

Untangling Knots Through Curve Repulsion

Joo-Hyun Paul Kim

March 5, 2023



What the curious folks ponder about

1 Introduction

2 Tangent-Point Energy

3 Gradient Flow

4 Curve Repulsion

Introduction

A Cool Knot

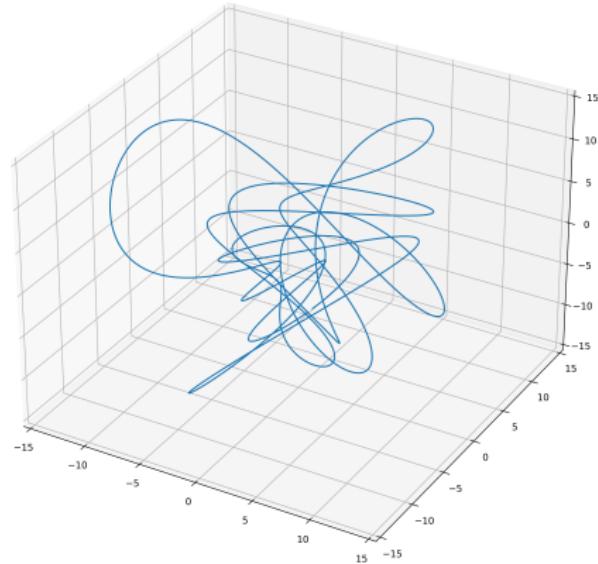


Figure: Imagine your earphones getting tangled like this...

Aim

- Finding a “homotopy” from a knot to an unknot.

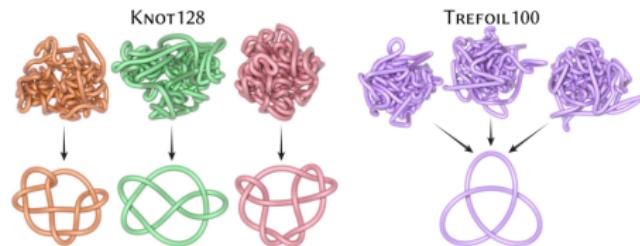


Figure: Unknots of test knots.[3]

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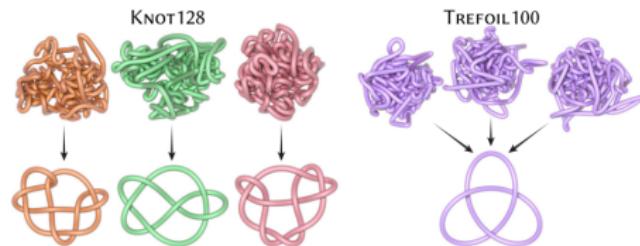


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- “Avoiding self-intersection”

General Strategy

- ① Define curve energy; penalizing the closeness of points on a curve.
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 - There is a freedom in choosing the “gradient” here.

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- ② Attempt to decrease the curve energy by continuously deforming the curve.
 - We evolve the curve according to the gradient flow equation.
 - There is a freedom in choosing the “gradient” here.
- ③ We expect the stationary state to be the “unknot”
 - Or at least a simpler state...

Tangent-Point Energy

Defining Curve Energy

Given an (arc-length parameterised) curve $\gamma : M \rightarrow \mathbb{R}^3$, we wish to assign energy of the form:

$$\mathcal{E}(\gamma) := \iint_{M^2} k(\gamma_x, \gamma_y) \, d\gamma_x \, d\gamma_y \quad (1)$$

such that

- \mathcal{E} is very high when two non-neighbouring points are very close.

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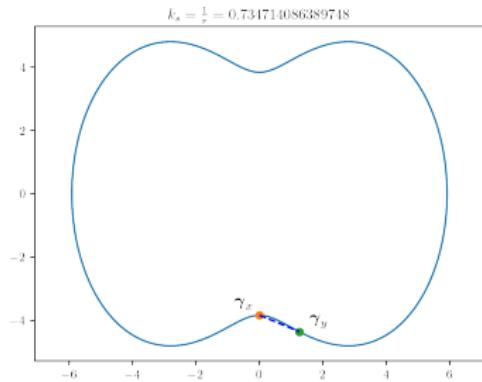
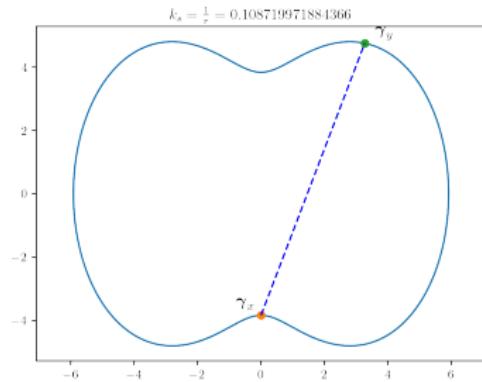
such that

- \mathcal{E} is very high when two non-neighbouring points are very close.

A naïve choice is $k(\gamma_x, \gamma_y) := \frac{1}{\|\gamma_x - \gamma_y\|}$

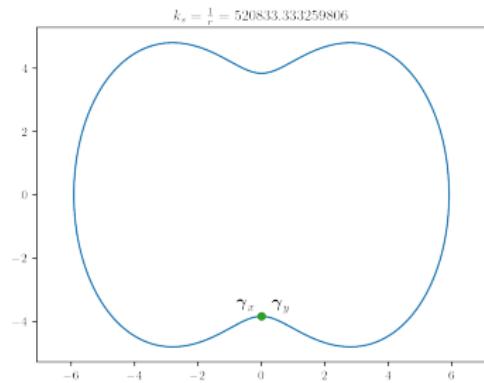
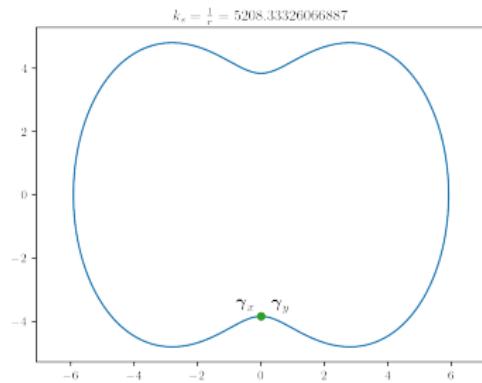
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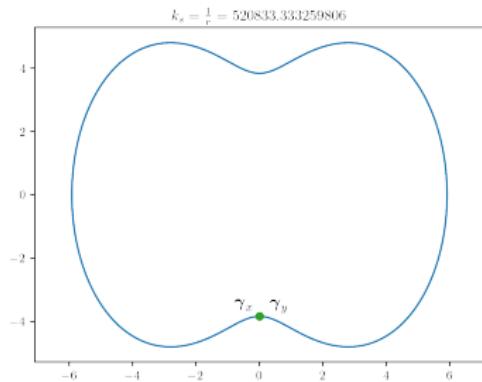
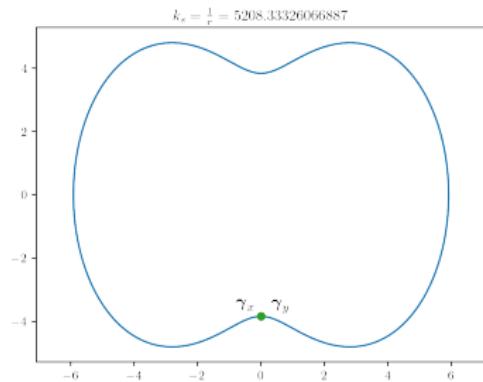
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This energy is not well-defined for a lot of curves!

Buck-Orloff Tangent-Point Energy

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Definition (Buck-Orloff Tangent-Point Energy)

For a smooth curve γ , define

$$\mathcal{E}(\gamma) := \iint_{M^2} k_4^2(\gamma_x, \gamma_y, \mathbf{T}_x) d\gamma_x d\gamma_y$$

where \mathbf{T}_x is the unit tangent vector at γ_x , with the kernel defined as

$$k_4^2(\mathbf{p}, \mathbf{q}, \mathbf{T}) := \frac{\|\mathbf{T} \wedge (\mathbf{p} - \mathbf{q})\|^2}{\|\mathbf{p} - \mathbf{q}\|^4}$$

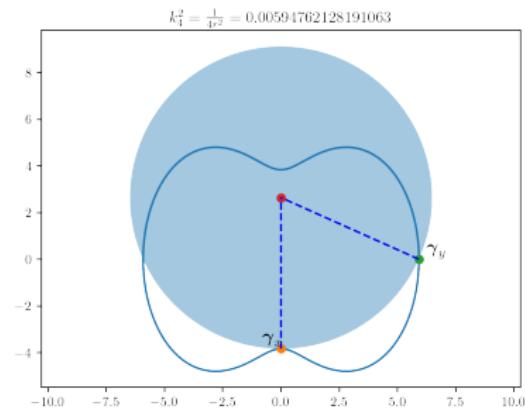
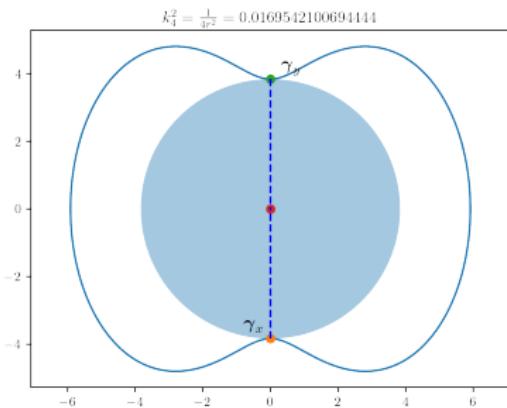
as **Buck-Orloff Tangent-Point Energy**.[1]

Intuition

What is the intuition behind the kernel $k_4^2(\mathbf{p}, \mathbf{q}, \mathbf{T}) := \frac{\|\mathbf{T} \wedge (\mathbf{p} - \mathbf{q})\|^2}{\|\mathbf{p} - \mathbf{q}\|^4}$?

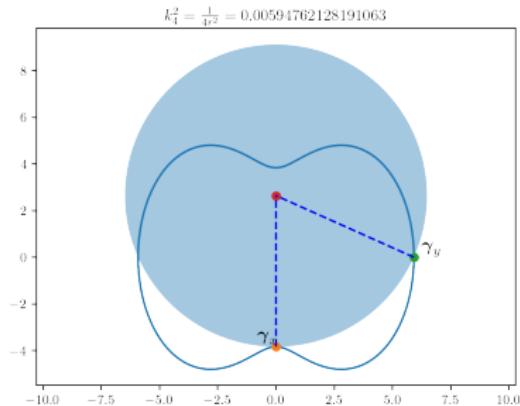
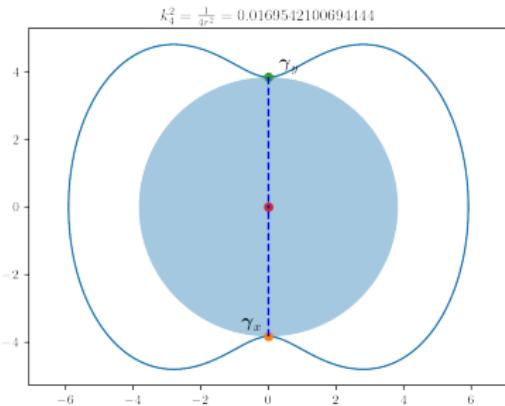
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Remark

Note that closer does not necessarily mean the kernel is larger.

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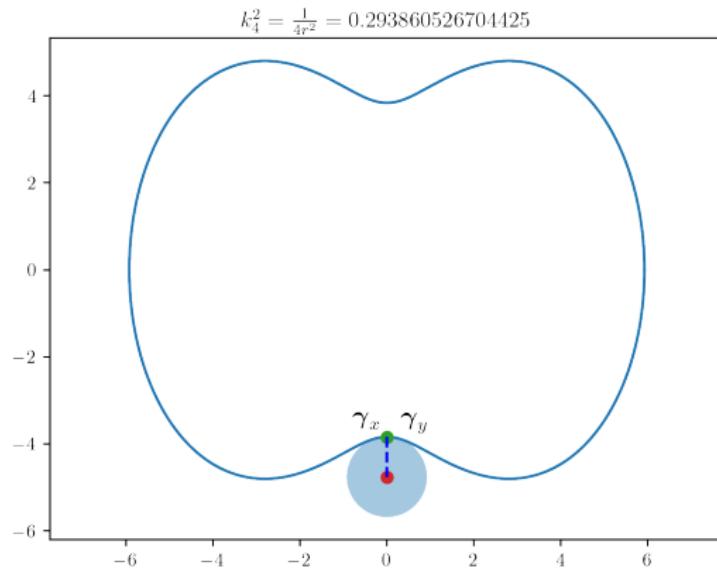


Figure: When two points are very close, the kernel converges to quarter of the squared curvature of the curve.

Example (Unit Circle)

$$\mathcal{E}(\gamma) = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \frac{||\mathbf{T}_x||^2 ||\gamma_x - \gamma_y||^2 - (\mathbf{T}_x \cdot (\gamma_x - \gamma_y))^2}{||\gamma_x - \gamma_y||^4} d\theta d\phi \quad (2)$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \frac{\sin^2 \frac{\theta-\phi}{2}}{(-1 + \cos(\theta - \phi))^2} d\theta d\phi \quad (3)$$

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Remark

Note that (4) suggests the order at “singularity” is inverse-square.

General Tangent-Point Energy

A more general form of tangent-point energy comes from Yu, Schumacher, and Crane [3]:

Definition (Generalised Tangent-Point Energy)

$$\mathcal{E}_\beta^\alpha(\gamma) := \iint_{M^2} \frac{\|\mathbf{T}_x \wedge (\gamma_x - \gamma_y)\|^\alpha}{\|\gamma_x - \gamma_y\|^\beta} d\gamma_x d\gamma_y$$

where $\alpha > 1$ and $\beta \in [\alpha + 2, 2\alpha + 1]$

Remark

When $\alpha = 2$ and $\beta = 4$, we are back to Buck-Orloff.

Gradient Flow

Motivating Gradient Flow

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Remark

cf) For minimising a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, one may consider

$$\frac{\partial \mathbf{x}}{\partial t} = -\nabla f(\mathbf{x})$$

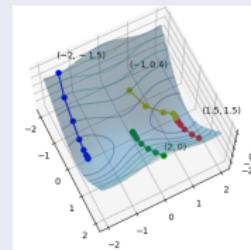


Figure: Steepest Descent [2]

Gradient in Space of Function?

(Gradient of Function)

$\nabla f(\mathbf{x})$ is such that for all $\mathbf{y} \in \mathbb{R}^n$

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(Gradient of Functional)

$\text{grad}_X E(f)$ is such that for all $g \in X$,

$$\langle \langle \text{grad}_X E, g \rangle \rangle_X = \frac{\partial}{\partial \epsilon} E(f + \epsilon g) \Big|_{\epsilon=0}$$

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Remark

For $X = L^2$, $(\text{grad}_X E)(f)$ is the first order variation from f with respect to ϵ .

Example: Dirichlet Energy in 1D

Definition (1D Dirichlet Energy)

For a differentiable function $f : \mathcal{I} \rightarrow \mathbb{R}$, define **1D Dirichlet energy**

$$E_D(f) := \int_{\mathcal{I}} |\nabla f(x)|^2 \, dx = \int_{\mathcal{I}} |f'(x)|^2 \, dx$$

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Remark

Dirichlet energy is high for function f that varies a lot, and minimised by any constant function.

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Then the gradient flow equation for Dirichlet energy becomes:

$$\frac{\partial f}{\partial t} = \Delta f \tag{6}$$

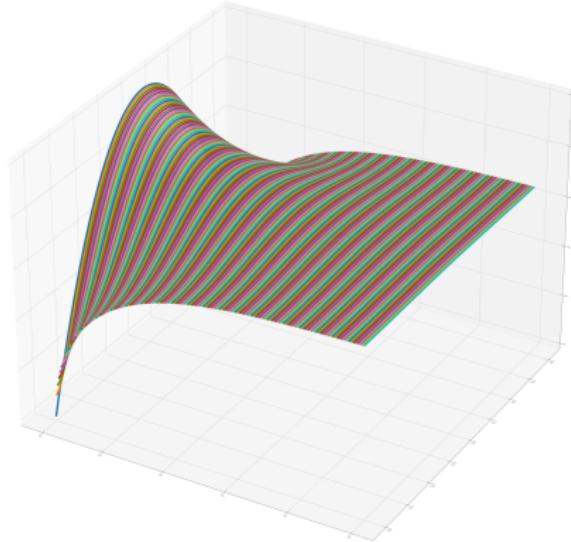


Figure: Solution to L^2 Gradient Flow Equation for Dirichlet Energy (AKA Heat Equation)

Dirichlet Gradient Flow in Other Function Space

Definition (Famous Inner Product Function Spaces)

- $L^p = \left\{ f \left| \left(\int |f|^p dx \right)^{1/p} < \infty \right. \right\}$
 - $\langle \langle f, g \rangle \rangle_{L^2} = \int fg dx$
- $p = 2$ Sobolev Space $H^k = \left\{ f \left| \sum_{i=0}^k \int |\mathcal{D}^{(i)} f|^2 dx < \infty \right. \right\}$
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- $L^2 = H^0: \frac{\partial f}{\partial t} = -\Delta f$
- $H^1: \frac{\partial f}{\partial t} = -f$
- $H^{-1}: \frac{\partial f}{\partial t} = -\Delta^2 f$
 - $\langle \langle f, g \rangle \rangle_{H^{-1}} = \langle \langle \Delta^{-1} f, \Delta^{-1} g \rangle \rangle_{H^1}$
- $H^2: \frac{\partial f}{\partial t} = \Delta^{-1} f$

Curve Repulsion

Gradient Flow on Tangent-Point Energy

We now take the tangent-point energy and gradient flow together:

$$\frac{\partial \gamma}{\partial t} = - \text{grad}_{L^2} \mathcal{E} \quad (7)$$

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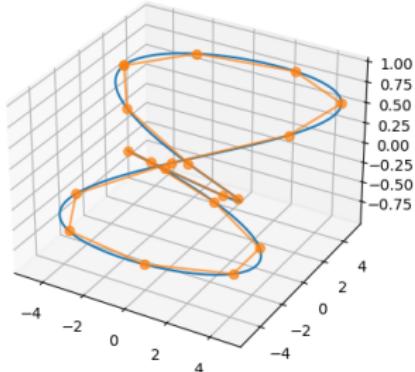


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For numerical solution, one discretises the curve. Enumerate each point as $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{J-1}$, and write Γ for the resulting polygonal curve.

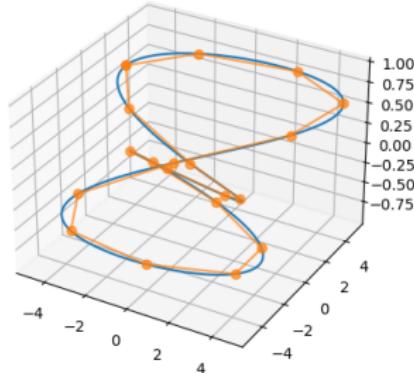


Figure: Discretised Curve

Subtlety

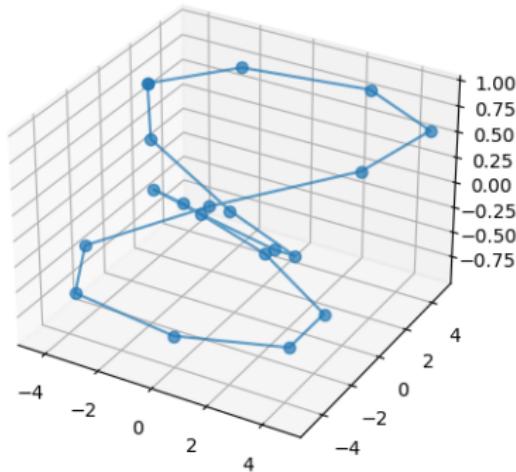


Figure: Tangent-point energy is not well-defined for polygonal curves.

Energy Discretisation

Note the form of tangent-point energy:

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Definition (Discretised Energy)

Given a closed curve with discretised points $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{J-1}$, define **discretised energy** E as:

$$E(\Gamma) := \sum_{\substack{i,j \in \{0,1,\dots,J-1\} \\ \text{dist}_{\text{geodesic}}(\Gamma(i), \Gamma(j)) > 1}} k_{i,j} \|\mathbf{x}_{i+1} - \mathbf{x}_i\| \|\mathbf{x}_{j+1} - \mathbf{x}_j\|$$

where the discretised kernel $k_{i,j}$ is given by:

$$\frac{1}{4} (k_\beta^\alpha(x_i, x_j, T_i) + k_\beta^\alpha(x_i, x_{j+1}, T_i) + k_\beta^\alpha(x_{i+1}, x_j, T_i) + k_\beta^\alpha(x_{i+1}, x_{j+1}, T_i))$$

Numerical Scheme

Original L^2 Gradient Flow Equation:

$$\frac{\partial \gamma}{\partial t} = - \text{grad}_{L^2} \mathcal{E} - \underbrace{\mathcal{C}}_{\text{Constraint}} \quad (9)$$

By forward difference, we arrived at an explicit Euler scheme:

$$\frac{\Gamma^{k+1} - \Gamma^k}{\Delta T} = - \underbrace{\frac{\partial E}{\partial \Gamma}}_{\text{Gradient from perturbing each point in each direction}} - \underbrace{\frac{\partial \mathcal{C}}{\partial \Gamma}}_{\text{Derived from Constraint}}$$

where we may further approximate the RHS by central difference scheme, for example:

$$\frac{\partial E}{\partial \Gamma_i} \approx \frac{E|_{\mathbf{x}_i+h} - E|_{\mathbf{x}_i-h}}{2h} \left(= \frac{\partial E}{\partial \Gamma_i} + O(h^2) \right)$$

Gradient Flow Demo

- Ellipse to Circle (ellipse.mp4)
- Saddle to Circle (saddle.mp4)

Bibliography

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