Initially $\vec{y} = D\vec{\theta}$

Q1 (a) Derivative filter is a linear filter.

$\therefore$ Can be represented as a matrix M_D

$$\therefore M_d\vec{y} = M_d(D\vec{\theta}) = (M_d D)\vec{\theta}$$

if $\vec{\theta}$ is k-sparse than $\vec{\theta}$ here is also k-sparse

$\therefore D \longrightarrow M_d D$ in (a)

(b) Denote the Rotation matrix which rotates anticlockwise by θ as M_θ

$\vec{y} = D\vec{\theta}$ if $D_{m\times n}$

Now construct $D' = \left[M_\alpha D \quad M_\beta D \right]_{m\times(2n)}$

If image is rotated by α then it would be sparse in $\vec{\theta}'$ corresponding to first n entries (to $M_\alpha D$)

and 0 in the later n entries.

Similary for rotation by angle β.

c) $I^i_{new}(x,y) = \alpha\left(I^i_{old}(x,y)\right)^2 + \beta\left(I^i_{old}(x,y)\right) + \gamma$ to images in S.

Intensity $\propto \vec{y}$ $(= D\vec{\theta})$

$$\therefore \vec{y}_{New} = \alpha\, \underbrace{\vec{y}\,(.*)\,\vec{y}}_{\text{elementwise multiplication}} + \beta\vec{y} + \gamma\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$$

Consider $\vec{y} .* \vec{y}$

$$\left(\sum_{j=1}^{k} \vec{d}_j\theta_j\right)\times\left(\sum_{l=1}^{k} \vec{d}_l\theta_l\right) = \sum_{\substack{1\le j\le k\\1\le l\le k}}\sum (\vec{d}_j .* \vec{d}_l)\,\theta_j\theta_l$$

$D_M = \cancel{D}$. Matrix formed by element wise products of all pairs of columns of D

$$D_3 = D' = \begin{array}{c} \\ \left[\begin{array}{ccc} \alpha D_M & \beta D & \begin{array}{c} \text{vector of 1's} \\ 1 \\ \vdots \\ 1 \end{array} \end{array} \right] \end{array} \begin{bmatrix} \vec{\theta}_M \\ \vec{\theta} \\ \vec{\gamma} \end{bmatrix}$$

$$\vec{\gamma} = \begin{bmatrix} \gamma \\ \cancel{\gamma} \\ \cancel{\gamma} \\ \cancel{\gamma} \end{bmatrix}$$

$\vec{\theta}$ k sparse

$\vec{\theta}_M$: vector whose entry is $\theta_j \theta_l$ $\quad 1 \le j \le K$, $1 \le l \le K$

Sparsity level in D' if original θ is k-sparse is $k^2 + k + 1$

d) Similar approach as in part a,

Apply the same blur filter to all columns of original Dictionary matrix D.

~~e)~~ $\vec{y} = D\vec{\theta}$ $\qquad \theta$ is k-sparse

$$b * \vec{y} = b * D\vec{\theta}$$

$$= b * \sum_i \vec{D}_i \theta_i \qquad \vec{D}_i : i^{th} \text{ column of } D$$

$$= \sum_i (b * \vec{D}_i)\, \theta_i$$

$$= D'\vec{\theta} \qquad [\vec{\theta} \text{ is k-sparse}]$$

e) For each blur kernel do as in 'd'
& then concatenate Dictionaries corresponding to all blur kernels

$$B = \{b_1, b_2, \cdots, b_{|B|}\} \qquad , \vec{b} = \sum_{i=1}^{|B|} c_i \vec{b}_i$$

$$D' = \left[c_1 b_1 * D \quad c_2 b_2 * D \quad \cdots \quad c_{|B|} b_{|B|} * D \right]$$

$\vec{y} = D\vec{\theta}$ $\qquad \vec{\theta}$ is k sparse

$\vec{y}' = D'\vec{\theta}'$ $\qquad \vec{\theta}'$ is $|B|k$ sparse

f) Radon Transform

$$R_\theta(f) = g(\rho, \theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, \delta(x\cos\theta + y\sin\theta - \rho)\, dx\, dy.$$

f) R_θ := Radon Transform Matrix corresponding to angle θ.

$$\vec{y} = D\vec{\theta}$$

S_6 is obtained from $R_\theta \vec{y}$

$$R_\theta \vec{y} = R_\theta (D\vec{\theta}) = (R_\theta D)\vec{\theta}$$

$D' = R_\theta D$ $\quad$ $\vec{\theta}$ remains k-sparse

g) $T_{\vec{c}}(\vec{y})$:= Translation operation on $\vec{y}$

Shifting origin to $\vec{c} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\vec{c}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $\quad$ $\vec{c}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

Translation is a linear operation ie. can be represented as a Matrix operation.

$$T_{\vec{c}}(\vec{y}_1 + \vec{y}_2) = T_{\vec{c}}(\vec{y}_1) + T_{\vec{c}}(\vec{y}_2)$$

$$T_{\vec{c}}(\lambda \vec{y}) = \lambda T_{\vec{c}}(\vec{y})$$

Consider $D' = [\, T_{\vec{c}_1} D \quad T_{\vec{c}_2} D \,]$

D' is dictionary for S_7