Homework #3 Solutions

1.

(1) (5%)

$$L_A = f(\mathbf{x}; A) = \frac{1}{\left(2\pi\sigma_\omega^2\right)^{M/2}} e^{-\frac{1}{2\sigma_\omega^2}\sum_{n=0}^{M-1}\left[x(n)-As(n)\right]^2}$$

$$\frac{\partial \ln L_A}{\partial A} = \frac{\partial}{\partial A}\left[-\frac{M}{2}\ln\left(2\pi\sigma_\omega^2\right) - \frac{1}{2\sigma_\omega^2}\sum_{n=0}^{M-1}\left[x(n)-As(n)\right]^2\right] = \frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1}\left[x(n)-As(n)\right]s(n) = 0$$

$$A_{ML} = \frac{\sum_{n=0}^{M-1} x(n)s(n)}{\sum_{n=0}^{M-1} s^2(n)}$$

(2) (5%)

$$f(\mathbf{x} \mid A) = \frac{1}{\left(2\pi\sigma_\omega^2\right)^{M/2}} e^{-\frac{1}{2\sigma_\omega^2}\sum_{n=0}^{M-1}\left[x(n)-As(n)\right]^2}$$

$$f(A) = \frac{1}{\sqrt{2\pi\sigma_A^2}} e^{-\frac{1}{2\sigma_A^2}\left(A-\bar{A}\right)^2}$$

$$\frac{\partial}{\partial A}\left[\ln f(\mathbf{x} \mid A) + \ln f(A)\right] = \frac{\partial}{\partial A}\begin{bmatrix} -\frac{M}{2}\ln\left(2\pi\sigma_\omega^2\right) - \frac{1}{2\sigma_\omega^2}\sum_{n=0}^{M-1}\left[x(n)-As(n)\right]^2 \\ -\frac{1}{2}\ln\left(2\pi\sigma_A^2\right) - \frac{1}{2\sigma_A^2}\left(A-\bar{A}\right)^2 \end{bmatrix}$$

$$= \frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1}\left[x(n)-As(n)\right]s(n) - \frac{1}{\sigma_A^2}\left(A-\bar{A}\right) = 0$$

$$A_{MAP}\left[\frac{1}{\sigma_A^2} + \frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1} s^2(n)\right] = \frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1} x(n)s(n) + \frac{1}{\sigma_A^2}\bar{A}$$

$$A_{MAP} = \frac{\frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1} x(n)s(n) + \frac{1}{\sigma_A^2}\bar{A}}{\frac{1}{\sigma_A^2} + \frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1} s^2(n)} = \frac{\sum_{n=0}^{M-1} x(n)s(n) + \frac{\sigma_\omega^2}{\sigma_A^2}\bar{A}}{\sum_{n=0}^{M-1} s^2(n) + \frac{\sigma_\omega^2}{\sigma_A^2}}$$

(3) (6%)

$$\frac{\partial}{\partial A}\left[\ln f(\mathbf{x} \mid A) + \ln f(A)\right] = \frac{\partial}{\partial A}\left[-\frac{M}{2}\ln\left(2\pi\sigma_\omega^2\right) - \frac{1}{2\sigma_\omega^2}\sum_{n=0}^{M-1}\left[x(n)-As(n)\right]^2 + \ln A - \ln\sigma_A^2 - \frac{A^2}{2\sigma_A^2}\right]$$

$$= \frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1}\left[x(n)-As(n)\right]s(n) + \frac{1}{A} - \frac{A}{\sigma_A^2} = 0$$

$$A_{MAP}^2\left[\frac{1}{\sigma_A^2}+\frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1}s^2(n)\right]-\frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1}x(n)s(n)A_{MAP}-1=0$$

$\because A\ge 0$

$$A_{MAP}=\frac{\frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1}x(n)s(n)+\sqrt{\frac{1}{\sigma_\omega^4}\left[\sum_{n=0}^{M-1}x(n)s(n)\right]^2+4\left[\frac{1}{\sigma_A^2}+\frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1}s^2(n)\right]}}{2\left[\frac{1}{\sigma_A^2}+\frac{1}{\sigma_\omega^2}\sum_{n=0}^{M-1}s^2(n)\right]}$$

$$=\frac{\sum_{n=0}^{M-1}x(n)s(n)+\sqrt{\left[\sum_{n=0}^{M-1}x(n)s(n)\right]^2+4\left[\frac{\sigma_\omega^4}{\sigma_A^2}+\sigma_\omega^2\sum_{n=0}^{M-1}s^2(n)\right]}}{2\left[\frac{\sigma_\omega^2}{\sigma_A^2}+\sum_{n=0}^{M-1}s^2(n)\right]}$$

2. (16%)
   LMMSE estimator = Wiener filter
   The normal equation: $\mathbf{Rf}=\mathbf{g}$ => $E\left(\mathbf{xx}^T\right)\mathbf{f}=E\left(A\mathbf{x}\right)$

$$\Rightarrow \begin{bmatrix} 1+\frac{\sigma^2}{\sigma_A^2} & \mathbf{1} \\ \mathbf{1} & 1+\frac{\sigma^2}{\sigma_A^2}\end{bmatrix}\begin{bmatrix} f_0 \\ \vdots \\ f_{M-1}\end{bmatrix}=\mathbf{1}_{M\times 1}$$

=> Subtraction between any two equations from the above equation set:

$$\frac{\sigma^2}{\sigma_A^2}f_i-\frac{\sigma^2}{\sigma_A^2}f_j=0,\ \forall\, i\ne j$$

$$\therefore f_0=f_1=\cdots=f_{M-1}$$

$$\therefore\left(1+\frac{\sigma^2}{\sigma_A^2}\right)f_0+(M-1)f_0=1 \quad \Rightarrow f_0=\frac{\sigma_A^2}{M\sigma_A^2+\sigma^2}$$

$$\therefore f_n=\frac{\sigma_A^2}{M\sigma_A^2+\sigma^2},\ n=0,1,\cdots,M-1$$

$$\therefore \hat{A}_{LMMSE}=\mathbf{f}^T\mathbf{x}=\frac{\sigma_A^2}{M\sigma_A^2+\sigma^2}\sum_{n=0}^{M-1}x(n)$$

3.

$$r(i-j)=E\{x(n-n_0-i+1)x(n-n_0-j+1)\}=E\{x(n-i)x(n-j)\}$$
$$=E\{[w(n-i)+aw(n-50-i)][w(n-j)+aw(n-50-j)]\}$$
$$=(1+a^2)\delta(i-j)+a\delta(i-j+50)+a\delta(i-j-50)$$

$$\mathbf{g}=\left[E\{aw(n-50)x(n-n_0)\} \quad \cdots \quad E\{aw(n-50)x(n-\hat{n}-L+1)\}\right]^T$$

$$E\{aw(n-50)x(n-n_0)\}=E\{aw(n-50)[w(n-n_0)+aw(n-n_0-50)]\}$$
$$=a\delta(n_0-50)+a^2\delta(\hat{n})$$

$$E\{aw(n-50)x(n-n_0-L+1)\}$$
$$=E\{aw(n-50)[w(n-n_0-L+1)+aw(n-n_0-L+1-50)]\}$$
$$=a\delta(n_0+L-1-50)+a^2\delta(n_0+L-1)$$

$$\mathbf{g}=\left[a\delta(n_0-50)+a^2\delta(n_0) \quad \cdots \quad a\delta(n_0+L-1-50)+a^2\delta(n_0+L-1)\right]^T$$

(1) (8%)

i. prediction distance, $n_0=30$,

$$\mathbf{g}=\left[a\delta(n_0-50)+a^2\delta(n_0) \quad \cdots \quad a\delta(n_0+L-1-50)+a^2\delta(n_0+L-1)\right]^T$$
$$=\left[a\delta(-20)+a^2\delta(30) \quad \cdots \quad a\delta(-6)+a^2\delta(44)\right]^T=\mathbf{0}$$

$$r(i-j)=(1+a^2)\delta(i-j)+a\delta(i-j+50)+a\delta(i-j-50)$$

$$\mathbf{Rc}=\mathbf{g}=\mathbf{0}\Rightarrow\mathbf{c}=\mathbf{0}\Rightarrow\hat{x}(n)=0$$

Prediction error: $e(n)=aw(n-50)-0=aw(n-50)$

ii. prediction distance, $n_0=40$,

$$\mathbf{g}=\left[a\delta(n_0-50)+a^2\delta(n_0) \quad \cdots \quad a\delta(n_0+L-1-50)+a^2\delta(n_0+L-1)\right]^T$$
$$=\left[a\delta(-10)+a^2\delta(40) \quad \cdots \quad a\delta(4)+a^2\delta(54)\right]^T$$
$$=\begin{bmatrix}0 & \cdots & 0 & a & 0 & 0 & 0 & 0\end{bmatrix}^T_{15\times 1}$$

$$r(i-j)=(1+a^2)\delta(i-j)+a\delta(i-j+50)+a\delta(i-j-50)$$

$$\mathbf{Rc}=\mathbf{g}\Rightarrow(1+a^2)\mathbf{Ic}=\begin{bmatrix}0 & \cdots & 0 & a & 0 & 0 & 0 & 0\end{bmatrix}^T_{15\times 1}$$

$$c(n)=\begin{cases}\dfrac{a}{1+a^2}, & n=11\\ 0, & n\neq 11\end{cases}$$

$$\hat{x}(n)=\frac{a}{1+a^2}x(n-40-11+1)=\frac{a}{1+a^2}x(n-50)$$

Prediction error:

$$e(n) = aw(n-50) - \hat{x}(n)$$
$$= aw(n-50) - \frac{a}{1+a^2}w(n-50) - \frac{a^2}{1+a^2}w(n-100)$$
$$= \frac{a^3}{1+a^2}w(n-50) - \frac{a^2}{1+a^2}w(n-100)$$

iii. prediction distance, $\hat{n} = 50$.

$$\mathbf{g} = \left[a\delta(n_0-50) + a^2\delta(n_0) \quad \cdots \quad a\delta(n_0+L-1-50) + a^2\delta(n_0+L-1)\right]^T$$
$$= \left[a\delta(0) + a^2\delta(50) \quad \cdots \quad a\delta(14) + a^2\delta(64)\right]^T$$
$$= \begin{bmatrix} a & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T_{15\times 1}$$
$$r(i-j) = (1+a^2)\delta(i-j) + a\delta(i-j+50) + a\delta(i-j-50)$$
$$\mathbf{Rc} = \mathbf{g} \Rightarrow \mathbf{Rc} = \mathbf{g} \Rightarrow (1+a^2)\mathbf{Ic} = \begin{bmatrix} a & \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T_{15\times 1}$$
$$c(n) = \begin{cases} \frac{a}{1+a^2}, & n = 1 \\ 0, & n \neq 1 \end{cases}$$
$$\hat{x}(n) = \frac{a}{1+a^2}x(n-50-1+1) = \frac{a}{1+a^2}x(n-50)$$

Prediction error:

$$e(n) = aw(n-50) - \hat{x}(n)$$
$$= aw(n-50) - \frac{a}{1+a^2}w(n-50) - \frac{a^2}{1+a^2}w(n-100)$$
$$= \frac{a^3}{1+a^2}w(n-50) - \frac{a^2}{1+a^2}w(n-100)$$

(2) (8%)

From the range of $\mathbf{g}$, $n_0 - 50 \le 0$ and $n_0 + L - 1 - 50 \ge 0$,

$50 - L + 1 \le n_0 \le 50 \Rightarrow 36 \le n_0 \le 50$

4. (16%)

$$\sigma_d^2 = E\left\{d^2(n)\right\} - E\left\{d(n)\right\}^2$$

$$= E\left\{\left(\sum_{i=0}^{\infty} h(i)w(n-i)\right)\left(\sum_{j=0}^{\infty} h(j)w(n-j)\right)\right\}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} h(i)h(j)E\left\{w(n-i)w(n-j)\right\}$$

$$= \sigma_w^2 \sum_{i=0}^{\infty} h^2(i)$$

$$g(i) = E\left\{x(n-i)d(n)\right\} = E\left\{x(n-i)\sum_{j=0}^{\infty} h(j)w(n-j)\right\}$$

$$= \sum_{i=0}^{\infty} h(j)E\left\{w(n-i)w(n-j)\right\}$$

$$= \sigma_w^2 h(i)$$

$$(\because E\left\{w(n-i)w(n-j)\right\} = \begin{cases} \sigma_w^2, i = j \\ 0, otherwise \end{cases})$$

$$\Rightarrow \mathbf{f}^* = \mathbf{R}^{-1}\mathbf{g} = \frac{1}{\sigma_w^2}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}\begin{bmatrix} \sigma_w^2 h(1) \\ \sigma_w^2 h(2) \\ \vdots \\ \sigma_w^2 h(L-1) \end{bmatrix} = \begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ h(L-1) \end{bmatrix}$$

By the definition of $J_{\min}$, we can obtain

$$J_{\min} = \sigma_d^2 - (\mathbf{f}^*)^T \mathbf{g} = \sigma_w^2 \sum_{i=0}^{\infty} h^2(i) - \begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ h(L-1) \end{bmatrix}^T \begin{bmatrix} \sigma_w^2 h(1) \\ \sigma_w^2 h(2) \\ \vdots \\ \sigma_w^2 h(L-1) \end{bmatrix}$$

$$= \sigma_w^2 \sum_{i=0}^{\infty} h^2(i) - \sigma_w^2 \sum_{i=1}^{L-1} h^2(i)$$

$$= \sigma_w^2 \sum_{i=L}^{\infty} h^2(i)$$

5.

(1) (5%)

$$r(i)=\sum_{n=-\infty}^{\infty} h(n)h(n+i)\,,\quad g(i)=\sum_{n=-\infty}^{\infty} h(n-i)\delta(n)=h(-i)$$

$$\Rightarrow r(0)=\sum_{n=-\infty}^{\infty} h^2(n)=1^2+(-1)^2+0.25^2=\frac{33}{16},$$

$$r(1)=\sum_{n=-\infty}^{\infty} h(n)h(n+1)=h(0)h(1)+h(1)h(2)=-\frac{5}{4},$$

$$g(0)=h(0)=1,\ \ g(1)=h(-1)=0$$

$$\therefore \mathbf{Rf}=\mathbf{g}\Rightarrow\begin{bmatrix} r(0) & r(1)\\ r(1) & r(0)\end{bmatrix}\begin{bmatrix} f(0)\\ f(1)\end{bmatrix}=\begin{bmatrix} g(0)\\ g(1)\end{bmatrix}\Rightarrow\begin{bmatrix} 33/16 & -5/4\\ -5/4 & 33/16\end{bmatrix}\begin{bmatrix} f(0)\\ f(1)\end{bmatrix}=\begin{bmatrix} 1\\ 0\end{bmatrix}$$

$$\Rightarrow \mathbf{f}^*=\mathbf{R}^{-1}\mathbf{g}=\frac{256}{689}\begin{bmatrix} 33/16 & 5/4\\ 5/4 & 33/16\end{bmatrix}\begin{bmatrix} 1\\ 0\end{bmatrix}=\begin{bmatrix} 528/689\\ 320/689\end{bmatrix}=\begin{bmatrix} 0.7663\\ 0.4644\end{bmatrix}$$

(2) (5%)

$$J_{\min}=\delta^2(n)-\mathbf{f}^T\mathbf{g}=1-\frac{528}{689}=\frac{161}{689}=0.2336$$

(3) (6%)

$$H(z)=1-z^{-1}+0.25z^{-2}$$

$$\begin{array}{r l} & 1+z^{-1}+\cdots \\ 1-z^{-1}+0.25z^{-2}\,\big) & \overline{1\qquad\qquad\qquad} \\ & \underline{1-z^{-1}+0.25z^{-2}} \\ & \quad z^{-1}+0.25z^{-2} \\ & \quad z^{-1}-z^{-2}+0.25z^{-3} \end{array}$$

Let $H'(z)$ be the truncated $1/H(z)$

$$H(z)H'(z)=\left(1-z^{-1}+0.25z^{-2}\right)\left(1+z^{-1}\right)=1+z^{-1}-z^{-1}-z^{-2}+0.25z^{-2}+0.25z^{-3}$$

$$=1-0.75z^{-2}+0.25z^{-3}$$

$$\Rightarrow \delta'(0)=1,\ \delta'(1)=0,\ \delta'(2)=-0.75,\ \delta'(3)=0.25$$

$$J=\left[\delta(n)-\delta'(n)\right]^2=(0.75)^2+(0.25)^2=\frac{9}{16}+\frac{1}{16}=\frac{10}{16}=0.625$$

$$\therefore J_{\min}=\frac{161}{689}<J=\frac{10}{16}$$

6.

(1) (10%)

$$x(n) = s(n) - 2s(n-1) + v(n)$$

$\mathbf{Rw} = \mathbf{g}$ => $E\left(\mathbf{x}_n \mathbf{x}_n^T\right)\mathbf{w} = E\left(\mathbf{x}_n s(n)\right)$ where $\mathbf{x}_n = [x(n) \;\; x(n-1)]^T$

$$=> \begin{bmatrix} 5.1 & -2 \\ -2 & 5.1 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \therefore \mathbf{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0.2317 \\ 0.0909 \end{bmatrix}$$

$$\therefore W(z) = 0.2317 + 0.0909 z^{-1}$$

(2) (10%)

$$H(z)W(z) = \left(a + bz^{-1} + az^{-2}\right)\left(w_0 + w_1 z^{-1} + w_2 z^{-2} + w_1 z^{-3} + w_0 z^{-4}\right)$$

Clearly, we need 6 delay units.