# Untangling Knots Through Curve Repulsion

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## Abstract

Curves are one of the fundamental objects in geometry and engineering, yet most analysis of curves often disregard their physical characteristics such as their spacial volume or uncrossability. One common situation that such physical characteristics become significant is when one attempts to untangle a knot. An approach to achieve this is to assign an "energy" to a curve such that this energy would increase when two points on "different sides" of a curve are closer, then one continuously deforms the curve to reduce this energy, the expectation being that the curve that achieves minimal energy must be the untangled knot. This dissertation explores numerical methods of achieving this.

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# Part I
# Introduction

Placeholder Text for Introduction

## 1 Preamble

### 1.1 Steepest Descent to Gradient Flow Equation

For minimising a differentiable function $f : E \subset \mathbb{R}^n \to \mathbb{R}$, there is a well-known method known as **steepest descent method** (SDM)[1].

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k \nabla f \left(\mathbf{x}^k\right) \tag{1}$$

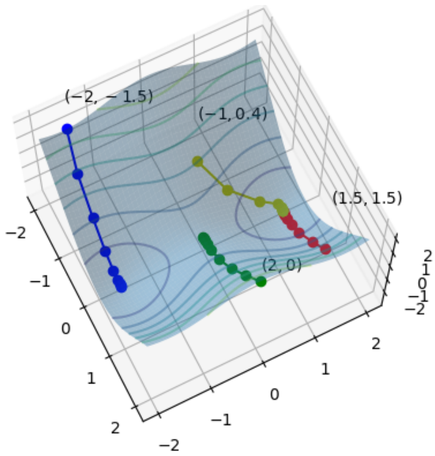


Figure 1: SDM applied to $f(x, y) = -3\cos x + \cos^2 y$ at different initial points.

Starting from the initial input point $\mathbf{x}^0$, at each iteration, input points $\left\{\mathbf{x}^k\right\}$ move in the direction of "steepest" decrease, with specified step size $\alpha_k > 0$, reducing the value at evaluation of $f$. Note that in general, this method is not guaranteed to find the minimiser. On the other hand, convergence is guaranteed under certain assumptions, for example, convexity and $L$-smoothness with a certain choice of step size $\alpha_k$.

Analogously, differential equation known describing the reduction process of a functional $F : \mathcal{F} \to \mathbb{R}$ (where $\mathcal{F}$ is some set of functions) can be motivated. Starting from (1), replacing $\mathbf{x}^k$ to $f_k$ and $\nabla f\left(\mathbf{x}^k\right)$ to $\operatorname{grad}_X F\left(f_k\right)$

$$f_{k+1} = f_k - \alpha_k \operatorname{grad}_X F\left(f_k\right) \tag{2}$$

Now think of $f_k$ as "snapshots" at certain time $t = t_k$. Without loss of generality, let $\alpha_k \equiv 1$.[1] Dividing (2) by time step $\Delta t := t_{k+1} - t_k$, and taking the limit as $\Delta t \to 0$, we acquire the **gradient flow equation**[2].

$$\frac{\partial F}{\partial t} = -\operatorname{grad}_X F \tag{3}$$

where index $k$ transforms to "time" variable $t$.

Note that grad of a functional is not defined yet. This depends on the function space $X$ of interest.

[1]This is justified by taking a differrent time scale; essentially nondimensionalisation.

Part II

# Curve Energy Reduction via Gradient Flow

# Part III
# Functional Reduction to Function Reduction

## References


[1] "Chapter 8: Primal and Dual Projected Subgradient Methods". In: *First-Order Methods in Optimization*, pp. 195–245. DOI: 10.1137/1.9781611974997.ch8. eprint: https://epubs.siam.org/doi/pdf/10.1137/1.9781611974997.ch8. URL: https://epubs.siam.org/doi/abs/10.1137/1.9781611974997.ch8.

[2] Chris Yu, Henrik Schumacher, and Keenan Crane. "Repulsive Curves". In: *ACM Transactions on Graphics* 40.2 (Apr. 2021), pp. 1–21. DOI: 10.1145/3439429.