# CS 5291: Stochastic Processes for Networking

## HW4

1. Suppose that $X_1$ and $X_2$ are two independent nonnegative continuous random variables with probability density functions $f_1(x) = \lambda_1 e^{-\lambda_1 x}$ and $f_2(x) = \lambda_2 e^{-\lambda_2 x}$

   (a) Derive the expected value $E[\min(X_1, X_2)]$.

   $E[min(X_1, X_2)]$

   $$= \int_0^\infty P(min(X_1, X_2) > x)dx$$

   $$= \int_0^\infty P(X_1 > X)P(X_2 > x)dx$$

   $$= \int_0^\infty e^{-(\lambda_1+\lambda_2)x}dx$$

   $$= \frac{1}{\lambda_1 + \lambda_2}\int_0^\infty (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)x}dx = \frac{1}{\lambda_1 + \lambda_2}$$

   (b) Derive the expected value $E[\max(X_1, X_2)]$.

   $E[max(X_1, X_2)]$

   $$= \int_0^\infty P(max(X_1, X_2) > x)dx$$

   $$= \int_0^\infty \big(1 - P(max(X_1, X_2) < x)\big)dx$$

   $$= \int_0^\infty (1 - P(X_1 < x)P(X_2 < x))dx$$

   $$= \int_0^\infty (1 - (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 x}))dx$$

   $$= \int_0^\infty \left(e^{-\lambda_1 x} + e^{-\lambda_2 x} - e^{-(\lambda_1+\lambda_2)x}\right)dx = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

2. A certain scientific theory supposes that mistakes in cell division occur according to a Poisson process with rate 2.5 per year, and that an individual dies when 196 such mistakes have occurred. Assuming this theory, find

   The lifetime of an individual follows the Erlang distribution with parameters

   $\lambda = 2.5 \; and \; k = 196$

(a) the mean lifetime of an individual

The mean lifetime of an individual $= \frac{196}{2.5} = 78.4$ years

(b) the variance of the lifetime of an individual

The variance of the lifetime of an individual$=\frac{196}{2.5^2} = 31.36$ years

(c) an approximate of the probability that an individual dies before age 67.2

**Hint**: Use the central limit theorem. Suppose the value of the complementary cumulative distribution function of the normal distribution, $Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} dz$, can be known by table lookup.

We use the central limit theorem to justify approximating the life distribution by a normal distribution with mean 78.4 and standard deviation $\sqrt{31.36} = 5.6$

$$P(L < 67.2) \fallingdotseq P\left(Z < \frac{67.2 - 78.4}{5.60}\right) = P(Z < -2) = Q(2) = 0.0227$$

3. Consider a Poisson process $N(t)$ is decomposed into two counting processes $N_1(t)$ and $N_2(t)$ with probability $p$ and $1-p$, respectively.

(a) Find the joint probability mass function $\Pr\{N_1(t) = n, N_2(t) = m\}$

$$\begin{aligned} P(N_1(t) = n, N_2(t) = m) &= P(N_1(t) = n, N(t) = n + m) \\ &= P(N(t) = n + m) \times P(N_1(t) = n | N(t) = n + m) \\ &= \frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!} \times C_n^{n+m} p^n (1-p)^m \\ &= \frac{e^{-\lambda t}(\lambda t)^{n+m}}{n!\,m!} p^n (1-p)^m \end{aligned}$$

(b) Prove that $N_1(t)$ and $N_2(t)$ are independent of each other. That is, $\Pr\{N_1(t) = n, N_2(t) = m\} = \Pr\{N_1(t) = n\} \cdot \Pr\{N_2(t) = m\}$.

$$P(N_1(t) = n) = \sum_{m=0}^{\infty} P\{N_1(t) = n, N_2(t) = m\}$$

$$= \frac{e^{-\lambda t}(\lambda pt)^n}{n!}\left(1 + \frac{(1-p)\lambda t}{1!} + \frac{\big((1-p)\lambda t\big)^2}{2!} + \frac{\big((1-p)\lambda t\big)^3}{3!} + \cdots\right)$$

$$= \frac{e^{-\lambda}\ (\lambda pt)^n}{n!} e^{(1-p)\lambda t} = \frac{e^{-\lambda pt}(\lambda pt)^n}{n!}$$

Similarly, $P(N_2(t) = m) = \frac{e^{-\lambda(1-p)t}(\lambda(1-p)t)^m}{m!}$

$$P(N_1(t) = n, N_2(t) = m) = \frac{e^{-\lambda t}(\lambda t)^{n+m}}{n!\,m!} p^n (1-p)^m$$

$$= \frac{e^{-\lambda pt}(\lambda pt)^n}{n!} \times \frac{e^{-\lambda(1-p)t}(\lambda(1-p)t)^m}{m!}$$

$$= P(N_1(t) = n) \times P(N_2(t) = m)$$

Hence, $N_1(t)$ and $N_2(t)$ are independent of each other.

4. On Friday night, the number of customers arriving at a lounge bar can be modeled by a Poisson process with intensity that twelve customers per hour.

(a) Find the probability that there are 2 customers between 21:00 and 21:40.

We know that $\lambda = 12$ and the interval between 21:00 and 21:40 has the length $\tau = \frac{2}{3}$ hours. If X is the number of arrivals in that interval, we can write $X \sim \text{Poisson}(8)$. Therefore,

$$P(X = 2) = \frac{e^{-8}(8)^2}{2!}$$

(b) Find the probability that there are 4 customers between 21:00 and 21:40 and 6 customers between 21:40 and 22:00.

Here, we have two non-overlapping intervals

$I_1 = (21{:}00, 21{:}40]$ and $I_2 = (21{:}40, 22{:}00]$.

Thus, we can write

$$P(\text{ 4 arrivals in } I_1 \textit{ and } \text{6 arrivals in } I_2\text{ })$$

$$= P(\text{ 4 arrivals in } I_1\text{ }) \cdot P(\text{ 6 arrivals in } I_2\text{ })$$

$$= \frac{e^{-8}(8)^4}{4!} \cdot \frac{e^{-4}(4)^6}{6!}$$