CS 5291: Stochastic Processes for Networking

HW5

1. Some components of a two-component system fail after receiving a shock. Shocks of three types arrive independently and in accordance with Poisson processes. Shocks of the first type arrive at a Poisson rate λ_1 and cause the first component to fail. Those of the second type arrive at a Poisson rate λ_2 and cause the second component to fail. The third type of shock arrives at a Poisson rate λ_3 and causes both components to fail. Let X_1 and X_2 denote the survival times for the two components. Show that the joint distribution of X_1 and X_1 is given by

$$P\{X_1 > s, X_2 > t\} = \exp\{-\lambda_1 s - \lambda_2 t - \lambda_3 \max(s,t)\}$$

Let T_i denote the arrival time of the first type i shock, i = 1, 2, 3.

$$\begin{aligned} P\{X_1 > s, X_2 > t\} &= P\{T_1 > s, T_3 > s, T_2 > t, T_3 > t\} \\ &= P\{T_1 > s, T_2 > t, T_3 > \max(s,t)\} \\ &= e^{-\lambda_1 s} e^{-\lambda_2 t} e^{-\lambda_3 \max(s,t)} \end{aligned}$$

2. Let $N(t)$ and $M(t)$ are two independent non-homogeneous Poisson process, with respective intensity function $\lambda(t)$ and $\mu(t)$. Let $N^*(t) = N(t) + M(t)$. Please use $\boldsymbol{o}(\cdot)$ to explain that why an event of $\{N^*(t)\}$ occurs at time t then the event at time t is from $\{N(t)\}$ process with the probability $\frac{\lambda(t)}{\lambda(t)+\mu(t)}$.

(Hint: P{the event is from N(t) given an event occurs in (t, t+h)})

$P\{\text{the event is from } N(t) \text{ given an event occurs in } (t, t+h)\}$

$=\frac{P\{\text{from } N(t),\ \text{not from } M(t)\}}{P\{\text{either from } N(t) \text{ or } M(t)\}}$

$=\frac{\lambda(t)h[1-\mu(t)h]}{\lambda(t)h+\mu(t)h-(o(h))}$ $\quad$ ($o(h)$ is negligible.)

$=\frac{\lambda(t)h}{\lambda(t)h+\mu(t)h}$

$=\frac{\lambda(t)}{\lambda(t)+\mu(t)}.$

3. Beverly has a radio that works on a single battery. As soon as the battery in use fails, Beverly immediately replaces it with a new battery. If the lifetime of a battery (in hours) is distributed uniformly over the interval (30, 60), then at what rate does Beverly have to change batteries?

If we let $N(t)$ denote the number of batteries that have failed by time t, then the rate at which Beverly replaces batteries is given by

$$\lim_{t\to\infty}\frac{N(t)}{t}=\frac{1}{\mu}=\frac{1}{45}$$

4. A worker sequentially works on jobs. Each time a job is completed, a new one is begun. Each job, independently, takes a random amount of time having distribution F to complete. However, independently of this, shocks occur according to a Poisson process with rate λ. Whenever a shock occurs, the worker discontinuous working on the present job and starts a new one. In the long run, at what rate are jobs completed?

A job completion constitutes a renewal. Let T denote the time between renewals. Let W be the time it takes to finish the next job. Let S be the time of the next shock.

$$\begin{aligned}
E[T|W=w] &= \int_0^\infty E[T|W=w,S=s]P\{S=s\}ds \\
&= \int_0^w (s+E[T])\lambda e^{-\lambda s}ds+\int_w^\infty w\lambda e^{-\lambda}\ ds \\
&= \int_0^w w\lambda e^{-\lambda s}ds+E[T]\left(1-e^{-\lambda w}\right)+we^{-\lambda} \\
&= -we^{-\lambda}\ +\frac{1-e^{-\lambda w}}{\lambda}+E[T](1-e^{-\lambda}\ )+we^{-\lambda} \\
&= (1-e^{-\lambda}\ )\left(E[T]+\frac{1}{\lambda}\right)
\end{aligned}$$

$$E[T]=E\big[E[T|W=w]\big]=\left(1-E[e^{-\lambda w}]\right)\left(E[T]+\frac{1}{\lambda}\right)$$

$$E[T]=\frac{\frac{1}{E[e^{-\lambda w}]}-1}{\lambda}=\frac{\frac{1}{\int_0^\infty e^{-\lambda}\ dF(w)}-1}{\lambda}$$

$$rate=\frac{1}{E[T]}=\frac{\lambda}{\frac{1}{\int_0^\infty e^{-\lambda w}dF(w)}-1}$$