CS 5291: Stochastic Processes for Networking

HW2

1. A fair coin is tossed until two tails occur successively. Find the expected number of the tosses required.
   Hint: Let
   $$X = \begin{cases} 0 \text{ if the first toss results in tails} \\ 1 \text{ if the first toss results in heads,} \end{cases}$$
   and condition on X.

   Let N be the number of tosses required;

   Then $E[N] = E\big[E[N|X]\big] = E[X|X=1]\,\mathrm{P}\{X=1\} + E[N|X=0]\,\mathrm{P}\{X=0\}$
   $$= (1+E[N]) \times \frac{1}{2} + \left(2 \times \frac{1}{2} + (2+E[N]) \times \frac{1}{2}\right) \times \frac{1}{2} = \frac{3}{4}E[N] + \frac{3}{2}$$
   $\rightarrow E[N] = 6$

2. A deck of 52 playing cards, containing all 4 aces, is randomly divided into 4 piles of 13 cards each. Define events E_1, E_2, E_3, and E_4 as follows:
   $$E_1 = \{\text{the first pile has exactly 1 ace}\},$$
   $$E_2 = \{\text{the second pile has exactly 1 ace}\},$$
   $$E_3 = \{\text{the third pile has exactly 1 ace}\},$$
   $$E_4 = \{\text{the fourth pile has exactly 1 ace}\}.$$
   Find $\mathrm{P}\{E_1, E_2, E_3, E_4\}$, the probability that each pile has an ace.
   Hint:
   $$\mathrm{P}\{E_1E_2 \dots E_n\} = \mathrm{P}\{E_1\}\,\mathrm{P}\{E_2|E_1\}\,\mathrm{P}\{E_3|E_1E_2\} \dots \mathrm{P}\{E_n|E_1 \dots E_{n-1}\}$$

   $$\mathrm{P}\{\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{E}_4\} = \frac{C_1^4 C_{12}^{48}}{C_{13}^{52}} \times \frac{C_1^3 C_{12}^{36}}{C_{13}^{39}} \times \frac{C_1^2 C_{12}^{24}}{C_{13}^{26}} \times 1$$

3. Two unbiased six-sided dice are thrown.
   (a) What is the probability that at least one lands on six?
   Let E be the event at least one lands on six.
   $$\mathrm{P}\{E\} = \frac{\text{number of ways to get } E}{\text{number of sample points}} = \frac{11}{36}$$

   (b) If the two dice land on the different values, what is the probability that at least one lands on six?

Let D be the event two dice land on the different values.

$P\{D\} = 1 - \text{Prob}\{\text{two dice land on the same value}\}$

$$= 1 - \frac{6}{36} = \frac{5}{6}$$

$$P\{E|D\} = \frac{P\{ED\}}{P\{D\}} = \frac{\frac{10}{36}}{\frac{5}{6}} = \frac{1}{3}$$

4. Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables, each having a uniform distribution over (0,1). Let $X = \text{maximum}(Y_1, Y_2, \ldots, Y_n)$.

(a) Show that the cumulative distribution function of n, $F_X(\cdot)$, is given by

$F_X(x) = x^n,\ 0 \le x \le 1$

If x is a real number between 0 and 1, then $X < x$ if and only if $Y_k < x$ for all k, and $P\{Y_k < x\} = x$ since x is uniform on (0,1). (The PDF is equal to 1, so the CDF is $\int_0^x 1dx = x$)

Because Y_k are independent, $P\{X < x\} = P\{Y_1 < x\} * P\{Y_2 < x\} * \ldots * P\{Y_n < x\} = x^n$.

(b) What is the probability density function of X?

The PDF of X is obtained by differentiating,

$f_X(x) = \frac{F_X(x)}{dx} = nx^{n-1},\ \ 0 \le x \le 1$

5. Derive the moment generating functions for the following random variables. Then, derive the expected value, second moment, and variance of each random variable.

(a) Uniform distribution

Uniform distribution: $f_X(x) = \frac{1}{b-a}$

$\phi(t) = E[e^{tX}]$

$$= \int_a^b e^{tx} \frac{1}{b-a}\, dx = \frac{1}{b-a} \frac{1}{t} \int_a^b t\, e^{tx} dx$$

$$= \frac{1}{t(b-a)} [e^{tx}|_a^b] = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

$$\phi'(t) = \frac{d\phi(t)}{dt} = \frac{1}{b-a}\left(\frac{be^{tb} - ae^{ta}}{t} - \frac{e^{tb} - e^{ta}}{t^2}\right)$$

$$\phi''(t) = \frac{1}{b-a}(\frac{2(e^{tb}-e^{ta})}{t^3} - \frac{2(be^{tb}-ae^{ta})}{t^2} + \frac{b^2e^{tb}-a^2e^{ta}}{t})$$

$$E[X] = \phi'^{(0)} = \lim_{s\to 0}\frac{1}{b-a}\left(\frac{be^{tb}-ae^{ta}}{t} - \frac{e^{tb}-e^{ta}}{t^2}\right)$$

$$= \frac{a+b}{2} \text{ (by L'Hospital's rule)}$$

$$E[X^2] = \phi'(0) = \frac{a^2+ab+b^2}{3}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{(b-a)^2}{12}$$

(b) Exponential distribution

Exponential distribution: $f_X(x) = \lambda e^{-\lambda x}$

$$\phi(t) = E[e^{tX}] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx = \lambda\int_0^\infty e^{-(\lambda-t)x}dx$$

$$= \lambda\frac{1}{\lambda-t}\int_0^\infty(\lambda-t)e^{-(\lambda-t)x}dx = \frac{\lambda}{\lambda-t}$$

$$\phi'(t) = \frac{\lambda}{(\lambda-t)^2}$$

$$\phi''(t) = \frac{2\lambda}{(\lambda-t)^3}$$

$$E[X] = \phi'(0) = \frac{1}{\lambda}$$

$$E[X^2] = \phi''(0) = \frac{2}{\lambda^2}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{1}{\lambda^2}$$

6. Derive the tightest Chernoff's Bound for the Poisson random variable X with PMF $P_X(n) = \frac{e^{-\lambda}\lambda^n}{n!}, n = 0,1,2,3,\ldots$

$$f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!} = e^{-\lambda}e^{e^t\lambda} = e^{-\lambda(1-e^t)}$$

$\mathrm{P}\{X \geq a\} \leq \min_t\left(e^{-ta}M_X(t)\right) = \min_t\left(e^{-ta-\lambda(1-e^t)}\right)$

Let $g(t) = e^{-ta-\ (1-e^t)}$

To find the minimum of $g(t)$, we need to solve $\frac{dg(t)}{dt} = 0$.

$$\frac{dg(t)}{dt} = 0 \Rightarrow (\lambda e^t - a)e^{-ta-\lambda(1-e^t)} = 0 \Rightarrow t = \ln\left(\frac{a}{\lambda}\right)$$

$$\mathrm{P}\{X \geq a\} \leq g\left(\ln\left(\frac{a}{\lambda}\right)\right) = e^{-a\ln\left(\frac{a}{\lambda}\right)-\lambda+a} = e^{-\lambda}e^{\ln\left(\left(\frac{\lambda}{a}\right)^a\right)}e^a = e^{-\lambda}\left(\frac{\lambda e}{a}\right)^a$$

7. With $K(t) = \ln(E[e^{tX}])$, show that $K'(0) = E[X], K''(0) = Var(X)$.

$$K'(t) = \frac{E[Xe^{tX}]}{E[e^{tX}]}$$

$$K''(t) = \frac{E[e^{tX}]E[X^2e^{tX}] - E^2[Xe^{tX}]}{E^2[e^{tX}]}$$

Hence,

$K'(0) = E[X]$,

$K''(0) = E[X^2] - E^2[X] = Var(X)$