CS 5291: Stochastic Processes for Networking

midterm solutions

1. (10%)

   $E[Y] = E\big[E[Y|X = x]\big]$

   Given a starting time $(X = x)$

   $$\Rightarrow E[Y|X=x] = \int_0^x y \cdot \frac{3}{x^3} y^2 dy = \frac{3}{x^3}\frac{1}{4}y^4 \Big|_0^x = \frac{3}{4}x$$

   $$E[Y] = E\big[E[Y|X=x]\big] = \int_0^1 E[Y|X=x] \cdot f_x(x)dx = \int_0^1 \frac{3}{4}x \cdot 1\, dx$$

   $$= \frac{3}{4} \cdot \frac{1}{2}x^2 \Big|_0^1 = \frac{3}{8}$$

2. (15%)

   (a)

   $$P\{x \geq a\} \leq \min_{t>0} e^{-ta} M_X(t) = \min_{t>0} e^{-ta-\lambda(1-e^t)}$$

   Let $g(t) = e^{-ta-\lambda(1-e^t)}$

   To find the minimum of $g(t)$, we need to solve $\frac{dg(t)}{dt} = 0$.

   $$\frac{dg(t)}{dt} = 0 \Rightarrow (\lambda e^t - a)e^{-ta-\lambda(1-e^t)} = 0 \Rightarrow t = \ln\frac{a}{\lambda}$$

   $$P\{X \geq a\} \leq g\left(ln\left(\frac{a}{\lambda}\right)\right) = e^{-a ln\left(\frac{a}{\lambda}\right)-\lambda+a} = e^{\ln\left(\frac{\lambda}{a}\right)^a} \cdot e^{-\lambda} \cdot e^a = e^{-\lambda}\left(\frac{\lambda e}{a}\right)^a$$

   (b)

   The Chebyshev's inequality is $P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$ for $k \geq 0$ and for any random variable *X*.

   Because *X* is Poisson distributed, its mean and variance are $E[X] = \lambda, VAR(X) = \lambda$. Therefore, $P(|X-\lambda| \geq b) \leq \frac{\lambda}{b^2}$.

3. (10%)

   (a)

$$P\{\text{no event in a time interval of length } T\} = P\{N(T) = 0\} = e^{-\lambda T} \cdot \frac{(\lambda T)^0}{0!}$$

$$= e^{-\lambda T}$$

(b)

Let W be waiting time and X be the time until the first car.

$$E[W] = \int_0^{\infty} E[W|X = x] \cdot \lambda e^{-\lambda x}\, dx = \int_0^{T} E[W|X = x] \cdot \lambda e^{-\lambda x}\, dx$$

$$= \int_0^T (x + E[W])\lambda e^{-\lambda x} dx = \int_0^T \lambda x e^{-\lambda x} dx + E[W] \int_0^T \lambda e^{-\lambda x} dx$$

$$= x \cdot \left(-e^{-\lambda x}\right)\Big|_0^T - \int_0^T 1 \cdot \left(-e^{-\lambda}\ \right) dx + E[W](1 - e^{-\lambda T})$$

$$= -Te^{-\lambda T} - \tfrac{1}{\lambda} e^{-\lambda T} + \tfrac{1}{\lambda} + E[W](1 - e^{-\lambda T})$$

$$\Rightarrow E[W]e^{-\lambda T} = -Te^{-\lambda T} - \tfrac{1}{\lambda} e^{-\lambda T} + \tfrac{1}{\lambda}$$

$$\Rightarrow E[W] = -T - \tfrac{1}{\lambda} + \tfrac{e^{\lambda T}}{\lambda}$$

4. (10%)

(a)

$$\mathrm{P}(N(t) = n, M(t) = m) = \mathrm{P}(N(t) = n, K(t) = n + m)$$

$$= \mathrm{P}(K(t) = n + m) \cdot \mathrm{P}(N(t) = n \mid K(t) = n + m)$$

$$= e^{-\lambda t} \tfrac{(\lambda t)^{n+m}}{(n+m)!} \cdot \tbinom{n+m}{n} p^n (1 - p)^m = e^{-\lambda t} \tfrac{(\lambda t)^{n+m}}{n!m!} \cdot p^n \cdot (1 - p)^m$$

(b)

$$\mathrm{P}(N(t) = n) = \sum_{m=0}^{\infty} P(N(t) = n, M(t) = m)$$

$$= e^{-\lambda t} \tfrac{(\lambda t)^{n+m}}{n!m!} \cdot p^n \cdot (1 - p)^m = e^{-\lambda t} \tfrac{(\lambda p t)^n}{n!} \cdot \sum_{m=0}^{\infty} \tfrac{[\lambda(1-p)t]^m}{m!}$$

$$= e^{-\lambda t} \tfrac{(\lambda p t)^n}{n!} \cdot e^{\lambda(1-p)t} = e^{-\lambda p t} \tfrac{(\lambda p t)^n}{n!}$$

5. (10%)

Let t as the time moment. t=0.5 means the 8:30am; t=1.5 means the 9:30am.
A good model for the preceding would be to assume that arrivals constitute a nonhomogeneous Poisson process with intensity function $\lambda(t)$ given by

$$\lambda(t)=\begin{cases}5t+5, & 0\le t\le 1\\ 10, & 1\le t\le 2\\ 0, & \text{otherwise}\end{cases}$$

Let $m(t)$ be the mean value function.

$$m(1.5)-m(0.5)=\int_{0.5}^{1.5}\lambda(t)\,dt=\int_{0.5}^{1}(5t+5)\,dt+\int_{1}^{1.5}10\,dt$$

$$=\frac{5}{2}t^2+5t\Big|_{0.5}^{1}+10t\Big|_{1}^{1.5}=\frac{75}{8}$$

Therefore, $P\{N(1.5)-N(0.5)=0\}=e^{-(m(1.5)-m(0.5))}=e^{-\frac{75}{8}}$

6. (10%)

(a)

$$\frac{1}{\mu_1+\mu_2+\mu_3}+\sum_{i=1}^{3}\frac{\mu_i}{\mu_1+\mu_2+\mu_3}\frac{1}{\mu_i}=\frac{4}{\mu_1+\mu_2+\mu_3}$$

(b)

$$\frac{1}{\mu_1+\mu_2+\mu_3}+\text{answer of Problem 9.a}=\frac{5}{\mu_1+\mu_2+\mu_3}$$

7. (15%)

Let $N(t)$ denote the number of customers arrive to a supermarket by time t with rate $\lambda=7+3=10$ customers per hour.

(a) Let $N_s(t)$ denote the number of sales made by time t. $N_s(t)$ is a Poisson process with rate $\lambda_s=0.3\times 10=3$.

$$\mathrm{P}\{N_s(1)\ge 1\}=1-\mathrm{P}\{N_s(1)=0\}=1-e^{-3\times 1}\times\frac{(3\times 1)^0}{0!}=1-\mathrm{e}^{-3}$$

(b) Let S_n denote the time n^{th} sale made, X_n denote the inter-arrival time between the n^{th} sale and the previous one.

$$\mathrm{E}[S_5]=\mathrm{E}[X_1+X_2+\cdots+X_5]=5\times\mathrm{E}[X_1]=5\times\frac{1}{3}=\frac{5}{3}$$

$$9\text{ A.M.}+\frac{5}{3}\text{ hours}=10{:}40\text{ A.M.}$$

(c) Let $N_{ms}(t)$ denote the number of sales made by male customers by time t with rate $\lambda = 3 \times 0.3 = 0.9$ sales per hour, and $N_{fs}(t)$ denote the number of sales made by female customers by time t with rate $\lambda = 7 \times 0.3 = 2.1$ sales per hour.

$$\mathrm{P}\{\text{2 items are bought by 1 male and 1 female customers in a certain period } t\}$$
$$= \mathrm{P}\{\text{1 item is bought by 1 male customer in a certain period } t\}$$
$$\times \mathrm{P}\{\text{1 item is bought by 1 female customer in a certain period } t\}$$
$$= \mathrm{P}\{N_{ms}(t) = 1\} \times \mathrm{P}\{N_{fs}(t) = 1\}$$
$$\therefore \mathrm{P}\{\text{2 items are bought by 1 male and 1 female customers in a certain period } t$$
$$|\text{2 items are sold in a certain period } t\}$$
$$= \frac{\mathrm{P}\{N_{ms}(t) = 1\} \times \mathrm{P}\{N_{fs}(t) = 1\}}{\mathrm{P}\{N_s(t) = 2\}} = \frac{\frac{e^{-0.9t}(0.9t)^1}{1!} \times \frac{e^{-2.1t}(2.1t)^1}{1!}}{\frac{e^{-3t}(3t)^2}{2!}} = \frac{21}{50}$$

8. (10%)

If the repair crew checks the machine while it is still working or before it fails, the crew does nothing.

It would be the next time the crew coming. Because of memoryless property, the time interval between two consecutive replacements is

$$\frac{1}{\lambda} + \frac{1}{\mu}$$

9. (10%)

Let X_n^1 denote the n-th interarrival time of the first Poisson process $N_1(t)$ and X_n^2 denote the n-th interarrival time of the second Poisson process $N_2(t)$.

We first attempt to deal with the case of $n = 1$, that is, we want to compute $\mathrm{P}\{S_1^1 < S_1^2\}$.

Since $S_1^1 = X_1^1$ and $S_1^2 = X_1^2$ are both exponentially distributed random variables with respective mean $1/\lambda_1$ and $1/\lambda_2$, we can get

$$\mathrm{P}\{S_1^1 < S_1^2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Next we attempt to deal with the case of $n = 2$. To compute $\mathrm{P}\{S_2^1 < S_1^2\}$, we use the law of total probability by conditioning on the relationship between S_1^1 and S_1^2 (that is, either $S_1^1 < S_1^2$ or $S_1^1 \geq S_1^2$):

$$\mathrm{P}\{S_2^1 < S_1^2\} = \mathrm{P}\{S_2^1 < S_1^2 \mid S_1^1 < S_1^2\} \cdot \mathrm{P}\{S_1^1 < S_1^2\} + \mathrm{P}\{S_2^1 < S_1^2 \mid S_1^1 \geq S_1^2\} \cdot \mathrm{P}\{S_1^1 \geq S_1^2\}$$

$$= \mathrm{P}\{S_2^1 < S_1^2 \mid S_1^1 < S_1^2\} \cdot \mathrm{P}\{S_1^1 < S_1^2\}$$

where the last equality holds true because the second term is zero. Representing a certain arrival time by the sum of inter-arrival times, $\mathrm{P}\{S_2^1 < S_1^2\}$ can be expressed as

$$\mathrm{P}\{S_2^1 < S_1^2\} = \{S_2^1 < S_1^2 \mid S_1^1 < S_1^2\} \cdot \mathrm{P}\{S_1^1 < S_1^2\}$$

$$= \mathrm{P}\{S_1^1 + X_2^1 < X_1^2 \mid S_1^1 < S_1^2\} \cdot \mathrm{P}\{S_1^1 < S_1^2\}$$

$$= \mathrm{P}\{X_2^1 < X_1^2\} \cdot \mathrm{P}\{S_1^1 < S_1^2\}$$

$$(\text{By memoryless property of the exp r.v.} X_1^2)$$

$$= \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right) = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^2$$

The above result can be generalized to any integer-valued *n*, for example, by mathematical induction:

$$\mathrm{P}\{S_n^1 < S_1^2\} = \mathrm{P}\{S_n^1 < S_1^2 \mid S_{n-1}^1 < S_1^2\} \cdot \mathrm{P}\{S_{n-1}^1 < S_1^2\}$$

$$= \mathrm{P}\{S_{n-1}^1 + X_n^1 < X_1^2 \mid S_{n-1}^1 < X_1^2\} \cdot \mathrm{P}\{S_{n-1}^1 < S_1^2\}$$

$$= \mathrm{P}\{X_n^1 < X_1^2\} \cdot \mathrm{P}\{S_{n-1}^1 < S_1^2\}$$

So we obtain

$$\mathrm{P}\{S_n^1 < S_1^2\} = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^n$$