1 INTRODUCTION

The purpose of this lab is to use GNURadio GUI to visualize the aliasing affects of digital signal sampling at various frequencies. The students will learn the full definition of the Nyquist-Shannon theorem and will be able to use a practical approach to determining when fold-back distortion occurs and how to prevent it.

1.1 Background

Distortion can occur when the constituent parts of a signal exceed the Nyquist-Shannon frequency and "fold-back" into the spectrum of interest. The solution to this is to over-sample the input signal well over the Nyquist-Shannon frequency, so that aliasing does not occur, or is at least mitigated to the point that it does not distort the signal to an appreciable degree.

1.2 Acronyms

Note in the remainder of this report we will have the following symbols:

$$f_a = alias\ frequency$$

$$f_{in} = input\ signal\ frequency$$

$$f_s = sampling\ frequency$$

$$N = any\ non\ zero\ integer$$

$$f_{out} = output\ signal\ frequency$$

$$FT = Fourier\ transfrom$$

2 PROCEDURE

2.1 Sinusoidal Wave

Students started the GNUCompanion application and executed Jay Patel's file Exercise_Sampling_and_Aliasing.grc. Upon executing this file the students had a GUI with which they could visualize 3 different types of waves: Cosine, Square and Triangle. Students were able to toggle the input signal frequency and the sampling frequency. For part 1, students first set f_s = 10kHz and sampled the sinusoidal wave to observe its frequency spectrum and waveform in the time domain. Students then adjusted the input frequency from 1 to 20kHz and observed the signal changes. Following this students documented the output frequency for odd input frequencies from 1kHz to 19kHz, then increased the sampling frequency to 40kHz and documented again.

2.2 Square Wave

Students then exited the waveform GUI and adjusted the offset from a default of 0 to -0.5 (for ease of visualizing the V_{p-p} of the waveform). Then students executed the same file mentioned in the last section and selected the "Sqaure Wave" option. After selection this option, students set values f_{in} = 1kHz and f_s = 20kHz to observe the spectrum. Students then set the f_{in} = 2kHz, f_{in} = 3kHz and observed for aliasing. Following this, students set f_s = 10kHz and repeated.

2.3 Triangle Wave

Students then changed the waveform to "Triangle Wave" and repeated the procedure for in the **Square Wave** section.

3 RESULTS & DISCUSSION

3.1 Question 1

As demonstrated in Fig. 1, the input signal has a frequency of 1kHz, highlighted in the red box. The sampled signal can be seen in the time domain to have a frequency of $\frac{1}{T} = \frac{1}{1ms} = 1kHz$, and in the frequency domain the familiar cosine Fourier Transform can be seen with the two impulses at the ±1kHz points.

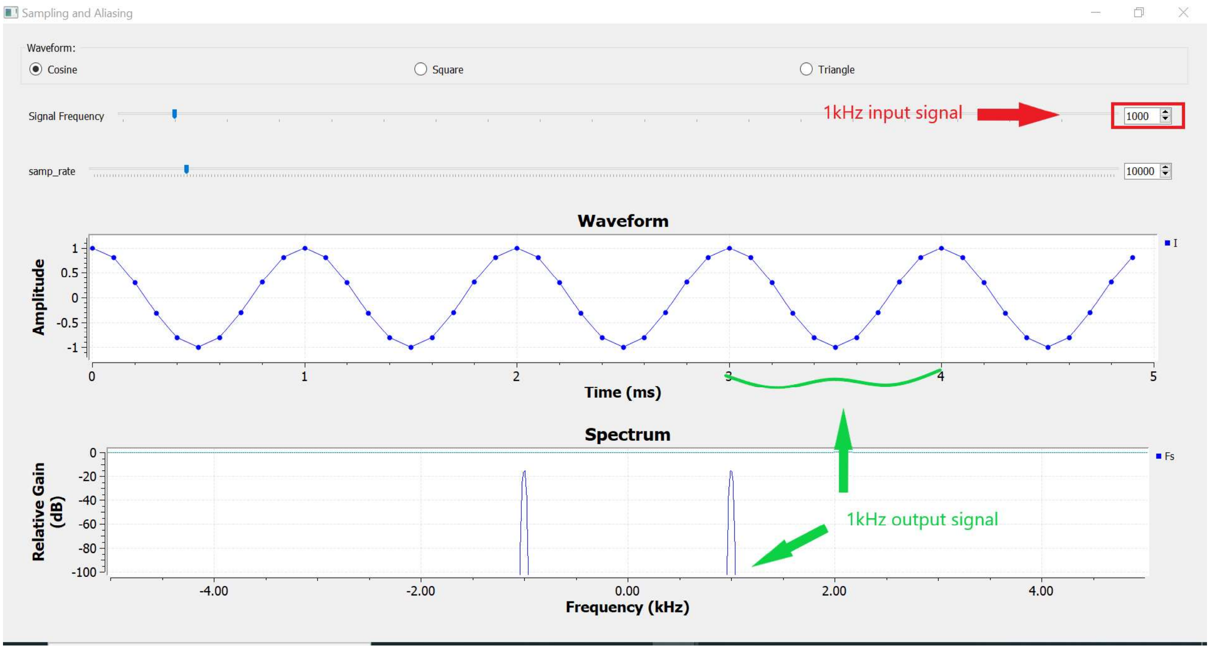

Figure 1: f_s = 10kHz, f_{in} = 1kHz, showing the input signal frequency matches the output signal frequency

3.2 Question 2

Determining the aliasing frequency is quite simple when we know f_s and f_{in}. The aliasing frequency is simply the absolute value of the input frequency subtracting any integer multiple of the sampling frequency, shown in equation 1 below:

$$f_a(N) = |f_{in} - N f_s| \tag{1}$$

Using code from Jay Patel, modified to show the 1kHz signal in blue, the 19kHz signal in red and the green points indicating the points where samples were taken at the given sample rate, the code snippet below produced Fig. 4:

```
# and plot it - like in Excel !
plt.plot(x,y,'b', label='1kHz signal')
plt.plot(x, y_2, 'b', label = '19kHz signal', color='r')
# add a point for the every tenth element, demonstrating sample frequency
for i in range(1,100,1):
  plt.plot(x[10*i],y[10*i], 'go')
plt.plot(x[10],y[10], 'go', label="Sample points")
#plt.text(x[432]+0.1, y[432], str(y[432]))
plt.legend(); plt.grid()
```

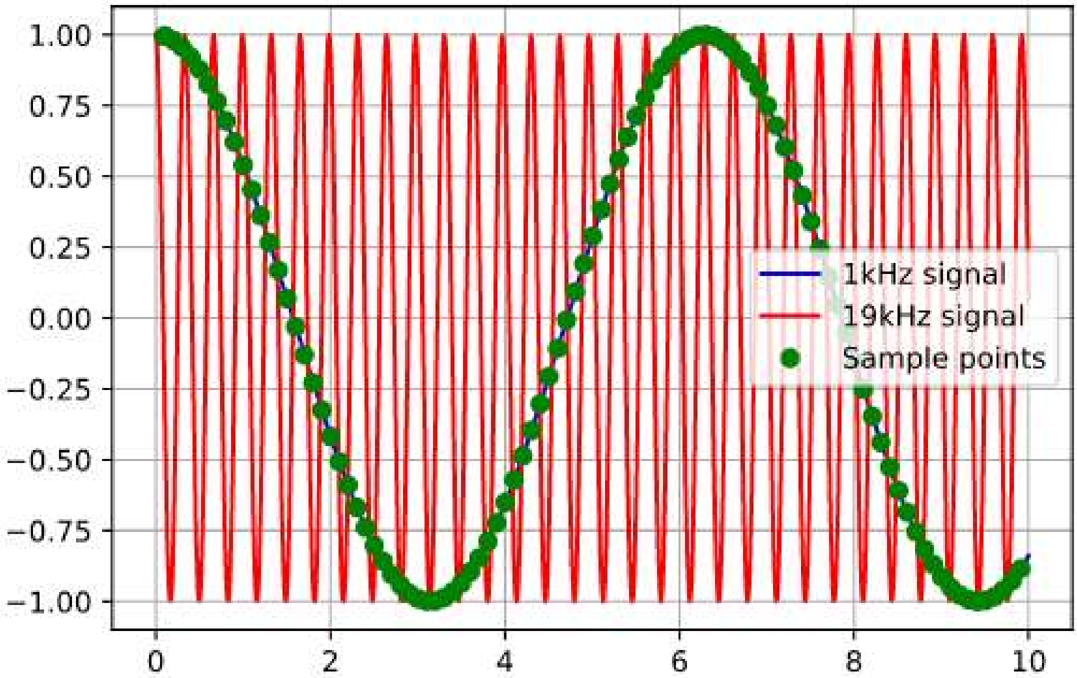

Figure 2: f_s = 10kHz, f_{in} = 19kHz, showing why aliasing frequency creates a signal similar to signal at f_{in} = 1kHz

What is observed in Fig. 4 can be explained mathematically by:

$$f_a(2) = |19kHz - 2(10kHz)| = |19kHz - 20kHz| = 1kHz \tag{2}$$

This is why the signal at 19kHz in Fig. 5 looks almost identical to the one at 1kHz in the GNURadio run in

Fig. 1.

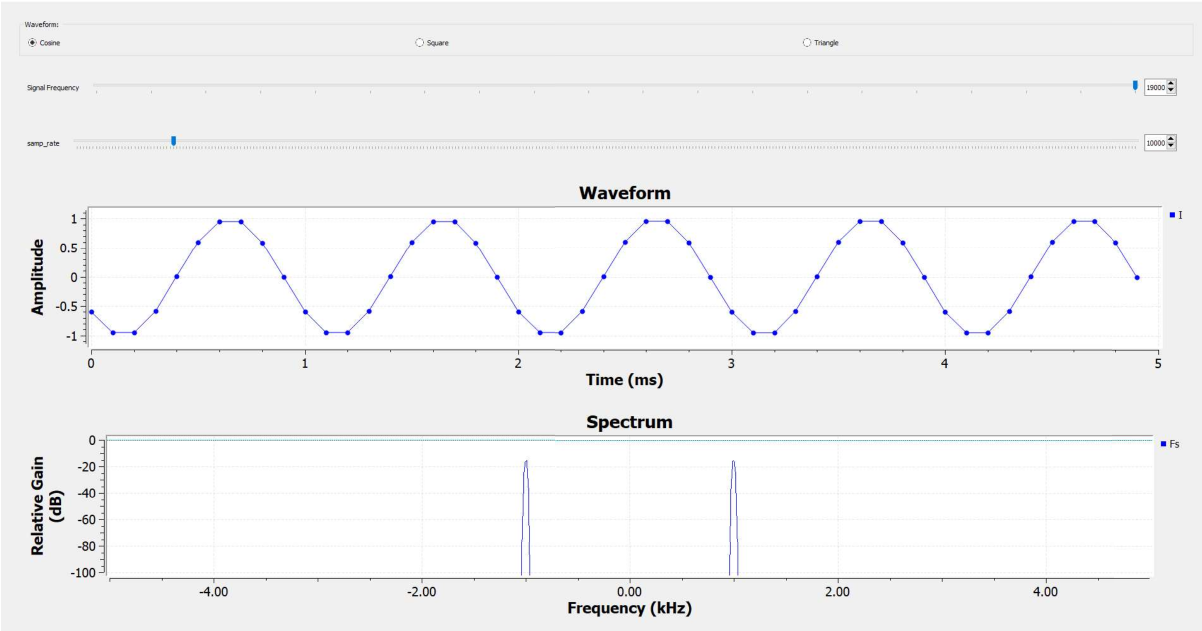

Figure 3: f_s = 10kHz, f_{in} = 19kHz, showing sample looks similar to signal at f_{in} = 1kHz

As the f_{in} varies from 1 to 19kHz, the waveform cycles through distorted patterns, to sharp triangle waveforms, to the 1kHz lower frequency signal again.

3.3 Question 3

Table 1: f_s = 10kHz, f_{in} varied from 1-19kHz at 2kHz steps, showing f_{out} for each step

f_{in} (kHz)	1	3	5	7	9	11	13	15	17	19
f_{out} (kHz)	1	3	5	3	1	1	3	5	3	1

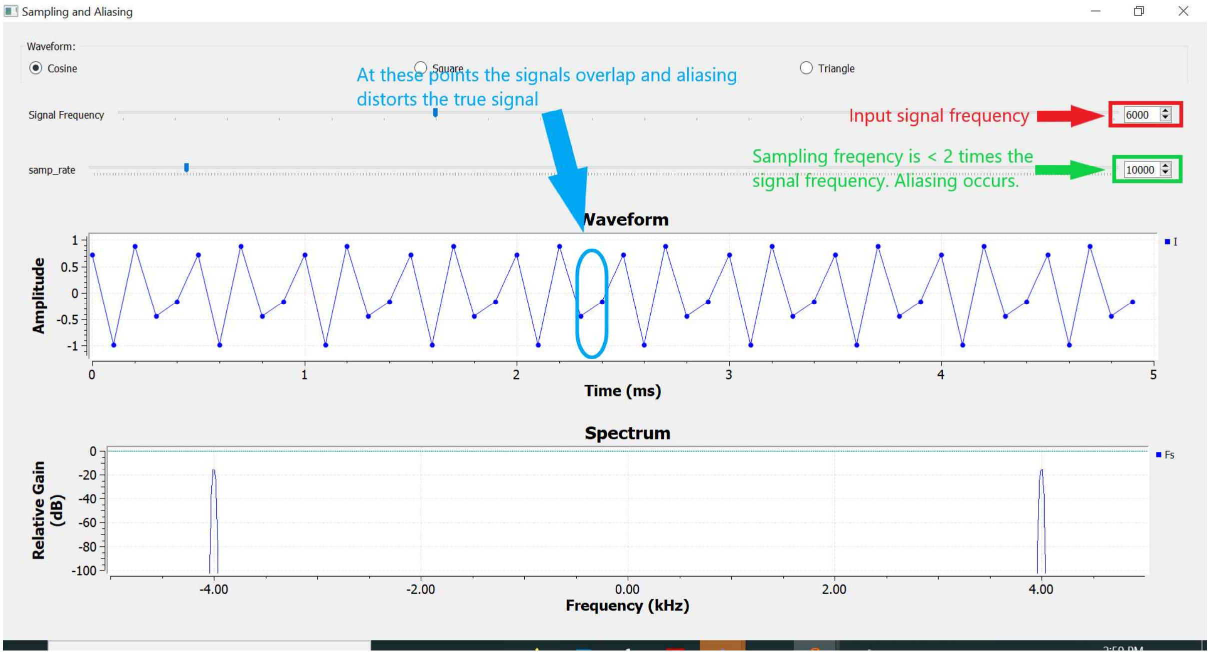

Figure 4: f_s = 10kHz, f_{in} = 6kHz, showing foldback effect

As observed in Table 1, the fold-back affect occurs at 7kHz (aliasing starts to happen at 6kHz, as seen in Fig. 3, but in 2kHz jumps, the value of interest is 7kHz). According to Table 1, fold-back occurs again at every $f + \frac{1}{T_s}$ interval. We have foldback at 7kHz, then that same foldback happens again at 7kHz + f_s = 7kHz + 10kHz = 17kHz, The cycle repeats at 9 and 19 kHz as well. We can see that for every value of output frequency, there is an identical frequency (or rather an aliased frequency that is indistinguishable) 10kHz down the table.

3.4 Question 4

Table 2: f_s = 40kHz, f_{in} varied from 1-19kHz at 2kHz steps, showing f_{out} for each step

f_{in} (kHz)	1	3	5	7	9	11	13	15	17	19
f_{out} (kHz)	1	3	5	7	9	11	13	15	17	19

There does not appear to be a fold-back affect occurring at f_s = 40kHz, because the maximum f_{in} value is 19kHz, which is less than half the Nyquist rate.

3.5 Question 5

Comparing step 3 and 4, it is demonstrated that the foldback effect occurs only when the $f_s < 2\times$ the max harmonic frequency present in the signal.

3.6 Question 6

Square waves comprise the odd harmonics of a sinusoidal wave that is of the same frequency of the square wave. Each of the harmonics should have an amplitude of $\frac{1}{k}$, where k= ±1 , ±3 , ±5 , etc.... For the square wave seen in Fig. 5, the frequency is 20kHz and the harmonics can be seen at ± 1 kHz,± 3 kHz,± 5kHz, ± 7kHz, and ± 9kHz. As labeled in Fig.6, each of the peaks in the FT are the amplitude of the first harmonic's peak divided by their respective k value. For example, let's look at the ± 5 kHz peak (k = 5) in Fig. 6 below. The peak of the ± 1 kHz peak is roughly -20dB, and the peak of the 5th harmonic peak is -32.45dB. With those values we can calculate to verify that the peak of the ± 5kHz peak does in fact equal $\frac{1}{5}$ the of the 1st harmonic peak:

$$gain_{1kHz} = 10^{(-20dB/20)} = 0.1$$

$$gain_{5kHz} = 10^{(-32.46dB/20)} = 0.023 \simeq \frac{0.1}{5} = 0.02$$

It is verified that the peak is $\frac{1}{5}$, as expected. The rest of the values of the peaks are shown in Table 3.

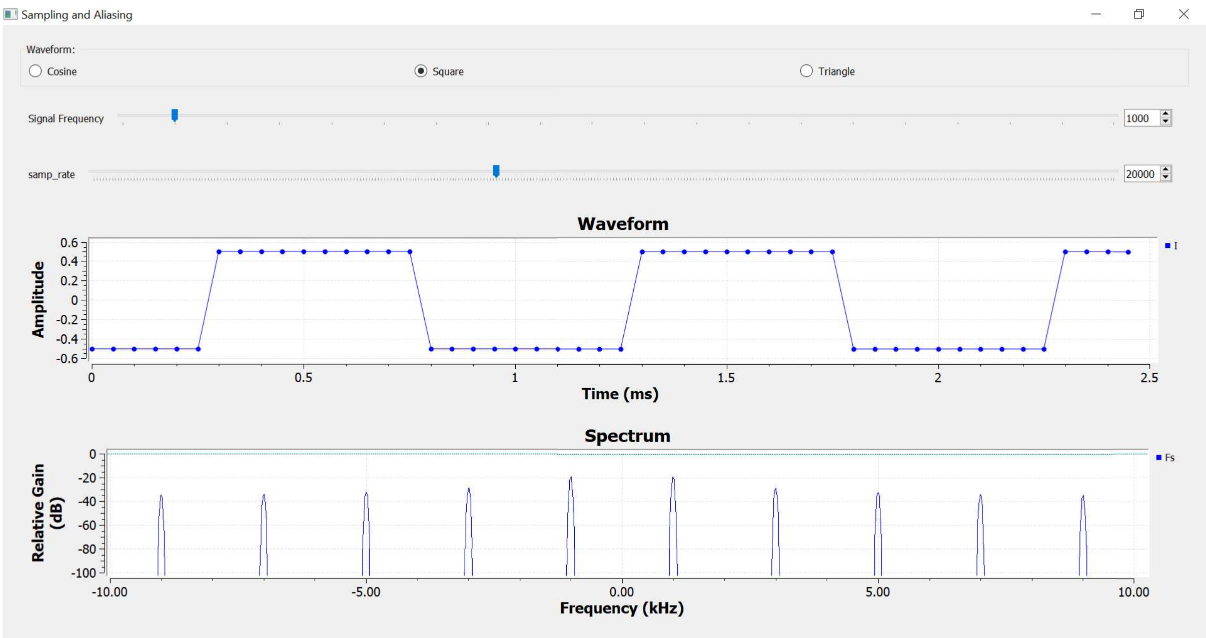

Figure 5: f_s = 20kHz, f_{in} = 1kHz , showing odd harmonics in FT of square wave

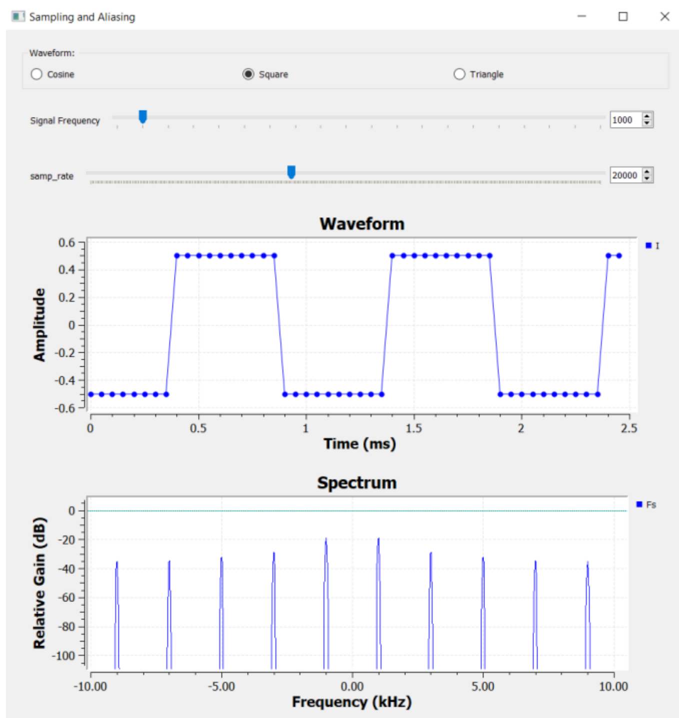

Figure 1: 1kHz 50% duty cycle square wave of 20kHz

At input frequency 1kHz and sampling rate 20kHz, in the frequency domain we see five different pairs of spikes at frequencies of +/- 1, +/- 3, +/- 5, +/- 7, +/- 9 kHz. The 50% square wave is ideal which means we only get odd harmonics and since we have 1kHz at the input frequency. Aliasing will occur if there are any frequency spikes at other frequencies, in this case we don’t observe any aliasing since all five spike pairs occur at the odd harmoncs. We can deduce the following equation for the harmonics:

$$nth\ harmonic\ frequency = n * 1kHz$$

Therefore, our harmonics are as follows:

$$1st\ harmonic = 1kHz$$
$$3rd\ harmonic = 3kHz$$
$$5th\ harmonic = 5kHz$$
$$7th\ harmonic = 7kHz$$
$$9th\ harmonic = 9kHz$$

7. **Keeping the sampling frequency at 20 kHz, use 2 kHz first & 3 kHz next as the frequency of the input signal. Can you find any aliasing effect on the spectrum of the output signal? If your answer is “Yes”, please explain how the spectrum aliasing happen for each input signal, 2 kHz and 3 kHz. If your answer is “No”, please give a reason and comment on your observations.**

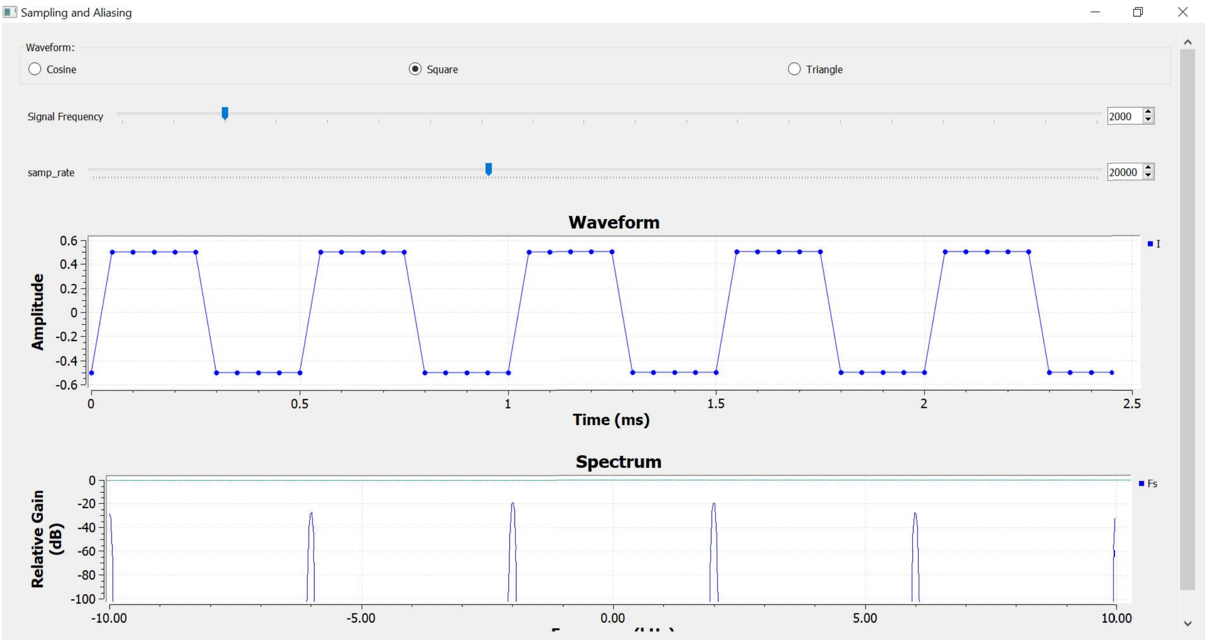

Figure 7: f_s = 20kHz, f_{in} = 2kHz , showing lack of aliasing in square wave spectrum

The 3kHz signal however, does appear to aliasing. The peaks one would expect in a 3kHz square wave spectrum are 3kHz(2k+1), where k = 0, ±1 , ±3 , ±5, etc..., but we have additional peaks here, highlighted with red dots in Fig. 8. These peaks are at frequencies ± 1 kHz,± 5 kHz, and± 7kHz.

Figure 8: f_s = 20kHz, f_{in} = 3kHz , showing aliasing in square wave spectrum

At 3kHz input frequency and 20kHz sampling rate we see five different pairs of spikes at frequencies of +/- 1, +/- 3, +/- 5, +/- 7, +/- 9 kHz. Since the input frequency is 3kHz we can deduce the following formula:

$$nth\ harmonic\ frequency = n * 3kHz$$

The harmonics are as follows:

$$1st\ harmonic = 3kHz$$
$$3rd\ harmonic = 9kHz$$
$$5th\ harmonic = 15kHz$$
$$7th\ harmonic = 21kHz$$
$$9th\ harmonic = 27kHz$$

We observe aliasing at +/- 1,+/-5 and +/-7kHz frequencies since those pairs of spikes do not correspond to any odd harmonic position listed above.

8. **Change the sampling frequency to 10 kHz, repeat step6 with the same 1 kHz, 50% square wave, pay attention to the behavior of the spikes and their frequencies in the spectrum of the sampled signal.**

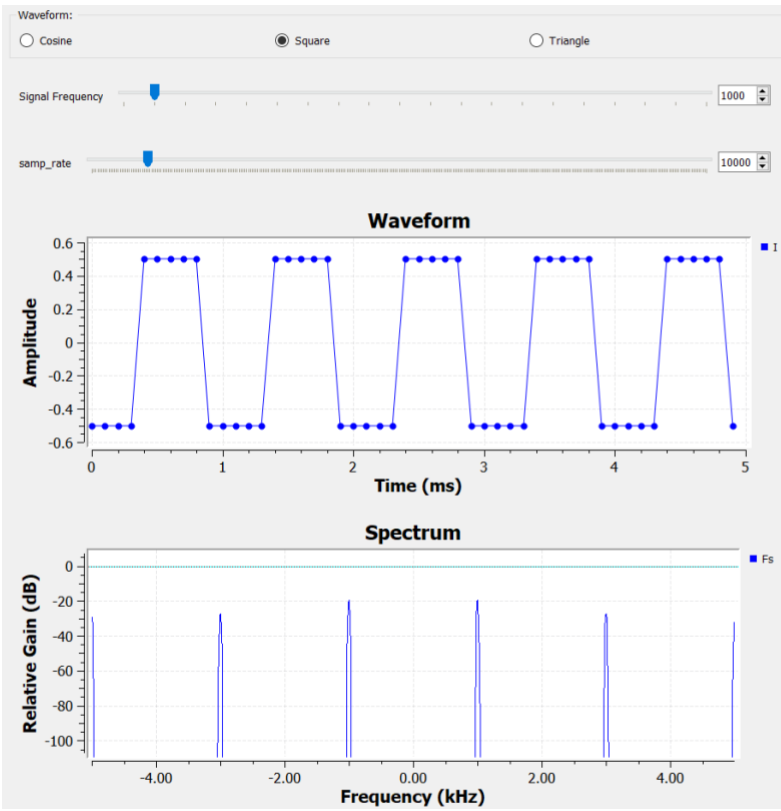

Figure 4: 1kHz 50% duty cycle square wave of 10kHz

At input frequency 1kHz and sampling rate 10kHz, in the frequency domain we see three different pairs of spikes at frequencies of +/- 1, +/- 3, +/- 5 kHz. The 50% square wave is ideal which means we only get odd harmonics and since we have 1kHz at the input frequency. Aliasing will occur if there are any frequency spikes at other frequencies, in this case we don’t

3.8 Question 8

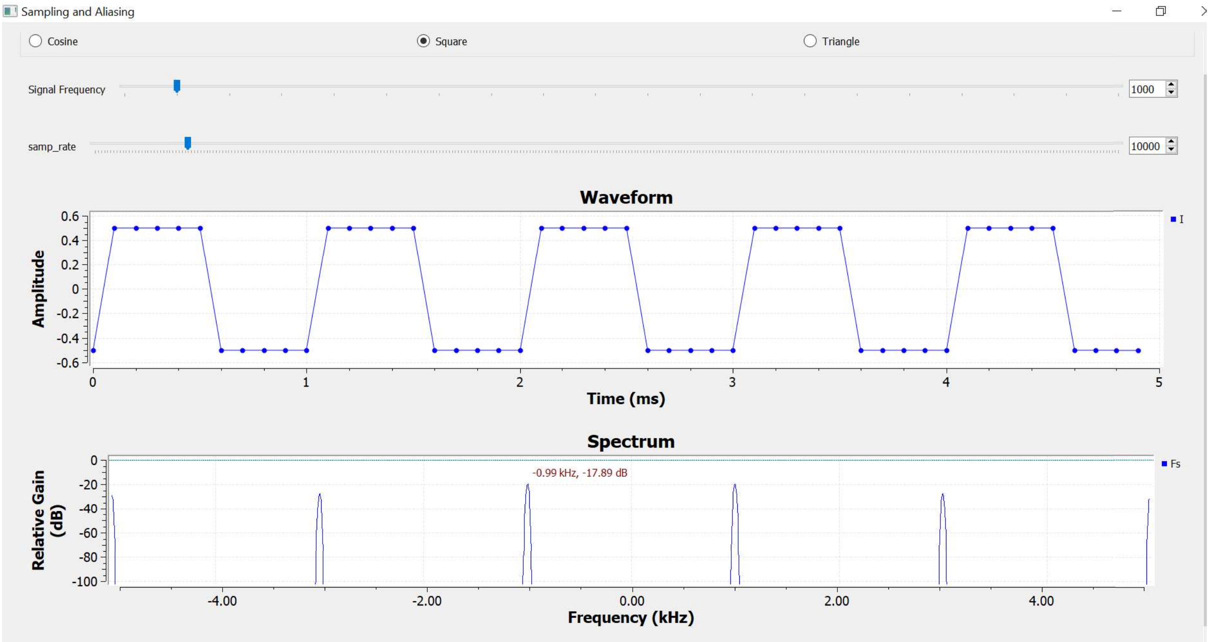

Figure 9: f_s = 10kHz, f_{in} = 1kHz , showing square wave spectrum

The square wave sampled using f_s = 10kHz with f_{in} = 1kHz appears to have all the peaks in the correct frequencies, but the bandwidth seems to have narrowed, cutting off at ± 5kHz in Fig. 9, unlike the 10kHz cutoff seen if Fig. 6, which was sampled using f_s = 20kHz.

observe any aliasing since all three spike pairs occur at the odd harmoncs. We can deduce the following equation for the harmonics:

$$nth\ harmonic\ frequency = n * 1kHz$$

Therefore, our harmonics are as follows:

$$1st\ harmonic = 1kHz$$
$$3rd\ harmonic = 3kHz$$
$$5th\ harmonic = 5kHz$$
$$7th\ harmonic = 7kHz$$

9. **Keeping the sampling frequency at 10 kHz, use 2 kHz first & 3 kHz next as the frequency of the signal. Can you find any aliasing effect on the spectrum of the sampled signal? If your answer is "Yes", please explain how the spectrum aliasing happens for each input signal, 2 kHz and 3 kHz. If your answer is "No", please give a reason and comment on your observations.**

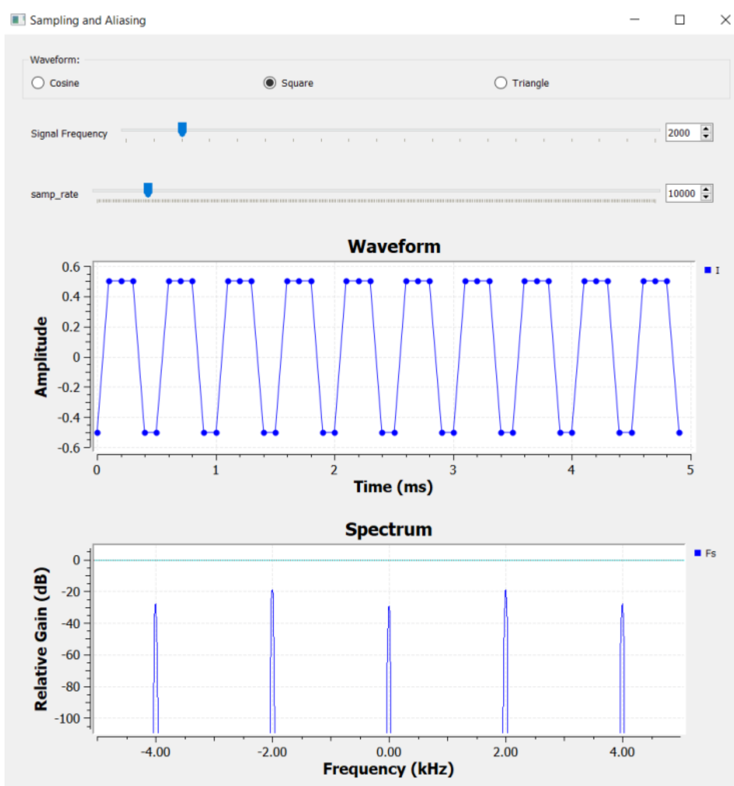

Figure 5: 2kHz 50% duty cycle square wave of 10kHz

At input frequency 2kHz and sampling rate 10kHz, in the frequency domain we see two different pairs of spikes at frequencies of +/- 2, +/- 4kHz and a single spike at 0kHz. The 50% square wave is ideal which means we only get odd harmonics and since we have 2kHz at the input frequency we can deduce the following equation for the harmonics. We observe aliasing at 0kHz and at +/- 4kHz since those peaks don't correspond to any odd harmonic.

$$nth\ harmonic\ frequency = n * 2kHz$$

Therefore, our harmonics are as follows:

$$1st\ harmonic = 2kHz$$
$$3rd\ harmonic = 5kHz$$
$$5th\ harmonic = 10kHz$$

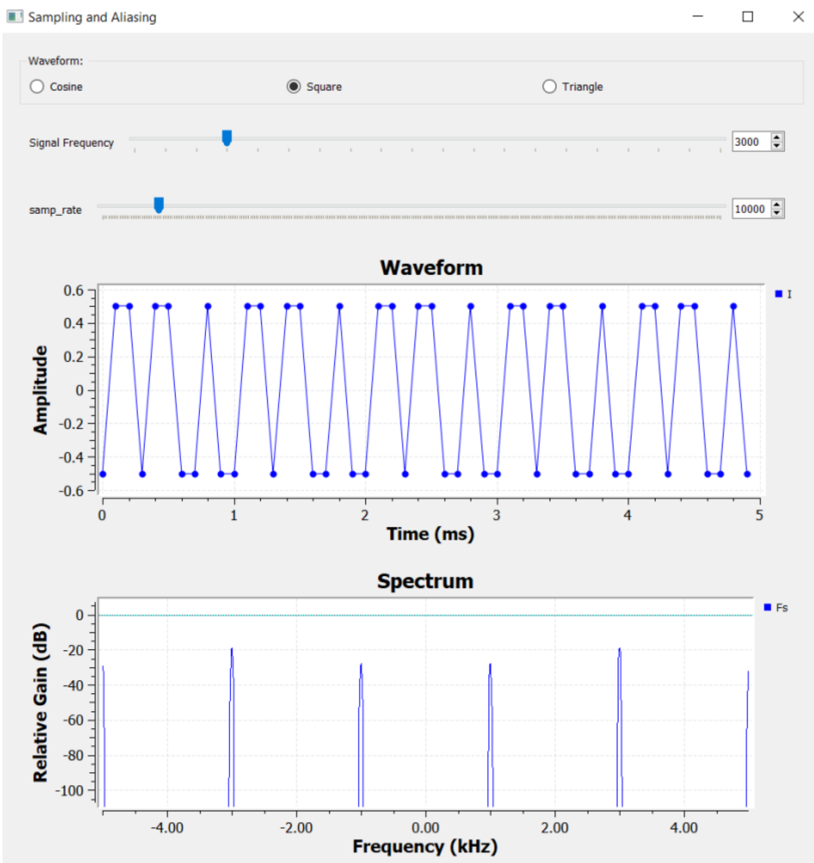

Figure 6: 3kHz 50% duty cycle square wave of 10kHz

At input frequency 3kHz and sampling rate 10kHz, in the frequency domain we see three different pairs of spikes at frequencies +/- 1, +/- 3 and +/- 5kHz. The 50% square wave is ideal which means we only get odd harmonics and since we have 2kHz at the input frequency we can deduce the following equation for the harmonics.

$$nth\ harmonic\ frequency = n * 3kHz$$

Therefore, our harmonics are as follows:

$$1st\ harmonic = 3kHz$$
$$3rd\ harmonic = 9kHz$$
$$5th\ harmonic = 15kHz$$

We observe aliasing at frequencies +/- 1, and +/- 5kHz since those spikes don't correspond to any odd harmonic.

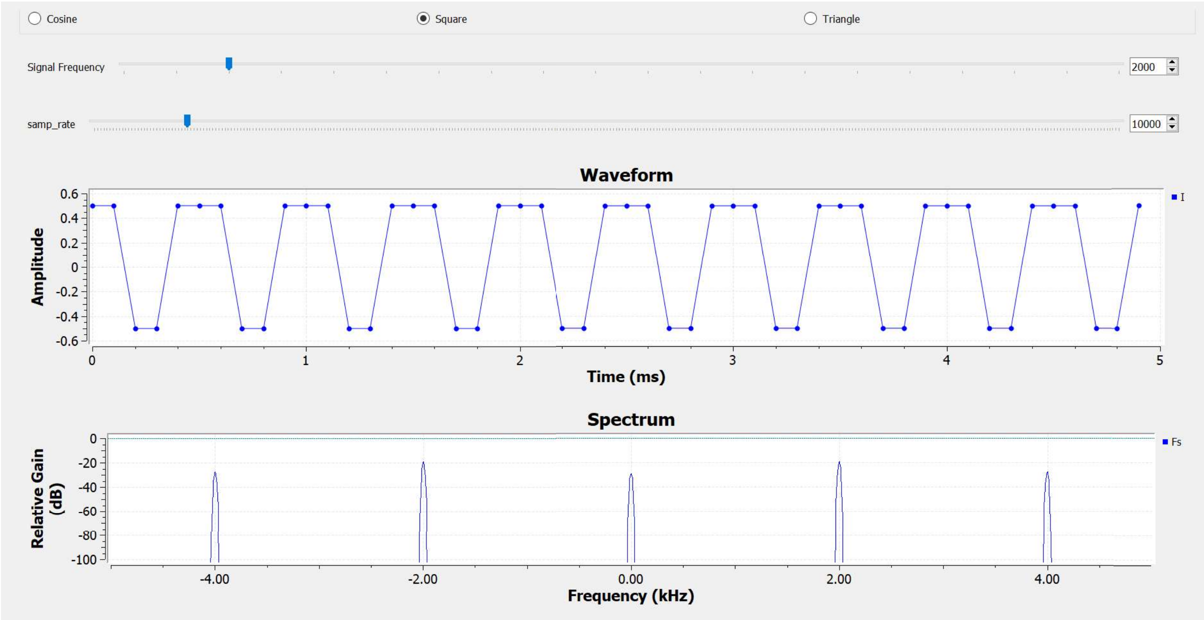

Figure 10: f_s = 10kHz, f_{in} = 2kHz , showing square wave spectrum

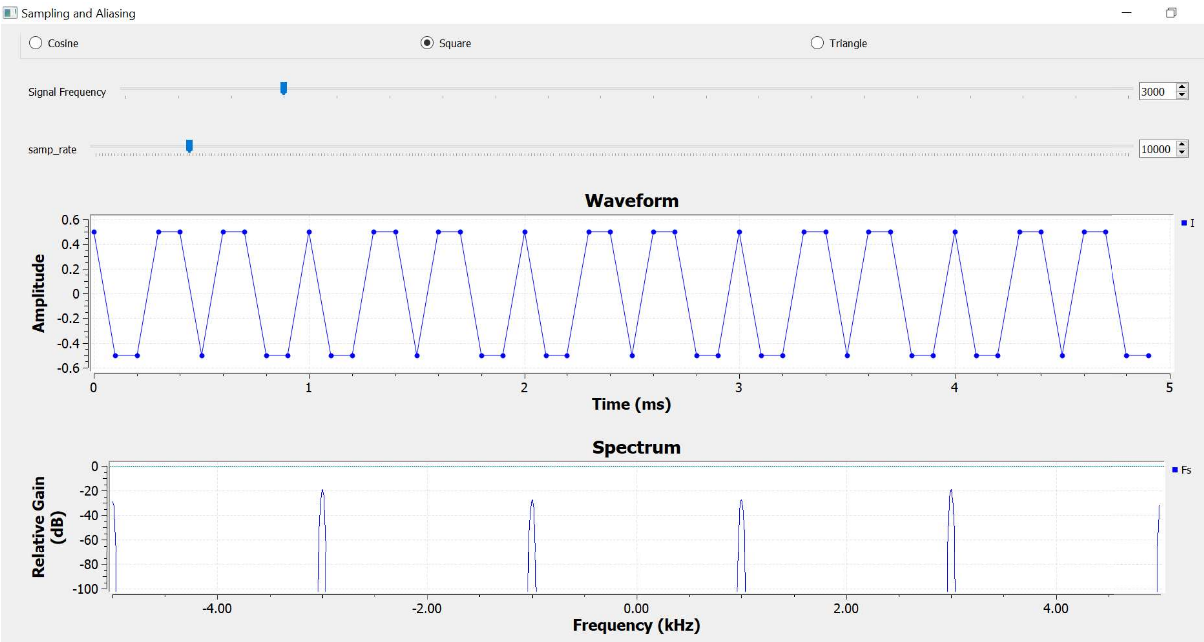

Figure 11: f_s = 10kHz, f_{in} = 3kHz , showing square wave spectrum

3.10 Question 10

The triangle wave is similar to the square wave in that it is represented by odd harmonics, but the amplitude of its harmonics are $\frac{1}{k^2}$ multiple of the first harmonic amplitude. Looking at the peaks in Fig. 12, it is clear that the peaks are not odd harmonics, and looking at Fig. 13, the peaks are on odd frequencies, but the amplitudes do not follow the $\frac{1}{k^2}$ theory. It is likely from these observations that there is aliasing present in both of these signals.

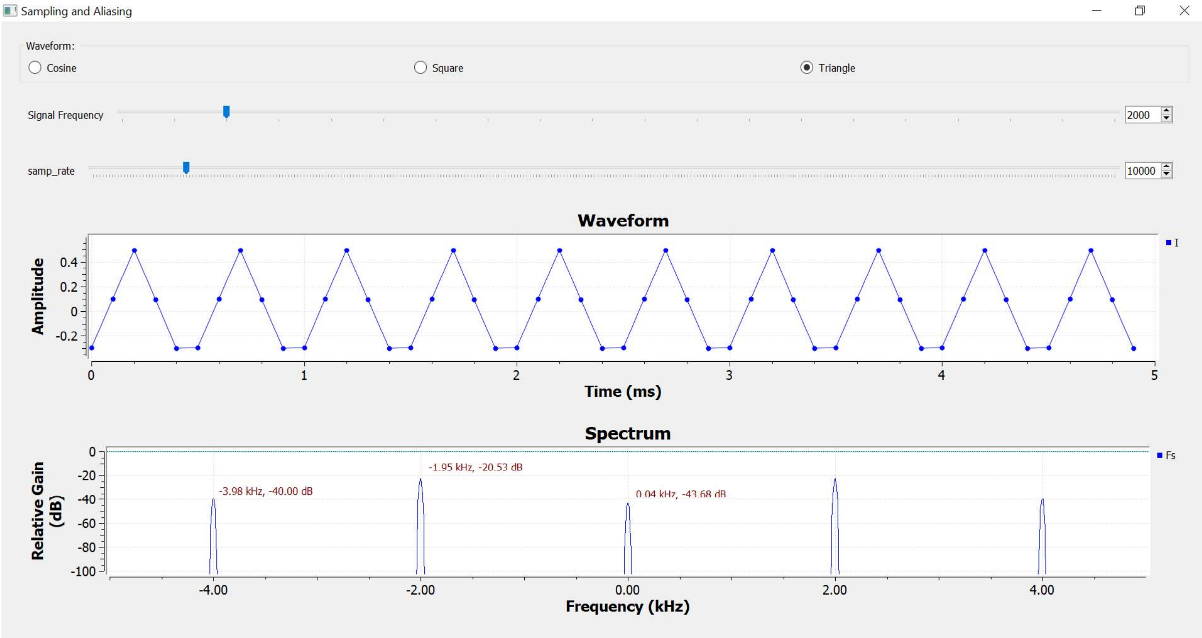

Figure 12: f_s = 10kHz, f_{in} = 2kHz , showing triangle wave spectrum

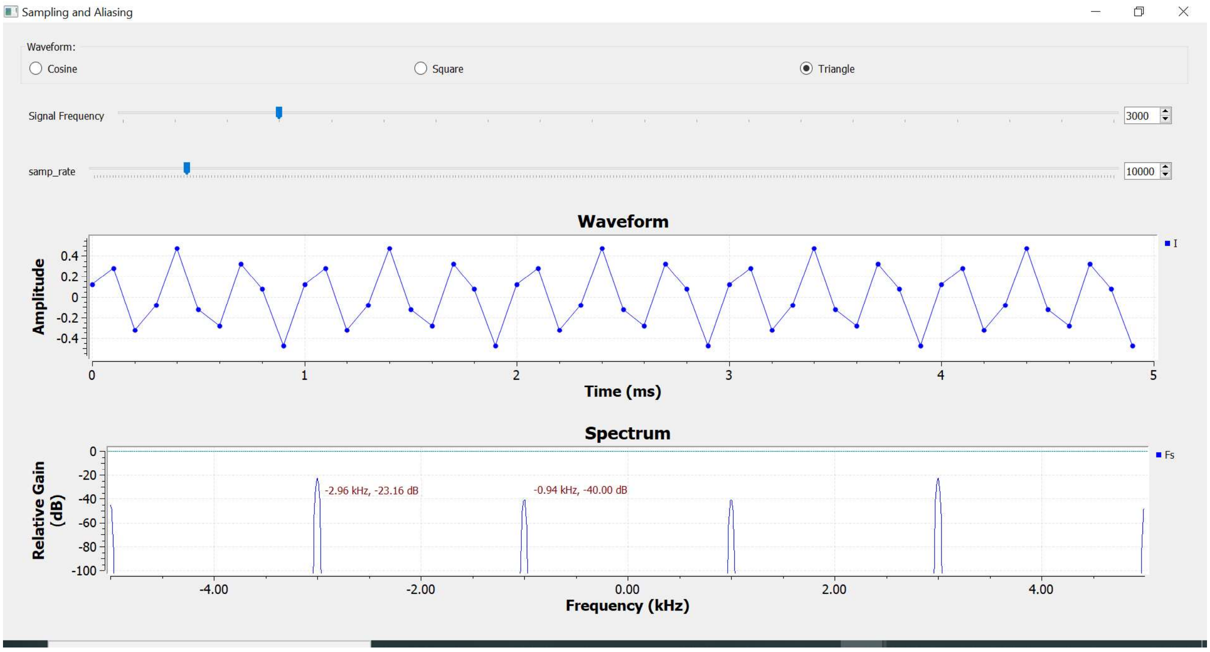

Figure 13: f_s = 10kHz, f_{in} = 3kHz , showing triangle wave spectrum

For a triangular wave with input frequency 2kHz and 10kHz sampling rate, we observe two pairs of spikes at frequencies +/- 2, +/- 4, and a single peak at 0kHz. The triangular wave will only have odd harmonics similar to the 50% square wave. Aliasing occurs if any spikes are present at any other frequencies. In the case of a 2kHz frequency, we have aliasing at 0kHz and +/- 4 kHz as those spikes don't correspond to any odd harmonic position.
As we have an input signal of 2kHz the harmonics should be at :

$$nth\ harmonic\ frequency = n * 2kHz$$

Therefore, our harmonics are as follows:

$$1st\ harmonic = 2kHz$$

$$3rd\ harmonic = 6kHz$$

$$5th\ harmonic = 10kHz$$

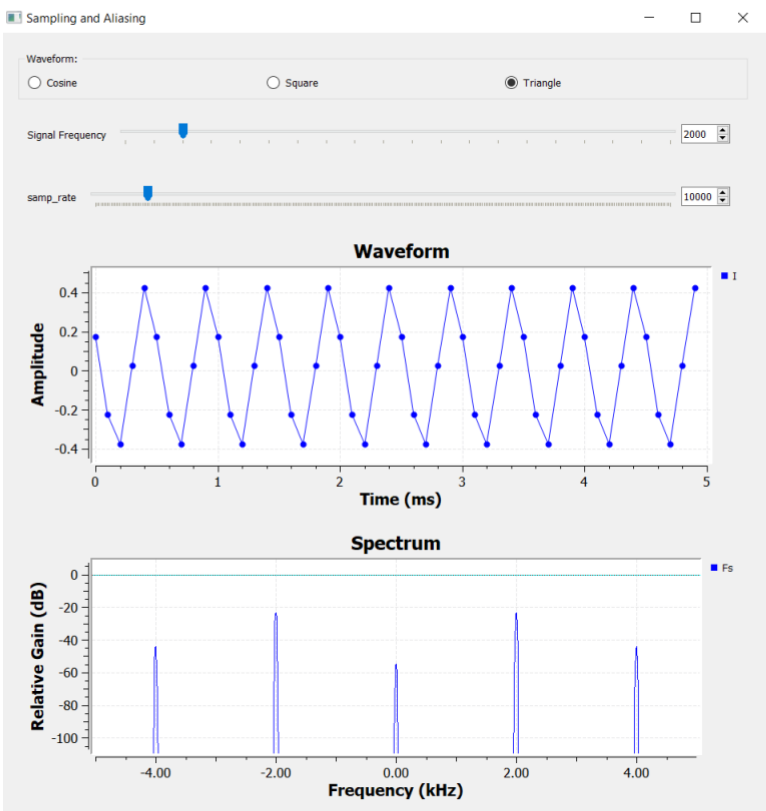

Figure 7: 2kHz triangle wave of 10kHz

For a triangular wave with input frequency of 3kHz and 10kHz sampling rate of 10kHz, we observe three pairs of spikes at frequencies +/- 1, +/- 3 and +/- 5kHz. Like the 50% square wave, the triangular wave will only have odd harmonics. Aliasing occurs if any spikes are present at any other frequencies. In the case of a 3kHz frequency, we observe aliasing at +/-1kHz and +/- 5 kHz, as those spikes don't correspond to an odd harmonic position.
As we have an input signal of 3kHz the harmonics should be at:

$$nth\ harmonic\ frequency = n * 3kHz$$

$$1st\ harmonic = 3kHz$$
$$3rd\ harmonic = 9kHz$$
$$5th\ harmonic = 15kHz$$

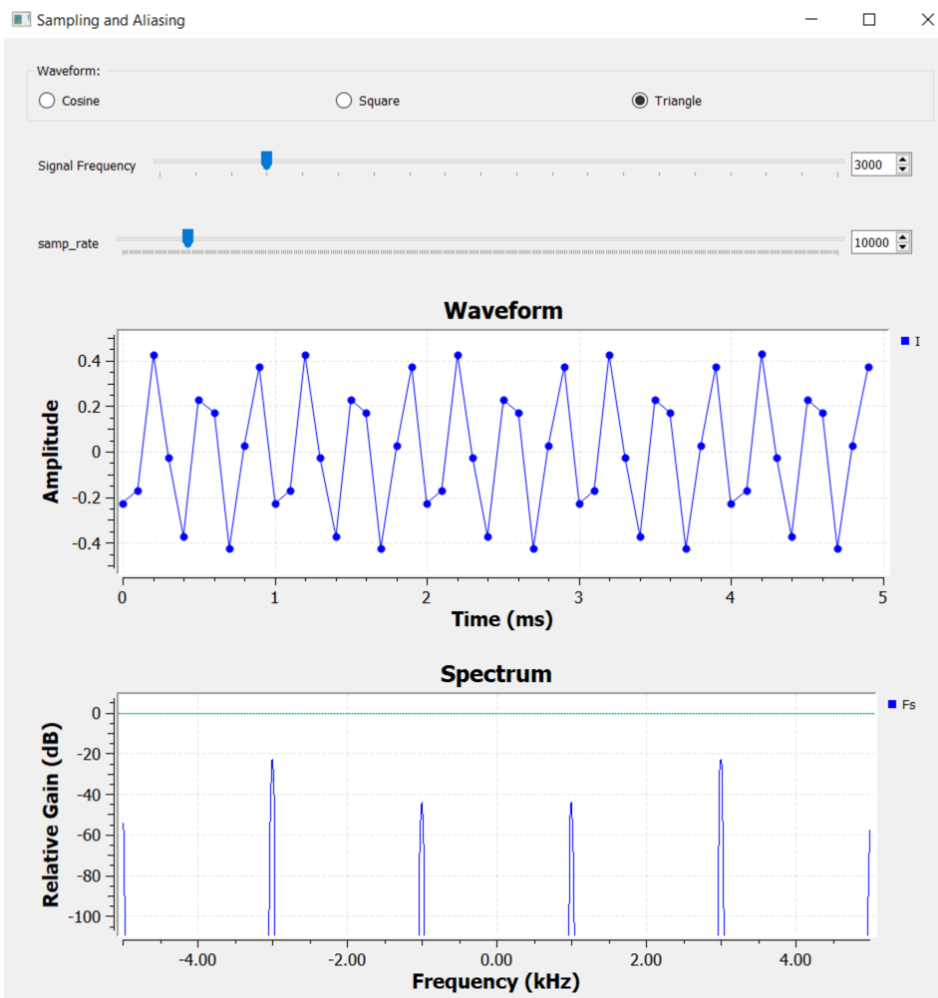

Figure 8:3kHz triangle wave of 10kHz

As we compare the 2-3kHz amplitude spectra of the triangular and square waves we observe that the frequency spikes occur at the same frequencies as well as aliasing however the amplitudes of the peaks are larger in the square waveforms than the triangular waveforms due to the Fourier series coefficients being larger in the square waveforms than the triangulat waveforms.