* mid 2019?

1. (a) $f_X(x) = kx, \quad 0 \le x \le 1 \quad E(X) = 1$?

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 kx\,dx = k \cdot \frac{1}{2}x^2 \Big|_0^1 = \frac{k}{2} = 1$$

$k = 2$

(b) T. driving time $\sim U(1, 1+X^2)$ $E(T) = ?$

given starting time $X = x$

$$E(T) = E(\underline{E(T|X=x)})$$

$$\hookrightarrow \int_1^{1+x^2} t \cdot \frac{1}{(1+x^2)-1}\,dt = \frac{1}{x^2} \cdot \frac{1}{2} t^2 \Big|_1^{1+x^2}$$

$$= \frac{1}{x^2} \cdot \frac{1}{2}\left[(1+x^2)^2 - 1\right] = \frac{1}{2}(2+x^2) = 1 + \frac{1}{2}x^2$$

$$\cancel{1 + 2x^2 + x^4 - 1}$$

$$\Rightarrow E(T) = E(E(T|X=x)) = \int_0^1 E(T|X=x) \cdot \overset{kx}{\overset{\uparrow}{f_X(x)}}\,dx$$

$$= \int_0^1 \left(1 + \frac{x^2}{2}\right) \cdot 2x\,dx = \int_0^1 x^3 + 2x\,dx = \frac{1}{4}x^4 + x^2 \Big|_0^1 = \frac{1}{4} + 1$$

$$= \frac{5}{4}$$

2. Markov inequality $\boxed{P(X \ge a) \le \frac{E(X)}{a}}$

$\boxed{M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\,dx} = e^{-\lambda(1-e^t)}$

Chebyshev's inequality $\boxed{P(|X-\mu| \ge k) \le \frac{\sigma^2}{k^2}}$

Chernoff's bound $\boxed{P(X \ge a) \le e^{-ta} \cdot M_X(t)}$

2. $P(X \ge a) \le \min_t \{\text{Chernoff}\}$

$$\min_t \{ e^{ta} \cdot \underbrace{M_X(t)}_{e^{-\lambda(1-e^t)}} \} \quad \lambda = a$$

let $g(t) = e^{ta} \cdot e^{-\lambda(1-e^t)} = e^{ta-\lambda(1-e^t)}$

$g'(t) = 0 \Rightarrow e^{ta-\lambda(1-e^t)} \cdot (a-$

$\boxed{P(X \ge a) \le e^{-ta} \cdot M_X(t)}$ find $\min_t$ Chernoff

$$g(t) = e^{-ta} \cdot e^{-\lambda(1-e^t)} = e^{-ta-\lambda+\lambda e^t}$$

$$g'(t) = 0 \Rightarrow \underbrace{e^{-ta-\lambda+\lambda e^t}}_{恆正} \cdot (-a+\lambda e^t) \cdot 1 = 0$$

$$\lambda e^t = a \quad \boxed{e^t = \frac{a}{\lambda}} \quad t = \ln(\frac{a}{\lambda})$$

$$g(\ln(\frac{a}{\lambda})) = e^{-a \cdot \ln(\frac{a}{\lambda})} \cdot e^{-\lambda + e^{\ln(\frac{a}{\lambda})}}$$

$$= (\frac{a}{\lambda})^{-a} \cdot e^{-\lambda} \cdot e^{\lambda \cdot \frac{a}{\lambda}} = (\frac{\lambda}{a})^{a} \cdot e^{a} \cdot e^{-\lambda}$$

$$= e^{-\lambda} (\frac{\lambda e}{a})^{a} \quad \#$$

Poisson

$M_X(t) = e^{-\lambda(1-e^t)}$? $M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} e^{tn} (e^{-\lambda} \frac{\lambda^n}{n!})$

$$= e^{-\lambda} \sum_{n=0}^{\infty} e^{tn} \cdot \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(e^t \lambda)^n}{n!}$$

$$\boxed{e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}}$$

$$= e^{-\lambda} \cdot e^{(e^t \lambda)} = e^{-\lambda(1-e^t)}$$

$$\boxed{e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots}$$

3. $\boxed{P\{N(t)=n\} = e^{-\lambda t}\cdot\frac{(\lambda t)^n}{n!}}$ $P\{N(1)=0\} = e^{-2}$

$e^{-\lambda}\cdot\frac{1}{0!} = e^{-\lambda} = e^{-2} \Rightarrow \lambda = 2$

(a) $E(S_{10}) = 10\cdot E(X) = 10\cdot\frac{1}{\lambda} = 5$ #

$\boxed{S_n \sim \text{Erlang}(n,\lambda) \quad E(S_n) = n\cdot\frac{1}{\lambda}, \quad var(S_n) = n\cdot\frac{1}{\lambda^2}}$

(b) $P\{N(2)=0\} = e^{-2\cdot 2}\cdot\frac{(2\cdot 2)^0}{0!} = e^{-4}$ #

(c) $E(N(4)-N(2) \mid N(1)=3) = E(N(2+2)-N(2))$ $\because$ indep. incre.

$= E(N(2))$ $\because$ stationary incre. $= \lambda t = 2\cdot 2 = 4$ #

(d) let $N'(t)$ be # of immigrants of Eng. descent until the t^{th} day. $\Rightarrow N'(t)$ is a Poisson decomposition with prob. $p=\frac{1}{5}$ $\Rightarrow$ Poisson w/ rate $\lambda p = \frac{1}{5}\lambda = \frac{2}{5}$

$P\{N'(4)=0\} = e^{-\lambda p t}\cdot\frac{0}{0!} = e^{-\frac{28}{5}}$ #

4. $P(N(\frac{1}{6})=3 \mid N(1)=3)$ $\boxed{P(A|B) = \frac{P(A\cap B)}{P(B)}}$

$= \frac{P(3 \text{ in } (0,\frac{1}{6}) \text{ 且 } 0 \text{ in } (\frac{1}{6},1))}{P(3 \text{ in } (0,1))} = \frac{P(N(\frac{1}{6})=3 \text{ 且 } N(\frac{5}{6})=0)}{P(N(1)=3)}$

$= \frac{e^{-\lambda\frac{1}{6}}\cdot\frac{(\frac{\lambda}{6})^3}{3!}\cdot e^{-\lambda\frac{5}{6}}\cdot\frac{0}{0!}}{e^{-\lambda}\cdot\frac{\lambda^3}{3!}} = \frac{e^{-\lambda(\frac{1}{6}+\frac{5}{6})}\cdot\frac{\lambda^3}{6^3}\cdot 3!}{e^{-\lambda}\cdot\lambda^3\cdot 3!}$

$= \frac{1}{6^3} = \frac{1}{36\ 6} = \frac{1}{216}$ #

2. derive tightest Chernoff for Poisson

1° Poisson MGF $\boxed{P\{X \geq a\} \leq e^{-ta} \cdot M_X(t)}$

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\,dx = \sum_x e^{tx} \cdot P(X=x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \cdot \lambda)^{x}}{x!} \qquad \boxed{e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$$

$k \quad k \quad k!$

$$= e^{-\lambda} \cdot e^{(e^t \cdot \lambda)} = e^{-\lambda(1-e^t)}$$

2° $P\{X \geq a\} \leq \min_t \{ e^{-ta} \cdot \underbrace{M_X(t)}_{= e^{-\lambda(1-e^t)}} \}$

let $g(t) = e^{-ta} \cdot e^{-\lambda(1-e^t)}$

$$g'(t) = 0 \Rightarrow \underbrace{e^{-ta-\lambda+\lambda e^t}}_{\text{恆正}} \cdot \underbrace{(-a+\lambda e^t)}_{=0} = 0$$

$\lambda e^t = a \quad e^t = \frac{a}{\lambda} \longleftrightarrow t = \ln(\frac{a}{\lambda})$ 有 min

$$\Rightarrow g(\ln(\tfrac{a}{\lambda})) = (e^t)^{-a} \cdot e^{-\lambda} \cdot e^{\lambda(e^t)}$$

$$= (\tfrac{a}{\lambda})^{-a} \cdot e^{-\lambda} \cdot e^{\lambda \cdot \frac{a}{\lambda}} = (\tfrac{\lambda}{a})^a e^{-\lambda} e^a$$

$= e^{-\lambda} (\frac{\lambda e}{\lambda})^a$ is the tightest bound $\divideontimes$

5. $E(\max(X_1, X_2)) = \int_0^{\infty} P\{\max(X_1, X_2) > x\}\,dx$

$$= \int_0^{\infty} 1 - P(\max(X_1, X_2) \leq x)\,dx = \int_0^{\infty} 1 - P(X_1 \leq x) P(X_2 \leq x)\,dx$$

$$-1 + e^{-\lambda_1 x} + e^{-\lambda_2 x} \divideontimes e^{-(\lambda_1 + \lambda_2)x}$$

$$= \int_0^{\infty} 1 - (1-e^{-\lambda_1 x})(1-e^{-\lambda_2 x})\,dx = \int_0^{\infty} e^{-\lambda_1 x} + e^{-\lambda_2 x} - e^{-(\lambda_1 + \lambda_2)x}\,dx$$

$$= \frac{(0-1)}{-\lambda_1} + \frac{(0-1)}{-\lambda_2} - \frac{(0-1)}{-(\lambda_1 + \lambda_2)} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

6. (a) $P\{N(T)=0\} = e^{-\lambda T} \cdot \frac{0}{0!} = e^{-\lambda T}$

※

(b) let W be waiting time, X be time until 1^{st} car

$$E(W) = E(E(W|X=x)) = \int_0^\infty E(W|X=x)\cdot \lambda e^{-\lambda x}\,dx$$

$$= \int_0^T (x + E(W))\lambda e^{-\lambda x}\,dx$$

$$= \int_0^T x\cdot \lambda e^{-\lambda x}\,dx + E(W)\cdot \underline{\int_0^T \lambda e^{-\lambda x}\,dx}$$

$$\hookrightarrow \lambda \left.\frac{e^{-\lambda x}}{-\lambda}\right|_0^T = \left.e^{-\lambda x}\right|_T^0 = 1 - e^{-\lambda T}$$

$$\underbrace{(1-(1-e^{-\lambda T}))}_{\substack{\| \\ e^{-\lambda T}}} E(W) = \boxed{\begin{array}{ccc} + & x & \lambda e^{-\lambda x}\,dx \\ - & 1 & -e^{-\lambda x} \\ + & 0 & +\frac{e^{-\lambda x}}{\lambda} \end{array}} = \left.x e^{-\lambda x}\right|_T^0 + \left.\frac{e^{-\lambda x}}{\lambda}\right|_T^0$$

$$= (0 - Te^{-\lambda T}) + \left(\frac{1}{\lambda} - \frac{e^{-\lambda T}}{\lambda}\right)$$

$$\Rightarrow E(W) = e^{\lambda T}\cdot\left(-Te^{-\lambda T} + \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda T}\right)$$

$$= -T + \frac{1}{\lambda}e^{\lambda T} - \frac{1}{\lambda} \qquad \therefore E(W) = -T - \frac{1}{\lambda} + \frac{1}{\lambda}e^{\lambda T}$$

※

7. Poisson $\underset{P(t)}{}$ $\xrightarrow{p} N(t)$, $\xrightarrow{1-p} M(t)$ counting $P(t) = N(t) + M(t) = n+m$

$$P\{N(t)=n,\ M(t)=m\} = P\{N(t)=n,\ P(t)=n+m\}$$

$$= P\{P(t)=n+m\}\ P\{N(t)=n \mid P(t)=n+m\}$$

$$= e^{-\lambda t}\cdot\frac{(\lambda t)^{n+m}}{(n+m)!}\cdot\binom{n+m}{n} p^n (1-p)^{m+n-n} \qquad \frac{(n+m)!}{n!\,m!}$$

$$= \frac{e^{-\lambda t}\cdot(\lambda t)^{n+m}}{n!\,m!} p^n (1-p)^m$$

※

(b) $P(N(t)=n) \xrightarrow[\text{prob.}]{\text{marginal}} \sum_{k=0}^{m} P(N(t)=n,\ M(t)=\cancel{m}\,k)$

$$\cancel{= \sum_{k=0}^{m} \frac{e^{-\lambda t}\cdot(\lambda t)^{n+k}}{(n+k)!} C\, p^{n}(1-p)^{n+k-n} = \sum_{k=0}^{m} \frac{e^{\lambda t}\cdot(\lambda t)^{n+k}}{(n+k)!} p^{n}(1-p)^{k}}$$

$$(\lambda t)^{n}(\lambda t)^{k}\cdot p^{n}(1-p)^{k}$$

$$= \sum_{k=0}^{m} \frac{e^{-\lambda t}\cdot(\lambda t)^{n+k}}{n!\,k!}\cdot p^{n}(1-p)^{k} \qquad \boxed{e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!}}$$

$$= e^{-\lambda t}\cdot\frac{(\lambda p t)^{n}}{n!}\cdot \underline{\sum_{k=0}^{m} \frac{[\lambda(1-p)t]^{k}}{k!}} \longrightarrow e^{\lambda(1-p)t} = e^{\lambda t}\cdot e^{-\lambda p t}$$

$$= \frac{e^{-\lambda p t}\cdot(\lambda p t)^{n}}{n!}$$

#

$$P(M(t)=m) = \sum_{k=0}^{n} P(M(t)=m,\ N(t)=k)$$

$$= \sum_{k=0}^{n} \frac{e^{-\lambda t}\cdot(\lambda t)^{m+k}}{m!\,k!} p^{k}(1-p)^{m} = \frac{e^{-\lambda t}\cdot(\lambda(1-p)t)^{m}}{m!}\cdot \underbrace{\sum_{k=0}^{n} \frac{(\lambda p t)^{k}}{k!}}_{e^{\lambda p t}}$$

$$= \frac{e^{-\lambda(1-p)t}\cdot(\lambda(1-p)t)^{m}}{m!}$$

#

$$\Rightarrow P(N(t)=n)\cdot P(M(t)=n) = \frac{e^{-\lambda t}\cdot e^{-\lambda p t}\cdot e^{\lambda p t}\cdot(\lambda t)^{n+m}\cdot p^{n}(1-p)^{m}}{n!\,m!}$$

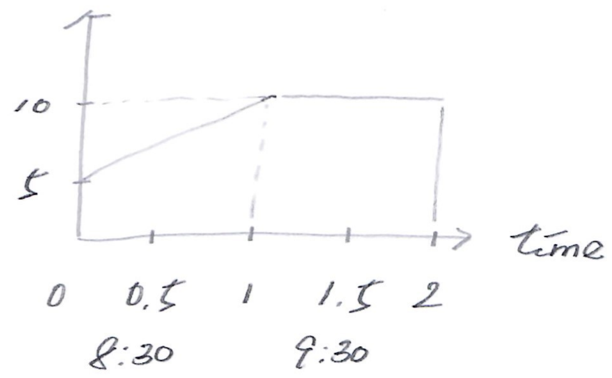
$$= \frac{e^{-\lambda t}\cdot(\lambda t)^{n+m}}{n!\,m!}\cdot p^{n}(1-p)^{m}$$

#

8. (a) non-homogeneous Poisson process is a good model

let t as time moment. $t=0.5 \to$ 8:30 a.m.

rate $\lambda(t) = \begin{cases} 5+5t, & 0 \le t < 1 \\ 10, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$ $\quad$. $t=1.5 \to$ 9:30 a.m.



\#

(b) expected # of arrivals in (0.5, 1.5)

$$E(N(1.5)-N(0.5)) = m(1.5)-m(0.5) = \int_{0.5}^{1.5} \lambda(t)\,dt$$

$$= \int_{0.5}^{1} 5+5t\,dt + \int_{1}^{1.5} 10\,dt = (5t+\frac{5}{2}t^2)\Big|_{0.5}^{1} + 10t\Big|_{1}^{1.5}$$

$$= 5(1-0.5) + \frac{5}{2}(\overset{0.75}{1-0.25}) + 10(0.5) = \frac{75}{8} = 9.375$$

$\frac{5}{2}$ $\frac{20}{8}$ $\quad \frac{5}{2}\cdot\frac{3}{4} = \frac{15}{8}$ $\quad \frac{40}{8}$ $\quad$ \#

prob. that no arrivals in (0.5, 1.5)

$$P\{N(1.5)-N(0.5)=0\} = e^{-(m(1.5)-m(0.5))} \cdot \frac{0}{0!}$$

$$= e^{-\frac{75}{8}} = 8.482 \cdot 10^{-5}$$

\#

$8 \overline{)75}$ = 9.375
72
30
24
20

9. let S_i be service time at server i, $i=1,2$

X be time until next arrival. (直到下个 arrival の時間間隔)

p be the proportion of customers that are served by both servers. (被 2个 server 服務の比例)

$$P = P\{X > S_1 + S_2\} = P\{X > S_1\} P\{X > S_1 + S_2 \mid X > S_1\}$$

$$= \frac{\mu_1}{\mu_1 + \lambda} \cdot \frac{\mu_2}{\mu_2 + \lambda}$$

#

10. $E(N(T)) = E(E(N(T) \mid T)) = E(\lambda T) = \lambda E(T)$

$$E(N^2(T)) = E(E(N^2(T) \mid T)) = E(\lambda T + \lambda^2 T^2) = \lambda E(T) + \lambda^2 E(T^2)$$

$$\because var(N(T))\big|_T = E(N^2(T))\big|_T - E(N(T))\big|_T^2 = \lambda T$$

$$\Rightarrow E(N^2(T) \mid T) = \lambda T + \lambda^2 E(T)^2 = \lambda T + \lambda^2 T^2$$

$$var(N(T)) = E(N^2(T)) - E(N(T))^2$$

$$= \lambda E(T) + \lambda^2 E(T^2) - \lambda^2 E(T)^2$$

$$= \lambda \cdot \mu + \lambda^2 (var(T))$$

$$= \lambda \mu + \lambda^2 \sigma^2$$

#

* mid 2018

1. $E(Y) = E(E(Y|X=x))$ given starting time $X=x$

$$\int_0^x y \cdot \frac{2}{x^2} y \, dy = \frac{2}{x^2} \cdot \frac{1}{3} y^3 \Big|_0^x = \frac{2}{3} x$$

$$= \int_0^1 E(Y|X=x) \cdot f_X(x) \, dx = \int_0^1 \frac{2}{3} x \cdot 2x \, dx = \frac{4}{3} \cdot \frac{1}{3} x^3 \Big|_0^1$$

$$= \frac{4}{9}$$ #

8. $P\{S_n^1 < S_1^2\} = ?$

1° $n=1$, $P(S_1^1 < S_1^2) = P(X_1^1 < X_1^2) = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^1$

2° $n=2$, $P(S_2^1 < S_1^2) = P(S_2^1 < S_1^2 \mid S_1^1 < S_1^2) \cdot P(S_1^1 < S_1^2)$

$$+ P(S_2^1 < S_1^2 \mid S_1^1 \geq S_1^2) \cdot \underbrace{P(S_1^1 \geq S_1^2)}_{0}$$

$$= P(S_2^1 < S_1^2 \mid S_1^1 < S_1^2) \cdot P(S_1^1 < S_1^2)$$

$$= P(X_1^1 + X_2^1 < X_1^2 \mid X_1^1 < X_1^2) \cdot P(X_1^1 < X_1^2)$$

$$= P(X_2^1 < X_1^2) \cdot P(X_1^1 < X_1^2) \quad \because \text{memoryless}$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^2$$

3° $P(S_n^1 < S_1^2) = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^n$ #

9. long-run proportion time chair 2 not empty $= \dfrac{E(\text{chair not empty})}{E(\text{cycle length})} = \dfrac{\frac{1}{\mu_2}}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\lambda}}$ #

10. X be 25种 type 的 coupon 任选 10, $X = X_1 + \cdots + X_{25}$

$$X_i = \begin{cases} 1, & \text{至少1張}\ i \\ 0, & \text{沒有} \end{cases}$$

$$E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = P(X_i = 1)$$

$$= 1 - P(10\text{張都沒}\ i) = 1 - \left(\frac{24}{25}\right)^{10}$$

$$\Rightarrow E(X) = E(X_1) + \cdots + E(X_{25}) = 25\left(1 - \left(\frac{24}{25}\right)^{10}\right)$$

#

11. (a) $$\frac{1}{\mu_1 + \mu_2 + \mu_3} + \frac{\sum_{i=1}^{3} \mu_i}{\mu_1 + \mu_2 + \mu_3} \cdot \frac{1}{\mu_i} = \frac{4}{\mu_1 + \mu_2 + \mu_3}$$

(b) $$\frac{1}{\mu_1 + \mu_2 + \mu_3} + (a) = \frac{5}{\mu_1 + \mu_2 + \mu_3}$$

#