CS 5291: Stochastic Processes for Networking

HW1

1. X is an exponentially distributed random variable with parameter λ. $Y = X^2$. Find the mathematical expectation of Y.

$$E[Y] = E[X^2] = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx$$

$$= x^2(-e^{-\lambda x})\Big|_0^{\infty} - \int_0^{\infty} 2x(-e^{-\lambda x})dx = \frac{2}{\lambda}\int_0^{\infty} x\lambda e^{-\lambda x}dx = \frac{2}{\lambda^2}$$

2. Suppose X is a non-negative and continuous random variable whose pdf is $f_X(x)$ and whose cdf is $F_X(x)$. Starting from the definition of the mathematical expectation, prove that $E[X] = \int_0^{\infty}(1 - F_X(x))dx$.

$$E[X] = \int_0^{\infty} xf(x)dx$$

$$\int_0^{\infty}[1 - F_X(x)]dx = x(1 - F_X(x))\Big|_0^{\infty} + \int_0^{\infty} xf(x)dx = \int_0^{\infty} xf(x)dx = E[X]$$

CS 5291: Stochastic Processes for Networking

HW2

1. A fair coin is tossed until two tails occur successively. Find the expected number of the tosses required.

   Hint: Let

$$X = \begin{cases} 0 \text{ if the first toss results in tails} \\ 1 \text{ if the first toss results in heads,} \end{cases}$$

   and condition on X.

   Let N be the number of tosses required;

   Then $E[N] = E\big[E[N|X]\big] = E[X|X=1]\ \mathrm{P}\{X=1\} + E[N|X=0]\ \mathrm{P}\{X=0\}$

$$= (1+E[N]) \times \frac{1}{2} + \left(2 \times \frac{1}{2} + (2+E[N]) \times \frac{1}{2}\right) \times \frac{1}{2} = \frac{3}{4} E[N] + \frac{3}{2}$$

   $\rightarrow E[N] = 6$

2. A deck of 52 playing cards, containing all 4 aces, is randomly divided into 4 piles of 13 cards each. Define events E_1, E_2, E_3, and E_4 as follows:

$$E_1 = \{\text{the first pile has exactly 1 ace}\},$$
$$E_2 = \{\text{the second pile has exactly 1 ace}\},$$
$$E_3 = \{\text{the third pile has exactly 1 ace}\},$$
$$E_4 = \{\text{the fourth pile has exactly 1 ace}\}.$$

   Find $\mathrm{P}\{E_1, E_2, E_3, E_4\}$, the probability that each pile has an ace.

   Hint:

$$\mathrm{P}\{E_1 E_2 \dots E_n\} = \mathrm{P}\{E_1\}\ \mathrm{P}\{E_2|E_1\}\ \mathrm{P}\{E_3|E_1 E_2\} \dots\ \mathrm{P}\{E_n|E_1 \dots E_{n-1}\}$$

$$\mathrm{P}\{\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{E}_4\} = \frac{C_1^4 C_{12}^{48}}{C_{13}^{52}} \times \frac{C_1^3 C_{12}^{36}}{C_{13}^{39}} \times \frac{C_1^2 C_{12}^{24}}{C_{13}^{26}} \times 1$$

3. Two unbiased six-sided dice are thrown.

   (a) What is the probability that at least one lands on six?

   Let E be the event at least one lands on six.

$$\mathrm{P}\{E\} = \frac{\text{number of ways to get } E}{\text{number of sample points}} = \frac{11}{36}$$

   (b) If the two dice land on the different values, what is the probability that at least one lands on six?

Let D be the event two dice land on the different values.

$P\{D\} = 1 - \text{Prob}\{\text{two dice land on the same value}\}$

$$= 1 - \frac{6}{36} = \frac{5}{6}$$

$$P\{E|D\} = \frac{P\{ED\}}{P\{D\}} = \frac{\frac{10}{36}}{\frac{5}{6}} = \frac{1}{3}$$

4. Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables, each having a uniform distribution over (0,1). Let $X = \text{maximum}(Y_1, Y_2, \ldots, Y_n)$.

(a) Show that the cumulative distribution function of n, $F_X(\cdot)$, is given by

$F_X(x) = x^n,\ 0 \le x \le 1$

If x is a real number between 0 and 1, then $X < x$ if and only if $Y_k < x$ for all k, and $P\{Y_k < x\} = x$ since x is uniform on (0,1). (The PDF is equal to 1, so the CDF is $\int_0^x 1dx = x$)

Because Y_k are independent, $P\{X < x\} = P\{Y_1 < x\} * P\{Y_2 < x\} * \ldots * P\{Y_n < x\} = x^n$.

(b) What is the probability density function of X?

The PDF of X is obtained by differentiating,

$f_X(x) = \frac{F_X(x)}{dx} = nx^{n-1},\ \ 0 \le x \le 1$

5. Derive the moment generating functions for the following random variables. Then, derive the expected value, second moment, and variance of each random variable.

(a) Uniform distribution

Uniform distribution: $f_X(x) = \frac{1}{b-a}$

$\phi(t) = E[e^{tX}]$

$$= \int_a^b e^{tx} \frac{1}{b-a}\, dx = \frac{1}{b-a} \frac{1}{t} \int_a^b t\, e^{tx} dx$$

$$= \frac{1}{t(b-a)} [e^{tx}|_a^b] = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

$$\phi'(t) = \frac{d\phi(t)}{dt} = \frac{1}{b-a} \left(\frac{be^{tb} - ae^{ta}}{t} - \frac{e^{tb} - e^{ta}}{t^2}\right)$$

$$\phi''(t) = \frac{1}{b-a}\left(\frac{2(e^{tb}-e^{ta})}{t^3} - \frac{2(be^{tb}-ae^{ta})}{t^2} + \frac{b^2e^{tb}-a^2e^{ta}}{t}\right)$$

$$E[X] = \phi'^{(0)} = \lim_{s\to 0}\frac{1}{b-a}\left(\frac{be^{tb}-ae^{ta}}{t} - \frac{e^{tb}-e^{ta}}{t^2}\right)$$

$$= \frac{a+b}{2} \text{ (by L'Hospital's rule)}$$

$$E[X^2] = \phi'(0) = \frac{a^2+ab+b^2}{3}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{(b-a)^2}{12}$$

(b) Exponential distribution

Exponential distribution: $f_X(x) = \lambda e^{-\lambda x}$

$$\phi(t) = E[e^{tX}] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx = \lambda\int_0^\infty e^{-(\lambda-t)x}dx$$

$$= \lambda\frac{1}{\lambda-t}\int_0^\infty(\lambda-t)e^{-(\lambda-t)x}dx = \frac{\lambda}{\lambda-t}$$

$$\phi'(t) = \frac{\lambda}{(\lambda-t)^2}$$

$$\phi''(t) = \frac{2\lambda}{(\lambda-t)^3}$$

$$E[X] = \phi'(0) = \frac{1}{\lambda}$$

$$E[X^2] = \phi''(0) = \frac{2}{\lambda^2}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{1}{\lambda^2}$$

6. Derive the tightest Chernoff's Bound for the Poisson random variable X with PMF $P_X(n) = \frac{e^{-\lambda}\lambda^n}{n!}, n = 0,1,2,3,\ldots$

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}e^{e^t\lambda} = e^{-\lambda(1-e^t)}$$

$\mathrm{P}\{X \geq a\} \leq \min_t \left(e^{-ta} M_X(t)\right) = \min_t \left(e^{-ta-\lambda(1-e^t)}\right)$

Let $g(t) = e^{-ta-\ (1-e^t)}$

To find the minimum of $g(t)$, we need to solve $\frac{dg(t)}{dt} = 0.$

$$\frac{dg(t)}{dt} = 0 \Rightarrow (\lambda e^t - a)e^{-ta-\lambda(1-e^t)} = 0 \Rightarrow t = ln\left(\frac{a}{\lambda}\right)$$

$$\mathrm{P}\{X \geq a\} \leq g\left(ln\left(\frac{a}{\lambda}\right)\right) = e^{-a\,ln\left(\frac{a}{\lambda}\right)-\lambda+a} = e^{-\lambda} e^{ln\left(\left(\frac{\lambda}{a}\right)^a\right)} e^a = e^{-\lambda}\left(\frac{\lambda e}{a}\right)^a$$

7. With $K(t) = ln(E[e^{tX}])$, show that $K'(0) = E[X], K''(0) = Var(X)$.

$$K'(t) = \frac{E[Xe^{tX}]}{E[e^{tX}]}$$

$$K''(t) = \frac{E[e^{tX}]E[X^2e^{tX}] - E^2[Xe^{tX}]}{E^2[e^{tX}]}$$

Hence,

$K'(0) = E[X]$,

$K''(0) = E[X^2] - E^2[X] = Var(X)$

CS 5291: Stochastic Processes for Networking

HW3

1. Assume X and Y are independent Poisson random variables with means $E[X] = \lambda_1$ and $E[Y] = \lambda_2$. (The probability mass function of Poisson distribution is $\frac{e^{-\lambda}\lambda^n}{n!}, n = 0, 1, \ldots$, where λ is the mean.)

   (a) Compute the conditional probability mass function $\mathrm{P}\{X = k \mid X + Y = n\}$. Is it the same as $\mathrm{P}\{X = k\}$?
   Hint: You can start by the definition of conditional probability.

$$\mathrm{P}\{X + Y = n\} = \sum_{i=0}^{n} \mathrm{P}\{X = i, Y = n - i\} = \sum_{i=0}^{n} \frac{(e^{-\lambda_1}\lambda_1^i)}{i!} \frac{e^{-\lambda_2}\lambda_2^{n-i}}{(n-i)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{n} \frac{\lambda_1^i \lambda_2^{n-i}}{i!\,(n-i)!} \frac{n!}{n!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n \sim \text{Possion } (\lambda_1 + \lambda_2)$$

$$\mathrm{P}\{X = k \mid X + Y = n\} = \frac{\mathrm{P}\{X = k, Y = n - k\}}{\mathrm{P}\{X + Y = n\}} = \frac{\frac{e^{-\lambda_1}\lambda_1^k}{k!} \frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}}{\frac{e^{-(\lambda_1+\lambda_2)}\,(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \frac{n!}{k!\,(n-k)!} \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} = \mathrm{C}_k^n \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$\mathrm{P}\{X = k\} = \frac{e^{-\lambda_1}\lambda_1^k}{k!}$$

$$\mathrm{P}\{X = k \mid X + Y = n\} \neq \mathrm{P}\{X = k\}$$

   (b) Find $E[X \mid X + Y = n]$. Is it the same as $E[X]$?

$$E[X|X + Y = n] = \sum_{x=0}^{\infty} x\, \mathrm{P}\{X = x \mid X + Y = n\}$$

$$\xrightarrow{\text{by } (a)} \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!} \frac{\lambda_1^x \lambda_2^{n-x}}{(\lambda_1+\lambda_2)^n} = \frac{1}{(\lambda_1+\lambda_2)^n} \sum_{x=1}^{n} x \frac{n!}{x!\,(n-x)!} \lambda_1^x \lambda_2^{n-x}$$

$$= \frac{n}{(\lambda_1+\lambda_2)^n} \sum_{x-1=0}^{n-1} \frac{(n-1)!}{(x-1)!\,(n-x)!} \lambda_1^x \lambda_2^{n-x}$$

$$\xrightarrow{\text{let } i=x-1} \frac{n}{(\lambda_1+\lambda_2)^n} \sum_{i=0}^{n-1} \frac{(n-1)!}{i!\,(n-i-1)!} \lambda_1^{i+1} \lambda_2^{n-i-1}$$

$$= \frac{n\lambda_1}{(\lambda_1+\lambda_2)^n} \sum_{i=0}^{n-1} C_i^{n-1} \lambda_1^i \lambda_2^{(n-1)-i} = \frac{n\lambda_1}{(\lambda_1+\lambda_2)^n} (\lambda_1+\lambda_2)^{n-1} = n\frac{\lambda_1}{\lambda_1+\lambda_2}$$

$E[X \mid X+Y=n] \neq E[X]$

(c) Find $E[E[X \mid X+Y]]$. Is it the same as $E[X]$?

$$E\big[E[X|X+Y]\big] = \sum_n E[X|X+Y=n]\, \mathrm{P}\{X+Y=n\}$$

$$\xrightarrow{\text{by } (a),(b)} \sum_n \left(n\frac{\lambda_1}{\lambda_1+\lambda_2}\right)\left(\frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n\right)$$

$$= \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right) \sum_n n\, e^{-(\lambda_1+\lambda_2)} \left(\frac{\lambda_1+\lambda_2}{n!}\right)^n$$

$$= \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)(\lambda_1+\lambda_2) = \lambda_1 = E[X]$$

2. A dog is trapped in a cave. It needs to choose one of two directions. If it goes to the left, then it will walk around in the cave for five minutes and will then return to its initial position. If it goes to the right, then with probability $\frac{1}{4}$ it will depart the cave after four minutes of traveling, and with probability $\frac{3}{4}$ it will return to its initial position after seven minutes of traveling. Assuming that the dog is at all times equally likely to go to the left or the right, what is the expected number of minutes that it will be trapped in the cave?

Let N denote the number of minutes in the cave. If R is the event the dog chooses its right, and L the event it chooses its left, we have by conditioning on the first direction chosen:

$$E(N) = \tfrac{1}{2}E(N|L) + \tfrac{1}{2}E(N|R)$$

$$= \frac{1}{2}[5 + E(N)] + \frac{1}{2}\left[\frac{1}{4}(4) + \frac{3}{4}\left(7 + E(N)\right)\right]$$

$$= \frac{7}{8}E(N) + \frac{45}{8} = 45$$

3. Suppose that two teams are playing a series of games, each of which is independently won by team A with probability p and by team B with probability $1 - p$. The winner of the series is the first team to win i games.

(a) If $i = 4$, find the probability that a total of 7 games are played.

$$C\binom{6}{3}p^3(1-p)^3 = 20p^3(1-p)^3$$

(b) From the question (a), show that this probability is maximized when $p = 0.5$.

Let $g(p) = 20p^3(1-p)^3 = -20p^6 + 60p^5 - 60p^4 + 20p^3$

$g'(p) = -120p^5 + 300p^4 - 240p^3 + 60p^2$

$g'(p) = -60p^2(p-1)^2(2p-1) = 0$

So $p = 0$ or 1 or 0.5, the maximum value occurs at one of those values.

$g(0.5) = 0.3125 > g(1) = g(0) = 0$

CS 5291: Stochastic Processes for Networking

HW4

1. Suppose that X_1 and X_2 are two independent nonnegative continuous random variables with probability density functions $f_1(x) = \lambda_1 e^{-\lambda_1 x}$ and $f_2(x) = \lambda_2 e^{-\lambda_2 x}$

   (a) Derive the expected value $E[\min(X_1, X_2)]$.

   $E[min(X_1, X_2)]$

   $$= \int_0^\infty P(min(X_1, X_2) > x)dx$$

   $$= \int_0^\infty P(X_1 > X)P(X_2 > x)dx$$

   $$= \int_0^\infty e^{-(\lambda_1+\lambda_2)x}dx$$

   $$= \frac{1}{\lambda_1+\lambda_2}\int_0^\infty (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x}dx = \frac{1}{\lambda_1+\lambda_2}$$

   (b) Derive the expected value $E[\max(X_1, X_2)]$.

   $E[max(X_1, X_2)]$

   $$= \int_0^\infty P(max(X_1, X_2) > x)dx$$

   $$= \int_0^\infty \big(1 - P(max(X_1, X_2) < x)\big)dx$$

   $$= \int_0^\infty (1 - P(X_1 < x)P(X_2 < x))dx$$

   $$= \int_0^\infty (1 - (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 x}))dx$$

   $$= \int_0^\infty \big(e^{-\lambda_1 x} + e^{-\lambda_2 x} - e^{-(\lambda_1+\lambda_2)x}\big)dx = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}$$

2. A certain scientific theory supposes that mistakes in cell division occur according to a Poisson process with rate 2.5 per year, and that an individual dies when 196 such mistakes have occurred. Assuming this theory, find

   The lifetime of an individual follows the Erlang distribution with parameters

   $\lambda = 2.5 \; and \; k = 196$

(a) the mean lifetime of an individual

The mean lifetime of an individual $= \frac{196}{2.5} = 78.4$ years

(b) the variance of the lifetime of an individual

The variance of the lifetime of an individual$=\frac{196}{2.5^2} = 31.36$ years

(c) an approximate of the probability that an individual dies before age 67.2

Hint: Use the central limit theorem. Suppose the value of the complementary cumulative distribution function of the normal distribution, $Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} dz$, can be known by table lookup.

We use the central limit theorem to justify approximating the life distribution by a normal distribution with mean 78.4 and standard deviation $\sqrt{31.36} = 5.6$

$$P(L < 67.2) \fallingdotseq P\left(Z < \frac{67.2 - 78.4}{5.60}\right) = P(Z < -2) = Q(2) = 0.0227$$

3. Consider a Poisson process $N(t)$ is decomposed into two counting processes $N_1(t)$ and $N_2(t)$ with probability p and $1-p$, respectively.

(a) Find the joint probability mass function $\Pr\{N_1(t) = n, N_2(t) = m\}$

$$\begin{aligned} P(N_1(t) = n, N_2(t) = m) &= P(N_1(t) = n, N(t) = n+m) \\ &= P(N(t) = n+m) \times P(N_1(t) = n | N(t) = n+m) \\ &= \frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!} \times C_n^{n+m} p^n (1-p)^m \\ &= \frac{e^{-\lambda t}(\lambda t)^{n+m}}{n!\, m!} p^n (1-p)^m \end{aligned}$$

(b) Prove that $N_1(t)$ and $N_2(t)$ are independent of each other. That is, $\Pr\{N_1(t) = n, N_2(t) = m\} = \Pr\{N_1(t) = n\} \cdot \Pr\{N_2(t) = m\}$.

$$P(N_1(t) = n) = \sum_{m=0}^{\infty} P\{N_1(t) = n, N_2(t) = m\}$$

$$= \frac{e^{-\lambda t}(\lambda pt)^n}{n!}\left(1 + \frac{(1-p)\lambda t}{1!} + \frac{\left((1-p)\lambda t\right)^2}{2!} + \frac{\left((1-p)\lambda t\right)^3}{3!} + \cdots\right)$$

$$= \frac{e^{-\lambda}\ (\lambda pt)^n}{n!} e^{(1-p)\lambda t} = \frac{e^{-\lambda pt}(\lambda pt)^n}{n!}$$

Similarly, $P(N_2(t) = m) = \frac{e^{-\lambda(1-p)t}(\lambda(1-p)t)^m}{m!}$

$$P(N_1(t) = n, N_2(t) = m) = \frac{e^{-\lambda t}(\lambda t)^{n+m}}{n!\,m!} p^n (1-p)^m$$

$$= \frac{e^{-\lambda pt}(\lambda pt)^n}{n!} \times \frac{e^{-\lambda(1-p)t}(\lambda(1-p)t)^m}{m!}$$

$$= P(N_1(t) = n) \times P(N_2(t) = m)$$

Hence, $N_1(t)$ and $N_2(t)$ are independent of each other.

4. On Friday night, the number of customers arriving at a lounge bar can be modeled by a Poisson process with intensity that twelve customers per hour.

(a) Find the probability that there are 2 customers between 21:00 and 21:40.

We know that $\lambda = 12$ and the interval between 21:00 and 21:40 has the length $\tau = \frac{2}{3}$ hours. If X is the number of arrivals in that interval, we can write $X \sim \text{Poisson}(8)$. Therefore,

$$P(X = 2) = \frac{e^{-8}(8)^2}{2!}$$

(b) Find the probability that there are 4 customers between 21:00 and 21:40 and 6 customers between 21:40 and 22:00.

Here, we have two non-overlapping intervals

$I_1 = (21{:}00, 21{:}40]$ and $I_2 = (21{:}40, 22{:}00]$.

Thus, we can write

$$P(\text{ 4 arrivals in } I_1 \textit{ and } \text{6 arrivals in } I_2\ )$$

$$= P(\text{ 4 arrivals in } I_1\ ) \cdot P(\text{ 6 arrivals in } I_2\ )$$

$$= \frac{e^{-8}(8)^4}{4!} \cdot \frac{e^{-4}(4)^6}{6!}$$