CS 5291: Stochastic Processes for Networking
Final Exam

(Total points: 110)

1. (10%) Consider a miner trapped in a room that contains three doors. Door 1 leads him to freedom after two days of travel; door 2 returns him to his room after a four-day journey; and door 3 returns him to his room after a six-day journey. Suppose at all times he is equally likely to choose any of the three doors, and let T denote the time it takes the miner to become free.
   (a) Define a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$ and a stopping time N such that
   $$T = \sum_{i=1}^{N} X_i$$
   Note: You may have to imagine that the miner continues to randomly choose doors even after he reaches safety.
   (b) Use Wald's equation to find $E[T]$.
2. (10%) For a renewal process $\{N(t), t \geq 0\}$ whose interarrival times have a common cdf $F_X(t)$ and pdf $f_X(t)$, $m(t) = F_X(t) + \int_0^t m(t-x) f_X(x) dx$ is a widely used formula that helps to obtain the mean-value function $m(t) = E[N(t)]$.
   (a) Prove $M(s) = \frac{X(s)}{s(1-X(s))}$, where $X(s)$ is the Laplace transform of the pdf $f_X(t)$ and $M(s)$ is the Laplace transform of $m(t)$.
   (b) If $m(t) = \frac{t}{2}, t \geq 0$, what is $P\{N(5) = 0\}$?
3. (10%) A worker sequentially works on jobs. Each time a job is completed, a new one is begun. Each job, independently, takes a random amount of time having distribution F to complete. However, independently of this, shocks occur according to a Poisson process with rate λ. Whenever a shock occurs, the worker discontinuous working on the present job and starts a new one. In the long run, at what rate are jobs completed?
4. (10%) A machine in use is replaced by a new machine either when it fails or when it reaches the age of T years. The lifetimes of machines are independent with a common cdf $F_X(t)$ and pdf $f_X(t)$. Find the long-run rate at which machines in use fail.
5. (10%) A truck driver regularly drives round trips from A to B and then back to A. Each time he drives from A to B, he drives at a fixed speed that (in miles per hour) is uniformly distributed between 40 and 60; each time he drives from B to A, he drives at fixed speed that is equally likely to be either 40 or 60. In the long run, what proportion of his driving time is spent going to B?
   [Hint:] Calculate the average time spent of both directions. Then, the proportion can be derived by using alternating renewal process.
6. (10%) A particle moves among $n+1$ verticesthat are situated on a circle in the following manner. At each step it moves one step either in the clockwise direction with probability p or the counterclockwise direction with probability $1-p$. Starting at a specified state, call it state 0, let T be the time of the first return to state 0. Find the expected value of T.
7. (10%) Specify the classes of the discrete-time Markov chain corresponding to the transition matrix below, and determine whether these classes are transient or

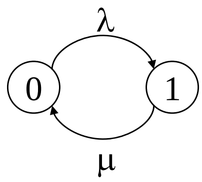
recurrent.

$$P = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

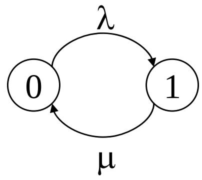
8. (10%) Compute the matrix below.

$$\begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 1/2 & 0 \end{pmatrix}^{\infty}$$

9. (10%) Consider the two-state continuous-time Markov chain (CTMC) shown below. Suppose $\lambda = 3$ and $\mu = 2$. What is the (long-run) fraction of time the process stays in state 0?



10. (10%) Consider the two-state continuous-time Markov chain (CTMC) shown below.



Denote the transition probability function of the CTMC that presently in state i will be in state j a time t later by $P_{ij}(t)$. Derive $P_{11}(t)$.

[Hint: Kolmogorov's forward/backward equation.]

11. (10%) Consider a (infinite-state) birth and death process in which the birth rates of all states are λ and the death rates of all states except state 0 are μ.
    (a) Regard the birth and death process as a continuous-time Markov chain. Write down its balance equations.
    (b) Assume the limiting probabilities, $P_0, P_1, \ldots$, exist. Solve them in terms of P_0.