CS 529100: Stochastic Processes for Networking

Final exam solutions

1. (5% each)

   (a)

   $X_i =$ amount of time he has to travel after his i^{th} choice (We will assume that he keeps on making choices even after becoming free). N is the number of choices he makes until becoming free. Specifically, $N = \min\{i | X_i = 2\}$.

   (b)

$$E[T] = E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X] = 3 \times \frac{1}{3}(2+4+6) = 12$$

2. (5% each)

   (a)

$$
\begin{aligned}
m(t) &= \int_0^t f_X(\tau)d\tau + \int_0^t m(t-x)f_X(x)dx \\
M(s) &= \int_0^\infty e^{-st}\int_0^t f_X(\tau)d\tau\, dt + \int_0^\infty e^{-st}\int_0^t m(t-x)f_X(x)dx\, dt \\
&= \int_0^\infty \int_\tau^\infty e^{-st} f_X(\tau)dt\, d\tau + \int_0^\infty \int_x^\infty e^{-st} m(t-x)f_X(x)dt\, dx \\
&= \int_0^\infty \frac{e^{-s\tau}}{s} f_X(\tau)d\tau + \int_0^\infty e^{-sx} f_X(x)\int_x^\infty e^{-s(t-x)}m(t-x)dt\, dx \\
&= \frac{X(s)}{s} + \int_0^\infty e^{-sx}f_X(x)dx \int_0^\infty e^{-s(u)}m(u)du \quad (\text{Let } u = t-x) \\
&= \frac{X(s)}{s} + X(s)M(s) \\
\rightarrow M(s) &= \frac{X(s)}{s(1-X(s))}
\end{aligned}
$$

   (b)

   By the one-to-one correspondence of $m(t)$ and F, it follows that $\{N(t), t \geq 0\}$ is a Poisson process with rate 1/2. Hence $P\{N(5) = 0\} = e^{-\frac{5}{2}}$.

3. (10%)

A job completion constitutes a renewal. Let T denote the time between renewals. Let W be the time it takes to finish the next job. Let S be the time of the next shock.

$$E[T|W=w]=\int_0^\infty E[T|W=w,S=s]P\{S=s\}ds=\int_0^w (s+E[T])\lambda e^{-\lambda s}ds+\int_w^\infty w\lambda e^{-\lambda s}ds$$

$$=\int_0^w w\lambda e^{-\lambda s}ds+E[T]\left(1-e^{-\lambda w}\right)+we^{-\lambda w}$$

$$=-we^{-\lambda w}+\frac{1-e^{-\lambda w}}{\lambda}+E[T]\left(1-e^{-\lambda w}\right)+xe^{-\lambda w}=\left(1-e^{-\lambda w}\right)\left(E[T]+\frac{1}{\lambda}\right)$$

$$E[T]=E\big[E[T|W=w]\big]=\left(1-E[e^{-\lambda W}]\right)\left(E[T]+\frac{1}{\lambda}\right)$$

$$E[T]=\frac{\frac{1}{E[e^{-\lambda W}]}-1}{\lambda}=\frac{\frac{1}{\int_0^\infty e^{-\lambda w}dF(w)}-1}{\lambda}$$

$$\text{rate}=\frac{1}{E[T]}=\frac{\lambda}{\frac{1}{\int_0^\infty e^{-\lambda w}dF(w)}-1}$$

4. (10%)

A machine failure constitutes a renewal. Let C denote the time between renewals. Let X denote the lifetime of a machine.

$$E[C]=\int_0^T xf_X(x)dx+\int_T^\infty (T+E[C])f_X(x)dx$$

$$=\int_0^T xf_X(x)dx+(T+E[C])(1-F_X(T))$$

$$E[C]=\frac{\int_0^T xf_X(x)dx+T(1-F_X(T))}{F_X(T)}$$

The long-run rate at which machines in use fail $=\dfrac{1}{E[C]}=\dfrac{F_X(T)}{\int_0^T xf_X(x)dx+T(1-F_X(T))}$

5. (10%)

"on": driving from A to B

"off": driving from B to A

Let d be the distance between A and B.

$$E[\text{on}]=\int_{40}^{60}\frac{d}{x}\times\frac{1}{20}dx=\frac{d}{20}(\ln(60)-\ln(40))=\frac{d}{20}\ln\left(\frac{3}{2}\right)$$

$$E[\text{off}]=\frac{d}{40}\times\frac{1}{2}+\frac{d}{60}\times\frac{1}{2}=\frac{d}{48}$$

The proportion of his driving time is spend going to B is $\dfrac{E[\text{on}]}{E[\text{on}]+E[\text{off}]}=\dfrac{\frac{d}{20}\ln\left(\frac{3}{2}\right)}{\frac{d}{20}\ln\left(\frac{3}{2}\right)+\frac{d}{48}}$

6. (10%)

The particle moving process is a Markov chain. The long-run proportion of time that the particle is in state 0 is $\frac{1}{n+1}$ since the particle will be in every state with equal probability.

Thus, $E[T] = n + 1$.

7. (10%)

Three classes:

{0,2}: recurrent class

{1}: transient class

{3,4}: recurrent class

8. (10%)

$$\begin{cases} \pi_0 = \pi_0 \times 0 + \pi_1 \times \dfrac{1}{4} + \pi_2 \times \dfrac{1}{2} \\ \pi_1 = \pi_0 \times \dfrac{1}{3} + \pi_1 \times \dfrac{1}{2} + \pi_2 \times \dfrac{1}{2} \\ \pi_2 = \pi_0 \times \dfrac{2}{3} + \pi_1 \times \dfrac{1}{4} + \pi_2 \times 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\rightarrow \begin{cases} \pi_0 = \dfrac{9}{35} \\ \pi_1 = \dfrac{16}{35} \\ \pi_2 = \dfrac{10}{35} \end{cases}$$

$$\begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 1/2 & 0 \end{pmatrix}^{\infty} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 \\ \pi_0 & \pi_1 & \pi_2 \\ \pi_0 & \pi_1 & \pi_2 \end{pmatrix} = \begin{pmatrix} 9/35 & 16/35 & 10/35 \\ 9/35 & 16/35 & 10/35 \\ 9/35 & 16/35 & 10/35 \end{pmatrix}$$

9. (10%)

$$\begin{cases} P_0 \times 3 = P_1 \times 2 \\ P_0 + P_1 = 1 \end{cases}$$

$$\rightarrow P_0 = \frac{2}{5} \quad P_1 = \frac{3}{5}$$

The fraction of time the process stays in state $0 = \dfrac{2}{5}$.

10. (10%)

Using Kolmogorov's forward equation:

$P'_{11}(t) = P_{10}(t)\lambda - P_{11}(t)\mu = \big(1 - P_{11}(t)\big)\lambda - P_{11}(t)\mu = -(\lambda + \mu)P_{11}(t) + \lambda$

$\rightarrow P'_{11}(t) + (\lambda + \mu)P_{11}(t) = \lambda$

Non-homogeneous solution:

$$P_{11}(t) = \frac{\lambda}{\lambda + \mu}$$

Homogeneous solution:

$P_{11}(t) = Ae^{-(\lambda+\mu)t}$

$$\rightarrow P_{11}(t) = Ae^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda + \mu}$$

$$\because P_{11}(0) = 1 \rightarrow A = \frac{\mu}{\lambda + \mu}$$

$$\therefore P_{11}(t) = \frac{\mu}{\lambda + \mu}e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda + \mu}$$

11. (10%)

(a)

$P_0\lambda = P_1\mu$

$P_i(\lambda + \mu) = P_{i-1}\lambda + P_{i+1}\mu \;\; , \;\; i = 1,2,\ldots$

(b)

$$\begin{cases} P_0 = P_0 \\ P_i = \left(\dfrac{\lambda}{\mu}\right)^i P_0, i = 1,2,\ldots \end{cases}$$