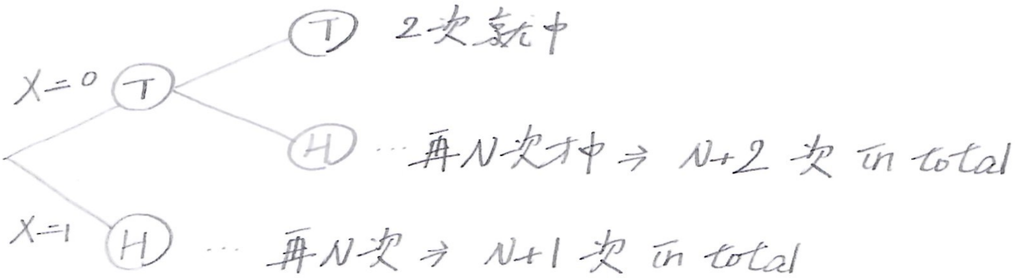
Stochastic Process for Networking HW2 110064533 陳劭珩

1. 試問預期至少需要擲多少次公正硬幣(fair coin)才會連續出現2次枚字(tails)?
(expected N tosses required)

$$X = \begin{cases} 0, & 1^{st} \text{ toss is tails} \\ 1, & 1^{st} \text{ toss is head} \end{cases}$$

利用 law of total expectation $\boxed{E(Y) = \sum_x E(Y \mid X=x) \cdot P(X=x)}$

sample space $S = \{(r_1, r_2, \ldots, r_n),\ n \geq 2\}$, $r_i \in \{\text{heads, tails}\}$



只知道 1^{st} 結果

$$\Rightarrow E(N) = E(E(N \mid X)) = \sum_x E(N \mid X=x) \cdot P(X=x)$$

$$= \underbrace{E(N \mid X=0)}_{\text{再拆成2項}} \cdot P(X=0) + E(N \mid X=1) \cdot P(X=1)$$

$$= \left[E(2)\frac{1}{2} + E(N+2)\frac{1}{2}\right]\frac{1}{2} + E(N+1)\frac{1}{2}$$

$$= 2 \cdot \frac{1}{4} + E(N)\frac{1}{4} + 2 \cdot \frac{1}{4} + E(N)\frac{1}{2} + 1 \cdot \frac{1}{2}$$

$$\Rightarrow E(N) = \frac{3}{4}E(N) + \frac{3}{2}$$

$$\Rightarrow \frac{1}{4}E(N) = \frac{3}{2}$$

$\Rightarrow E(N) = 6$ ∴ the expected number of the tosses required is 6.

✻

2. 一副牌52張，有4張Ace，隨机分4个pile，1个pile 13張

定義4个event $E_1=\{1^{st}$ pile 有剛好 1張Ace$\}$

$E_2=\{2^{nd}$ 〃 $\}$

$E_3=\{3^{rd}$ 〃 $\}$

$E_4=\{4^{th}$ 〃 $\}$

試求 $P\{E_1E_2E_3E_4\}$?

$\Rightarrow P(E_1E_2E_3E_4)=P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3)$

$$P(E_1)=\frac{C^4_1\cdot C^{48}_{12}}{C^{52}_{13}}=\frac{(4張Ace任选1)\cdot(扣除Ace的48張任选12)}{(52張任选13)}$$

$$P(E_2|E_1)=\frac{C^3_1\cdot C^{36}_{12}}{C^{39}_{13}}$$

$$P(E_3|E_1E_2)=\frac{C^2_1\cdot C^{24}_{12}}{C^{26}_{13}}$$

$$P(E_4|E_1E_2E_3)=\frac{C^1_1C^{12}_{12}}{C^{13}_{13}}=1$$

$$\Rightarrow P(E_1E_2E_3E_4)=\frac{C^4_1C^{48}_{12}}{C^{52}_{13}}\cdot\frac{C^3_1C^{36}_{12}}{C^{39}_{13}}\cdot\frac{C^2_1C^{24}_{12}}{C^{26}_{13}}\cdot 1$$

$$=\frac{4\cdot\frac{48!}{12!36!}\cdot 3\frac{36!}{12!24!}\cdot 2\cdot\frac{24!}{12!12!}}{\frac{52!}{13!39!}\cdot\frac{39!}{13!26!}\cdot\frac{26!}{13!13!}}=\frac{4!\cdot 48!\,(13!)^4}{52!\cdot(12!)^4}$$

$$=\frac{24\cdot 13^4}{49\cdot 50\cdot 51\cdot 52}=\frac{2197}{20825}$$

$\doteqdot 0.1055$ ※

3. 擲2顆公正6面骰子，試問 (a) 至少1個6の机率？ (b) 2面值不同の情況下，至少1個6の机率？

(a) $P_{\text{至少1個6}} = 1 - P_{\text{都沒6}} = 1 - \frac{5}{6} \cdot \frac{5}{6} = \frac{11}{36}$ #

(b) $P(\underset{A}{\text{至少1個6}} \mid \underset{B}{\text{2面不同}}) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6} \cdot \frac{5}{6} \cdot 2!}{1 - \frac{6}{36}} = \frac{1}{3}$ #

4. $Y_1, Y_2, \ldots, Y_n$ iid r.v. $\sim U[0,1]$. $X = \max\{Y_1 \ldots Y_n\}$

(a) 試証 cdf $F_X(x) = x^n$, $0 \le x \le 1$

由定義 $\boxed{F_X(x) \triangleq P\{X \le x\}} = P\{\max\{Y_1, Y_2, \ldots, Y_n\} \le x\}$

$= P\{Y_1 \le x\} \cdot P\{Y_2 \le x\} \cdots P\{Y_n \le n\}$ ($\because$ iid)

$= F_{Y_1}(x) \cdots F_{Y_n}(x) = \int_0^x \frac{1}{1-0} dx \cdots \int_0^x \frac{1}{1-0} dx$

$= x \cdots x = x^n$ 得証 #

(b) 試問 pdf $f_X(x) = ?$

$\boxed{f_X(x) = \frac{d}{dx} F_X(x)} = \frac{d}{dx} x^n = n x^{n-1}$ #

Note. $X \sim U[a,b]$ is continuous r.v.

pdf $f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$

mean $E(X) = \frac{a+b}{2}$

cdf $F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x < b \\ 1, & x \ge b \end{cases}$

variance $var(X) = \frac{(b-a)^2}{12}$

mgf $M_X(t) = E(e^{tX}) = \frac{e^{tb} - e^{ta}}{t \cdot (b-a)}$

$P(c \le x \le d) = F(d) - F(c)$

$= \int_c^d \frac{1}{b-a} dx = \frac{d-c}{b-a}$

5. 推導 (a) Uniform distribution 跟 (b) Exponential distribution 的 mgf $M_X(t)$，並利用 mgf 導出 mean, 2^{nd} moment, variance.

(a) consider a continuous r.v. $X \sim U[a,b]$

$1°$ $\boxed{\phi(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tX} \cdot f_X(x)dx} = \int_a^b e^{tX} \frac{1}{b-a} dx$

$= \frac{1}{b-a} \cdot \int_a^b e^{tX} dx = \frac{1}{b-a} \cdot \frac{1}{t} e^{tX} \Big|_a^b = \frac{e^{tb} - e^{ta}}{(b-a)t}$

$= \frac{1}{b-a} \cdot t^{-1}(e^{tb} - e^{ta})$ #

$2°$ $\phi'(t) = \frac{1}{b-a}\left[(-t^{-2})(e^{tb} - e^{ta}) + t^{-1}(be^{tb} - ae^{ta})\right]$

$\phi''(t) = \frac{1}{b-a}\Big[2t^{-3}(e^{tb} - e^{ta}) - t^{-2}(be^{tb} - ae^{ta})$
$- t^{-2}(be^{tb} - ae^{ta}) + t^{-1}(b^2e^{tb} - a^2e^{ta})\Big]$ #

$3°$ $E(X) = \phi'(0)$

$E(X^2) = \phi''(0)$

$4°$ $var(X) = E(X^2) - E(X)^2 = \phi''(0) - [\phi'(0)]^2$

(b) $X \sim Exp(\lambda)$, $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$

$1°$ $\phi(t) = \int_{-\infty}^{\infty} e^{tX} f_X(x) dx$

$= \int_0^{\infty} e^{tX} \cdot \lambda e^{-\lambda x} dx$

$= \int_0^{\infty} \lambda e^{-(\lambda - t)x} dx$

$= \frac{\lambda e^{-(\lambda-t)x}\Big|_0^{\infty}}{-(\lambda - t)}$

$= \frac{\lambda(0-1)}{-(\lambda-t)}$

$= \frac{\lambda}{\lambda - t}$, $\lambda > t$ #

$2°$ $\phi'(t) = \frac{0 - \lambda(-1)}{(\lambda-t)^2} = \lambda(\lambda-t)^{-2}$

$\phi''(t) = -2\lambda(\lambda-t)^{-3}(-1) = 2\lambda(\lambda-t)^{-3}$ #

$3°$ $E(X) = \phi'(0) = \lambda^{-1}$, $E(X^2) = \phi''(0) = 2\lambda^{-2}$

$4°$ $var(X) = E(X^2) - E^2(X) = \frac{2}{\lambda^2} - (\frac{1}{\lambda})^2 = \lambda^{-2} = \frac{1}{\lambda^2}$ #

6. derive the tightest Chernoff's bound for the Poisson r.v. X with PMF $P_X(n)=\frac{e^{-\lambda}\cdot\lambda^n}{n!}$, $n=0,1,2,\ldots$

Chernoff's bound $\boxed{P\{X\geq a\}\leq e^{-ta}\cdot M_X(t)}$ $\forall\, t>0$

令 $g(t)\triangleq e^{-ta}\cdot M_X(t)$, $t>0$ 即 tightest bound $=\min\limits_{t>0} g(t)$

1°

$$\boxed{M_X(t)=\sum_x e^{tx}\cdot p(X=x)}=\sum_{n=0}^{\infty}e^{t\cdot n}\cdot\frac{e^{-\lambda}\cdot\lambda^n}{n!}$$

$$=e^{-\lambda}\cdot\sum_{n=0}^{\infty}\frac{e^{tn}\cdot\lambda^n}{n!}=e^{-\lambda}\cdot\sum_{n=0}^{\infty}\frac{(e^t\cdot\lambda)^n}{n!}\quad\left(\because\ \boxed{e^{\lambda}=\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}}\right)$$

$$=e^{-\lambda}\cdot e^{(e^t\lambda)}=e^{\lambda(e^t-1)}$$

2°

$$\Rightarrow g(t)=e^{-ta}\cdot M_X(t)=e^{-ta}\cdot e^{\lambda(e^t-1)}=e^{\lambda e^t-\lambda-ta},\quad t>0$$

$$\xrightarrow{\text{極值發生在一階導數為}0}g'(t)=(\lambda e^t-a)\underbrace{e^{\lambda e^t-\lambda-ta}}\ ,\quad t>0$$

$\therefore\ \lambda e^t-a=0\quad e^t=\frac{a}{\lambda}\quad t=\ln\frac{a}{\lambda}$ 時 $\xrightarrow{\text{恆正}}$ $g(t)$ 有極值

3°

$t=\ln\frac{a}{\lambda}$ 代回 $g(t)$ 求解

$$g(\ln\tfrac{a}{\lambda})=e^{\lambda e^{\ln\frac{a}{\lambda}}-\lambda-a\ln\frac{a}{\lambda}}=e^{\lambda\cdot\frac{a}{\lambda}-\lambda-(\ln\frac{a}{\lambda})^a}$$

$$=e^{a-\lambda-(\ln\frac{a}{\lambda})^a}=e^a\cdot e^{-\lambda}\cdot e^{-(\ln\frac{a}{\lambda})^a}$$

$$e^{(\ln\frac{a}{\lambda})^{-a}}=\left(\frac{a}{\lambda}\right)^{-a}=\left(\frac{\lambda}{a}\right)^a$$

$$=e^a\cdot e^{-\lambda}\left(\frac{\lambda}{a}\right)^a$$

$$=e^{-\lambda}\cdot\left(\frac{\lambda e}{a}\right)^a$$

$\therefore$ tightest bound $=e^{-\lambda}\left(\frac{\lambda e}{a}\right)^a$

#

7. with $K(t) = \ln\left[E(e^{tX})\right]$, show that $K'(0) = E(X)$

$K''(0) = \mathrm{var}(X)$

$M_X(t) = E(e^{tX})$ $\qquad \boxed{e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$

$$e^{tX} = \sum_{n=0}^{\infty} \frac{(tX)^n}{n!} = 1 + t \cdot X + \frac{t^2 X^2}{2!} + \cdots + \frac{t^n \cdot X^n}{n!}$$

$$\Rightarrow M_X(t) = E\left(\sum_{n=0}^{\infty} \frac{t^n X^n}{n!}\right) = 1 + t \cdot E(X) + \frac{t^2}{2!} E(X^2) + \cdots + \frac{t^n}{n!} E(X^n)$$

$$= 1 + t \cdot m_1 + \frac{t^2}{2!} m_2 + \cdots + \frac{t^n}{n!} m_n = 1 + \sum_{n=1}^{\infty} \frac{m_n t^n}{n!}$$

n^{th} moment $m_n = E(X^n) = M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t) \Big|_{t=0}$

1^{st} $m_1 = M_X'(0) = 0 + m_1 + 0 \cdots = m_1$

2^{nd} $m_2 = M_X''(0) = 0 + 0 + \frac{2!}{2!} m_2 + 0 \cdots = m_2$

$$\Rightarrow M_X(t) = 1 + \sum_{n=1}^{\infty} \frac{m_n t^n}{n!} = e^{K(t)} \quad (\because \text{已知 } K(t) = \ln M_X(t))$$

$$1 + \sum_{n=1}^{\infty} \frac{m_n t^n}{n!} = 1 + K(t) + \frac{K^2(t)}{2!} + \cdots + \frac{K^n(t)}{n!}$$

$K(t) = \ln(M_X(t))$ $\qquad K'(t) = \frac{d}{dt} K(t) = \frac{d}{dt} \ln(M_X(t)) = \frac{1}{M_X(t)} \cdot \frac{d}{dt} M_X(t)$

$$\Rightarrow K'(0) = \frac{d}{dt} K(t) \Big|_{t=0} = \frac{M_X'(0)}{M_X(0)} = \frac{m_1}{1} = m_1 = E(X) \quad \#$$

$$K''(t) = \frac{d}{dt}\left(\frac{\frac{d}{dt} M_X(t)}{M_X(t)}\right)\Bigg|_{t=0} = \frac{\left(\frac{d^2}{dt^2} M_X(t)\right) \cdot M_X(t) - \frac{d}{dt} M_X(t) \cdot \frac{d}{dt} M_X(t)}{(M_X(t))^2}\Bigg|_{t=0}$$

$$\Rightarrow K''(0) = \frac{M_X''(0) \cdot M_X(0) - (M_X'(0))^2}{(M_X(0))^2} = \frac{m_2 \cdot 1 - (m_1)^2}{1^2} = m_2 - m_1^2$$

$$= E(X^2) - E(X)^2 = \mathrm{var}(X)$$

$\#$