Gradient Flow to Continuous Optimization via Fourier Series

Paul Kim

February 9, 2023

We've been concerned about minimizing the energy functional of the form:

$$\mathcal{E}(\boldsymbol{\gamma}) \coloneqq \int_{C_{\boldsymbol{\gamma}}} \int_{C_{\boldsymbol{\gamma}}} k\left(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2\right) \mathrm{d}\boldsymbol{\gamma}_1 \, \mathrm{d}\boldsymbol{\gamma}_2 \tag{1}$$

where $\boldsymbol{\gamma} : E \to \mathbb{R}^3$ is a parameterization function of a closed curve on the interval $E = [0, 2\pi)$ (without loss of generality).

Note that we assume $\boldsymbol{\gamma}$ to be a periodic function of period 2π.

1 Multidimensional Fourier Series

1.1 1D Fourier Series

Given a continuous 1D 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ (where we only need to define f on $[0, 2\pi)$), there exists a Fourier series representation:

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right) && (2) \\ &= \sum_{n=-\infty}^{\infty} c_n e^{inx} && (3) \end{aligned}$$

where the coefficients $\{a_n\}, \{b_n\}, \{c_n\}$ are given by

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) \, \mathrm{d}x && \in \mathbb{R} && (4) \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) \, \mathrm{d}x && \in \mathbb{R} && (5) \\ c_n &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, \mathrm{d}x && \in \mathbb{C} && (6) \end{aligned}$$

Fourier convergence theorem states that the rate of convergence is $O\left(\frac{1}{n^{p+1}}\right)$ where f has the first jump discontinuity in the p_{th} derivative.[1]

[1] Lecture note

1.2 Multidimensional Extension

For a vector valued function of dimension N (which we will take $N = 3$ for our case), we have Fourier series representation in each of the coordinates.

For $\mathbf{f} : \mathbb{R} \to \mathbb{R}^N$, we write its Fourier series as:

$$\mathbf{f}(x) = \frac{1}{2} \begin{pmatrix} a_{1,0} \\ a_{2,0} \\ \vdots \\ a_{N,0} \end{pmatrix} + \sum_{n=1}^{\infty} \begin{pmatrix} a_{1,n} & b_{1,n} \\ a_{2,n} & b_{2,n} \\ \vdots & \\ a_{N,n} & b_{N,n} \end{pmatrix} \begin{pmatrix} \cos(nx) \\ \sin(nx) \end{pmatrix} \tag{7}$$

$$= \sum_{n=-\infty}^{\infty} \begin{pmatrix} c_{1,n} \\ c_{2,n} \\ \vdots \\ c_{N,n} \end{pmatrix} e^{-inx} \tag{8}$$