Dirichlet Energy in Different Spaces

December 3, 2022

1 Relevant Definitions

Definition 1 (Gradient Flow)**.** Given an energy (functional) $\mathcal{E}(f)$, *gradient flow* is:

$$\frac{d}{dt}f = -\operatorname{grad}\mathcal{E}(f) \tag{1}$$

Definition 2 (Differential)**.** *Differential* $d\mathcal{E}$ describes change in $\mathcal{E}$ due to[1] u:

$$d\mathcal{E}|_f(u) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathcal{E}(f+\epsilon u) - \mathcal{E}(f)\right) \tag{2}$$

Definition 3 (Gradient)**.** *Gradient* of $\mathcal{E}$ is the unique function $\operatorname{grad}\mathcal{E}$ such that,

$$\langle\langle \operatorname{grad}\mathcal{E}, u\rangle\rangle_V = d\mathcal{E}(u) \tag{3}$$

2 Dirichlet Energy Example

Definition 4 (Dirichlet Energy)**.** Dirichlet Energy is defined as:

$$\mathcal{E}_D(f) \coloneqq \frac{1}{2}\int_\Omega |\operatorname{grad} f(x)|^2\,\mathrm{d}x = \frac{1}{2}\langle\langle \nabla f, \nabla f\rangle\rangle_{L^2} = -\frac{1}{2}\langle\langle \Delta f, f\rangle\rangle_{L^2} \tag{4}$$

Computing the differential $d\mathcal{E}_D|_f(u)$ of Dirichlet energy using (2),

$$\begin{aligned}
d\mathcal{E}_D|_f(u) &= -\frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\langle\langle \Delta(f+\epsilon u), f+\epsilon u\rangle\rangle_{L^2} - \langle\langle \Delta f, f\rangle\rangle_{L^2}\right) && (5)\\
&= -\frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\epsilon\langle\langle \Delta f, u\rangle\rangle_{L^2} + \epsilon\langle\langle \Delta u, f\rangle\rangle_{L^2} + \epsilon^2\langle\langle u, u\rangle\rangle_{L^2}\right) && (6)\\
&= -\frac{1}{2}(\langle\langle \Delta f, u\rangle\rangle_{L^2} + \langle\langle \Delta u, f\rangle\rangle_{L^2}) && (7)\\
&= -\frac{1}{2}(\langle\langle \Delta f, u\rangle\rangle_{L^2} + \langle\langle \Delta f, u\rangle\rangle_{L^2}) && (8)\\
&= -\langle\langle \Delta f, u\rangle\rangle_{L^2} && (9)
\end{aligned}$$

where the last step is by integration by parts (twice)[2].

[1] analogous in traditional vector space would be "in the direction of" u

[2] This trick will be used all over the place when constructing gradients. See 3.1 in the Appendix

2.1 Gradient Flow in L^2

By (3), we may spot from (9) that the gradient can be written as

$$\operatorname{grad} \mathcal{E}_{L^2} = -\Delta f \tag{10}$$

Hence, the gradient flow equation (1) can be written as:

$$\frac{d}{dt} f = \Delta f \tag{11}$$

which is the heat equation.

2.2 Gradient Flow in H^1

We may express the differential (9) as

$$\begin{aligned} d\mathcal{E}_D|_f(u) &= -\langle\langle \Delta f, u \rangle\rangle_{L^2} && (12)\\ &= \langle\langle \nabla f, \nabla u \rangle\rangle_{L^2} && (13)\\ &= \langle\langle f, u \rangle\rangle_{H^1} && (14) \end{aligned}$$

So by (3), the gradient in H^1 can be written as

$$\operatorname{grad} \mathcal{E}_{H^1} = f \tag{15}$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt} f = -f \tag{16}$$

which describes exponential decay.

2.3 Gradient Flow in H^{-1}

We may express the differential (9) as

$$\begin{aligned} d\mathcal{E}_D|_f(u) &= -\langle\langle \Delta f, u \rangle\rangle_{L^2} && (17)\\ &= \langle\langle \nabla f, \nabla u \rangle\rangle_{L^2} && (18)\\ &= \langle\langle \nabla^3 f, \nabla^{-1} u \rangle\rangle_{L_2} && (19)\\ &= \langle\langle \Delta f, \Delta^{-1} u \rangle\rangle_{H^1} && (20)\\ &= \langle\langle \Delta^2 f, u \rangle\rangle_{H^{-1}} && (21) \end{aligned}$$

So by (3), the gradient in H^{-1} can be written as

$$\operatorname{grad} \mathcal{E}_{H^{-1}} = \Delta^2 f \tag{22}$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt} f = -\Delta^2 f \tag{23}$$

2.4 Gradient Flow in H^2

We may express the differential (9) as

$$d\mathcal{E}_D|_f(u) = -\langle\langle \Delta f, u \rangle\rangle_{L^2} \tag{24}$$
$$= -\langle\langle f, \Delta u \rangle\rangle_{L^2} \tag{25}$$
$$= -\langle\langle \Delta^{-1} f, u \rangle\rangle_{H^2} \tag{26}$$

So by (3), the gradient in H^2 can be written as

$$\operatorname{grad} \mathcal{E}_{H^2} = -\Delta^{-1} f \tag{27}$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt} f = \Delta^{-1} f \tag{28}$$

3 Appendix

3.1 A (Somewhat) Careful Justification for IBP

Suppose we are concerned with the integral $I := \langle\langle \Delta u, f \rangle\rangle_{L^2} = \int_R (\Delta u) f \, \mathrm{d}\mathbf{x}$, where R is a bounded region with boundary ∂R.

3.1.1 IBP (on Gradient Operator)

One could show that

$$\langle\langle \Delta u, f \rangle\rangle_{L^2} = -\langle\langle \nabla u, \nabla f \rangle\rangle_{L^2} = -\int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} \tag{29}$$

using the identity[3]:

$$\nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \tag{30}$$

Note that

$$\begin{align} \langle\langle \Delta u, f \rangle\rangle_{L^2} &= -\int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} \tag{31} \\ &= \int_R \nabla \cdot (f \nabla u) \, \mathrm{d}\mathbf{x} - \int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} \tag{32} \\ &= \oint_{\partial R} f \frac{du}{dn} \, \mathrm{d}s - \int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} \tag{33} \end{align}$$

(Need the first term of last line to be zero. . .)

[3] This is also known as Green's Identity in some literature