Functional Analysis: Gradient

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1 Gradient on $\mathbb{R}^n$

For $f : \mathbb{R}^n \to \mathbb{R}$, we may define the gradient ∇f as:

Definition 1. $\nabla f =: v$ such that

$$\frac{\partial}{\partial \epsilon} f\left(x + \epsilon y\right)|_{\epsilon=0} = \langle v, y \rangle_{\mathbb{R}^n} \tag{1}$$

2 Gradient on L^2

Similarly, for $E : L^2 \to \mathbb{R}$ (a functional), we define the gradient ∇E as:

Definition 2. ∇E such that

$$\frac{\partial}{\partial \epsilon} E\left[f + \epsilon g\right]|_{\epsilon=0} = \langle \nabla E, g \rangle_{L^2} \tag{2}$$

2.1 Dirichlet Energy

For Dirichlet energy defined by $E(f) = \int_{\mathbb{R}} |\nabla f|^2 \, \mathrm{d}x$,

$$\frac{\partial}{\partial \epsilon} E\left(f + \epsilon g\right)|_{\epsilon=0} = \int_{\mathbb{R}} \frac{\delta E}{\delta f} g \tag{3}$$

2.2 Tangent Point Energy

For our tangent point energy defined by $E(\gamma) = \int_{M^2} (\cdot) \, \mathrm{d}x_\gamma \, \mathrm{d}y_\gamma$

$$\frac{\partial}{\partial \epsilon} E(\gamma + \epsilon \delta)|_{\epsilon=0} = \int_{M^2} \frac{\delta E}{\delta \gamma} \delta \tag{4}$$

3 Gradient on Integer Sobolev Spaces

For integer Sobolev spaces H^k, inner product is defined by $\langle f, g \rangle_{H^k} = \sum_{i=0}^{k} \langle D^i f, D^i g \rangle$. Write the L^2 gradient as h: $\frac{\partial}{\partial \epsilon} \mathcal{E}(f + \epsilon g)|_{\epsilon=0} = \langle h, g \rangle_{L^2} =: \mathcal{D}$

3.1 Gradient on H^1

$$\begin{aligned}
\mathcal{D} &= \langle \nabla^2 (\Delta^{-1}) h, g \rangle_{L^2} && (5)\\
&= \langle -\nabla \left(\Delta^{-1}\right) h, \nabla g \rangle_{L^2} && (6)\\
&= \langle -\Delta^{-1} h, g \rangle_{H^1} && (7)
\end{aligned}$$

So, $\operatorname{grad}_{H^1} \mathcal{E} = -\Delta^{-1} h$

3.2 Gradient on H^{-1}

Note that $\langle f, g \rangle_{H^{-1}} = \langle \Delta^{-1} f, \Delta^{-1} g \rangle_{H^1} = \langle \nabla \left(\Delta^{-1} f\right), \nabla \left(\Delta^{-1} g\right) \rangle_{L^2}$ Now note,

$$\begin{aligned}
\mathcal{D} &= \langle h, g \rangle_{L^2} && (8)\\
&= \langle -\nabla h, \nabla \left(\Delta^{-1} g\right) \rangle_{L^2} && (9)\\
&\overset{f=-\Delta h}{=} \langle -\Delta h, g \rangle_{H^{-1}} && (10)
\end{aligned}$$

So, $\operatorname{grad}_{H^{-1}} \mathcal{E} = -\Delta h$

3.3 Gradient on H^2

$$\begin{aligned}
\mathcal{D} &= \langle h, g \rangle_{L^2} && (11)\\
&= \langle \Delta^{-1} h, \Delta g \rangle_{L^2} && (12)\\
&= \langle \Delta \left(\Delta^{-2} h\right), \Delta g \rangle_{L^2} && (13)\\
&= \langle \Delta^{-2} h, g \rangle_{H^2} && (14)
\end{aligned}$$

So, $\operatorname{grad}_{H^2} \mathcal{E} = \Delta^{-2} h$