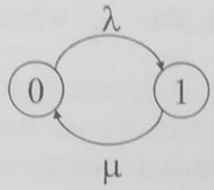
1. (10pt) There are two coins. Coin 1 comes up heads with probability 0.7 and coin 2 with probability 0.4. A coin is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other coin (until it comes up tails and then we start flipping the other coin). If we start flipping with coin 1, what is the probability that coin 2 is used on the third flip?

2. (10pt) Consider the two-state continuous-time Markov chain shown below where λ and μ represent two constants. We try to derive $P_{11}(t)$, which is the probability transition function from state 1 to state 1 in a period of t time units.



(a) Write down a differential equation for $P_{11}(t)$, by using Kolmogorov's forward equation $P'_{ij}(t) = \sum_{k \neq j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j$. The differential equation must only contain constants and $P_{11}(t)$.

(b) Find out $P_{11}(t)$ by solving the differential equation.

3. (10pt) Specify the classes of the discrete-time Markov chain corresponding to the transition matrix below, and determine whether these classes are transient or recurrent.

$$P = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

4. (10pt) A particle moves among $n+1$ vertices that are situated on a circle in the following manner. At each step it moves one step either in the clockwise direction with probability p or the counterclockwise direction with probability $1-p$. Starting at a specified state, call it state 0, let T be the time of the first return to state 0. Find the expected value of T.

5. (10pt) Compute M^{∞}, where the matrix M is defined as:

$$M = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

6. (10pt) A continuous-time Markov chain (CTMC) is a continuous-time stochastic process having the Markovian property. Define the probability transition function, $P_{ij}(t)$, as the probability that a process presently in state i will be in state j t time units later. Prove or disapprove the following equation for any CTMC: For all $s \geq 0, t \geq 0$,

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s)$$

7. (10pt) Tom has a gas oven, which is inspected by inspectors according to a renewal process whose inter-arrival times are distributed uniformly over the interval from 1 year to 2.5 years. If the inspectors find out that maintenance of the oven has not been performed for more than one year, Tom has to pay a fine of 210 dollars. Tom's strategy is to perform maintenance each time exactly one year after the visit of the inspectors. Compute the (average) long-run amount of fine Tom has to pay per year.

8. (10pt) Suppose that a system in use has a built-in function, RAND(), which generates a uniform random variable taking on values in $(0,1)$. In addition, the system can run basic arithmetic operations including addition, subtraction, multiplication, division, exponential and logarithm.
   (a) Develop a simple algorithm to generate a random variable with pdf $12t(1-t)^2, 0<t<1$.
   (b) Given a random number generator $\mathrm{U}(0,1)$. Please develop a simple algorithm that generates the Erlang-distributed random variable S_n, whose pdf is $\lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}, t>0$.

9. (10%) Consider a taxi station where taxis and customers arrive in accordance with Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are present. However, an arriving customer that does not find a taxi waiting leaves. Find
   (a) The average number of taxis waiting, and
   (b) The proportion of arriving customers that get taxis.

10. (10%) A dog is trapped in a cave. It needs to choose one of two directions. If it goes to the left, it will walk around in ~~the~~ cave for five minutes and will then return to its initial position. If it goes to the right, it will depart the cave after four minutes of traveling. Assume that the dog is at all times equally likely to go to the left or the right. Let T denote the time duration that it will be trapped in the cave. Find the expected value of the trapped duration, $E[T]$.

11. (10%) There are three machines, all of which are needed for a system to work. Machine i functions for an exponential time with rate λ_i before it fails, $i=1,2,3$. When a machine fails, the system is shut down and repair begins on the failed machine. The time to fix machine 1 is exponential with rate 5; the time to fix machine 2 is uniform on $(0,4)$; and the time to fix machine 3 is a gamma random variable with parameters $n=3$ and $\lambda=2$. Once a failed machine is repaired, it is as good as new and all machines are restart. What proportion of time is the system working?