

# Untangling Knots Through Curve Repulsion



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## Abstract

Curves are one of the fundamental objects in geometry and engineering, yet most analysis of curves often disregard their physical characteristics such as their spatial volume or uncrossability. One common situation that such physical characteristics become significant is when one attempts to untangle a knot. An approach to achieve this is to assign an “energy” to a curve such that this energy would increase when two points on “different sides” of a curve are closer, then one continuously deforms the curve to reduce this energy, the expectation being that the curve that achieves minimal energy must be the untangled knot. This dissertation explores numerical methods of achieving this.

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## 1 Introduction

Shape optimisation is an important idea in engineering, relevant in anywhere from aircraft designs to packaging ramen noodle. One of the simplest and most fundamental shapes to consider is a curve. While curves are simple objects in theory, they prove to be quite difficult to analyse in practice with realistic physics. Even in absence of other objects, one must consider resilience to bending, stretching, and especially, impenetrability against itself. With these physical factors in mind, untangling a knot like the one shown (Figure 1) becomes a very complicated process, especially in a computer simulation. In this dissertation, we explore numerical methods to achieve this.

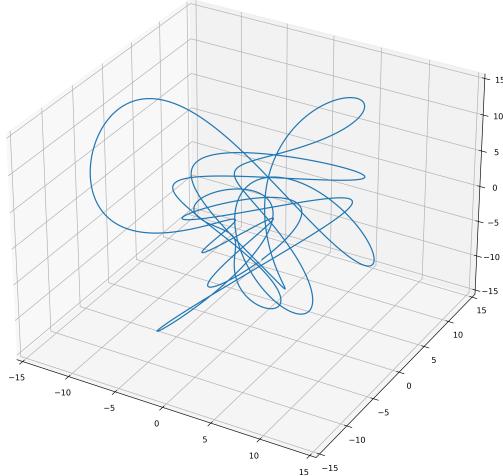


Figure 1: A complicated knot in  $\mathbb{R}^3$

The main idea is to *assign energy that penalises “physical entanglement”*. Given a parameterised curve  $\gamma : M \rightarrow \mathbb{R}^3$  ( $M$  being the domain of the parameter, often an interval), one defines some *curve energy*  $\mathcal{E}$  of the form:

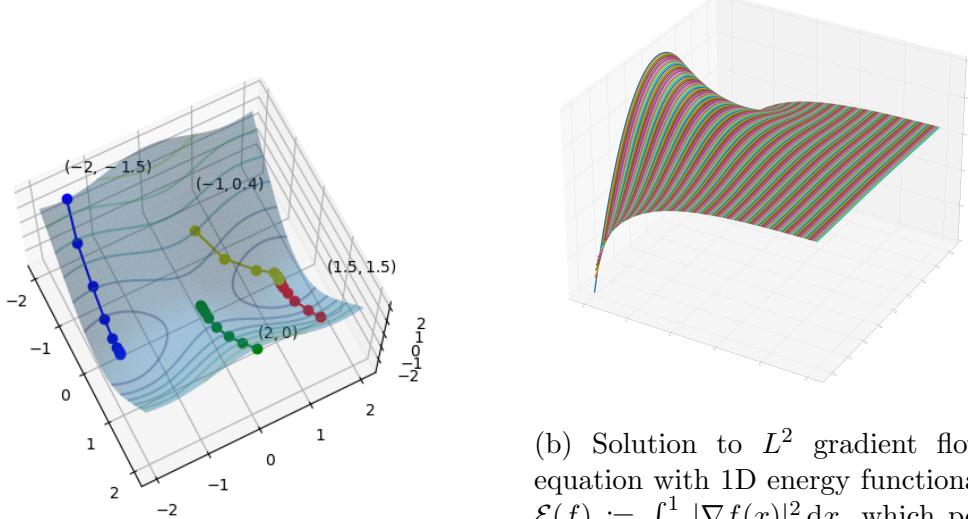
$$\mathcal{E}(\gamma) := \iint_{M^2} k(\gamma_x, \gamma_y) d\gamma_x d\gamma_y \quad (1)$$

where  $k(\gamma_1, \gamma_2) \geq 0$  is the *curve energy kernel* such that  $k \rightarrow +\infty$  as  $|\gamma_1 - \gamma_2| \rightarrow 0^+$ .

A naïve choice of  $k$  satisfying this condition is  $k_S(\gamma_1, \gamma_2) := \frac{1}{|\gamma_1 - \gamma_2|}$ . However, it turns out that  $k_S \sim O\left(\frac{1}{|\gamma_1 - \gamma_2|}\right)$  as  $|\gamma_1 - \gamma_2| \rightarrow 0$  (consider neighbouring points), meaning the  $\mathcal{E}$  diverges all continuous curves  $\gamma$  of nonzero measure. This seemingly ill-defined energy, however, may not be the end of story in terms of numerics, and will be explored later.

A more analytically sensible choice of  $k$  would be the tangent-point kernel introduced in a paper by Buck and Orloff[2] and later generalised by Yu, Schumacher, and Crane[3].

The next part of the idea is to *reduce  $\mathcal{E}$  by continuously deforming the curve* based on a descent method until it reaches a stationary curve, at which, we expect it to be the “unknot” of the original curve. Note that by construction of  $\mathcal{E}$ , if the curve is to self-intersect,  $\mathcal{E}$  increases, and the descent method encourages the curve to repel, preventing the self-intersection.



(a) SDM applied to  $f(x, y) = -3 \cos x + \cos^2 y$  at different initial points.

(b) Solution to  $L^2$  gradient flow equation with 1D energy functional  $\mathcal{E}(f) := \int_{-1}^1 |\nabla f(x)|^2 dx$ , which penalises variation in function. Note that the solution converges to a function with no variation.

Figure 2: Gradient flow can be understood as a continuous analogue of steepest descent.

## 2 Gradient Flow Equation

Since we pose the problem as continuous reduction of some functional, we need an applicable framework which fits our intention. In our case, **gradient flow equation** seems to be appropriate.

### 2.1 Motivation of Gradient Flow Equation

For minimising a differentiable function  $f : E \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , there is a well-known method known as **steepest descent method** (SDM)[1].

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k \nabla f(\mathbf{x}^k) \quad (2)$$

Starting from the initial input point  $\mathbf{x}^0$ , at each iteration, input points  $\{\mathbf{x}^k\}$  move in the direction of “steepest” decrease, with specified step size  $\alpha_k > 0$ , reducing the value at evaluation of  $f$ . Note that in general, this method is not guaranteed to find the minimiser as shown in Figure (2a). On the other hand, convergence is guaranteed under certain assumptions, for example, convexity and  $L$ -smoothness<sup>1</sup> with a certain choice of step size  $\alpha_k$ .

Analogously, differential equation known describing the reduction process of a functional  $F : \mathcal{X} \rightarrow \mathbb{R}$  (where  $\mathcal{X}$  is an inner product function space) can be moti-

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<sup>1</sup> $\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$  for some constant  $L > 0$

vated. Starting from (2), replacing  $\mathbf{x}^k$  by  $f_k$  and  $\nabla f(\mathbf{x}^k)$  by  $\text{grad}_{\mathcal{X}} F(f_k)$

$$f_{k+1} = f_k - \alpha_k \text{grad}_{\mathcal{X}} F(f_k) \quad (3)$$

Now think of  $f_k$  as “snapshots” at certain time  $t = t_k$ . Without loss of generality, let  $\alpha_k \equiv 1$ .<sup>2</sup> Dividing (3) by time step  $\Delta t := t_{k+1} - t_k$ , and taking the limit as  $\Delta t \rightarrow 0$ , we acquire the **gradient flow equation**[3].

$$\frac{\partial f}{\partial t} = - \text{grad}_{\mathcal{X}} F(f) \quad (4)$$

where index  $k$  transforms to “time” variable  $t$ .

Note that grad of a functional is not defined yet. This depends on the inner product function space  $\mathcal{X}$  (eg.  $L^2$ ,  $H^1$ , …) of interest.

## 2.2 Gradient of Functional

Gradient of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  characterises the first-order variation in a certain direction, that is,

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \quad \nabla f(\mathbf{x}) \cdot \mathbf{y} = \frac{\partial}{\partial \epsilon} f(\mathbf{x} + \epsilon \mathbf{y})|_{\epsilon=0} \quad (5)$$

While this is not a conventional definition of function gradient, it is still an equivalent definition.

One could analogously construct the definition of gradient of a functional.

**Definition** (Gradient of Functional). For a functional  $F : \mathcal{X} \rightarrow \mathbb{R}$ , define functional gradient  $\text{grad}_{\mathcal{X}} F(f)$  as:

$$\langle \text{grad}_{\mathcal{X}} F(f), g \rangle_{\mathcal{X}} = \frac{\partial}{\partial \epsilon} F(f + \epsilon g)|_{\epsilon=0} \quad (6)$$

where  $\langle \cdot, \cdot \rangle_{\mathcal{X}}$  is the inner product defined over the inner product function space  $\mathcal{X}$ .

Common inner product spaces include  $L^2$ ,  $H^1$ , which are defined in Appendix A.

*Remark.* By assuming time-independence of  $f$  in the gradient flow equation (4), we get a stationary state equation  $\text{grad}_{\mathcal{X}} F(f) = 0$ , which implies  $\frac{\partial}{\partial \epsilon} F(f + \epsilon g)|_{\epsilon=0} = 0$  for all  $g \in \mathcal{X}$ . This shows that the stationary state of the gradient flow equation is precisely the solution to the Euler-Lagrange equation.

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<sup>2</sup>This is justified by taking a different time scale; essentially nondimensionalisation. Also  $\alpha_k \sim O(1)$  as  $k \rightarrow \infty$  (eg.  $\alpha = L^{-1}$  for  $L$ -smooth optimisation) is in fact a realistic choice.

### 2.2.1 Gradient on Integer Sobolev Spaces

Given  $L^2 = H^0$  gradient, one could express Sobolev gradients of other orders in terms of it. For functional  $F$ , write  $h := \text{grad}_{L^2} F$  for gradient of  $F$  in  $L^2$ , that is, by definition

$$\frac{\partial}{\partial \epsilon} F(f + \epsilon g) |_{\epsilon=0} = \langle h, g \rangle_{L^2} =: \mathcal{D} \quad (7)$$

By integration by parts,

$$\begin{aligned} \mathcal{D} &= \langle \Delta \Delta^{-1} h, g \rangle_{L^2} \\ &= \langle -\nabla \Delta^{-1} h, \nabla g \rangle_{L^2} \\ &= \langle -\Delta^{-1} h, g \rangle_{H^1} \end{aligned}$$

So one concludes that

$$\text{grad}_{H^1} F = -\Delta^{-1} h \quad (8)$$

Similarly, one could acquire gradient in  $H^2$  by

$$\begin{aligned} \mathcal{D} &= \langle \Delta^{-1} h, \Delta g \rangle_{L^2} \\ &= \langle \Delta(\Delta^{-2} h), \Delta g \rangle_{L^2} \\ &= \langle \Delta^{-2} h, g \rangle_{H^2} \end{aligned}$$

Hence,

$$\text{grad}_{H^2} F = \Delta^{-2} h \quad (9)$$

One could take it even further and define gradient in  $H^{-1}$ ,

$$\begin{aligned} \mathcal{D} &= \langle h, \Delta \Delta^{-1} g \rangle_{L^2} \\ &= \langle -\nabla h, \nabla(\Delta^{-1} g) \rangle_{L^2} \\ &= \langle -\Delta h, g \rangle_{H^{-1}} \end{aligned}$$

So,

$$\text{grad}_{H^{-1}} F = -\Delta h \quad (10)$$

*Remark.* From (8), (9), and (10), one may deduce that choosing the right Sobolev space for a functional may make the gradient flow equation much easier to solve. For example, if  $\text{grad}_{L^2} F(f) = h = -\Delta f$ , instead of solving  $L^2$  gradient flow equation  $\frac{\partial}{\partial t} f = \Delta f$ , solving  $H^1$  gradient flow equation  $\frac{\partial}{\partial t} f = -f$  is much trivial.

## 3 Tangent-Point Energy

Given a curve  $\gamma : M \rightarrow \mathbb{R}^3$ , a sensible choice of curve energy is the tangent-point energy[3].

**Definition** (Tangent-Point Energy). For a continuously differentiable parameterised curve  $\gamma : M \rightarrow \mathbb{R}^3$ , define **tangent-point energy** as:

$$\mathcal{E}_\beta^\alpha(\gamma) := \iint_{M^2} k_\beta^\alpha(\gamma_x, \gamma_y, \mathbf{T}_x) d\gamma_x d\gamma_y \quad (11)$$

where  $\mathbf{T}_x := \frac{d\gamma_x}{dt} / \left| \frac{d\gamma_x}{dt} \right|$  is the unit tangent vector at  $\gamma_x$  along the curve, and **tangent-point kernel** is given as:

$$k_\beta^\alpha(\mathbf{p}, \mathbf{q}, \mathbf{T}) := \frac{|\mathbf{T} \wedge (\mathbf{p} - \mathbf{q})|^\alpha}{|\mathbf{p} - \mathbf{q}|^\beta} \quad (12)$$

$\alpha$  and  $\beta$  are parameters one could choose, but for tangent-point energy to be well-defined, one may choose them to satisfy  $\alpha > 1$  and  $\beta \in [\alpha + 2, 2\alpha + 1]$ .

Note that choosing  $\alpha = 2$  and  $\beta = 4$  results in the scaled version of the original tangent-point energy by Buck and Orloff[2].

The geometric intuition of  $k_4^2(\gamma_x, \gamma_y, \mathbf{T}_x)$  (kernel of ‘‘Buck-Orloff tangent-point energy’’) is that the kernel evaluates to  $\frac{1}{4r^2}$  where  $r$  is the radius of the smallest circle drawn that is tangent at  $\gamma_x$  and crosses through  $\gamma_y$  as shown in Figure 3.

The choice of parameters  $\alpha$  and  $\beta$  changes the behaviour of the tangent-point energy as stated in the following lemma.

**Lemma 1.** *Tangent-point energy  $\mathcal{E}_\beta^\alpha$  defined as (11) is scale invariant with respect to the curve if and only if  $\beta = \alpha + 2$ . Moreover, if  $\beta > \alpha + 2$ , then  $\mathcal{E}_\beta^\alpha$  scales inversely with the curve.*

*Proof.* Take a parameterised curve  $\gamma : M \rightarrow \mathbb{R}^3$  and  $\Gamma := c\gamma$ , a curve scaled by factor  $c > 0$  of  $\gamma$ . Note that the unit tangent vector is identical for  $\gamma$  and  $\Gamma$ , that is,  $\mathbf{T}_x := \frac{d\gamma_x}{dt} / \left| \frac{d\gamma_x}{dt} \right| = \frac{d\Gamma_x}{dt} / \left| \frac{d\Gamma_x}{dt} \right|$

Then,

$$\frac{k_\beta^\alpha(\gamma_x, \gamma_y, \mathbf{T}_x)}{k_\beta^\alpha(\Gamma_x, \Gamma_y, \mathbf{T}_x)} = \frac{|\gamma_x - \gamma_y|^\alpha}{|\Gamma_x - \Gamma_y|^\beta} = c^{\beta-\alpha} \quad (13)$$

Also note that

$$d\Gamma = c d\gamma \quad (14)$$

So, we deduce from (11),

$$\mathcal{E}_\beta^\alpha(\Gamma) = c^{\alpha-\beta+2} \mathcal{E}_\beta^\alpha(\gamma) \quad (15)$$

Hence,  $\mathcal{E}_\beta^\alpha$  is scale invariant with respect to the curve if and only if  $\alpha - \beta + 2 = 0$ , and if  $\beta > \alpha + 2$ ,  $\mathcal{E}_\beta^\alpha$  scales as  $O\left(\frac{1}{c^{\beta-(\alpha+2)}}\right)$  as  $c \rightarrow \infty$ .  $\blacksquare$

*Remark.* If  $\beta < \alpha + 2$ , then the energy scales with the size of the curve, meaning that the energy is trivially minimised by scaling down the curve to a singularity, which is not desirable in our context.

One could also justify the condition on  $\alpha$  and  $\beta$  by the following lemma.



Figure 3: Intuition behind  $k_4^2$

**Lemma 2.** Given  $\alpha$  and  $\beta$ , the singularity of the kernel at a point and another point of which is arc-length  $\epsilon > 0$  away from it is of order  $O(\epsilon^{2\alpha-\beta})$ , that is,  $k_\beta^\alpha(\gamma(s), \gamma(s+\epsilon), \mathbf{T}(s)) = O(\epsilon^{2\alpha-\beta})$  as  $\epsilon \rightarrow 0$ . Moreover, if  $2\alpha = \beta$ , then the kernel converges to  $(\frac{\kappa}{2})^\alpha$  as the two points get closer, where  $\kappa$  is the curvature of the curve at the point.

*Proof.* For  $\gamma(s) = (x(s), y(s), z(s))$  parameterised by arc-length, one recognises that the tangent vector at this point is  $\mathbf{T} = \gamma'(s)$ . Note that  $\|\gamma'(s)\| = \|\mathbf{T}\| = 1$ .

By Taylor expansion:

$$\gamma(s+\epsilon) = \gamma(s) + \epsilon\gamma'(s) + \frac{1}{2}\epsilon^2\gamma''(s) + O(\epsilon^3) \quad (16)$$

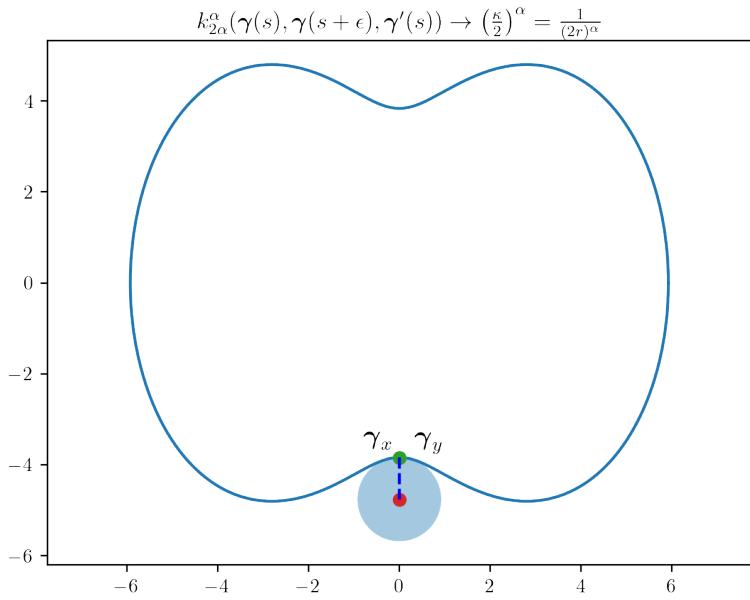
Then around  $\epsilon = 0$ ,

$$\begin{aligned} k_\beta^\alpha(\gamma(s), \gamma(s+\epsilon), \gamma'(s)) \\ &= \frac{\|\gamma'(s) \wedge (\gamma(s+\epsilon) - \gamma(s))\|^\alpha}{\|\gamma(s+\epsilon) - \gamma(s)\|^\beta} \\ &= \frac{\|\gamma'(s) \wedge (\epsilon\gamma'(s) + \frac{1}{2}\epsilon^2\gamma''(s) + O(\epsilon^3))\|^\alpha}{\|\epsilon\gamma'(s) + O(\epsilon^2)\|^\beta} \\ &= \frac{\epsilon^{2\alpha-\beta} \|\gamma'(s) \wedge \gamma''(s) + O(\epsilon)\|^\alpha}{2^\alpha \|\gamma'(s) + O(\epsilon)\|^\beta} \quad \because \gamma'(s) \wedge \gamma'(s) = \mathbf{0} \\ &= \frac{\epsilon^{2\alpha-\beta}}{2^\alpha} \left( \frac{\|\gamma'(s)\|^\alpha \|\gamma''(s)\|^\alpha}{\|\gamma'(s)\|^\beta} + O(\epsilon) \right) \quad \because \gamma'(s) \perp \gamma''(s) \\ &= \frac{\epsilon^{2\alpha-\beta}}{2^\alpha} (\kappa^\alpha + O(\epsilon)) \quad \because \|\gamma'(s)\| = \|\mathbf{T}\| = 1 \\ &\quad \text{and } \|\gamma''(s)\| = \kappa \end{aligned}$$

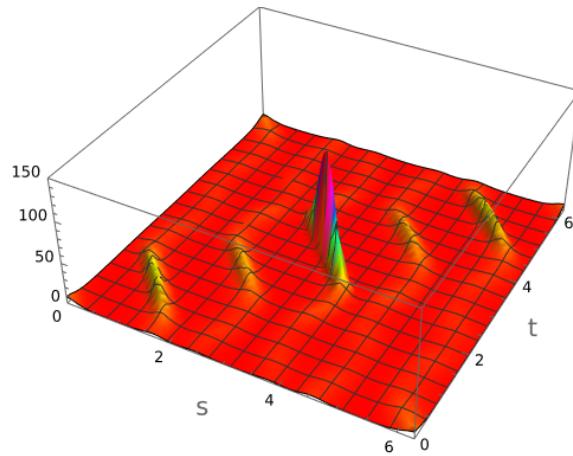
So with arc-length perturbation of  $\epsilon$ , the order of singularity of the kernel is  $O(\epsilon^{2\alpha-\beta})$ . In particular, if  $2\alpha - \beta = 0$ , the kernel converges to  $(\frac{\kappa}{2})^\alpha$ , as demonstrated in Figure 4a. ■

From lemma 1 and lemma 2, the well-definedness condition of tangent-point energy immediately follows from arguing by considering integrability of functions with isolated poles of order lower than 1. (See Figure 5)

**Corollary 3.** If  $\alpha > 1$  and  $\beta \in [\alpha + 2, 2\alpha + 1]$ , tangent-point energy  $\mathcal{E}_\beta^\alpha$  is well-defined.



(a)  $k_{2\alpha}^{\alpha}(\gamma(s), \gamma(s+\epsilon), \gamma'(s))$  converges to  $\left(\frac{\kappa}{2}\right)^{\alpha}$ .



(b)  $k_{2\alpha}^{\alpha}$  heat map

Figure 4: Behaviours of  $\beta = 2\alpha$  kernel



Figure 5: Heat map of  $k_{4.5}^2$ : Even with singularities, because their order is less than 1, the kernel is integrable, hence  $\mathcal{E}_{4.5}^2$  is well-defined.

# Appendices

## A Definitions of Important Inner Product Spaces

(For our purposes, we focus our attention to real domain.) Here are some of the notable inner product spaces.

### A.1 $L^2$ Space

**Definition ( $L^2$  Space).** For interval  $I \subset \mathbb{R}$ ,  $L^2$  space over is defined as

$$L^2 = \left\{ f : I \rightarrow \mathbb{R} \mid \left( \int_I |f|^2 d\mu \right)^{1/2} < \infty \right\} \quad (17)$$

$L^2$  inner product is defined as

$$\forall f, g \in L^2 \quad \langle f, g \rangle_{L^2} = \int_I f g d\mu \quad (18)$$

### A.2 $H^k$ Space

To define Sobolev (inner product) spaces (denoted  $H^k$  where  $k \in \mathbb{N} \cup \{0\}$ ) one must define weak derivative operator  $D$ :

**Definition (Weak Derivative).** For  $u$  locally integrable on  $I$ , if there exists  $w$  such that for all infinitely smooth  $v : I \rightarrow \mathbb{R}$  with compact support,

$$\int_I w v d\mu = - \int_I u v' d\mu \quad (19)$$

then  $w$  is said to be **weak derivative** of  $u$ , and one writes  $Du = w$ .

Weak derivative extends the definition of conventional derivative, and is equivalent to the conventional derivative for smooth functions.

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