Homework #2 Solutions

1. (16%)

Real and even:

$$
\begin{aligned}
R\left(e^{j\omega}\right) &= \sum_{i=-\infty}^{\infty} r(i)e^{-j\omega i} \\
&= \sum_{i=-\infty}^{\infty} \underbrace{\underbrace{r(i)}_{even}\underbrace{\cos\left(\omega i\right)}_{even}}_{even} + j\sum_{i=-\infty}^{\infty} \underbrace{\underbrace{r(i)}_{even}\underbrace{\sin\left(\omega i\right)}_{odd}}_{odd} = \sum_{i=-\infty}^{\infty} r(i)\cos\left(\omega i\right) + 0 \\
&\therefore R^*\left(e^{j\omega}\right) = R\left(e^{j\omega}\right),\ R\left(e^{-j\omega}\right) = R\left(e^{j\omega}\right) \Rightarrow real\ \ \&\ \ even
\end{aligned}
$$

Nonnegative:

Define $X_N(\omega) = \mathcal{F}\left\{w_N(i)x(i)\right\}$, where $x(i)$ is a WSS random sequence, $\mathcal{F}\{\cdot\}$ means Fourier transform, and $w_N(i)$ is a rectangular window function which is expressed as

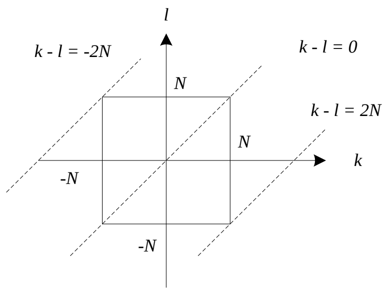
$$
w_N(i) = \begin{cases} 1, & |i| \le N \\ 0, & \text{otherwise} \end{cases}
$$

Using the above definition, we have

$$
\frac{1}{2N+1} E\left\{\left|X_N(\omega)\right|^2\right\} \ge 0
$$

$$
\begin{aligned}
\frac{1}{2N+1} E\left\{\left|X_N(\omega)\right|^2\right\} &= \frac{1}{2N+1} E\left\{\sum_{k=-N}^{N}\sum_{l=-N}^{N} x(k)x^*(l)e^{-j\omega k}e^{j\omega l}\right\} \\
&= \frac{1}{2N+1}\sum_{k=-N}^{N}\sum_{l=-N}^{N} E\left\{x(k)x^*(l)\right\}e^{-j\omega(k-l)} \\
&= \frac{1}{2N+1}\sum_{m=-2N}^{2N}\left(2N+1-|m|\right)r(m)e^{-j\omega m} \qquad (m = k-l) \\
&= \sum_{m=-2N}^{2N}\left(1-\frac{|m|}{2N+1}\right)r(m)e^{-j\omega m}
\end{aligned}
$$

The third equality is because all the term along the diagonal $m = k - l$ are equal, that m ranges from $-2N$ to $2N$, and that there $2N+1-|m|$ terms along the diagonal $m = k - l$.



The Fourier transform of $r(m)$ can be found when $N \to \infty$, i.e.,

$$R\left(e^{j\omega}\right) = \lim_{N\to\infty} \sum_{m=-2N}^{2N} r(m)\left(1-\frac{|m|}{2N+1}\right)e^{-j\omega m} = \sum_{m=-\infty}^{\infty} r(m)e^{-j\omega m} \geq 0$$

(Ref.: H. Stark and J. W. Woods, *Probability and Random Processes with Applications to Signal Processing*. 3rd ed., Upper Saddle River, NJ: Prentice-Hall, 2002.)

2. (16%)

$x(n) = x(n-1) - 0.25x(n-2) + w(n)$

$$H(z) = \frac{X(z)}{W(z)} = \frac{1}{1-z^{-1}+0.25z^{-2}} = \frac{1}{(0.5z^{-1}-1)^2}$$

$$R_{xx}(z) = H(z)H(z^{-1})\sigma_w^2 = \frac{\sigma_w^2}{(0.5z^{-1}-1)^2(0.5z-1)^2}$$

$$= \sigma_w^2 \frac{4z^{-2}}{(0.5z^{-1}-1)^2(1-2z^{-1})^2} = \sigma_w^2 \frac{4z^{-2}}{(1-0.5z^{-1})^2(1-2z^{-1})^2} = \sigma_w^2 G(z)$$

$$= \sigma_w^2 \{\frac{c_1}{(1-0.5z^{-1})^2} + \frac{c_2}{1-0.5z^{-1}} + \frac{c_3}{(1-2z^{-1})^2} + \frac{c_4}{1-2z^{-1}}\}$$

$$c_1 = (1-0.5z^{-1})^2 G(z)\Big|_{z^{-1}=2} = \frac{16}{9}$$

$$c_2 = \frac{1}{1!}(-2)\frac{d}{dz^{-1}}\{(1-0.5z^{-1})^2 G(z)\}\Big|_{z^{-1}=2} = -2\frac{d}{dz^{-1}}\frac{4z^{-2}}{(1-2z^{-1})^2}\Big|_{z^{-1}=2}$$

$$\overset{x=z^{-1}}{=} -2\frac{d}{dx}\frac{4x^2}{(1-2x)^2}\Big|_{x=2}$$

$$= -2[\frac{8x}{(1-2x)^2} - 2(-2)\frac{4x^2}{(1-2x)^3}]\Big|_{x=2} = -2[\frac{16}{9} - \frac{64}{27}] = \frac{32}{27}$$

$$c_3 = (1-2z^{-1})^2 G(z)\Big|_{z^{-1}=0.5} = \frac{16}{9}$$

$$c_4 = \frac{1}{1!}(-2^{-1})\frac{d}{dz^{-1}}\{(1-2z^{-1})^2 G(z)\}\Big|_{z^{-1}=0.5} = -0.5\frac{d}{dz^{-1}}\frac{4z^{-2}}{(1-0.5z^{-1})^2}\Big|_{z^{-1}=0.5}$$

$$\overset{x=z^{-1}}{=} -0.5\frac{d}{dx}\frac{4x^2}{(1-0.5x)^2}\Big|_{x=0.5}$$

$$= -0.5[\frac{8x}{(1-0.5x)^2} - 2(-0.5)\frac{4x^2}{(1-0.5x)^3}]\Big|_{x=0.5} = -0.5[\frac{64}{9} + \frac{64}{27}] = -\frac{128}{27}$$

$$G(z) = \frac{16/9}{(1-0.5z^{-1})^2} + \frac{32/27}{1-0.5z^{-1}} + \frac{16/9}{(1-2z^{-1})^2} - \frac{128/27}{1-2z^{-1}}$$

With the following transform pairs

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}},\ \text{ROC}:\ |z|>|a|;\ \ -a^n u[-n-1] \xleftrightarrow{z} \frac{1}{1-az^{-1}},\ \text{ROC}:\ |z|<|a|$$

$$(n+1)a^n u[n] \xleftrightarrow{z} \frac{1}{(1-az^{-1})^2},\ \text{ROC}:\ |z|>|a|$$

$$-(n+1)a^n u[-n-2] \xleftrightarrow{z} \frac{1}{(1-az^{-1})^2},\ \text{ROC}:\ |z|<|a|$$

we can find the autocorrelation function from the different regions of convergence as follows. (We can directly determine the ROC such that the Fourier transform pair of the autocorrelations and the power spectral densities exist. That is, ROC: $0.5<|z|<2$.)

(1) ROC: $2<|z|$

$$g(i)=\frac{16}{9}(i+1)(\frac{1}{2})^i u[i]+\frac{32}{27}(\frac{1}{2})^i u[i]+\frac{16}{9}(i+1)2^i u[i]-\frac{128}{27}2^i u[i]$$

$\because g(0)=0$ and the property of autocorrelation $r_{xx}(0)\ge |r_{xx}(i)|$

$\therefore$ It cannot be an autocorrelation function.

(2) ROC: $|z|<0.5$

$$g(i)=\frac{16}{9}(-i-1)(\frac{1}{2})^i u[-i-2]-\frac{32}{27}(\frac{1}{2})^i u[-i-1]$$
$$+\frac{16}{9}(-i-1)2^i u[-i-2]+\frac{128}{27}2^i u[-i-1]$$

$\because g(0)$ is undefined. $\therefore$ It cannot be an autocorrelation function.

(3) ROC: $0.5<|z|<2$

$$g(i)=\frac{16}{9}(i+1)(\frac{1}{2})^i u[i]+\frac{32}{27}(\frac{1}{2})^i u[i]+\frac{16}{9}(-i-1)2^i u[-i-2]+\frac{128}{27}2^i u[-i-1]$$

$$g(0)=\frac{16}{9}+\frac{32}{27}=\frac{80}{27}$$

$$g(1)=\frac{16}{9}\times 2\times\frac{1}{2}+\frac{32}{27}\times\frac{1}{2}=\frac{64}{27};\ g(-1)=\frac{128}{27}\times\frac{1}{2}=\frac{64}{27}$$

$$g(2)=\frac{12}{9}+\frac{8}{27}=\frac{44}{27};\ g(-2)=\frac{16}{9}\times\frac{1}{4}+\frac{128}{27}\times\frac{1}{4}=\frac{44}{27}$$

$$\vdots$$

$$\therefore r_{xx}(i)=\sigma_w^2 g(i)$$

$$g(i)=\frac{16}{9}(-i-1)(\frac{1}{2})^i u[-i-2]-\frac{32}{27}(\frac{1}{2})^i u[-i-1]$$
$$+\frac{16}{9}(-i-1)2^i u[-i-2]+\frac{128}{27}2^i u[-i-1]$$

$\because$ $g(0)$ is undefined. $\therefore$ It cannot be an autocorrelation function.

3. (16%)

(1) (8%)

$$|H(e^{j\omega})|^2=\frac{(1-z_1e^{-j\omega})}{(e^{-j\omega}-z_1)}\cdot\frac{(1-z_1e^{j\omega})}{(e^{j\omega}-z_1)}=\frac{1-z_1e^{-j\omega}-z_1e^{j\omega}+z_1^2}{1-z_1e^{-j\omega}-z_1e^{j\omega}+z_1^2}=1$$

(2) (8%)

$$|H(e^{j\omega})|^2=\frac{(1-z_ae^{-j\omega})(1-z_a^*e^{-j\omega})}{(e^{-j\omega}-z_a)(e^{-j\omega}-z_a^*)}\cdot\frac{(1-z_a^*e^{j\omega})(1-z_ae^{j\omega})}{(e^{j\omega}-z_a^*)(e^{j\omega}-z_a)}$$
$$=\frac{|1-z_ae^{-j\omega}|^2|1-z_a^*e^{-j\omega}|^2}{|e^{-j\omega}-z_a|^2|e^{-j\omega}-z_a^*|^2}=\frac{|1-z_ae^{-j\omega}|^2|1-z_a^*e^{-j\omega}|^2}{|1-z_ae^{j\omega}|^2|1-z_a^*e^{j\omega}|^2}=1$$

Assume that there are M imaginary number pairs in the z_i, $i=N_1+1,\ldots,N$, i.e., there are $N-N_1-2M$ real numbers. Then $H_{ap}(z)$ can be expressed as follows:

$$H_{ap}(z)=\frac{(1-z_{N_1+1}z^{-1})\cdots(1-z_Nz^{-1})}{(z^{-1}-z_{N_1+1})\cdots(z^{-1}-z_N)}$$
$$=H_{i,1}(z)\cdots H_{i,M}(z)H_{r,1}(z)\cdots H_{r,N-N_1-2M}(z)$$

where $H_{i,m}(z),\ m=1,\ldots,M$ and $H_{r,n}(z),\ n=1,\ldots,(N-N_1-2M)$ are the transfer functions which contain the imaginary and real roots, respectively.

Using the results from (1) and (2),

$$|H_{ap}(e^{j\omega})|^2=|H_{i,1}(e^{j\omega})\cdots H_{i,M}(e^{j\omega})H_{r,1}(e^{j\omega})\cdots H_{r,N-N_1-2M}(e^{j\omega})|^2$$
$$=|H_{i,1}(e^{j\omega})|^2\cdots|H_{i,M}(e^{j\omega})|^2|H_{r,1}(e^{j\omega})|^2\cdots|H_{r,N-N_1-2M}(e^{j\omega})|^2=1$$

4. (16%)

$$Var\{\hat{x}(n)\}=\frac{1}{M^2}\sum_{m=1}^{M}Var\{y_m(n)\}=\frac{1}{M^2}\sum_{m=1}^{M}Var\{x(n)+v_m(n)\}$$
$$=\frac{1}{M^2}\sum_{m=1}^{M}Var\{v_m(n)\}=\frac{M\sigma^2}{M^2}=\frac{\sigma^2}{M}$$

5. (16%)

(1) (8%)

Unbiased: $E\{\hat{\mu}-\mu\} = E\{\hat{\mu}\}-\mu = \frac{1}{M}\sum_{i=0}^{M-1} E\{x(i)\}-\mu = \frac{1}{M}M\mu-\mu = 0$

Consistent: $\lim_{M\to\infty} mse(\hat{\mu}) = \lim_{M\to\infty}\left(Var\{\hat{\mu}\}+B(\hat{\mu})\right) = \lim_{M\to\infty} Var\{\hat{\mu}\}$

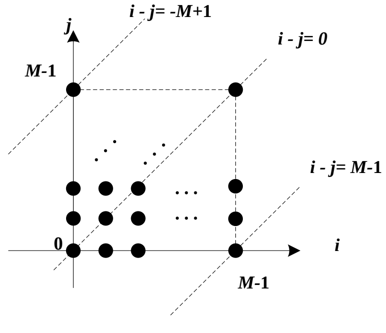
(2) (8%)

$$E\{\hat{\mu}\} = \frac{1}{M}\sum_{i=0}^{M-1} E\{w(i)+aw(i-1)\} = \frac{1}{M}\cdot 0 = 0$$
$$E\{x(n)\} = E\{w(n)+aw(n-1)\} = 0 + a\cdot 0 = 0 \Rightarrow \therefore E\{\hat{\mu}-E\{x(n)\}\} = 0$$

Consistent:

$$
\begin{aligned}
Var\{\hat{\mu}\} &= E\{\hat{\mu}^2\} - \left(E\{\hat{\mu}\}\right)^2 \\
&= E\left\{\left(\frac{1}{M}\sum_{i=0}^{M-1}\left[w(i)+aw(i-1)\right]\right)^2\right\} \\
&= \frac{1}{M^2}\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} E\left\{\left[w(i)+aw(i-1)\right]\left[w(j)+aw(j-1)\right]\right\} \\
&= \frac{1}{M^2}\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} E\{w(i)w(j)+aw(i)w(j-1) \\
&\qquad\qquad + aw(i-1)w(j)+a^2w(i-1)w(j-1)\} \\
&= \frac{1}{M^2}\sum_{i=0}^{M-1}\sum_{j=0}^{M-1}\left[\delta(i-j)+a\delta(i-j+1)+a\delta(i-j-1)+a^2\delta(i-j)\right] \\
&= \frac{1}{M^2}\left[\underbrace{M}_{i-j=0}+\underbrace{a(M-1)}_{i-j=-1}+\underbrace{a(M-1)}_{i-j=1}+\underbrace{a^2M}_{i-j=0}\right] = \frac{(a+1)^2}{M} - \frac{2a}{M^2}
\end{aligned}
$$

Therefore, $\lim_{M\to\infty} Var\{\hat{\mu}\} \to 0$.

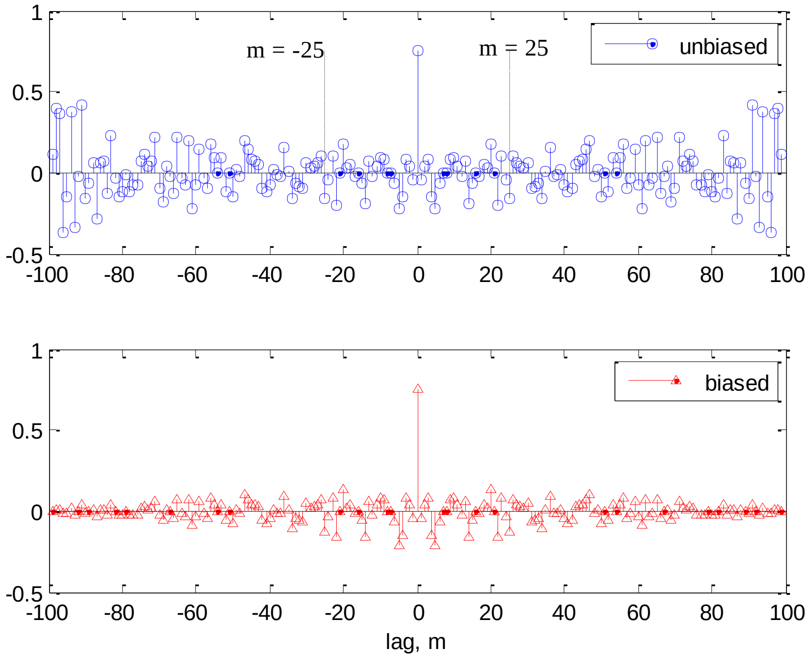


6. (20%)

(1) (10%)

In Fig. 1, the estimated results of the biased and unbiased estimators are given. The high variability phenomenon for unbiased estimator is obvious when $m > M/4$.

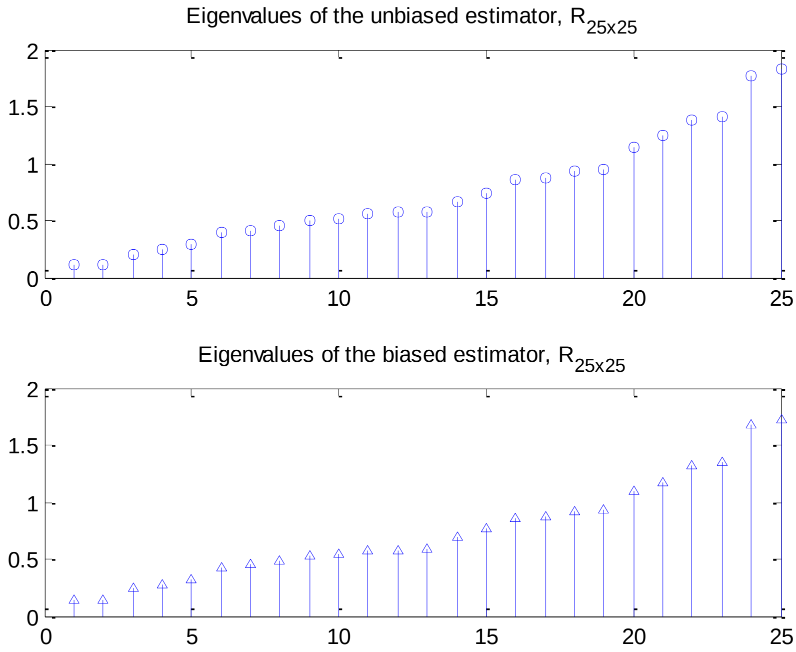
Note: Since the estimated autocorrelation is a discrete signal, we often use MATLAB function **`stem`** to show the figure instead of the function **`plot`**.



■ **Figure 1**. Estimation autocorrelation of both types.

(2) (10%)

The most convenient method for verifying a positive semi-definite matrix is to check if all its eigenvalues are nonnegative. We use the simulated data from part (1) ($\hat{r}(m)$ and $r'(m)$, $m = 0 \sim 24$) to generate a 25×25 autocorrelation matrix and find its eigenvalues. Fig. 2 shows all the eigenvalues are all positive (even use unbiased estimator). The reason is the differences between two estimators in this range are not obvious.

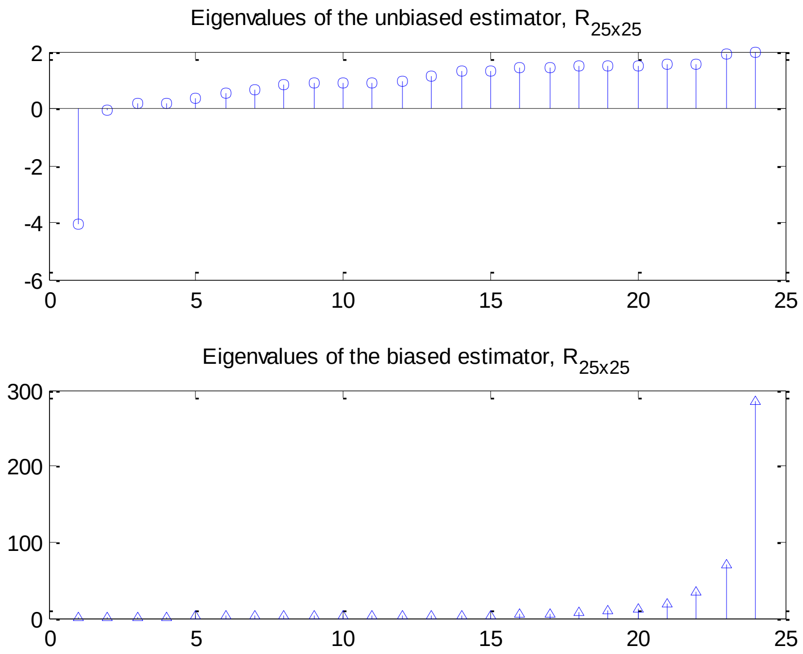


■ **Figure 2**. All the eigenvalues of estimated autocorrelation matrices.

But this case cannot give any promise to ensure that the unbiased estimator can conform to the positive semi-definite property. One counterexample is a random signal with Normal distribution and the ratio of mean and variance is very large, e.g. mean/variance = 15, *N*(15, 1). That is, this signal can be considered almost like a deterministic signal and its autocorrelation matrix is almost a rank one matrix which contains a very large eigenvalue. Its autocorrelation function is almost a constant, i.e.,

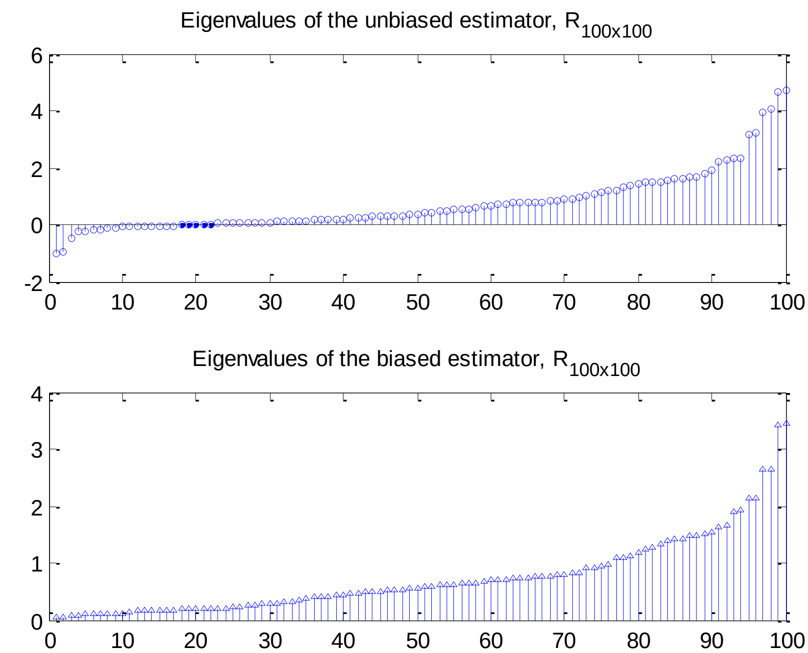
$$\hat{r}(m) = \frac{1}{M-|m|}\sum_{n=0}^{M-|m|-1} x(n)x(n+|m|) \cong \frac{1}{M-|m|}\sum_{n=0}^{M-|m|-1} x^2(n) = x^2(n)$$
$$\left(x(n) \approx x(n+|m|) \text{ since the variance is very small}\right)$$

Then we find some eigenvalues are negative from Fig. 3.



■ **Figure 3**. All the eigenvalues of autocorrelation matrix except the *largest* one.

We use the simulated data from part (1) ($\hat{r}(m)$ and $r'(m)$, $m = 0 \sim 99$) to generate a 100×100 autocorrelation matrix and find its eigenvalues. Fig. 4 shows all the eigenvalues generated from the autocorrelation of the biased estimator are all positive, but that of the unbiased estimator are not. Therefore, we usually use the biased estimator to replace the true autocorrelation.



■ **Figure 4**. All the eigenvalues of estimated autocorrelation matrices (matrix size: 100×100).

The followed descriptions will show that why the eigenvalues of the matrix from biased estimator are all nonnegative.

Suppose that there are M data points $x[0], x[1], \ldots, x[M-1]$. To estimate a $P \times P$ correlation matrix (for $P \le M$) form the matrix

$$\mathbf{X} = \begin{bmatrix} x[0] & 0 & \cdots & 0 \\ x[1] & x[0] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x[P-1] & x[P-2] & \cdots & x[0] \\ x[P] & x[P-1] & \cdots & x[1] \\ \vdots & \vdots & & \vdots \\ x[M-1] & x[M-2] & \cdots & x[M-P] \\ 0 & x[M-1] & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x[M-1] \end{bmatrix}_{(M+P-1)\times P}$$

The correlation matrix estimate is then given by

$$\mathbf{R}' = \frac{1}{M}\mathbf{X}^T\mathbf{X} \quad \left(\text{or} \quad \mathbf{R}' = \frac{1}{M}\mathbf{X}^H\mathbf{X} \quad \text{for complex}\right)$$

Then the terms in this matrix are exactly those given in

$$r'(m) = \frac{1}{M}\sum_{n=0}^{M-|m|-1} x(n)x(n+|m|)$$

and that the matrix is Toeplitz. This estimate is positive semi-definite since if $\mathbf{a}$ is any vector of size P,

$$\mathbf{a}^T\mathbf{R}'\mathbf{a} = \frac{1}{M}\mathbf{a}^T\mathbf{X}^T\mathbf{X}\mathbf{a} = \frac{1}{M}|\mathbf{X}\mathbf{a}|^2 \ge 0$$