# Functional Analysis: Gradient

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## 1 Gradient on $\mathbb{R}^n$

For $f : \mathbb{R}^n \to \mathbb{R}$, we may define the gradient $\nabla f$ as:

**Definition 1.** $\nabla f =: v$ such that

$$\frac{\partial}{\partial \epsilon} f\left(x + \epsilon y\right)|_{\epsilon=0} = \langle v, y \rangle_{\mathbb{R}^n} \tag{1}$$

## 2 Gradient on $L^2$

Similarly, for $E : L^2 \to \mathbb{R}$ (a functional), we define the gradient $\nabla E$ as:

**Definition 2.** $\nabla E$ such that

$$\frac{\partial}{\partial \epsilon} E\left[f + \epsilon g\right]|_{\epsilon=0} = \langle \nabla E, g \rangle_{L^2} \tag{2}$$

### 2.1 Dirichlet Energy

For Dirichlet energy defined by $E(f) = \int_{\mathbb{R}} |\nabla f|^2 \, \mathrm{d}x$,

$$\frac{\partial}{\partial \epsilon} E\left(f + \epsilon g\right)|_{\epsilon=0} = \int_{\mathbb{R}} \frac{\delta E}{\delta f} g \tag{3}$$

### 2.2 Tangent Point Energy

For our tangent point energy defined by $E(\gamma) = \int_{M^2} (\cdot) \, \mathrm{d}x_\gamma \, \mathrm{d}y_\gamma$

$$\frac{\partial}{\partial \epsilon} E(\gamma + \epsilon\delta)|_{\epsilon=0} = \int_{M^2} \frac{\delta E}{\delta \gamma} \delta \tag{4}$$