Quiz 1

Student ID: Name:

(a). The continuous random variable X is exponentially distributed with parameter λ. Its cumulative distribution function is

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

Another random variable Y is defined as $Y = 8X$.

Please find $\mathrm{E}[Y]$.

$$E[Y] = E[8X] = \int_0^\infty 8x\lambda e^{-\lambda}\ dx = 8\int_0^\infty x\lambda e^{-\lambda x} dx$$

$$= 8[x(-e^{-\lambda}\ )|_0^\infty - \int_0^\infty (-e^{-\lambda x})dx] = 8\int_0^\infty e^{-\lambda x} dx = \frac{8}{\lambda}$$

(b). Suppose X has the following probability mass function:

$$\mathrm{p}(0) = 0.1, \quad \mathrm{p}(1) = 0.6, \quad \mathrm{p}(2) = 0.3$$

Please Calculate $\mathrm{E}[X^2]$.

Letting $\mathrm{Y} = \mathrm{X}^2$, we have that Y is a random variable that can take on one of the values $0^2, 1^2,\ 2^2$ with respective probabilities

$$p_Y(0) = P\{Y = 0^2\} = 0.1$$

$$p_Y(1) = P\{Y = 1^2\} = 0.6$$

$$p_Y(4) = P\{Y = 2^2\} = 0.3$$

Hence,

$$\mathrm{E}[Y] = \mathrm{E}[X^2] = 0 \cdot (0.1) + 1 \cdot (0.6) + 4 \cdot (0.3) = 1.8$$

Quiz 2

Student ID: Name:

For an exponential random variable X, $X \sim \mathrm{Exp}(\lambda)$, we know its moment generating function is $\phi(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$. What is the third moment of X? (That is, what is $E[X^3]$?)

$$\phi(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} \cdot \lambda e^{-\lambda x} dx = \int_0^{\infty} \lambda \cdot e^{-(\lambda - t)x} dx$$

$$= -\left(\frac{\lambda}{\lambda - t}\right) e^{-(\lambda - t)x}\Big|_0^{\infty} = \frac{\lambda}{\lambda - t}, \qquad t < \lambda$$

$$\phi'(t) = \frac{\lambda}{(\lambda - t)^2}$$

$$\phi''(t) = \frac{2\lambda}{(\lambda - t)^3}$$

$$\phi'''(t) = \frac{6\lambda}{(\lambda - t)^4}$$

$$E[X^3] = \phi'''(0) = \frac{6}{\lambda^3}$$

Quiz 3

Student ID: Name:

1. X_1 and X_2 are independent exponential random variables, each with rate λ. The random variable Y is defined as $\mathrm{Y} = \min(\mathrm{X}_1, \mathrm{X}_2)$.
   Please find $\mathrm{E}[\mathrm{Y}]$.

$$\mathrm{Y} \sim Exp(\lambda + \lambda) = Exp(2\lambda)$$

$$\mathrm{E}[\mathrm{Y}] = \frac{1}{2\lambda}$$

2. Let $\mathrm{X} \sim \mathrm{Exponential}(\lambda)$. Use Chebyshev's inequality to find an upper bound of $\mathrm{P}(|X - E[X]| \geq \mathrm{b})$.

   Since $\mathrm{X} \sim \mathrm{Exponential}(\lambda)$, we have $\mathrm{E}[\mathrm{X}] = \frac{1}{\lambda}$ and $\mathrm{Var}[\mathrm{X}] = \frac{1}{\lambda^2}$.

   Using Chebyshev's inequality, we have

$$\mathrm{P}(|X - E[X]| \geq \mathrm{b}) \leq \frac{Var(X)}{b^2}$$

$$= \frac{1}{\lambda^2 b^2}$$

Quiz 4

Student ID: Name:

1. Let $\{N(t), t \geq 0\}$ be a Poisson process with $P\{N(1) = 0\} = e^{-3}$. Let S_n denote the time of the n-th event. Find $E[N(4) - N(2)|N(1) = 3]$.

$$\mathrm{e}^{-\lambda\times 1}\frac{(\lambda \times 1)^0}{0!} = e^{-3} \Longrightarrow \lambda = 3$$

$$\mathrm{E[N(4) - N(2)|N(1) = 3]} = E[N(2)] = \lambda \times t = 3 \times 2 = 6$$

2. On Friday night, the number of customers arriving at a lounge bar can be modeled by a Poisson process with intensity that twelve customers per hour.
   (a) Find the probability that there are 2 customers between 21:00 and 21:40.
   (b) Find the probability that there are 4 customers between 21:00 and 21:40 and 6 customers between 21:40 and 22:00.

(a) We know that $\lambda = 12$ and the interval between 21:00 and 21:40 has the length $\tau = \frac{2}{3}$ hours. If X is the number of arrivals in that interval, we can write $\mathrm{X} \sim \mathrm{Poisson}(8)$. Therefore,

$$\mathrm{P(X = 2)} = \frac{e^{-8}(8)^2}{2!}$$

(b) Here, we have two non-overlapping intervals $I_1 = (21{:}00, 21{:}40]$ and $I_2 = (21{:}40, 22{:}00]$. Thus, we can write

$$\mathrm{P(\ 4\ arrivals\ in\ } I_1 \ and\ \mathrm{6\ arrivals\ in\ } I_2\ )$$

$$= \mathrm{P(\ 4\ arrivals\ in\ } I_1\ ) \cdot P(\ \mathrm{6\ arrivals\ in\ } I_2\ )$$

$$= \frac{e^{-8}(8)^4}{4!} \cdot \frac{e^{-4}(4)^6}{6!}$$