(1) $\Phi_{m\times n} = \left[\vec{c}_1 \ \vec{c}_2 \cdots \vec{c}_n\right]_{m\times n}$

$(\vec{c}_i)_{m\times 1}$ and $\|\vec{c}_i\|_2 = 1$

a) $D2 = \Phi\Phi^T$

$D1 = \Phi^T\Phi$

$= \Phi^T_{n\times m}\Phi_{m\times n}$

$= \begin{bmatrix} \vec{c}_1^{\,T} \\ \vec{c}_2^{\,T} \\ \vdots \\ \vec{c}_n^{\,T} \end{bmatrix}_{n\times m} \left[\vec{c}_1 \ \vec{c}_2 \cdots \vec{c}_n\right]_{m\times n}$

$= \begin{bmatrix} 1 & \vec{c}_1^{\,T}\vec{c}_2 & \cdots & \vec{c}_1^{\,T}\vec{c}_n \\ \vec{c}_2^{\,T}\vec{c}_1 & 1 & & \vec{c}_2^{\,T}\vec{c}_n \\ \vdots & & \ddots & \\ \vec{c}_n^{\,T}\vec{c}_1 & \vec{c}_n^{\,T}\vec{c}_2 & \cdots & 1 \end{bmatrix}$

$D1 = D1^T$ (symmetric matrix)

Clearly $\text{trace}(D1) = n$

b) $D2 = \left[\vec{c}_1 \ \vec{c}_2 \cdots \vec{c}_n\right]_{m\times n} \begin{bmatrix} \vec{c}_1^{\,T} \\ \vec{c}_2^{\,T} \\ \vdots \\ \vec{c}_n^{\,T} \end{bmatrix}_{n\times m}$

$=$

Consider vectorised $D2 = D2^{(v)}_{m^2\times 1}$ Note that (v) means vectorised

vectorised $I = I^{(v)}_{m^2\times 1}$

$\langle D2^{(v)}, I^{(v)}\rangle \le \|I^{(v)}\|_2 \, \|D2^{(v)}\|_2$

$tr(D_2) \le \sqrt{m}\sqrt{tr(D2D2^T)}$

Since $D2$ is symmetric

$tr(D2\,D2^T) = \|D_2\|_F^2 = \|D2^{(v)}\|_2^2$

$D2 = D2^T$

hence D2 is a symmetric matric (non negative Eigen Values)

$\therefore D2\, D2^T = (D2)^2$

$\therefore$ ~~If λ_i is an Eigen~~

Aliter

$\therefore \lambda$ is an Eigen Value of $D2 \Rightarrow \lambda^2$ is Eigen Value of $D2\,D2^T = D2^2$

Applying Jensen's inequality to $f(x) = x^2$ $E[f(x)] > f[E(x)]$

with x. as Eigenvalues of D2 we get

$$\frac{\sum_{i=1}^{m} \lambda_i^2}{m} > \left(\frac{\sum_{i=1}^{m} \lambda_i}{m}\right)^2 \quad \text{—(a)}$$

Rearranging we get

$$\sum_{i=1}^{m} \lambda_i \le \sqrt{m} \sqrt{\sum_{i=1}^{m} \lambda_i^2}$$

Since Trace of matrix is sum of its Eigen Values

(b) follows

c) $$D2 = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1m} & C_{2m} & & C_{nm} \end{bmatrix}_{m\times n} \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1m} \\ C_{21} & C_{22} & \cdots & C_{2m} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nm} \end{bmatrix}_{n\times m}$$

- Let $A := D2_{m\times n}$

then $$a_{ij} = \sum_{k=1}^{n} C_{ki} C_{kj}$$

$$D2\,D2^{T} = A^{2} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

Let d_i be i^{th} diagonal element

$$d_i = \sum_{l=1}^{m} a_{il}\, a_{li}$$

$$\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_m^2$$

$$d_i = \sum_{l=1}^{m} \left(\sum_{k_1=1}^{n} C_{k_1 i} C_{k_1 l} \right) \left(\sum_{k_2=1}^{n} C_{k_2 l} C_{k_2 i} \right)$$

$$\sum_{i=1}^{m} d_i = \sum_{i=1}^{m} \sum_{l=1}^{m} \left(\sum_{k_1=1}^{n} C_{k_1 i} C_{k_1 l} \right) \left(\sum_{k_2=1}^{n} C_{k_2 l} C_{k_2 i} \right)$$

$$= \sum_{i=1}^{m} \sum_{l=1}^{m} \left\{ \sum_{k_1=1}^{n} C_{k_1 i}^2 C_{k_1 l}^2 + 2 \sum_{1 \le j_1 < j_2 \le n} (C_{j_1 i} C_{j_2 l})(C_{j_1 l} C_{j_2 i}) \right.$$

Recall that $\|C_k\|_2 = 1 \quad \forall k$

$$= \sum_{i=1}^{m} \left(\sum_{k_1=1}^{n} C_{k_1 i}^2 \underbrace{\sum_{l=1}^{n} C_{k_1 l}^2}_{1} + \sum_{\substack{j_1 \ne j_2 \\ 1 \le j_1, j_2 \le n}} \sum_{l=1}^{m} C_{j_1 i} C_{j_2 l} C_{j_1 l} C_{j_2 i} \right)$$

$$= \sum_{k_1=1}^{n} \sum_{i=1}^{n} C_{k_1 i}^2 + \sum_{\substack{j_1 \ne j_2 \\ 1 \le j_1, j_2 \le n}} \sum_{i=1}^{m} \sum_{l=1}^{m} \cancel{C_{j_1 l} C_{j_2 l}} \, C_{j_1 i} C_{j_1 l} C_{j_2 i} C_{j_2 l}$$

$$= \sum_{k_1=1}^{n} 1 + \sum_{\substack{j_1 \ne j_2 \\ 1 \le j_1, j_2 \le n}} \cancel{\vec{C}_j^T C_l} \left(\vec{C}_{j_1}^T \vec{C}_{j_2} \right)^2$$

$$= n + \sum_{i,j,\, i \ne j} \left(\vec{C}_i^T \vec{C}_j \right)^2$$

(d) $\text{trace}(D2) = \sum_{i=1}^{n} \sum_{k=1}^{n} C_{ki} C_{kj}$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} C_{ki}^2 \qquad (\text{sy}$$

$$= \sum_{k=1}^{n} 1$$

$$= n$$

Using (b) & (c) we get.

$$n^2 \leq m\left(n + \sum_{i,j,i\neq j} \left(\Phi_i^t \Phi_j\right)^2\right)$$

(e) $|\Phi_i^T \Phi_j| \leq \mu$ -6

$$\therefore\ n^2 \leq m(n + n(n-1)\mu^2)$$

$$n \leq m(1 + (n-1)\mu^2)$$

$$\sqrt{\frac{n-m}{m(n-1)}} \leq \mu$$

(non-zero)

(f) → When all Eigen Values of D2 are same (here α)

$$\Phi \Phi^T = I_{m\times m}\, \alpha \qquad \alpha \in R^+$$

Then the Jensen inequality used in (b) will achieve equality. -(a)

→ Furthermore,

$i \neq j$, $\Phi_i^t \Phi_j$ has to be $\pm\mu$ -6