

# Untangling Knots Through Curve Repulsion



Department of Mathematics  
University of Oxford  
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## Abstract

Curves are one of the fundamental objects in geometry and engineering, yet most analysis of curves often disregard their physical characteristics such as their spatial volume or uncrossability. One common situation that such physical characteristics become significant is when one attempts to untangle a knot. An approach to achieve this is to assign an “energy” to a curve such that this energy would increase when two points on “different sides” of a curve are closer, then one continuously deforms the curve to reduce this energy, the expectation being that the curve that achieves minimal energy must be the untangled knot. This dissertation explores numerical methods of achieving this.

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## Part 1

# Introduction

Shape optimisation is an important idea in engineering, relevant in anywhere from aircraft designs to packaging ramen noodle. One of the simplest and most fundamental shapes to consider is a curve. While curves are simple objects in theory, they prove to be quite difficult to analyse in practice with realistic physics. Even in absence of other objects, one must consider resilience to bending, stretching, and especially, impenetrability against itself. With these physical factors in mind, untangling a knot like the one shown (Figure 1) becomes a very complicated process, especially in a computer simulation. In this dissertation, we explore numerical methods to achieve this.

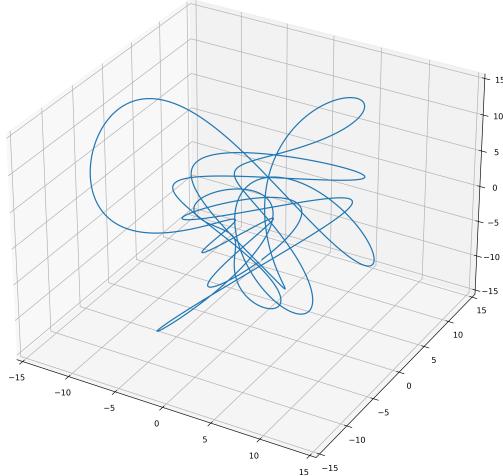


Figure 1: A complicated knot in  $\mathbb{R}^3$

The main idea is to *assign energy that penalises “physical entanglement”*. Given a parameterised curve  $\gamma : M \rightarrow \mathbb{R}^3$  ( $M$  being the domain of the parameter, often an interval), one defines some *curve energy*  $\mathcal{E}$  of the form:

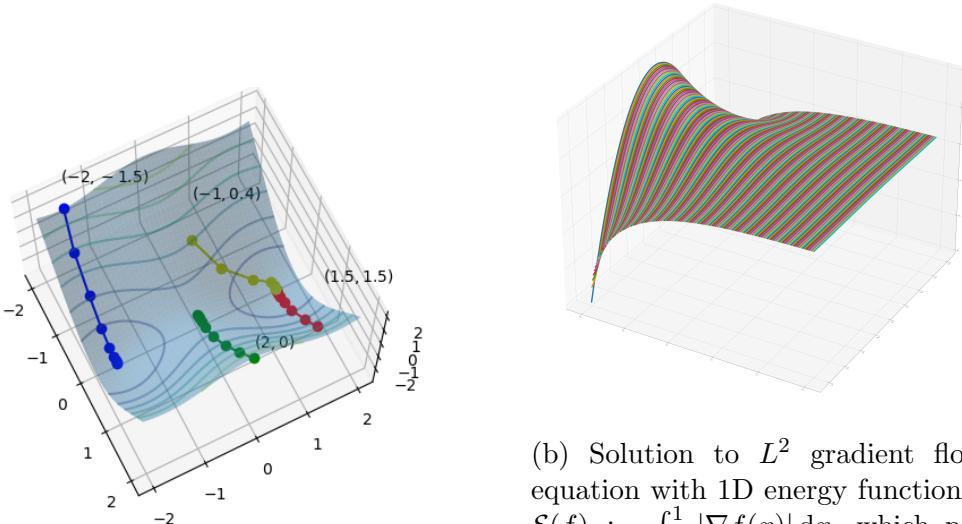
$$\mathcal{E}(\gamma) := \iint_{M^2} k(\gamma_x, \gamma_y) d\gamma_x d\gamma_y \quad (1)$$

where  $k(\gamma_1, \gamma_2) \geq 0$  is the *curve energy kernel* such that  $k \rightarrow +\infty$  as  $|\gamma_1 - \gamma_2| \rightarrow 0^+$ .

A naïve choice of  $k$  satisfying this condition is  $k_S(\gamma_1, \gamma_2) := \frac{1}{|\gamma_1 - \gamma_2|}$ . However, it turns out that  $k_S \sim O\left(\frac{1}{|\gamma_1 - \gamma_2|}\right)$  as  $|\gamma_1 - \gamma_2| \rightarrow 0$  (consider neighbouring points), meaning the  $\mathcal{E}$  diverges all continuous curves  $\gamma$  of nonzero measure. This seemingly ill-defined energy, however, may not be the end of story in terms of numerics, and will be explored later.

A more analytically sensible choice of  $k$  would be the tangent-point kernel introduced in a paper by Buck and Orloff[1] and later generalised by Yu, Schumacher, and Crane[3].

The next part of the idea is to *reduce  $\mathcal{E}$  by continuously deforming the curve* based on a descent method until it reaches a stationary curve, at which, we expect it to be the “unknot” of the original curve. Note that by construction of  $\mathcal{E}$ , if the curve is to self-intersect,  $\mathcal{E}$  increases, and the descent method encourages the curve to repel, preventing the self-intersection.



(a) SDM applied to  $f(x, y) = -3 \cos x + \cos^2 y$  at different initial points.

(b) Solution to  $L^2$  gradient flow equation with 1D energy functional  $\mathcal{E}(f) := \int_{-1}^1 |\nabla f(x)| dx$ , which penalises variation in function. Note that the solution converges to a function with no variation.

## Part 2

# Curve Energy Reduction via Gradient Flow

## 2.1 Steepest Descent to Gradient Flow Equation

For minimising a differentiable function  $f : E \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , there is a well-known method known as **steepest descent method** (SDM)[2].

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k \nabla f(\mathbf{x}^k) \quad (2)$$

Starting from the initial input point  $\mathbf{x}^0$ , at each iteration, input points  $\{\mathbf{x}^k\}$  move in the direction of “steepest” decrease, with specified step size  $\alpha_k > 0$ , reducing the value at evaluation of  $f$ . Note that in general, this method is not guaranteed to find the minimiser. On the other hand, convergence is guaranteed under certain assumptions, for example, convexity and  $L$ -smoothness with a certain choice of step size  $\alpha_k$ .

Analogously, differential equation known describing the reduction process of a functional  $F : \mathcal{F} \subset \mathcal{X} \rightarrow \mathbb{R}$  (where  $\mathcal{X}$  is a function space) can be motivated. Starting from (2), replacing  $\mathbf{x}^k$  by  $f_k$  and  $\nabla f(\mathbf{x}^k)$  by  $\text{grad}_{\mathcal{X}} F(f_k)$

$$f_{k+1} = f_k - \alpha_k \text{grad}_{\mathcal{X}} F(f_k) \quad (3)$$

Now think of  $f_k$  as ‘‘snapshots’’ at certain time  $t = t_k$ . Without loss of generality, let  $\alpha_k \equiv 1$ .<sup>1</sup> Dividing (3) by time step  $\Delta t := t_{k+1} - t_k$ , and taking the limit as  $\Delta t \rightarrow 0$ , we acquire the **gradient flow equation**[3].

$$\frac{\partial F}{\partial t} = - \operatorname{grad}_{\mathcal{X}} F \quad (4)$$

where index  $k$  transforms to ‘‘time’’ variable  $t$ .

Note that grad of a functional is not defined yet. This depends on the function space  $\mathcal{X}$  (eg.  $L^2$ ,  $H^1$ , ...) of interest.

## 2.2 Tangent-Point Energy

Given a curve  $\gamma : M \rightarrow \mathbb{R}^3$ , a sensible choice of curve energy is the tangent-point energy[3].

**Definition.** For a continuously differentiable parameterised curve  $\gamma : M \rightarrow \mathbb{R}^3$ , define **tangent-point energy** as:

$$\mathcal{E}_\beta^\alpha(\gamma) := \iint_{M^2} k_\beta^\alpha(\gamma_x, \gamma_y, \mathbf{T}_x) d\gamma_x d\gamma_y \quad (5)$$

where  $\mathbf{T}_x := \frac{d\gamma_x}{dt} / \left| \frac{d\gamma_x}{dt} \right|$  is the unit tangent vector at  $\gamma_x$  along the curve, and **tangent-point kernel** is given as:

$$k_\beta^\alpha(\mathbf{p}, \mathbf{q}, \mathbf{T}) := \frac{|\mathbf{T} \wedge (\mathbf{p} - \mathbf{q})|^\alpha}{|\mathbf{p} - \mathbf{q}|^\beta} \quad (6)$$

$\alpha$  and  $\beta$  are parameters one could choose, but for tangent-point energy to be well-defined, one may choose them to satisfy  $\alpha > 1$  and  $\beta \in [\alpha + 2, 2\alpha + 1]$ .

Note that choosing  $\alpha = 2$  and  $\beta = 4$  results in the original tangent-point energy by Buck and Orloff[1].

The choice of parameters  $\alpha$  and  $\beta$  changes the behaviour of the tangent-point energy as stated in the following lemma.

**Lemma 1.** *Tangent-point energy  $\mathcal{E}_\beta^\alpha$  defined as (5) is scale invariant with respect to the curve if and only if  $\beta = \alpha + 2$ . Moreover, if  $\beta > \alpha + 2$ , then  $\mathcal{E}_\beta^\alpha$  scales inversely with the curve.*

*Proof.* Take a parameterised curve  $\gamma : M \rightarrow \mathbb{R}^3$  and  $\Gamma := c\gamma$ , a curve scaled by factor  $c > 0$  of  $\gamma$ . Note that the unit tangent vector is identical for  $\gamma$  and  $\Gamma$ , that is,  $\mathbf{T}_x := \frac{d\gamma_x}{dt} / \left| \frac{d\gamma_x}{dt} \right| = \frac{d\Gamma_x}{dt} / \left| \frac{d\Gamma_x}{dt} \right|$

Then,

$$\frac{k_\beta^\alpha(\gamma_x, \gamma_y, \mathbf{T}_x)}{k_\beta^\alpha(\Gamma_x, \Gamma_y, \mathbf{T}_x)} = \frac{|\gamma_x - \gamma_y|^\alpha}{|\Gamma_x - \Gamma_y|^\beta} = c^{\beta-\alpha} \quad (7)$$

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<sup>1</sup>This is justified by taking a different time scale; essentially nondimensionalisation.

Also note that

$$d\Gamma = c d\gamma \quad (8)$$

So, we deduce from (5),

$$\mathcal{E}_\beta^\alpha(\boldsymbol{\Gamma}) = c^{\alpha-\beta+2} \mathcal{E}_\beta^\alpha(\boldsymbol{\gamma}) \quad (9)$$

Hence,  $\mathcal{E}_\beta^\alpha$  is scale invariant with respect to the curve if and only if  $\alpha - \beta + 2 = 0$ , and if  $\beta > \alpha + 2$ ,  $\mathcal{E}_\beta^\alpha$  scales at the rate of  $O\left(\frac{1}{c^{\beta-(\alpha+2)}}\right)$  as  $c \rightarrow \infty$ .  $\blacksquare$

# Part 3

# Functional Reduction to Function Reduction

## References

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