Untangling Knots Through Curve Repulsion

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What the curious folks ponder about

Introduction

A Cool Knot

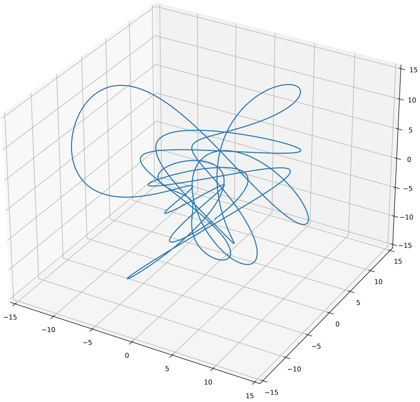

Figure: Imagine your earphones getting tangled like this. . .

Aim

- Finding a "homotopy" from a knot to an unknot.

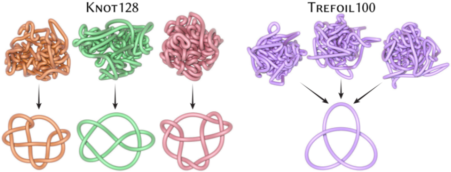

Figure: Unknots of test knots.[2]

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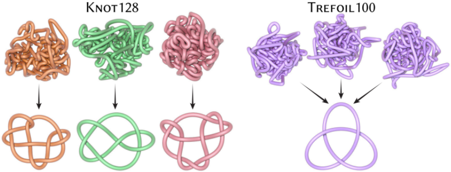

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- "Avoiding self-intersection"

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   - We evolve the curve according to the gradient flow equation.
   - There is a freedom in choosing the "gradient" here.
3. We expect the stationary state to be the "unknot"
   - Or at least a simpler state...

Tangent-Point Energy

Defining Curve Energy

Given an (arc-length parameterised) curve $\gamma : M \to \mathbb{R}^3$, we wish to assign energy of the form:

$$\mathcal{E}(\gamma) := \iint_{M^2} k\left(\gamma_x, \gamma_y\right) \, \mathrm{d}\gamma_x \, \mathrm{d}\gamma_y \tag{1}$$

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A naïve choice is $k\left(\gamma_x, \gamma_y\right) := \frac{1}{||\gamma_x - \gamma_y||}$

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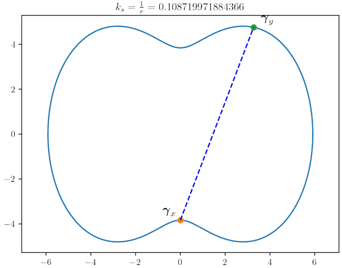

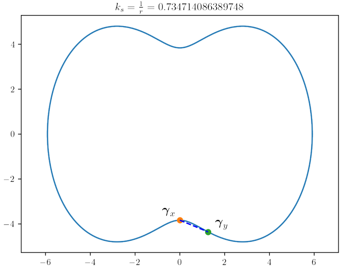

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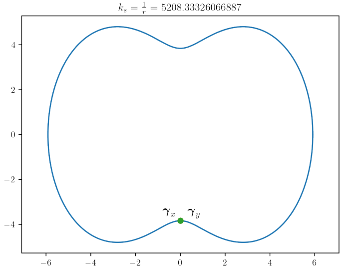

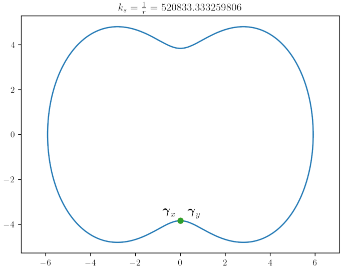

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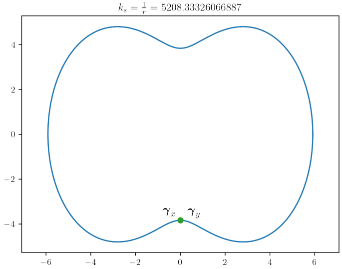

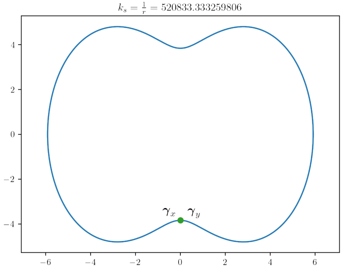

This energy is not well-defined for a lot of curves!

Buck-Orloff Tangent-Point Energy

- From the simple energy, need a way to suppress the infinite contribution of the “singularity”.

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Definition (Buck-Orloff Tangent-Point Energy)

For a smooth curve γ, define

$$\mathcal{E}(\gamma) := \iint_{M^2} k_4^2 \left(\gamma_x, \gamma_y, \mathbf{T}_x\right) \, \mathrm{d}\gamma_x \, \mathrm{d}\gamma_y$$

where $\mathbf{T}_x$ is the unit tangent vector at γ_x, with the kernel defined as

$$k_4^2 (\mathbf{p}, \mathbf{q}, \mathbf{T}) := \frac{||\mathbf{T} \wedge (\mathbf{p} - \mathbf{q})||^2}{||\mathbf{p} - \mathbf{q}||^4}$$

as **Buck-Orloff Tangent-Point Energy**.[1]

Intuition

What is the intuition behind the kernel $k_4^2\left(\mathbf{p},\mathbf{q},\mathbf{T}\right) := \frac{||\mathbf{T}\wedge(\mathbf{p}-\mathbf{q})||^2}{||\mathbf{p}-\mathbf{q}||^4}$?

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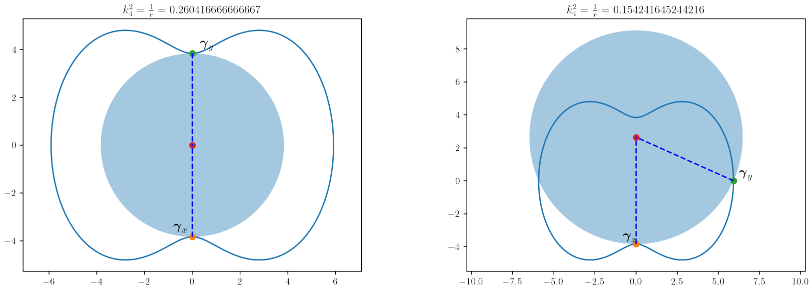

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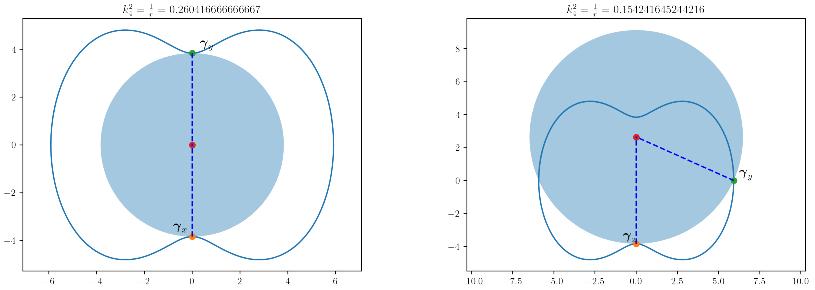

Remark

Note that closer does not necessarily mean the kernel is larger.

Intuition

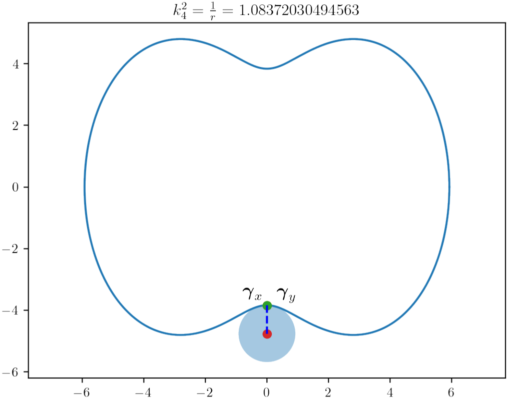

Figure: When two points are very close, the kernel converges to the curvature of the curve.

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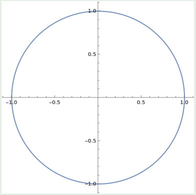

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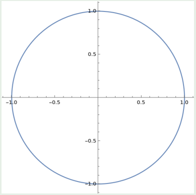

Then write:

$$\begin{cases} \gamma_x(\theta) = (\cos\theta, \sin\theta, 0) \\ \gamma_y(\phi) = (\cos\phi, \sin\phi, 0) \\ \mathbf{T}_x(\theta) = (-\sin\theta, \cos\theta, 0) \end{cases}$$

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Substituting to Buck-Orloff Tangent-Point energy formula:

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Using a few identities:

$$\mathcal{E}(\gamma) = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \frac{||\mathbf{T}_x||^2 ||\gamma_x - \gamma_y||^2 - \left(\mathbf{T}_x \cdot (\gamma_x - \gamma_y)\right)^2}{||\gamma_x - \gamma_y||^4} \, \mathrm{d}\theta \, \mathrm{d}\phi$$

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Remark

Note that (4) suggests the order at “singularity” is inverse-square.

General Tangent-Point Energy

A more general form of tangent-point energy comes from Yu, Schumacher, and Crane [2]:

Definition (Generalised Tangent-Point Energy)

$$\mathcal{E}^{\alpha}_{\beta}(\gamma) := \iint_{M^2} \frac{||\mathbf{T}_x \wedge (\gamma_x - \gamma_y)||^{\alpha}}{||\gamma_x - \gamma_y||^{\beta}} \, d\gamma_x \, d\gamma_y$$

where $\alpha > 1$ and $\beta \in [\alpha + 2, 2\alpha + 1)$

Remark

When $\alpha = 2$ and $\beta = 4$, we are back to Buck-Orloff.

Gradient Flow

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Remark

cf) For minimising a function $f : \mathbb{R}^n \to \mathbb{R}$, one may consider

$$\frac{\partial \mathbf{x}}{\partial t} = -\nabla f(\mathbf{x})$$

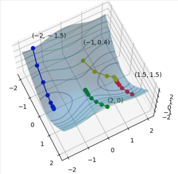

Figure: Steepest Descent

Gradient in Space of Function?

(Gradient of Function)
$\nabla f(\mathbf{x})$ is such that for all $\mathbf{y} \in \mathbb{R}^n$

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$\operatorname{grad}_X E(f)$ is such that for all $g \in X$,

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Remark

For $X = L^2$, $(\operatorname{grad}_X E)(f)$ is the first order variation from f with respect to ϵ.

Example: Dirichlet Energy in 1D

Definition (1D Dirichlet Energy)

For a differentiable function $f : \mathcal{I} \to \mathbb{R}$, define **1D Dirichlet energy**

$$E_D(f) := \int_{\mathcal{I}} |\nabla f(x)|^2 \, \mathrm{d}x = \int_{\mathcal{I}} |f'(x)|^2 \, \mathrm{d}x$$

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Remark

Dirichlet energy is high for function f that varies a lot, and minimised by any constant function.

Example: L^2 Gradient Flow of Dirichlet Energy

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Lemma (L^2 Gradient of Dirichlet Energy)

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(for suitable boundary condition if finite interval)

Then the gradient flow equation for Dirichlet energy becomes:

$$\frac{\partial f}{\partial t} = \Delta f \tag{6}$$

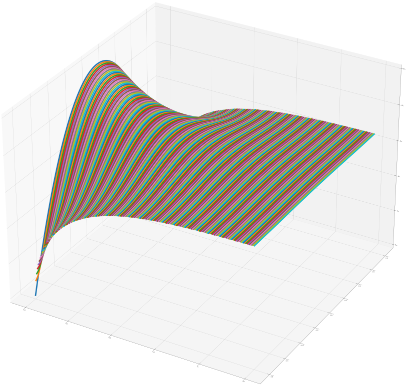

Figure: Solution to L^2 Gradient Flow Equation for Dirichlet Energy (AKA Heat Equation)

Dirichlet Gradient Flow in Other Function Space

Definition (Famous Inner Product Function Spaces)

- $L^p = \left\{ f \,\middle|\, \left(\int |f|^p \,\mathrm{d}x\right)^{1/p} < \infty \right\}$
  - $\langle\langle f, g \rangle\rangle_{L^2} = \int fg \,\mathrm{d}x$
- $p = 2$ Sobolev Space $H^k = \left\{ f \,\middle|\, \sum_{i=0}^{k} \int \left|\mathcal{D}^{(i)} f\right|^2 \,\mathrm{d}x < \infty \right\}$

- L^2: $\frac{\partial f}{\partial t} = -\Delta f$
- H^1: $\frac{\partial f}{\partial t} = -f$
- H^{-1}: $\frac{\partial f}{\partial t} = -\Delta^2 f$
  - $\langle\langle f, g \rangle\rangle_{H^{-1}} = \langle\langle \Delta^{-1} f, \Delta^{-1} g \rangle\rangle_{H^1}$
- H^2: $\frac{\partial f}{\partial t} = \Delta^{-1} f$

Bibliography

[1] Gregory Buck and Jeremey Orloff. "A simple energy function for knots". In: *Topology and its Applications* 61.3 (Feb. 1995), pp. 205–214. DOI: 10.1016/0166-8641(94)00024-w.

[2] Chris Yu, Henrik Schumacher, and Keenan Crane. "Repulsive Curves". In: *ACM Transactions on Graphics* 40.2 (Apr. 2021), pp. 1–21. DOI: 10.1145/3439429.