CS 5291: Stochastic Processes for Networking

HW2

1. A fair coin is tossed until two tails occur successively. Find the expected number of the tosses required.
   Hint: Let
   $$X = \begin{cases} 0 \text{ if the first toss results in tails} \\ 1 \text{ if the first toss results in heads,} \end{cases}$$
   and condition on X.

2. A deck of 52 playing cards, containing all 4 aces, is randomly divided into 4 piles of 13 cards each. Define events E_1, E_2, E_3, and E_4 as follows:
   $$E_1 = \{\text{the first pile has exactly 1 ace}\},$$
   $$E_2 = \{\text{the second pile has exactly 1 ace}\},$$
   $$E_3 = \{\text{the third pile has exactly 1 ace}\},$$
   $$E_4 = \{\text{the fourth pile has exactly 1 ace}\}.$$
   Find $\mathrm{P}\{E_1, E_2, E_3, E_4\}$, the probability that each pile has an ace.
   Hint:
   $$\mathrm{P}\{E_1 E_2 \dots E_n\} = \mathrm{P}\{E_1\}\ \mathrm{P}\{E_2|E_1\}\ \mathrm{P}\{E_3|E_1 E_2\} \dots\ \mathrm{P}\{E_n|E_1 \dots E_{n-1}\}$$

3. Two unbiased six-sided dice are thrown.
   (a) What is the probability that at least one lands on six?
   (b) If the two dice land on different values, what is the probability that at least one lands on six?

4. Let $Y_1, Y_2, \dots, Y_n$ be independent random variables, each having a uniform distribution over (0,1). Let $X = \text{maximum}(Y_1, Y_2, \dots, Y_n)$.
   (a) Show that the cumulative distribution function of n, $F_X(\cdot)$, is given by
   $F_X(x) = x^n,\ 0 \le x \le 1$
   (b) What is the probability density function of *X*?

5. Derive the moment generating functions for the following random variables. Then, derive the expected value, second moment, and variance of each random variable.
   (a) Uniform distribution
   (b) Exponential distribution

6. Derive the tightest Chernoff's Bound for the Poisson random variable X with PMF $P_X(n) = \frac{e^{-\lambda}\lambda^n}{n!}, n = 0,1,2,3, \dots$

7. With $K(t) = ln(E[e^{tX}])$, show that $K'(0) = E[X], K''(0) = Var(X)$.