Gradient Flow to Continuous Optimization via Fourier Series

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We've been concerned about minimizing the energy functional of the form:

$$\mathcal{E}(\boldsymbol{\gamma}) := \int_{C_{\boldsymbol{\gamma}}} \int_{C_{\boldsymbol{\gamma}}} k\left(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2\right) \, \mathrm{d}\boldsymbol{\gamma}_1 \, \mathrm{d}\boldsymbol{\gamma}_2 \tag{1}$$

where $\boldsymbol{\gamma} : E \to \mathbb{R}^3$ is a parameterization function of a closed curve on the interval $E = [0, 2\pi)$ (without loss of generality).

Note that we assume $\boldsymbol{\gamma}$ to be a periodic function of period 2π.

1 Multidimensional Fourier Series

1.1 1D Fourier Series

Given a continuous 1D 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ (where we only need to define f on $[0, 2\pi)$), there exists a Fourier series representation:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(nx\right) + b_n \sin\left(nx\right)\right) \tag{2}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{3}$$

where the coefficients $\{a_n\}, \{b_n\}, \{c_n\}$ are given by

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos\left(nx\right) \mathrm{d}x \qquad \in \mathbb{R} \tag{4}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin\left(nx\right) \mathrm{d}x \qquad \in \mathbb{R} \tag{5}$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \mathrm{d}x \qquad \in \mathbb{C} \tag{6}$$

Fourier convergence theorem states that the rate of convergence is $O\left(\frac{1}{n^{p+1}}\right)$ where f has the first jump discontinuity in the pth derivative.[1]

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[1] Lecture note

1.2 Multidimensional Extension

For a vector valued function of dimension N (which we will take $N = 3$ for our case), we have Fourier series representation in each of the coordinates.

For $\mathbf{f} : \mathbb{R} \to \mathbb{R}^N$, we write its Fourier series as:

$$\mathbf{f}(x) = \frac{1}{2} \begin{pmatrix} a_{1,0} \\ a_{2,0} \\ \vdots \\ a_{N,0} \end{pmatrix} + \sum_{n=1}^{\infty} \begin{pmatrix} a_{1,n} & b_{1,n} \\ a_{2,n} & b_{2,n} \\ \vdots & \\ a_{N,n} & b_{N,n} \end{pmatrix} \begin{pmatrix} \cos(nx) \\ \sin(nx) \end{pmatrix} \tag{7}$$

$$= \sum_{n=-\infty}^{\infty} \begin{pmatrix} c_{1,n} \\ c_{2,n} \\ \vdots \\ c_{N,n} \end{pmatrix} e^{-inx} \tag{8}$$

where the coefficients $\{a_{i,n}\}, \{b_{i,n}\}, \{c_{i,n}\}$ are given by

$$a_{i,n} = \frac{1}{\pi} \int_0^{2\pi} f_i(x) \cos(nx) \, \mathrm{d}x \tag{9}$$

$$b_{i,n} = \frac{1}{\pi} \int_0^{2\pi} f_i(x) \sin(nx) \, \mathrm{d}x \tag{10}$$

$$c_{i,n} = \frac{1}{2\pi} \int_0^{2\pi} f_i(x) e^{-inx} \, \mathrm{d}x \tag{11}$$

for $i = 1, 2, \cdots, N$

2 Gradient Flow to Continuous Optimization

For minimization of $\mathcal{E}$, the original approach is to use the gradient flow equation:

$$\frac{\partial \mathcal{E}}{\partial t} = -\operatorname{grad}_X E \tag{12}$$

where grad_X is gradient on inner product space X.

This is needed as a description of reduction of functional $\mathcal{E}$ was needed. For a solution, one would could use a numerical method to solve the differential equation by sampling at different points on the curve.

However, consider expressing $\boldsymbol{\gamma}(t)$ as a 3D Fourier series as:

$$\boldsymbol{\gamma}(t) = \sum_{n=-\infty}^{\infty} \mathbf{c}_n e^{-int} \tag{13}$$

One could approximate this to J terms:

$$\boldsymbol{\gamma}_N(t) = \sum_{n=-J}^{J} \mathbf{c}_n e^{-int} \tag{14}$$

(Noting $||\boldsymbol{\gamma} - \boldsymbol{\gamma}_J|| = O\left(\frac{1}{J^{p+1}}\right)$)

Then now the reduction process of $\mathcal{E}\left(\gamma\right)$ can be approximated by the reduction process of $\mathcal{E}\left(\gamma_J\right)$. It is paramount to note that $\mathcal{E}\left(\gamma_J\right) : \mathbb{R}^{3(2J+1)} = \mathbb{R}^{6J+3} \rightarrow \mathbb{R}$ where the parameters are the Fourier coefficients in 3D. The idea is now to consider this a standard optimiation problem:

$$\min_{[\mathbf{c}_{-J}, \cdots, \mathbf{c}_J]} \mathcal{E}\left(\gamma_J\right) \tag{15}$$

which can be attempted by standard techniques such as SDM, bArmijo linesearch, etc.