Simple Energy

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When tangent-point energy was used evolve the curve, it turned out that in a discrete setting, it is minimized when the points are colinear.

Instead, it might be worth investigating the simplest of curve energies.

NOTE: The "simple energy" is not scale invariant $\alpha < 2$, which results in the phenomenon of "reduction to singularity" when minimizing the energy.

1 Simple Energy

For a closed curve $C = \gamma(t)$ over $t \in M$, define **simple energy** as:

$$\mathcal{E}^{\alpha}(\gamma) \coloneqq \iint_{M^2} \frac{1}{|\gamma(x) - \gamma(y)|^{\alpha}} \, \mathrm{d}\gamma_x \, \mathrm{d}\gamma_y \tag{1}$$

where $\alpha > 0$.

For $\alpha \geq 1$, this energy is ill-defined in an analytic framework. However, in a discrete scheme inspired by this, there is a natural way to make this well-defined.

Note that this energy is scale-invariant if and only if $\alpha = 2$. This may be an important fact. When $\alpha < 2$, this energy scales as $k^{2-\alpha}$ where k is a scale factor to a given curve (eg. $\Gamma = k\gamma$), meaning that the simple energy is minimized when the curve reduces to singularity (which is not relevant) On the other hand, if $\alpha > 2$, then by the same reason, energy reduces as the curve expands.

For our purposes, if simple energy is to be used, <u>α needs to be taken at least 2</u>.

2 Discrete Simple Energy

Definition 1. Given a closed, non-intersection polygonal curve $\Gamma \coloneqq (x_0, x_1, \cdots, x_{J-1})$ $(x_0 = x_J)$ define **discrete simple energy of first kind** as:

$$E_1^{\alpha}(\Gamma) = \sum_{i=0}^{J-1} \sum_{j \neq i} \frac{1}{|x_i - x_j|^{\alpha}} |x_i - x_{i+1}||x_j - x_{j+1}| \tag{2}$$

where $\alpha > 0$.

This is the simplest energy to compute, but it may not be symmetric.

To address the issue of non-symmetricness, a simple modification can be done:

Definition 2. Given a closed, non-intersection polygonal curve $\Gamma := (x_0, x_1, \cdots, x_{J-1})$ $(x_0 = x_J)$ define **discrete simple energy of second kind** as:

$$E_2^{\alpha}(\Gamma) = \sum_{i=0}^{J-1} \sum_{|i-j|>1} k_{i,j}^{\alpha} |x_i - x_{i+1}||x_j - x_{j+1}| \tag{3}$$

where

$$k_{i,j}^{\alpha} = \frac{1}{4} \left(\frac{1}{|x_i - x_j|^{\alpha}} + \frac{1}{|x_i - x_{j+1}|^{\alpha}} + \frac{1}{|x_{i+1} - x_j|^{\alpha}} + \frac{1}{|x_{i+1} - x_{j+1}|^{\alpha}} \right) \tag{4}$$

and $\alpha > 0$. Note that the indexing must be done cyclically, so $|i - j| > 1$ is modulo J.