$$E[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$$

$\frac{\delta E}{\delta u}$ defined by:

$$\underbrace{\frac{d}{d\varepsilon} E[u+\varepsilon v]\Big|_{\varepsilon=0}}_{D_v E[u]} = \int_\Omega \frac{\delta E}{\delta u} v \, dx$$

Compute:

$$E[u+\varepsilon v] = \frac{1}{2}\int |\nabla u + \varepsilon \nabla v|^2$$

$$\therefore \frac{d}{d\varepsilon} E[u+\varepsilon v] = \int (\nabla u + \varepsilon \nabla v) \cdot \nabla v$$

$$\therefore \frac{d}{d\varepsilon} E[u+\varepsilon v]\Big|_{\varepsilon=0} = \int_\Omega \nabla u \cdot \nabla v \, dx$$

$$= -\int_\Omega \Delta u \, v \, dx + \int_{\partial\Omega} (\nabla u \cdot \hat{n}) v \, ds$$

$$\therefore \frac{\delta E}{\delta u} = -\Delta u \quad \text{if } \underbrace{\nabla u \cdot \hat{n} \equiv 0 \text{ on } \partial\Omega}_{\text{NATURAL B.Cs}}$$

So, in L^2,

$D_v E[u] = \langle -\Delta u, v \rangle_{L^2}$ if $\nabla u \cdot \hat{n} \equiv 0$ on $\partial\Omega$,

so $\partial_t u = \Delta u$.

In H_0^1,

$D_v E[u] = \cdots = \int_{\Omega} \nabla u \cdot \nabla v = \langle u, v \rangle_{H_0^1}$,

so no B.C.s,

but H_0^1 is not H^1, H_0^1 is the closure of C_0^∞ in H^1, &

$C_0^\infty = \{ f \in C^\infty \text{ s.t. } f \equiv 0 \text{ on } \partial\Omega \}$.