Continuation semantics i[1]

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1 Roadmap

1.1 Goals for the first block

- To be able to understand Barker & Shan's tower notation, and how to translate from towers to flat representations, and vice versa.

Add more goals

2 Some notation conventions

Generally speaking, I'll be assuming Heim & Kratzer 1998 as background, but I'll depart from their notation slightly.

Expressions in the meta-language will be typeset in sans serif.

$$[\![[_{\text{DP}}\ \text{John}]]\!] \coloneqq \underbrace{\mathsf{John}}_{\text{individual}}$$

Seeing as its primitive, we'll treat white-space as function application, e.g.:

(1) $(\lambda x\ .\ \mathsf{left}\ x)\ \mathsf{paul} = \mathsf{left}\ \mathsf{paul}$

Function application associates to the *left*:

(2) $(\lambda x\ .\ \lambda y\ .\ y\ \mathsf{likes}\ x)\ \mathsf{paul}\ \mathsf{sophie} \equiv ((\lambda x\ .\ \lambda y\ .\ y\ \mathsf{likes}\ x)\ \mathsf{paul})\ \mathsf{sophie}$

We'll write *types* in a `fixed width font`. We have our familiar primitive types...

(3) `type` ≔ `e` | `t` | `s` | ...

[1] 24.979: Topics in semantics

Getting high:

scope, projection, and evaluation order

...and of course *function types*. Unlike Heim & Kratzer (1998), who use ⟨.⟩ as the constructor for a function type, we'll be using the (more standard (outside of linguistics!)) arrow constructor (→):

(4) $\langle \mathtt{e}, \mathtt{t} \rangle \equiv \mathtt{e} \to \mathtt{t}$

The constructor for function types *associates to the right*:

(5) $\mathtt{e} \to \mathtt{e} \to \mathtt{t} \equiv \mathtt{e} \to (\mathtt{e} \to \mathtt{t})$

3 *The Partee triangle*

(6) $\mathsf{LIFT} x \coloneqq \lambda k \ . \ k \ x$ $\qquad \mathsf{LIFT} : \mathtt{e} \to (\mathtt{e} \to \mathtt{t}) \to \mathtt{t}$

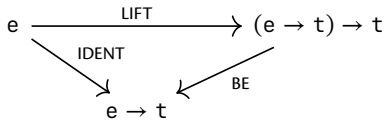
(7) $\mathsf{IDENT} \ x \coloneqq \lambda y \ . \ y = x$ $\qquad \mathsf{IDENT} : \mathtt{e} \to \mathtt{e} \to \mathtt{t}$

(8) $\mathsf{BE} \ Q \coloneqq \lambda x \ . \ Q \ (\lambda y \ . \ y = x)$ $\qquad \mathsf{BE} : ((\mathtt{e} \to \mathtt{t}) \to \mathtt{t}) \to \mathtt{e} \to \mathtt{t}$

Commutative diagrams

The *Partee triangle* is a *commutative diagram*. We say that a diagram *commutes* if, when there are multiple paths between two points, those paths are equivalent. The equivalence in (10) is therefore expressed by the triangle.

(9) The Partee triangle[2]



[2] Partee 1986

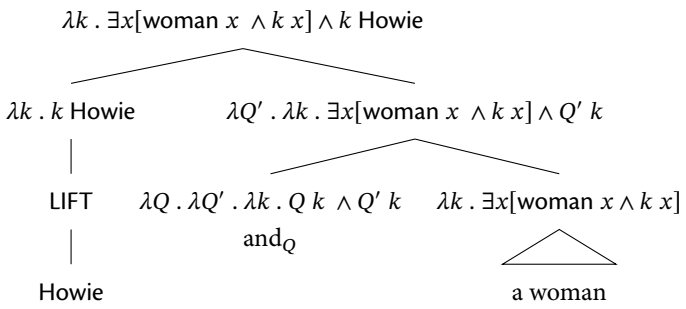
(10) $\mathsf{ident} \equiv \mathsf{BE} \circ (\uparrow)$

Observation: expressions quantificational and non-quantificational DPs can be coordinated:

(11) [Howie and a woman] entered the club

LIFT allows something that, by virtue of its quantificational nature, is an *inherent* scope-taker, to combine with something that *isn't*:[3]

(12)



3.1 Generalizing the triangle

Based on the way in which LIFT and friends are defined, in theory we could replace e with *any* type. Let's give a more general statement of LIFT and friends as polymorphic functions:

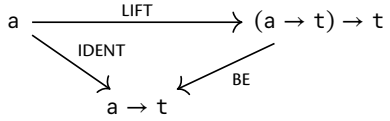
(13) $\mathsf{LIFT}x \coloneqq \lambda k\ .\ k\ x$ $\qquad\qquad$ $\mathsf{LIFT} : \mathtt{a} \to (\mathtt{a} \to \mathtt{t}) \to \mathtt{t}$

(14) $\mathsf{IDENT}\ x \coloneqq \lambda y\ .\ y = x$ $\qquad\qquad$ $\mathsf{IDENT} : \mathtt{a} \to \mathtt{a} \to \mathtt{t}$

(15) $\mathsf{BE}\ Q \coloneqq \lambda x\ .\ Q\ (\lambda y\ .\ y = x)$ $\qquad\qquad$ $\mathsf{BE} : ((\mathtt{a} \to \mathtt{t}) \to \mathtt{t}) \to \mathtt{a} \to \mathtt{t}$

The diagram, of course, still commutes:

(16) The (generalized) Partee triangle



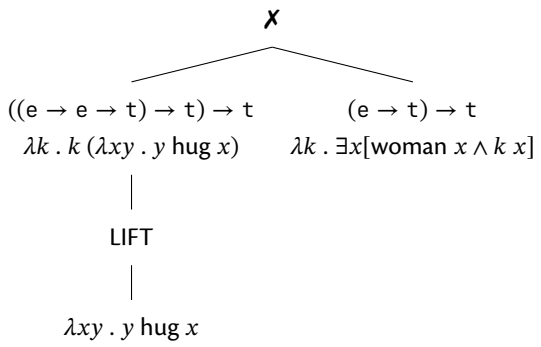
Why might a polymorphic LIFT be useful? Recall that we used a typed instantiation of LIFT in order to allow a quantificational thing to combine with a non-quantificational thing. Polymorphic LIFT allows us to type-lift, e.g., a function that takes multiple arguments:

(17) $\mathsf{LIFT}\ (\lambda xy\ .\ y\ \mathsf{hug}\ x) = \overbrace{\lambda k\ .\ k\ (\lambda xy\ .\ y\ \mathsf{hug}\ x)}^{((\mathtt{e}\to\mathtt{e}\to\mathtt{t})\to\mathtt{t})\to\mathtt{t}}$

One tantalizing possibility is that this allow us to combine a non-quantificational transitive verb with a quantificational DP.

[3] If you're familiar with Partee & Rooth 2012 you'll notice that the *and* that coordinates quantificational DPs (written here as and_Q), is just the result of applying their *generalized conjunction* rule. We'll return to generalized conjunction, and the connection to coninuations in §7.1.

(18)



Unfortunately, assuming the usual inventory of composition rules (i.e., *function application*, *predicate modification*, and *predicate abstraction*), we're stuck.[4] So, **let's invent a new one**.

Here's the intuition we're going to pursue. Let's look again at the types. One way of thinking about what LIFT as follows: it takes an a-type thing and adds a "wrapper". Quantificational DPs, on the other hand come "pre-wrapped".

- LIFT ⟦hug⟧ : (⌈(e → e → t)⌉ → t) → t
- ⟦a woman⟧ : (⌈e⌉ → t) → t

If we look at the wrapped-up types, we see a *function* from individuals, and an individual – namely, two things that can combine via function application.

What we need to accomplish is the following:

- Unwrap lifted *hug*.
- Unwrap the *a woman*.
- Use function application to combine the unwrapped values.
- Finally, wrap the result back up! Think of the quantificational meaning as being like a taco – it isn't really a taco without the wrapper, therefore we don't want to throw the wrapper away.

In order to accomplish this, we'll define a new composition rule: Scopal Function Application (SFA). We're going to define SFA in terms of Function Application (FA); we haven't been explicit about how FA is defined yet, so let's do that now. We'll write FA as the infix operator A.[5]

[4] Other, more exotic composition rules such as *restrict* won't help either. Take my word for this!

[5]

different ways of making function application bidirectional

(19) Function Application (FA) (def.)

a. $f\ \mathsf{A}\ x \coloneqq f\ x$ $\qquad\qquad \mathsf{A} : (\mathsf{a} \to \mathsf{b}) \to \mathsf{a} \to \mathsf{b}$

b. $x\ \mathsf{A}\ f \coloneqq f\ x$ $\qquad\qquad \mathsf{A} : \mathsf{a} \to (\mathsf{a} \to \mathsf{b}) \to \mathsf{b}$

Here, function application is made bidirectional by *overloading* – we've defined forwards and backwards application, and given them the same function name.

Scopal Function Application (SFA)

We'll write SFA as the infix operator S. Note that, since A is overloaded already, and S is defined in terms of A, S gets overloaded too.

(20) Scopal Function Application (SFA) (def.)

$m\ \mathsf{S}\ n \coloneqq \lambda k\ .\ m\ (\lambda a\ .\ n\ (\lambda b\ .\ k\ (a\ \mathsf{A}\ b)))$

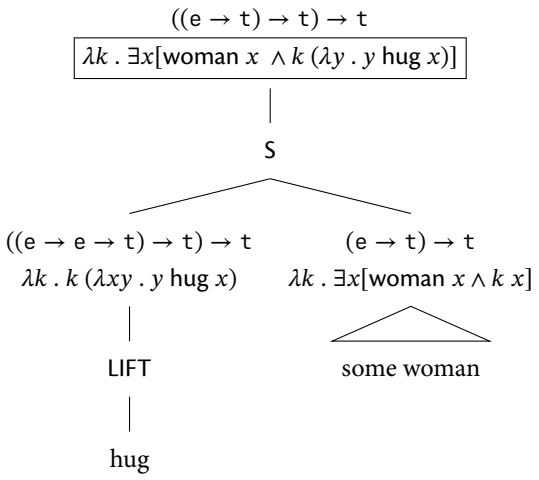
$\mathsf{S} : (((\mathsf{a} \to \mathsf{b}) \to \mathsf{t}) \to \mathsf{t}) \to ((\mathsf{a} \to \mathsf{t}) \to \mathsf{t}) \to (\mathsf{b} \to \mathsf{t}) \to \mathsf{t}$

$\mathsf{S} : ((\mathsf{a} \to \mathsf{t}) \to \mathsf{t}) \to (((\mathsf{a} \to \mathsf{b}) \to \mathsf{t}) \to \mathsf{t}) \to (\mathsf{b} \to \mathsf{t}) \to \mathsf{t}$

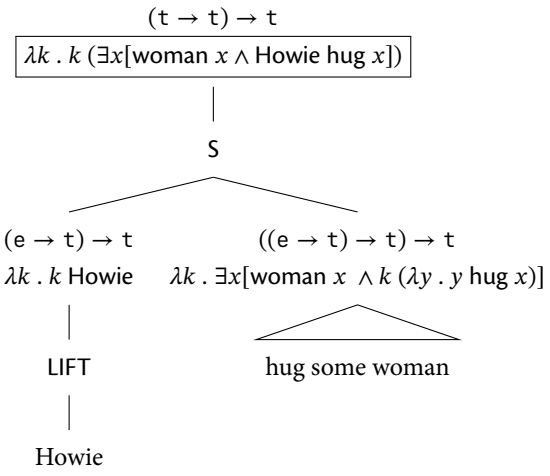
Now, let's illustrate how SFA plus generalized LIFT allows to compose a quantificational thing with non-quantificational things, using a simple example sentence:

(21) Howie hugged some woman.

(22) Step 1: compose *some woman* with LIFT-ed *hug*.



(23) Step 2: compose the resulting VP-denotation with LIFT-ed *Howie*



We're now tantalizingly close to deriving the right kind of object for the sentential meaning (namely, something of type t). Only, what we have is something of type (t → t) → t, i.e., a truth value in a "wrapper". How do we get back the wrapped up value? We saturate the *k* argument with the identity function. We'll call this operation LOWER.[6]

(24) LOWER (def.)

LOWER $m \coloneqq m\ id$ LOWER : ((a → a) → a) → a

Applying LOWER to the final value in (23) gives us a type t sentential meaning.

(25) LOWER (λk . k ($\exists x$[Howie hug x]))

$= (\lambda k$. $k(\exists x$[Howie hug x])) id

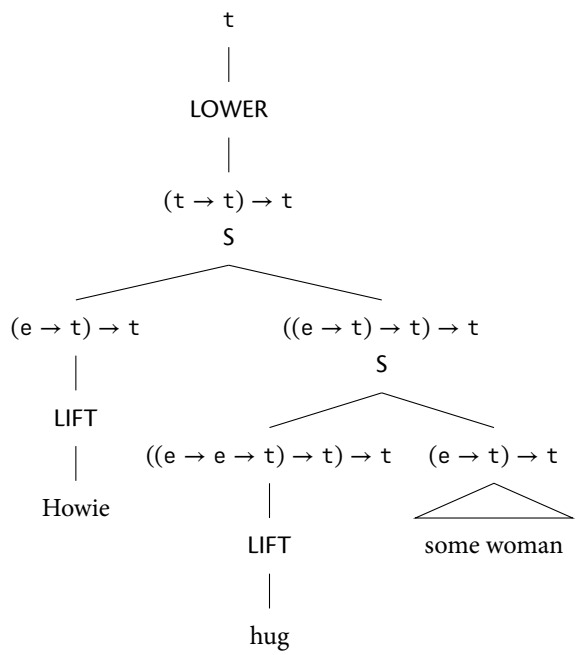
$= id$ ($\exists x$[Howie hug x])

$= \exists x$[Howie hug x]

Let's zoom back out and see how all the pieces fit together by looking at the *graph of the derivation*.

[6] Here we've given LOWER the maximally polymorphic type compatible with the function definition; in fact, all we need is (LOWER : ((t → t) → t) → t).

(26)



So far, we've provided an account of how quantificational things compose with non-quantificational things, by making use of...

- ...an independently motivated type-shifting rule (LIFT)...
- ...a way to apply LIFT-ed values (S)...
- ...and a way to get an ordinary value back from a LIFT-ed value (LOWER).

This seems pretty nice, but as I'm sure you've noticed, things are quickly going to get pretty cumbersome with more complicated sentences, especially with multiple quantifiers. Before we go any further, let's introduce some notational conveniences.

4 Towers

We've been using the metaphor of a *wrapper* for thinking about what LIFT does to an ordinary semantic value. Let's make this a bit more transparent by introducing a new *type constructor* for LIFT-ed values.[7]

(27) $\mathsf{K_t\ a := (a \to t) \to t}$

[7] A *type constructor* is just a function from a type to a new type – here, it's a rule for taking any type a and returning the type of the corresponding LIFT-ed value.

- Quantificational DPs are therefore of type $\mathsf{K_t\ e}$ (inherently).
- LIFT takes something of type a, and lifts it into something of type $\mathsf{K_t\ a}$.

Rather than dealing with *flat* expressions of the simply-typed lambda calculus, which will become increasingly difficult to reason about, we'll follow Barker & Shan 2014 in using *tower notation*.[8]

Let's look again at the meaning of a quantificational DP. The k argument which acts as the *wrapper* is called the *continuation argument*.

(28) $[\![\text{some woman}]\!] \coloneqq \lambda k\ .\ \exists x[\mathsf{woman}\ x\ \wedge k\ x]$

(29) $[\![\text{some woman}]\!] \coloneqq \dfrac{\exists x[\mathsf{woman}\ x \wedge []]}{x}$

(30) $\mathsf{LIFT}\ ([\![\text{hug}]\!]) = \dfrac{[]}{\lambda xy\ .\ y\ \mathsf{hug}\ x}$

In general:

(31) $\dfrac{f\ []}{x} \coloneqq \lambda k\ .\ f\ (k\ x)$

We can use tower notation for types too:

(32) $\dfrac{\mathsf{b}}{\mathsf{a}} \coloneqq (\mathsf{a} \to \mathsf{b}) \to \mathsf{b}$

We can now redefine our type constructor $\mathsf{K_t}$, and our type-shifting operations using our new, much more concise, tower notation. These will be our canonical definitions from now on. We'll also start abbreviating a LIFT-ed value a as $a^{\uparrow}$ and a LOWER-ed value b as $b^{\downarrow}$.

(33) The continuation type constructor $\mathsf{K_t}$ (def.)

$$\mathsf{K_t\ a} \coloneqq \frac{\mathsf{t}}{\mathsf{a}}$$

[8] To my mind, one of Barker & Shan's central achievements is simply the introduction of an accessible notational convention for reasoning about the kinds of lifted meanings we're using here.

(34) LIFT (def.)[9]

$$a^{\uparrow} := \frac{[]}{a} \qquad (\uparrow) : \mathsf{a} \to \mathsf{K_t\ a}$$

(35) Scopal Function Application (SFA) (def.)[10]

$$\frac{f\ []}{x}\ \mathsf{S}\ \frac{g\ []}{y} := \frac{f\ (g\ [])}{x\ \mathsf{A}\ y} \qquad \mathsf{S} : \mathsf{K_t\ (a \to b) \to K_t\ a \to K_t\ b}$$

(36) LOWER (def.)[11]

$$\left(\frac{f\ []}{p}\right)^{\downarrow} \qquad (\downarrow) : \mathsf{K_t\ t \to t}$$

In order to see the tower notation in action, let's go through an example involving multiple quantifiers, and show how continuation semantics derives the surface scope reading:

(37) Some knight chased every thief.

First, we combine *every thief* with lifted *chase* via SFA:

(38)

$$\boxed{\frac{\forall x[\text{thief } x \to []]}{\lambda y\ .\ y \text{ chase } x}}$$

S

$$\frac{[]}{\lambda xy.y \text{ chase } x} \qquad \frac{\forall x[\text{thief } x \to []]}{x}$$

Next, the (boxed) VP value combines with *some knight* via SFA:

(39)

$$\boxed{\frac{\exists y[\text{knight } y \wedge \forall x[\text{thief } x \to []]]}{y \text{ chase } x}}$$

S

$$\frac{\exists y[\text{knight } y \wedge []]}{y} \qquad \frac{\forall x[\text{thief } x \to []]}{\lambda y\ .\ y \text{ chase } x}$$

chase every thief

Finally, the resulting tower is collapsed via *lower*:

(40)

$$\boxed{\exists y[\text{knight } y \wedge \forall x[\text{thief } x \to y \text{ chase } x]]}$$

$$\left(\frac{\exists y[\text{knight } y \wedge \forall x[\text{thief } x \to []]]}{y \text{ chase } x}\right)^{\downarrow}$$

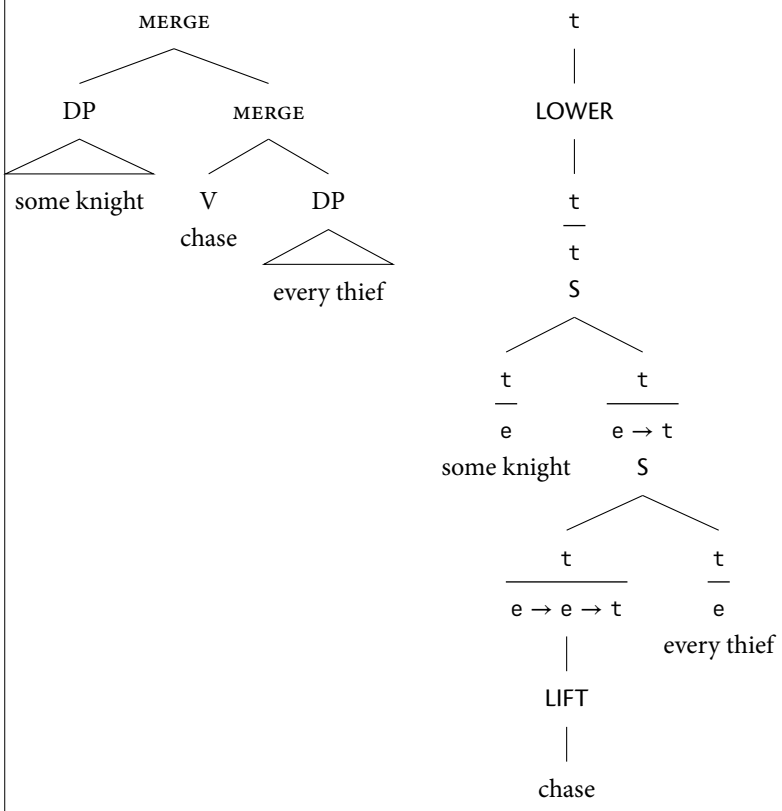
Let's zoom out and look at the *graph of the syntactic derivation* alongside the

[9] Thinking in terms of towers, LIFT takes a value a and returns a "trivial" tower, i.e., a tower with an empty top-story.

[10] SFA takes two scopal values – one with a function on the bottom floor, and the other with an argument on the bottom floor – and combines them by (i) doing function application on the bottom floor, and (ii) *sequencing* the scope-takers.

[11] LOWER *collapses the tower*, applying whatever is on the top story to whatever is on the bottom story.

graph of the semantic derivation.



5 *Scopal ambiguities*

5.1 *Interaction between scope-takers and other operators*

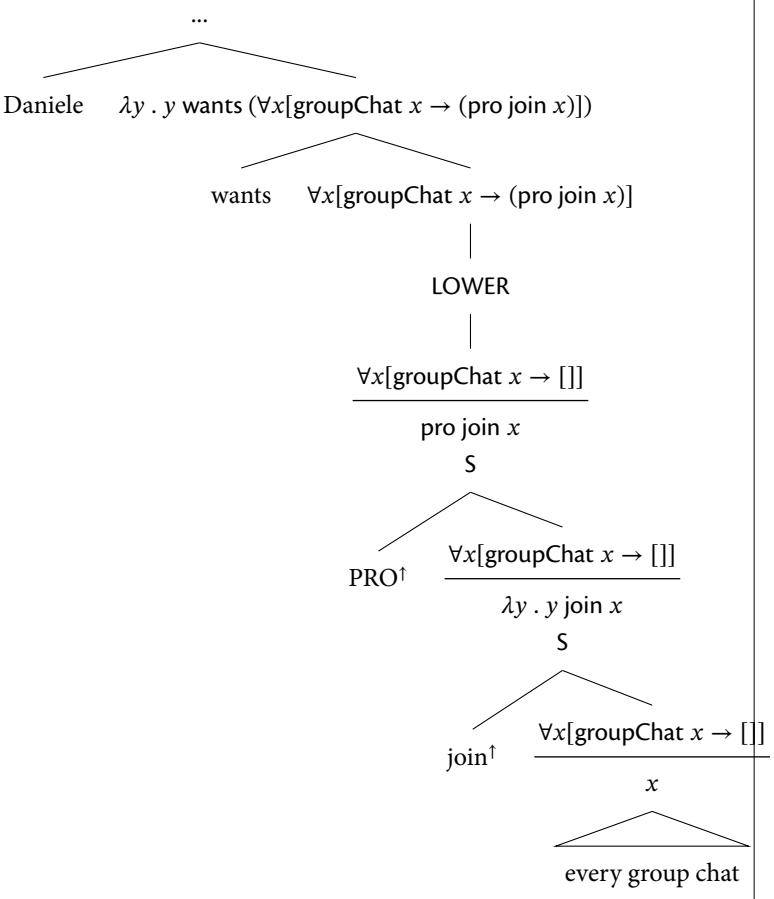
Right now, we have a theory which is very good at deriving the surface scope reading of a sentence with multiple quantifiers. It can also derive some ambiguities that arise due to interactions of quantifiers and scopally *immobile* expressions, such as intensional predicates. Consider, e.g., the interaction between a universal quantifier and the desire verb *want*.

(41) Daniele wants to join every group chat. want > ∀; ∀ > want

(41) is ambiguous: (i) if *every* takes scope below *want*, it is true if Dani has a desire about joining every group chat, (ii) if *every* takes scope over *want*, it is true if every group chat is s.t. Dani has a desire to join in.

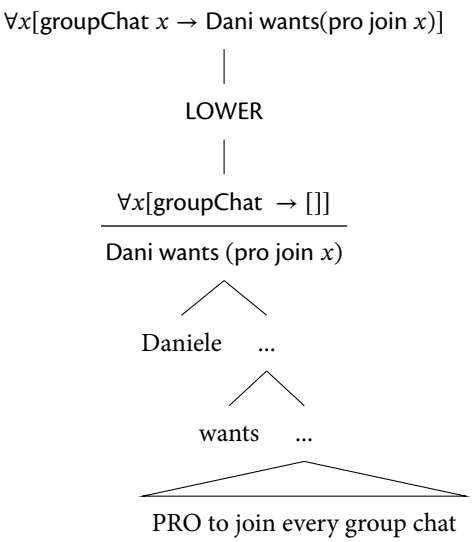
We actually already have everything we need in order to account for this. In a nutshell, the two readings correspond to an application of LOWER either below or above the intensional predicate.[12]

(42) Daniele wants to join every group chat. want > ∀



(43) Daniele wants to join every group chat. ∀ > want

[12] For a fully-fledged treatment of the examples below, we would of course need to systematically replace `t` in our fragment to some intensional type.



We can schematize what's going on here by boxing the point of the derivation at which LOWER applies:

- Daniele (wants $\boxed{\text{pro}^{\uparrow}\ \text{S}\ (\text{join}^{\uparrow}\ \text{S}\ \text{everyGroupChat})}^{\downarrow}$)
- $\boxed{\text{Daniele}^{\uparrow}\ \text{S}\ (\text{wants}^{\uparrow}\ \text{S}\ (\text{pro}^{\uparrow}\ \text{S}\ (\text{join}^{\uparrow}\ \text{S}\ \text{everyGroupChat})))}^{\downarrow}$

If you're more familiar with a treatment of scope-taking in terms of quantifier raising, then you can think of LOWER as being the correlate of the landing site of QR; semantic composition proceeds via S up until we encounter the landing site, at which point we switch back to "vanilla" semantic composition via A.

5.2 Scope rigidity

As we've seen however, when we're dealing with multiple scopally *mobile* expressions (such as quantifiers), continuation semantics derives surface scope readings by default. It is, therefore, well-suited to languages such as German and Japanese, which have been argued to display *scope-rigidity* (modulo semantic reconstruction amongst other potential exceptions).

The following example is from Kuroda (1970).

add ref

(44) a. *Dareka-ga subete-no hon-o yonda.*
someone-NOM all-GEN book-ACC read.
"Someone read all the books." ∃ > ∀; ✗ ∀ > ∃

b. *Subete-no hon-o dareka-ga yonda.*
all-GEN book-ACC someone-NOM read.
"Someone read all the books" ∀ > ∃; ∃ > ∀

There are also, of course, famous environments in English where we observe apparent scope-rigidity, such as the double-object construction (scope rigidity in the double-object construction is usually described as *scope freezing*).

(45) a. Daniele sent a syntactician every sticker. ∃ > ∀; ✗ ∀ > ∃
b. Daniele sent a sticker to every syntactician. ∃ > ∀; ∀ > ∃

Bobaljik & Wurmbrand (2012) posit a (violable) economy condition in order to express the preference for surface scope observed in many languages. Bobaljik & Wurmbrand make architectural assumptions that we aren't necessarily committed to here, but the spirit of Scope Transparency (ScoT) is very much in line with a continuation semantics for quantifiers, where surface scope is the default, and inverse scope will only be achievable via additional application of the type-shifting rules posited.

(46) Scope Transparency (ScoT)
If the order of two elements at LF is $A > B$, then the order at PF is $A > B$.

5.3 Deriving inverse scope

Recall that LIFT is a *polymorphic function* – it lifts a value into a trivial tower:

(47) $a^{\uparrow} := \dfrac{[]}{a}$

Since LIFT is polymorphic, in principle it can apply to any kind of value – even a tower! Let's flip back to lambda notation to see what happens.

(48) $[\![\text{everyone}]\!] := \lambda k \,.\, \forall x[k\ x]$

(49) $[\![\text{everyone}]\!]^{\uparrow} = \lambda l \,.\, l\ (\lambda k \,.\, \forall x[k\ x])$

Going back to tower notation, lifting a tower adds a trivial third story:[13] Fol-

[13] In fact, via successive application of LIFT, we can generate an $n-$story tower.

lowing Charlow (2014), when we apply LIFT to a tower, we'll describe the operation as *external lift* (although, it's worth bearing in mind that this is really just our original LIFT function).

One important thing to note is that, when we externally lift a tower, the quantificational part of the meaning always remains on the same story relative to the bottom floor. Intuitively, this reflects the fact that, ultimately, LIFT alone isn't going to be enough to derive quantifier scope ambiguities.

(50) $\left(\dfrac{\forall x[]}{x}\right) = \dfrac{[]}{\dfrac{\forall x[]}{x}}$

The extra ingredient we're going to need, is the ability to sandwhich an empty story into the *middle* of our tower, pushing the quantificational part of the meaning to the very top. This is *internal lift* (⇑⇑).[14]

(52) *Internal lift* (def.)
a. $(⇑⇑) : \mathsf{K_t}\ \mathsf{a} \rightarrow \mathsf{K_t}\ (\mathsf{K_t}\ \mathsf{a})$
b. $m^{⇑⇑} \coloneqq \lambda k\ .\ m\ (\lambda x\ .\ k\ x^{⇑})$

It's much easier to see what internal lift is doing by using the tower notation. We can also handily compare its effects to those of *external* lift.

(53) *Internal lift* (tower ver.)

$$\left(\frac{f\ []}{x}\right)^{⇑⇑} \coloneqq \frac{\dfrac{f\ []}{[]}}{x}$$

(54) *External lift* (tower ver.)

$$\left(\frac{f\ []}{x}\right)^{⇑} \coloneqq \frac{[]}{\dfrac{f\ []}{x}}$$

Armed with *internal* and *external* lifting operations, we now have everything we need to derive inverse scope. We'll start with a simple example (55).

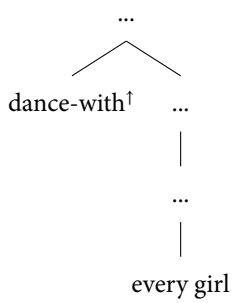
The trick is: we *internally* lift the quantifier that is destined to take wide scope. That's it!

(55) A boy danced with every girl. $\forall > \exists$

[14] I can tell what you're thinking: "seriously? Another *darn* type-shifter? How many of these bad boys are we going to need?!". Don't worry, I got you. Even thought we've defined internal lift here as a primitive operation, it actually just follows from our existing machinery. Concretely, *internal lift* is really just *lifted* LIFT (so many lifts!). Lifted LIFT applies to its argument via S.

(51) $(\mathsf{LIFT}^{⇑})\ \mathsf{S}\ \dfrac{f\ []}{x} = \dfrac{\dfrac{f\ []}{[]}}{x}$

(56) Step 1: internally lift *every girl*.



6 Scope islands and RESET

7 Continuations beyond DPs

7.1 Generalized (con/dis)junction

8 Indexed continuations

Let's look again at the "shape" of a continuation type:

(57) $\mathtt{K_r} \coloneqq (\mathtt{a} \to \mathtt{r}) \to \mathtt{r}$

We can consider a more general version of this type (see Wadler 1994).[15]

(58) $\mathtt{K^i_r} \coloneqq (\mathtt{a})$

generalized conjunction as continuised conjunction.

References

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Bobaljik, Jonathan David & Susi Wurmbrand. 2012. Word Order and Scope: Transparent Interfaces and the ¾ Signature. *Linguistic Inquiry* 43(3). 371–421.

[15] I'm using the variable names *i* and *r* as mnemonics for *intermediate type* and *return type* respectively.

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A Continuations from a categorical perspective

The triple $(\mathsf{K_t}, \uparrow, \mathsf{S})$ is an *applicative functor*, a highly influential notion in the literature on functional programming (Mcbride & Paterson 2008); for applications in linguistic semantics see Kiselyov 2017, Charlow 2018.

Cite my applicatives paper here

Technically speaking, an applicative functor is a type constructor (here $\mathsf{K_t}$), together with two functions $(\uparrow) : \mathsf{a} \to \mathsf{K_t\ a}$ and $(\mathsf{S} : \mathsf{K_t}\ (\mathsf{a} \to \mathsf{b}) \to \mathsf{K_t\ a} \to \mathsf{K_t\ b})$ obeying the following laws:

Check the applicative laws

(59) **Homomorphism**
$f^{\uparrow}\ \mathsf{S}\ x^{\uparrow} \equiv (f\ x)^{\uparrow}$

(60) **Interchange**
$(\lambda k\ .\ k\ x)^{\uparrow}\ \mathsf{S}\ m \equiv m \mathsf{S}\ x^{\uparrow}$

(61) **Identity**
$id^{\uparrow}\ \mathsf{S}\ m \equiv m$

(62) **Composition**
$(\circ)^{\uparrow}\ \mathsf{S}\ u\ \mathsf{S}\ v\ \mathsf{S}\ w \equiv u\ \mathsf{S}\ (v\ \mathsf{S}\ w)$

You can verify for yourselves that the triple $(\mathsf{K_t}, \uparrow, \mathsf{S})$ obeys the applicative laws – we'll call it the *continuation applicative*.

A related, more powerful abstraction from the functional programming literature is *monads*. There is a growing body of literature in linguistic semantics that explicitly makes use of monads (Charlow 2014,). A monad, like an applicative functor, is defined as a triple consisting of a type constructor and two functions. Monads are strictly speaking more powerful than applicative functors; that is to say, if you have a monad you are guaranteed to have an applicative functor, but not vice versa.

cite others

In, e.g., Barker & Shan (2014), and much of the existing literature in linguistic semantics making use of continuations, they are presented in their applicative guise; in the functional programming literature however (and especially in `haskell`) continuations are more widely used in their monadic guise. A monad is a triple $(\mathsf{K_t}, (\uparrow), \mu)$ – the crucial addition here is *join* (μ).

(63) *join* (def.)

a. $\mu : \mathsf{K_t}\ (\mathsf{K_t}\ \mathsf{a}) \to \mathsf{K_t}\ \mathsf{a}$

b. $m^{\mu} \coloneqq \lambda k\ .\ m\ (\lambda c\ .\ c\ k)$

In tower terms, *join* takes a two-level tower and sequences effects from the top story down:

(64) $$\frac{\frac{f\ []}{g\ []}}{x}^{\mu} = \frac{f\ (g\ [])}{x}$$

Interestingly, it looks like we can define join just in terms of operations from the applicative instance:

(65) $m^{\mu} = m \circ (\uparrow)$

I leave a demonstration of this fact as an exercise.

B LaTeX dojo

Here is the macro I use to typeset towers in LaTeX. Declare this in your preamble. You'll need the `booktabs` and `xparse` packages too.

```
\NewDocumentCommand\semtower{mm}{
  \begin{tabular}[c]{@{\,}c@{\,}}
    \(#1\)
```

```
      \\
      \midrule
      \(#2\)
      \\
    \end{tabular}
  }
```

A simple two-level tower can now be typeset as follows:

```
    $$\semtower{f []}{x}$$
```

Resulting in:

$$\frac{f\ []}{x}$$