Polynomial Roots

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Here are formulas of roots of polynomial functions:

1. Solution of $ax + b$ is:

$$r = -\frac{b}{a} \tag{1}$$

2. Solution of $ax^2 + bx + c$ is:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \equiv \frac{-2c}{b \pm \sqrt{b^2 - 4ac}} \tag{2}$$

3. Solution of $ax^3 + bx^2 + cx + d$ is:

$$r_1 = -\frac{1}{3a}\left[b + \sqrt[3]{\frac{2b^3 - 9abc + 27a^2d \pm \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}}{2}} + \frac{b^2 - 3ac}{\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d \pm \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}}{2}}}\right]$$

$$r_2 = -\frac{1}{3a}\left[b + \left(\frac{-1 + \sqrt{-3}}{2}\right)\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d \pm \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}}{2}} + \frac{b^2 - 3ac}{\left(\frac{-1 + \sqrt{-3}}{2}\right)\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d \pm \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}}{2}}}\right] \tag{3}$$

$$r_3 = -\frac{1}{3a}\left[b + \left(\frac{-1 - \sqrt{-3}}{2}\right)\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d \pm \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}}{2}} + \frac{b^2 - 3ac}{\left(\frac{-1 - \sqrt{-3}}{2}\right)\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d \pm \sqrt{\left(2b^3 - 9abc + 27a^2d\right)^2 - 4\left(b^2 - 3ac\right)^3}}{2}}}\right]$$

4. Solution of $x^4 + ax^3 + bx^2 + cx + d$ is:

$$r_1 = -\frac{a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} - \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} - \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}} - \frac{-a^3 + 4ab - 8c}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}}}} \tag{4}$$

$$r_2 = -\frac{a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} + \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} - \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}} - \frac{-a^3 + 4ab - 8c}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}}}} \tag{5}$$

$$r_3 = -\frac{a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} - \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} - \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}} + \frac{-a^3 + 4ab - 8c}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}}}} \tag{6}$$

$$r_4 = -\frac{a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} + \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} - \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}} + \frac{-a^3 + 4ab - 8c}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}}}} \tag{7}$$