

Curve Repulsion - 1

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1 Theory behind Discretization



Figure 1: Discretization process

We now consider a discretization of a curve. Index the points on the curve by $\mathcal{I} = \{1, 2, \dots, M\}$, such that points are given by $\{\gamma_1, \gamma_2, \dots, \gamma_M\}$

Using the similar notation to the paper by Yu, Schumacher, and Crane, for edge $I = \{\gamma_i, \gamma_j\} \in E$ and for function $u : \mathbb{R}^3 \rightarrow \mathbb{R}$

- $l_I := |\gamma_i - \gamma_j|$
- $T_I := \frac{\gamma_j - \gamma_i}{l_I}$
- $\mathbf{x}_I := \frac{\gamma_i + \gamma_j}{2}$
- $u_I := \frac{u_i + u_j}{2}$
- Syntactic sugar: $u_i \equiv u(\gamma_i)$
- $u[I] := \begin{pmatrix} u_i \\ u_j \end{pmatrix}$

1.1 Discrete Energy

The naïve discretization of $\mathcal{E}_\beta^\alpha := \iint_{M^2} k_\beta^\alpha(\gamma(x), \gamma(y), T(x)) \, dx_\gamma \, dy_\gamma$ where $k_\beta^\alpha(p, q, T) := \frac{|T \times (p - q)|^\alpha}{|p - q|^\beta}$ is given by

$$\sum_{I \in E} \sum_{J \in E} \int_{\bar{I}} \int_{\bar{J}} k_\beta^\alpha(\gamma(x), \gamma(y), T_I) \, dx_\gamma \, dy_\gamma \quad (1)$$

However, in a polygonal curve (hence the discretized curve), (1) is ill-defined.



Figure 2: Near each vertex, the integrand is unbounded.

So resolve this by removing the two neighboring edges.¹ Also approximate the kernel by the average of the kernel evaluated at each pair of appropriate edges (total: 4)

$$\hat{\mathcal{E}}_{\beta}^{\alpha} := \sum_{I,J \in E, I \cap J = \emptyset} \left(\hat{k}_{\beta}^{\alpha} \right)_{I,J} l_I l_J \quad (2)$$

$$\left(\hat{k}_{\beta}^{\alpha} \right)_{I,J} := \frac{1}{4} \sum_{i \in J, j \in J} k_{\beta}^{\alpha} (\gamma_i, \gamma_j, T_I) \quad (3)$$

2 Discrete Gradient Flow in L^2 for Closed Loop

Suppose a curve is discretized as position vectors: x_1, x_2, \dots, x_M (and $x_{M+1} := x_1$).

Also denote the edge from x_i to x_{i+1} as I_i (as opposed to the previous section).

¹In the limit, the contribution from this removed edge goes to zero.

The **discretized energy** E can be expressed as:

$$E = \sum_{i=1}^M \sum_{\substack{j=1 \\ |j-i|>1}} k_{i,j} \|x_{i+1} - x_i\| \|x_{j+1} - x_j\| \quad (4)$$

$$k_{i,j} = \frac{1}{4} (k_{\beta}^{\alpha}(x_i, x_j, T_i) + k_{\beta}^{\alpha}(x_i, x_{j+1}, T_i) + k_{\beta}^{\alpha}(x_{i+1}, x_j, T_i) + k_{\beta}^{\alpha}(x_{i+1}, x_{j+1}, T_i)) \quad (5)$$

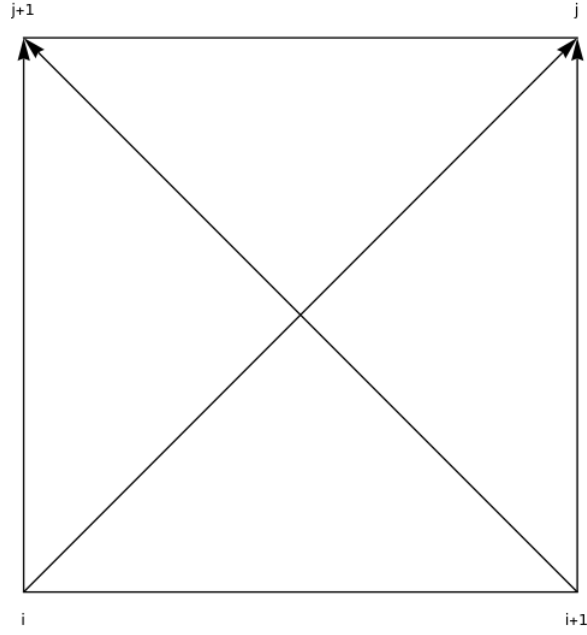


Figure 3: Kernel $k_{i,j}$ computation

Gradient flow equation in L^2 is given by:

$$\dot{x}_i = -\frac{\partial E}{\partial x_i} \quad (6)$$