Advanced Image Processing

Part VII:

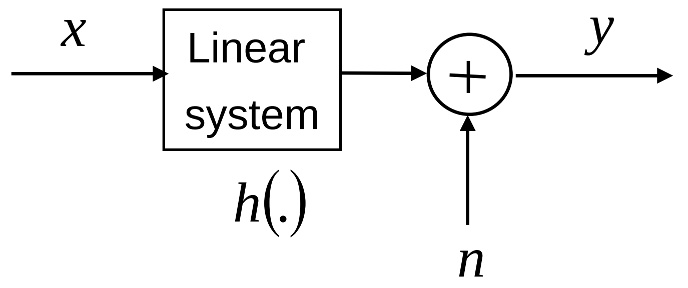
Image Restoration

Sviatoslav Voloshynovskiy

Roadmap:

1. Inverse Filtering (ML and LSE)
2. Deterministic Regularization (smoothness)
3. Iterative Methods and Regularization via Truncation
4. MAP Restoration
5. Penalized ML and Choice of Regularization Parameter and Functional
6. Thresholding Estimators for Restoration
7. Blind Deconvolution
8. Robust Restoration

1. Inverse Filtering: Mathematical Model

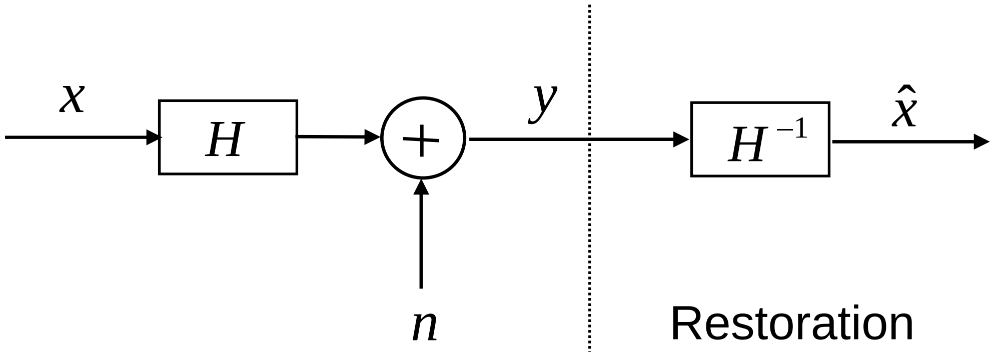


$$\boxed{y = Hx + n}$$

2-D convolution: $$y(n_1, n_2) = \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{M-1} h(n_1 - k_1, n_2 - k_2)\, x(k_1, k_2)$$

Frequency Domain: $$Y(m_1, m_2) = H(m_1, m_2)\, X(m_1, m_2)$$

1. Inverse Filtering



Restoration

$$\boxed{y = Hx + n}$$

$$\boxed{\hat{x} = H^{-1} y}$$

Assuming $\sigma_n^2 \to 0$

<u>ML and LSE solution:</u>

$$\hat{x}_{ML} = \arg\max_{\hat{x}} \log\left[p_{Y|X}(y|x)\right] = \arg\min_{\hat{x}} - \log\left[p_{Y|X}(y|x)\right]$$

$$\hat{x}_{ML} = \arg\min_{\hat{x}} \frac{1}{2\sigma_n^2} \left\| y - Hx \right\|^2 \Rightarrow \boxed{\hat{x} = \left(H^T H\right)^{-1} H^T y}$$

1. Inverse Filtering: Frequency Domain

$$\xrightarrow{\;y\;} \boxed{H^{-1}} \xrightarrow{\;\hat{x}\;}$$

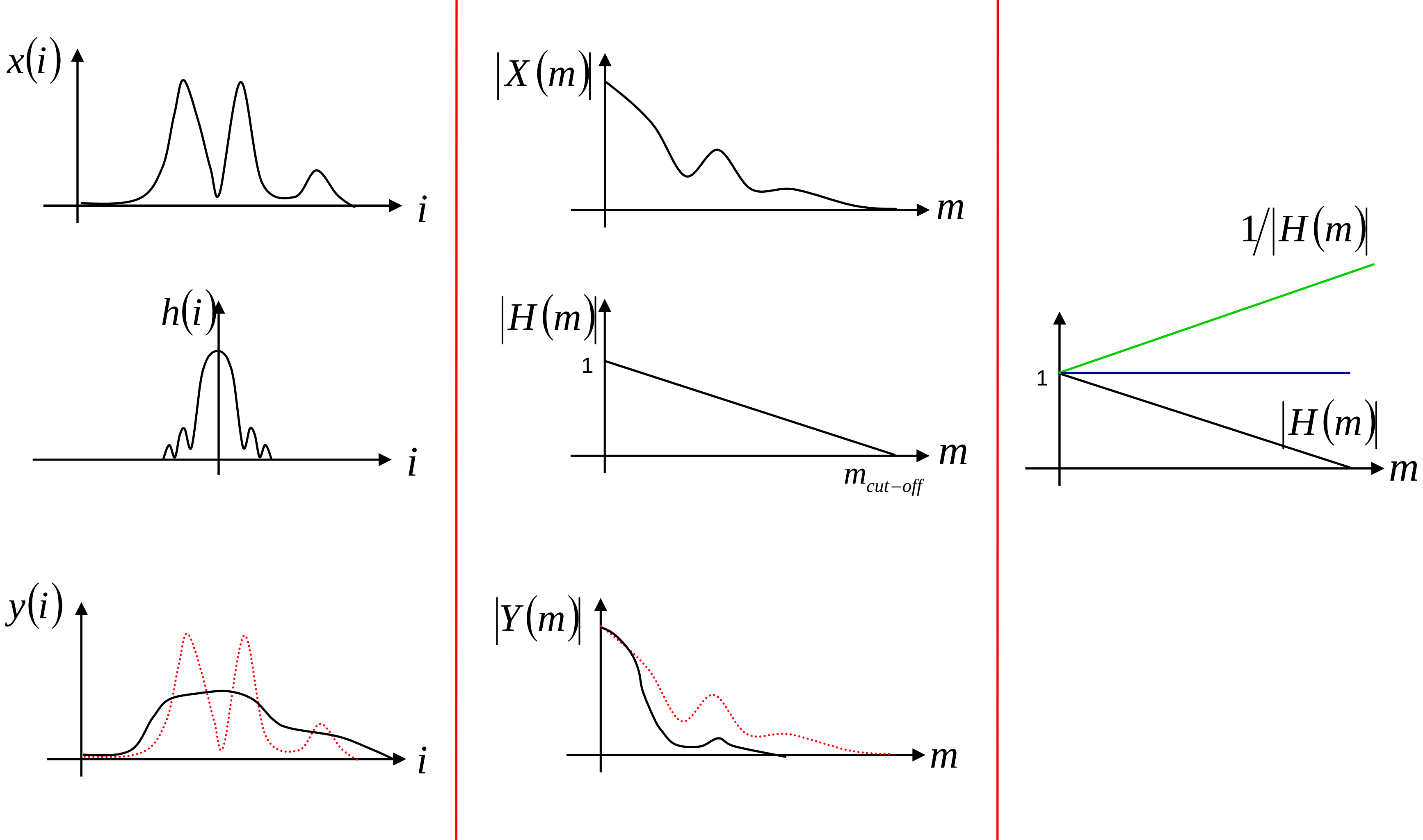
$$\boxed{\hat{x} = H^{-1} y}$$

$$Y(m_1, m_2) = H(m_1, m_2) X(m_1, m_2) + N(m_1, m_2)$$

$$H_{inv}(m_1, m_2) = 1/H(m_1, m_2) \Rightarrow H_{inv}(m_1, m_2) H(m_1, m_2) = 1$$

$$\begin{aligned}\hat{F}(m_1, m_2) &= H_{inv}(m_1, m_2) Y(m_1, m_2) \\ &= H_{inv}(m_1, m_2)\left[H(m_1, m_2) X(m_1, m_2) + N(m_1, m_2)\right] \\ &= X(m_1, m_2) + \underline{H_{inv}(m_1, m_2) N(m_1, m_2)}\end{aligned}$$

1. Inverse Filtering: Frequency Domain



1. Blurred images [A. Bovik, Ch.3.6]

Original Image

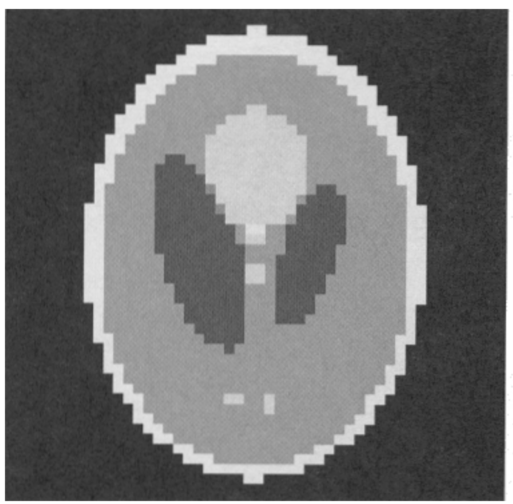
Motion blur (L=7)

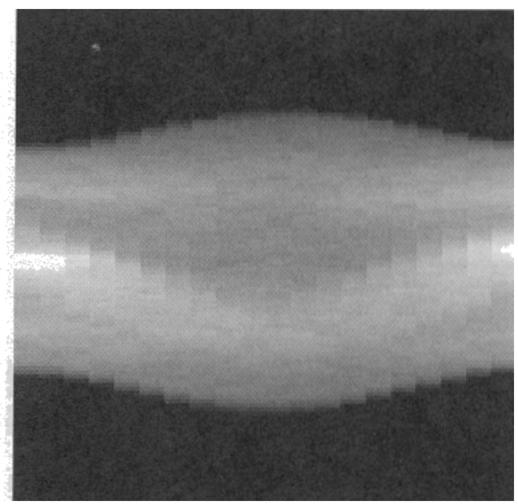
AWGN: 30 dB

Original

Tomographic Image

50x50





Project data with

20 angles and 125

samples per angle

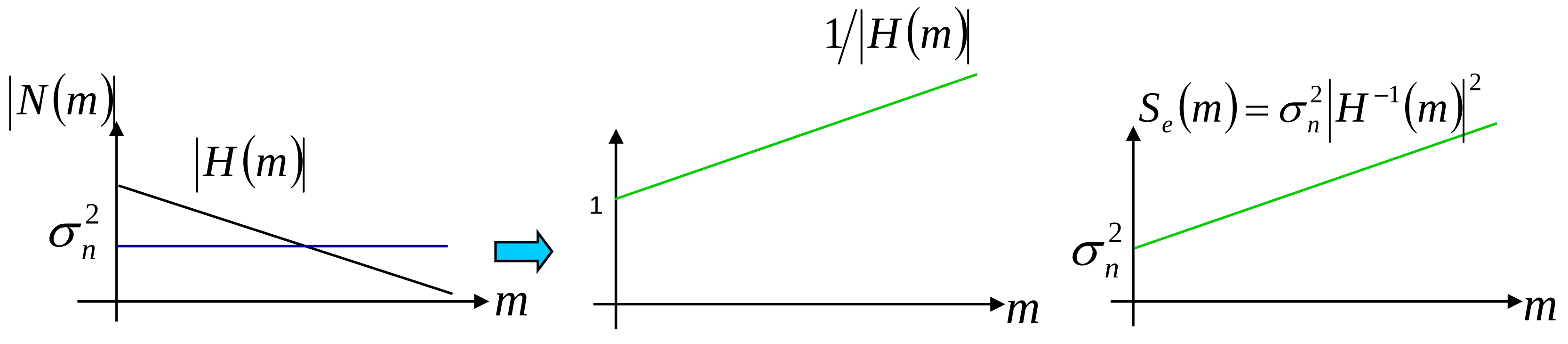
AWGN: 30 dB

1. Inverse Filtering: Frequency Domain

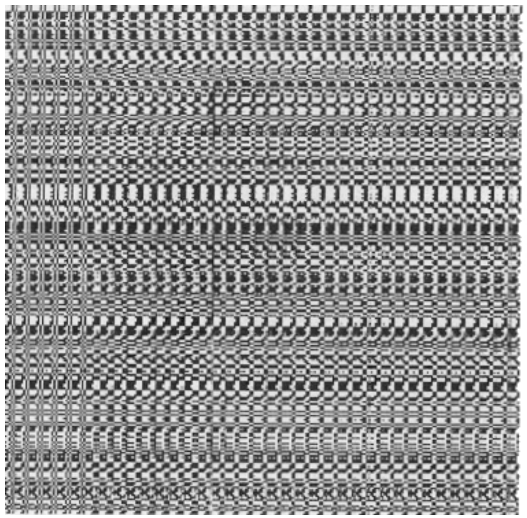
If n is AWGN with variance σ_n^2, then the spectral density of the reconstruction error:

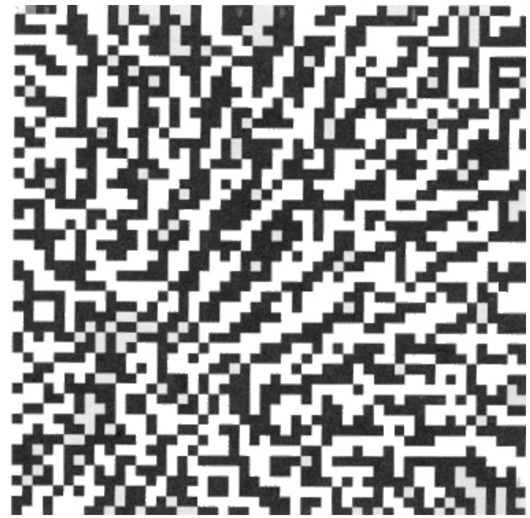
$$S_e(m_1, m_2) = \sigma_n^2 \left| H^{-1}(m_1, m_2) \right|^2$$

The reconstruction error is very large at high frequencies.



1. Inverse Filtering: The Need for Regularization





Have it something common with the original images?

1. Inverse Filtering: Problems

Several problems with the inverse filtering:

- the inverse filter may not exist (singularities in motion blur and defocusing);
- the excessive noise amplification at high frequencies may cause the complete degradation of the image

  Reason: the power spectra of the blurred image is typically highest at low frequencies and rolls off significantly for higher ones.

  The power spectrum of AWGN, on the other hand, typically is distributed uniformly: $S_e(m) = \sigma_n^2 \left|H^{-1}(m)\right|^2 = 10^{-3}/10^{-6} = 10^3 >> \left|X(m)\right|$

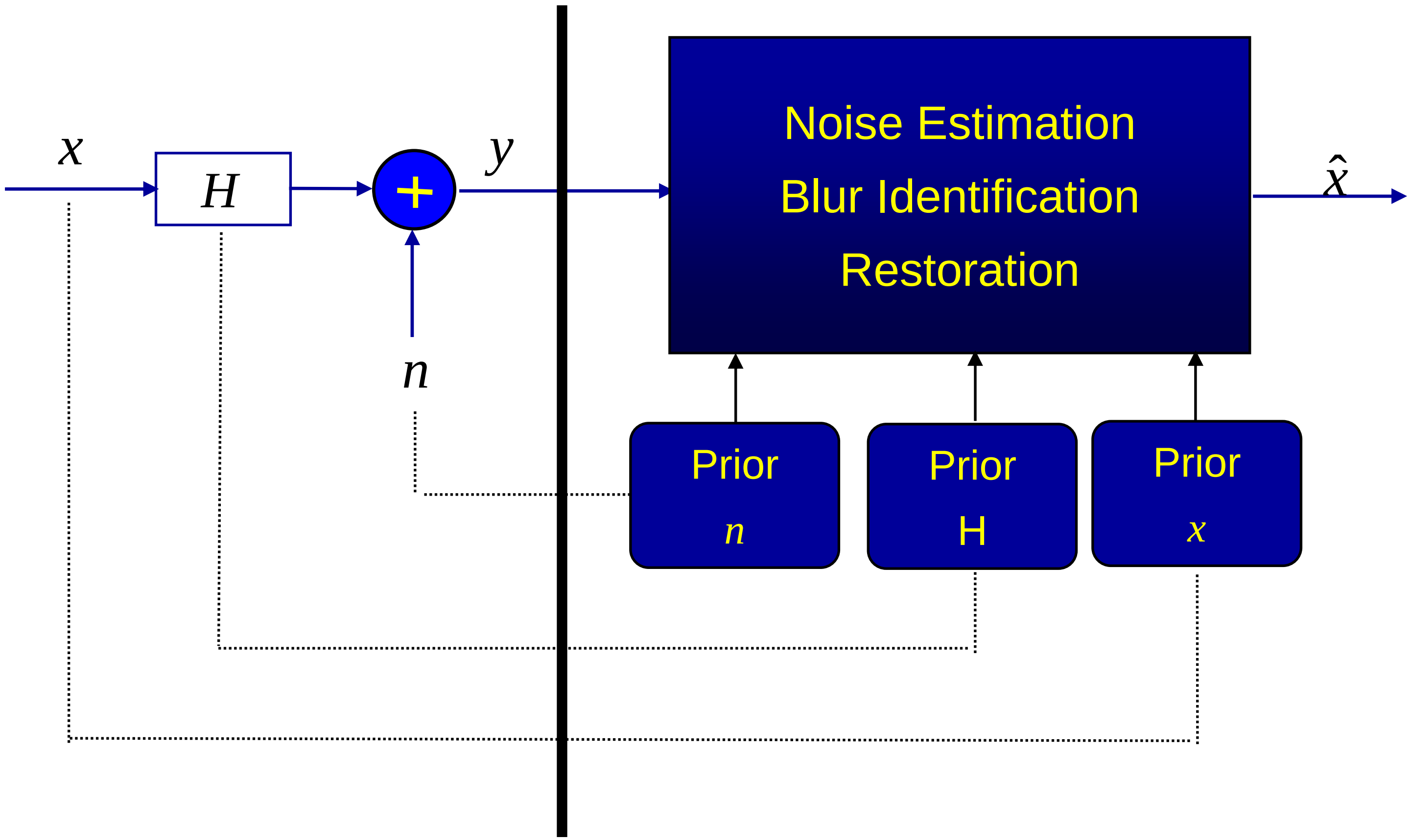
  Therefore, the obtained solution is useless.
- Computational aspect of high dimensional matrix inversion.

1. Restoration: General Conclusions

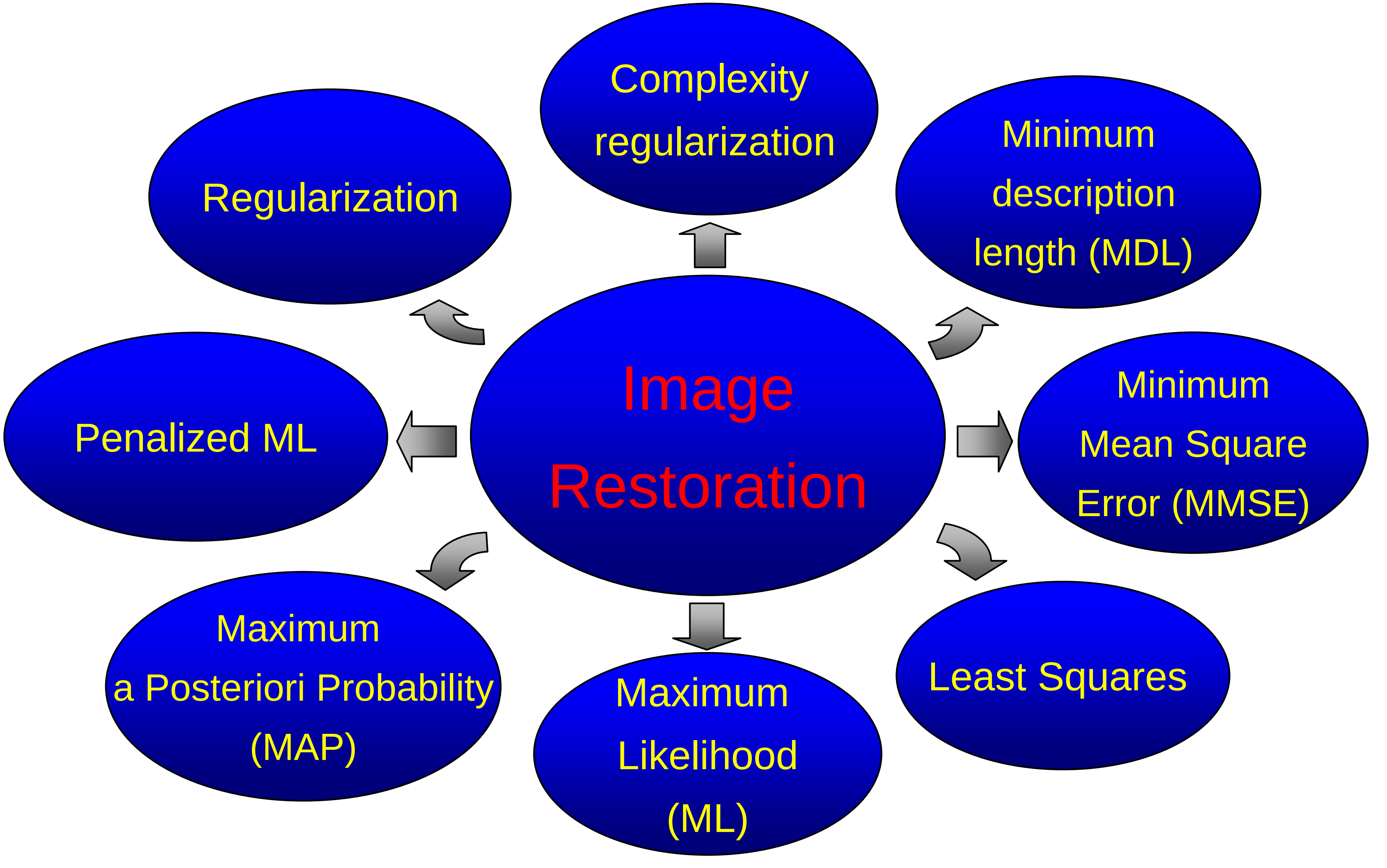
- The image restoration is ill-posed:
  - additive noise;
  - singularity of inverse operator.
- The solution may not be unique (due to low-pass character of imaging systems only a small fraction of the image spectrum is available for restoration - band-limited extrapolation).

  It requires some prior knowledge about imaging process and image.
- The solution has to be stable to the observed data variations and to the prior ambiguity about blur operator and noise.

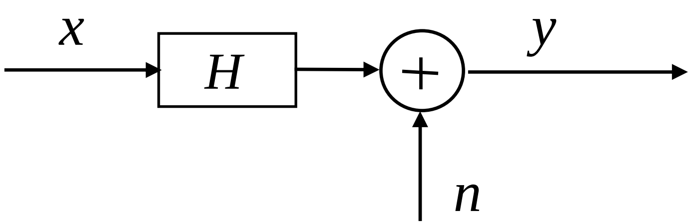
1. General Framework of Image Restoration



1. General Framework of Image Restoration



2. Tikhonov Regularization (deterministic)



- We have seen that the ML (LSE)

$$\hat{x}_{ML} = \arg\min_{\hat{x}} \|y - Hx\|^2$$

is unstable under even small amount of noise in y.

- To stabilize the solution a prior information about image is used. The regularized estimator is defined as follows:

$$\boxed{\hat{x}_{REG} = \arg\min_{\hat{x}\in\aleph} \|y - Hx\|^2 + \lambda\phi(x)}$$

Regularization parameter $\lambda \geq 0$

Regularization functional

2. Regularization: Generalized Framework

- Main contributions: V. Ivanov (1962) and D. Philips (1962)

  Generalized concept: Russian mathematician A. Tikhonov (1963)

- Principle assumption: the smoothness and the energy of real signals and images should be bounded.

  The measure of smoothness: derivative.

  Bounding factor: energy.

Energy is measured as L_2-norm :

$$E_x = \|x\|^2 = \int |x(t)|^2 dt \Longrightarrow$$ Therefore, quadratic functionals

2. Regularization: Energy Constraint

- The LSE with the constrain on the energy leads to the minimization problem:

$$\hat{x}_{REG_0} = \arg\min_{\hat{x}} \|y - Hx\|^2 \quad \text{Subject to: } E_x = \|x\|^2 \le E$$

- This problem can be reformulated according to the method of Langrange multipliers:

$$\hat{x}_{REG_0} = \arg\min_{\hat{x}\in\aleph} \|y - Hx\|^2 + \lambda\|x\|^2$$

- Solution:

$$\hat{x}_{REG_0} = \left(H^T H + \lambda I\right)^{-1} H^T y$$

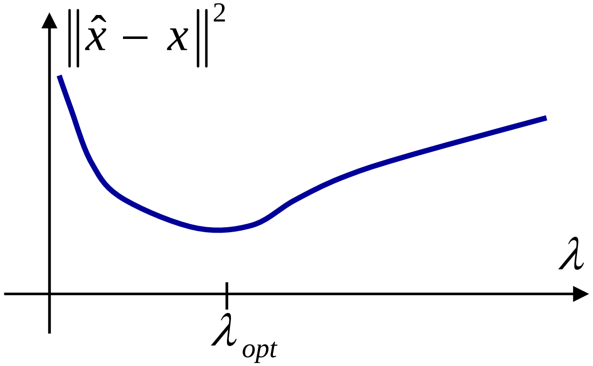
$$\hat{X}_{REG_0}(f) = \frac{H^*(f)}{\left(|H(f)|^2 + \lambda\right)} Y(f)$$

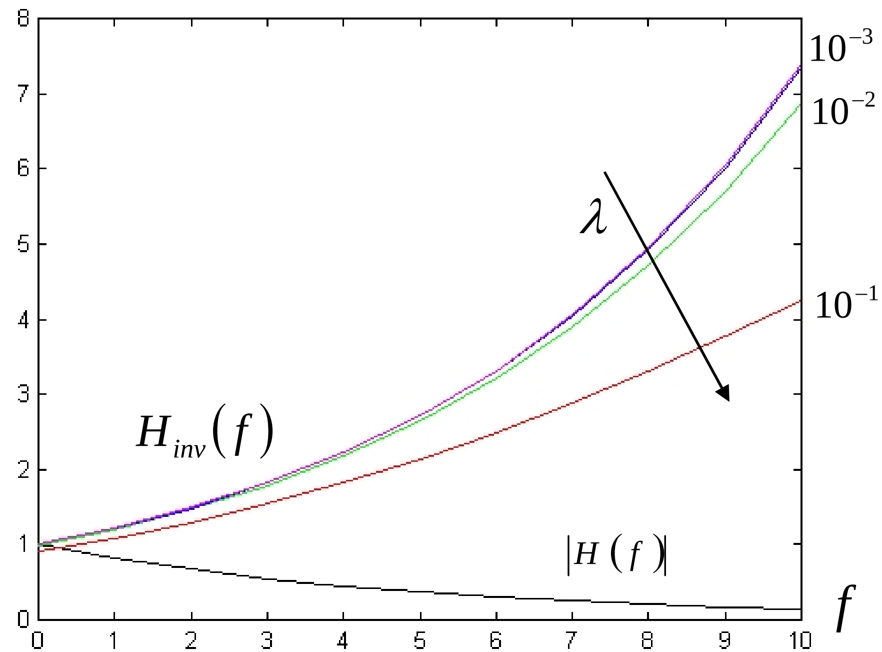
2. Regularization: Energy Constraint

$$\hat{X}_{REG_0}(f) = \frac{H^*(f)}{\left(|H(f)|^2 + \lambda\right)} Y(f)$$

$$H_{inv}(f)$$

Optimal regularization parameter





2. Regularization: First-order Derivative

- The LSE with the constraint on the constrained first-order derivative leads to the minimization problem:

$$\hat{x}_{REG_1} = \arg\min_{\hat{x}} \|y - Hx\|^2 \quad \text{Subject to: } E_x = \|Cx\|^2 \leq E$$

- Cx is the first order derivative: $Cx = x'(t) \Leftrightarrow -fX(f)$

$$\hat{x}_{REG_1} = \arg\min_{\hat{x}\in\aleph} \|y - Hx\|^2 + \lambda\|Cx\|^2$$

- Solution:

$$\hat{x}_{REG_1} = \left(H^T H + \lambda C^T C\right)^{-1} H^T y$$

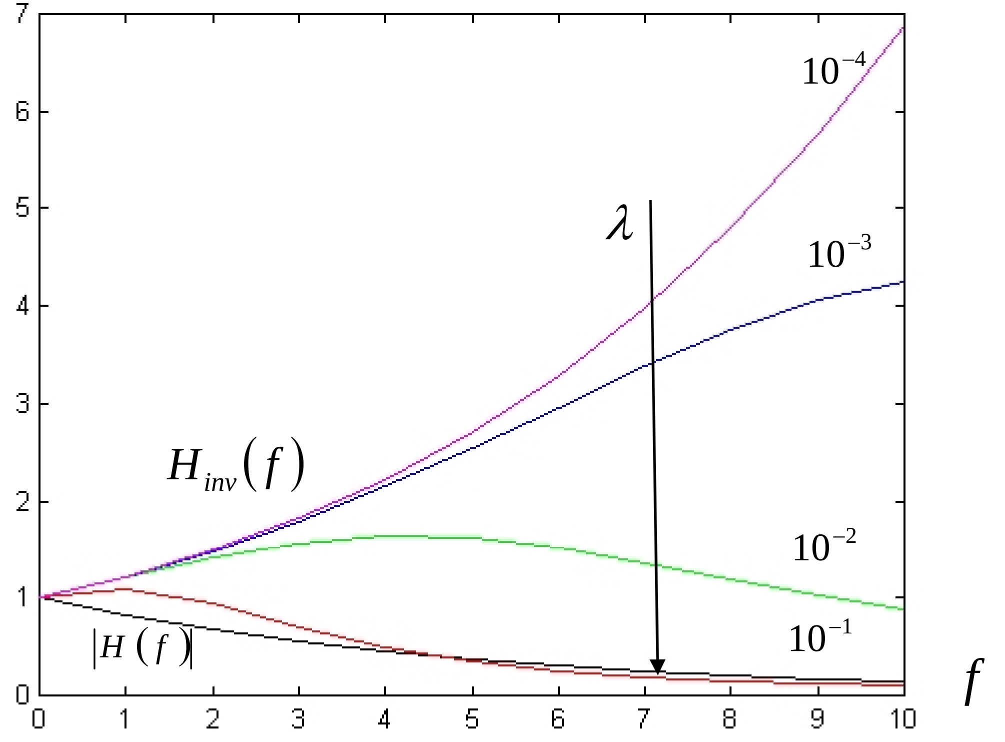
$$\hat{X}_{REG_1}(f) = \frac{H^*(f)}{\left(|H(f)|^2 + \lambda|C(f)|^2\right)} Y(f)$$

2. Regularization: First-order Derivative

$$\hat{X}_{REG_1}(f) = \left(\frac{H^*(f)}{\left(|H(f)|^2 + \lambda |C(f)|^2\right)} \right) Y(f)$$

$H_{inv}(f)$

$$|C(f)|^2 = f^2$$



2. Regularization: Second-order Derivative

- The LSE with the constraint on the second-order derivative leads to the minimization problem (Philips):

$$\hat{x}_{REG_2} = \arg\min_{\hat{x}} \|y - Hx\|^2 \quad \text{Subject to: } E_x = \|Cx\|^2 \le E$$

- Cx is the second order derivative (gradient): $Cx = x''(t) \Leftrightarrow -f^2 X(f)$

$$\hat{x}_{REG_2} = \arg\min_{\hat{x}\in\aleph} \|y - Hx\|^2 + \lambda\|Cx\|^2$$

- Solution:

$$\hat{x}_{REG_2} = \left(H^T H + \lambda C^T C\right)^{-1} H^T y$$

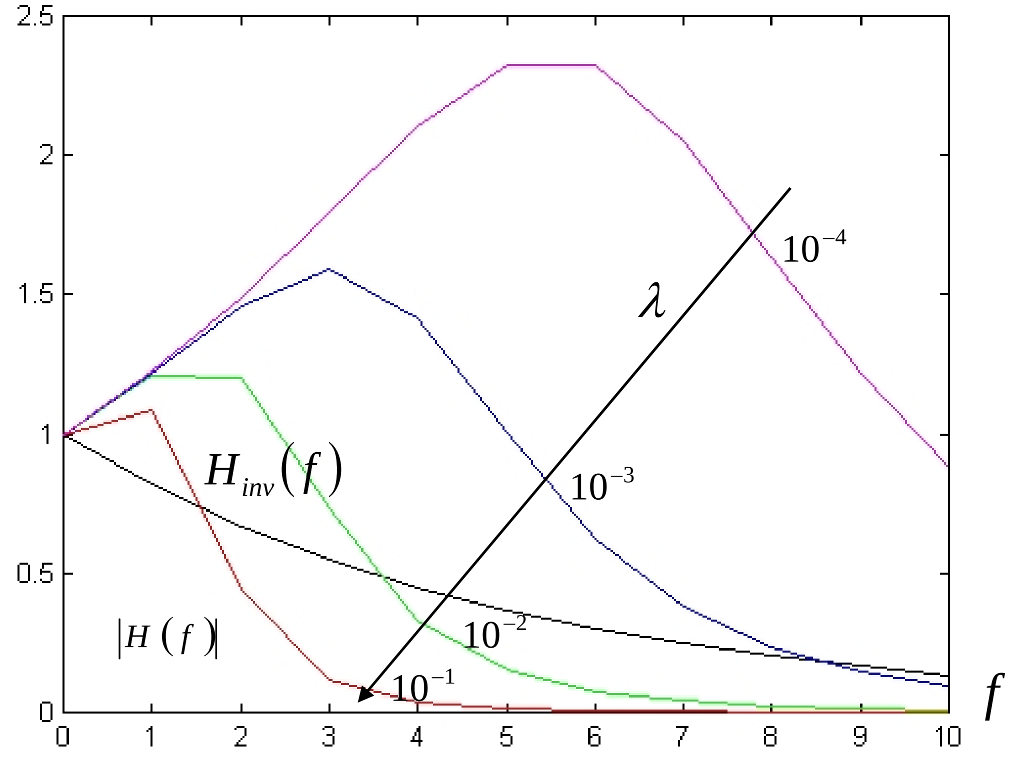
$$\hat{X}_{REG_1}(f) = \frac{H^*(f)}{\left(|H(f)|^2 + \lambda |C(f)|^2\right)} Y(f)$$

2. Regularization: Second-order Derivative

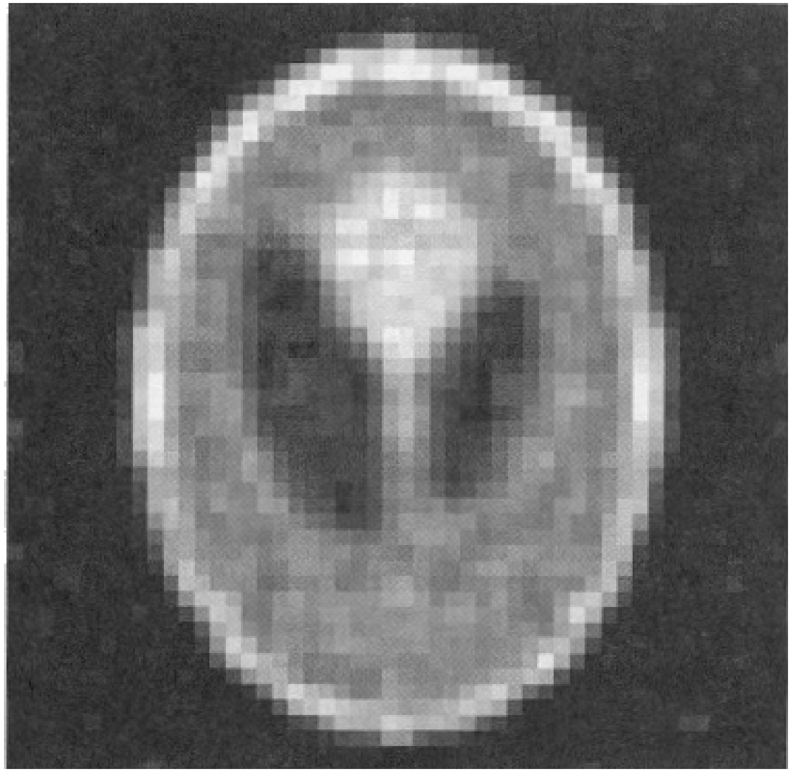
$$\hat{X}_{REG_2}(f) = \frac{H^*(f)}{\left(|H(f)|^2 + \lambda |C(f)|^2\right)} Y(f)$$

$H_{inv}(f)$

$$|C(f)|^2 = f^4$$



2. Tikhonov Regularization: C - gradient



Problem: ringing artifacts near edges

2. Tikhonov Regularization

- The regularization functional includes all derivatives combined in some weighted fashion. It is very powerful concept from the approximation point of view since it allows to capture different local image features.
- All smoothness priors considered above are the particular cases of Tikhonov regularization.
- Moreover, the introduction of adaptive estimation of the weights leads to the very flexible framework of locally adaptive regularization.

$$\hat{x}_{TIKH} = \arg\min_{\hat{x}\in\aleph} \|y - Hx\|^2 + \lambda \|C_{TIKH}\, x\|^2$$

$$\|C_{TIKH}\, x\|^2 = a_0 \|x\|^2 + \sum_{i=1}^{q} a_i \left\| \frac{\partial x}{\partial t_i} \right\|^2 + \sum_{i,j=1}^{q} a_{i,j} \left\| \frac{\partial x}{\partial t_i \partial t_j} \right\|^2$$

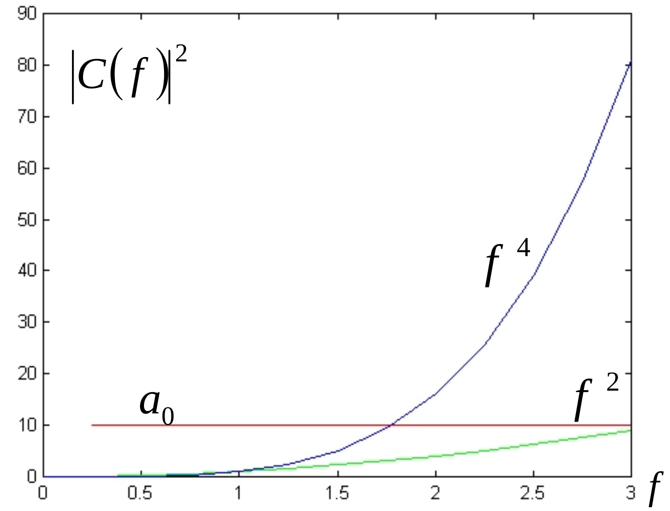
2. Tikhonov Regularization

Solution:

$$\hat{x}_{TIKH} = \left(H^T H + \lambda C_{TIKH}{}^T C_{TIKH}\right)^{-1} H^T y$$

$$\hat{X}_{TIKH}(f) = \frac{H^*(f)}{\left(|H(f)|^2 + \lambda |C_{TIKH}(f)|^2\right)} Y(f)$$

$$|C(f)|^2 = a_0 + \sum_{i=1}^{q} a_i f_i^2 + \sum_{i,j=1}^{q} a_{i,j} (f_i f_j)^2$$

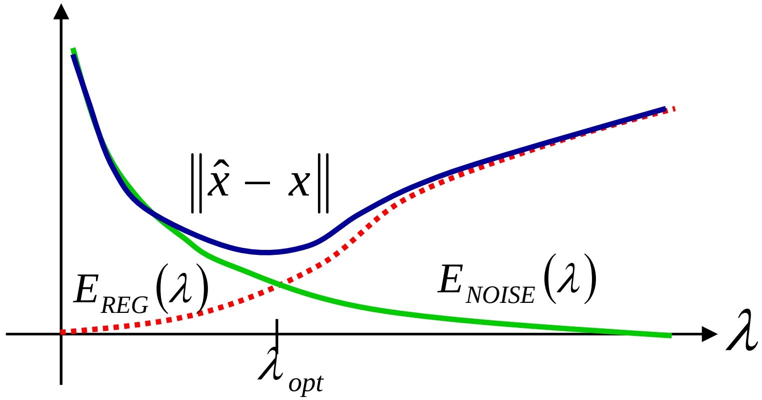


2. Tikhonov Regularization: Error of Restoration

$$\hat{x}_{REG_2} = \left(H^T H + \lambda C^T C\right)^{-1} H^T y \qquad \hat{X}_{REG}(f) = \frac{H^*(f)}{\left(|H(f)|^2 + \lambda |C(f)|^2\right)} Y(f)$$

$$\|x - \hat{x}_{REG}\| \le \int \left| \frac{\lambda |C(f)|^2}{|H(f)|^2 + \lambda |C(f)|^2} \right| |X(f)| df + \int \left| \frac{\lambda H^*(f)}{|H(f)|^2 + \lambda |C(f)|^2} \right| |N(f)| df$$

$$\|x - \hat{x}_{REG}\| \le E_{REG}(\lambda) + E_{NOISE}(\lambda)$$



R. Lagendijk & J. Biemond

2. Selection of Regularization Parameter

- Selection of regularization parameter:

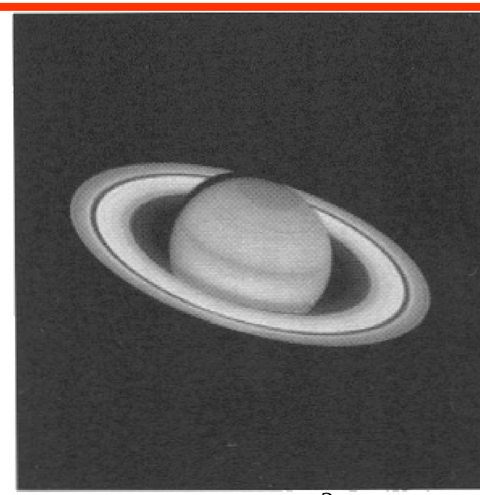
tradeoff between fidelity to the data and fidelity to some set of priors.

$$\hat{x}_{REG_2} = \arg\min_{\hat{x}\in\aleph} \left\| y - Hx \right\|^2 + \lambda \left\| Cx \right\|^2$$

- Methods of regularization parameter selection:
  - based on visual inspection of restored image;
  - the discrepancy principle;
  - based on some knowledge of the noise;
  - the L-curve;
  - generalized cross-validation;
  - statistical estimation theory (see later MAP restoration).

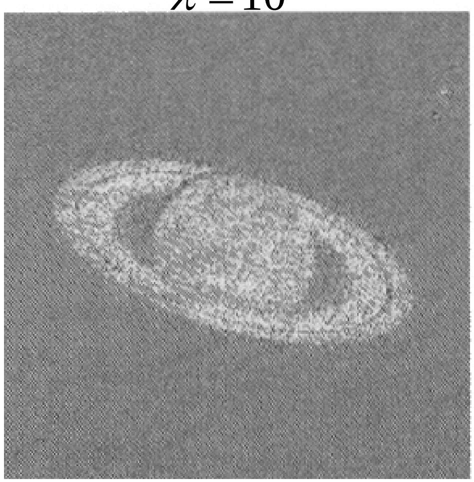
2. Regularisation Parameter Selection based on Visual Inspection [Bertero&Boccacci]

$\lambda = 10^{-5}$

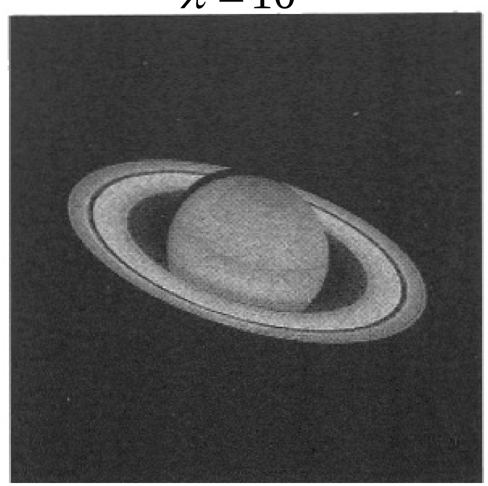


$\lambda = 10^{-2}$

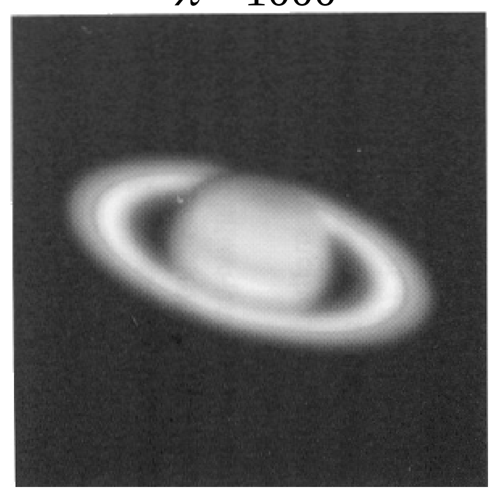
$\lambda = 1000$



$\lambda = 0$



$\lambda = 10^{-3}$



$\lambda = 1$

2. Regularization Parameter: Discrepancy Principle

- The Discrepancy principle introduced by the Russian mathematician Morozov.

- It assumes the prior knowledge of noise energy (variance):

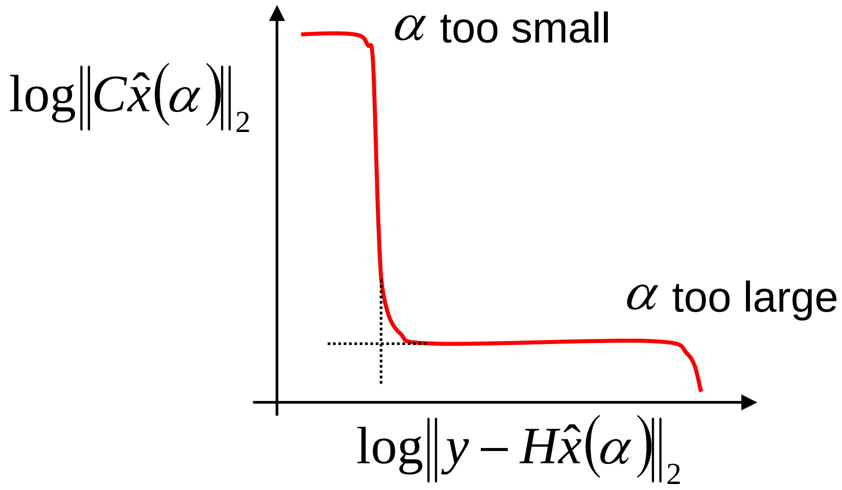
$$\|n\|_2^2 = E_n = \sigma_n^2$$

- Formally, the regularization parameter is chosen as that value for which the residual norm achieves the equality:

$$\|y - H\hat{x}(\lambda)\|_2^2 \le E_n = \sigma_n^2$$

2. Regularization Parameter: L-curve

- L-curve introduced by the Swedish mathematician P. Hansen:
- L-curve visualizes the tradeoff between $\left\| y - H\hat{x}(\lambda) \right\|$ and $\left\| C\hat{x}(\lambda) \right\|$



$$\lambda_{opt} = \frac{\left\| y - H\hat{x}(\lambda) \right\|_2^2}{\left\| C\hat{x}(\lambda) \right\|_2^2} \Rightarrow\Rightarrow\Rightarrow\Rightarrow \frac{\sigma_n^2}{\sigma_x^2}$$

3. Iterative Solution

- Van Citter's Method, 1930s or simple iteration, or steepest descent:

(solution of a system of linear equations $y = Hx$)

$$\boxed{\begin{aligned} \hat{x}^k &= \beta y \\ \hat{x}^{k+1} &= \hat{x}^k + \beta\left(y - H\hat{x}^k\right) \\ &= \beta y + \left(I - \beta H\right)\hat{x}^k = \beta y + R\hat{x}^k \end{aligned}}$$

- Convergence:

$$\hat{x}^k = \beta \sum_{r=0}^{k} R^r y \Rightarrow \hat{x}^k = \beta\left(I - R\right)^{-1}\left(I - R^{k+1}\right)y \Rightarrow \lim_{k\to\infty} R^{k+1} y = 0$$

$$\Rightarrow \lim_{k\to\infty} x^{k+1} = \beta\left(I - R\right)^{-1} y = H^{-1} y$$ Inverse filter

3. Iterative Solution

- The solution after *k* iterations in the frequency domain:

$$\hat{x}^k = \Im^{-1}\left\{\sum_m \boxed{\frac{1}{H(m)}}\left(1-\left(1-\beta H(m)\right)^{k+1}\right)Y(m)\right\}$$

Part of inverse filter

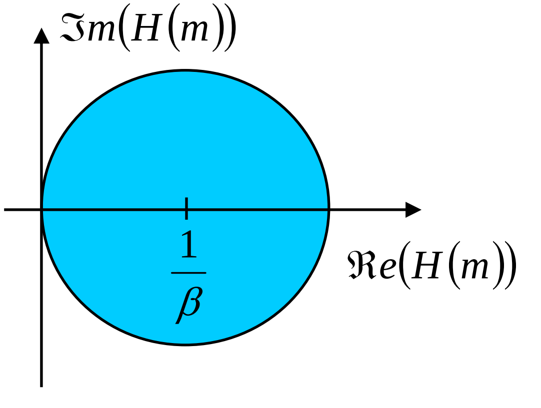
- From this the convergence condition:

$$\boxed{\left|1-\beta H(m)\right| < 1, \forall m}$$

- This is equivalent to the condition:

$$\boxed{\Re e\left(\beta H(m)\right) > 0}$$

This condition is not satisfied for the motion blur and defocusing.

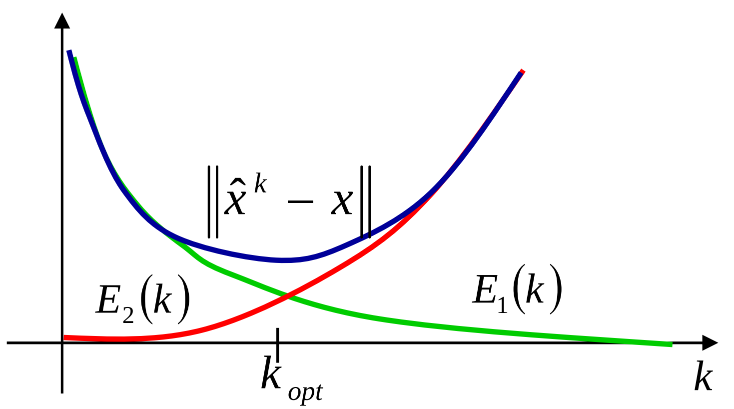


3. Iterative Solution: Regularization via Truncation

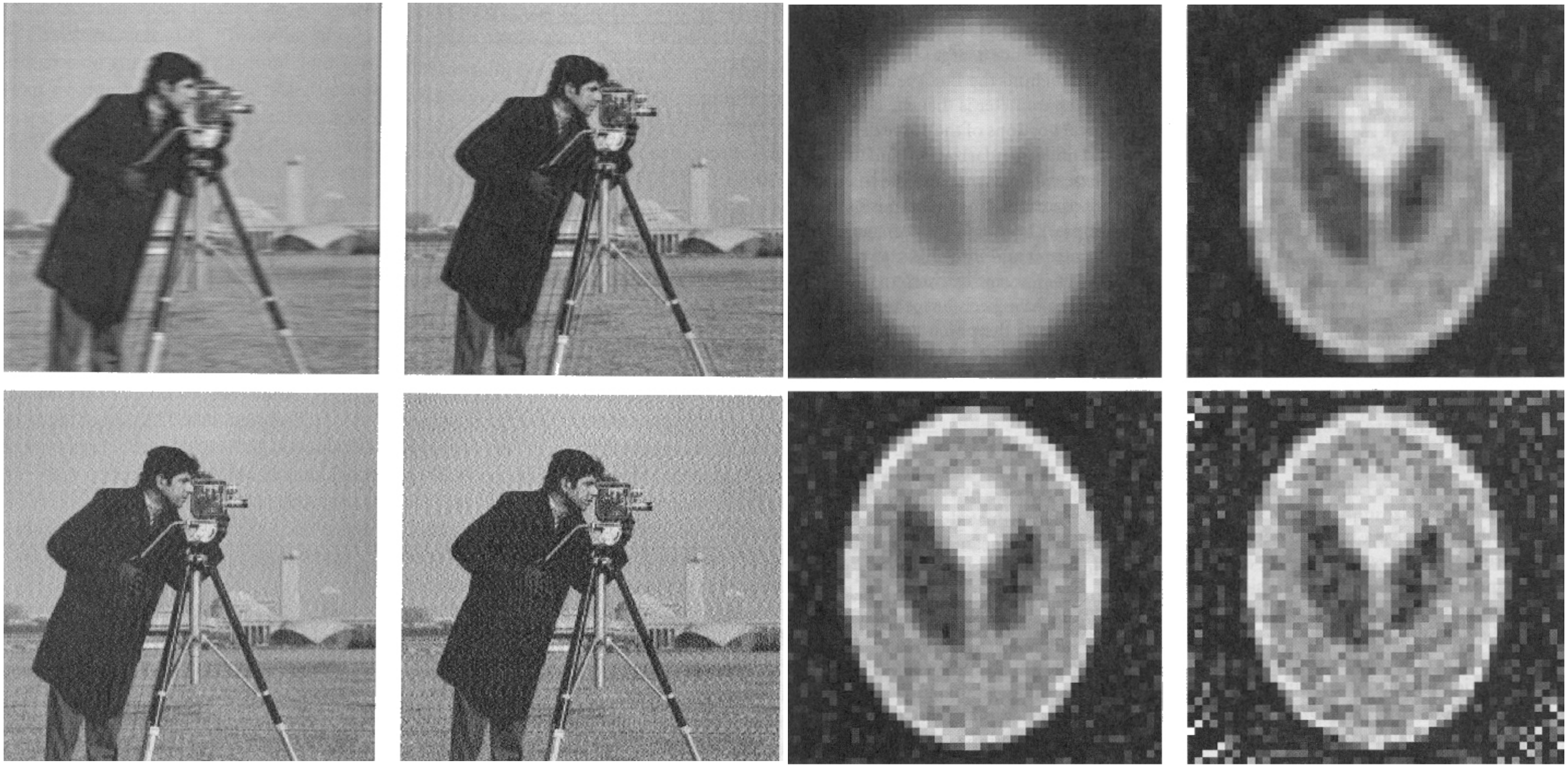
- The error of restoration in the presence of noise:

$$\left\|\hat{x}^{k}-x\right\| \le \sum_{m}\left|1-\beta H(m)\right|^{k+1}\left|X(m)\right| + \sum_{m}\left|1-\left(1-\beta H(m)\right)^{k+1}\right| \frac{\left|N(m)\right|}{\left|H(m)\right|}$$

$$\left\|\hat{x}-x\right\| \le E_1(k) + E_2(k)$$



3. Regularization via Truncation



3. Iterative Solution

- Van Citter's Method with reblurring or ML-solution:

  equivalent problem: minimize $\Phi(x) = \|y - Hx\|^2$

  - To accelerate convergence for motion blur and defocusing the reblurring is used:

$$\boxed{H^T y = H^T H x}$$

$$\boxed{\hat{x}^{k+1} = \hat{x}^k + \beta H^T\left(y - H\hat{x}^k\right) = \beta H^T y + \left(I - \beta H^T H\right)\hat{x}^k}$$

$$\boxed{0 < \left|1 - \beta\left|H(m)\right|^2\right| < 1} \xrightarrow{\text{Equivalently}} \left|H(m)\right| \neq 0$$

3. Iterative Solution with Regularization

$$\hat{x}_{REG} = \arg\min_{\hat{x}\in\aleph} \Phi(x) = \|y - Hx\|^2 + \lambda\|Cx\|^2$$

$$\hat{x}^{k+1} = \hat{x}^k - \beta\nabla\Phi\left(\hat{x}^k\right)$$

$$\nabla\Phi(x) = -H^T(y - Hx) + \lambda C^T Cx$$

$$\hat{x}^{k+1} = \hat{x}^k + \beta\left(H^T\left(y - H\hat{x}^k\right) - \lambda C^T C\hat{x}^k\right)$$

$$\hat{x}^{k+1} = \beta H^T y + \left(I - \beta\left(H^T H + \lambda C^T C\right)\right)\hat{x}^k$$

$$\hat{x}^{k+1} = \boxed{\left(I - \beta\lambda C^T C\right)}\hat{x}^k + \beta H^T\left(y - H\hat{x}^k\right)$$

Smoothing filter

3. Iterative Solution with Regularization

- The regularized solution after *k* iterations in the frequency domain

$$\hat{x}^k = \mathfrak{I}^{-1}\left\{ \sum_m \boxed{\frac{H^*(m)}{|H(m)|^2 + \lambda |C(m)|^2}} \left(1 - \left(1 - \beta\left(|H(m)|^2 + \lambda |C(m)|^2\right)\right)^{k+1} \right) Y(m) \right\}$$

Part of regularized solution

- From this the convergence condition:

$$\boxed{\left|1 - \beta\left(|H(m)|^2 + \lambda |C(m)|^2\right)\right| < 1, \ \forall m}$$

3. Iterative Solution with POCS

- The iterative methods with regularization are extremely powerful tool of image restoration.
- Among many advantages such as:
  - computational simplicity;
  - absence of matrix inversion;
  - high level of parallelism;
  - a possibility to obtain the regularized solution

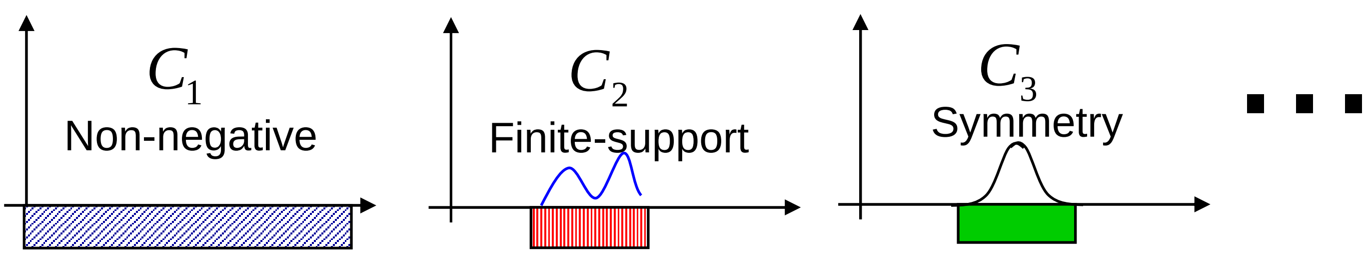
they make possible to easily incorporate a lot of prior constrains on the solution in those cases when the purely stochastic methods cannot be used due to the lack of the adequate stochastic prior models or at least the solution of the problem is extremely difficult and the results are not tractable.

3. Iterative Solution with POCS

- More over such methods might provide the unique possibilities such as band-limited extrapolation and geometrical priors that lead to highly non-linear image restoration methods:

$$\hat{x}^{k+1} = P\left[\left(I - \beta\lambda C^T C\right)\hat{x}^k + \beta H^T\left(y - H\hat{x}^k\right)\right]$$

$$P_i\left[\hat{x}\right] = \begin{cases} x, \ x \in C_i, \\ h, \ x \notin C_i. \end{cases} \qquad P = \left\{P_1, P_2, \ldots, P_M\right\}$$



4. MAP Restoration: Gaussian Prior

$$\hat{x}_{MAP} = \arg\min_{x \in \aleph} \left\{ -\ln p(y|x) - \ln p(x) \right\}$$

Prior Models:

$$n \sim N(0, R_n) \Rightarrow -\ln p(y|x) = \frac{1}{2} \|y - Hx\|^2_{R_n^{-1}}$$

$$x \sim N(0, R_x) \Rightarrow -\ln p(x) = \frac{1}{2} \|x\|^2_{R_x^{-1}}$$

$$\boxed{\hat{x}_{MAP} = \arg\min_{x \in \aleph} \left\{ \|y - Hx\|^2_{R_n^{-1}} + \|x\|^2_{R_x^{-1}} \right\}}$$

$$\boxed{\hat{x}_{MAP} = \left(H^T R_n^{-1} H + \lambda R_x^{-1}\right)^{-1} H^T R_n^{-1} y}$$

The solution is equivalent to Wiener filer and the MMSE.

4. MAP Restoration and Relationship to Regularization

$$\hat{x}_{MAP} = \arg\min_{x \in \aleph} \left\{ -\ln p(y|x) - \ln p(x) \right\}$$

$$n \sim N(0, \sigma_n^2 I) \Rightarrow -\ln p(y|x) = \frac{1}{2}\|y - Hx\|^2 \qquad R_n = \sigma_n^2 I$$

$$x \sim N(0, \sigma_x^2 I) \Rightarrow -\ln p(x) = \frac{1}{2}\|x\|^2 \qquad R_x = \sigma_x^2 I$$

$$\boxed{\hat{x}_{MAP} = \arg\min_{x \in \aleph} \left\{ \|y - Hx\|_2^2 + \frac{\sigma_n^2}{\sigma_x^2}\|x\|_2^2 \right\}}$$

The solution is equivalent to Tikhonov regularization

$$\boxed{\hat{x}_{REG_0} = \left(H^T H + \lambda I\right)^{-1} H^T y \quad \text{with} \quad \lambda = \frac{\sigma_n^2}{\sigma_x^2} \quad \text{and} \quad C = I}$$

!

5. Choice of Penalizing Functional

- The choice of regularization functional plays the fundamental role in the design of image restoration methods. Clearly the regularization functional captures our a prior knowledge about the image

- Therefore, it should be very carefully chosen considering different factors such as:
  - the smoothness of homogeneous areas of the image;
  - possibility of allowing sharp transitions between different regions;
  - it should be convex to guarantee the existence of a unique solution or at least to facilitate the numerical optimization.

5. Penalized ML

- To stabilize the solution a prior information about image is used as well.
- The difference with the regularization approach consists in the possibility to choose any penalizing functional (and not necessary quadratic).

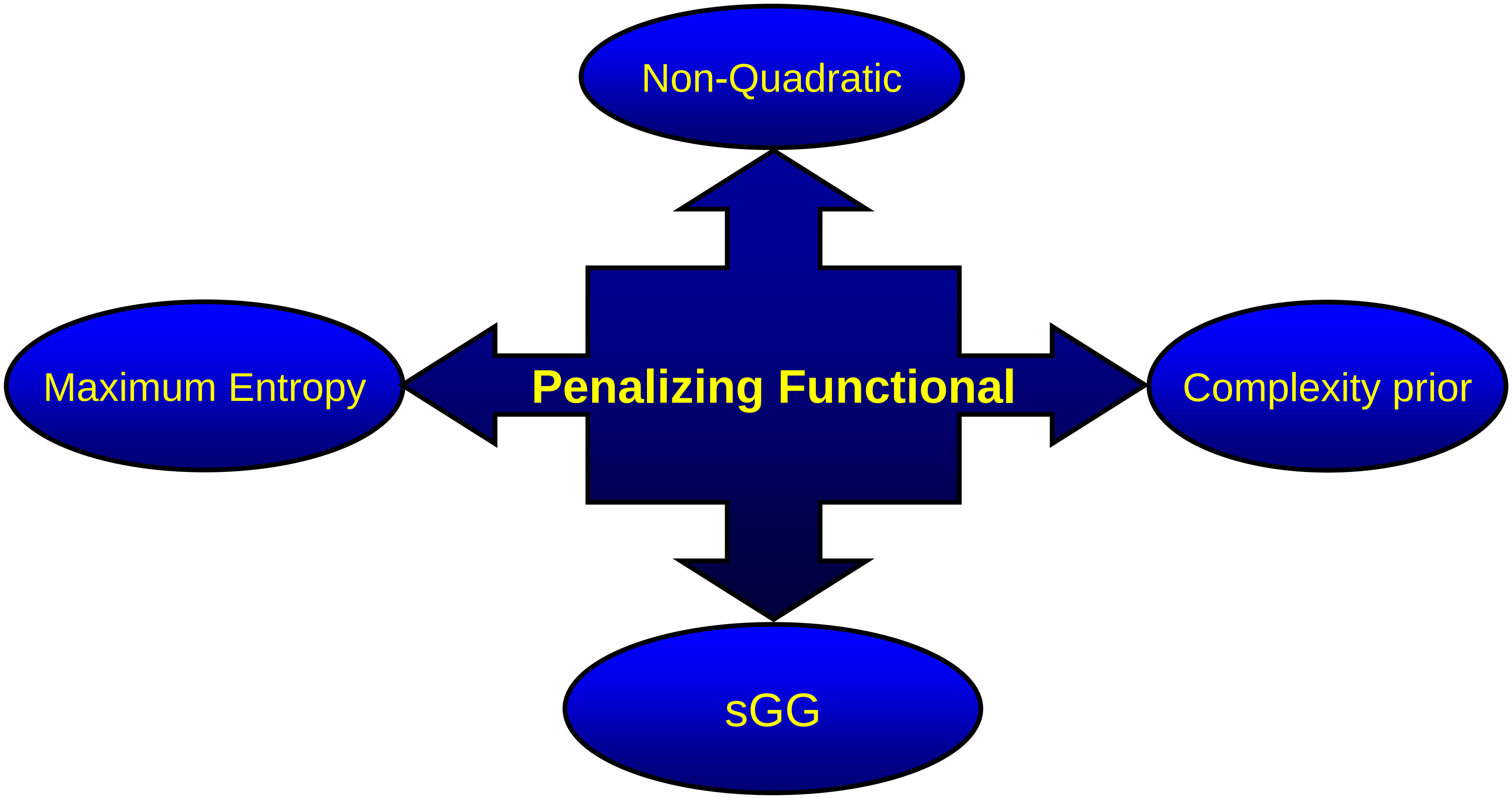
$$\boxed{\hat{x}_{PML} = \arg\min_{x \in \aleph} \left\{ -\ln p(y|x) + \lambda \phi(x) \right\}}$$

Penalizing functional

- ML: $\lambda = 0$
- MAP: $-\lambda\phi(x) = \ln p(x) = \ln e^{-\lambda\phi(x)}$ - exponential family of priors
- Tikhonov Regularization: $-\ln p(y|x) = \frac{1}{2\sigma_n^2}\|y - Hx\|^2$

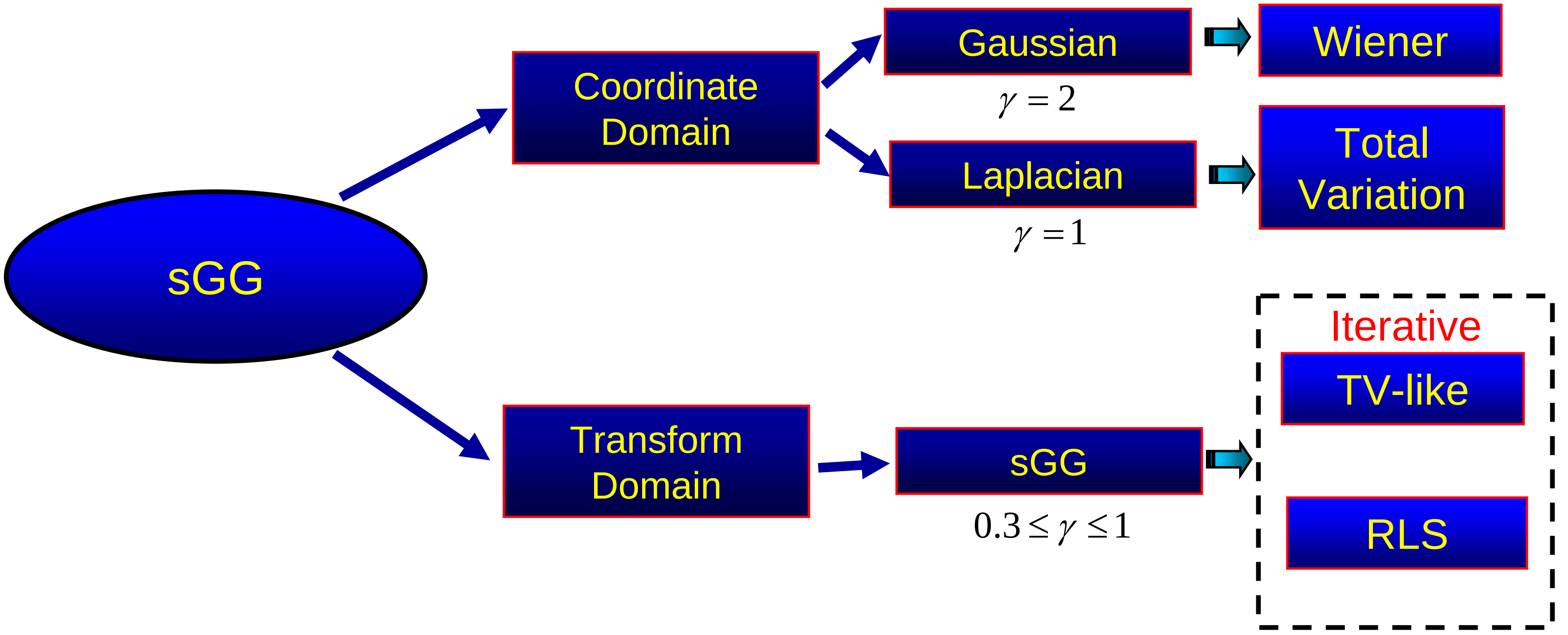
$$-\ln p(x) = \frac{1}{2\sigma_x^2}\|Cx\|^2$$

5. Choice of Penalizing Functional



5. Choice of Penalizing Functional : sGG Family

SGG image prior can be utilized in both the coordinate domain and in the transform domain.



5. Choice of Penalizing Functional : sGG Family

$$y = Hx + n \xrightarrow{\text{DWT}} Wy = \left(WHW^T\right)Wx + Wn \longrightarrow \tilde{y} = \tilde{H}\tilde{x} + \tilde{n}$$

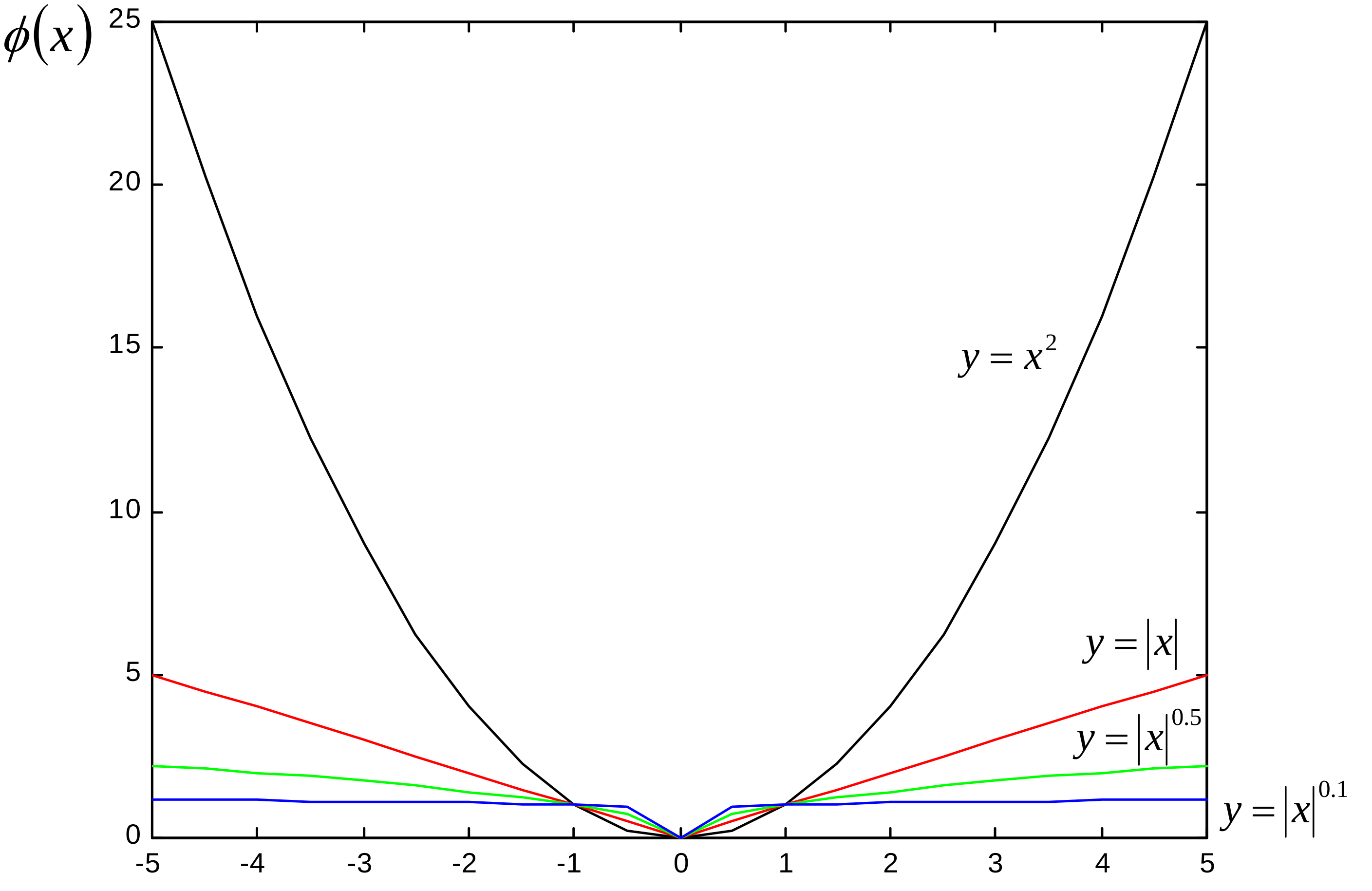
In wavelet domain	Via transform
	Cx $x - \overline{x}$
$0.3 < \gamma < 1$	

$$\phi(x) = \left[\eta(\gamma)\left|\frac{x}{\sigma_n}\right|\right]^{\gamma}$$

$$\hat{x}_{MAP} = \arg\max_{\hat{x}\in\aleph}\left[\ln p_{Y|X}(y|x) + \ln p_X(x)\right] =$$

$$= \arg\min_{\hat{x}\in\aleph}\left[\frac{1}{2\sigma_n^2}\left\|y - Hx\right\|_2^2 + \phi(\underbrace{x - \overline{x}}_{Cx})\right]$$

5. Choice of Penalizing Functional : sGG Family



5. Total Variation Restoration (Laplacian image prior)

L.I.Rudin, S.Osher, and E.Fatemi, Nonlinear total variation based noise removal algorithms, Physica D., vol. 60, pp. 259-269, 1992.

$$\boxed{\hat{x}_{TV} = \arg\min_{\hat{x}\in\aleph} \left[\|y - Hx\|_2^2 + \lambda \|Cx\|_1 \right]}$$

- Solution via approximation of ℓ_1- norm putting $\beta \to 0$:

$$\|Cx\|_1 \approx \sum_{i=1}^{N} \sqrt{\left|[Cx]_i\right|^2 + \beta}$$

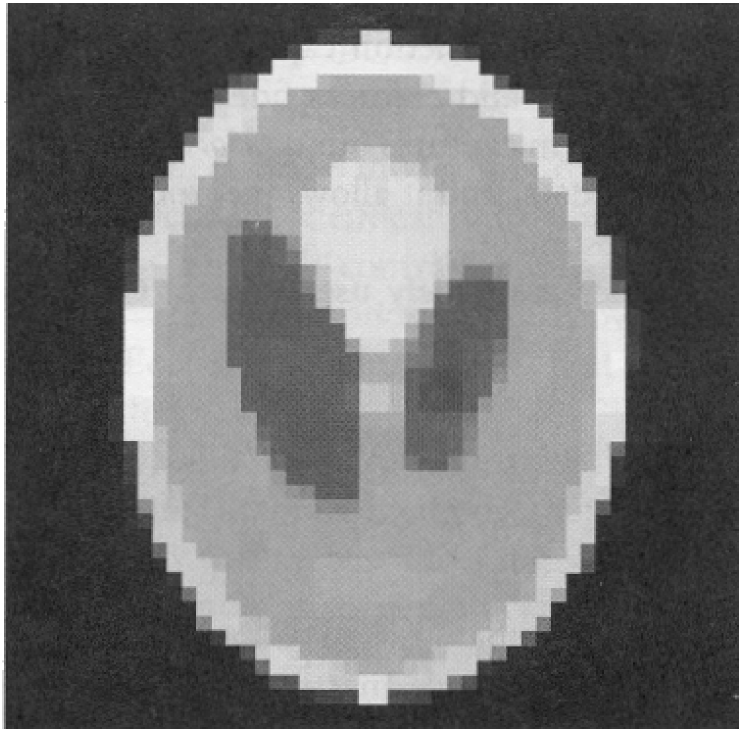
$$\left(H^T H + \lambda C^T W_\beta(\hat{x}_{TV})C\right)^{-1} \hat{x}_{TV} = H^T y$$

$$W_\beta(\hat{x}_{TV}) = \frac{1}{2} diag\left[\frac{1}{\sqrt{\left|[Cx]_i\right|^2 + \beta}}\right]$$

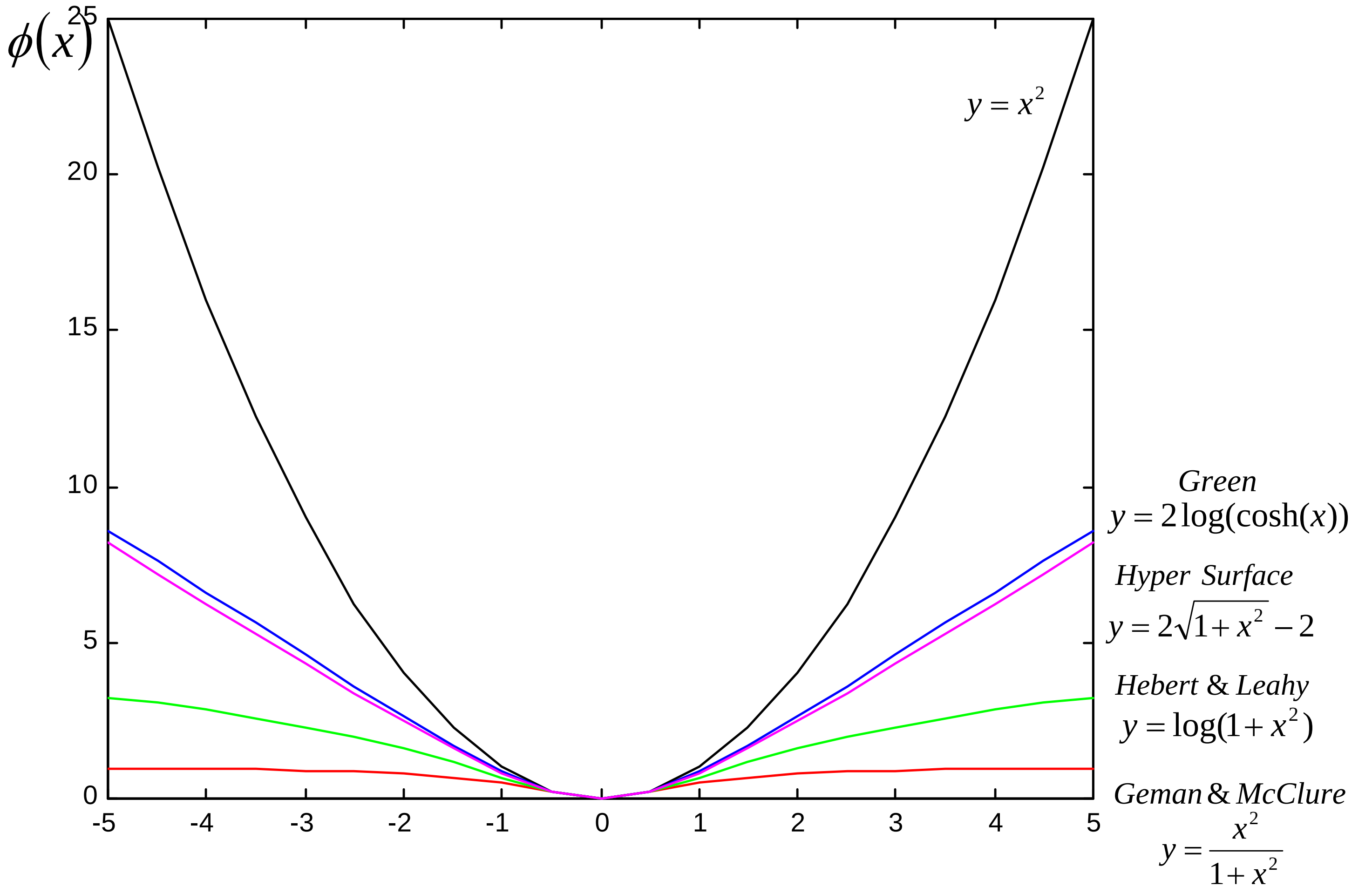
Iterative Solution

$$\boxed{\left(H^T H + \lambda C^T W_\beta(\hat{x}^k)C\right)^{-1} \hat{x}^{k+1} = H^T y}$$

5. Total Variation Restoration (Laplacian image prior)



5. Non-quadratic functions



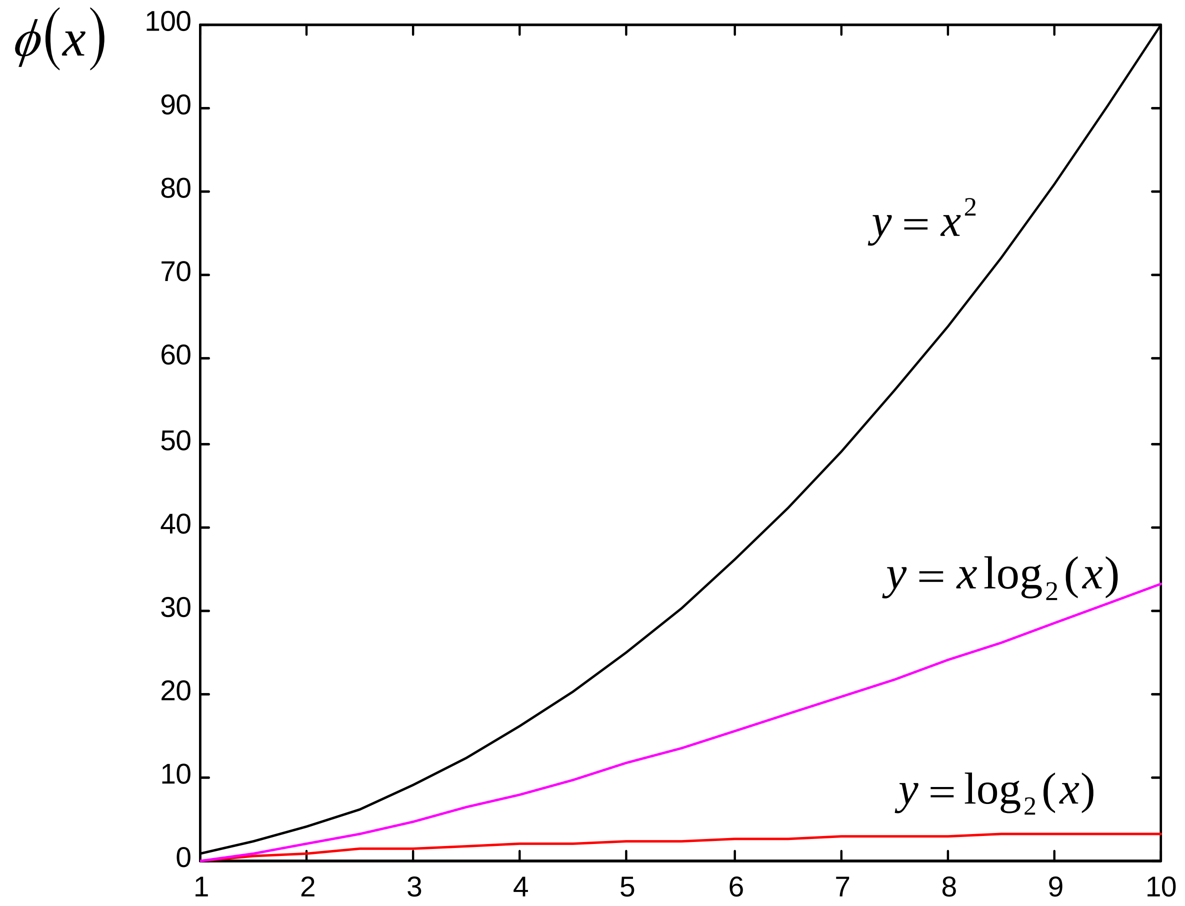
5. Maximum Entropy and Information Priors

$$\phi(x) = -\sum_{pixels} x \log_2 x$$

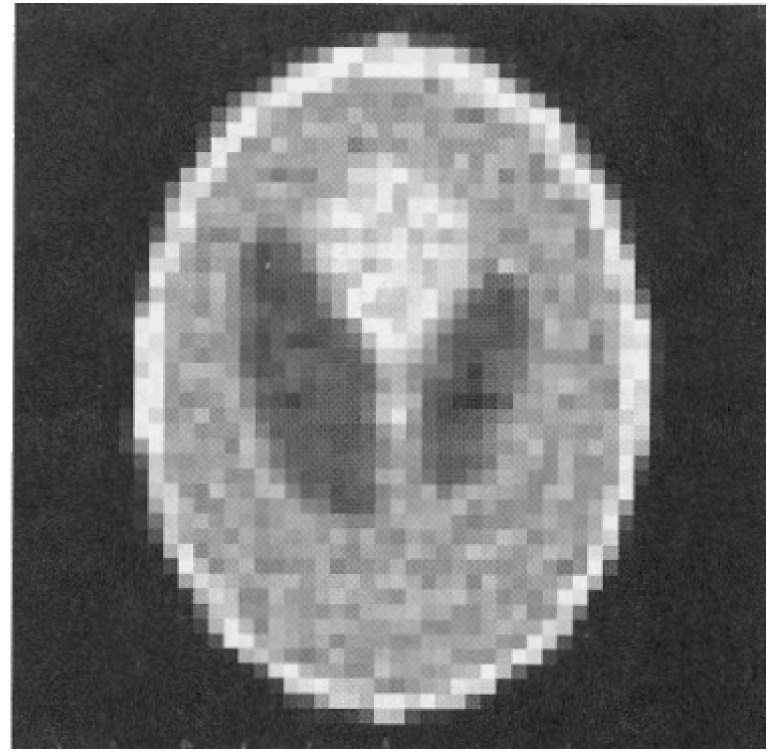
Frieden, 1978

$$\phi(x) = -\sum_{pixels} \log_2 x$$

Burg, 1978



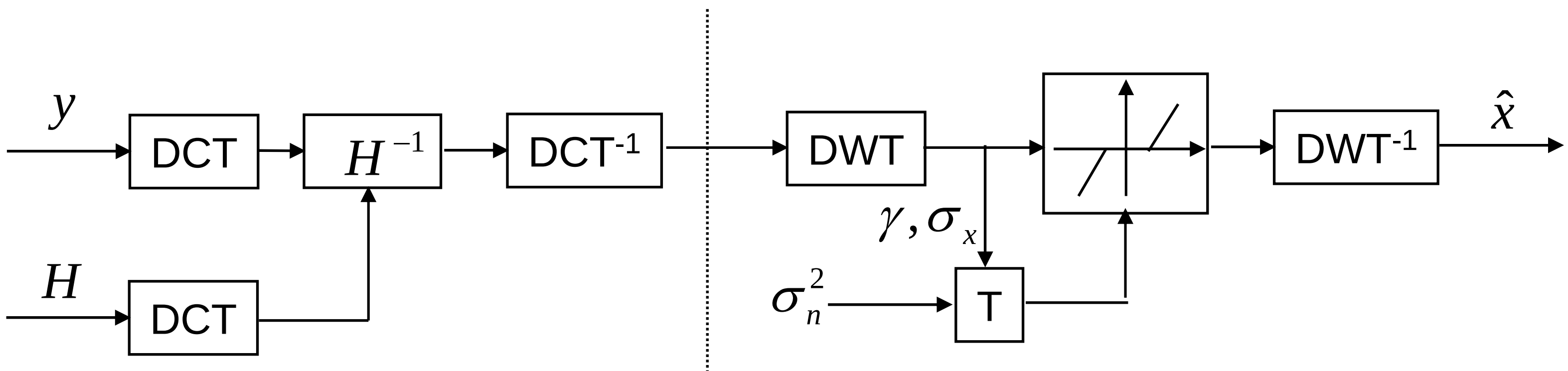
5. Maximum Entropy Restoration



$$\hat{x}_{ME} = \arg\min_{\hat{x} \in \aleph} \left[\left\| y - Hx \right\|_2^2 + \lambda \sum_{i=1}^{N} x_i \log(x_i) \right]$$

6. Thresholding Estimators for Restoration

Separated Restoration and Denoising



Restoration:

Inversion of blur assuming nonull coefficients

Denoising:

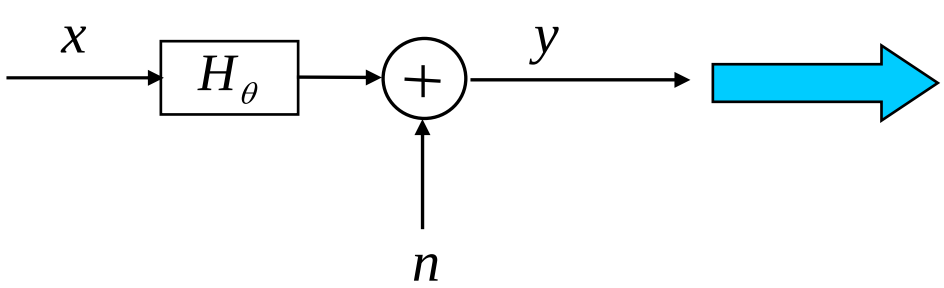
via thresholding

J. Kalifa and S. Mallat, Bayesian inference in wavelet based methods, chapter Minimax restoration and deconvolution, Springer, 1999.

7. Blind Deconvolution

- Blind deconvolution is applied in the situation when the type of blur or some blur parameters are unknown.
- Assume that the blur filter H depends on some unknown parameter θ.
  - Gaussian blur: θ =standard deviation of Gaussian
  - motion blur: θ= motion vector.
  - Defocusing: θ = radius of defocusing in pill-box model.

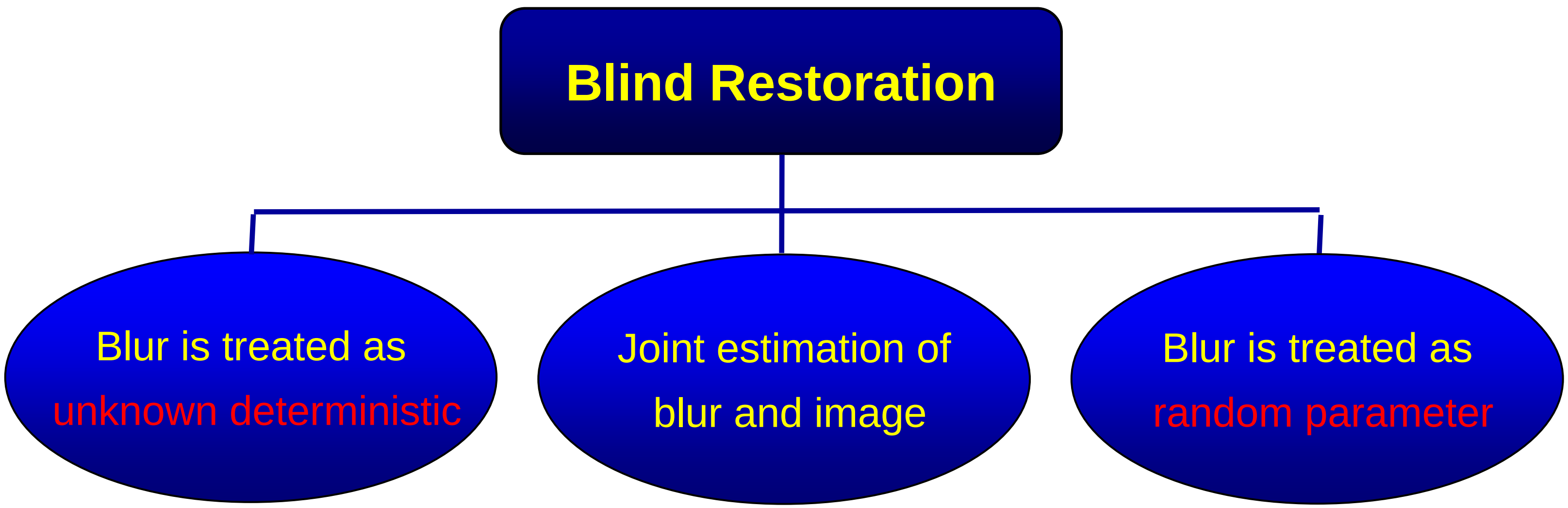
The degradation model becomes:



It is necessary to estimate both θ and the image x.

7. Blind Deconvolution: Stochastic Framework

$$x \longrightarrow \boxed{p\left(y \middle| x,\theta\right)} \longrightarrow y$$



Plug-in-methods

Bayesian estimation

7. Blind Deconvolution: Deterministic Blur

If θ is treated as an unknown deterministic parameter, then one can first estimate θ using the method of maximum likelihood (ML):

$$\boxed{\hat{\theta}_{ML} = \arg\max_{\theta} p(y|\theta) = \arg\max_{\theta} \int p(y|x,\theta)p(x)dx}$$ marginalization

(also called “empirical Bayes” [Archer, 1995])

and then plug this estimate in a MAP or Bayesian criterion, as in the standard (nonblind) restoration problems.

Sometimes the plug-in estimate is not $\hat{\theta}_{ML}$, but some other estimate. Under regularity conditions, one may expect $\hat{\theta}_{ML}$ to be an accurate estimator of θ. In this situation, the plug-in methods works well.

7. Blind Deconvolution: Joint Estimation

Joint estimation of θ and x:

$$\boxed{\left(\hat{x}_{MAP}, \hat{\theta}_{MAP}\right) = \arg\max_{(x,\theta)} p\left(y \middle| x, \theta\right) p(x)}$$

Expectation Maximization (EM) algorithm has been successfully used for solving the above problem

[K.L. Lagendijk and K. Biemond, Iterative Identification and Restoration of Images, Kluwer, Boston, 1991, Ch.7.].

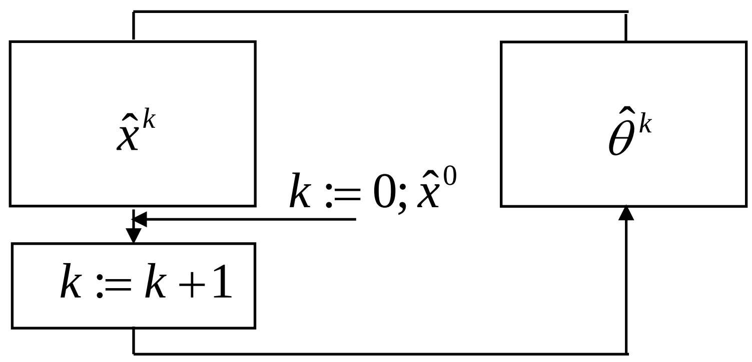
7. Blind Deconvolution: Bayesian Estimation

If a prior $p(\theta)$ is available, then the estimation problem may be formulated as a MAP estimation problem:

$$\boxed{\left(\hat{x}_{MAP}, \hat{\theta}_{MAP}\right) = \arg\max_{(x,\theta)} p\left(y \middle| x, \theta\right) p(x) p(\theta)}$$

where it is assumed that θ and x are independent.

It is also assumed that θ is a nuisance (not important) parameter.

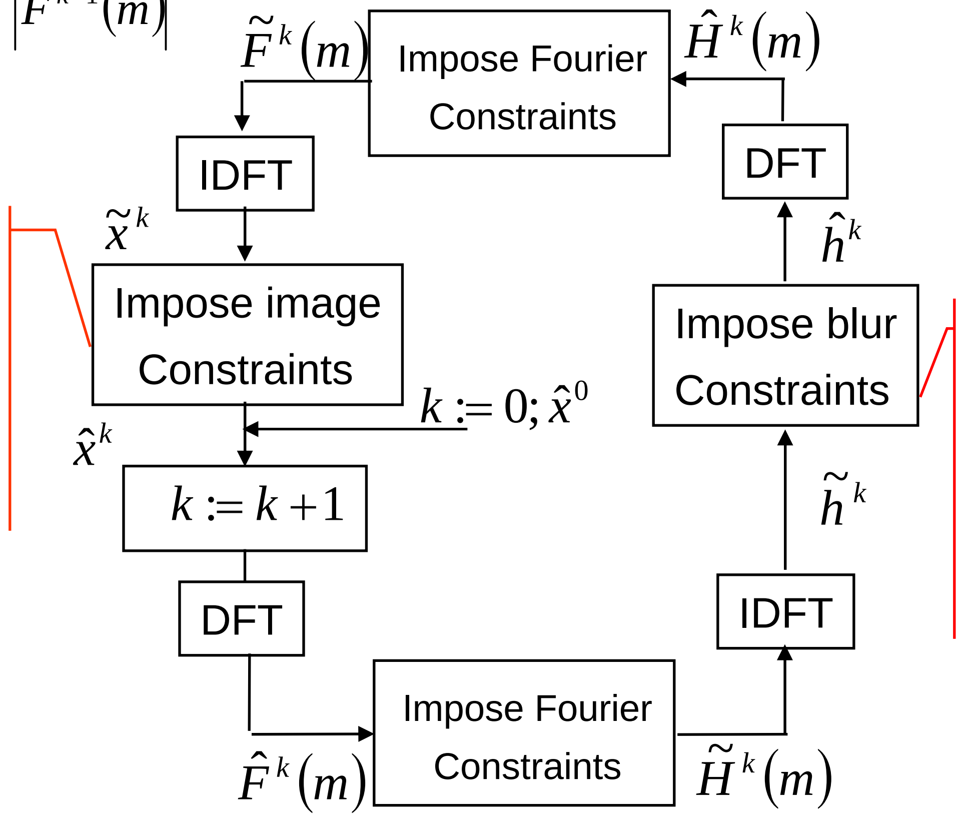


7. Blind Deconvolution: Iterative Constraint Minimization

$$\tilde{F}^{k}(m)=\frac{G(m)\hat{H}^{*k-1}(m)}{\left|\hat{H}^{k-1}(m)\right|^{2}+\alpha/\left|\hat{F}^{k-1}(m)\right|^{2}}$$

$$\tilde{H}^{k}(m)=\frac{G(m)\hat{F}^{*k-1}(m)}{\left|\hat{F}^{k-1}(m)\right|^{2}+\alpha/\left|\tilde{H}^{k-1}(m)\right|^{2}}$$

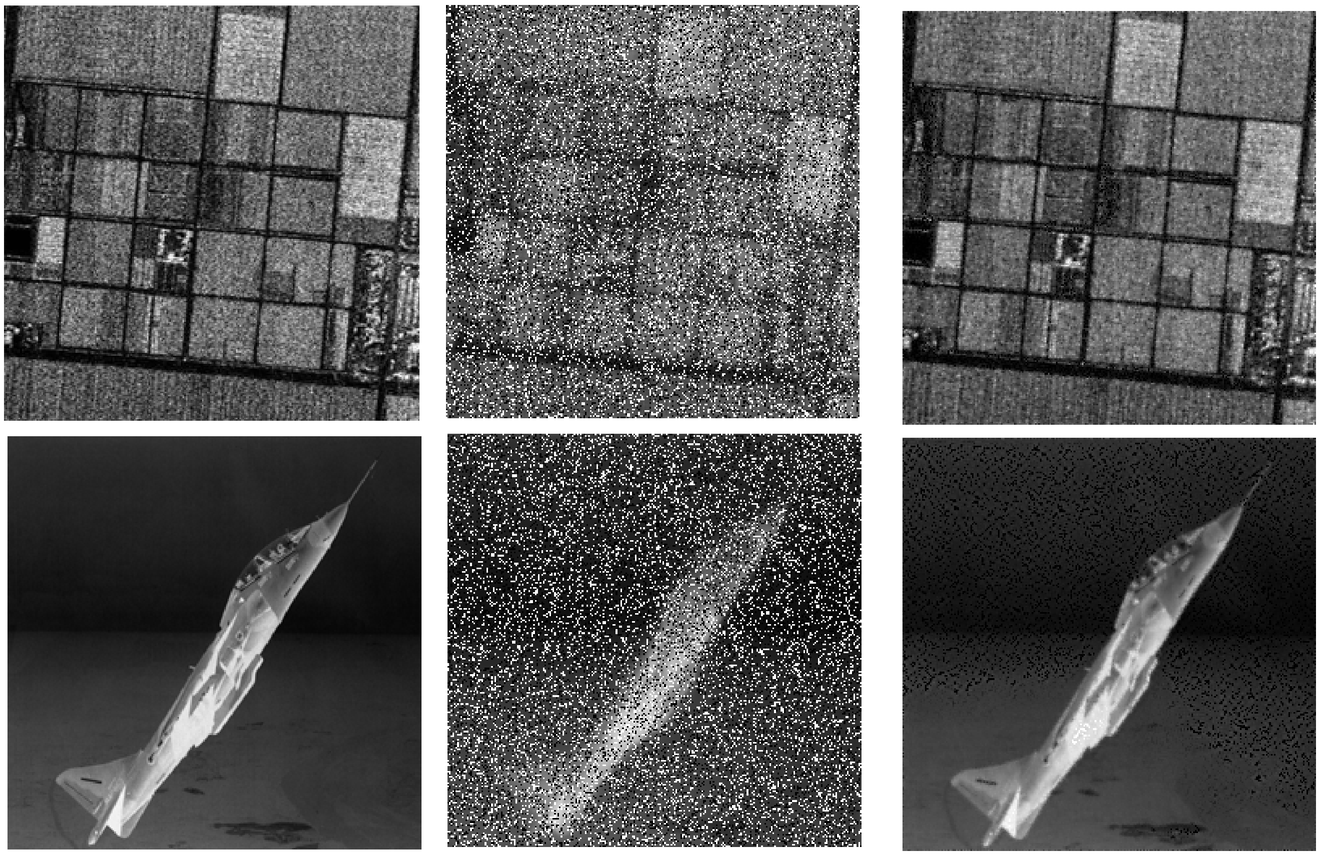
- nonnegativity
- finite support
- bounds on amplitude
- smoothness



- nonnegativity
- finite support
- bounds on amplitude
- smoothness
- shape of blur

D. Kundur, D. Hatizanakos, Blind Image Restoration, IEEE Sigan Proc. Magazine, May, 1997.

8. Robust Image Restoration



8. Robust Image Restoration

R=9

R=50

Defocused Image

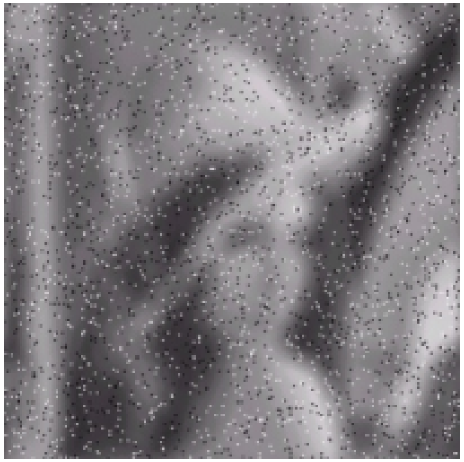
Restored Image,

Tikhonov Regularization

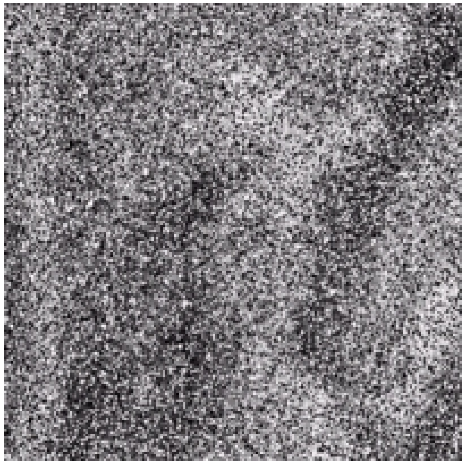
Penalized ML

8. Robust Image Restoration

5%



50%



Defocused Image, R=9

Restored Image,

Tikhonov Regularization

Penalized ML