Subject panel: Mathematical Methods and Applications

Suggested title of dissertation:

Untangling Knots Through Curve Repulsion

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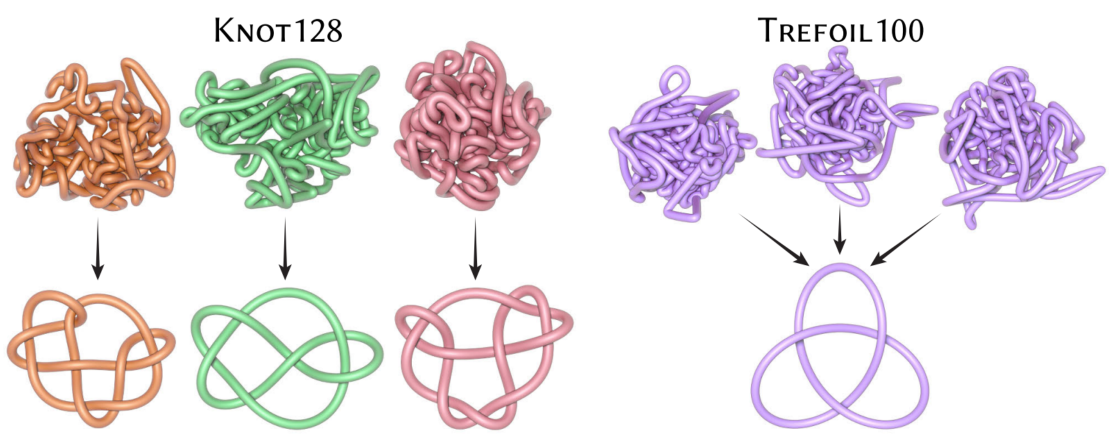
Description of proposal:

Shape optimisation plays a crucial role in many applications. In the case of curves, common problems are to model the space-filling growth of a curve that is constrained to a given surface or volume [YSC21], to simplify the geometry of a knot [BO95] or a surface [YBSC21] while avoiding self-intersections, or to compute minimal surfaces [PP93].

This project will explore a family of numerical methods for curve optimisations based on curve repulsion. The basic idea is to assign an *energy* to each curve, and then continuously deform the curve in a way that minimises this energy.



Knot simplification [YSC21].

Given a parametrised curve $\gamma : [0, L] \to \mathbb{R}^3$, its energy is defined as

$$E = \int_0^L \int_0^L k\big(\gamma(t), \gamma(s)\big) \mathrm{d}t \, \mathrm{d}s, \tag{1}$$

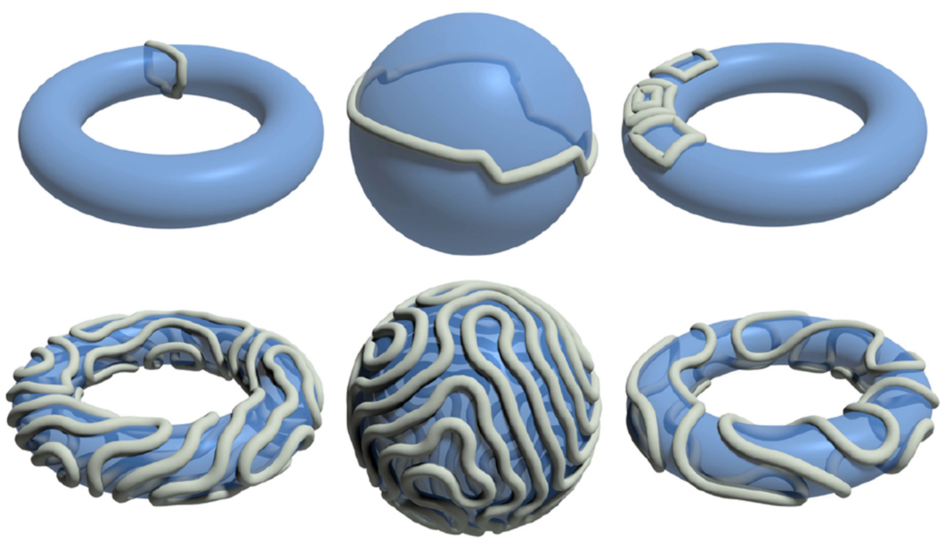
for a given *kernel* k. Intuitively, we might want to assign a kernel such as

$$k(x, y) = \frac{1}{|y - x|}, \tag{2}$$

but this is singular for neighbouring points in the curve. Other choices are available, such as the *tangent-point energy* [BO95], which only penalises close but non-neighbouring points.

Once an energy is chosen, it can be minimised by constructing its *gradient flow*, a differential equation which captures the gradient descent of the energy E. Given an initial curve γ, its evolution under this gradient flow will always lead to curves with the same or with lower energies.

Crucially, a single energy has many gradient flows, depending on the choice of geometry. Curve optimisation is often done in the spaces L^2, H^1, H^2; each space prioritises a different aspect of the minimisation. Recent works have employed fractional Sobolev spaces H^s, choosing the exponent s to match properties of the energy E so that the resulting gradient flow can be solved numerically with efficient methods [YSC21].



Constrained curve growth [YSC21].

Possible avenues of investigation:

1. implementing a curve optimisation scheme in L^2;
2. implementing a curve optimisation scheme in H^1;
3. implementing a curve optimisation scheme in H^2;
4. implementing a curve optimisation scheme in a fractional Sobolev space H^s;
5. validating and benchmarking any of the implementations against other existing algorithms, e.g. KnotPlot [SB02].

Pre-requisite knowledge

Essential: Programming skills in Julia, Python, or similar.

Recommended: A7 Numerical Analysis, B6.1 Numerical Solution of Partial Differential Equations.

Useful: B4.1 Functional Analysis I, B4.2 Functional Analysis II, B5.2 Applied Partial Differential Equations.

Further references

[BO95] Gregory Buck and Jeremey Orloff. A simple energy function for knots. *Topology and its Applications*, 61(3):205–214, February 1995.

[PP93] Ulrich Pinkall and Konrad Polthier. Computing discrete minimal surfaces and their conjugates. *Experimental Mathematics*, 2(1):15–36, January 1993.

[SB02] Robert G. Scharein and Kellogg S. Booth. Interactive knot theory with KnotPlot. In *Multimedia Tools for Communicating Mathematics*, pages 277–290. Springer Berlin Heidelberg, 2002.

[YBSC21] Chris Yu, Caleb Brakensiek, Henrik Schumacher, and Keenan Crane. Repulsive surfaces. July 2021.

[YSC21] Chris Yu, Henrik Schumacher, and Keenan Crane. Repulsive curves. *ACM Transactions on Graphics*, 40(2):1–21, April 2021.