109064554
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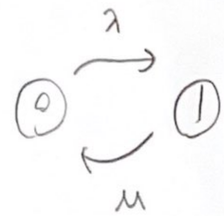
1. $P_{11}(t) = ?$



(a)

$$P'_{ij}(t) = \sum_{k \neq j} P_{ik}(t)\, q_{kj} - P_{ij}(t)\, v_j$$

$$P'_{11}(t) = P_{10}(t) \cdot \lambda - P_{11}(t) \cdot \mu$$

$$= [1 - P_{11}(t)] \cdot \lambda - P_{11}(t) \cdot \mu$$

$$= -(\lambda + \mu) P_{11}(t) + \lambda$$

$$P'_{11}(t) + (\lambda + \mu) P_{11}(t) = \lambda$$

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(b)

$$P'_{11}(t) + (\lambda + \mu) P_{11}(t) = \lambda$$

Non-homogeneous solution:

$$\frac{\lambda}{\lambda + \mu}$$

Homogeneous solution:

$$A e^{-(\lambda + \mu) t}$$

$$P_{11}(t) = A e^{-(\lambda + \mu) t} + \frac{\lambda}{\lambda + \mu}$$

因 $P_{11}(0) = 1 = A + \dfrac{\lambda}{\lambda + \mu}$

所以 $A = \dfrac{\mu}{\lambda + \mu}$

$$\Rightarrow P_{11}(t) = \frac{\mu}{\lambda + \mu} e^{-(\lambda + \mu) t} + \frac{\lambda}{\lambda + \mu}$$

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2. clockwise p, counterclockwise $1-p$

(n+1)頂點

Starting at state 0

T: time of the first return to state 0

$E[T]=?$

long-run proportion of time that the particle is at state 0 is $\frac{1}{n+1}$, the particle will be in every state with equal probability

$\Rightarrow E[T]=n+1$ #

3. $M^{\infty}=?$

$$M=\begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$[\pi_0\ \pi_1\ \pi_2]=[\pi_0\ \pi_1\ \pi_2]\begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{cases} \pi_0=\frac{1}{4}\pi_1+\frac{1}{2}\pi_2 \\ \pi_1=\frac{1}{3}\pi_0+\frac{1}{2}\pi_1+\frac{1}{2}\pi_2 \\ \pi_2=\frac{2}{3}\pi_0+\frac{1}{4}\pi_1 \\ \pi_0+\pi_1+\pi_2=1 \end{cases}$$

$$\pi_2=\frac{2}{3}\left(\frac{1}{4}\pi_1+\frac{1}{2}\pi_2\right)+\frac{1}{4}\pi_1$$

$$=\frac{1}{6}\pi_1+\frac{1}{3}\pi_2+\frac{1}{4}\pi_1$$

$$\frac{2}{3}\pi_2=\frac{5}{12}\pi_1 \qquad \pi_1=\frac{2}{3}\pi_2\cdot\frac{12}{5}=\frac{8}{5}\pi_2$$

$$\pi_0=\frac{1}{4}\cdot\frac{8}{5}\pi_2+\frac{1}{2}\pi_2$$

$$=\frac{2}{5}\pi_2+\frac{1}{2}\pi_2=\frac{4+5}{10}\pi_2=\frac{9}{10}\pi_2$$

$$\frac{9}{10}\pi_2+\frac{8}{5}\pi_2+\pi_2=1$$

$$\frac{9+16+10}{10}\pi_2=1$$

$$\frac{35}{10}\pi_2=1$$

$$\pi_2=\frac{10}{35},\ \pi_1=\frac{8}{5}\cdot\frac{10}{35}=\frac{16}{35},\ \pi_0=\frac{9}{10}\cdot\frac{10}{35}=\frac{9}{35}$$

$$M^{\infty}=\begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \\ \pi_0 & \pi_1 & \pi_2 \\ \pi_0 & \pi_1 & \pi_2 \end{bmatrix}=\begin{bmatrix} \frac{9}{35} & \frac{16}{35} & \frac{10}{35} \\ \frac{9}{35} & \frac{16}{35} & \frac{10}{35} \\ \frac{9}{35} & \frac{16}{35} & \frac{10}{35} \end{bmatrix}$$ #

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4. Prove : For all $s \geq 0$, $t \geq 0$

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s) \ ?$$

$$i \xrightarrow{\text{time } t} k \xrightarrow{\text{time } s} j$$

$$P_{ij}(t+s) = P\{X(t+s) = j \mid X(0) = i\}$$

$$= \sum_{k=0}^{\infty} P\{\underbrace{X(t+s)}_{A} = j, \underbrace{X(t)}_{B} = k \mid X(0) = i\}$$

$$= \sum_{k=0}^{\infty} P\{X(t) = k \mid X(0) = i\} \cdot P\{X(t+s) = j \mid X(t) = k, X(0) = i\}$$

$$= \sum_{k=0}^{\infty} P_{ik}(t) \cdot P_{kj}(s)$$

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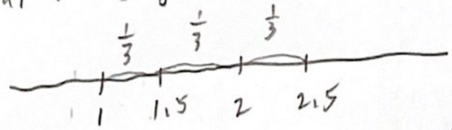
5. 1 year to 2.5 year 210 dollars



C: length of the inspection cycle.

$E[R] = 210 \times \frac{1}{3} = 70$

$E[C] = \frac{7}{4}$

long-run amount of fine $= \dfrac{70}{\frac{7}{4}}$

$= 70 \cdot \frac{4}{7} = 40$ dollars

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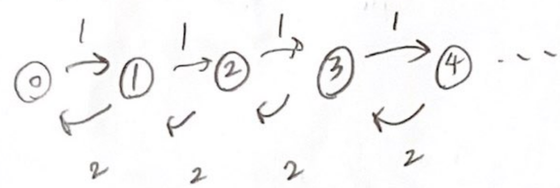
6. taxi ~ poisson with rate = 1

customer ~ poisson with rate = 2



average # of taxis waiting = 1

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7. $\frac{1}{2}$ left — return — 5 mins

$\frac{1}{2}$ right — depart — 4 mins

T: time duration that it will be trapped in the cave

$E[T] = ?$

$$E[T] = \frac{1}{2}(5 + E[T]) + \frac{1}{2} \cdot 4$$

$$= \frac{5}{2} + \frac{E[T]}{2} + 2$$

$$\frac{1}{2}E[T] = \frac{9}{2}$$

$$E[T] = 9 \text{ mins}$$

#

8

fix
- machine 1 exp with rate 5
- machine 2 uniform (0,4)
- machine 3 gamma, $n=3, \lambda=2$

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"on" → system working

"off" → system isn't working

$$E[on] = \frac{1}{\lambda_1+\lambda_2+\lambda_3}$$

$$E[off] = \sum_{i=1}^{3} E[\text{repair time for machine} \mid \text{machine } i \text{ fails}] \cdot P(\text{machine } i \text{ fails})$$

$$= \frac{1}{5} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} + \frac{0+4}{2} \cdot \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} + \frac{3}{2} \cdot \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}$$

$$= \frac{\lambda_1}{5(\lambda_1+\lambda_2+\lambda_3)} + \frac{2\lambda_2}{\lambda_1+\lambda_2+\lambda_3} + \frac{3\lambda_3}{2(\lambda_1+\lambda_2+\lambda_3)}$$

system is working $= \dfrac{E[on]}{E[on]+E[off]}$

$$= \frac{\frac{1}{\lambda_1+\lambda_2+\lambda_3}}{\frac{1}{\lambda_1+\lambda_2+\lambda_3} + \frac{\lambda_1}{5(\lambda_1+\lambda_2+\lambda_3)} + \frac{2\lambda_2}{\lambda_1+\lambda_2+\lambda_3} + \frac{3\lambda_3}{2(\lambda_1+\lambda_2+\lambda_3)}}$$

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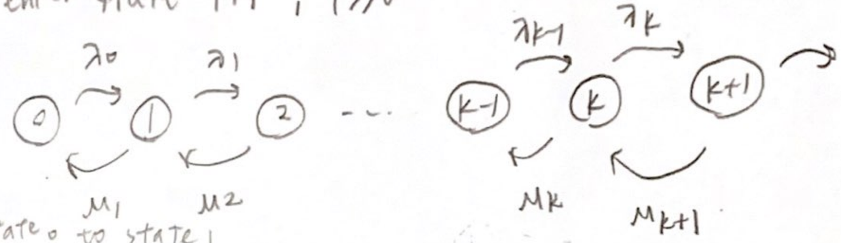
1. $\lambda_i = (i+1)\lambda$

$\mu_i = i\mu \quad i \geq 0$

T_i : the time, starting from state i, it takes for the process to enter state $i+1$, $i \geq 0$



from state 0 to state 1

(a) $E[T_0] = ? = \mu_0 1$

離開 state 0 的 rate $= \lambda_0$

離開 state 1 的 rate $= \mu_1 + \lambda_1$

$E[T_0] \dfrac{\mu_1 + \lambda_1}{\lambda_0}$

#

Prove that $E[T_i] = \dfrac{1}{\lambda_i} + \dfrac{\mu_i}{\lambda_i} E[T_{i-1}]$ for $i \geq 1$

(b)

state 1 ~ state 2

$i=1 \quad E[T_1] = \dfrac{1}{\lambda_1} + \dfrac{\mu_1}{\lambda_1} E[T_0]$

離開 state 1 rate $= \lambda_1 + \mu_1$

離開 state 1 流到 state 2 $= \dfrac{\lambda_1}{\lambda_1 + \mu_1}$

—

10. $X_i \sim$ uniformly dist $(0,1)$ 109064054

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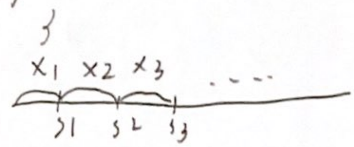
$m(t)$: mean-value fun of renewal process

$m(t) = F(t) + \int_0^t m(t-x) f(x)\, dx$



(a) $m(t) = E[N(t)] = F(t) + \int_0^t m(t-x) f(x)\, dx$

$= \frac{t-0}{1-0} + \int_0^t m(t-x) \cdot 1\, dx$

$= t + \int_0^t m(t-x)\, dx$

$m(t) = t + \int_0^t m(t-x)\, dx$

$m'(t) = 1 + m(t)$

$m'(t) - m(t) = 1$ #

(b) homogeneous solution: $m(t) = e^{st}$

$m'(t) - m(t) = 0$

$s e^{st} - e^{st} = 0$

$(s-1) e^{st} = 0$, $s = 1$

non-homogeneous solution:

$m'(t) - m(t) = 1$

$m(t) = -1$

general solution: $m(t) = Ae^t - 1$

$m(0) = A - 1 = 0 \Rightarrow A = 1$

$\Rightarrow m(t) = e^t - 1$ 當 $0 < t < 1$

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11.

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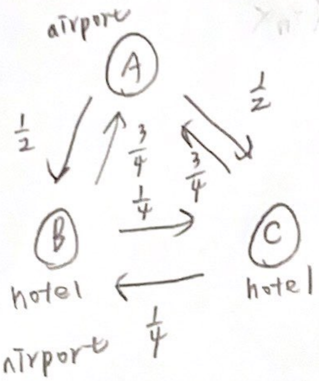
印翊境



begin at airport

$P\{X_0=A\}=1$

$P\{$ at hotel B at time 3 $\}=?$

$$P=\begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & \frac{1}{2} & \frac{1}{2} \\ B & \frac{3}{4} & 0 & \frac{1}{4} \\ C & \frac{3}{4} & \frac{1}{4} & 0 \end{array}$$

$\frac{3}{8}+\frac{2}{8}$

$\frac{3}{8}+\frac{1}{16}\cdot\ \frac{}{16}\quad \frac{3}{8}$

$\frac{3}{8}$

$\frac{3}{8}+\frac{1}{16}=\frac{7}{16}$

$$P^2=\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}=\begin{bmatrix} \frac{6}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{3}{16} & \frac{7}{16} & \frac{6}{16} \\ \frac{3}{16} & \frac{6}{16} & \frac{7}{16} \end{bmatrix}$$

$$P^3=\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix}\begin{bmatrix} \frac{6}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{3}{16} & \frac{7}{16} & \frac{6}{16} \\ \frac{3}{16} & \frac{6}{16} & \frac{7}{16} \end{bmatrix}=\begin{array}{c|ccc} & A & B & C \\ \hline A & \frac{6}{32} & \frac{13}{32} & \frac{13}{32} \\ B & \frac{39}{64} & \frac{12}{64} & \frac{13}{64} \\ C & \frac{39}{64} & \frac{13}{64} & \frac{12}{64} \end{array}$$

$P\{X_3=B \mid X_0=A\}$

$=\frac{13}{32}$ #