Untangling Knots Through Curve Repulsion

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UNIVERSITY OF OXFORD

What the curious folks ponder about

Introduction

A Cool Knot

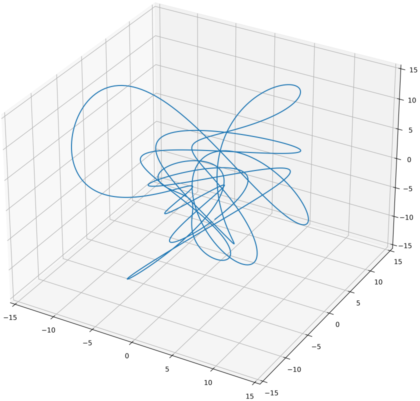

Figure: Imagine your earphones getting tangled like this...

Aim

- Finding a "homotopy" from a knot to an unknot.

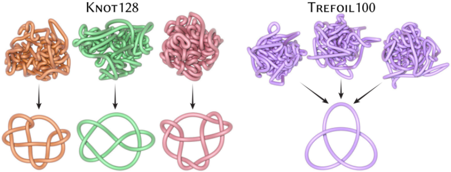

Figure: Unknots of test knots.[2]

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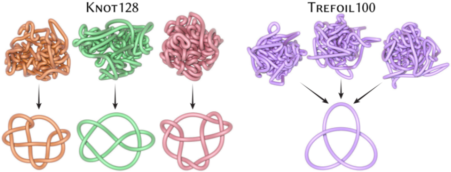

Figure: Unknots of test knots.[2]

- "Avoiding self-intersection"

General Strategy

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    - There is a freedom in choosing the “gradient” here.

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2. Attempt to decrease the curve energy by continuously deforming the curve.
    - We evolve the curve according to the gradient flow equation.
    - There is a freedom in choosing the “gradient” here.
3. We expect the stationary state to be the “unknot”
    - Or at least a simpler state…

Tangent-Point Energy

Defining Curve Energy

Given an (arc-length parameterised) curve $\gamma : M \to \mathbb{R}^3$, we wish to assign energy of the form:

$$\mathcal{E}(\gamma) := \iint_{M^2} k\left(\gamma_x, \gamma_y\right)\, \mathrm{d}\gamma_x\, \mathrm{d}\gamma_y \tag{1}$$

such that

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A naïve choice is $k\left(\gamma_x, \gamma_y\right) := \frac{1}{||\gamma_x - \gamma_y||}$

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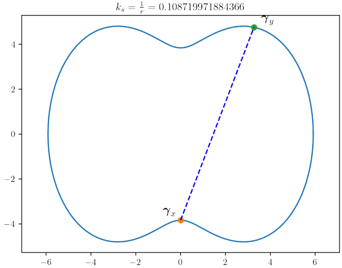

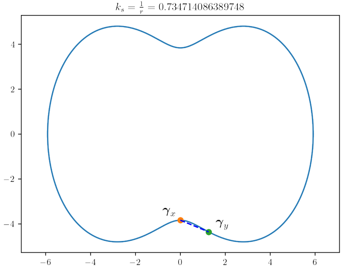

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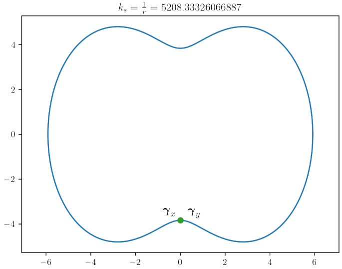

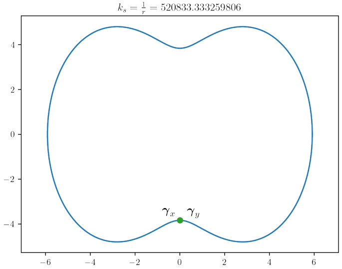

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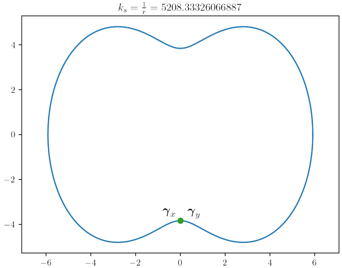

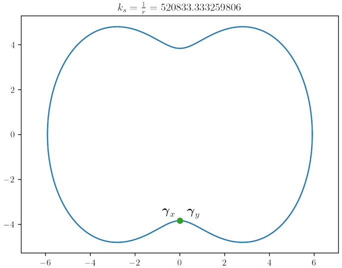

This energy is not well-defined for a lot of curves!

Buck-Orloff Tangent-Point Energy

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Definition (Buck-Orloff Tangent-Point Energy)

For a smooth curve γ, define

$$\mathcal{E}(\gamma) := \iint_{M^2} k_4^2 \left(\gamma_x, \gamma_y, \mathbf{T}_x\right) \, \mathrm{d}\gamma_x \, \mathrm{d}\gamma_y$$

where $\mathbf{T}_x$ is the unit tangent vector at γ_x, with the kernel defined as

$$k_4^2(\mathbf{p}, \mathbf{q}, \mathbf{T}) := \frac{||\mathbf{T} \wedge (\mathbf{p} - \mathbf{q})||^2}{||\mathbf{p} - \mathbf{q}||^4}$$

as **Buck-Orloff Tangent-Point Energy**.[1]

Intuition

What is the intuition behind the kernel $k_4^2\left(\mathbf{p},\mathbf{q},\mathbf{T}\right) := \frac{||\mathbf{T}\wedge(\mathbf{p}-\mathbf{q})||^2}{||\mathbf{p}-\mathbf{q}||^4}$?

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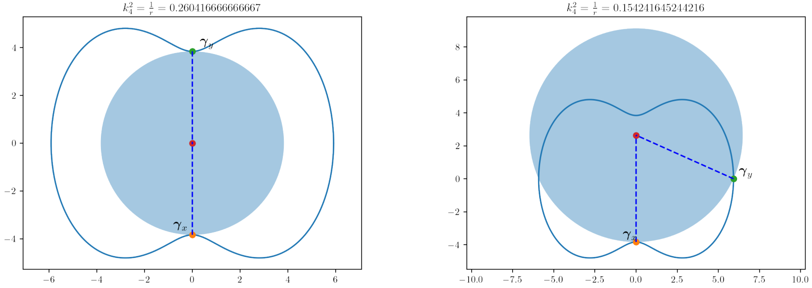

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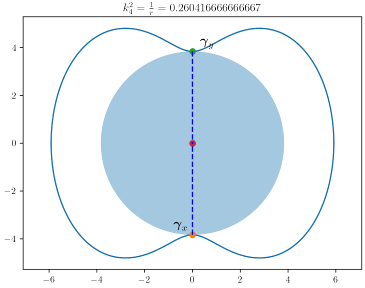

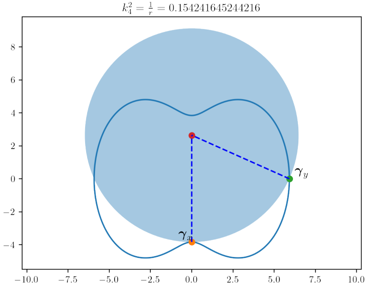

Remark

Note that closer does not necessarily mean the kernel is larger.

Intuition

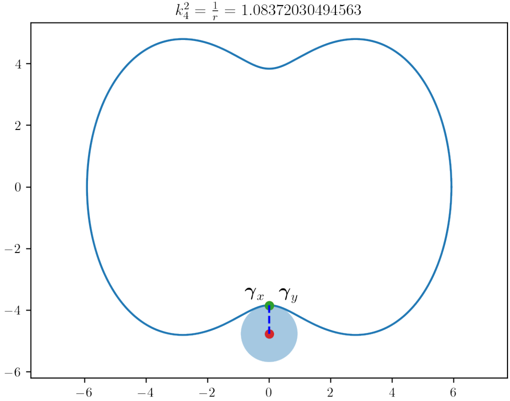

Figure: When two points are very close, the kernel converges to the curvature of the curve.

Example: Buck-Orloff Tangent-Point Energy of a Circle

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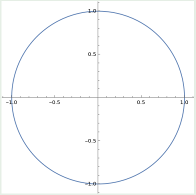

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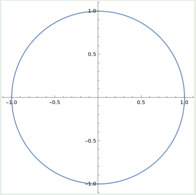

Then write:

$$\begin{cases} \gamma_x(\theta) = (\cos\theta, \sin\theta, 0) \\ \gamma_y(\phi) = (\cos\phi, \sin\phi, 0) \\ \mathbf{T}_x(\theta) = (-\sin\theta, \cos\theta, 0) \end{cases}$$

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Example (Cont.)

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Substituting to Buck-Orloff Tangent-Point energy formula:

$$\mathcal{E}(\gamma) := \iint_{M^2} \frac{||\mathbf{T}_x \wedge (\gamma_x - \gamma_y)||^2}{||\gamma_x - \gamma_y||^4} \, d\gamma_x \, d\gamma_y$$

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Using a few identities:

$$\mathcal{E}(\gamma) = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \frac{||\mathbf{T}_x||^2 ||\gamma_x - \gamma_y||^2 - \left(\mathbf{T}_x \cdot (\gamma_x - \gamma_y)\right)^2}{||\gamma_x - \gamma_y||^4} \, \mathrm{d}\theta \, \mathrm{d}\phi$$

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$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \frac{\sin^2 \frac{\theta-\phi}{2}}{\left(-1 + \cos(\theta - \phi)\right)^2} \, d\theta \, d\phi \tag{3}$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \frac{\sin^2 \frac{\theta-\phi}{2}}{\sin^4 \frac{\theta-\phi}{2}} \, d\theta \, d\phi \tag{4}$$

$$= \pi^2 \tag{5}$$

Example: Buck-Orloff Tangent-Point Energy of a Circle

Example (Cont.)

$$\begin{aligned}
\mathcal{E}(\gamma) &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi} \frac{||\mathbf{T}_x||^2||\gamma_x-\gamma_y||^2-\left(\mathbf{T}_x\cdot(\gamma_x-\gamma_y)\right)^2}{||\gamma_x-\gamma_y||^4}\,\mathrm{d}\theta\,\mathrm{d}\phi && (2)\\
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&= \pi^2 && (5)
\end{aligned}$$

Remark

Note that (4) suggests the order at "singularity" is inverse-square.

General Tangent-Point Energy

A more general form of tangent-point energy comes from Yu, Schumacher, and Crane [2]:

Definition (Generalised Tangent-Point Energy)

$$\mathcal{E}^{\alpha}_{\beta}(\gamma) := \iint_{M^2} \frac{||\mathbf{T}_x \wedge (\gamma_x - \gamma_y)||^{\alpha}}{||\gamma_x - \gamma_y||^{\beta}} \, d\gamma_x \, d\gamma_y$$

where $\alpha > 1$ and $\beta \in [\alpha + 2, 2\alpha + 1)$

Remark

When $\alpha = 2$ and $\beta = 4$, we are back to Buck-Orloff

Bibliography

[1] Gregory Buck and Jeremey Orloff. "A simple energy function for knots". In: *Topology and its Applications* 61.3 (Feb. 1995), pp. 205–214. DOI: 10.1016/0166-8641(94)00024-w.

[2] Chris Yu, Henrik Schumacher, and Keenan Crane. "Repulsive Curves". In: *ACM Transactions on Graphics* 40.2 (Apr. 2021), pp. 1–21. DOI: 10.1145/3439429.