Handling Ordinal Predictors in Regression Models via Monotonic Effects

Paul Bürkner & Emmanuel Charpentier

EAM Conference 2018

Linear Regression

Assume that the predictor term η is a linear combination of the predictor variables multiplied by the regression coefficents:

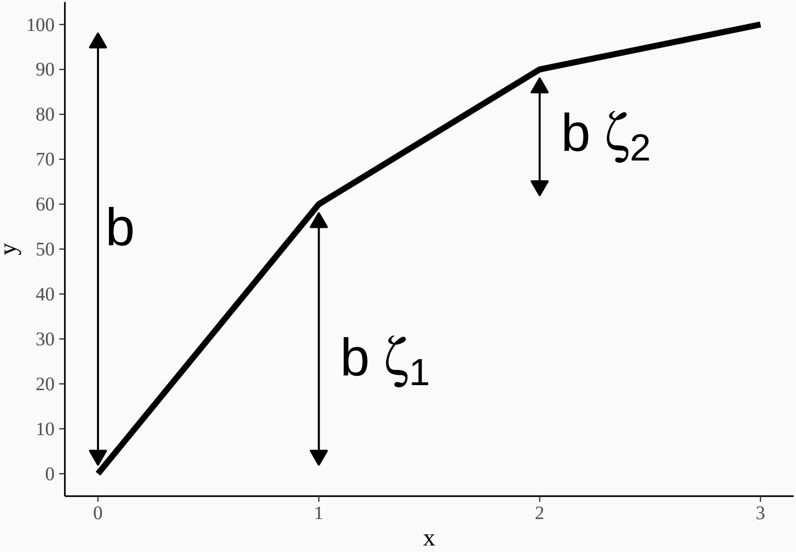
$$\eta = b_0 + \sum_{k=1}^{K} b_k x_k$$

Predictors x_k may be

- continuous variables
- coded categorical variables

What about ordinal predictors?

Monotonic Effects: Idea



Monotonic Effects: Mathematical Formulation

Monotonic regression of an ordinal predictor $x \in \{0, ..., C\}$:

$$\eta = b_0 + b \sum_{i=1}^{x} \zeta_i$$

- For notational convenience: $\sum_{i=1}^{0} \zeta_i = 0$
- Parameter ζ is a simplex: $\zeta_i \in [0, 1]$ and $\sum_{i=1}^{C} \zeta_i = 1$
- Parameter b may be any real value

Implications:

- Effect of x is either monotonically increasing or decreasing
- b indicates the direction and scale of the effect
- Categories are equidistant if and only if $\zeta_i = 1/C$

Monotonic Effects: Interactions

Ordinary Regression model including the interaction of z and x:

$$\eta = b_0 + b_1\, z + b_2\, x + b_3\, z\, x$$

Generalization to monotonic effects:

- Define $\mathrm{mo}(x, \zeta) := \sum_{i=1}^{x} \zeta_i$ for brevity
- Replace x with $\mathrm{mo}(x, \zeta)$:

$$\eta = b_0 + b_1\, z + b_2 \,\mathrm{mo}(x, \zeta_{b_2}) + b_3\, z \,\mathrm{mo}(x, \zeta_{b_3})$$

- Relation of ζ_{b_2} and ζ_{b_3} determines the type of monotonicty
- x is (conditionally) monotonic for all z if $\zeta_{b_2} = \zeta_{b_3}$

Monotonic Effects in a Bayesian Framework

Priors on b:

- Any reasonable prior for regression coefficients
- For instance: $b \sim \mathcal{N}(0, s)$ for a fixed standard deviation s

Prior on ζ:

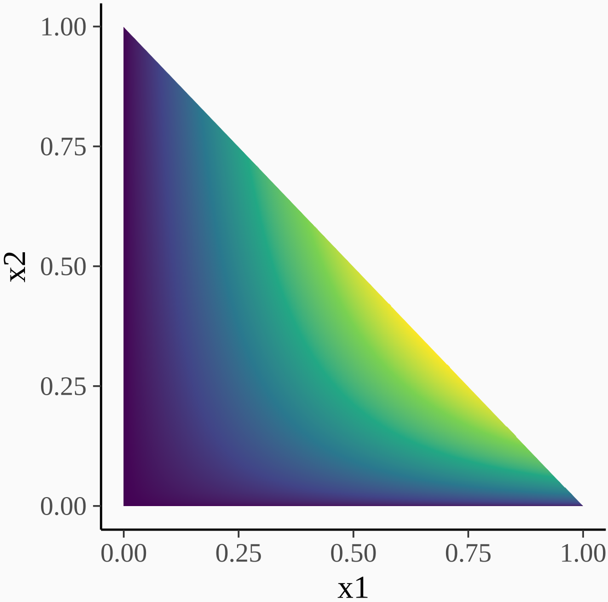
- Dirichlet prior: $\zeta \sim \mathcal{D}(\alpha)$
- α: Concentration parameter of the same length as ζ

Let $\alpha_0 = \sum_{i=1}^{C} \alpha_i$, then:

$$\mathbb{E}(\zeta_i) = \frac{\alpha_i}{\alpha_0}$$

$$\mathrm{SD}(\zeta_i) = \sqrt{\frac{\alpha_i(\alpha_0 - \alpha_i)}{(\alpha_0^2(\alpha_0 + 1)}}$$

Dirichlet Prior: Visualization for $\alpha = (2, 1.5, 1)$



Monotonic effects in the R package brms

brms is a comprehensive framework for Bayesian regression models

Stan is used for the model fitting behind the scenes

Monotonic effects are fully built into the formula syntax of brms

Monotonic effect of x on y:

```
y ~ mo(x)
```

Main effects and interaction of x and z:

```
y ~ mo(x) * z
```

Varying effect of x over group g:

```
y ~ mo(x) + (mo(x) | g)
```

Case Study: Measures of Chronic Widespread Pain (CWP)

Objective: Predict subjective physical health by measures of CWP

Examples for CWP measures:

- Impairments in walking
- Impairments in moving around

Scale from 0 ('no problem') to 4 ('complete problem')

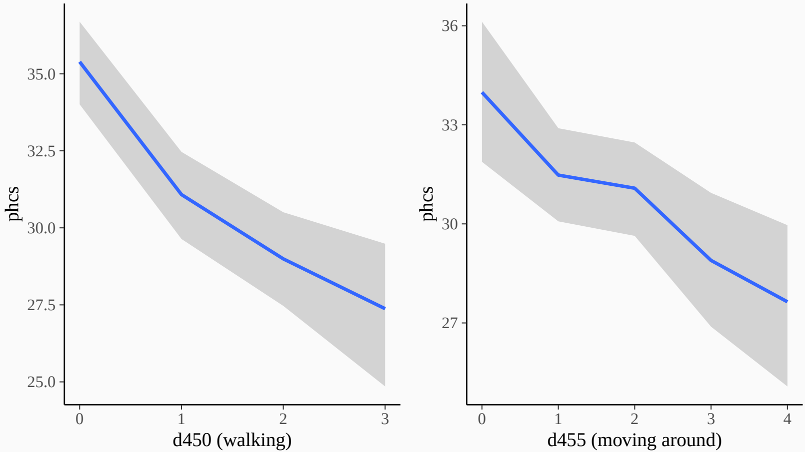
Data provided in the **ordPens** package

For details see Gertheiss, Hogger, Oberhauser, & Tutz (2011)

Plausible assumption: CWP measures have **monotonic effects**

Case Study: Model Specification

```r
library(brms)
fit1 <- brm(phcs ~ mo(d450) + mo(d455), data = cwp)
```



Other Approaches for Modelling Ordinal Predictors

Categorical isotonic regression:

- Estimate group means of ordinal categories such that $\mu_0 < \mu_1 < ... < \mu_C$
- Equivalent to monotonic effects in simple cases
- Harder to penalize via priors

Penalized regression (Gertheiss & Tutz, 2009):

- Apply dummy coding on the ordinal variable
- Penalize larger differences between adjacent categories via

$$J(b) = \sum_{i=1}^{C} (b_i - b_{i-1})^2$$

- Closely related to regression splines
- No monotonicity constraint

Learn More about Monotonic Effects and brms

Manuscript draft:
https://github.com/paul-buerkner/monotonic-effects-paper

Vignette in brms: `vignette("brms_monotonic")`

Documentation of the formula syntax: `?brmsformula`

Papers about brms: Bürkner (2017) and Bürkner (2018)

Forums: http://discourse.mc-stan.org/

Email: paul.buerkner@gmail.com

Twitter: @paulbuerkner

References

Bürkner, P.-C. (2017). brms: An R package for Bayesian multilevel models using Stan. *Journal of Statistical Software*, *80*(1), 1–28. https://doi.org/10.18637/jss.v080.i01

Bürkner, P.-C. (2018). Advanced Bayesian multilevel modeling with the R package brms. *The R Journal*, 1–15.

Gertheiss, J., Hogger, S., Oberhauser, C., & Tutz, G. (2011). Selection of ordinally scaled independent variables with applications to international classification of functioning core sets. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, *60*(3), 377–395.

Gertheiss, J., & Tutz, G. (2009). Penalized regression with ordinal predictors. *International Statistical Review*, *77*(3), 345–365.

Appendix

Counter Example to the Conditional Monotonicity

Model: $\eta = b_0 + b_1\, z + b_2\, \mathrm{mo}(x, \zeta_{b_2}) + b_3\, z\, \mathrm{mo}(x, \zeta_{b_3})$