② From RIP equation

$$(1-\delta_s)\,\|\theta\|_2^2 \le \|A\theta\|_2^2 \le (1+\delta_s)\,\|\theta\|_2^2 \qquad \theta : \text{s-sparse}$$

Writing down again the matrix A^TA we obtained in Q1

$$\vec{\theta}_s^{\,T} A^T A \vec{\theta}_s = \vec{\theta}_s^{\,T} \begin{bmatrix} 1 & \vec{A}_1^T\vec{A}_2 & \vec{A}_1^T\vec{A}_3 & \cdots & \vec{A}_1^T\vec{A}_n \\ \vec{A}_2^T\vec{A}_1 & 1 & \vec{A}_2^T\vec{A}_3 & \cdots & \vec{A}_2^T\vec{A}_n \\ \vdots & & & & \\ \vdots & & & & \\ \vec{A}_n^T\vec{A}_1 & \vec{A}_n^T\vec{A}_2 & \cdots & & 1 \end{bmatrix}_{n\times n} \vec{\theta}_s$$

$\vec{\theta}_s^{\,T}$ and $\vec{\theta}_s$ are effectively picking about $|S|$ rows & $|S|$ columns indexed by their non-zero entries.

wlog let $\|\vec{\theta}_s\| = 1$ ☆

The matrix $A_S^T A_S$ is symmetric

Hence $\exists$ an orthogonal Matrix M such that $A_s^T A_s = M^T D M$
where D is a diagonal matrix formed by Eigen vectors of $A_s^T A_s$

$\therefore \cdot \cancel{\vec{v}^T M^T D}$

Let $\|\vec{v}\| = 1$

$$\vec{v}^T A_S^T A_S \vec{v} = \vec{v}^T M^T D M \vec{v} = \vec{u}^T D \vec{u} = \sum_{i=1}^{|S|} \lambda_i u_i^2$$

$\vec{u} = M\vec{v}$

$\|\vec{u}\|^2 = u^T u$
$= \vec{v}^T M^T M \vec{v}$
$= \vec{v}^T \vec{v}$ $\quad (\because M$ is orthogonal$)$
$= 1$ $\quad (\because \|\vec{v}\| = 1)$

Clearly this is maximised when $u_m = 1$
where $m = \operatorname{argmax}_i \lambda_i$

and minimised when $u_j = 1$
where $j = \operatorname{argmin}_i \lambda_i$

$\|A\theta\|_2^2 \le (1+\delta_s)\,\|\theta\|_2^2$ $\quad ; \quad$ $(1-\delta_s)\,\|\theta\|_2^2 \le \|A\theta\|_2^2$

$\therefore \lambda_{max} \le 1+\delta_s$ $\quad ; \quad$ $1-\delta_s < \lambda_{min}$

Here we had relax the constraint that θ is s-sparse and set of values taken by $\|A\theta\|_2^2$ (θ-unconstrained) is a superset of values taken by $\|A\theta\|_2^2$ (θ-s-sparse)

$$\therefore \delta_s = \max(1-\lambda_{min},\ \lambda_{max}-1)$$

where λ_{min} & λ_{max} are over all submatrices indexed by $|S|$