Physica A: Statistical Mechanics and its Applications
Volume 536, 15 December 2019, 122494

On a more general fractional integration by parts formulae and applications

Thabet Abdeljawad [a], Abdon Atangana [b], J.F. Gómez-Aguilar [c], Fahd Jarad [d]

https://doi.org/10.1016/j.physa.2019.122494

Highlights

- We develop fractional integration by parts for Riemann–Liouville, Liouville–Caputo, Caputo–Fabrizio and Atangana–Baleanu operators.
- We provided examples without loss of generality for the case of Caputo–Fabrizio.
- For the classical case our formulae of fractional integration by parts results in the previously obtained integration by parts.

Abstract

The integration by part comes from the product rule of classical differentiation and integration. The concept was adapted in fractional differential and integration and has several applications in control theory. However, the formulation in fractional calculus is the classical integral of a fractional derivative of a product of a fractional derivative of a given function f and a function g. We argue that, this formulation could be done using only fractional operators; thus, we develop fractional integration by parts for fractional integrals, Riemann–Liouville, Liouville–Caputo, Caputo–Fabrizio and Atangana–Baleanu fractional derivatives. We allow the left and right fractional integrals of order $\alpha > 0$ to act on the integrated terms instead of the usual integral and then make use of the fractional type Leibniz rules to formulate the integration by parts by means of new generalized type fractional operators with binomial coefficients defined for analytic functions. In the case $\alpha = 1$, our formulae of fractional integration by parts results in previously obtained integration by parts in fractional calculus. The two disciplines or branches of mathematics are built differently, while classical differentiation is built with the concept of rate of change of a given function, a fractional differential operator is a convolution.

Keywords

Fractional calculus; New integration by parts; Convolution; Binomial coefficients; Fractional derivatives

1. Introduction

The concept of fractional calculus although being nowadays considered as a branch of mathematics and applied mathematics, has been constructed using mostly some concepts of classical differentiation and integration. See for instance recently the imposition in fractional calculus of index law and Leibniz law that are fundamentally laws from classical calculus. For historical sake, within the field of differential and integration, the integration by part is the process that finds the classical integral of a product of two functions in terms of the integral of their derivative and anti-derivative. This process is commonly used to transform the antiderivative of a product of functions into an antiderivative for which a solution can be more obvious to be calculated. In general,

$$\int_a^b v du = [vu]_a^b - \int_a^b u dv, \tag{1}$$

or for a more compact form, we have

$$\int v du = vu - \int u dv, \tag{2}$$

then the above integration by part was adopted in fractional calculus, where

$$\int_a^b {}_0^C\mathscr{D}_t^\alpha f(t) \cdot g(t)\, dt. \tag{3}$$

The above formulation has been used for theoretical and practical purposes in the last decade with great success. One of the application of this mathematical formula is perhaps the optimal control in control theory. Nevertheless it is very strange that within the concept of fractional differential integration, the classical integral is used with fractional derivative. The question one will ask at this stage is that, did we accepted to use the classical integral due to simplification?. In this paper, believe that within the framework of fractional calculus, one must exclusively use fractional operators, therefore, we replace the classical integral with fractional integral to provide alternative formula of integration by part with Riemann–Liouville, Liouville–Caputo, Caputo–Fabrizio and Atangana–Baleanu fractional derivatives[1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23].

2. Mathematical formulations

Definition 1 Left and Right Riemann–Liouville Fractional Integrals

Let f be defined and integrable on an interval $[a,b]$ and $Re(\alpha) > 0$. Then[24], [25], [26]

1. The left Riemann–Liouville integral of order α starting at t, where $a \le t < b$, is defined by

$$\left({}_tI^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_t^x (x-s)^{\alpha-1} f(s)\, ds. \tag{4}$$

2. The right Riemann–Liouville integral of order α ending at t, where $a < t \le b$, is defined by

$$\left(I_t^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^t (s-x)^{\alpha-1} f(s)\, ds. \tag{5}$$

Definition 2 Left and Right Riemann–Liouville Fractional Derivatives

Let f be defined on an interval $[a,b]$,$n = [\alpha] + 1$ and $0 < \alpha \le 1$. Then[24], [25], [26]

1. The left Riemann–Liouville fractional derivative of order α starting at t, where $a \le t < ba \le t < b$ is defined by

$$\left({}_tD^\alpha f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_t^x (x-s)^{n-\alpha-1} f(s)\, ds. \tag{6}$$

2. The right Riemann–Liouville fractional derivative of order α ending at t, where $a < t \le b$, is defined by $a < t \le b$ is defined by

$$\left(D_t^\alpha f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^t (s-x)^{n-\alpha-1} f(s)\, ds. \tag{7}$$

If $\alpha < 1$ then $n = 1$ and if $\alpha = 1$ then $n = 2$ and hence $\left({}_tD^1 f\right)(x) = f'(x)$ and $\left(D_t^1 f\right)(x) = -f'(x)$.

Definition 3 Left and Right Liouville–Caputo Fractional Derivatives

Let f be defined and absolutely continuous on an interval $[a,b]$ ($f \in AC[a,b]$) and $0 < \alpha \le 1$. Then[26]

1. The left Caputo fractional derivative of order α starting at t, where $a \le t < b$, is defined by

$$\left({}_t^CD^\alpha f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_t^x (x-s)^{-\alpha} f'(s)\, ds, \tag{8}$$

where $\left({}_t^CD^1 f\right)(x) = f'(x)$.

2. The right Caputo fractional derivative of order α ending at t, where $a < t \le b$ is defined by

$$\left({}^CD_t^\alpha f\right)(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^t (s-x)^{-\alpha} f'(s)\, ds, \tag{9}$$

where $\left({}_t^CD^1 f\right)(x) = -f'(x)$.

Lemma 1

Let $a, b \in \mathbb{R}$ such that $a < b$. If $\alpha, \beta \in \mathbb{C}$ such that $Re(\alpha) > 0, \quad Re(\beta) > 0$. Then [26]

1. $\left({}_aI^\alpha (t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (x-a)^{\alpha+\beta-1}$.
2. $\left(I_b^\alpha (b-t)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (b-x)^{\alpha+\beta-1}$.

Let $a, b \in \mathbb{R}$ such that $a < b$ and f a function defined on $[a,b]$. Then the action of the Q-operator is given by $Qf(t) = f(a+b-t)$. It is known and easily can be checked that the following are valid for a function f defined on $[a,b]$ and $Re(\alpha) > 0$:

1. $Q_aI^\alpha f(t) = ({}_aI^\alpha f)(a+b-t) = I_b^\alpha Qf(t)$.

2. $Q_a D^\alpha f(t) = ({}_a D^\alpha f)(a+b-t) = D_b^\alpha Q f(t)$.

3. $QQf(t) = f(t)$.

4. $(-1)^k Q \frac{d^k}{dt^k} f(t) = \frac{d^k}{dt^k} Q f(t)$.

Definition 4

The Caputo-Fabrizio derivative is defined as [27]

$$ {}_0^{CF}\mathscr{D}_t^\alpha \{f(t)\} = \frac{M(\alpha)}{n-\alpha} \int_0^t f^{(n)}(\tau) \exp\left[-\frac{\alpha(t-\tau)}{n-\alpha}\right] d\tau, \quad n-1 < \alpha \le n, \tag{10} $$

where $M(\alpha)$ is a normalization function.

Theorem 1

Let f andg be continuous function such that ${}_0^{CF}\mathscr{D}_t^\alpha f(t)$ exist and $g(t) \quad {}_0^{CF}\mathscr{D}_t^\alpha f(t)$ is Caputo–Fabrizio integrable then, we propose the following integration by part formula

$$ {}_a^{CF}I_t^\alpha \left[{}_0^{CF}\mathscr{D}_t^\alpha f(t) \cdot g(t) \right]. \tag{11} $$

However

$$ {}_a^{CF}I_t^\alpha \left[g(t) \cdot {}_0^{CF}\mathscr{D}_t^\alpha f(t)\right] = \frac{1-\alpha}{M(\alpha)} g(t) \cdot {}_0^{CF}\mathscr{D}_t^\alpha f(t) + \frac{\alpha}{M(\alpha)} \int_a^t g(\tau) \cdot {}_0^{CF}\mathscr{D}_\tau^\alpha f(\tau) \quad d\tau. \tag{12} $$

$$ = \frac{1-\alpha}{M(\alpha)} g(t) \left({}_0^{CF}\mathscr{D}_t^\alpha f(t)\right) + \frac{\alpha}{M(\alpha)} \left[G(t) \ {}_0^{CF}\mathscr{D}_\tau^\alpha - \int_0^t G(\tau) \left({}_0^{CF}\mathscr{D}_\tau^\alpha f(\tau)\right)' d\tau\right], \tag{13} $$

where G is the anti-derivative of g.

Lemma 2

Let $\Re(\alpha) \ge 0$ and$n = [\Re(\alpha)] + 1$. Iff $\in AC^n[a,b]$, where$0 < a < b < \infty$. Then[26]

$$ \left({}_a^C D^\alpha f\right)(x) = \left({}_a D^\alpha f\right)(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha}, \tag{14} $$

$$ \left({}^C D_b^\alpha f\right)(x) = \left(D_b^\alpha f\right)(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-x)^{k-\alpha} \tag{15} $$

.

Lemma 3 The Product Rule for Riemann–Liouville Fractional Integrals and Derivatives

For functions f and g such that $f(x), g(x)$ and $f(x)g(x)$ are in the form $(x-a)^\mu h(x)$ ($(b-x)^\mu h(x)$ in the right case), $Re(\mu) > -1$ with$h(x)$ is analytic on a domain $D \subset \mathbb{C}$ containing a (b in the right case) we have[25], [28], [29], [30], [31]

- *For $\beta \in \mathbb{C}$ and $x \in \mathbb{R} - \{a\}$*

$$ {}_a D^\beta (f(x) g(x)) = \sum_{k=0}^\infty \binom{\beta}{k} \left({}_a D^{\beta-k} f\right)(x) g^{(k)}(x) \tag{16} $$

.

- *For $\alpha \in \mathbb{C}, \quad Re(\alpha) > 0$ we have*

$$ {}_a I^\alpha (f(x) g(x)) = \sum_{k=0}^\infty \binom{-\alpha}{k} \left({}_a I^{\alpha+k} f\right)(x) g^{(k)}(x). \tag{17} $$

Above, Eq.(17) is a particular case of Eq.(16) when $\beta = -\alpha$ is negative.

Remark 1

Applying the Q-operator, which acts as $Qf(x) = f(a+b-x)$, so that $QQf(x) = f(x)$, to the identities (16), (17) and making use of the facts that $(Q_a I^\alpha f)(x) = I_b^\alpha (Qf)(x)$ and $(Q_a D^\alpha f)(x) = D_b^\alpha (Qf)(x)$, we obtain respectively

$$ D_b^\beta (f(x) g(x)) = \sum_{k=0}^\infty \binom{\beta}{k} \left(D_b^{\beta-k} f\right)(x) (-1)^k g^{(k)}(x), \tag{18} $$

and

$$ I_b^\alpha (f(x) g(x)) = \sum_{k=0}^\infty \binom{-\alpha}{k} \left(I_b^{\alpha+k} f\right)(x) (-1)^k g^{(k)}(x), \quad Re(\alpha) > 0. \tag{19} $$

We also have made use of that $Qf^{(k)}(x) = (-1)^k \frac{d^k}{dx^k}(Qf)(x)$.

Definition 5

For $Re(\beta) > 0, \quad 1 \le p \le \infty$ we define the following function spaces[26]

$$_aI^\beta(L_p) = \{f : f = {_aI^\beta}\varphi, \quad \varphi \in L_p(a,b)\}, \tag{20}$$

and

$$I_b^\beta(L_p) = \left\{f : f = I_b^\beta \psi, \quad \psi \in L_p(a,b)\right\}. \tag{21}$$

Remark 2

It is well-known that the space $C[a,b]$ of continuous functions on $[a,b]$ is a subspace of $_aI^\beta(L_p)$ and $I_b^\beta(L_p)$. In fact, for $f \in C[a,b]$ we have $_aI^\beta {_aD^\beta} f(x) = f(x)$ and $I_b^\beta D_b^\beta f(x) = f(x)$.

Lemma 4

[26]

Let $\beta > 0, p, q \ge 1$, and $\frac{1}{p} + \frac{1}{q} \le 1 + \beta$ ($p, q \ne 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \beta$). Then

1. *If $f \in L_p(a,b)$ and $g \in L_q(a,b)$, then*

$$\int_a^b f(x)\left({_aI^\beta} g\right)(x)\,dx = \int_a^b g(x)\left(I_b^\beta f\right)(x)\,dx. \tag{22}$$

2. *If $f \in I_b^\beta(L_p)$ and $g \in {_aI^\beta}(L_q)$, then*

$$\int_a^b f(x)\left({_aD^\beta} g\right)(x)\,dx = \int_a^b g(x)\left(D_b^\beta f\right)(x)\,dx. \tag{23}$$

Lemma 5 Integration by Parts for Liouville–Caputo Fractional Derivatives

Let $0 < \beta < 1$ and $f \in L_p(a,b)$ and $g \in AC[a,b]$. Then[32]

$$\int_a^b f(t)\left({_a^CD^\beta} g\right)(t)\,dt = \int_a^b g(t)\left(D_b^\beta f\right)(t)\,dt + \left(I_b^{1-\beta} f\right)(t)\,g(t)\big|_a^b, \tag{24}$$

and

$$\int_a^b f(t)\left({^CD_b^\beta} g\right)(t)\,dt = \int_a^b g(t)\left({_aD^\beta} f\right)(t)\,dt - \left({_aI^{1-\beta}} f\right)(t)\,g(t)\big|_a^b. \tag{25}$$

Now, we may state the higher order case as well. The following result which may be also gathered from Eq.(24) and Lemma2, is a particular case ($\rho = 1$) of Corollary 1 in[32].

Lemma 6

Let $0 < \beta \le 1$, and $f \in L_p(a,b)$, $g \in AC[a,b]$. Then[32]

$$\int_a^b f(t)\left({_aD^\beta} g\right)(t)\,dt = \int_a^b g(t)\left(D_b^\beta f\right)(t)\,dt + g(b)\left(I_b^{1-\beta} f\right)(b^-). \tag{26}$$

The following result can be deduced from Eq.(25) and Lemma2 and is a particular case ($\rho = 1$) of Corollary 2 in[32].

Lemma 7

Let $0 < \beta \le 1$, and $f \in L_p(a,b)$, $g \in AC[a,b]$. Then,

$$\int_a^b f(t)\left(D_b^\beta g\right)(t)\,dt = \int_a^b g(t)\left({_aD^\beta} f\right)(t)\,dt + g(a)\left({_aI^{1-\beta}} f\right)(a^+). \tag{27}$$

3. Integration by parts formula

In this section, we attempt to provide the derivation of integration by part where the classical integral operator is replaced by fractional integral.

Let $t \in (0,b]$ we get

$$
\begin{aligned}
&{}_0I^{\alpha}\left(v(t)\left({}_0D^{\alpha}u\right)(t)\right)\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)\left({}_0D^{\alpha}u\right)(s)\,ds,\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t D_t^{\alpha}\left[(t-s)^{\alpha-1}v(s)\right]u(s)\,ds,\\
&\quad \text{Using Lemma 2.5 in [26].}\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t D_t^{\alpha}\left[QQ_s(t-s)^{\alpha-1}v(s)\right]u(s)\,ds,\qquad Qf(s)=\\
&\quad f(t-s)\qquad \text{shifting operator.}\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t Q_0D^{\alpha}\left[(t-(t-s))^{\alpha-1}v\right](s)\,ds,\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t Q_0D^{\alpha}\left[(t-t+s)^{\alpha-1}v\right](s)\,ds,\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t Q\left[\sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{\partial^k}{\partial s^k}(t-t+s)^{\alpha-1}{}_0D^{\alpha-k}v(s)\right]u(s)\,ds\\
&.\ \text{We have used the Fractional Leibniz rule (2.202)}\qquad in\\
&\quad [25],\\
&=\frac{1}{\Gamma(\alpha)}\\
&\int_0^t Q\left[\sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)}(t-t+s)^{\alpha-k-1}{}_0D^{\alpha-k}v(s)\right]u(s)\,ds,\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t\left[\sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)}(t-s)^{\alpha-k-1}\right.\\
&\quad\left. D_t^{\alpha-k}Qv(s)\right]u(s)\,ds,\\
&=\frac{1}{\Gamma(\alpha)}\int_0^t\left[\sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)}(t-s)^{\alpha-k-1}{}_0D^{\alpha-k}v(t-s)\right]u(s)\,ds,
\end{aligned}
$$

the relation between the left and right derivatives and Q-operator van be seen in [24].

The Riemann–Liouville fractional derivative in the integrand can be changed by Liouville–Caputo fractional derivative and the zero can be changed by any number a. The orders of the integral and derivative can be different.

4. More general integration by parts for fractional integrals and derivatives

Theorem 2 Fractional Integration by Parts for Riemann–Liouville Fractional Integrals

Let $\beta>0, p,q\geq 1$, and $\frac{1}{p}+\frac{1}{q}\leq 1+\beta$ ($p,q\neq 1$ in the case $\frac{1}{p}+\frac{1}{q}=1+\beta$). Let $\alpha>0$.

1. *If $(b-x)^{\alpha-1}f(x)\in L_p(a,b)$ such that f is analytic on $[a,b]$ and $g\in L_q(a,b)$, then we have*

$$
\begin{aligned}
&{}_aI^{\alpha}\left[f(x)\,{}_aI^{\beta}g(x)\right](t)\\
&=\int_a^t g(s)\left(\sum_{k=0}^{\infty}\binom{-\beta}{k}\frac{(-1)^k}{\Gamma(\alpha+\beta+k)}(t-s)^{k+\alpha+\beta-1}f^{(k)}(s)\right)ds.
\end{aligned}
\tag{28}
$$

2. *If $(x-a)^{\alpha-1}f(x)\in L_p(a,b)$ and $g\in L_q(a,b)$, then we have*

$$
\begin{aligned}
&I_b^{\alpha}\left[f(x)\,I_b^{\beta}g(x)\right](t)\\
&=\int_t^b g(s)\left(\sum_{k=0}^{\infty}\frac{\binom{-\beta}{k}}{\Gamma(\alpha+\beta+k)}(s-t)^{k+\alpha+\beta-1}f^{(k)}(s)\right)ds\\
&.
\end{aligned}
\tag{29}
$$

Proof

1.

From definition and by Eq.(22) of Lemma 4 with $t > a$ in place of b, we have

$$
\begin{aligned}
&{}_aI^{\alpha}\left[f(x)\,{}_aI^{\beta}g(x)\right](t)\\
&= \tfrac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\left({}_aI^{\beta}g\right)(s)\,ds,\\
&= \tfrac{1}{\Gamma(\alpha)}\int_a^t \quad I_t^{\beta}\left[(t-s)^{\alpha-1}f(s)\right]g(s)\,ds.
\end{aligned} \tag{30}
$$

By applying the fractional Leibniz rule (19) to the right hand side of Eq.(30) we see that

$$
\begin{aligned}
&{}_aI^{\alpha}\left[f(x)\,{}_aI^{\beta}g(x)\right](t)\\
&= \tfrac{1}{\Gamma(\alpha)}\int_a^t\Big[\textstyle\sum_{k=0}^{\infty}\binom{-\beta}{k}\\
&\quad I_t^{\beta+k}(t-s)^{\alpha-1}(-1)^k f^{(k)}(s)\Big]g(s)\,ds,\\
&= \tfrac{1}{\Gamma(\alpha)}\int_a^t\Big[\textstyle\sum_{k=0}^{\infty}\binom{-\beta}{k}\quad \frac{(-1)^k}{\Gamma(\alpha+\beta+k)}(t-s)^{\alpha+\beta+k-1}f^{(k)}(s)\Big]g(s)\,ds.
\end{aligned} \tag{31}
$$

Above we have used the second part of Lemma 1.

2. The proof is similar to that in the first part. However, we make use of the product rule (17) and the first part of Lemma 1. Alternatively, the proof can be achieved by applying the Q-operators to Eq.(28) and making use of its properties. In fact, we shall have

$$
\begin{aligned}
&I_b^{\alpha}\left[Qf(x)\,I_b^{\beta}Qg(x)\right](t)\\
&= \int_a^{a+b-t} g(s)\left(\textstyle\sum_{k=0}^{\infty}\frac{(-1)^k\binom{-\beta}{k}}{\Gamma(\alpha+\beta+k)}(a+b-t-s)^{k+\alpha+\beta-1}f^{(k)}(s)\right)ds.
\end{aligned}
$$

If in the above equation, we replace $f(x)$ by $Qf(x)$ and $g(x)$ by $Qg(x)$ then we have

$$
\begin{aligned}
&I_b^{\alpha}\left[f(x)\,I_b^{\beta}g(x)\right](t) =\\
&\int_a^{a+b-t} g(a+b-s)\\
&\left(\textstyle\sum_{k=0}^{\infty}\frac{(-1)^k\binom{-\beta}{k}}{\Gamma(\alpha+\beta+k)}(a+b-t-s)^{k+\alpha+\beta-1}(-1)^k f^{(k)}(a+b-s)\right)ds.
\end{aligned} \tag{32}
$$

Finally, the identity (29) will follow by using the change of variable $u = a+b-s$.□

Corollary 1 The Classical Integration by Parts for Fractional Integrals is a Particular Case

1 *If in Eq.(28) of Theorem 2, we set $\alpha = 1$ and $t = b$, then we obtain Eq.(22) of Lemma 4.*

2 *If in Eq.(29) of Theorem 2, we set $\alpha = 1$ and $t = a$, then we obtain Eq.(22) of Lemma 4 with the change of rules of f and g.*

Proof

We prove only the first part. The proof of the second part is similar where we use the Taylor expansion of $f(t)$ about s inside ${}_aI^{\beta}f(s)$ instead, and by writing $(t-s)^k = (-1)^k(s-t)^k$.

By expanding $f(t)$ about s and integrating term by term, we see that

$$
\begin{aligned}
&I_b^{\beta}f(s) = \tfrac{1}{\Gamma(\beta)}\int_s^b (t-s)^{\beta-1}f(t)\,dt,\\
&= \tfrac{1}{\Gamma(\beta)}\int_s^b (t-s)^{\beta-1}\left(\textstyle\sum_{k=0}^{\infty}\frac{f^{(k)}(s)}{k!}(t-s)^k\right)dt,\\
&= \textstyle\sum_{k=0}^{\infty}\frac{f^{(k)}(s)}{\Gamma(\beta)\Gamma(k+1)}\int_s^b (t-s)^{k+\beta-1}\,dt,\\
&= \textstyle\sum_{k=0}^{\infty}\frac{f^{(k)}(s)}{\Gamma(\beta)\Gamma(k+1)(k+\beta)}(b-s)^{k+\beta}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\int_a^b g(s)\,I_b^{\beta}f(s)\,ds\\
&= \int_a^b g(s)\left(\textstyle\sum_{k=0}^{\infty}\frac{f^{(k)}(s)}{\Gamma(\beta)\Gamma(k+1)(k+\beta)}(b-s)^{k+\beta}\right)ds.
\end{aligned} \tag{33}
$$

On the other hand, if we set $\alpha = 1$ and $t = b$ in Eq.(28) of Theorem 2, we see that

$$
\begin{aligned}
&{}_aI^{1}\left[f(x)\,{}_aI^{\beta}g(x)\right](b)\\
&= \int_a^b g(s)\left(\textstyle\sum_{k=0}^{\infty}\binom{-\beta}{k}\frac{(-1)^k}{\Gamma(1+\beta+k)}(b-s)^{k+\beta}f^{(k)}(s)\right)ds.
\end{aligned} \tag{34}
$$

By expanding the binomial coefficient in Eq.(34), we noting that

$$\frac{\Gamma(1-\beta)(-1)^k}{\Gamma(1-\beta-k)\Gamma(\beta+k)} = \frac{1}{\Gamma(\beta)},$$

for any $k = 0, 1, \ldots$, and we conclude that Eqs. (33), (34) are the same. □

The next integration by parts will generalize Lemma 5.

Theorem 3 The Fractional Integration by Parts for Liouville–Caputo Fractional Derivatives

Let $\alpha > 0, 0 < \beta \leq 1$. Then

1. *If $(b-x)^{\alpha-1} f(x) \in L_p(a,b)$ such that f is analytic on $[a,b]$ and $g \in AC[a,b]$. Then,*

$$\begin{aligned} {}_aI^{\alpha}\left[f(x) \quad {}_a^CD^{\beta} g(x)\right](t) &= R_{\alpha,\beta,t}(s)\, g(s)\Big|_a^t \\ &- \int_a^t R'_{\alpha,\beta,t}(s)\, g(s)\, ds, \qquad a < t \leq b, \end{aligned} \tag{35}$$

where $R_{\alpha,\beta,t}(s) = \sum_{k=0}^{\infty} \frac{(-1)^k \binom{\beta-1}{k} (t-s)^{k+\alpha-\beta}}{\Gamma(\alpha+k+1-\beta)} f^{(k)}(s)$.

2. *If $(b-x)^{\alpha-1} f(x) \in L_p(a,b)$ such that f is analytic on $[a,b]$ and $g \in AC[a,b]$. Then,*

$$\begin{aligned} {}_aI^{\alpha}\left[f(x) \quad {}_a^CD^{\beta} g(x)\right](t) &= R_{\alpha,\beta,t}(s)\, g(s)\Big|_a^t \\ &+ \int_a^t T_{\alpha,\beta,t}(s)\, g(s)\, ds, \qquad a < t \leq b, \end{aligned} \tag{36}$$

where $T_{\alpha,\beta,t}(s) = \sum_{k=0}^{\infty} \frac{\binom{\beta}{k} (t-s)^{k+\alpha-\beta-1} (-1)^k}{\Gamma(\alpha-\beta+k)} f^{(k)}(s)$.

3. *If $(x-a)^{\alpha-1} f(x) \in L_p(a,b)$ such that f is analytic on $[a,b]$ and $g \in AC[a,b]$. Then,*

$$\begin{aligned} I_b^{\alpha}\left[f(x) \quad {}^CD_b^{\beta} g(x)\right](t) &= \int_t^b L'_{\alpha,\beta,t}(s)\, g(s)\, ds \\ &- L_{\alpha,\beta,t}(s)\, g(s)\Big|_t^b, \qquad a \leq t < b, \end{aligned} \tag{37}$$

where $L_{\alpha,\beta,t}(s) = \sum_{k=0}^{\infty} \frac{\binom{\beta-1}{k} (s-t)^{k+\alpha-\beta}}{\Gamma(\alpha+k+1-\beta)} f^{(k)}(s)$.

4. *If $(x-a)^{\alpha-1} f(x) \in L_p(a,b)$ such that f is analytic on $[a,b]$ and $g \in AC[a,b]$. Then, If*

$$\begin{aligned} I_b^{\alpha}\left[f(x) \quad {}^CD_b^{\beta} g(x)\right](t) &= \int_t^b M_{\alpha,\beta,t}(s)\, g(s)\, ds \\ &- L_{\alpha,\beta,t}(s)\, g(s)\Big|_t^b, \qquad a \leq t < b, \end{aligned} \tag{38}$$

where $M_{\alpha,\beta,t}(s) = \sum_{k=0}^{\infty} \frac{\binom{\beta}{k} (s-t)^{k+\alpha-\beta-1}}{\Gamma(\alpha-\beta+k)} f^{(k)}(s)$.

Proof

1. From the definition of Liouville–Caputo fractional derivative, the integration by parts (22) in Lemma 4, the product rule (19), and ordinary integration by parts, we have

$$\begin{aligned} &{}_aI^{\alpha}\left[f(x) \quad {}_a^CD^{\beta} g(x)\right](t) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, {}_aI^{1-\beta} g'(s)\, ds, \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t I_t^{1-\beta}\left[(t-s)^{\alpha-1} f(s)\right](s)\, g'(s)\, ds, \\ &= \int_a^t R_{\alpha,\beta,t}(s)\, g'(s)\, ds, \\ &= R_{\alpha,\beta,t}(s)\, g(s)\Big|_a^t - \int_a^t R'_{\alpha,\beta,t}(s)\, g(s)\, ds. \end{aligned}$$

2. By the integration by parts (24) and the fractional product rule (18), we have

$$\begin{aligned} {}_aI^{\alpha}\left[f(x) \quad {}_a^CD^{\beta} g(x)\right](t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \\ &\quad {}_a^CD^{\beta} g(s)\, ds, \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t D_t^{\beta}\left[(t-s)^{\alpha-1} f(s)\right](s)\, g(s)\, ds, \\ &= R_{\alpha,\beta,t}(s)\, g(s)\Big|_a^t + \int_a^t T_{\alpha,\beta,t}(s)\, g(s)\, ds. \end{aligned}$$

3. The proof is analogous to part (1), however, we make use the fractional product rule (17) instead. An alternative proof can executed by using the action of the Q-operator on (35).

4. The proof is analogous to part (2), however, we make use the fractional product rule (16) and the integration by parts (25) instead. An alternative proof can executed by using the action of the Q-operator on Eq.(36).□

Remark 3

By the help of some binomial and Gamma function identities it can be shown that $T_{\alpha,\beta,t}(s) = -R'_{\alpha,\beta,t}(s)$ and $T_{1,\beta,b}(s) = D_b^{\beta} f(s)$. Hence, parts (1) and (2) in Theorem3 above are generalizations to the integration by parts (24) in Lemma5.

Similarly, it can be shown that $M_{\alpha,\beta,t}(s) = L'_{\alpha,\beta,t}(s)$ and $T_{1,\beta,a}(s) = {}_aD^{\beta} f(s)$. Hence, parts (3) and (4) in Theorem3 above are generalizations to the integration by parts (25) in Lemma5.

Definition 6

Let $\alpha > 0$, $0 < \beta \leq 1$ and f be an analytic function on a domain containing the interval $[a,b]$. Then, we define the following generalized fractional operators:

1. The right generalized fractional integral ending at t:

$$(I_{\alpha,\beta,t} f)(s) = \frac{1}{\Gamma(\alpha)} I_t^{\beta}\left[(t-s)^{\alpha-1} f(s)\right](s), \quad s \in [a,b], \quad a < t \leq b. \tag{39}$$

2. The right generalized fractional derivative ending at t:

$$(D_{\alpha,\beta,t} f)(s) = \frac{1}{\Gamma(\alpha)} D_t^{\beta}\left[(t-s)^{\alpha-1} f(s)\right](s), \quad s \in [a,b], \quad a < t \leq b. \tag{40}$$

3. The left generalized fractional integral starting at t:

$$({}_{\alpha,\beta,t} I f)(s) = \frac{1}{\Gamma(\alpha)} {}_tI^{\beta}\left[(s-t)^{\alpha-1} f(s)\right](s), \quad s \in [a,b], \quad a \leq t < b. \tag{41}$$

4. The left generalized fractional derivative starting at t:

$$({}_{\alpha,\beta,t} D f)(s) = \frac{1}{\Gamma(\alpha)} {}_tD^{\beta}\left[(s-t)^{\alpha-1} f(s)\right](s), \quad s \in [a,b], \quad a \leq t < b. \tag{42}$$

Remark 4

From Theorem3 and Remark3 above we have the following observations:

1. $$(I_{\alpha,\beta,t} f)(s) = (R_{\alpha,1-\beta,t})(s) = \sum_{k=0}^{\infty} \frac{(-1)^k \binom{-\beta}{k} (t-s)^{k+\alpha+\beta-1}}{\Gamma(\alpha+k+\beta)} f^{(k)}(s), \quad s \in [a,b], \quad a < t \leq b, \tag{43}$$

   and in particular $(I_{1,\beta,b} f)(s) = \left(I_b^{\beta} f\right)(s)$.

2. $$(D_{\alpha,\beta,t} f)(s) = T_{\alpha,\beta,t}(s) = \sum_{k=0}^{\infty} \frac{\binom{\beta}{k} (t-s)^{k+\alpha-\beta-1} (-1)^k}{\Gamma(\alpha-\beta+k)} f^{(k)}(s), \quad s \in [a,b], \quad a < t \leq b, \tag{44}$$

   in particular $(D_{1,\beta,b} f)(s) = \left(D_b^{\beta} f\right)(s)$, and $(D_{\alpha,\beta,t} f)(s) = -\frac{d}{ds}(I_{\alpha,1-\beta,t} f)(s)$.

3. $$({}_{\alpha,\beta,t} I f)(s) = (L_{\alpha,1-\beta,t})(s) = \sum_{k=0}^{\infty} \frac{\binom{-\beta}{k} (s-t)^{k+\alpha+\beta-1}}{\Gamma(\alpha+k+\beta)} f^{(k)}(s), \quad s \in [a,b], \quad a \leq t < b, \tag{45}$$

   and in particular $({}_{1,\beta,a} I f)(s) = \left({}_aI^{\beta} f\right)(s)$.

4. $$({}_{\alpha,\beta,t} D f)(s) = M_{\alpha,\beta,t}(s) = \sum_{k=0}^{\infty} \frac{\binom{\beta}{k} (s-t)^{k+\alpha-\beta-1}}{\Gamma(\alpha-\beta+k)} f^{(k)}(s), \quad s \in [a,b], \quad a \leq t < b, \tag{46}$$

   in particular $({}_{1,\beta,a} D f)(s) = \left({}_aD^{\beta} f\right)(s)$, and $({}_{\alpha,\beta,t} D f)(s) = \frac{d}{ds}({}_{\alpha,1-\beta,t} I f)(s)$.

Theorem 4 The Fractional Integration by Parts for Riemann–Liouville Fractional Derivatives

Let $\alpha > 0, 0 < \beta \le 1$. Then

1. *If $(b-x)^{\alpha-1} f(x) \in L_p(a,b)$ such that f is analytic on $[a,b]$ and $g \in AC[a,b]$. Then,*

$$ {}_aI^{\alpha}\left[f(x)\,{}_aD^{\beta}g(x)\right](t) = \left(I_{\alpha,1-\beta,t}f\right)(t^-)\,g(t) + \int_a^t \left(D_{\alpha,\beta,t}f\right)(s)\,g(s)\,ds, \qquad a < t \le b, \tag{47} $$

2. *If $(x-a)^{\alpha-1} f(x) \in L_p(a,b)$ such that f is analytic on $[a,b]$ and $g \in AC[a,b]$. Then,*

$$ I_b^{\alpha}\left[f(x)\,D_b^{\beta}g(x)\right](t) = \left({}_{\alpha,1-\beta,t}If\right)(t^+)\,g(t) + \int_t^b \left({}_{\alpha,\beta,t}Df\right)(s)\,g(s)\,ds, \qquad a \le t < b. \tag{48} $$

Proof

1. By Eq.(14) in Lemma2 with $n=1$ and the integration by parts (36), we have

$$ \begin{aligned} &{}_aI^{\alpha}\left[f(x)\,{}_aD^{\beta}g(x)\right](t) \\ &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\left[{}_a^CD^{\beta}g(s) - \frac{g(a)(s-a^{-\beta})}{\Gamma()1-\beta}\right]ds, \\ &= \left(I_{\alpha,1-\beta,t}f\right)(s)\,g(s)\Big|_a^t + \int_a^t g(s)\left(D_{\alpha,\beta,t}f\right)(s)\,ds, \\ &-\left(I_{\alpha,1-\beta,t}f\right)(a)\,g(a), \\ &= \left(I_{\alpha,1-\beta,t}f\right)(t^-)\,g(t) + \int_a^t \left(D_{\alpha,\beta,t}f\right)(s)\,g(s)\,ds. \end{aligned} \tag{49} $$

2. The proof is similar to the proof of the first part. We make use of Eq.(15) in Lemma2 with $n=1$ and the integration by parts (38).□

Remark 5

- Theorem4 above generalizes Lemma 6, Lemma 7. In fact,
  1. The integration by part formula (47) results in Lemma6 when $\alpha = 1$ and $t = b$.
  2. The integration by part formula (48) results in Lemma7 when $\alpha = 1$ and $t = a$.
- Notice that although f is analytic on $[a,b]$ in the case $\alpha \ne 1$, the boundary terms $\left(I_{\alpha,1-\beta,t}f\right)(t^-)$ and $\left({}_{\alpha,1-\beta,t}If\right)(t^+)$ are not necessarily zero. In the case $\alpha = 1$ the correspondent terms $\left(I_b^{1-\beta}f\right)(b^-)$ and $\left({}_aI^{1-\beta}f\right)(a^+)$ are also not necessary zero, despite the nonexistence of the singular kernel measured by $\alpha - 1$. This is due to that the function f is only requested to be in $L_p(a,b)$ and not necessarily continuous or analytic.

5. Numerical approximation of the new fractional integration by parts

5.1. Riemann–Liouville–Caputo fractional derivative

We present the numerical representation of the integration by part for the Riemann–Liouville fractional derivatives.

Theorem 5

Let f andg be two continuous functions such that ${}_0^{RL}\mathscr{D}_t^{\alpha}f(t)$ exists, then

$$ {}_0^{RL}I_t^{\alpha}\left\{\left[g(t)\;{}_0^{RL}\mathscr{D}_t^{\alpha}f(t)\right]\right\}_{t=t_n} \tag{50} $$

$$ \begin{aligned} &= \frac{1}{\Gamma(\alpha)}\int_0^{t_n} g(t)\;{}_0^{RL}\mathscr{D}_t^{\alpha}f(t)\,(t_n-t)^{\alpha-1}dt, \\ &= \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\left\{g(t_{j+1})\;{}_0^{RL}\mathscr{D}_t^{\alpha}f(t_{j+1}) + g(t_j)\;{}_0^{RL}\mathscr{D}_t^{\alpha}f(t_j)\right\}(t_n-t)^{\alpha-1}dt. \end{aligned} \tag{51} $$

Proof

Without loss of generality we evaluate

$$ {}_0^{RL}\mathscr{D}_t^{\alpha}f(t_{j+1}) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t f(\lambda)(t-\lambda)^{-\alpha}dt\Big|_{t_{j+1}}, = \frac{1}{\Gamma(1-\alpha)}\frac{F(t_{j+1})-F(t_j)}{\Delta t}. \tag{52} $$

$$ {}_0^{RL}\mathscr{D}_t^{\alpha}f(t_{j+1}) = \frac{1}{\Gamma(1-\alpha)}\frac{\sum_{\lambda=0}^{j+1}f(t_\lambda)\delta_{j,\lambda}^{\alpha} - \sum_{k=0}^{j}f(t_k)\delta_{j,k}^{\alpha}}{\Delta t}, \tag{53} $$

such that replacing the above in Eq.(51), we have the following

$$= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Gamma(\alpha)} \frac{1}{\Delta(t)} \tag{54}$$
$$\sum_{j=0}^{n} \left\{ \int_{t_j}^{t_{j+1}} g\left(t_{j+1}\right) \left(\sum_{\lambda=0}^{j+1} f\left(t_\lambda\right) \delta_{j,\lambda}^{\alpha} - \sum_{k=0}^{j} f\left(t_k\right) \right) \delta_{j,k}^{\alpha} + \right.$$
$$\left. + \, g\left(t_j\right) \left(\sum_{i=0}^{j} f\left(t_i\right) \delta_{j,i}^{\alpha} - \sum_{\lambda=1}^{j-1} f\left(t_\lambda\right) \delta_{\lambda,i}^{\alpha} \right) \right\} \left(t_n - t\right)^{\alpha-1} dt,$$
$$= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Gamma(\alpha)} \frac{1}{\Delta(t)} \tag{55}$$
$$\sum_{j=0}^{n} \left\{ g\left(t_{j+1}\right) \left[\sum_{\lambda=0}^{j+1} f\left(t_\lambda\right) \delta_{j,\lambda}^{\alpha} - \sum_{k=0}^{j} f\left(t_k\right) \delta_{j,k}^{\alpha} \right] + \right.$$
$$\left. + \left[g\left(t_j\right) \left(\sum_{i=0}^{j} f\left(t_i\right) \delta_{j,i}^{\alpha} - \sum_{\lambda=1}^{j-1} f\left(t_\lambda\right) \delta_{\lambda,i}^{\alpha} \right) \right] \right\} \delta_{n,j}^{\alpha}.$$

The approximation with Riemann–Liouville integral and Liouville–Caputo derivative is given by

$$ {}_0^{RL}I_t^{\alpha} \left\{ \left[{}_0^{C}\mathscr{D}_t^{\alpha} g\left(t\right) \right] \right\} \tag{56}$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^{t_n} g\left(t\right) \, {}_0^{C}\mathscr{D}_t^{\alpha} f\left(t\right) \left(t - \tau\right)^{\alpha-1} d\tau,$$

at a given $t = t_n$, we have the following

$$ {}_0^{RL}I_t^{\alpha} \left\{ \left[g\left(t\right) \, {}_0^{C}\mathscr{D}_t^{\alpha} f\left(t\right) \right] \right\}_{t=t_n} \tag{57}$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^{t_n} g\left(t\right) \, {}_0^{C}\mathscr{D}_t^{\alpha} f\left(t\right) \left(t_n - t\right)^{\alpha-1} dt,$$
$$= \frac{1}{2\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left\{ g\left(t_{j+1}\right) \, {}_0^{C}\mathscr{D}_t^{\alpha} f\left(t_{j+1}\right) \right. \tag{58}$$
$$\left. + \, g\left(t_j\right) \, {}_0^{C}\mathscr{D}_t^{\alpha} f\left(t_j\right) \right\} \left(t_n - t\right)^{\alpha-1} dt,$$
$$= \frac{1}{2\Gamma(\alpha)}$$
$$\sum_{j=0}^{n-1} \left[g\left(t_{j+1}\right) \, {}_0^{C}\mathscr{D}_t^{\alpha} f\left(t_{j+1}\right) + g\left(t_j\right) \, {}_0^{C}\mathscr{D}_t^{\alpha} f\left(t_j\right) \right]$$
$$\int_{t_j}^{t_{j+1}} \left(t_n - t\right)^{\alpha-1} dt,$$
$$= \frac{1}{2\Gamma(\alpha)} \sum_{j=0}^{n-1} \left[g\left(t_{j+1}\right) \, {}_0^{C}\mathscr{D}_t^{\alpha} f\left(t_{j+1}\right) + g\left(t_j\right) \, {}_0^{C}\mathscr{D}_t^{\alpha} f\left(t_j\right) \right] \delta_{n,j}^{\alpha}.$$

Without loss of generality we consider the following

$$ {}_0^{C}\mathscr{D}_t^{\alpha} f\left(t_{j+1}\right) = \frac{1}{\Gamma(1-\alpha)} \int_{t_j}^{t_{j+1}} \frac{d}{dt} f\left(t\right) f(t_{j+1} - t)^{-\alpha} dt \tag{59}$$
$$= \frac{1}{\Gamma(1-\alpha)} \sum_{\lambda=0}^{j-1} \frac{f^{\lambda+1} - f^{\lambda}}{\Delta t} \delta_{\lambda,j}^{\alpha}.$$

Putting all equations together, we have

$$= \frac{1}{2\Gamma(\alpha)} \tag{60}$$
$$\sum_{j=0}^{n-1} \left[g\left(t_{j+1}\right) \frac{1}{\Gamma(1-\alpha)} \sum_{\lambda=0}^{j+1} \frac{f^{\lambda+1} - f^{\lambda}}{\Delta t} \delta_{\lambda,j}^{\alpha} + g\left(t\right) \sum_{\lambda=0}^{j} \frac{f^{\lambda+1} - f^{\lambda}}{\Delta t} \delta_{\lambda,j}^{\alpha} \right] \delta_{\lambda,j}^{\alpha}.$$

□

5.2. Caputo–Fabrizio fractional derivative

We present in this section the numerical representation of the new integration by part. Let us start with Caputo–Fabrizio fractional derivative

$$ {}_0^{CF}I_t^{\alpha} \left\{ \left[{}_0^{CF}\mathscr{D}_t^{\alpha} f\left(t\right) \right] g\left(t\right) \right\} = \frac{1-\alpha}{M(\alpha)} \left({}_0^{CF}\mathscr{D}_t^{\alpha} f\left(t\right) \right) g\left(t\right) \tag{61}$$
$$+ \frac{\alpha}{M(\alpha)} \int_0^t g\left(\tau\right) \, {}_0^{CF}\mathscr{D}_\tau^{\alpha} f\left(\tau\right) d\tau,$$
$$= \frac{1-\alpha}{M(\alpha)} g\left(t\right) \left({}_0^{CF}\mathscr{D}_t^{\alpha} f\left(t\right) \right)$$
$$+ \frac{\alpha}{M(\alpha)} \left[G\left(t\right) \, {}_0^{CF}\mathscr{D}_\tau^{\alpha} - \int_0^t G\left(\tau\right) \left({}_0^{CF}\mathscr{D}_\tau^{\alpha} f\left(\tau\right) \right)' d\tau \right],$$

at the point t_n, we have

$$ {}_0^{CF}I_t^{\alpha} \left[\left({}_0^{CF}\mathscr{D}_\tau^{\alpha} f\left(t_n\right) \right) g\left(t_n\right) \right] \tag{62}$$
$$= \frac{1-\alpha}{M(\alpha)} g\left(t_n\right) \, {}_0^{CF}\mathscr{D}_\tau^{\alpha} f\left(t_n\right) +$$
$$+ \frac{\alpha}{M(\alpha)} \left\{ \left[G\left(t_n\right) \, {}_0^{CF}\mathscr{D}_\tau^{\alpha} f\left(t_n\right) \right] - \int_0^{t_n} G\left(\tau\right) \left({}_0^{CF}\mathscr{D}_\tau^{\alpha} f\left(\tau\right) \right)' d\tau \right\},$$

where,

$$ {}_0^{CF}\mathscr{D}_t^{\alpha} f\left(t_n\right) = \frac{M(\alpha)}{1-\alpha} \int_0^{t_n} \exp\left[-\frac{\alpha}{1-\alpha} \left(t_n - \tau\right) \right] f'\left(\tau\right) d\tau, \tag{63}$$
$$= \sum_{j=0}^{n} \frac{M(\alpha)}{1-\alpha} \frac{f^{n+1} - f^{n}}{\Delta t} \int_{t_n}^{t_{n+1}}$$
$$\exp\left[-\frac{\alpha}{1-\alpha} \left(t_n - \tau\right) \right] f'\left(\tau\right) d\tau,$$
$$= \sum_{j=0}^{n} \frac{M(\alpha)}{1-\alpha} \frac{f^{n+1} - f^{n}}{\Delta t} \delta_{n,j}^{\alpha},$$

and

$$\int_0^{t_n} G(t)\left[{}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\right]^2 dt$$
$$= \sum_{j=0}^{n} G(t_n)\frac{{}_0^{CF}\mathscr{D}_t^{\alpha} f(t_{n+1}) - {}_0^{CF}\mathscr{D}_t^{\alpha} f(t_n)}{\Delta t},$$
$$= \sum_{j=0}^{n} G(t_n)\left[{}_0^{CF}\mathscr{D}_t^{\alpha} f(t_{n+1}) - {}_0^{CF}\mathscr{D}_t^{\alpha} f(t_n)\right], \tag{64}$$

such that the numerical approximation should be

$${}_0^{CF}I_t^{\alpha}\left\{\left[g(t_n)\ {}_0^{CF}\mathscr{D}_t^{\alpha} f(t_n)\right]\right\}$$
$$= g(t)\sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\delta_{n,j}^{\alpha}\left[\frac{1-\alpha}{M(\alpha)}g(t_n) + G(t_n)\frac{\alpha}{M(\alpha)}\right] -$$
$$-\sum_{j=0}^{n} G(t_n)\left[{}_0^{CF}\mathscr{D}_t^{\alpha} f(t_{n+1}) - {}_0^{CF}\mathscr{D}_t^{\alpha} f(t_n)\right], \tag{65}$$

$${}_0^{CF}I_t^{\alpha}\left\{\left[g(t_n)\ {}_0^{CF}\mathscr{D}_t^{\alpha} f(t_n)\right]\right\}$$
$$= \sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\delta_{n,j}^{\alpha}\left[\frac{1-\alpha}{M(\alpha)}g(t_n) + G(t_n)\right] -$$
$$-\sum_{j=0}^{n} G(t_j)\left[\frac{M(\alpha)}{1-\alpha}\sum_{k=0}^{j+1}\frac{f^{k+1}-f^k}{\Delta t}\delta_{j,k}^{\alpha} - \frac{M(\alpha)}{1-\alpha}\sum_{l=0}^{j}\frac{f^{l+1}-f^l}{\Delta t}\delta_{j,l}^{\alpha}\right], \tag{66}$$

Theorem 6

Let $f(t)$ and $g(t)$ two continuous functions such that ${}_0^{CF}I_t^{\alpha}\left[g(t)\ {}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\right]$ exists, then the numerical approximation is given as

$${}_0^{CF}I_t^{\alpha}\left\{\left[g(t_n)\ {}_0^{CF}\mathscr{D}_t^{\alpha} f(t_n)\right]\right\}$$
$$= \sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\delta_{n,j}^{\alpha}\left[\frac{1-\alpha}{M(\alpha)}g(t_n) + G(t_n)\right] -$$
$$-\sum_{j=0}^{n} G(t_j)\left\{\frac{M(\alpha)}{1-\alpha}\left[\sum_{k=0}^{j+1}\frac{f^{k+1}-f^k}{\Delta t}\delta_{j,k}^{\alpha} - \sum_{l=0}^{j}\frac{f^{l+1}-f^l}{\Delta t}\delta_{j,l}^{\alpha}\right]\right\}$$
$$+ R_{n,j}^{\alpha}, \tag{67}$$

where $R_{n,j}^{\alpha}$ is

$$\left|R_{n,j}^{\alpha}\right| < \frac{M(\alpha)}{\alpha}\left|1+G(t_n)\right|\left|\frac{d^2 f}{dt^2}\right|\left|1-\exp\left[-\frac{\alpha}{1-\alpha}\right]\right.$$
$$+ t_n\left|\frac{d^2}{dt^2}\ {}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\right|\left|G(t)\right|. \tag{68}$$

Proof

Let $f(t)$ and $g(t)$ be continuous in $[a,b]$ such that $G' = g(t)$ exists, then

$${}_0^{CF}I_t^{\alpha}\left[{}_0^{CF}\mathscr{D}_t^{\alpha} f(t_n)\,g(t_n)\right] = \frac{1-\alpha}{M(\alpha)}g(t_n)$$
$$+ \frac{\alpha}{M(\alpha)}\int_0^{t_n} g(t)\ {}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\,dt,$$
$$= G(t_n)\ {}_0^{CF}\mathscr{D}_t^{\alpha} f(t_n) - \int_0^{t_n} G(t)\left({}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\right)' dt, \tag{69}$$

$${}_0^{CF}\mathscr{D}_t^{\alpha} f(t_n)$$
$$= \frac{M(\alpha)}{1-\alpha}\int_0^{t_n}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-\tau)\right]\frac{d}{dt}f(t)\,dt, \tag{70}$$

$$= \frac{M(\alpha)}{1-\alpha}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\frac{f^{j+1}-f^j}{\Delta t}\delta_{n,j}^{\alpha}$$
$$+ \frac{M(\alpha)}{1-\alpha}\int_0^{t_n}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-\tau)\right]\frac{d^2}{dt^2}f(t)\,dt, \tag{71}$$

$$= \frac{M(\alpha)}{1-\alpha}\sum_{j=0}^{n}\int_{t_j}^{t_{j+1}}\frac{f^{j+1}-f^j}{\Delta t}\delta_{n,j}^{\alpha} + R_{n,j}^{\alpha}, \tag{72}$$

where,

$$\left|R_{n,j}^{\alpha}\right| = \frac{M(\alpha)}{1-\alpha}\int_0^{t_n}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-t)\right]\frac{d^2}{dt^2}f(t)\,dt, \tag{73}$$

$$G(t_n)\ {}_0^{CF}\mathscr{D}_t^{\alpha} f(t_n) = G(t_n)\sum_{j=0}^{n}\frac{f^{j+1}-f^j}{\Delta t}\delta_{n,j}^{\alpha} + R_{n,j}^{\alpha,1}, \tag{74}$$

where,

$$R_{n,j}^{\alpha,1} = G(t_n)\frac{M(\alpha)}{1-\alpha}\int_0^{t_n}\exp\left[-\frac{\alpha}{1-\alpha}(t_n-t)\right]\frac{d^2}{dt^2}f(t)\,dt, \tag{75}$$

$$\int_0^{t_n}\left[{}_0^{CF}\mathscr{D}_t^{\alpha} f(t_n)\right]'$$
$$= \sum_{j=0}^{n} G(t_j)\left[\frac{{}_0^{CF}\mathscr{D}_t^{\alpha} f(t_{j+1}) - {}_0^{CF}\mathscr{D}_t^{\alpha} f(t_j)}{\Delta}dt\right] + R_{n,j}^{\alpha,2}, \tag{76}$$

where,

$$R_{n,j}^{\alpha,2} = \int_0^{t_n}\frac{d^2}{dt^2}f(t)\ {}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\,G(t)\,dt, \tag{77}$$

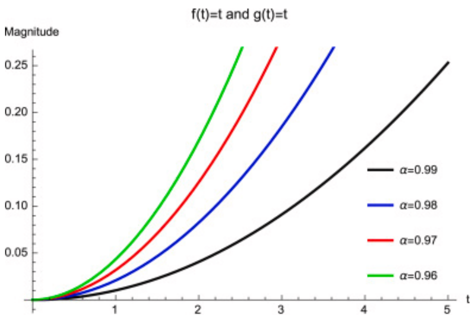
thus, from the above equation

$$R_n^{\alpha} = \int_0^{t_n} \frac{d}{dt}\, {}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\, G(t)\, dt \tag{78}$$
$$+ G(t_n) \frac{M(\alpha)}{1-\alpha} \int_0^{t_n} \exp\left[-\frac{\alpha}{1-\alpha}(t_n - t)\right] dt +$$
$$+ \frac{M(\alpha)}{1-\alpha} \int_0^{t_n} \exp\left[-\frac{\alpha}{1-\alpha}(t_n - t)\right] \frac{d^2}{dt^2} f(t)\, dt,$$
$$= \frac{M(\alpha)}{1-\alpha}(1 + G(t_n)) \int_0^{t_n} \exp\left[-\frac{\alpha}{1-\alpha}(t_n - t)\right] \frac{d^2}{dt^2} f(t)\, dt \tag{79}$$
$$+ \int_0^{t_n} \frac{d^2}{dt^2}\, {}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\, G(t)\, dt,$$
$$(R_n^{\alpha}) \leq \frac{M(\alpha)}{\alpha} |1 + G(t_n)| \left|\frac{d^2}{dt^2} f(t)\right| \left|1 - \exp\left[-\frac{\alpha}{1-\alpha} t_n\right]\right| \tag{80}$$
$$+ t_n \left|\frac{d^2}{dt^2}\, {}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\right| |G(t)|. \quad \square$$

Example 1

Let $f(t) = t$ and $g(t) = t$, then using the formula of integration by part, we obtain the following

$${}_0^{CF}I_t^{\alpha}\left[g(t)\, {}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\right] = \frac{1-\alpha}{M(\alpha)} g(t)\, {}_0^{CF}\mathscr{D}_t^{\alpha} f(t) \tag{81}$$
$$+ \frac{\alpha}{M(\alpha)} \int_0^t g(\tau)\, {}_0^{CF}\mathscr{D}_\tau^{\alpha} f(\tau)\, d\tau,$$
$$= \frac{1-\alpha}{M(\alpha)} t \frac{\alpha-1}{\alpha} \left[\exp\left[-\frac{\alpha}{1-\alpha} t\right] - 1\right] \tag{82}$$
$$+$$
$$\frac{\alpha}{M(\alpha)} \left\{ \frac{\alpha-1}{\alpha} \left[-\frac{t^2}{\alpha} + (\alpha - 1) \left(\alpha - 1 + \exp\left[-\frac{\alpha}{1-\alpha} t\right] \right) (1 + (t-1)\alpha) \right] \right\}.$$



Download : Download high-res image (275KB)

Download : Download full-size image

Fig. 1. Numerical simulation for Eq.(82) for several values of α, arbitrarily chosen.

Fig. 1 shows numerical simulations for Eq.(82) with $\alpha = 0.99$, $\alpha = 0.98$, $\alpha = 0.97$ and $\alpha = 0.96$, arbitrarily chosen.

Example 2

Let $f(t) = t$ and $g(t) = t^n$ for $\forall \quad n \geq 1$, then $G(t) = \frac{t^{n+1}}{n+1}$.

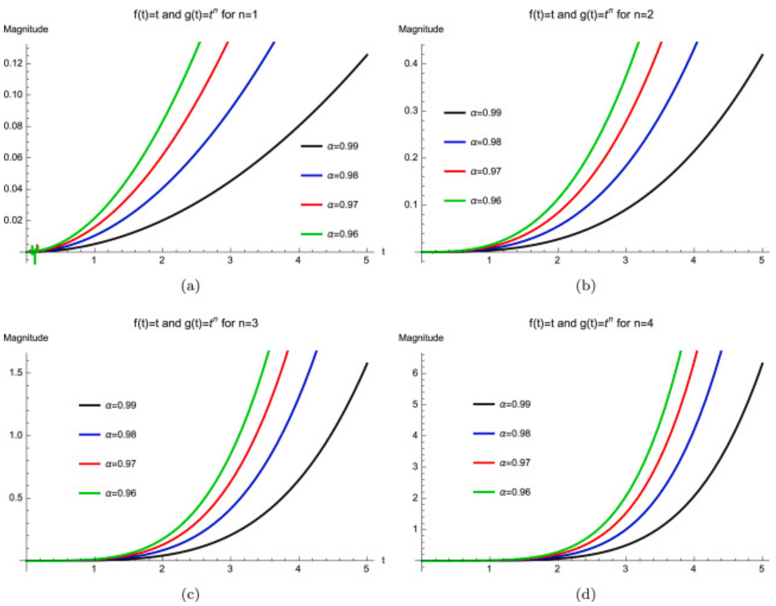
$${}_0^{CF}I_t^{\alpha}\left[g(t)\, {}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\right] \tag{83}$$
$$= \frac{1-\alpha}{M(\alpha)} t^n \frac{\alpha-1}{\alpha} \left[\exp\left[-\frac{\alpha}{1-\alpha} t\right] - 1\right] +$$
$$+ \frac{\alpha-1}{M(\alpha)} t^n \left[-\frac{t}{n+1} + t \left(-\frac{t\alpha}{\alpha-1}\right)^{-1-n} \left(\Gamma(n+1) - \Gamma\left(1 + n - \frac{\alpha}{1-\alpha} t\right) \right) \right].$$

Figs. 2(a)–2(d) show numerical simulations for Eq.(83) with $\alpha = 0.99$, $\alpha = 0.98$, $\alpha = 0.97$ and $\alpha = 0.96$, arbitrarily chosen.

Example 3

Let $f(t) = \sin(t)$ and $g(t) = t^n$ for $\forall \quad n \geq 1$, then using the formula of integration by part, we obtain the following

$${}_0^{CF}I_t^{\alpha}\left[g(t)\, {}_0^{CF}\mathscr{D}_t^{\alpha} f(t)\right] \tag{84}$$
$$= \frac{(1-\alpha)(\alpha-1)}{M(\alpha)} \frac{t^n}{2\alpha^2 - 2\alpha + 1} \left\{ \exp\left[\frac{\alpha}{\alpha-1} t\right] \alpha - \alpha \cos(t) + (\alpha - 1) \sin(t) \right\} +$$
$$+$$
$$\frac{\alpha}{M(\alpha)} \frac{1}{1 + 2(\alpha-1)\alpha} t^n \, (\alpha$$
$$-1) \left\{ -(\alpha t) \text{ Hypergeometric PFQ} \left[\left\{\frac{1}{2}, \frac{n}{2}\right\}, \left\{\frac{1}{2}, \frac{3}{2} + \frac{n}{2}\right\}, -\frac{t^2}{4} \right] \right\}$$
$$+$$
$$+ (\alpha - 1) \left[\left(-\frac{\alpha}{\alpha-1} t\right)^{-n} \left(-n\Gamma(n) + \Gamma\left(1 + n, -\frac{\alpha}{\alpha-1} t\right) \right) \right] +$$
$$+ \frac{t^2}{2+n} \text{Hypergeometric PFQ} \left[\left\{1 + \frac{n}{2}\right\}, \left\{\frac{3}{2}, 2 + \frac{n}{2}\right\}, -\frac{t^2}{4} \right].$$

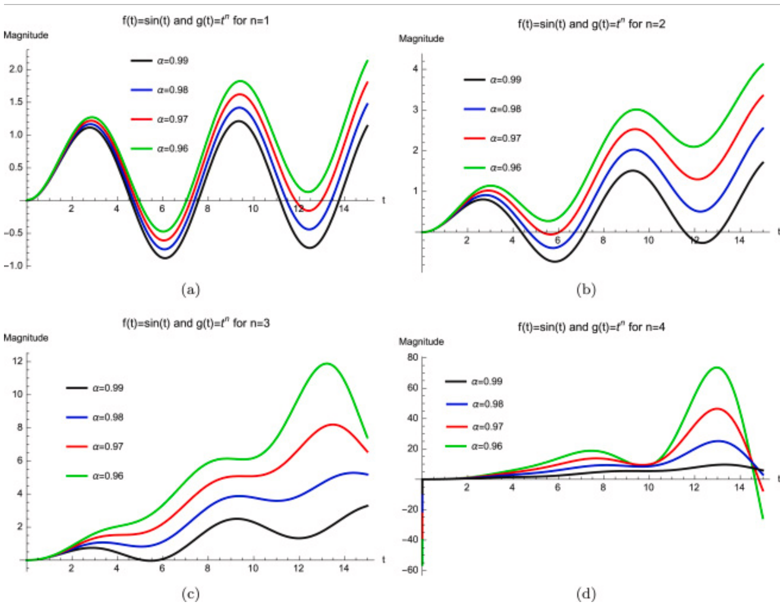


Download : Download high-res image (670KB)

Download : Download full-size image

Fig. 2. Numerical simulation for Eq.(83) for several values of α, arbitrarily chosen.

Figs. 3(a)–3(d) show numerical simulations for Eq.(84) with $\alpha = 0.99$, $\alpha = 0.98$, $\alpha = 0.97$ and $\alpha = 0.96$, arbitrarily chosen.



Download : Download high-res image (810KB)

Download : Download full-size image

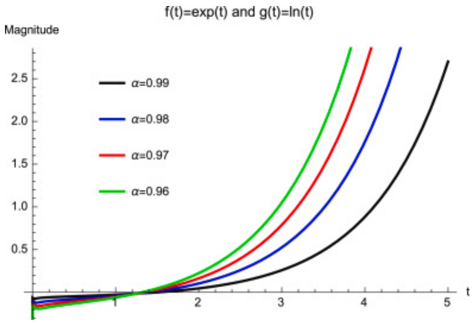
Fig. 3. Numerical simulation for Eq.(84) for several values of α, arbitrarily chosen.

Example 4

Let $f(t) = \exp(t)$ and $g(t) = \ln(t)$, then using the formula of integration by part, we obtain the following

$$
\begin{aligned}
&{}_0^{CF}I_t^{\alpha}\left[g(t)\ {}_0^{CF}\mathscr{D}_t^{\alpha}f(t)\right] \\
&= \tfrac{1-\alpha}{M(\alpha)}\ln(t)\exp(t)\exp\left[\tfrac{t}{\alpha-1}\right](\alpha-1)+ \\
&\quad + \tfrac{\alpha}{M(\alpha)}(\alpha-1)\{-\text{Euler Gamma}+\text{exp Integral}[1,-t] \\
&-\ln(-t)+\ln(t)-\exp(t)\cdot\ln(t)\}+ \\
&\quad + \\
&\tfrac{\alpha-1}{\alpha}\left\{\text{Euler Gamma}+\text{exp Integral}\left[1,-\tfrac{t\alpha}{\alpha-1}\right]+\right. \\
&\left.\exp\left[\tfrac{\alpha}{\alpha-1}t\right]\ln(t)+\ln\left[-\tfrac{\alpha}{\alpha-1}t\right]\right\}.
\end{aligned}
\tag{85}
$$

Fig.4 shows numerical simulations for Eq.(85) with $\alpha=0.99$, $\alpha=0.98$, $\alpha=0.97$ and $\alpha=0.96$, arbitrarily chosen.



Download : Download high-res image (280KB)

Download : Download full-size image

Fig. 4. Numerical simulation for Eq.(85) for several values of α, arbitrarily chosen.

Example 5

Let $f(t)=\tan(t)$ and $g(t)=t$. The exact solution of the following formula ${}_0^{CF}I_t^{\alpha}\left[\ {}_0^{CF}\mathscr{D}_t^{\alpha}f(t)\cdot g(t)\right]$ cannot be obtained, then we propose the following numerical approximation

$$
\begin{aligned}
&{}_0^{CF}I_t^{\alpha}\left[g(t_n)\ {}_0^{CF}\mathscr{D}_t^{\alpha}f(t_n)\right] = \tfrac{1-\alpha}{M(\alpha)}t_n \\
&\times\tfrac{1}{\alpha(2+i)-2}\left\{\exp\left[-\tfrac{\alpha}{1-\alpha}t_n\right]\alpha\cdot\right. \\
&\cdot\text{Hypergeometric}_2F_1\left[1,\tfrac{2-(2+i)\alpha}{2-2\alpha},\tfrac{4-(4+i)\alpha}{2-2\alpha},-1\right]+ \\
&+\exp[2it_n]\,\alpha\cdot\text{Hypergeometric}_2F_1\left[1,\tfrac{i\alpha}{2(\alpha-1)},\tfrac{2-(2+i)\alpha}{2-2\alpha},-1\right] \\
&+ \\
&(-2+(2+i)\alpha)\Big(i+ \\
&\exp\left[\tfrac{\alpha}{\alpha-1}t_n\right]\text{Hypergeometric}_2F_1\left[1,\tfrac{i\alpha}{2(\alpha-1)},\tfrac{2-(2+i)\alpha}{2-2\alpha},-1\right]--i \\
&\quad\left.\text{Hypergeometric}_2F_1\left[1,\tfrac{i\alpha}{2(\alpha-1)},\tfrac{2-(2+i)\alpha}{2-2\alpha},-\exp[2it_n]\right]+\tan(t_n)\Big)\right\}
\end{aligned}
\tag{86}
$$

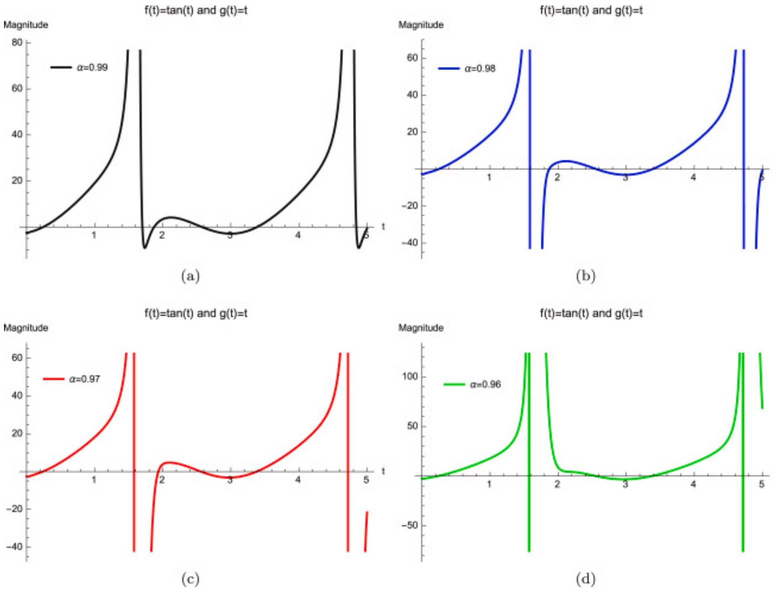
$$
\begin{aligned}
&+ \\
&\tfrac{\alpha}{M(\alpha)} \\
&\textstyle\sum_{j=0}^{n}\left\{\tfrac{1}{(2+i)\alpha-2}\left[\exp\left[-\tfrac{\alpha}{1-\alpha}t_j\right]\alpha\text{Hypergeometric}_2F_1\left[1,\tfrac{2-(2+i)\alpha}{2-2\alpha},\tfrac{4-(4+i)\alpha}{2-2\alpha},-1\right]++i^{j+1}\alpha\right.\right. \\
&\quad\left.\text{Hypergeometric}_2F_1\left[1,\tfrac{2-(2+i)\alpha}{2-2\alpha},\tfrac{4-(4+i)\alpha}{2-2\alpha},-\exp[2it_j]\right]\right] \\
&+ \\
&+((2+i)\alpha-2)\Big[i \\
&\exp\left[-\tfrac{\alpha}{1-\alpha}t_j\right]\text{Hypergeometric}_2F_1\left[1,\tfrac{i\alpha}{2(\alpha-1)},\tfrac{2-(2+i)\alpha}{2-2\alpha},-1\right]--i \\
&\quad\left.\text{Hypergeometric}_2F_1\left[1,\tfrac{i\alpha}{2(\alpha-1)},\tfrac{2-(2+i)\alpha}{2-2\alpha},1-\exp[2it_j]\right]\Big]+\tan(t_j)\right\} \\
&.
\end{aligned}
\tag{87}
$$

Figs. 5(a)–5(d) and 6 show numerical simulations for Eq.(87) with $\alpha=0.99$, $\alpha=0.98$, $\alpha=0.97$ and $\alpha=0.96$, arbitrarily chosen.

5.3. Atangana–Baleanu fractional derivative

The integration by parts for the Atangana–Baleanu derivative in the Riemann–Liouville sense (ABR) is given by [33]

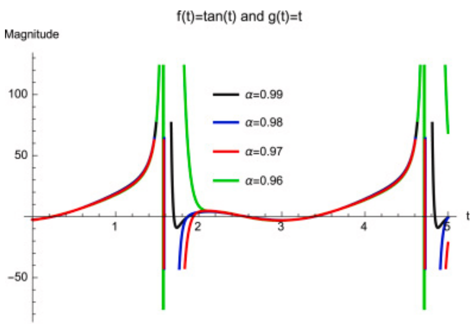
$$
\begin{aligned}
&{}_0^{AB}I_t^{\alpha}\left[g(t)\ {}_0^{ABR}\mathscr{D}_t^{\alpha}f(t)\right] = \tfrac{1-\alpha}{AB(\alpha)}g(t)\ {}_0^{ABR}\mathscr{D}_t^{\alpha}f(t) \\
&+\tfrac{\alpha}{\Gamma(\alpha)AB(\alpha)}\int_0^t g(\tau)\ {}_0^{ABR}\mathscr{D}_\tau^{\alpha}f(\tau)(t-\tau)^{-\alpha}d\tau.
\end{aligned}
\tag{88}
$$



Download : Download high-res image (463KB)

Download : Download full-size image

Fig. 5. Numerical simulation for Eq.(87) for several values of α, arbitrarily chosen.



Download : Download high-res image (250KB)

Download : Download full-size image

Fig. 6. Numerical simulation for Eq.(87) for several values of α, arbitrarily chosen.

So that at $t=t_n$, we have

$$\begin{aligned} &{}_0^{AB}I_{t_n}^{\alpha}\left[g(t_n)\;{}_0^{ABR}\mathscr{D}_{t_n}^{\alpha}f(t_n)\right] \\ &= \tfrac{1-\alpha}{AB(\alpha)}g(t_n)\;{}_0^{ABR}\mathscr{D}_{t_n}^{\alpha}f(t_n) \\ &+ \tfrac{\alpha}{\Gamma(\alpha)AB(\alpha)}\int_0^{t_n}(t_n-\tau)^{-\alpha}g(\tau)\;{}_0^{ABR}\mathscr{D}_{\tau}^{\alpha}f(\tau)\,d\tau, \end{aligned} \tag{89}$$

here we present the discretization for the ABR derivative

$$\begin{aligned} &{}_0^{ABR}\mathscr{D}_t^{\alpha}f(t) \\ &= \tfrac{AB(\alpha)}{\Delta t(1-\alpha)}\left(\sum_{j=0}^{n}f(t_j)\,\delta_{n,j}^{\alpha}-\sum_{\lambda=0}^{n-1}f(t_\lambda)\,\delta_{n,\lambda}^{\alpha}\right). \end{aligned} \tag{90}$$

Therefore the numerical approximation of integration by parts of ABR and AB integral is given by

$$\begin{aligned} &{}_0^{AB}I_{t_n}^{\alpha}\left\{\left[g(t)\;{}_0^{ABR}\mathscr{D}_t^{\alpha}f(t)\right]\right\} \\ &= \tfrac{1-\alpha}{AB(\alpha)}g(t_n)\,\tfrac{AB(\alpha)}{\Delta t(1-\alpha)}\left(\sum_{j=0}^{n}f(t_j)\,\delta_{n,j}^{\alpha}-\sum_{\lambda=0}^{n}f(t_\lambda)\,\delta_{n,\lambda}^{\alpha}\right)+ \\ &+ \\ &\tfrac{\alpha}{AB(\alpha)\Gamma(\alpha)}\,\tfrac{1}{\Gamma(\alpha)}\,\tfrac{1}{\Delta t} \\ &\sum_{j=0}^{n}\left\{\left[g(t_{j+1})\sum_{\lambda=1}^{j+1}f(t_\lambda)\,\delta_{n,\lambda}^{\alpha}-\sum_{n=0}^{j}f(t_n)\,\delta_{j,n}^{\alpha}\right]+\right. \\ &\quad\left.+\;g(t_j)\left[\sum_{i=1}^{j}f(t_i)\,\delta(i,j)^{\alpha}-\sum_{i=1}^{j-1}f(t_i)\,\delta(j,i)^{\alpha}\right]\right\}\delta(n,j)^{\alpha}. \end{aligned} \tag{91}$$

The above is obtained following the decomposition for the RL integration by parts. Now, Following the same routine to obtain the integration by parts for ABC derivative and AB integral, we have

$$
\begin{aligned}
&{}_{0}^{AB}I_{t_n}^{\alpha}\left\{\left[g(t)\ {}_{0}^{ABC}\mathscr{D}_t^{\alpha}f(t)\right]\right\} \\
&=\frac{1-\alpha}{AB(\alpha)}g(t_n)\frac{AB(\alpha)}{1-\alpha}\frac{1}{\Delta t}\sum_{j=0}^{n}\frac{f(t_{j+1}-f(t_j))}{\Delta t}\delta_{n,j}^{\alpha}+ \\
&\quad+\frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\sum_{j=0}^{n}g(t_j)\left(\sum_{\lambda=0}^{n}\frac{f(t_{\lambda+1}-f(t_\lambda))}{\Delta t}\delta_{\lambda,j}^{\alpha}\right)\delta_{j,n}^{\alpha}.
\end{aligned}
\tag{92}
$$

6. Conclusions

The concept of integration by part is as old as the concept of calculus and has been used in all field of science, more particularly in applied mathematics and applied sciences. This concept although very basic as is being taught in high school has been found to be very useful and applicable, no wonder why several researchers are interested on its use and application. We can state precisely that, the concept if employed in variational calculus especially when deriving the Euler–Lagrange equation. The concept raised from the classical calculus was adopted or extended in the case of calculus with non-integer order. Nevertheless, in the case of fractional calculus, the evaluation is done using the classical integral with a fractional derivative. While this has been proven to be very useful in the papers already published in the literature, one will ask the question why use the classical integral in the case of non-conventional calculus. To answer that question, in this work, we have replace the classical integral to fractional integral and provide analytically results and also we provide the numerical approximation of such formula for each differential and integral operators including Riemann–Liouville–Caputo, Caputo–Fabrizio and Atangana–Baleanu fractional operators. We provided examples without loss of generality for the case of Caputo–Fabrizio considering several values of fractional order α. The numerical results proves that the fractional operators can capture more complexities than classical operators. It is worth noting that at limit case when $\alpha = 1$, we recover classical differential and integral operators. There is no doubt that this new approach will open doors and new insight within the field of fractional differentiation and integration.

Acknowledgements

The first author would like to thank Prince Sultan University, Saudi Arabia for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17. José Francisco Gómez Aguilar acknowledges the support provided by CONACyT, Mexico: cátedras CONACyT para jóvenes investigadores 2014 and SNI-CONACyT, Mexico .

Recommended articles

References

[1] Owolabi K.M.
Numerical approach to fractional blow-up equations with Atangana–Baleanu derivative in Riemann–Liouville sense
Math. Model. Nat. Phenom., 13 (1) (2018), pp. 1-7
Google Scholar ↗

[2] Kumar D., Singh J., Baleanu D., Rathore S.
Analysis of a fractional model of the Ambartsumian equation
Eur. Phys. J. Plus, 133 (7) (2018), pp. 1-16
Google Scholar ↗

[3] Singh J., Secer A., Swroop R., Kumar D.
A reliable analytical approach for a fractional model of advection–dispersion equation
Nonlinear Eng., 8 (1) (2019), pp. 107-116
View article ↗ CrossRef ↗ View in Scopus ↗ Google Scholar ↗

[4] Singh J., Kumar D., Baleanu D.
On the analysis of fractional diabetes model with exponential law
Adv. Difference Equ., 2018 (1) (2018), pp. 1-23
View article ↗ CrossRef ↗ Google Scholar ↗

[5] Owolabi K.M.
Modelling and simulation of a dynamical system with the Atangana–Baleanu fractional derivative
Eur. Phys. J. Plus, 133 (1) (2018), pp. 1-15
Google Scholar ↗

[6] Singh J., Kumar D., Baleanu D.
New aspects of fractional Biswas-Milovic model with Mittag-Leffler law
Math. Model. Nat. Phenom., 14 (3) (2019), pp. 1-13
Google Scholar ↗

[7] Kumar D., Tchier F., Singh J., Baleanu D.

An efficient computational technique for fractal vehicular traffic flow
Entropy, 20 (4) (2018), pp. 1-25
Google Scholar ↗

[8] Owolabi K.M.
Analysis and numerical simulation of multicomponent system with Atangana–Baleanu fractional derivative
Chaos Solitons Fractals, 115 (2018), pp. 127-134
View PDF View article View in Scopus ↗ Google Scholar ↗

[9] Singh J., Kumar D., Baleanu D., Rathore S.
On the local fractional wave equation in fractal strings
Math. Methods Appl. Sci., 1 (2019), pp. 1-13
Google Scholar ↗

[10] Owolabi K.M.
Numerical patterns in system of integer and non-integer order derivatives
Chaos Solitons Fractals, 115 (2018), pp. 143-153
View PDF View article View in Scopus ↗ Google Scholar ↗

[11] Owolabi K.M.
Numerical patterns in reaction–diffusion system with the Caputo and Atangana–Baleanu fractional derivatives
Chaos Solitons Fractals, 115 (2018), pp. 160-169
View PDF View article View in Scopus ↗ Google Scholar ↗

[12] Gul T., Khan M.A., Khan A., Shuaib M.
Fractional-order three-dimensional thin-film nanofluid flow on an inclined rotating disk
Eur. Phys. J. Plus, 133 (12) (2018), pp. 1-15
View in Scopus ↗ Google Scholar ↗

[13] Khan M.A., Ullah S., Okosun K.O., Shah K.
A fractional order pine wilt disease model with Caputo-Fabrizio derivative
Adv. Difference Equ., 2018 (1) (2018), pp. 1-14
Google Scholar ↗

[14] Khan M.A., Ullah S., Farooq M.
A new fractional model for tuberculosis with relapse via Atangana–Baleanu derivative
Chaos Solitons Fractals, 116 (2018), pp. 227-238
View article ↗ CrossRef ↗ Google Scholar ↗

[15] Singh J.
A new analysis for fractional rumor spreading dynamical model in a social network with Mittag-Leffler law
Chaos, 29 (1) (2019), pp. 1-9
View PDF View article Google Scholar ↗

[16] Singh J., Kumar D., Hammouch Z., Atangana A.
A fractional epidemiological model for computer viruses pertaining to a new fractional derivative
Appl. Math. Comput., 316 (2018), pp. 504-515
View PDF View article View in Scopus ↗ Google Scholar ↗

[17] Singh J., Kumar D., Baleanu D., Rathore S.
An efficient numerical algorithm for the fractional Drinfeld–Sokolov–Wilson equation
Appl. Math. Comput., 335 (2018), pp. 12-24
View PDF View article Google Scholar ↗

[18] Ullah S., Khan M.A., Farooq M.
A fractional model for the dynamics of TB virus
Chaos Solitons Fractals, 116 (2018), pp. 63-71
View PDF View article View in Scopus ↗ Google Scholar ↗

[19] Singh J., Kumar D., Qurashi M.A., Baleanu D.
A novel numerical approach for a nonlinear fractional dynamical model of interpersonal and romantic relationships
Entropy, 19 (7) (2017), pp. 1-7
Google Scholar ↗

[20] Ullah S., Khan M.A., Farooq M.
Modeling and analysis of the fractional HBV model with Atangana–Baleanu derivative
Eur. Phys. J. Plus, 133 (8) (2018), pp. 1-13
View PDF ↗ CrossRef ↗ View in Scopus ↗ Google Scholar ↗

[21] Kumar D., Singh J., Baleanu D.
Analysis of regularized long-wave equation associated with a new fractional operator with Mittag-Leffler type kernel
Physica A, 492 (2018), pp. 155-167
View PDF View article Google Scholar ↗

[22] Kumar D., Singh J., Baleanu D.
A new analysis of the Fornberg–Whitham equation pertaining to a fractional derivative with Mittag-Leffler-type kernel
Eur. Phys. J. Plus, 133 (2) (2018), pp. 1-17
Google Scholar ↗

[23] Ullah S., Khan M.A., Farooq M.
A new fractional model for the dynamics of the hepatitis B virus using the Caputo-Fabrizio derivative
Eur. Phys. J. Plus, 133 (6) (2018), pp. 1-17
View in Scopus ↗ Google Scholar ↗

[24] Samko G., Kilbas A.A., Marichev S.
Fractional Integrals and Derivatives: Theory and Applications
Gordon and Breach, Yverdon (1993)
Google Scholar ↗

[25] Podlubny I.
Fractional Differential Equations
Academic Press, San Diego CA (1999)
Google Scholar ↗

[26] Kilbas A.A., Srivastava M.H., Trujillo J.J.
Theory and Application of Fractional Differential Equations, North Holland Mathematics Studies, vol. 204 (2006)
Google Scholar ↗

[27] Caputo M., M. Fabrizio
A new definition of fractional derivative without singular kernel
Progr. Fract. Differ. Appl, 1 (2015), pp. 1-13
View in Scopus ↗ Google Scholar ↗

[28] Osler T.J.
Leibniz rule for fractional derivatives generalised and an application to infinite series
SIAM J. Appl. Math., 18 (1970), pp. 658-674
View article ↗ CrossRef ↗ Google Scholar ↗

[29] Osler T.J.
The fractional derivative of a composite function
SIAM J. Math. Anal., 1 (1970), pp. 288-293
View article ↗ CrossRef ↗ View in Scopus ↗ Google Scholar ↗

[30] Osler T.J.
Fractional derivatives and Leibniz rule
Amer. Math. Monthly, 78 (1971), pp. 645-649
View article ↗ CrossRef ↗ View in Scopus ↗ Google Scholar ↗

[31] Osler T.J.
The integral analog of the Leibniz rule
Math. Comput., 26 (1972), pp. 903-915
Google Scholar ↗

[32] Jarad F., Abdeljawad T.
Variational principles in the frame of certain generalized fractional derivatives
Discrete Contin. Dyn. Syst. Ser. S (2019)
preprint

Google Scholar ↗

[33] Atangana A., Baleanu D.
New fractional derivative with non-local and non-singular kernal
Therm. Sci., 20 (2) (2016), pp. 757-763
View in Scopus ↗ Google Scholar ↗