Homework #1 Solutions

1. (16%)

Let $\mathbf{x}=\begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}^T$ and $\mathbf{v}=\begin{bmatrix}v_1 & v_2 & v_3 & v_4\end{bmatrix}^T$. **x** is a linear transform of **v**, i.e.,

$$\mathbf{x}=\mathbf{A}\mathbf{v}=\begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 1 & 1 & -1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\end{bmatrix}\mathbf{v}.$$

The correlation matrix of **x** is

$$\mathbf{R}_x=\mathbf{A}E\left\{\mathbf{v}\mathbf{v}^H\right\}\mathbf{A}^H=\mathbf{A}\mathbf{R}_v\mathbf{A}^H=\mathbf{A}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 4\end{bmatrix}\mathbf{A}^H=\begin{bmatrix}10 & 0 & -2 & -4\\ 0 & 10 & 4 & 2\\ -2 & 4 & 10 & 0\\ -4 & 2 & 0 & 10\end{bmatrix}$$

Thus, (x_1,x_2) and (x_3,x_4) are uncorrelated.

2. (16%)

By Cauchy-Schwarz inequality, ($\left(E\{XY\}\right)^2\le E\left\{X^2\right\}E\left\{Y^2\right\}$)

$$\left(E\left\{x(n)y(n+i)\right\}\right)^2\le E\left\{x^2(n)\right\}E\left\{y^2(n+i)\right\}$$

$$\underset{x(n)\text{ and }y(n)\text{ are joint WSS.}}{\Rightarrow} r_{xy}{}^2(i)\le r_{xx}(0)r_{yy}(0)\Rightarrow\left|r_{xy}(i)\right|\le\sqrt{r_{xx}(0)r_{yy}(0)}$$

3. (16%)

$$r_{\omega\omega}(m)=\sigma_\omega^2\delta(m),\ \sigma_\omega^2=E\left\{\omega^2(n)\right\}$$

(1) (8%)

$$h(m)=\delta(m)+a\delta(m-1),\ h(-m)=\delta(m)+a\delta(m+1)$$

$$\begin{aligned}r_{xx}(m)&=h(m)*h(-m)*r_{\omega\omega}(m)\\&=\left[\delta(m)+a\delta(m-1)\right]*\left[\delta(m)+a\delta(m+1)\right]*\sigma_\omega^2\delta(m)\\&=\left[\delta(m)+a\delta(m+1)+a\delta(m-1)+a^2\delta(m)\right]*\sigma_\omega^2\delta(m)\\&=\left[\left(1+a^2\right)\delta(m)+a\delta(m+1)+a\delta(m-1)\right]\sigma_\omega^2\end{aligned}$$

(2) (8%)

$$g(m) = h(m) * h(-m) = \sum_{k=-\infty}^{\infty} h(k) h(m+k)$$
$$= \sum_{k=-\infty}^{\infty} a^{2k} u[k] a^{2(m+k)} u[m+k] = \sum_{k=-\infty}^{\infty} a^{4k} a^{2m} u[m+k]$$
$$= \frac{a^{2|m|}}{1-a^4}$$

$$r_{xx}(m) = g(m) * \sigma_\omega^2 \delta(m) = \frac{a^{2|m|}}{1-a^4} \sigma_\omega^2$$

4. (16%)

$$r(m) = E\{x(n)x(n-m)\}$$
$$= E\{[w(n) + b_1 w(n-1)][w(n-m) + b_1 w(n-m-1)]\}$$
$$r(0) = E\{w^2(n)\} + b_1^2 E\{w^2(n-1)\} = \sigma_w^2 (1 + b_1^2)$$
$$r(1) = b_1 E\{w^2(n-1)\} = b_1 \sigma_w^2$$
$$\Rightarrow \left|\frac{r(1)}{r(0)}\right| = \left|\frac{b_1}{1+b_1^2}\right| \triangleq |f(b_1)|$$
$$f'(b_1) = \frac{1-b_1^2}{(1+b_1^2)^2} = 0 \Rightarrow b_1 = \pm 1$$
$$\frac{r(1)}{r(0)} = \begin{cases} -0.5 \text{ (minimum) when } b_1 = -1 \\ \ \ 0.5 \text{ (maximum) when } b_1 = 1 \end{cases} \Rightarrow \left|\frac{r(1)}{r(0)}\right| \le 0.5$$

5. (16%)

(1) (8%)

$$R_{zz}(m,n) = E\{[x(m) + y(m)][x(n) + y(n)]\}$$
$$= R_{xx}(m,n) + R_{xy}(m,n) + R_{yx}(m,n) + R_{yy}(m,n)$$

(a) (4%)

If $x(n)$ and $y(n)$ are jointly WSS, then

$$r_{zz}(i) = r_{xx}(i) + r_{xy}(i) + r_{yx}(i) + r_{yy}(i)$$

where $i = m - n$. Taking the Fourier transform of the above expression, we obtain $R_{zz}(e^{j\omega}) = R_{xx}(e^{j\omega}) + R_{xy}(e^{j\omega}) + R_{yx}(e^{j\omega}) + R_{yy}(e^{j\omega})$.

(b) (4%)

If $x(n)$ and $y(n)$ are orthogonal, $r_{xy}(i) = r_{yx}(i) = 0$. Then $r_{zz}(i) = r_{xx}(i) + r_{yy}(i)$

and $R_{zz}(e^{j\omega}) = R_{xx}(e^{j\omega}) + R_{yy}(e^{j\omega})$.

(2) (8%)

Setting $i = 0$ and the result from (1)(b),

$r_{zz}(0) = r_{xx}(0) + r_{yy}(0) \Rightarrow E\{z^2(n)\} = E\{x^2(n)\} + E\{y^2(n)\}$

6. (20%)

$$R_{\omega\omega}\left(e^{j\omega}\right) = F\left\{r_{\omega\omega}(m)\right\} = F\left\{E\left\{\omega(n)\omega(n+m)\right\}\right\} = 1$$

$$h(n) = \delta(n) + 0.5\delta(n-1) \xrightarrow{F} H\left(e^{j\omega}\right) = 1 + \frac{1}{2}e^{-j\omega}$$

$$g(n) = \delta(n) - 0.5\delta(n-1) \xrightarrow{F} G\left(e^{j\omega}\right) = 1 - \frac{1}{2}e^{-j\omega}$$

(1) (8%)

$$\begin{aligned}
R_{yy}\left(e^{j\omega}\right) &= \left|H\left(e^{j\omega}\right)\right|^2 \left|G\left(e^{j\omega}\right)\right|^2 R_{\omega\omega}\left(e^{j\omega}\right) \\
&= H\left(e^{j\omega}\right)H^*\left(e^{j\omega}\right)G\left(e^{j\omega}\right)G^*\left(e^{j\omega}\right)R_{\omega\omega}\left(e^{j\omega}\right) \\
&= \left(1+\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{2}e^{j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{j\omega}\right)R_{\omega\omega}\left(e^{j\omega}\right) \\
&= \left(\frac{5}{4}+\cos(\omega)\right)\left(\frac{5}{4}-\cos(\omega)\right)R_{\omega\omega}\left(e^{j\omega}\right) \\
&= \left(\frac{25}{16}-\cos^2(\omega)\right)R_{\omega\omega}\left(e^{j\omega}\right) = \left(\frac{25}{16}-\cos^2(\omega)\right)
\end{aligned}$$

(2) (8%)

$$\begin{aligned}
y(n) &= \left(\delta(n)+0.5\delta(n-1)\right)*\omega(n) - \left(\delta(n)-0.5\delta(n-1)\right)*\omega(n) \\
&= \omega(n)+0.5\omega(n-1)-\omega(n)+0.5\omega(n-1) = \omega(n-1) \\
&= \delta(n-1)*\omega(n) = t(n)*\omega(n)
\end{aligned}$$

$$t(n) = \delta(n-1) \xrightarrow{F} T\left(e^{j\omega}\right) = e^{-j\omega}$$

$$\begin{aligned}
R_{yy}\left(e^{j\omega}\right) &= \left|T\left(e^{j\omega}\right)\right|^2 R_{\omega\omega}\left(e^{j\omega}\right) \\
&= R_{\omega\omega}\left(e^{j\omega}\right) = 1
\end{aligned}$$