CS 5291: Stochastic Processes for Networking

HW3

1. Assume X and Y are independent Poisson random variables with means $E[X] = \lambda_1$ and $E[Y] = \lambda_2$. (The probability mass function of Poisson distribution is $\frac{e^{-\lambda}\lambda^n}{n!}, n = 0, 1, \ldots$, where λ is the mean.)

   (a) Compute the conditional probability mass function $\mathrm{P}\{X = k \mid X + Y = n\}$. Is it the same as $\mathrm{P}\{X = k\}$?
   Hint: You can start by the definition of conditional probability.

$$\mathrm{P}\{X+Y=n\} = \sum_{i=0}^{n} \mathrm{P}\{X=i, Y=n-i\} = \sum_{i=0}^{n} \frac{(e^{-\lambda_1}\lambda_1^i)}{i!} \frac{e^{-\lambda_2}\lambda_2^{n-i}}{(n-i)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{n} \frac{\lambda_1^i \lambda_2^{n-i}}{i!\,(n-i)!} \frac{n!}{n!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n \sim \text{Possion } (\lambda_1+\lambda_2)$$

$$\mathrm{P}\{X=k \mid X+Y=n\} = \frac{\mathrm{P}\{X=k, Y=n-k\}}{\mathrm{P}\{X+Y=n\}} = \frac{\frac{e^{-\lambda_1}\lambda_1^k}{k!}\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \frac{n!}{k!\,(n-k)!}\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} = \mathrm{C}_k^n \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$\mathrm{P}\{X=k\} = \frac{e^{-\lambda_1}\lambda_1^k}{k!}$$

$$\mathrm{P}\{X=k \mid X+Y=n\} \neq \mathrm{P}\{X=k\}$$

   (b) Find $E[X \mid X + Y = n]$. Is it the same as $E[X]$?

$$E[X|X+Y=n] = \sum_{x=0}^{\infty} x\, \mathrm{P}\{X=x \mid X+Y=n\}$$

$$\xrightarrow{\text{by }(a)} \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!}\frac{\lambda_1^x\lambda_2^{n-x}}{(\lambda_1+\lambda_2)^n} = \frac{1}{(\lambda_1+\lambda_2)^n}\sum_{x=1}^{n} x \frac{n!}{x!\,(n-x)!}\lambda_1^x\lambda_2^{n-x}$$

$$= \frac{n}{(\lambda_1+\lambda_2)^n} \sum_{x-1=0}^{n-1} \frac{(n-1)!}{(x-1)!\,(n-x)!} \lambda_1^x \lambda_2^{n-x}$$

$$\xrightarrow{\text{let } i=x-1} \frac{n}{(\lambda_1+\lambda_2)^n} \sum_{i=0}^{n-1} \frac{(n-1)!}{i!\,(n-i-1)!} \lambda_1^{i+1} \lambda_2^{n-i-1}$$

$$= \frac{n\lambda_1}{(\lambda_1+\lambda_2)^n} \sum_{i=0}^{n-1} C_i^{n-1} \lambda_1^i \lambda_2^{(n-1)-i} = \frac{n\lambda_1}{(\lambda_1+\lambda_2)^n} (\lambda_1+\lambda_2)^{n-1} = n\frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$E[X \mid X+Y=n] \neq E[X]$$

(c) Find $E[E[X \mid X+Y]]$. Is it the same as $E[X]$?

$$E\big[E[X|X+Y]\big] = \sum_n E[X|X+Y=n]\ \mathrm{P}\{X+Y=n\}$$

$$\xrightarrow{\text{by } (a),(b)} \sum_n \left(n\frac{\lambda_1}{\lambda_1+\lambda_2}\right)\left(\frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n\right)$$

$$= \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)\sum_n n\,e^{-(\lambda_1+\lambda_2)}\left(\frac{\lambda_1+\lambda_2}{n!}\right)^n$$

$$= \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)(\lambda_1+\lambda_2) = \lambda_1 = E[X]$$

2. A dog is trapped in a cave. It needs to choose one of two directions. If it goes to the left, then it will walk around in the cave for five minutes and will then return to its initial position. If it goes to the right, then with probability $\frac{1}{4}$ it will depart the cave after four minutes of traveling, and with probability $\frac{3}{4}$ it will return to its initial position after seven minutes of traveling. Assuming that the dog is at all times equally likely to go to the left or the right, what is the expected number of minutes that it will be trapped in the cave?

Let N denote the number of minutes in the cave. If R is the event the dog chooses its right, and L the event it chooses its left, we have by conditioning on the first direction chosen:

$$E(N) = \tfrac{1}{2}E(N|L) + \tfrac{1}{2}E(N|R)$$

$$= \frac{1}{2}[5 + E(N)] + \frac{1}{2}\left[\frac{1}{4}(4) + \frac{3}{4}\left(7 + E(N)\right)\right]$$

$$= \frac{7}{8}E(N) + \frac{45}{8} = 45$$

3. Suppose that two teams are playing a series of games, each of which is independently won by team A with probability p and by team B with probability $1 - p$. The winner of the series is the first team to win i games.

(a) If $i = 4$, find the probability that a total of 7 games are played.

$$C\binom{6}{3}p^3(1-p)^3 = 20p^3(1-p)^3$$

(b) From the question (a), show that this probability is maximized when $p = 0.5$.

Let $g(p) = 20p^3(1-p)^3 = -20p^6 + 60p^5 - 60p^4 + 20p^3$

$g'(p) = -120p^5 + 300p^4 - 240p^3 + 60p^2$

$g'(p) = -60p^2(p-1)^2(2p-1) = 0$

So $p = 0$ or 1 or 0.5, the maximum value occurs at one of those values.

$g(0.5) = 0.3125 > g(1) = g(0) = 0$