Chapter 6 Applications of Adaptive Filtering

I. Adaptive Noise Cancellation (ANC)

1) Introduction

(a) ANC is concerned with the enhancement of noise corrupted signals. The great strength of ANC is that in contrast to other enhancement techniques no *a priori* knowledge of signal or noise is required for the method to be applied.

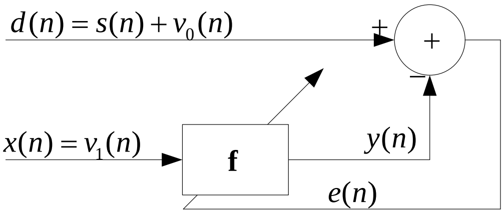
(b) In particular, noise canceling is a variation of optimal filtering which is designed to use a secondary or reference input. This reference input should contain little or no signal but should contain a noise measurement which is correlated (in some unknown way) with the noise component of the main or primary input.

(c) The ANC method is based on the principle of adaptively filtering this reference input to produce a replica of the noise which can then be subtracted from the primary measurement.

(i) The desired input is replaced by the primary measurement.

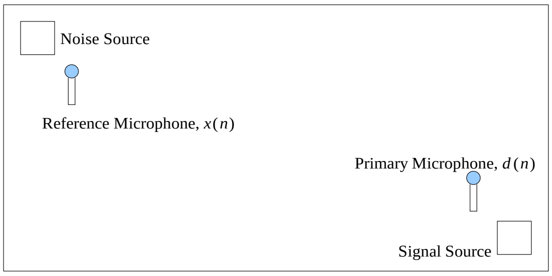
(ii) The system input is replaced by the reference.

(iii) The system output is the error signal.



■ **Figure 6.1** Adaptive noise canceling system. $d(n)$ is the primary measurement. The reference input $x(n)$ contains noise $v_1(n)$ which is correlated with $v_0(n)$ but not with $s(n)$.

<u>*Example* 1</u>: ANC in an acoustic enclosure



■ **Figure 6.2** Symbolic representation of ANC in an acoustic enclosure. Noise propagates to primary microphone and corrupts signal measurements. ANC uses a second microphone close to the noise source to form a reference.

2) Analysis of the ANC

(a) Minimize the mean-squared error

From Fig. 6.1, the error signal is

$$\begin{aligned} e(n) &= d(n) - y(n) \\ &= s(n) + v_0(n) - y(n) \end{aligned} \tag{1}$$

The mean-squared error is

$$E\{e^2(n)\} = E\{s^2(n)\} + E\{[v_0(n) - y(n)]^2\} + 2E\{s(n)[v_0(n) - y(n)]\} \tag{2}$$

where $s(n)$ is assumed that is uncorrelated with $v_0(n)$ and $v_1(n)$. Hence, for a fixed filter, $s(n)$ is also uncorrelated with $y(n) = f(n) * v_1(n)$. Therefore,

$$E\{e^2(n)\} = E\{s^2(n)\} + E\{[v_0(n) - y(n)]^2\} \tag{3}$$

It is clear that $E\{e^2(n)\}$ is minimized when $y(n) = v_0(n)$, and hence $e(n) = s(n)$, the desired result.

(b) The solution to the Normal Equations for the infinite two-sided least-squares (LS) filter

From chapter 3, the frequency response of the filter by the solution of the normal equations for the infinite two-sided LS filter is

$$F(e^{j\omega}) = \frac{R_{xd}(e^{j\omega})}{R_{xx}(e^{j\omega})} \tag{4}$$

where $R_{xd}(e^{j\omega})$ is the cross-spectrum of the input and desired signals, and $R_{xx}(e^{j\omega})$ is the power spectrum of the input signal.

Given that in the noise canceling system of Fig. 6.1

$$\begin{aligned} d(n) &= s(n) + v_0(n) \\ x(n) &= v_1(n) \end{aligned} \tag{5}$$

then

$$F(e^{j\omega}) = \frac{R_{v_1v_0}(e^{j\omega})}{R_{v_1v_1}(e^{j\omega})} \tag{6}$$

The noises $v_0(n)$ and $v_1(n)$ are, by assumption, correlated in some unknown sense.

(i) The transmission path relating $v_0(n)$ and $v_1(n)$ can be modeled by a linear system, $h(n)$. That is,

$$v_0(n) = h(n) * v_1(n) \tag{7}$$

The expressions relating the input and output spectra for a system $h(n)$ with stationary input is $R_{xy}(e^{j\omega}) = H(e^{j\omega}) R_{xx}(e^{j\omega})$. Therefore,

$$R_{v_1v_0}(e^{j\omega}) = H(e^{j\omega}) R_{v_1v_1}(e^{j\omega}) \tag{8}$$

Substituting (8) into (6) gives

$$F(e^{j\omega}) = H(e^{j\omega}) \tag{9}$$

The adaptive filter models the unknown transmission path, $h(n)$.

In the time domain, the filter output is

$$y(n) = f(n) * v_1(n) = h(n) * v_1(n) = v_0(n) \tag{10}$$

Hence, finally

$$e(n) = s(n) + v_0(n) - v_0(n) = s(n) \tag{11}$$

and perfect cancellation is the result.

Note:

1. In practice, the requirement for a perfect reference measurement is rarely met.
2. Returning to Fig. 6.2, it is obvious that if the noise can be transmitted from the reference to primary positions, then equally the signal can 'leak' in the opposite direction.

(ii) Assuming signal leakage can be modeled through a linear transmission path, $g(n)$, the ANC model of (5) becomes

$$\begin{aligned} d(n) &= s(n) + v_0(n) \\ x(n) &= g(n) * s(n) + v_1(n) = g(n) * s(n) + g(n) * v_0(n) \end{aligned} \tag{12}$$

Now, we define the **signal-to-noise density ratio** of a measurement $y(n) = s(n) + v(n)$ by

$$\rho_y(e^{j\omega}) = \frac{R_{ss}(e^{j\omega})}{R_{vv}(e^{j\omega})} \tag{13}$$

In terms of this definition, the performance of the ANC system with inputs (12) is derived as

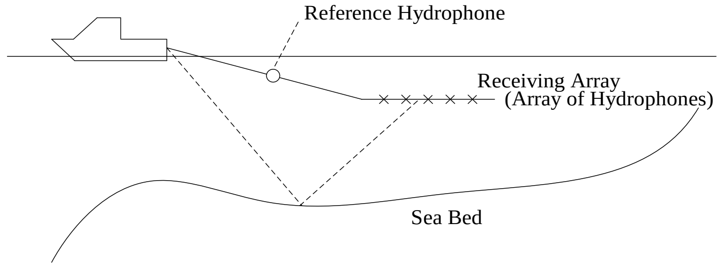
$$\rho_{out}(e^{j\omega}) = \frac{1}{\rho_{ref}(e^{j\omega})} \tag{14}$$

where $\rho_{out}(e^{j\omega})$ and $\rho_{ref}(e^{j\omega})$ are the signal-to-noise density ratios for the output and reference, respectively.

Note:

The result is intuitively reasonable. The effect of signal leakage is to degrade the ANC process. The more leakage, the better the reference SNR and the worse the result.

<u>Example</u> <u>2</u>: Own-ship noise cancellation in a towed-array system



■ **Figure 6.3 '**Own-ship' noise cancellation with a towed array. A ship tows an array of hydrophones. Received signals are corrupted by noise from the tow-ship. An ANC system constructed by using a reference derived from an extra hydrophone close to the tow-ship.

The ANC model is

$$\begin{aligned} d(n) &= s(n) + h(n) * i(n) \\ x(n) &= i(n) \end{aligned} \tag{15}$$

where $i(n)$ is the interference which measured at the reference position and it will not be an exact replica of that at the hydrophones. $h(n)$ is a system representing the effects of the transmission path on the interference. If $h(n)$ is linear, then the noise canceling system can be employed.

The noise spectrum contains both narrowband and broadband components and is dominated by a number of sinusoidal components with slowly varying amplitude and phase. A model for the problem is thus given by

$$\begin{aligned} d(n) &= s(n) + \sum_{i=0}^{M-1} B_i \sin(\omega_i nT + \theta_i) \\ x(n) &= \sum_{i=0}^{M-1} A_i \sin(\omega_i nT + \phi_i) \end{aligned} \tag{16}$$

When $M = 1$,

$$\begin{aligned} d(n) &= s(n) + B_0 \sin(\omega_0 nT + \theta_0) \\ x(n) &= A_0 \sin(\omega_0 nT + \phi_0) \end{aligned} \tag{17}$$

We recall from chapter 4, the transfer function of the LMS adaptive filter subjected to a sinusoidal input is (given for small α)

$$\frac{E(z)}{D(z)} \approx \frac{(1-e^{j\omega_o T}z^{-1})(1-e^{-j\omega_o T}z^{-1})}{[1-(1-\frac{\alpha L A_0^2}{4})e^{j\omega_o T}z^{-1}][1-(1-\frac{\alpha L A_0^2}{4})e^{-j\omega_o T}z^{-1}]} \tag{18}$$

which is a notch filter, centered on the interference frequency ω_0 .

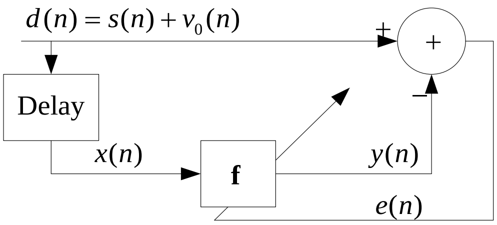
Note:

The overall effect of applying the ANC to a system with a single corrupting sinusoid is to form a notch at the interference frequency whose bandwidth is controlled by α , L, and A_0^2 . Assume L and A_0^2 be fixed,

1. When α is small, the filter removes the interference without impacting the spectrum at adjacent frequencies.
2. As α increases, the interference is still attenuated but the broader notch also removes part of the signal spectrum. ■

<u>*Example* 3</u>: ANC without an external reference measurement

In some situations, principally when the interference is periodic or at least 'pseudo-periodic' and the signal is broadband (as in the own-ship noise example above), it is possible to construct an ANC system without an external reference.



■ **Figure 6.4** ANC with reference generated by delay. Pseudo-periodic interference $v_0(n)$ corrupts broadband signal components. Periodic interference components remain correlated in the reference after delay Δ, so the signal components are decorrelated.

A model for the problem is thus given by

$$\begin{aligned} d(n) &= s(n) + v_0(n) \\ x(n) &= d(n-\Delta) \end{aligned} \tag{19}$$

The approach is to use a delay which is sufficient to decorrelate the signal component, that is

$$E\{s(n)s(n-\Delta)\} \approx 0 \tag{20}$$

Note:

1. The delay will not decorrelate the periodic interference components. In other words, the delay element can decorrelate the signal part, but not for the interference part.
2. The system efficiency depends principally on how well the assumption of zero signal correlation between primary and reference measurements holds up.
3. The system performance will always be inferior to that of the ANC system with an ideal reference due to the action of the filter on the signal component of the input.

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II. Adaptive Line Enhancement (ALE)

1) Introduction

ALE is a development of the ANC method. The adaptive algorithm is directed towards the problem of enhancing one or more narrowband signals ('spectral lines') of unknown and possibly drifting amplitudes and frequencies which are embedded in broadband noise.

<u>*Example*</u> <u>4</u>: Use passive sonar to detect low-level 'target' signature.

Targets — ships or submarines

Signatures — acoustic emissions from propulsion systems and auxiliary machinery.

The signature typically consists of a number of narrowband components ('spectral lines') superimposed on a broadband system.

(a) The amplitude of the signature at the receiving platform is typically very low relative to these noise sources and thus the measured Signal-to-Noise Ratio (SNR) in such problems is very low. A useful idealization of this problem, therefore, involves a model for the sampled received signal $s'(n)$, as

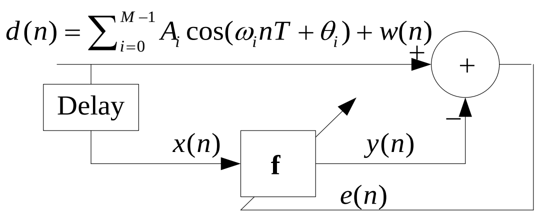
$$s'(n) = \sum_{i=0}^{M-1} A_i \cos(\omega_i nT + \theta_i) + w(n) \tag{21}$$

where A_i, ω_i, and θ_i are the amplitudes, frequencies, and phase angles, respectively, associated with the sinusoidal components. $w(n)$ is the white noise measurement with variance σ_w^2.

(b) Fig. 6.5 depicts the ALE system. The system uses the measured signal as desired response and a delayed version of itself as input. The principle is

$$\begin{aligned} E\{w(n)w(n-\Delta)\} &\approx 0 \\ E\{s(n)s(n-\Delta)\} &\neq 0 \end{aligned} \tag{22}$$

where Δ is a finite delay.



■ **Figure 6.5** Adaptive Line Enhancement (ALE) system. ALE structure is similar to ANC but the roles of signal and noise are reversed – the delay, Δ is intended to decorrelate broadband noise but will not decorrelate narrowband signals.

When functioning in an ideal delay, the adaptive filter output is an enhanced version of the sinusoidal components.

2) Fixed optimal solution to the ALE problem

For the sake of simplicity, $M = 1$. Equation (21) becomes

$$s'(n) = A_0 \cos(\omega_0 nT + \theta_0) + w(n) \tag{23}$$

As usual, we restrict attention to linear processing, and hence we generate an output $y(n)$ which is obtained by applying a linear filter to the measured signal $s'(n)$

$$y(n) = f(n) * s'(n) = y_s(n) + y_w(n) \tag{24}$$

where $y_s(n)$ and $y_w(n)$ are the signal and noise components of the output, respectively.

(a) If the aim of processing is purely to detect presence of signal, we may maximize the output SNR at each sample. The output SNR is defined as

$$SNR_{out} = \frac{\text{output signal power}}{\text{output noise power}} = \frac{E\{y_s^2(n)\}}{E\{y_w^2(n)\}} \tag{25}$$

The optimal linear solution for this problem is provided by the so-called matched filter. For a single sinusoid at frequency ω_0 embedded in white noise, the matched filter can be shown to have the form (see Appendix)

$$f(i) = C\cos(\omega_0 iT + \gamma) \tag{26}$$

where C is a constant and γ is a phase shift which, when added to the phase of the input sinusoid, produces a coherent signal output. That is, γ is chosen to ensure peak signal output at each instant n.

Note:

1 The matched filter solution produces the peak SNR at each sample, but does not preserve the signal waveform in the output. If, in addition to detection, we are interested in **classification**, then preservation of the waveform shape is important.

2 The matched filter solution can only be constructed given prior knowledge of the sinusoidal frequency ω_0.

3 The value C is arbitrary in the sense that its magnitude has no impact on the output SNR.

4 γ is dependent on n and therefore so is $f(i)$. We suppress this dependence here to avoid confusion with the adaptive filter which follows.

(b) The ALE method, introduced in 1975, is a technique designed to

(i) approximate the SNR gain obtained by the matched filter solution for this problem,

(ii) preserve the waveform,

(iii) and no prior knowledge of sinusoidal frequencies, amplitudes or phases, or even of the number of narrowband components present is required.

3) The Least-Squares (LS) Solution for the Line Enhancer

As usual, the adaptive system attempts to iterate towards the LS optimum, and to track variation as they occur.

(a) The input signals may be written

$$\begin{aligned} d(n) &= s'(n) = \sum_{i=0}^{M-1} A_i \cos(\omega_i nT + \theta_i) + w(n) \\ x(n) &= s'(n-\Delta) \end{aligned} \tag{27}$$

where Δ is a delay.

The fixed LS finite causal filter for the single sinusoid model of (23) is given by

$$f^*(i) = B\cos(\omega_0 iT + \phi),\ i = 0,\ 1,\ldots,\ L-1 \tag{28}$$

where $\phi = \Delta\omega_0 T$ is a phase angle which ensures coherent cancellation and

$$B = \frac{2\sigma_s^2/L}{2\sigma_w^2/L + \sigma_s^2} \tag{29}$$

Proof:

The normal equations

$$\sum_{i=0}^{L-1} r(j-i) f^*(i) = g(j),\ j = 0,\ 1,\ldots,\ L-1 \tag{30}$$

where

$$g(j) = E\{d(n)x(n-j)\} = E\{s'(n)s'(n-j-\Delta)\} = r(j+\Delta) \tag{31}$$

The correlation of $s'(n)$ by the assumption that θ_0 is an initially random phase is given by

$$r(m) = \sigma_s^2 \cos(\omega_0 mT) + \sigma_w^2 \delta(m) \tag{32}$$

where $\sigma_s^2 = A_0^2/2$ is the power of the sinusoid and σ_w^2 is the power of the noise.

Substituting $r(m)$ and the assumed form for $f^*(m)$ as given by (28) into the normal equations gives

$$\begin{aligned} &\sum_{i=0}^{L-1} [\sigma_w^2 \delta(j-i) + \sigma_s^2 \cos(\omega_0 (j-i)T)] B\cos(\omega_0 iT + \phi) \\ &= \sigma_w^2 \delta(j+\Delta) + \sigma_s^2 \cos(\omega_0 (j+\Delta)T),\ j = 0,\ 1,\ldots,\ L-1 \end{aligned} \tag{33}$$

Recognizing that with $\Delta > 0$, $\delta(j+\Delta) = 0$ for $j \ge 0$, we have

$$\begin{aligned} &\sigma_w^2 B\cos(\omega_0 jT + \phi) + \sigma_s^2 B \sum_{i=0}^{L-1} \cos(\omega_0 (j-i)T)\cos(\omega_0 iT + \phi) \\ &= \sigma_s^2 \cos(\omega_0 (j+\Delta)T) \end{aligned} \tag{34}$$

Now, expanding the second term of the left-hand side of this equation into sums and differences, and assuming that

$$\sum_{i=0}^{L-1} \cos(2\omega_0 iT + \phi) \to 0 \tag{35}$$

that is, assuming the filter length 'spans' an integer number of period at frequency $2\omega_0 T$, we have

$$[\sigma_w^2 B + \frac{\sigma_s^2 BL}{2}]\cos(\omega_0 jT + \phi) = \sigma_s^2 \cos(\omega_0 (j + \Delta)T),\ j = 0,\ 1, \ldots,\ L-1 \quad (36)$$

A solution to this set of equations is provided by

$$\Delta\omega_0 T = \phi \quad (37)$$

and

$$B = \frac{\sigma_s^2}{\sigma_w^2 + \sigma_s^2 L/2} = \frac{2\sigma_s^2/L}{2\sigma_w^2/L + \sigma_s^2} \quad (38)$$

The solution of (28), with B given by (29) is unique because the matrix **R** with elements given by (32) is positive definite.

Note: When $f^*(n)$ is convolved with $x(n) = \cos(\omega_0(n-\Delta)T)$, the phase angle ϕ acts to produce a coherent output as expected. ■

(b) B can also be written

$$B = \frac{\sigma_s^2/\sigma_w^2}{2/L + \sigma_s^2/\sigma_w^2} \times \frac{2}{L} \quad (39)$$

The input SNR, defined as the ratio of the power of the sinusoid to that of the noise, is given by

$$SNR_{in} = \frac{A_0^2}{2\sigma_w^2} = \frac{\sigma_s^2}{\sigma_w^2} \quad (40)$$

In terms of the input SNR, we have

$$B = \frac{SNR_{in}}{2/L + SNR_{in}} \times \frac{2}{L} \quad (41)$$

(c) Comparing the result of (28) with the matched filter of (26), the general form of the two filters is similar, though for the LS filter, B is determined by (41).

Note:

1 In the LS case, the phase angle is not dependent on the sample index. (waveform preservation)

2 The size of the constant B does not affect the output SNR, but produce imperfect signal cancellation in the error.

3 The great advantage of the LS method, apart from preserving the waveform, is the lack of prior knowledge required. All that is needed for the LS solution are the auto- and cross-correlation coefficients of (32) and (31), which can themselves be estimated from the data.

(d) The gain in SNR which may be obtained with the LS filter. (Assume $\theta_0 = 0$, since the phase has no impact on the SNR of the solution.)

The output of the LS filter is

$$\begin{aligned} y(n) &= f(n) * s(n) \\ &= B\cos(\omega_0 nT) * [A_0 \cos(\omega_0 nT) + w(n)] \\ &= \frac{BA_0 L}{2}\cos(\omega_0 nT) + B\sum_{i=0}^{L-1} w(n-i)\cos(\omega_0 iT) \\ &= y_s(n) + y_w(n) \end{aligned} \tag{42}$$

The output noise power

$$E\{y_w^2(n)\} = [B^2 \sum_{i=0}^{L-1} \cos^2(\omega_0 iT)]\sigma_w^2 = \frac{B^2\sigma_w^2 L}{2} \tag{43}$$

The output SNR is

$$SNR_{out} = \frac{E\{y_s^2(n)\}}{E\{y_w^2(n)\}} = \frac{A_0^2 L}{2\sigma_w^2} = \frac{L\sigma_s^2}{\sigma_w^2} = L \times SNR_{in} \tag{44}$$

(e) The LS result can be generalized to the case of a set of *M* sinusoids in white noise, with solution for $f^*(n)$ given by

$$f^*(n) = \sum_{i=0}^{M-1} B_i \cos(\omega_i nT + \phi_i) \tag{45}$$

For *L* sufficiently large, the B_i's can be approximated by

$$B_i = (\frac{\sigma_i^2}{\sigma_w^2}) \frac{2/L}{2/L + \sigma_i^2/\sigma_w^2} \tag{46}$$

where $\sigma_i^2 = A_i^2/2$. The ϕ_i's act to cancel the delay at each frequency and thus ensure coherent cancellation in the error.

4) The performance of the ALE

(a) Convergence of the filter and nonuniform convergence effects

(i) The single sinusoid problem with inputs in the form of

$$\begin{aligned} d(n) &= s(n) \\ x(n) &= s(n-\Delta) \end{aligned} \tag{47}$$

With

$$s(n) = A_0\cos(\omega_0 nT + \theta_0) + w(n) \tag{48}$$

we have

$$r(m) = \frac{A_0^2}{2}\cos(\omega_0 mT) + \sigma_w^2 \delta(m) \tag{49}$$

Then the correlation matrix **R** has the form

$$\mathbf{R} = \sigma_w^2 \mathbf{I} + \mathbf{R}_c \tag{50}$$

where $\mathbf{R}_c$ is Toeplitz with elements

$$r_c(i) = \frac{A_0^2}{2}\cos(\omega_0 iT) \tag{51}$$

(ii) Following the analysis of chapter 4, we define the mean error vector

$$\mathbf{u}_n = E\{\mathbf{f}_n\} - \mathbf{f}^* \tag{52}$$

By employing the decomposition $\mathbf{R} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$, where $\mathbf{Q}$ is the orthonormal modal matrix, we can decouple

$$E\{\mathbf{f}_{n+1}\} = (\mathbf{I} - \alpha\mathbf{R})E\{\mathbf{f}_n\} + \alpha\mathbf{g} \tag{53}$$

Writing $\mathbf{u}'_n = \mathbf{Q}^T\mathbf{u}_n$, we have

$$\mathbf{u}'_{n+1} = (\mathbf{I} - \alpha\mathbf{\Lambda})\mathbf{u}'_n \tag{54}$$

The decoupled error equation for the *i*th coefficient is therefore

$$u'_{n+1}(i) = (1 - \alpha\lambda_i)u'_n(i), \quad i = 0,\ 1,\ 2, \ldots,\ L-1 \tag{55}$$

where λ_i is the *i*th eigenvalue of $\mathbf{R} = \sigma_w^2\mathbf{I} + \mathbf{R}_c$ and

$$\lambda_i = \sigma_w^2 + \lambda_i^{(c)},\ i = 0,\ 1, \ldots,\ L-1 \tag{56}$$

where $\lambda_i^{(c)}$ is the *i*th eigenvalue of $\mathbf{R}_c$.

We know that the iteration of (55) is stable and that $E\{\mathbf{f}_{n+1}\}$ converges to $\mathbf{f}^*$ provided

$$0 < \alpha < \frac{2}{\lambda_{\max}} \tag{57}$$

where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{R}$. Or

$$0 < \alpha < \frac{2}{\sigma_w^2 + \lambda_{\max}^{(c)}} \tag{58}$$

where $\lambda_{\max}^{(c)}$ is the largest eigenvalue of $\mathbf{R}_c$.

We also know that convergence is characterized by time-constants

$$t_i = \frac{1}{\alpha\lambda_i} \tag{59}$$

(b) The correlation matrix $\mathbf{R}_c$ may be written as

$$\mathbf{R}_c = \frac{A_0^2\mathbf{P}}{2} \tag{60}$$

where $\mathbf{P}$ is an $L \times L$ matrix with coefficients $p(m) = \cos(\omega_0 mT)$.

It can be shown that $\mathbf{P}$ has eigenvalues given by

$$\begin{aligned} \lambda_1 &= \frac{1}{2}[L + \frac{\sin(\omega_0 LT)}{\sin(\omega_0 T)}] \\ \lambda_2 &= \frac{1}{2}[L - \frac{\sin(\omega_0 LT)}{\sin(\omega_0 T)}] \end{aligned} \tag{61}$$

and $\lambda_3 = \lambda_4 = \cdots = \lambda_L = 0$.

Note: For $L > 2$, $\mathbf{R}_c$ does not have full rank. The matrix $\mathbf{R}$ has eigenvalues $\sigma_w^2 + \lambda_i^{(c)}$ which are strictly positive and hence $\mathbf{R}$ does have full rank.

If $\omega_0 = \pi i/(LT)$ for any integer i, then $\sin(\omega_0 LT)/\sin(\omega_0 T) = 0$ and $\lambda_1 = \lambda_2 = L/2$. In fact this is a reasonable approximation for the eigenvalues for most frequencies ω_0, and using this result the eigenvalues of $\mathbf{R}_c$ are given by

$$\lambda_1^{(c)} = \lambda_2^{(c)} = \frac{A_0^2 L}{4},\ \lambda_3^{(c)} = \lambda_4^{(c)} = \cdots = \lambda_L^{(c)} = 0 \tag{62}$$

From (58), stability in the mean requires that

$$0 < \alpha < \frac{2}{\sigma_w^2 + A_0^2 L/4} \tag{63}$$

and the uncoupled modes now adapt with time-constants t_i

$$t_i = \frac{1}{\alpha(\sigma_w^2 + \lambda_i^{(c)})} \tag{64}$$

That is,

$$t_{1,2} = \frac{1}{\alpha(\sigma_w^2 + A_0^2 L/4)} \tag{65}$$

and for the remaining modes

$$t_i = \frac{1}{\alpha\sigma_w^2},\ i = 3,\ 4,\ldots,\ L \tag{66}$$

The convergence is certainly non-uniform.

(c) Another interesting point is that the growth (learning) and delay (forgetting) time constants generally differ.

In particular, if a sinusoid present in signal $s(n)$ is suddenly removed, the presence of this component in the filter response decays at a rate determined by $t = 1/(\alpha\sigma_w^2)$, since this is now driving this mode of the filter. With

$$\frac{1}{\alpha\sigma_w^2} > \frac{1}{\alpha(\sigma_w^2 + A_0^2 L/4)} \tag{67}$$

the decay of the signal component is slower than the growth.

(d) The generalization of these results in the case of M sinusoids with independent starting phases is straightforward. The eigenvalues of the correlation matrix become

$$\lambda_{2i}^{(c)} = \lambda_{2i+1}^{(c)} = \frac{A_i^2 L}{4},\ i = 0,\ 1,\ldots,\ M-1 \tag{68}$$

and

$$\lambda_i^{(c)} = 0,\ i = 2M,\ 2M+1,\ldots,\ L \tag{69}$$

where A_i is the amplitude of sinusoid i, and where we assume $M < L/2$.

(e) For input $x(n)$, with correlation matrix $\mathbf{R} = \sigma_w^2 \mathbf{I} + \mathbf{R}_c$ the result is

$$t_{2i} = t_{2i+1} = \frac{1}{\alpha(\sigma_w^2 + A_i^2 L/4)},\ i = 0,\ 1, \ldots,\ M-1 \tag{70}$$

$$t_i = \frac{1}{\alpha \sigma_w^2},\ i = 2M,\ 2M+1, \ldots,\ L \tag{71}$$

Again, convergence is non-uniform.

Examination of (70) shows that the modes associated with the sinusoids will converge at different rates with the time-constant reducing as the power of the component increases.

Thus, we observe that the speed of convergence of a particular mode increases as the spectral power of the associated component increases.

As we saw in chapter 4, a better limit for overall stability of the adaptive algorithm is given by

$$0 < \alpha < \frac{2}{tr(\mathbf{R})} \tag{72}$$

or, with eigenvalues of the form (68)

$$0 < \alpha < \frac{2}{L(\sigma_w^2 + \frac{1}{4}\sum_{i=0}^{M-1} A_i^2)} \tag{73}$$

Note, however, in the line enhancement where the SNR is often very low, it is necessary to use much lower values for α than suggested by these limits in order to control misadjustment effects.

(f) Misadjustment

From chapter 4, for small α the steady-state mean-squared error J_∞ may be approximated by

$$J_\infty \cong J_{\min}[1 + \frac{\alpha L}{2}(\text{power of the input})] \tag{74}$$

In general, input SNR is usually very low in ALE problems, often less than 0 dB. Thus $J_{\min}$ is typically a large proportion of the total input power. Hence, misadjustment is generally a significant factor in ALE problems.

For a fixed filter length L, $J_{\min}$ is fixed. Similarly, L controls the resolution of the filter and is therefore constrained by the operational requirements of the system.

For a fixed input power, therefore, we can only reduce the misadjustment by reducing α.

$$\alpha \downarrow\ \Rightarrow \begin{cases} \text{convergence rate} \downarrow \\ \text{the tracking capability of the ALE problem} \downarrow \end{cases}$$

(g) Simulations

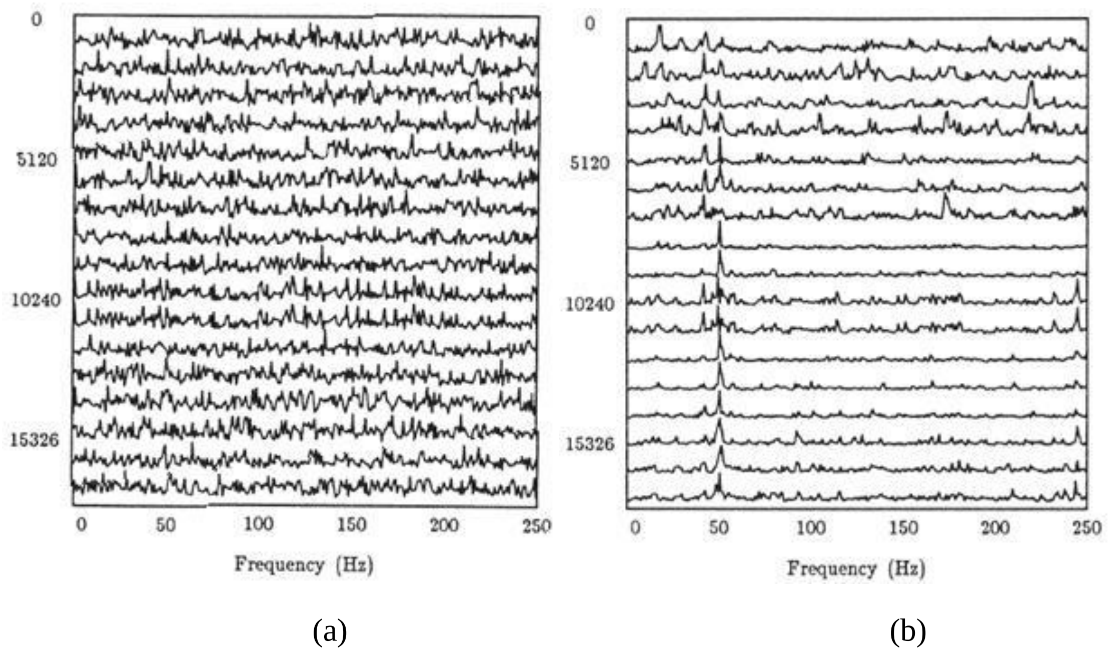
(i) Simulation scenario

- The data employed in these trials consisted of fixed and linearly swept sinusoids embedded in Gaussian white noise.
- The sample rate = 500 Hz.
- The frequency displays are calculated from raw, nonoverlapping, 512 point Discrete Fourier Transforms. These transforms are taken sequentially from the data at intervals of 1024 points.

(ii) Single sinusoid with frequency 50 Hz + white noise (overall SNR = –23.1 dB)

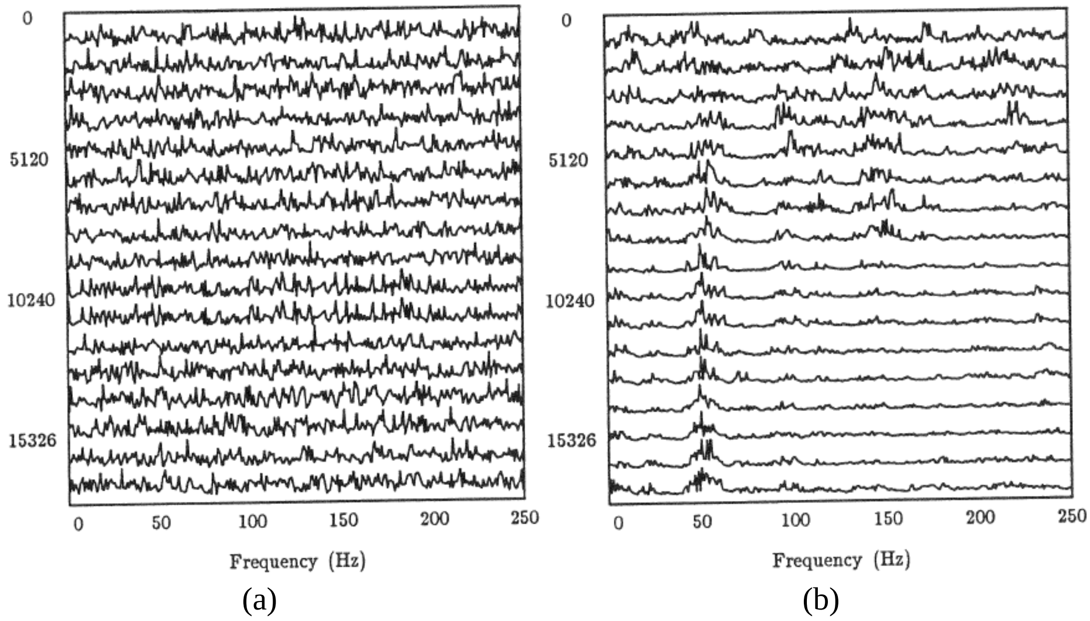
- Filter length, $L = 200$; step size, $\alpha = 10^{-12}$

  Fig. 6.6 demonstrates that enhancement is attained for a sufficiently small value of the adaptation constant α, though the improvement may not become apparent for a significant number of iterations.



■ **Figure 6.6** Adaptive line enhancement trial: single sinusoid plus white noise. (a) The decomposition of the input; (b) the corresponding display for the output.

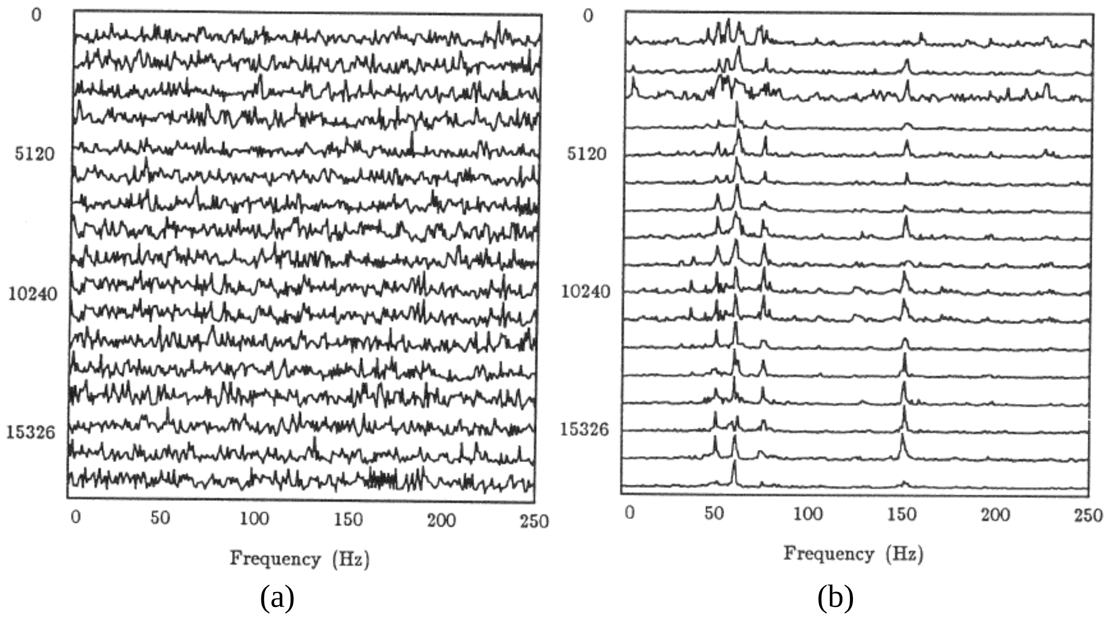
- Filter length, L = 40; step size, $\alpha = 10^{-12}$
  $L \downarrow \Rightarrow$ The resolution of the narrowband signal in the output $\downarrow$.
  Note that the enhancement achieved is essentially independent of f_0.



■ **Figure 6.7** Adaptive line enhancement trial: single sinusoid plus white noise. (a) The decomposition of the input; (b) the corresponding display for the output.

(iii) Multiple sinusoids (with four separate sinusoids at frequencies 50, 60, 75, and 150 Hz) + white noise
(overall SNR = –15.4 dB)

- Filter length, L = 200; step size, $\alpha = 10^{-9}$
  The only limitation in such a trial would be the ability or inability of the filter to resolve closely spaced sinusoids, which is determined by the filter length.

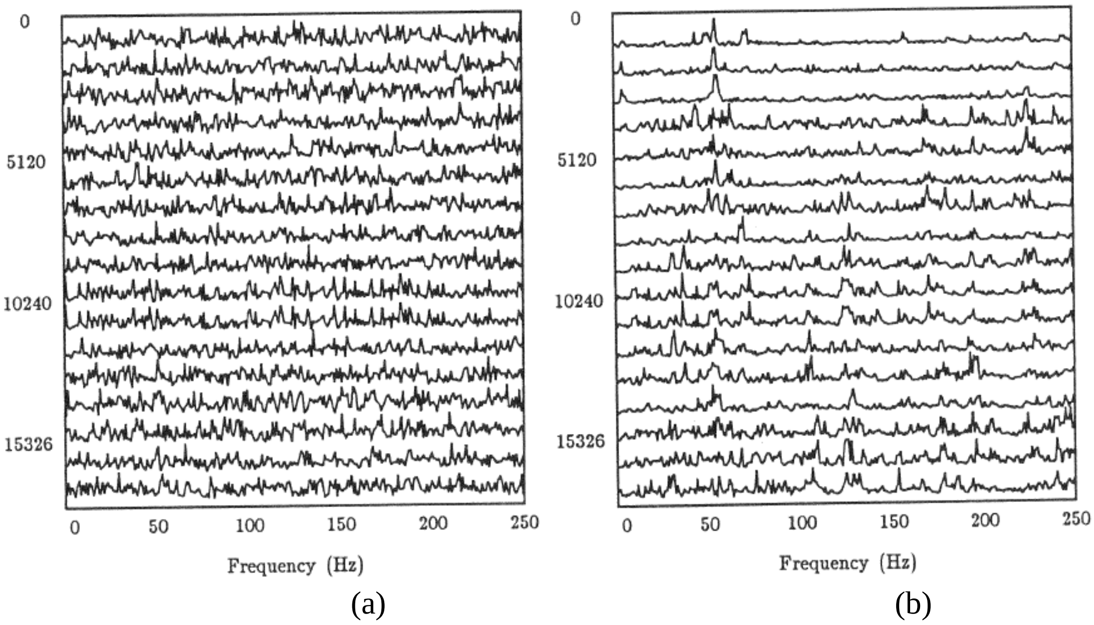


■ **Figure 6.8** Adaptive line enhancement trial: multiple sinusoids plus white noise. (a) The decomposition of the input; (b) the corresponding display for the output.

(iv) Swept sinusoid + noise (overall SNR = –23.1 dB)

- Filter length, L = 200; step size, α = 2.5×10^{-11}

  This simulation indicates one of the major limitations of the ALE method — tracking ability at these extremely low SNR's is very limited. In Fig. 6.9, even the very low sweep range (sweep from 50 to 100 Hz over 20,000 samples) is sufficient to badly impair enhancement.



■ **Figure 6.9** Adaptive line enhancement trial: swept sinusoid plus white noise. (a) The decomposition of the input; (b) the corresponding display for the output.

(h) Referring back to Fig. 6.5, we may observe one of the principle weakness of ALE — at low SNR's the error is almost totally noise. Hence, misadjustment is a major problem.

$$\alpha\downarrow \quad \text{or} \quad L\downarrow \quad \Rightarrow \text{misadjustment} \downarrow$$

$$\Downarrow \qquad\qquad \Downarrow$$

very slow adaptation ↓ filter resolution ↓

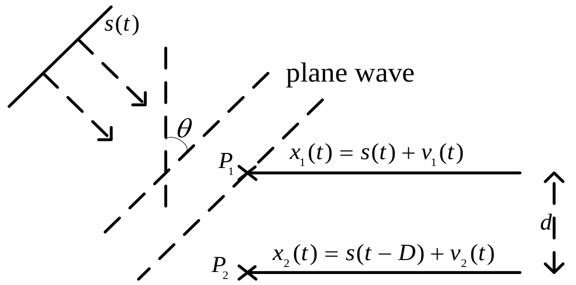
III. Adaptive Filters for Time-Delay Estimation

1) Introduction to Time-Delay Estimation

We consider the use of the LMS adaptive filter for the estimation of the time-delay between two measured signals.

Example 5: Radar, sonar, geophysics, and biomedical signal analysis

In the simplest arrangement, measurements $x_1(t)$ and $x_2(t)$ are made at sensors P_1 and P_2, say, separated by a distance d (see Fig. 6.10).



■ **Figure 6.10** Plane wave signal $s(t)$ incident at sensors P_1 and P_2 separated by (vertical) distance d from an angle θ. Measurement noises $v_1(t)$ and $v_2(t)$ are assumed to be present.

A simple model for the received signal is

$$\begin{aligned} x_1(t) &= s(t) + v_1(t) \\ x_2(t) &= s(t - D) + v_2(t) \end{aligned} \tag{75}$$

where $s(t)$ is the signal, which is derived from some distant source, and where we assume the signal can be represented as traveling through the medium at constant speed along plane waveforms.

Here, $v_1(t)$ and $v_2(t)$ are additive noise terms measured at the receiver, and D is the delay. For convenience, $v_1(t)$ and $v_2(t)$ are assumed to be zero-mean, stationary, and mutually uncorrelated.

$$E\{s(t)v_1(t)\} = 0;\ E\{s(t)v_2(t)\} = 0 \tag{76}$$

The delay

$$D = \frac{d}{c}\sin\theta \tag{77}$$

where θ is the arrival or 'bearing' angle to the source, and c is the propagation velocity of the signal through the medium.

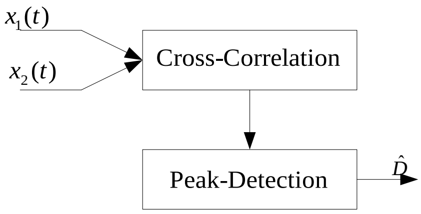
Therefore, the estimation of θ is reduced to the estimation of D given the noise measurements $x_1(t)$ and $x_2(t)$.

(a) One approach to the problem consists of simply computing the cross-correlation between $x_1(t)$ and $x_2(t)$ and taking the peak value as the estimate of the time-delay. The correlation for the stationary continuous signals $x_1(t)$ and $x_2(t)$ is

$$r_{x_1x_2}(\tau) = E\{x_1(t)x_2(t+\tau)\} \tag{78}$$

where τ is the correlation lag.

The peak value for the correlation lag $\tau = \hat{D}$, say, provides the required estimate of the delay.



■ **Figure 6.11** Schematic of delay estimation using cross-correlation. Delay estimate $\hat{D}$ is taken as the peak value of the short-time correlation between $x_1(t)$ and $x_2(t)$.

Referring to the model of (75), the nature of this peak will be strongly dependent on the properties of $s(t)$, and the measurement noises $v_1(t)$ and $v_2(t)$.

(b) The generalized correlation method

The generalized or weighted cross-correlation is defined as

$$r_{y_1y_2}(\tau) = w(\tau) * r_{x_1x_2}(\tau) \tag{79}$$

where $w(\tau)$ is a weighting function.

A number of estimators can be accommodated within this framework corresponding to different weighting function $w(\tau)$. For example, $w_{ML}(\tau)$ is the optimal Maximum Likelihood (ML) weighting. The ML weighting function is expressed in the frequency domain in terms of the magnitude squared coherence (MSC) function between $x_1(t)$ and $x_2(t)$. The function is defined by

$$MSC = |\gamma_{12}(\omega)|^2 = \frac{|R_{x_1x_2}(\omega)|^2}{R_{x_1x_1}(\omega)R_{x_2x_2}(\omega)} \tag{80}$$

where $R_{x_ix_i}(\omega)$ and $R_{x_ix_j}(\omega)$ are the power and cross spectra, respectively, for signals $x_i(t)$ and $x_j(t)$.

Note: The notation of the spectra for continuous and discrete signals are $R(\omega)$ and $R(e^{j\omega})$.

With these definitions for MSC and spectra, the ML weighting can be shown to have the form [2]

$$W_{ML}(\omega) = \frac{|\gamma_{12}(\omega)|^2}{|R_{x_1x_2}(\omega)|[1-|\gamma_{12}(\omega)|^2]} \tag{81}$$

where $|\gamma_{12}(\omega)|^2 < 1$.

This weighting may be expanded using the signal model of (75) as

$$W_{ML}(\omega)=\frac{R_{ss}(\omega)\big/[R_{v_1v_1}(\omega)R_{v_2v_2}(\omega)]}{1+R_{ss}(\omega)\big/R_{v_2v_2}(\omega)+R_{ss}(\omega)\big/R_{v_1v_1}(\omega)} \tag{82}$$

where the derivation of this equation depends on the assumed properties for the correlations of $s(t)$, $v_1(t)$, and $v_2(t)$ given above.

If the noises at each sensor have identical spectra $R_{vv}(\omega)$, (82) becomes

$$W_{ML}(\omega)=\frac{R_{ss}(\omega)\big/R_{vv}^2(\omega)}{1+2R_{ss}(\omega)\big/R_{vv}(\omega)} \tag{83}$$

Two points are apparent from (83):

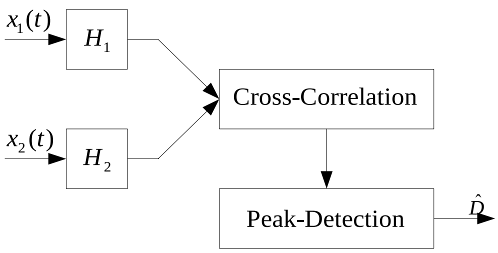
(i) Even in this simplified case, to effect the optimal weighting we need prior knowledge of both signal and noise spectra.

(ii) The solution is fixed, having no flexibility in the face of time-varying delays caused by source and/or receiver motion.

(c) The method given above and other generalized cross-correlation methods can be viewed as a prefiltering operation followed by cross-correlation in which individual inputs are filtered or weighted with transforms $H_1(\omega)$ and $H_2(\omega)$, prior to the correlation operation.

The weighting function in (79) is then equivalent to

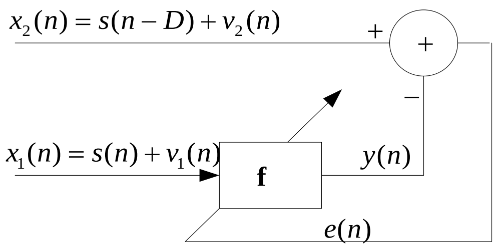
$$W(\omega)=H_1(\omega)H_2^*(\omega) \tag{84}$$



■ **Figure 6.12** Schematic of generalized cross-correlation for delay estimation. Raw correlation scheme has been augmented by pre-filters $H_1(\omega)$ and $H_2(\omega)$ applied to $x_1(t)$ and $x_2(t)$, respectively.

2) The LMS Time-Delay Estimator (LMSTDE)

One approach to the time-delay estimation problem that is particularly attractive when limited *a priori* information about signal and noise spectra is available, or when the delay estimate is subject to variation, is to utilize an adaptive filter. The LMSTDE is a scheme which attempts to do just that (see Fig. 6.13). Referring to the figure, the two inputs $x_1(t)$ and $x_2(t)$ are sampled at intervals T, say, giving $x_1(n)$ and $x_2(n)$.



■ **Figure 6.13** Adaptive filtering for time-delay estimation. **f** acts to cancel the delay between $x_1(n)$ and $x_2(n)$. The peak in **f** gives the delay estimate. Here D is taken as an integer multiple of the sample interval.

(a) The LMS update formula becomes

$$\mathbf{f}_{n+1} = \mathbf{f}_n + \alpha e(n)\mathbf{x}_n^{(1)} \tag{85}$$

where $\mathbf{x}_n^{(1)} = [x_1(n) \quad x_1(n-1) \quad \cdots \quad x_1(n-L+1)]^T$.

The desired input is formed from $x_2(n)$ and thus the error is

$$e(n) = x_2(n) - y(n) = x_2(n) - \mathbf{f}_n^T \mathbf{x}_n^{(1)} \tag{86}$$

and where, as usual $\mathbf{f}_n = [f_n(0) \quad \cdots \quad f_n(L-1)]^T$ and α is the adaptation constant.

(b) The delay estimate is taken as the peak value from the filter response. The major impact of this change is that if the time-delay does not correspond exactly to an integer multiple of the sample interval, then the peak in the filter response cannot correspond to the exact delay. To mitigate this problem, an interpolation between adjacent samples in the filter response is used to improve the delay estimate.

(c) From Fig. 6.13, we have

$$e(n) = s(n-D) + v_2(n) - f(n) * [s(n) + v_1(n)] \tag{87}$$

Given $v_1(n)$ and $v_2(n)$ mutually uncorrelated, it seems apparent that a good solution would be $f(n) = \delta(n-D)$ producing

$$e(n) = v_2(n) - v_1(n-D) \tag{88}$$

with

$$E\{e^2(n)\} = \sigma_2^2 + \sigma_1^2 \tag{89}$$

where σ_1^2 and σ_2^2 are the variances of the noise terms $v_1(n)$ and $v_2(n)$, respectively.

(d) When the input signals are statistically stationary, the steady-state behavior of the LMS filter may be approximated by the infinite two-sided Wiener filter. For the input signals $x_1(n)$ and $x_2(n)$, we have

$$F(e^{j\omega}) = \frac{R_{x_1x_2}(e^{j\omega})}{R_{x_1x_1}(e^{j\omega})} \tag{90}$$

In the noise-free case ($v_1(n) = v_2(n) = 0$), (90) reduces to

$$F(e^{j\omega}) = \frac{R_{ss}(e^{j\omega})e^{-j\omega D}}{R_{ss}(e^{j\omega})} = e^{-j\omega D} \tag{91}$$

and hence

$$f(n) = \delta(n-D) \tag{92}$$

In other cases, the noise modifies the response as

$$F(e^{j\omega}) = \frac{R_{ss}(e^{j\omega})e^{-j\omega D}}{R_{ss}(e^{j\omega}) + R_{v_1v_1}(e^{j\omega})} \tag{93}$$

so that the energy in the filter is spread and the time-domain resolution is reduced.

<u>*Example* 6</u>: LMSTDE

Input signal: $x_1(n) = s(n) + v_1(n)$ and $x_2(n) = s(n-D) + v_2(n)$

There are four forms for $s(n)$:

(i) $s(n) = w(n)$ is a zero-mean Gaussian iid sequence

(ii) $s(n) = h_1(n) * w(n)$

(iii) $s(n) = h_2(n) * w(n)$

(iv) $s(n) = h_3(n) * w(n)$

where $h_1(n)$, $h_2(n)$, and $h_3(n)$ are unit gain linear phase bandpass filters with nominal passbands:

$$\begin{aligned} H_1(e^{j2\pi fT}) &= 1,\ 150 \le f \le 350 \text{ Hz} \\ H_2(e^{j2\pi fT}) &= 1,\ 200 \le f \le 300 \text{ Hz} \\ H_3(e^{j2\pi fT}) &= 1,\ 225 \le f \le 275 \text{ Hz} \end{aligned} \tag{94}$$

The bandwidths are given relative to a sampling rate of 1000 Hz. The delay D was set to 10 samples. Thus we are concerned with the performance of the LMSTDE with broadband and with bandpass signals.

(1) $s(n) = w(n)$, filter length = 21, $\alpha = 0.01$

The optimal LS solution is $\mathbf{f}^* = \mathbf{R}^{-1}\mathbf{g}$. The correlation matrix is diagonal with $r(0) = \sigma_w^2$ and the cross-correlation vector has a single nonzero element $g(D) = \sigma_w^2$. The LS solution is

$$\mathbf{f}^* = \mathbf{R}^{-1}\mathbf{g} = \frac{1}{\sigma_w^2}[0,...,0,\sigma_w^2,0,...,0]^T \tag{95}$$

$$f^*(i) = \begin{cases} 1, & i = D \\ 0, & i \neq D \end{cases}$$

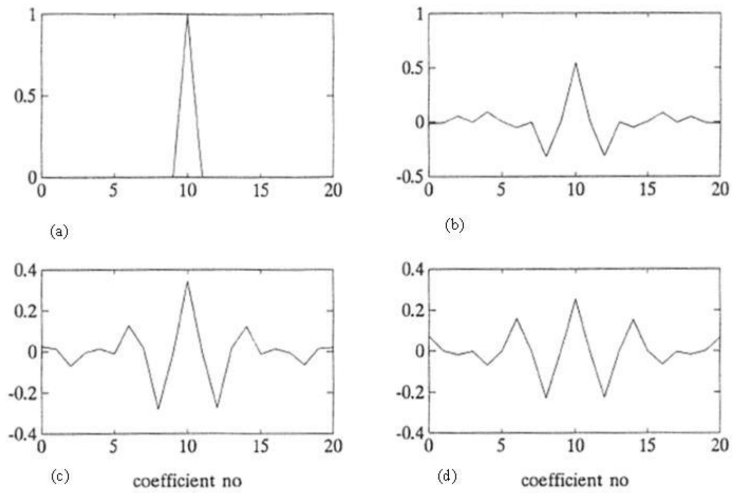
The performance of the filter is illustrated in Fig. 6.14(a) which shows the final set of filter coefficients after a run of 1000 iterations. Fig. 6.15(a) shows the track of $f_n(10)$ for n = 0, 1,…, 999, confirming the expected convergence.

(2) The final sets of filter coefficients obtained for the bandpass inputs from (ii), (iii), and (iv) are shown in Fig. 6.14(b), (c), and (d), respectively.

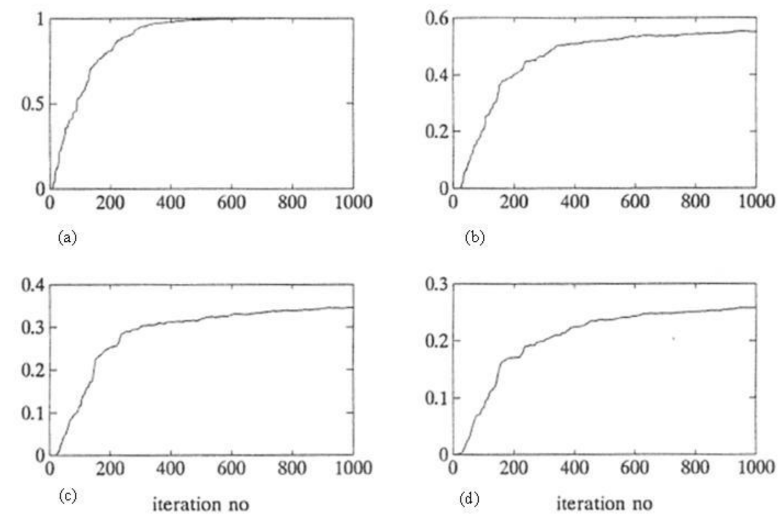
Bandwidth$\downarrow$ $\Rightarrow$Energy within the filter becomes less concentrative.

$\Rightarrow$ The peak becomes less and less distinct and hence more ambiguous.

The limit occurs when the input is a sinusoid. In this case, the filter response is also sinusoidal and the peak value is not unique.



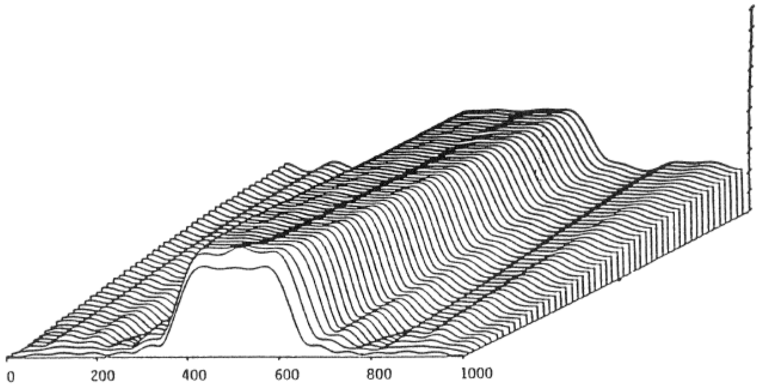
■ **Figure 6.14** Filter vectors for four signal conditions: (a) white input, (b) bandpass white input, bandwidth 200 Hz, (c) bandpass white noise, bandwidth 100Hz, (d) bandpass white noise, bandwidth 50 Hz. Delay = 10, f_s = 1000 for all cases.



■ **Figure 6.15** $f_n(10)$ for n = 0, 1,…, 999, for four signal conditions: (a) white input, (b) bandpass white input, bandwidth 200 Hz, (c) bandpass white noise, bandwidth 100Hz, (d) bandpass white noise, bandwidth 50 Hz. Delay = 10, f_s = 1000 for all cases.

(3) We may use bandpass signals such as these to illustrate the eigenvalue effect in the LMS filter. We recall from the ALE section that the eigenvalue disparity effect is such that we expect the filter to respond more quickly to stronger frequency components in the input.

In Fig. 6.16, each line (in time) is a Fourier transform of an impulse response, obtained by zero-padding the response to 256 points.



■ **Figure 6.16** Time-frequency decomposition of the LMS adaptive filter response. Each line represents the amplitude spectrum obtained by zero-padding the response to 256 points.

As expected, we see a rapid growth in the filter within the passband for the data. The out-of-band response, however, develops much more slowly, clearly illustrating slower modes of convergence associated with those frequency components. Even so, we should note that in this noise-free environment the delay estimate obtained by taking the peak value is correct within the first few iterations.

(e) In a noisy environment ($v_1(n)$, $v_2(n) \neq 0$),

the LS filter ≠ the ML estimator

$\Rightarrow$ It cannot achieve the minimum variance for the delay estimator as expressed by the CRLB.

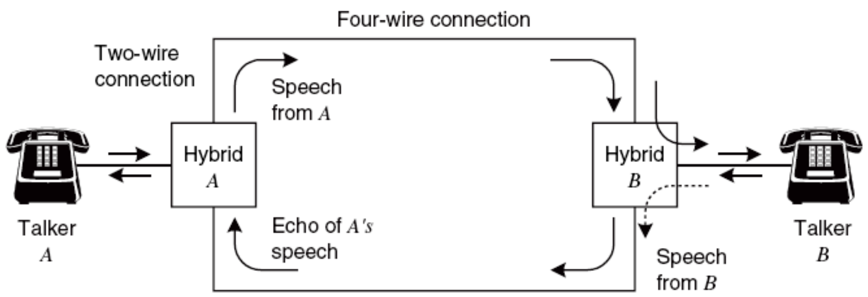
If the fixed processor does not achieve the CRLB, the adaptive scheme cannot be expected to do so. On the other hand, the advantages of the adaptive implementation, as in other application, are the very limited *a priori* knowledge required, the simplicity of the implementation, and the ability to track time-vary input conditions.

IV. Some Applications of Adaptive Filtering in Communications

1) Echo Cancellation in Voiced Channels

(a) A significant problem in long-distance telephone links is the generation of echoes. Echoes can arise in a number of ways but are primarily caused by impedance mismatch at the point where the two-wire local subscriber loop becomes a four-wire trunk.

In two-wire transmission, speech moving in both directions is superimposed. That is, all the speech is carried by the same wire pair. In the four-wire trunk, speech moving in each direction is carried separately. The four-wire arrangement is necessary if amplifiers, switches, and other devices are to be used.



■ **Figure 6.17** Schematic of a telephone communication system. Echoes are created by energy reflected due to impedance mismatch at the hybrids [1], [3].

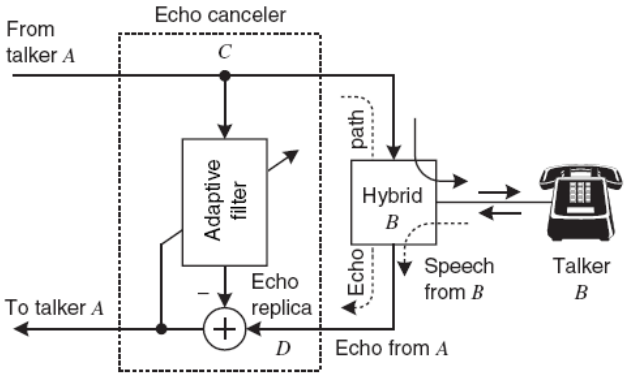
The hybrid is intended to direct arriving energy from the four-wire section into the local two-wire loop. Unfortunately, because of impedance mismatches some of this energy is returned to the outgoing leg of the four-wire system and is transmitted along with the outgoing speech. Consequently, after a transmission delay, this speech is returned to the original speaker and is received as an echo.

(b) The impact of echoes depends on the length of the loop, and hence on the two-way propagation delay.

(i) For short delays (≤ 50ms), the echo will have no subjective impact.

(ii) Transmission over long paths, for example with satellite transmission, may involve delays in excess of 0.5s, however. Such echoes are very disquieting for the speaker and can make conversation difficult to conduct.

(c) Cancel the echoes using an adaptive filter

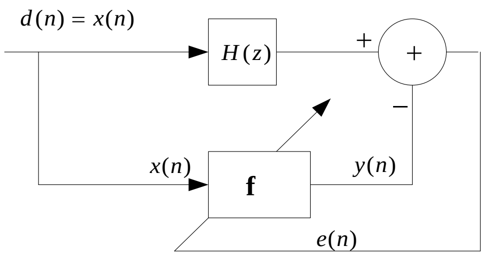


■ **Figure 6.18** Echo cancellation in telephone communications. The canceller models the reflection at the hybrid and cancels echoes [1], [3].

If we imagine temporarily that speaker 2 is silent, then the role of the adaptive filter is to identify the transfer function of the echo generation process within the hybrid. If this is successful, the filter output will equal the echo, which will therefore be cancelled in the error.

(d) The use of the adaptive filter in a system identification mode

In fact, the use of the adaptive filter in a system identification mode is not unique to the echo cancellation problem. The basic configuration is shown in Fig. 6.19. It is assumed that measurements of the input and output for the unknown system are available.



■ **Figure 6.19** Adaptive filtering for system identification.

For the system identification problem, when the desired signal is exactly matched by the adaptive filter output, this situation is referred to as **input-output equivalence**. Even if the unknown system and the adaptive filter are input-output equivalent, however, it is not certain that the filter coefficients will match the unknown system. There are several reasons why this can occur:

(1) The nature of the unknown system may be different from that of the adaptive filter.

Example 7:

$$\begin{cases} h(i): \text{ a recursive implementation} \\ f_n(i): \text{ a tapped delay line} \end{cases}$$

$$\begin{cases} h(i): \text{ linear or nonlinear} \\ f_n(i): \text{ linear} \end{cases}$$

(2) Even if the structure of $h(i)$ and $f_n(i)$ are the same, the order (that is length) of the filters may differ.

(3) The excitation may not be broadband. If this occurs, the adaptive filter can only model the unknown system at frequencies which are excited at the input. At other frequencies, there is no input, no output, and therefore no information available for the identification.

(4) The measurement of input, output, or both may be corrupted by noise.

The point is that even in the face of any or all of these problems, the adaptive filter may still be able to match the system output. That is, it may be input-output equivalent, because of the extra flexibility provided by the time-variation in the adaptive filter.

(e) Even the transfer function of the hybrid is nonlinear, this does not necessarily stop the LMS from effective cancellation, however, because as we have noted the time-variation within the adaptive filter provides an extra degree of freedom which can enable the filter to mimic nonlinear effects.

(f) The echo cancellation system is complicated by the presence in $d(n)$ of speech from speaker 2, $s_2(n)$, say. While this is uncorrelated with $x(n)$ and therefore should have limited impact on the adaptive solution, $s_2(n)$ will contribute to $e(n)$ and therefore will produce a non-zero minimum mean-squared error.

$\Rightarrow$ The system will exhibit misadjustment.

(i) To minimize this, the echo canceller should converge as rapidly as possible after the connection is made, before speaker 2 starts to talk.

(ii) More sophisticated cancellers usually have a built-in switch whose objective is to detect the onset of (local) speech. Once such speech is detected, adaptation can be frozen until a further period of silence occurs.

2) Adaptive Differential Pulse Code Modulation (ADPCM)

(a) Consider a communication channel in which digital data is transmitted using B bits to represent each sample. For a uniform sample rate f_s, the total bit rate

$$R = f_s B \quad \text{bits/sec (bps)} \tag{96}$$

(1) $R\uparrow \;\Rightarrow$ the cost of transmission $\uparrow$

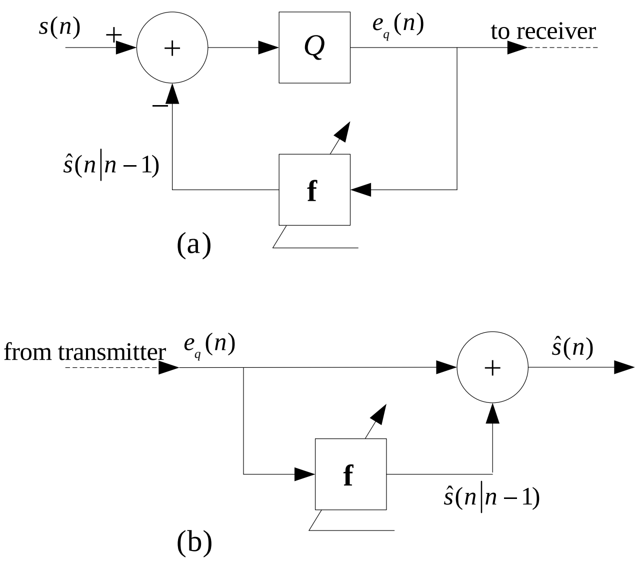
(2) The sample rate is fixed by the bandwidth of the signal, hence only through reduction of B reduction in data rate can be obtained.

(3) $B\uparrow \;\Rightarrow$ keep quantization noise at reasonable level
The problem, therefore, is to reduce B while maintaining acceptable quality of the received data.

(b) The ADPCM systems are based upon the following premise:
The number of bits required to transmit a signal with a given fidelity is proportional to the power of the signal transmitted.

(1) This principle is exploited in DPCM in which the difference between adjacent samples is transmitted, rather than the data itself.
(2) Speech has a fairly high short-term correlation; consequently the power of the difference is substantially lower than that of the original signal.
(3) At the next level of sophistication, a linear prediction of the signal is used (sometimes with just a single predictor coefficient) and the error is transmitted.
(4) In ADPCM, an adaptive prediction of the signal is employed and it is the error or residual which is transmitted.
(5) Because the power of the prediction error is less than that of the speech, less bits are required for transmission.

(c) An ADPCM system has adaptive predictions at the transmitter and the receiver.



■ **Figure 6.20** ADPCM system. (a) Transmitter; (b) receiver. Coefficients **f** are driven by quantized error $e_q(n)$. Note that a single sample delay is implicit in the filtering system.

(1) A prediction, $\hat{s}(n|n-1)$ is formed using previous predictions and the prediction error

$$e(n) = s(n) - \hat{s}(n|n-1) \tag{97}$$

And $e(n)$ is quantized to B bits

$$e_q(n) = Q\{e(n)\} \tag{98}$$

where $e_q(n)$ is the quantized error signal. The prediction $\hat{s}(n|n-1)$ has the form

$$\hat{s}(n|n-1) = \mathbf{f}_{n-1}^T \mathbf{e}_q(n-1) \tag{99}$$

where $\mathbf{e}_q(n) = [e_q(n), e_q(n-1), \ldots, e_q(n-L+1)]^T$.

(2) At the receiver, an identical prediction scheme is used to reconstruct $s(n)$ from the transmitted $e_q(n)$. The reconstruction is completed by forming the sum

$$\hat{s}(n) = \hat{s}(n|n-1) + e_q(n) \tag{100}$$

It is easy to see that if no quantization is present (that is if $e_q(n) = e(n)$), then $\hat{s}(n) = s(n)$ and the transmission is perfect. Moreover, this result holds irrespective of the nature of the prediction. This illustrates one of the important spectral features of this particular application — the prediction is essentially secondary to the quantization.

(i) In the absence of quantization the predictor becomes irrelevant.

(ii) In general, the quantization scheme is the primary determinant of the performance of the system.

(d) An LMS type predictor for an L point FIR implementation is provided by the equations:

$$\hat{s}(n|n-1) = \mathbf{f}_{n-1}^T \mathbf{e}_q(n-1) \tag{101}$$

$$e(n) = s(n) - \hat{s}(n|n-1) \tag{102}$$

$$e_q(n) = Q\{e(n)\} \tag{103}$$

$$\mathbf{f}_n = \mathbf{f}_{n-1} + \alpha \mathbf{e}_q(n-1) e_q(n) \tag{104}$$

(1) Instead the receiver computes the reconstructed signal $\hat{s}(n)$ via (100). It is apparent that given no channel errors and the same initial conditions, the adaptive filter coefficients at both transmitter and receiver are identical for all iterations (They operate on identical input data.)

(2) Note that, if one or more bit errors do occur in transmission, the transmitted and received $e_q(n)$ sequences will no longer be identical. Hence, the transmitted and received filter coefficients will also diverge.

(3) More disturbingly, the adaptive filter at the receiver has no feedback mechanism by which such errors can be corrected. This is because the sequence $e_q(n)$ is provided at the end of the transmission process and is not generated by the receiver. For this and other reasons, ADPCM systems almost never employ the simple LMS type update of (104). In practice, the algorithm is normalized and a leakage factor is incorporated.

Appendix
Matched Filter for Sinusoid in White Noise

Consider the signal

$$x(n) = A_0 \cos(\omega_0 nT + \theta_0) + w(n) = s(n) + w(n) \tag{A1}$$

where $s(n) = A_0 \cos(\omega_0 nT + \theta_0)$, and $w(n)$ is a zero-mean iid noise sequence with unit variance. When such a signal is filtered using a linear filter **f**, the output Signal-to-Noise Ratio (SNR_{out}) may be defined by

$$\begin{aligned} SNR_{out} &= \frac{\text{output signal power}}{\text{output noise power}} \\ &= \frac{(\mathbf{f}^T \mathbf{s}_n)^2}{E\{(\mathbf{f}^T \mathbf{w}_n)^2\}} = \frac{\mathbf{f}^T(\mathbf{s}_n \mathbf{s}_n^T)\mathbf{f}}{\mathbf{f}^T\mathbf{f}} \end{aligned} \tag{A2}$$

where $\mathbf{s}_n = [A_0 \cos(\omega_0 nT + \theta_0), ..., A_0 \cos(\omega_0 (n-L+1)T + \theta_0)]^T$ is the signal vector, and

where $\mathbf{w}_n = [w(n), w(n-1), ..., w(n-L+1)]^T$ is a vector of noise samples. The maximization of SNR_{out} is therefore equivalent to solving the constrained problem:

$$\max \ \{\mathbf{f}^T(\mathbf{s}_n \mathbf{s}_n^T)\mathbf{f}\} \text{ subject to } \mathbf{f}^T\mathbf{f} = k \tag{A3}$$

where k is a constant. Using Lagrange multipliers, we may write a modified functional J as

$$J = \mathbf{f}^T \mathbf{R}_s \mathbf{f} - \lambda(\mathbf{f}^T\mathbf{f} - k) \tag{A4}$$

where $\mathbf{R}_s = \mathbf{s}_n \mathbf{s}_n^T$ and λ is the Lagrange multiplier.

Differentiating (A4) gives

$$2\mathbf{R}_s \mathbf{f} - 2\lambda \mathbf{f} = 0 \tag{A5}$$

and hence

$$(\mathbf{R}_s - \lambda \mathbf{I})\mathbf{f} = 0 \tag{A6}$$

Note that λ is an eigenvalue of $\mathbf{R}_s = \mathbf{s}_n \mathbf{s}_n^T$ and **f** is the corresponding eigenvector. Also, from (A6)

$$\mathbf{f}^T \mathbf{R}_s \mathbf{f} - \lambda \mathbf{f}^T \mathbf{f} = 0 \tag{A7}$$

or

$$\lambda = \frac{\mathbf{f}^T \mathbf{R}_s \mathbf{f}}{\mathbf{f}^T \mathbf{f}} = SNR_{out} \tag{A8}$$

where the second equality follows from (A2). Clearly from (A8) $SNR_{\max}$ corresponds to $\lambda_{\max}$, the largest eigenvalue, and the optimum filter is given by the corresponding eigenvector, viz

$$SNR_{\max} = \lambda_{\max} = \frac{\mathbf{f}^{*T}\mathbf{R}_s\mathbf{f}^*}{\mathbf{f}^{*T}\mathbf{f}^*} \tag{A9}$$

Now, from (A6)

$$(SNR_{\max})\mathbf{f}^* = \mathbf{R}_s\mathbf{f}^* \tag{A10}$$

so that combining with (A9) we have

$$\frac{\mathbf{f}^{*T}\mathbf{s}_n\mathbf{s}_n^T\mathbf{f}^*}{\mathbf{f}^{*T}\mathbf{f}^*}\mathbf{f}^* = \mathbf{s}_n\mathbf{s}_n^T\mathbf{f}^* \tag{A11}$$

Canceling the scalar terms $\mathbf{s}_n^T\mathbf{f}^*$, we may write

$$\frac{\mathbf{f}^{*T}\mathbf{s}_n}{\mathbf{f}^{*T}\mathbf{f}^*}\mathbf{f}^* = \mathbf{s}_n \tag{A12}$$

but

$$\frac{\mathbf{f}^{*T}\mathbf{s}_n}{\mathbf{f}^{*T}\mathbf{f}^*} = \frac{1}{c_1} \tag{A13}$$

a scalar value. Hence, from (A12)

$$\mathbf{f}^* = c_1\mathbf{s}_n \tag{A14}$$

With components

$$f^*(i) = c_1A_0\cos[\omega_0(n-i)T + \theta_0],\ i = 0,\ 1, \ldots,\ L-1 \tag{A15}$$

or

$$f^*(i) = C\cos[\omega_0 iT + \gamma],\ i = 0,\ 1, \ldots,\ L-1 \tag{A16}$$

where $C = c_1A_0$, $\gamma = -\theta_0 - \omega_0 nT$. Also, the magnitude of C has no impact on SNR_{out}, but to satisfy a particular constraint $\mathbf{f}^{*T}\mathbf{f}^* = k$, we have from (A14)

$$c_1^2\mathbf{s}_n^T\mathbf{s}_n = k \tag{A17}$$

or

$$c_1^2 = \frac{k}{\mathbf{s}_n^T\mathbf{s}_n} = \frac{2k}{A_0^2L} \tag{A18}$$

Note:

Strictly, the matched filter $\mathbf{f}^*$, depends on n through the phase angle γ. We have suppressed this dependence in the notation used here in the interests of avoiding confusion with the adaptive filter.

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