

# Gradient Flow to Continuous Optimization via Fourier Series

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We've been concerned about minimizing the energy functional of the form:

$$\mathcal{E}(\gamma) := \int_{C_\gamma} \int_{C_\gamma} k(\gamma_1, \gamma_2) \, d\gamma_1 \, d\gamma_2 \quad (1)$$

where  $\gamma : E \rightarrow \mathbb{R}^3$  is a parameterization function of a closed curve on the interval  $E = [0, 2\pi)$  (without loss of generality).

Note that we assume  $\gamma$  to be a periodic function of period  $2\pi$ .

## 1 Multidimensional Fourier Series

### 1.1 1D Fourier Series

Given a continuous 1D  $2\pi$ -periodic function  $f : \mathbb{R} \rightarrow \mathbb{R}$  (where we only need to define  $f$  on  $[0, 2\pi)$ ), there exists a Fourier series representation:

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \quad (2)$$

$$= \sum_{i=-\infty}^{\infty} c_n e^{inx} \quad (3)$$

where the coefficients  $\{a_n\}, \{b_n\}, \{c_n\}$  are given by

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) \, dx \quad \in \mathbb{R} \quad (4)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) \, dx \quad \in \mathbb{R} \quad (5)$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, dx \quad \in \mathbb{C} \quad (6)$$

Fourier convergence theorem states that the rate of convergence is  $O\left(\frac{1}{n^{p+1}}\right)$  where  $f$  has the first jump discontinuity in the  $p_{\text{th}}$  derivative.<sup>1</sup>

### 1.2 Multidimensional Extension

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<sup>1</sup>Lecture note