CS 5291: Stochastic Processes for Networking

HW6

1. Let $X_1, X_2, \ldots$ be independent with

$$P\{X_i = 1\} = p = 1 - P\{X_i = 0\}, i \geq 1.$$

Define

$$N_1 = \min\{n: X_1 + X_2 + \cdots + X_n = 5\}$$

$$N_2 = \begin{cases} 3, & X_1 = 0 \\ 5, & X_1 = 1 \end{cases}$$

$$N_3 = \begin{cases} 3, & X_4 = 0 \\ 2, & X_4 = 1 \end{cases}$$

(a) Which of the N_i are stopping times for the sequence $X_1, \ldots$? An important result, known as *Wald's equation* states that if $X_1, X_2, \ldots$ are independent and identically distributed and have a finite mean $E[X]$, and if N is a stopping time for this sequence having a finite mean, then

$$E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X]$$

N_1, N_2 are stopping times.

N_3 is not. $\because$ N_3 depends on X_4

(b) What does Wald's equation tell us about the stopping times in part (a)? That is, please derive $E[N], E[X]$, or $E[\sum_{i=1}^{N} X_i]$ by using Wald's equation.

$E[X] = 0 \cdot P\{X_i = 0\} + 1 \cdot P\{X_i = 1\} = p$

For N_1, $E[\sum_{i=1}^{N_1} X_i] = 5, E[X] = p, \rightarrow E[N_1] = \frac{5}{p}$

For N_2, $E[N_2] = 3(1-p) + 5p = 3 + 2p$, $E[X] = p$,

$\rightarrow E[\sum_{i=1}^{N_2} X_i] = p(3 + 2p)$

2. Consider a miner trapped in a room that contains three doors. Door 1 leads him to freedom after two days of travel; door 2 returns him to his room after a four-day journey; and door 3 returns him to his room after a six-day journey. Suppose at all times he is equally likely to choose any of the three doors, and let

T denote the time it takes the miner to become free.

(a) Define a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$ and a stopping time N such that

$$T = \sum_{i=1}^{N} X_i$$

Note: You may have to imagine that the miner continues to randomly choose doors even after he reaches safety.

X_i = amount of time he has to travel after his i^{th} choice (We will assume that he keeps on making choices even after becoming free). N is the number of choices he makes until becoming free. $N = \min\{i | X_i = 2\}$.

(b) Use Wald's equation to find $\mathrm{E}[T]$.

$$\mathrm{E}[T] = \mathrm{E}\left[\sum_{i=1}^{N} X_i\right] = \mathrm{E}[N]\mathrm{E}[X] = 3 \times \frac{1}{3}(2+4+6) = 12$$

(c) Compute $\mathrm{E}[\sum_{i=1}^{N} X_i \,|N = n]$ and note that it is not equal to $\mathrm{E}[\sum_{i=1}^{N} X_i]$.

$$\mathrm{E}\left[\sum_{i=1}^{N} X_i | N = n\right] = \frac{n-1}{2}(4+6) + 2 = 5n - 3$$

(d) Use part (c) for a second derivation of $\mathrm{E}[T]$.

$$\mathrm{E}\left[\sum_{i=1}^{N} X_i\right] = \mathrm{E}\left[\mathrm{E}\left[\sum_{i=1}^{N} X_i | N = n\right]\right] = \mathrm{E}[5N - 3] = 15 - 3 = 12$$