Theorem 5 (Slides)

Consider $\vec{y} = \Phi \vec{x} + \vec{\eta}$
$= \Phi \Psi \vec{\theta} + \vec{\eta}$
$= A\vec{\theta} + \vec{\eta}$

Aim: Estimate $\vec{x}$ via $\vec{\theta}$

Let $\vec{\theta}_s$ be a sub-vector of $\vec{\theta}$ with S largest elements.

If $S < 0.5\left(1 + \frac{1}{\mu(A)}\right)$, sol^n θ^* to P2 yields following error bounds

sufficient condition for unique solution

$$\|\theta - \theta^*\|_2 \leq C_0(\varepsilon + \varepsilon') + C_1\|\vec{\theta} - \vec{\theta}_s\|_1, \quad \|\eta_2\| \leq \varepsilon$$

P2: $\min \|\vec{\theta}\|_1$ s.t. $\|\vec{y} - A\vec{\theta}\|_2 \leq \varepsilon'$, $\varepsilon \leq \varepsilon'$

Theorem 1 (Paper) BP DeNoising

index set corresponding to non-zero entries.

Let $\vec{z} = A\vec{x} + \vec{n}$, $\|\vec{n}\|_2 \leq \varepsilon$, $X = \mathrm{supp}_{n_X}(\vec{x})$.

If $n_X < \frac{1}{2}\left(1 + \frac{1}{\mu_a}\right)$ (2) is met, then sol^n $\hat{\vec{x}}$ to the convex program

(BPDN) minimize $\|\tilde{\vec{x}}\|_1$ subject to $\|\vec{z} - A\tilde{\vec{x}}\|_2 \leq \eta$

with $\varepsilon \leq \eta$ satisfies

$$\|\vec{x} - \hat{\vec{x}}\|_2 \leq C_0(\varepsilon + \eta) + C_1\|\vec{x} - \vec{x}_X\|_1$$

where both (non-negative) constants C_0 and C_1 depend on μ_a and n_X

Notation: $\mu_a = \max_{k,\ell,k\neq\ell} |\vec{a}_k^T \vec{a}_\ell|$

Define $M \times M$ diagonal (projection) matrix P_S for the set $S \subseteq \{1, \dots, M\}$ as:

$$[P_S]_{k,\ell} = \begin{cases} 1 & k = \ell \text{ and } k \in S \\ 0 & \text{otherwise.} \end{cases}$$

$\vec{m}_T = P_T \vec{m}$

Matrix M_T is obtained from M by retaining columns of M with indices in T

For $x \in \mathbb{R}$ $[x]^+ = \max\{x, 0\}$

Appendix A Proof of Theorem 1

Let $\vec{h} = \hat{\vec{x}} - \vec{x}$

$\hat{\vec{x}}$: sol^n of BPDN

$\vec{x}$: vector to be recovered

also define $\vec{h}_0 = P_{\mathcal{X}} \vec{h}$ $\qquad \mathcal{X} = \text{supp}_{n_x}(\vec{x})$

$\text{supp}_{n_x}(\vec{x})$: support set associated with best n_x-sparse approximation.

$$\hat{\mathcal{X}} = \text{supp}_{n_x}(\vec{x}) = \underset{\tilde{\mathcal{X}} \in \Sigma_{n_x}}{\text{argmin}} \|\vec{x} - \vec{x}_{\tilde{\mathcal{X}}}\|$$

Σ_{n_x}: contains all support sets of size n_x

A.1.1 Cone constraint

Let $e_0 = 2\|\vec{x} - \vec{x}_{\mathcal{X}}\|_1 = 2\|\vec{x}_{\mathcal{X}^c}\|_1$

$\vec{x}_{\mathcal{X}} = P_{\mathcal{X}} \vec{x}$

$$\|\vec{h} - \vec{h}_0\|_1 \leq \|\vec{h}_0\|_1 + e_0 \qquad \text{A.1}$$

$\|\hat{x}\|_1 \leq \|x\|_1$ $\quad$ ($\because$ BPDN returns sol^n with $\min \|\vec{x}\|_1$)

$\therefore \|\hat{x}_{\mathcal{X}}\|_1 + \|\hat{x}_{\mathcal{X}^c}\|_1 \leq \|x\|_1$

$\therefore \|\vec{h}_0 + \vec{x}_{\mathcal{X}}\|_1 + \|\vec{x}_{\mathcal{X}^c} + (\vec{h} - \vec{h}_0)\| \leq \|x\|_1$

$|\vec{a}| - |\vec{b}| \leq |\vec{a} + \vec{b}|$

so $\pm(|\vec{a}| - |\vec{b}|) \leq |\vec{a} + \vec{b}|$

$\therefore -\|\vec{h}_0\|_1 + \|\vec{x}_{\mathcal{X}}\|_1 + \|\vec{h} - \vec{h}_0\|_1 - \|x_{\mathcal{X}^c}\|_1 \leq \|x\|_1$

$\therefore \|\vec{h} - \vec{h}_0\|_1 \leq \|x_{\mathcal{X}^c}\|_1 + \|x\|_1 - \|\vec{x}_{\mathcal{X}}\|_1 + \|\vec{h}_0\|_1$

$\therefore \|\vec{h} - \vec{h}_0\|_1 \leq \|\vec{h}_0\|_1 + 2\|x_{\mathcal{X}^c}\|_1$

Date
Page

(A·2) $\|\vec{h}\|_1 \leq 2\|\vec{h}_o\|_1 + e_o$

$$\|\vec{h}\|_1 = \|\vec{h} - \vec{h}_o + \vec{h}_o\|_1$$
$$\leq \|\vec{h}_o\|_1 + \|\vec{h} - \vec{h}_o\|_1 \quad \text{Triangle inequality}$$
$$\leq 2\|\vec{h}_o\|_1 + e_o \quad \text{From A·1}$$

A·1·2 Tube constraint

$$\|A\vec{h}\| = \|A\hat{x} - \vec{y} - (A\vec{x} - \vec{y})\|_2$$
$$\leq \|A\hat{x} - y\|_2 + \|Ax - y\|_2 \quad \text{From Minkowski's inequality.}$$
$$\leq \eta + \varepsilon$$

→ BPDN minimize $\|\tilde{x}\|_1$ subject to $\|\vec{z} - A\vec{\tilde{x}}\|_2 \leq \eta$

$$\therefore \|\vec{y} - A\vec{\hat{x}}\| \leq \eta$$

→ $\vec{y} = A\vec{x} + \vec{n}$

$$\|\vec{y} - A\vec{x}\|_2 = \|\vec{n}\|_2$$
$$\leq \varepsilon \quad \text{[From initial assumptions of Theorem 1]}$$

A·1·3 Coherence Based RIP

$\vec{h}_o$ is n_x-sparse

$$\delta_s \leq \mu(n_x - 1)$$

(A·4) $$(1 - \mu_a(n_x-1))\|\vec{h}_o\|_2^2 \leq \|A\vec{h}_o\|_2^2 \leq (1 + \mu_a(n_x-1))\|\vec{h}_o\|_2^2 \quad \text{From HW 1 Q1}$$

in this paper $\vec{x}^T \leftrightarrow \vec{x}^H$

A·2 Bounding the error $\|\vec{h}_o\|_2$ on the signal support

$$|h^H A^H A h_o| + |(h-h_o)^H A^H A h_o| \geq |(h_o - h + h)^H A^H A h_o| \quad \Delta\text{-inequality}$$

$$\Rightarrow |h^H A^H A h_o| \geq |h_o^H A^H A h_o| - |(h-h_o)^H A^H A h_o|$$
$$\geq \underbrace{(1-\mu_a(n_x-1))\|h_o\|_2^2}_{\text{from A·4}} - \left|\sum_{k\in\mathcal{X}} \sum_{\ell\in\mathcal{X}^c} [h_o^H]_k a_k^H a_\ell [h]_\ell\right|$$
$$\geq (1 - \mu_a(n_x-1))\|h_o\|_2^2 - \underbrace{\mu}_{\max} \underbrace{\|h_o\|_1}_{|\mathcal{X}|} \underbrace{\|h - h_o\|_1}_{\|\mathcal{X}^c\|}$$

Date ______
Page ______

$\geq (1-\mu_a(n_x-1))\,\|h_o\|_2^2 - \mu\|h_o\|_1(\|h_o\|_1+e)$ From A·1 = (A·7)

/* $\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2 \leq \frac{\sum_{i=1}^{n} x_i^2}{n}$ By convexity of $f(x)=x^2$

We use this for bounding $\|h_o\|_1$ by $\|h_o\|_2$

$\left(\frac{\|h_o\|_1}{\cancel{|x|}\, n_x}\right)^2 \leq \frac{\|h_o\|_2^2}{\cancel{|x|}\, n_x} \Rightarrow \cancel{\|h_o\|_1 \leq \sqrt{|x|}\,\|h_o\|_2}$

$\vec{h}_o = P_x \vec{h}$

$\Rightarrow \|h_o\|_1 \leq \sqrt{n_x}\,\|h_o\|_2$

*/

Use the coment above in A·7 to get

$= (1-\mu_a(2n_x-1))\,\|\vec{h}_o\|_2^2 - \mu_a\sqrt{n_x}\,\|\vec{h}_o\|_2 e_o$

If $1-\mu_a(2n_x-1)>0$ and $\|\vec{h}_o\|>0$ than we can upper bound as follows

$$\|\vec{h}_o\|_2 \leq \frac{|h^H A^H A h_o| + \mu_a\sqrt{n_x}\,\|\vec{h}\|_2 e_o}{(1-\mu_a(2n_x-1))\,\|h_o\|_2}$$

$$\leq \frac{\|Ah\|_2\,\|Ah_o\|_2 + \mu_a\sqrt{n_x}\,\|h_o\|_2 e_o}{(1-\mu_a(2n_x-1))\,\|h_o\|_2}$$ (Cauchy-Schwartz Inequality)

$$\leq \frac{(\eta+\varepsilon)\sqrt{1+\mu_a(n_x-1)}\,\cancel{\|h_o\|_2} + \mu_a\sqrt{n_x}\,\cancel{\|h_o\|_2} e_o}{(1-\mu_a(2n_x-1))\,\cancel{\|h_o\|_2}}$$ From Tube Constraint A·1·2 & RIP

$$= \frac{(\eta+\varepsilon)\sqrt{1+\mu_a(n_x-1)} + \mu_a\sqrt{n_x}}{(1-\mu_a(2n_x-1))}$$

A·3 Bounding the Recovery error $\|\vec{h}\|_2$

$$\|Ah\|_2^2 = h^H A^H A h$$

$$= \sum_{k,l} [h^H]_k\, a_k^H a_l\, [h]_l$$

$$= \sum_k \underbrace{\|a_k\|_2^2}_{1}\, |[h]_k|^2 + \sum_{k,l,k\neq l} [h^H]_k\, a_k^H a_l\, [h]_l$$

$|a_k^H a_l| \leq \mu$

$$\geq \sum_k |[h]_k|^2 - \mu_a \sum_{k,l,k\neq l} |[h^H]_k [h]_l|$$

Splitting summation

$$= \cancel{\sum_x}\, \|h\|_2^2 + \mu_a \sum_k |[h]_k|^2 - \mu_a \sum_{k,l} |[h^H]_k [h]_l|$$

$$= (1+\mu_a)\|\vec{h}\|_2^2 - \mu_a\|\vec{h}\|_1^2$$

$$\|\vec{h}\|_1^2 = \left(\sum_k |[h]_k|\right)\left(\sum_k |[h]_k|\right)$$

$$= \sum_{k,l} |[h^H]_k [h]_l|$$

So

$$\|\vec{h}\|_2^2 \le \frac{\|Ah\|_2^2 + \mu_a\|h\|_1^2}{1+\mu_a}$$

$$\le \frac{(\eta+\varepsilon)^2 + \mu_a(2\|\vec{h}_0\|_1 + e_0)^2}{1+\mu_a}$$ Using Tube Constraints & A.2

$$\|\vec{h}\|_2 \le \frac{\sqrt{(\eta+\varepsilon)^2 + \mu_a(2\|\vec{h}_0\|_1 + e_0)^2}}{\sqrt{1+\mu_a}}$$

$$\le \frac{(\eta+\varepsilon) + \sqrt{\mu_a}(2\|\vec{h}_0\|_1 + e_0)}{\sqrt{1+\mu_a}}$$

$x, y > 0$

than $x^2+y^2 \le (x+y)^2$

$\sqrt{x^2+y^2} \le x+y$

By convexity of $f(x)=x^2$ we have $\|\vec{h}_0\|_1 \le \sqrt{n_x}\|\vec{h}_0\|_2$

$$\|\vec{h}\|_2 \le \frac{(\eta+\varepsilon) + \sqrt{\mu_a}(2\sqrt{n_x}\|\vec{h}_0\|_2 + e_0)}{\sqrt{1+\mu_a}}$$

We have from A.13 $\|\vec{h}_0\|_2 \le \dfrac{(\varepsilon+\eta)\sqrt{1+\mu_a(n_x-1)} + \mu_a\sqrt{n_x}\,e_0}{1-\mu_a(2n_x-1)}$

$$\cancel{\|\vec{h}\|_2 \le (\varepsilon+\eta) \quad 1-\mu_a(2n_x-1) + 2\sqrt{\mu_a n_x}\sqrt{1+\mu_a(n_x-1)}}$$

$$\sqrt{1+\mu_a}\;\cancel{\|\vec{h}\|_2 \le} (\eta+\varepsilon)\left[1 + \frac{2\sqrt{n_x\mu_a}\sqrt{1+\mu_a(n_x-1)}}{\cancel{1-\mu_a(2n_x-1)}}\right]$$

$$\frac{2n_x\mu_a\sqrt{\mu_a}\,e_0 + \sqrt{\mu_a}\,e_0}{1-\mu_a(2n_x-1)}$$

$$\sqrt{\mu_a}\,e_0\left[\frac{2n_x\mu_a + 1}{1-\mu_a(2n_x-1)}\right]$$

$$\sqrt{\mu_a}\,e_0\left[\frac{1+\mu_a}{1-\mu_a(2n_x-1)}\right]$$

$$\frac{\sqrt{\mu_a}e_0(1+\mu_a)}{1-\mu_a(2n_x-1)} = \frac{2n_x\mu_a\sqrt{\mu_a}e_0 + \sqrt{\mu_a}e_0}{1-\mu_a(2n_x-1)}$$

$$\sqrt{1+\mu_a}\,\|\vec{h}\|_2 \leq (\eta+\varepsilon) + 2\sqrt{n_x\mu_a}\left\{\frac{(\eta+\varepsilon)\sqrt{1+\mu_a(n_x-1)}+\mu_a\sqrt{n_x}e_0}{1-\mu_a(2n_x-1)}\right\} + \sqrt{\mu_a}\,e_0$$

$$\|\vec{h}\|_2 \leq (\eta+\varepsilon)\left[\frac{1-\mu_a(2n_x-1)+2\sqrt{n_x\mu_a}\sqrt{1+\mu_a(n_x-1)}}{\sqrt{1+\mu_a}\,(1-\mu_a(2n_x-1))}\right] + \frac{\sqrt{\mu_a}e_0(1+\mu_a)}{\sqrt{1+\mu_a}\,(1-\mu_a(2n_x-1))}$$

$$\leq (\eta+\varepsilon)\left[\frac{1-\mu_a(2n_x-1)+2\sqrt{n_x\mu_a}\sqrt{1+\mu_a(n_x-1)}}{\sqrt{1+\mu_a}\,(1-\mu_a(2n_x-1))}\right] + \frac{e_0\sqrt{\mu_a^2+\mu_a}}{(1-\mu_a(2n_x-1))}$$

$$= (\eta+\varepsilon)\,\varepsilon_0 + c_1\|\vec{x}-\vec{x}_x\|_1$$

$$1-\mu_a(2n_x-1)>0$$

$$\Rightarrow n_x < \frac{1}{2}\left(\frac{1}{\mu_a}+1\right)$$

Denominator & hence the RHS should surely be positive. as LHS is strictly non-negative. This is the simplest condition one may impose.