}\sum_{i=1}^{N}\frac{f_r(\mathbf{x},\vec{\omega}_i',\vec{\omega})L_i(\mathbf{x},\vec{\omega}_i')\cos\theta}{\mathrm{pdf}(\vec{\omega}_i')} = \rho_d(\mathbf{x})\frac{1}{N}\sum_{i=1}^{N}V(\vec{\omega}_i') \ .$$