Dirichlet Energy in Different Spaces

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1 Relevant Definitions

Definition 1 (Gradient Flow)**.** Given an energy (functional) $\mathcal{E}(f)$, *gradient flow* is:

$$\frac{d}{dt}f = -\operatorname{grad}\mathcal{E}(f) \tag{1}$$

Definition 2 (Differential)**.** *Differential* $d\mathcal{E}$ describes change in $\mathcal{E}$ due to[1] u:

$$d\mathcal{E}|_f(u) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathcal{E}(f+\epsilon u) - \mathcal{E}(f)\right) \tag{2}$$

Definition 3 (Gradient)**.** *Gradient* of $\mathcal{E}$ is the unique function $\operatorname{grad}\mathcal{E}$ such that,

$$\langle\langle \operatorname{grad}\mathcal{E}, u\rangle\rangle_V = d\mathcal{E}(u) \tag{3}$$

2 Dirichlet Energy Example

Definition 4 (Dirichlet Energy)**.** Dirichlet Energy is defined as:

$$\mathcal{E}_D(f) \coloneqq \frac{1}{2}\int_\Omega |\operatorname{grad} f(x)|^2\,\mathrm{d}x = \frac{1}{2}\langle\langle \nabla f, \nabla f\rangle\rangle_{L^2} = -\frac{1}{2}\langle\langle \Delta f, f\rangle\rangle_{L^2} \tag{4}$$

Computing the differential $d\mathcal{E}_D|_f(u)$ of Dirichlet energy using (2),

$$\begin{aligned}
d\mathcal{E}_D|_f(u) &= -\frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\langle\langle \Delta(f+\epsilon u), f+\epsilon u\rangle\rangle_{L^2} - \langle\langle \Delta f, f\rangle\rangle_{L^2}\right) && (5)\\
&= -\frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\epsilon\langle\langle \Delta f, u\rangle\rangle_{L^2} + \epsilon\langle\langle \Delta u, f\rangle\rangle_{L^2} + \epsilon^2\langle\langle u, u\rangle\rangle_{L^2}\right) && (6)\\
&= -\frac{1}{2}\left(\langle\langle \Delta f, u\rangle\rangle_{L^2} + \langle\langle \Delta u, f\rangle\rangle_{L^2}\right) && (7)\\
&= -\frac{1}{2}\left(\langle\langle \Delta f, u\rangle\rangle_{L^2} + \langle\langle \Delta f, u\rangle\rangle_{L^2}\right) && (8)\\
&= -\langle\langle \Delta f, u\rangle\rangle_{L^2} && (9)
\end{aligned}$$

where the last step is by integration by parts (twice)[2].

[1] analogous in traditional vector space would be "in the direction of" u

[2] This trick will be used all over the place when constructing gradients. See 4.1 in the Appendix

2.1 Gradient Flow in L^2

By (3), we may spot from (9) that the gradient can be written as

$$\operatorname{grad} \mathcal{E}_{L^2} = -\Delta f \tag{10}$$

Hence, the gradient flow equation (1) can be written as:

$$\frac{d}{dt} f = \Delta f \tag{11}$$

which is the heat equation.

2.2 Gradient Flow in H^1

We may express the differential (9) as

$$\begin{aligned} d\mathcal{E}_D|_f(u) &= -\langle\langle \Delta f, u \rangle\rangle_{L^2} && (12)\\ &= \langle\langle \nabla f, \nabla u \rangle\rangle_{L^2} && (13)\\ &= \langle\langle f, u \rangle\rangle_{H^1} && (14) \end{aligned}$$

So by (3), the gradient in H^1 can be written as

$$\operatorname{grad} \mathcal{E}_{H^1} = f \tag{15}$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt} f = -f \tag{16}$$

which describes exponential decay.

2.3 Gradient Flow in H^{-1}

We may express the differential (9) as

$$\begin{aligned} d\mathcal{E}_D|_f(u) &= -\langle\langle \Delta f, u \rangle\rangle_{L^2} && (17)\\ &= \langle\langle \nabla f, \nabla u \rangle\rangle_{L^2} && (18)\\ &= \langle\langle \nabla^3 f, \nabla^{-1} u \rangle\rangle_{L_2} && (19)\\ &= \langle\langle \Delta f, \Delta^{-1} u \rangle\rangle_{H^1} && (20)\\ &= \langle\langle \Delta^2 f, u \rangle\rangle_{H^{-1}} && (21) \end{aligned}$$

So by (3), the gradient in H^{-1} can be written as

$$\operatorname{grad} \mathcal{E}_{H^{-1}} = \Delta^2 f \tag{22}$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt} f = -\Delta^2 f \tag{23}$$

2.4 Gradient Flow in H^2

We may express the differential (9) as

$$
\begin{aligned}
d\mathcal{E}_D|_f(u) &= -\langle\langle \Delta f, u\rangle\rangle_{L^2} && (24)\\
&= -\langle\langle f, \Delta u\rangle\rangle_{L^2} && (25)\\
&= -\langle\langle \Delta^{-1} f, u\rangle\rangle_{H^2} && (26)
\end{aligned}
$$

So by (3), the gradient in H^2 can be written as

$$\operatorname{grad} \mathcal{E}_{H^2} = -\Delta^{-1} f \tag{27}$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt} f = \Delta^{-1} f \tag{28}$$

3 Numerically Solving Gradient Flow Equations

3.1 Gradient Flow in L^2

For L^2, given the gradient flow equation (11), we may write down explicit Euler scheme:

$$\frac{f_j^{m+1} - f_j^m}{\Delta t} = \frac{f_{j+1}^m - 2 f_j^m + f_{j-1}^m}{(\Delta x)^2} \tag{29}$$

which can be rewritten as

$$f_j^{m+1} = f_j^m + \mu \left(f_{j+1}^m - 2 f_j^m + f_{j-1}^m \right) \tag{30}$$

where $\mu := \frac{\Delta t}{(\Delta x)^2}$ is called the *CFL number*.

This method is known to have consistency error

$$T_j^m = \mathcal{O}\left((\Delta x)^2 + \Delta t \right) \tag{31}$$

3.2 Gradient Flow in H^1

Given the gradient flow equation (16), we may write down explicit Euler scheme:

$$\frac{f_j^{m+1} - f_j^m}{\Delta t} = -f_j^m \tag{32}$$

which can be rewritten as

$$f_j^{m+1} = f_j^m - (\Delta t) f_j^m \tag{33}$$

3.3 Gradient Flow in H^{-1}

Given the gradient flow equation (23), we may write down explicit Euler scheme:

$$\frac{f_j^{m+1} - f_j^m}{\Delta t} = -\frac{f_{j+2}^m - 4 f_{j+1}^m + 6 f_j^m - 4 f_{j-1}^m + f_{j-2}^m}{(\Delta x)^4} \tag{34}$$

3.4 Gradient Flow in H^2

Given the gradient flow equation (28), it can be rewritten as

$$\frac{d}{dt}\Delta f = f \tag{35}$$

we may write down explicit Euler scheme:

$$\frac{1}{\Delta t}\left(\frac{f_{j+1}^{m+1} - 2f_j^{m+1} + f_{j-1}^{m+1}}{(\Delta x)^2} - \frac{f_{j+1}^{m} - 2f_j^{m} + f_{j-1}^{m}}{(\Delta x)^2}\right) = f_j^m \tag{36}$$

which can be rewritten as

$$\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} f_{j-1}^{m+1} \\ f_j^{m+1} \\ f_{j+1}^{m+1} \end{pmatrix} = \begin{pmatrix} 1 & -2+\Delta t(\Delta x)^2 & 1 \end{pmatrix}\begin{pmatrix} f_{j-1}^{m} \\ f_j^{m} \\ f_{j+1}^{m} \end{pmatrix} \tag{37}$$

Assuming Dirichlet boundary condition $f_0^m = a$ and $f_J^m = b$, we may write this as a matrix equation:

$$\begin{pmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ \vdots & & & & & & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{J-4} \\ f_{J-3} \\ f_{J-2} \\ f_{J-1} \end{pmatrix}^{m+1}$$

$$= \begin{pmatrix} -2+\Delta t(\Delta x)^2 & 1 & & & & & \\ 1 & -2+\Delta t(\Delta x)^2 & 1 & & & & \\ & 1 & -2+\Delta t(\Delta x)^2 & 1 & & & \\ \vdots & & & & & & \\ & & & & 1 & -2+\Delta t(\Delta x)^2 & 1 \\ & & & & & 1 & -2 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{J-4} \\ f_{J-3} \\ f_{J-2} \\ f_{J-1} \end{pmatrix}^{m}$$

It might be worth noting that

$$\underbrace{\begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}^{-1}}_{n\times n} = -\frac{1}{n+1}\underbrace{\begin{pmatrix} n & (n-1) & (n-2) & \cdots & 1 \\ (n-1) & 2(n-1) & 2(n-2) & \cdots & 2 \\ (n-2) & 2(n-2) & 3(n-2) & \cdots & 3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}}_{\text{Symmetric Matrix}} \tag{38}$$

This method has consistency error

$$T_j^m = \mathcal{O}\left((\Delta x)^2 + (\Delta t)^2\right) \tag{39}$$

4 Appendix

4.1 A (Somewhat) Careful Justification for IBP

Suppose we are concerned with the integral $I := \langle\langle \Delta u, f \rangle\rangle_{L^2} = \int_R (\Delta u) f \, \mathrm{d}\mathbf{x}$, where R is a bounded region with boundary ∂R.

4.1.1 IBP (on Gradient Operator)

One could show that

$$\langle\langle \Delta u, f \rangle\rangle_{L^2} = -\langle\langle \nabla u, \nabla f \rangle\rangle_{L^2} = -\int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} \tag{40}$$

using the identity[3]:

$$\nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \tag{41}$$

Note that

$$\begin{aligned}
\langle\langle \Delta u, f \rangle\rangle_{L^2} &= -\int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} && (42)\\
&= \int_R \nabla \cdot (f \nabla u) \, \mathrm{d}\mathbf{x} - \int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} && (43)\\
&= \oint_{\partial R} f \frac{du}{dn} \, \mathrm{d}s - \int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} && (44)
\end{aligned}$$

(Need the first term of last line to be zero…)

[3]This is also known as Green's Identity in some literature