CS 5291: Stochastic Processes for Networking

HW7

1. If the mean-value function of the renewal process $\{N(t), t \geq 0\}$ is given by $m(t) = \frac{t}{2}, t \geq 0$, What is $P\{N(5) = 0\}$?

   By the one-to-one correspondence of $m(t)$ and F, it follows that $\{N(t), t \geq 0\}$ is a Poisson process with rate 1/2. Hence $\mathrm{P}\{\mathrm{N}(5) = 0\} = \mathrm{e}^{-\frac{5}{2}}$

2. Let $U_1, U_2, \ldots$ be independent uniform (0,1) random variables, and define N by
$$N = \min\{n: U_1 + U_2 + \cdots + U_n > 1\}$$
   What is $E[N]$?

   Let $N(x) = min\{n: U_1 + U_2 + \cdots + U_n > x\}$ and $g(x) = E[N(x)]$.
$$E[N(x)|U_1 = t] = \begin{cases} 1, & if\ t > x \\ 1 + g(x-t), & if\ t \leq x \end{cases}$$
   Consider $\mathrm{x} \leq 1$,
$$g(x) = \int_0^1 E[N(x)|U_1 = t]P\{U_1 = t\}dt$$
$$= \int_0^x \big(1 + g(x-t)\big)dt + \int_x^1 1dt$$
$$= 1 + \int_0^x g(x-t)dt$$
$$= 1 + \int_0^x g(t)dt$$
   $\to g'(x) = g(x)$
   $\to g(x) = Ae^x$
   $\because g(0) = 1 \to A = 1$
   $\therefore g(x) = e^x$
   $E[N] = g(1) = e$

3. There are three machines, all of which are needed for a system to work. Machine i functions for an exponential time with rate λ_i before it fails, $i = 1,2,3$. When a machine fails, the system is shut down and repair begins on the failed machine. The time to fix machine 1 is exponential with rate 5; the time to fix machine 2 is

uniform on (0,4); and the time to fix machine 3 is a gamma random variable with parameters $n = 3$ and $\lambda = 2$. Once a failed machine is repaired, it is as good as new and all machines are restart. What proportion of time is the system working?

Hint: *It is an alternating renewal process. Therefore, we can derive the mean off time by conditioning on which machine fails. Then the proportion can be derived by the observation of alternating renewal process.*

"on": system is working

"off": system is shut down for repairing machines

$$E[on] = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$E[off] = \sum_{i=1}^{3} E[repair\ time|machine\ i\ fails]P\{machine\ i\ fails\}$$

$$= \frac{1}{5} \times \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{0+4}{2} \times \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{3}{2} \times \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$= \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}\left(\frac{\lambda_1}{5} + 2\lambda_2 + \frac{3}{2}\lambda_3\right)$$

$$\frac{\mathrm{E[on]}}{E[on] + E[off]} = \frac{1}{1 + \frac{\lambda_1}{5} + 2\lambda_2 + \frac{3}{2}\lambda_3}$$

4. Suppose that whether or not it rains today depends on previous weather conditions through the last three days. If it has rained for the past three days, then it will rain today with probability 0.8; if it did not rain for any of the past three days, then it will rain today with probability 0.2; and in any other case the weather today will, with probability 0.6, be the same as the weather yesterday. Determine P for this Markov chain.

PS. *Let the state space be {RRR,RRD,RDR,RDD,DRR,DRD,DDR,DDD}, where R denotes rain day and D denotes dry day. For example, RDD means that it rained in three days ago and it did not rain in past two days. State changes from RDD to DDD means that it does not rain today.*

The state space is $\{\mathrm{RRR, RRD, RDR, RDD, DRR, DRD, DDR, DDD}\}$.

$$P = \begin{bmatrix} 0.8 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{bmatrix}$$

5. A Markov chain $\{X_n, \mathrm{n} > 0\}$ with states 0, 1, 2, has the transition probability matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

If $\mathrm{P}\{X_0 = 0\} = \mathrm{P}\{X_0 = 1\} = \frac{1}{4}$, find $\mathrm{E}[X_3]$.

Cubing the transition probability matrix, we obtain P^3:

$$\begin{bmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \frac{2}{9} & \frac{13}{36} \end{bmatrix}$$

Thus,

$$\mathrm{E}[X_3] = P(X_3 = 1) + 2\mathrm{P}(X_3 = 2)$$

$$= \frac{1}{4}P_{01}^3 + \frac{1}{4}P_{11}^3 + \frac{1}{2}P_{21}^3 + 2[\frac{1}{4}P_{02}^3 + \frac{1}{4}P_{12}^3 + \frac{1}{2}P_{22}^3]$$