Quiz 1.

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(a). The continuous random variable X is exponentially distributed with parameter λ. Its cumulative distribution function is

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

$+$	$8x$	$e^{-\lambda x}\,dx$
$-$	8	$\frac{1}{-\lambda}e^{-\lambda x}$
$+$	0	$\frac{1}{\lambda^2}e^{-\lambda x}$

$Y \triangleq g(X) = 8X$ Another random variable Y is defined as $Y = 8X$.

Please find $\mathrm{E}[Y]$.

$$E(Y) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx = \int_0^{\infty} 8x \cdot \lambda e^{-\lambda x}\,dx = \lambda \left[\frac{1}{\lambda} 8x e^{-\lambda x}\Big|_0^{\infty} - \frac{1}{\lambda^2} \cdot 8 e^{-\lambda x}\Big|_0^{\infty} \right]$$

$$= \lambda \cdot \left[0 - \frac{8}{\lambda^2} \cdot (0 - 1) \right] = \frac{8}{\lambda} \#$$

(b). Suppose X has the following probability mass function:

$$\mathrm{p}(0) = 0.1, \quad \mathrm{p}(1) = 0.6, \quad \mathrm{p}(2) = 0.3$$

Please Calculate $\mathrm{E}[X^2]$.

$E(X) = 0 \cdot 0.1 + 1 \cdot 0.6 + 2 \cdot 0.3 = 1.2$

$E(X^2) = Var(X) + E(X)^2$

$$E(X^2) = \sum_x g(X) \cdot P(X = x) = 0 + 1^2 \cdot 0.6 + 2^2 \cdot 0.3$$

$$= 0.6 + 1.2 = 1.8 \#$$

Quiz 2

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For an exponential random variable X, $X \sim \mathrm{Exp}(\lambda)$, we know its moment generating function is $\phi(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$. What is the third moment of X? (That is, what is $\mathrm{E}[X^3]$?)

$$MGF\ \phi(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\,dx = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x}\,dx$$

$$= \frac{\lambda}{-(\lambda - t)} \cdot \underbrace{e^{-(\lambda - t)x}\Big|_0^{\infty}}_{(0-1)} = \frac{\lambda}{\lambda - t} = \lambda(\lambda - t)^{-1}$$

1^{st} $\phi'(t) = \lambda(-1)(\lambda - t)^{-2}(-1) = \lambda(\lambda - t)^{-2}$

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2^{nd} $\phi''(t) = \lambda(-2)(\lambda - t)^{-3}(-1) = 2\lambda(\lambda - t)^{-3}$

3^{rd} $\phi'''(t) = 2\lambda(-3)(\lambda - t)^{-4}(-1) = \frac{6\lambda}{(\lambda - t)^4}$

$$E(X^3) = \phi'''(0) = \frac{6}{\lambda^3}$$

\#

Quiz 3

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$\lambda e^{-\lambda x}$

3. X_1 and X_2 are independent exponential random variables, each with rate λ. The random variable Y is defined as $Y = \min(X_1, X_2)$. Please find $E[Y]$.

$E(Y) = \int_{-\infty}^{\infty} P(Y > x)\,dx = \int_0^{\infty} P(\min(X_1, X_2) > x) = \int_0^{\infty} P(X_1 > x)P(X_2 > x)\,dx$

$= \int_0^{\infty} e^{-\lambda x} \cdot e^{-\lambda x}\,dx = \int_0^{\infty} e^{-2\lambda x}\,dx = \frac{-1}{2\lambda} e^{-2\lambda x}\Big|_0^{\infty} = \frac{1}{2\lambda}$ #

4. Let $X \sim \mathrm{Exponential}(\lambda)$. Use Chebyshev's inequality to find an upper bound of $P(|X - E[X]| \geq b)$.

$P(|X - E(X)| \geq b)$, $E(X) = \frac{1}{\lambda}$ $var(X) = \frac{1}{\lambda^2} = E(X^2) - E(X)^2$

$= P(|X - E(X)|^2 \geq b^2) = P\left(\frac{|X - E(X)|^2}{b^2} \geq 0\right)$

upper bound $= \frac{var(X)}{b^2} = \frac{1}{\lambda^2 b^2}$ #

Quiz 4

$\boxed{P\{N(t) = n\} = e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}}$ Student ID: 110064533 Name: 陳劭珩 100

1. Let $\{N(t), t \geq 0\}$ be a Poisson process with $P\{N(1) = 0\} = e^{-3}$. Let S_n denote the time of the n-th event. Find $E[N(4) - N(2) | N(1) = 3]$.

$\boxed{E(N(t)) = \lambda t}$

∵ indep. incre. ∵ stationary incre.

$\Rightarrow E(N(4) - N(2) | N(1) = 3) = E(N(2+2) - N(2)) = E(N(2)) = \lambda t = 3 \cdot 2 = 6$ #

$e^{-\lambda t} \cdot \frac{1}{0!} = e^{-3}$ $\lambda t = 3$ $\lambda = 3$

2. On Friday night, the number of customers arriving at a lounge bar can be modeled by a Poisson process with intensity that twelve customers per hour.

12 per hr

   (a) Find the probability that there are 2 customers between 21:00 and 21:40.

   (b) Find the probability that there are 4 customers between 21:00 and 21:40 and 6 customers between 21:40 and 22:00.

$\lambda t_1 = 12 \cdot \frac{4}{6} = 8$

$= 970.9073$

$\frac{2^{12} \cdot 2^{12}}{2^3 \cdot 3 \cdot 3 \cdot 2^3 \cdot 2 \cdot 15}$ 3 135

(a) $P\{2 \text{ within } \frac{40}{60} hr\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!} = e^{-8} \cdot \frac{8^2}{2!} = 32 e^{-8}$ #

(b) $\Rightarrow P\{4 \text{ within } \frac{4}{6} hr\} \cdot P\{6 \text{ within } \frac{2}{6} hr\} = \left(e^{-8} \frac{8^4}{4!}\right) \cdot \left(e^{-4} \cdot \frac{4^6}{6!}\right)$ #

$12 \cdot \frac{4}{6} = 8$ $12 \cdot \frac{2}{6} = 4$

$= \frac{2^{17}}{135} e^{-8} e^{-4}$