

Untangling Knots Through Curve Repulsion

Joo-Hyun Paul Kim

March 2, 2023



What the curious folks ponder about

1 Introduction

2 Tangent-Point Energy

3 Gradient Flow

Introduction

A Cool Knot

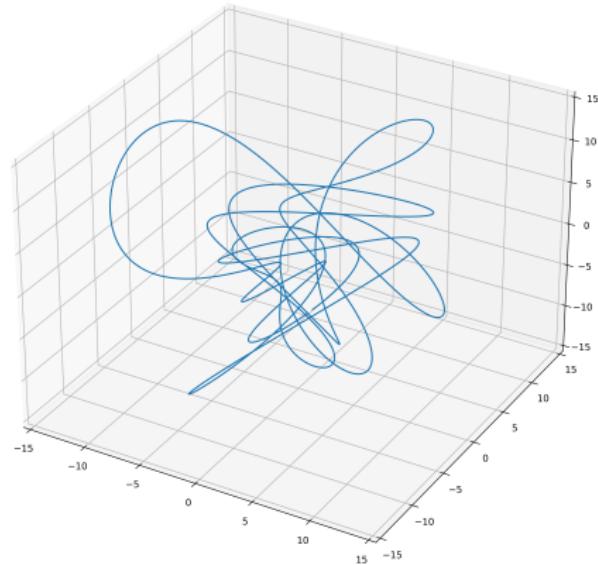


Figure: Imagine your earphones getting tangled like this...

Aim

- Finding a “homotopy” from a knot to an unknot.

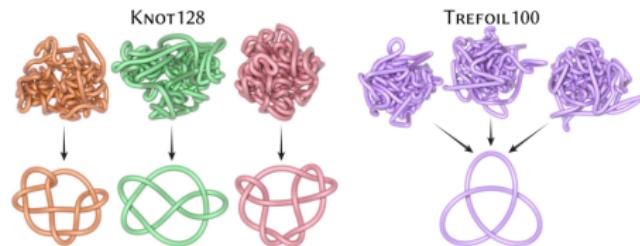


Figure: Unknots of test knots.[3]

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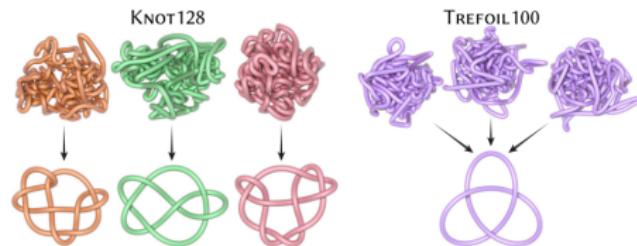


Figure: Unknots of test knots.[3]

- “Avoiding self-intersection”

General Strategy

- ① Define curve energy; penalizing the closeness of points on a curve.
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- ② Attempt to decrease the curve energy by continuously deforming the curve.
 - We evolve the curve according to the gradient flow equation.
 - There is a freedom in choosing the “gradient” here.
- ③ We expect the stationary state to be the “unknot”
 - Or at least a simpler state...

Tangent-Point Energy

Defining Curve Energy

Given an (arc-length parameterised) curve $\gamma : M \rightarrow \mathbb{R}^3$, we wish to assign energy of the form:

$$\mathcal{E}(\gamma) := \iint_{M^2} k(\gamma_x, \gamma_y) \, d\gamma_x \, d\gamma_y \quad (1)$$

such that

- \mathcal{E} is very high when two non-neighbouring points are very close.

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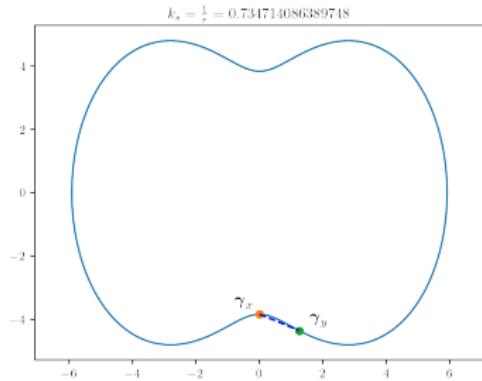
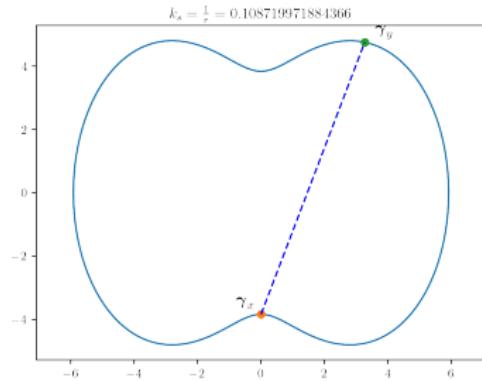
such that

- \mathcal{E} is very high when two non-neighbouring points are very close.

A naïve choice is $k(\gamma_x, \gamma_y) := \frac{1}{\|\gamma_x - \gamma_y\|}$

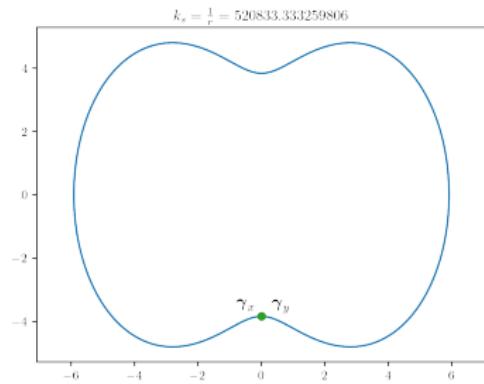
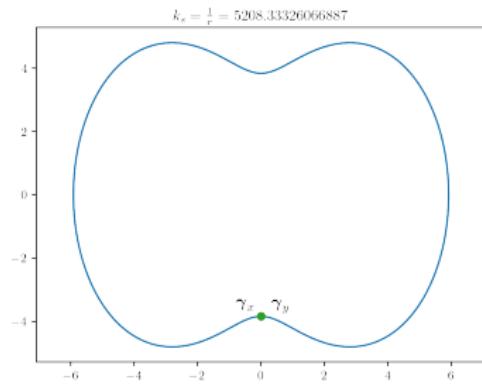
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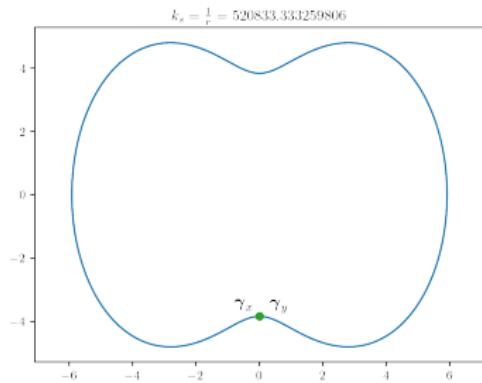
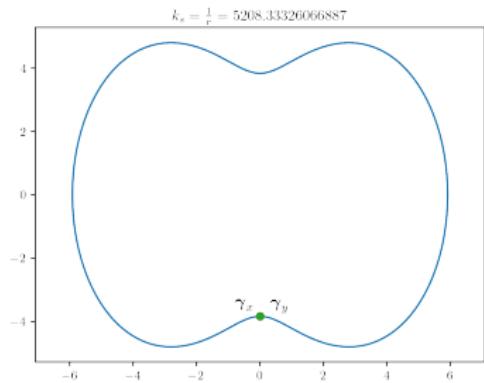
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This energy is not well-defined for a lot of curves!

Buck-Orloff Tangent-Point Energy

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Definition (Buck-Orloff Tangent-Point Energy)

For a smooth curve γ , define

$$\mathcal{E}(\gamma) := \iint_{M^2} k_4^2(\gamma_x, \gamma_y, \mathbf{T}_x) d\gamma_x d\gamma_y$$

where \mathbf{T}_x is the unit tangent vector at γ_x , with the kernel defined as

$$k_4^2(\mathbf{p}, \mathbf{q}, \mathbf{T}) := \frac{\|\mathbf{T} \wedge (\mathbf{p} - \mathbf{q})\|^2}{\|\mathbf{p} - \mathbf{q}\|^4}$$

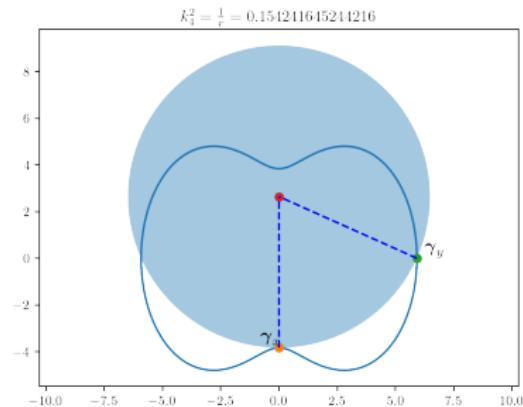
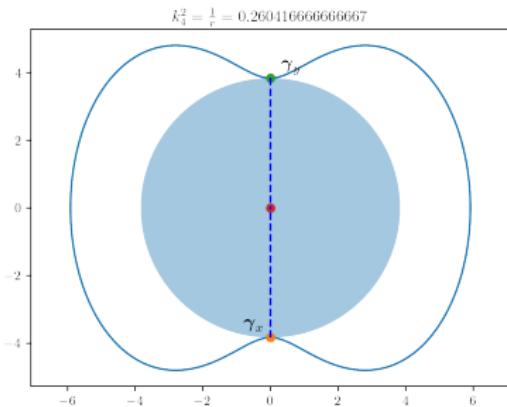
as **Buck-Orloff Tangent-Point Energy**.[1]

Intuition

What is the intuition behind the kernel $k_4^2(\mathbf{p}, \mathbf{q}, \mathbf{T}) := \frac{\|\mathbf{T} \wedge (\mathbf{p} - \mathbf{q})\|^2}{\|\mathbf{p} - \mathbf{q}\|^4}$?

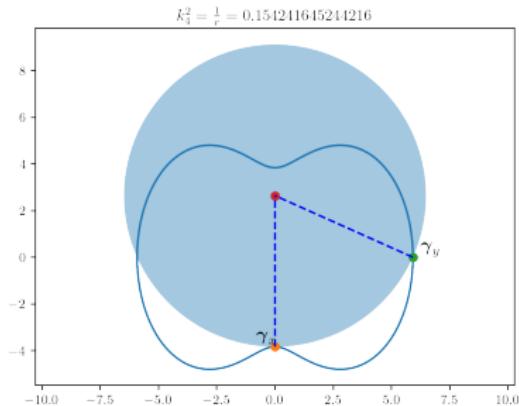
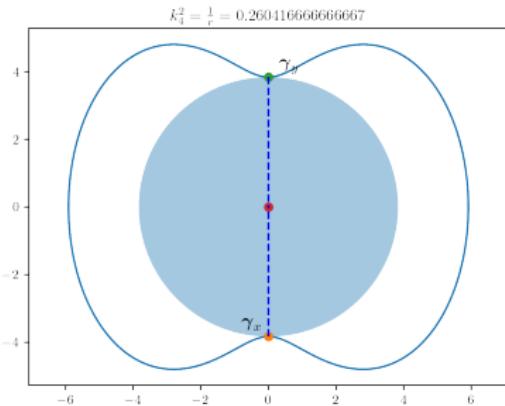
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Remark

Note that closer does not necessarily mean the kernel is larger.

Intuition

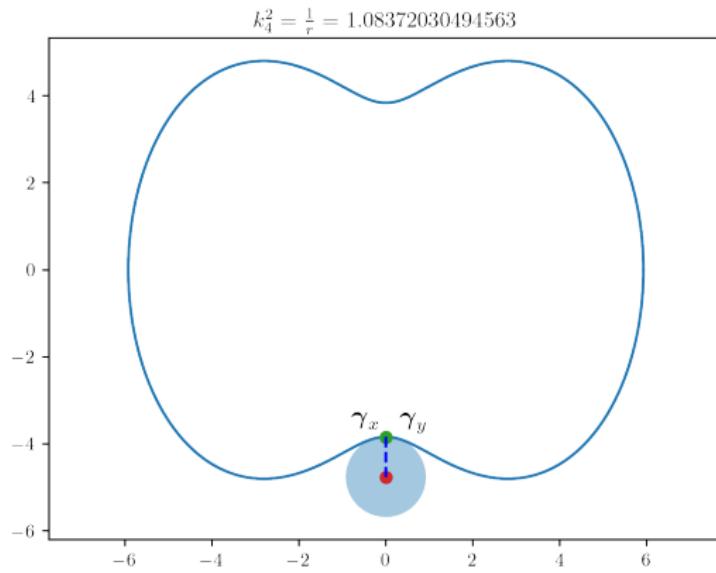


Figure: When two points are very close, the kernel converges to the curvature of the curve.

Example: Buck-Orloff Tangent-Point Energy of a Circle

Example

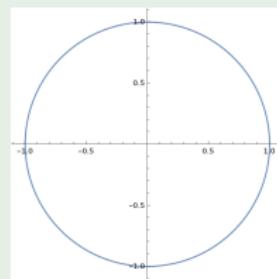
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$$\gamma(t) = (\cos t, \sin t, 0)$$

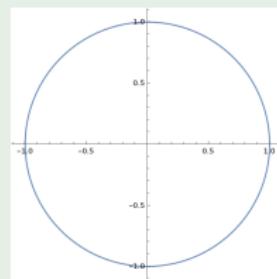


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Then write:

$$\begin{cases} \gamma_x(\theta) = (\cos \theta, \sin \theta, 0) \\ \gamma_y(\phi) = (\cos \phi, \sin \phi, 0) \\ \mathbf{T}_x(\theta) = (-\sin \theta, \cos \theta, 0) \end{cases}$$

Example (Cont.)

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Substituting to Buck-Orloff Tangent-Point energy formula:

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Using a few identities:

$$\mathcal{E}(\gamma) = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \frac{\|\mathbf{T}_x\|^2 \|\gamma_x - \gamma_y\|^2 - (\mathbf{T}_x \cdot (\gamma_x - \gamma_y))^2}{\|\gamma_x - \gamma_y\|^4} d\theta d\phi$$

Example (Cont.)

$$\mathcal{E}(\gamma) = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \frac{||\mathbf{T}_x||^2 ||\gamma_x - \gamma_y||^2 - (\mathbf{T}_x \cdot (\gamma_x - \gamma_y))^2}{||\gamma_x - \gamma_y||^4} d\theta d\phi \quad (2)$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \frac{\sin^2 \frac{\theta-\phi}{2}}{(-1 + \cos(\theta - \phi))^2} d\theta d\phi \quad (3)$$

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Remark

Note that (4) suggests the order at “singularity” is inverse-square.

General Tangent-Point Energy

A more general form of tangent-point energy comes from Yu, Schumacher, and Crane [3]:

Definition (Generalised Tangent-Point Energy)

$$\mathcal{E}_\beta^\alpha(\gamma) := \iint_{M^2} \frac{||\mathbf{T}_x \wedge (\gamma_x - \gamma_y)||^\alpha}{||\gamma_x - \gamma_y||^\beta} d\gamma_x d\gamma_y$$

where $\alpha > 1$ and $\beta \in [\alpha + 2, 2\alpha + 1]$

Remark

When $\alpha = 2$ and $\beta = 4$, we are back to Buck-Orloff.

Gradient Flow

Motivating Gradient Flow

A simple method of minimising a (differentiable) function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the steepest descent [2]

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k \nabla f(\mathbf{x}^k) \quad (6)$$

where $\alpha^k > 0$.

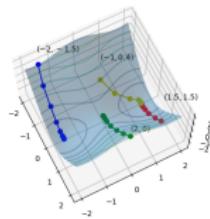


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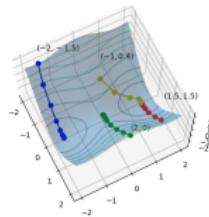


Figure: Steepest Descent

For motivation, take $\alpha^k = \alpha \equiv \text{const.}$

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In the case of functional $\mathcal{E} : X \rightarrow \mathbb{R}$, analogously write steepest descent step:

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Take limit as $\Delta t \rightarrow 0$, then scale time variable to nondimensionalise to get the gradient flow equation.

Definition (Gradient Flow Equation)

$$\frac{\partial f}{\partial t} = -\operatorname{grad}_X f$$

But what is $\operatorname{grad}_X f$?

Gradient in Space of Function?

(Gradient of Function)

$\nabla f(\mathbf{x})$ is such that for all $\mathbf{y} \in \mathbb{R}^n$

$$\nabla f(\mathbf{x}) \cdot \mathbf{y} = \frac{\partial}{\partial \epsilon} f(\mathbf{x} + \epsilon \mathbf{y}) \Big|_{\epsilon=0}$$

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$\text{grad}_X E(f)$ is such that for all $g \in X$,

$$\langle \langle \text{grad}_X E, g \rangle \rangle_X = \frac{\partial}{\partial \epsilon} E(f + \epsilon g) \Big|_{\epsilon=0}$$

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Remark

For $X = L^2$, $(\text{grad}_X E)(f)$ is the first order variation from f with respect to ϵ .

Example: Dirichlet Energy in 1D

Definition (1D Dirichlet Energy)

For a differentiable function $f : \mathcal{I} \rightarrow \mathbb{R}$, define **1D Dirichlet energy**

$$E_D(f) := \int_{\mathcal{I}} |\nabla f(x)|^2 \, dx = \int_{\mathcal{I}} |f'(x)|^2 \, dx$$

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Remark

Dirichlet energy is high for function f that varies a lot, and minimised by any constant function.

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Lemma (L^2 Gradient of Dirichlet Energy)

$$\text{grad}_{L^2} E_D(f) = -\Delta f$$

(for suitable boundary condition if finite interval)

Then the gradient flow equation for Dirichlet energy becomes:

$$\frac{\partial f}{\partial t} = \Delta f \tag{7}$$

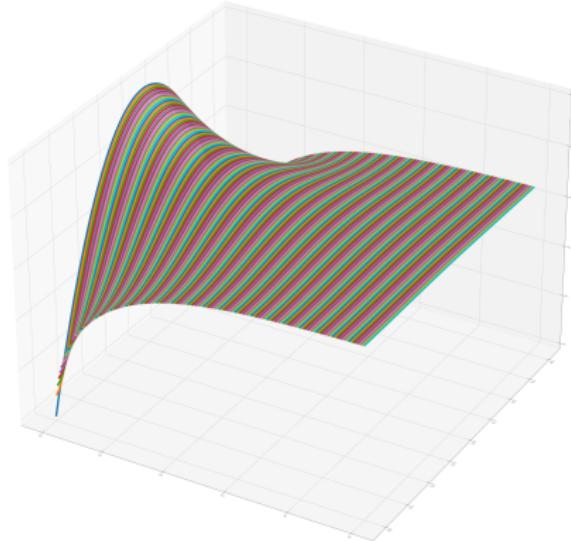


Figure: Solution to L^2 Gradient Flow Equation for Dirichlet Energy (AKA Heat Equation)

Bibliography

- [1] Gregory Buck and Jeremey Orloff. “A simple energy function for knots”. In: *Topology and its Applications* 61.3 (Feb. 1995), pp. 205–214. doi: [10.1016/0166-8641\(94\)00024-w](https://doi.org/10.1016/0166-8641(94)00024-w).
- [2] “Chapter 8: Primal and Dual Projected Subgradient Methods”. In: *First-Order Methods in Optimization*, pp. 195–245. doi: [10.1137/1.9781611974997.ch8](https://doi.org/10.1137/1.9781611974997.ch8). eprint: <https://epubs.siam.org/doi/pdf/10.1137/1.9781611974997.ch8>. URL: <https://epubs.siam.org/doi/abs/10.1137/1.9781611974997.ch8>.
- [3] Chris Yu, Henrik Schumacher, and Keenan Crane. “Repulsive Curves”. In: *ACM Transactions on Graphics* 40.2 (Apr. 2021), pp. 1–21. doi: [10.1145/3439429](https://doi.org/10.1145/3439429).