

# Dirichlet Energy in Different Spaces

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## 1 Relevant Definitions

**Definition 1** (Gradient Flow). Given an energy (functional)  $\mathcal{E}(f)$ , *gradient flow* is:

$$\frac{d}{dt}f = -\text{grad } \mathcal{E}(f) \quad (1)$$

**Definition 2** (Differential). *Differential*  $d\mathcal{E}$  describes change in  $\mathcal{E}$  due to<sup>1</sup>  $u$ :

$$d\mathcal{E}|_f(u) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\mathcal{E}(f + \epsilon u) - \mathcal{E}(f)) \quad (2)$$

**Definition 3** (Gradient). Given a space  $X$ , *gradient* of  $\mathcal{E}$  is the unique function  $\text{grad}_X \mathcal{E}$  such that,

$$\langle \text{grad}_X \mathcal{E}, u \rangle_V = d\mathcal{E}(u) \quad \forall u \in X \quad (3)$$

## 2 Dirichlet Energy Example

**Definition 4** (Dirichlet Energy). Dirichlet Energy is defined as:

$$\mathcal{E}_D(f) := \frac{1}{2} \int_{\Omega} |\text{grad } f(x)|^2 dx = \frac{1}{2} \langle \nabla f, \nabla f \rangle_{L^2} = -\frac{1}{2} \langle \Delta f, f \rangle_{L^2} \quad (4)$$

Computing the differential  $d\mathcal{E}_D|_f(u)$  of Dirichlet energy using (2),

$$d\mathcal{E}_D|_f(u) = -\frac{1}{2} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\langle \Delta(f + \epsilon u), f + \epsilon u \rangle_{L^2} - \langle \Delta f, f \rangle_{L^2}) \quad (5)$$

$$= -\frac{1}{2} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\epsilon \langle \Delta f, u \rangle_{L^2} + \epsilon \langle \Delta u, f \rangle_{L^2} + \epsilon^2 \langle u, u \rangle_{L^2}) \quad (6)$$

$$= -\frac{1}{2} (\langle \Delta f, u \rangle_{L^2} + \langle \Delta u, f \rangle_{L^2}) \quad (7)$$

$$= -\frac{1}{2} (\langle \Delta f, u \rangle_{L^2} + \langle \Delta f, u \rangle_{L^2}) \quad (8)$$

$$= -\langle \Delta f, u \rangle_{L^2} \quad (9)$$

where the last step is by integration by parts (twice)<sup>2</sup>.

<sup>1</sup>analogous in traditional vector space would be “in the direction of”  $u$

<sup>2</sup>This trick will be used all over the place when constructing gradients. See 4.1 in the Appendix

## 2.1 Gradient Flow in $L^2$

By (3), we may spot from (9) that the gradient can be written as

$$\text{grad } \mathcal{E}_{L^2} = -\Delta f \quad (10)$$

Hence, the gradient flow equation (1) can be written as:

$$\frac{d}{dt}f = \Delta f \quad (11)$$

which is the heat equation.

## 2.2 Gradient Flow in $H^1$

We may express the differential (9) as

$$d\mathcal{E}_D|_f(u) = -\langle \Delta f, u \rangle_{L^2} \quad (12)$$

$$= \langle \nabla f, \nabla u \rangle_{L^2} \quad (13)$$

$$= \langle f, u \rangle_{H^1} \quad (14)$$

So by (3), the gradient in  $H^1$  can be written as

$$\text{grad } \mathcal{E}_{H^1} = f \quad (15)$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt}f = -f \quad (16)$$

which describes exponential decay.

## 2.3 Gradient Flow in $H^{-1}$

We may express the differential (9) as

$$d\mathcal{E}_D|_f(u) = -\langle \Delta f, u \rangle_{L^2} \quad (17)$$

$$= \langle \nabla (\Delta f), \nabla (\Delta^{-1}u) \rangle_{L^2} \quad (18)$$

$$= \langle \Delta f, \Delta^{-1}u \rangle_{H^1} \quad (19)$$

$$= \langle \Delta^2 f, u \rangle_{H^{-1}} \quad (20)$$

So by (3), the gradient in  $H^{-1}$  can be written as

$$\text{grad } \mathcal{E}_{H^{-1}} = \Delta^2 f \quad (21)$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt}f = -\Delta^2 f \quad (22)$$

## 2.4 Gradient Flow in $H^2$

We may express the differential (9) as

$$d\mathcal{E}_D|_f(u) = -\langle \Delta f, u \rangle_{L^2} \quad (23)$$

$$= -\langle f, \Delta u \rangle_{L^2} \quad (24)$$

$$= -\langle \Delta^{-1} f, u \rangle_{H^2} \quad (25)$$

So by (3), the gradient in  $H^2$  can be written as

$$\text{grad } \mathcal{E}_{H^2} = -\Delta^{-1} f \quad (26)$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt} f = \Delta^{-1} f \quad (27)$$

## 3 Numerically Solving Gradient Flow Equations

### 3.1 Gradient Flow in $L^2$

For  $L^2$ , given the gradient flow equation (11), we may write down explicit Euler scheme:

$$\frac{f_j^{m+1} - f_j^m}{\Delta t} = \frac{f_{j+1}^m - 2f_j^m + f_{j-1}^m}{(\Delta x)^2} \quad (28)$$

which can be rewritten as

$$f_j^{m+1} = f_j^m + \mu (f_{j+1}^m - 2f_j^m + f_{j-1}^m) \quad (29)$$

where  $\mu := \frac{\Delta t}{(\Delta x)^2}$  is called the *CFL number*.

This method is known to have consistency error

$$T_j^m = \mathcal{O}((\Delta x)^2 + \Delta t) \quad (30)$$

### 3.2 Gradient Flow in $H^1$

Given the gradient flow equation (16), we may write down explicit Euler scheme:

$$\frac{f_j^{m+1} - f_j^m}{\Delta t} = -f_j^m \quad (31)$$

which can be rewritten as

$$f_j^{m+1} = f_j^m - (\Delta t) f_j^m \quad (32)$$

### 3.3 Gradient Flow in $H^{-1}$

Given the gradient flow equation (22), we may write down explicit Euler scheme:

$$\frac{f_j^{m+1} - f_j^m}{\Delta t} = -\frac{f_{j+2}^m - 4f_{j+1}^m + 6f_j^m - 4f_{j-1}^m + f_{j-2}^m}{(\Delta x)^4} \quad (33)$$

### 3.4 Gradient Flow in $H^2$

Given the gradient flow equation (27), it can be rewritten as

$$\frac{d}{dt}\Delta f = f \quad (34)$$

we may write down explicit Euler scheme:

$$\frac{1}{\Delta t} \left( \frac{f_{j+1}^{m+1} - 2f_j^{m+1} + f_{j-1}^{m+1}}{(\Delta x)^2} - \frac{f_{j+1}^m - 2f_j^m + f_{j-1}^m}{(\Delta x)^2} \right) = f_j^m \quad (35)$$

which can be rewritten as

$$\begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} f_{j-1}^{m+1} \\ f_j^{m+1} \\ f_{j+1}^{m+1} \end{pmatrix} = \begin{pmatrix} 1 & -2 + \Delta t(\Delta x)^2 & 1 \end{pmatrix} \begin{pmatrix} f_{j-1}^m \\ f_j^m \\ f_{j+1}^m \end{pmatrix} \quad (36)$$

Assuming Dirichlet boundary condition  $f_0^m = a$  and  $f_J^m = b$ , we may write this as a matrix equation:

$$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ \vdots & & & & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{J-4} \\ f_{J-3} \\ f_{J-2} \\ f_{J-1} \end{pmatrix}^{m+1} = \begin{pmatrix} -2 + \Delta t(\Delta x)^2 & 1 & & & & \\ 1 & -2 + \Delta t(\Delta x)^2 & 1 & & & \\ & 1 & -2 + \Delta t(\Delta x)^2 & 1 & & \\ \vdots & & & & \ddots & \\ & & & 1 & -2 + \Delta t(\Delta x)^2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{J-4} \\ f_{J-3} \\ f_{J-2} \\ f_{J-1} \end{pmatrix}^m$$

It might be worth noting that

$$A_n^{-1} := \underbrace{\begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}}_{n \times n}^{-1} = -\frac{1}{n+1} \underbrace{\begin{pmatrix} n & (n-1) & (n-2) & \cdots & 1 \\ (n-1) & 2(n-1) & 2(n-2) & \cdots & 2 \\ (n-2) & 2(n-2) & 3(n-2) & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}}_{\text{Symmetric Matrix}} \quad (37)$$

In terms of practical computation, it might be worth investigating how the condition number grows as the size of  $A_n^{-1}$  (which is equivalent to the condition

number of  $A_n$ ) grows. To do this, we could investigate eigenvalues<sup>3</sup> A (potentially) useful fact about this matrix is that the characteristic equation can be written recursively.

$$\begin{cases} |A_n - \lambda I| =: J_n = -\delta J_{n-1} - J_{n-2} \\ J_1 = -\delta \\ J_2 = \delta^2 - 1 \end{cases} \quad (38)$$

where  $\delta := \lambda + 2$ .<sup>4</sup>

This method has consistency error

$$T_j^m = \mathcal{O}\left((\Delta x)^2 + \Delta t\right) \quad (39)$$

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<sup>3</sup>Analytically, solutions to the characteristic equation.

<sup>4</sup>The term *continuant* might be interesting to look at

## 4 Appendix

### 4.1 A (Somewhat) Careful Justification for IBP

Suppose we are concerned with the integral  $I := \langle \Delta u, f \rangle_{L^2} = \int_R (\Delta u) f \, d\mathbf{x}$ , where  $R$  is a bounded region with boundary  $\partial R$ .

#### 4.1.1 IBP (on Gradient Operator)

One could show that

$$\langle \Delta u, f \rangle_{L^2} = -\langle \nabla u, \nabla f \rangle_{L^2} = - \int_R \nabla u \cdot \nabla f \, d\mathbf{x} \quad (40)$$

using the identity<sup>5</sup>:

$$\nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \quad (41)$$

Note that

$$\langle \Delta u, f \rangle_{L^2} = - \int_R \nabla u \cdot \nabla f \, d\mathbf{x} \quad (42)$$

$$= \int_R \nabla \cdot (f \nabla u) \, d\mathbf{x} - \int_R \nabla u \cdot \nabla f \, d\mathbf{x} \quad (43)$$

$$= \oint_{\partial R} f \frac{du}{dn} \, ds - \int_R \nabla u \cdot \nabla f \, d\mathbf{x} \quad (44)$$

(Need the first term of last line to be zero...)

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<sup>5</sup>This is also known as Green's Identity in some literature