Untangling Knots Through Curve Repulsion

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What the curious folks ponder about

Introduction

A Cool Knot

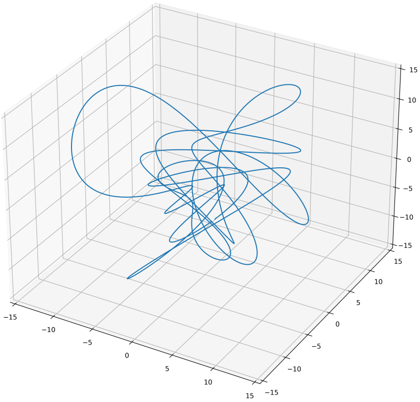

Figure: Imagine your earphones getting tangled like this. . .

Aim

- Finding a “homotopy” from a knot to an unknot.

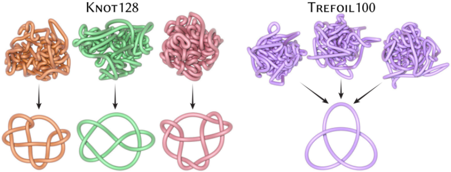

Figure: Unknots of test knots.[2]

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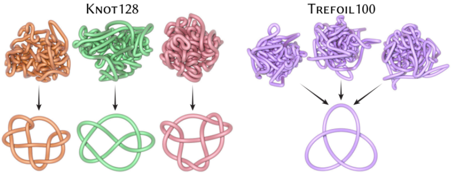

Figure: Unknots of test knots.[2]

- “Avoiding self-intersection”

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    - There is a freedom in choosing the "gradient" here.
3. We expect the stationary state to be the "unknot"
    - Or at least a simpler state...

Tangent-Point Energy

Defining Curve Energy

Given an (arc-length parameterised) curve $\gamma : M \to \mathbb{R}^3$, we wish to assign energy of the form:

$$\mathcal{E}(\gamma) := \iint_{M^2} k\left(\gamma_x, \gamma_y\right) \, \mathrm{d}\gamma_x \, \mathrm{d}\gamma_y \tag{1}$$

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A naïve choice is $k\left(\gamma_x, \gamma_y\right) := \frac{1}{||\gamma_x - \gamma_y||}$

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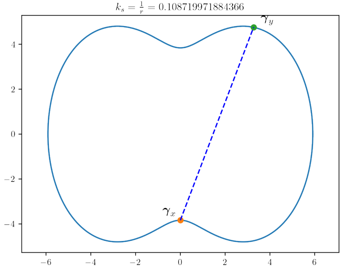

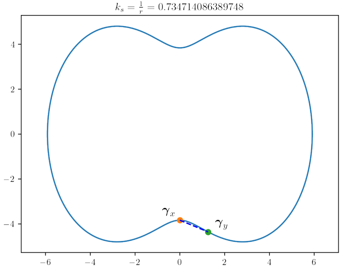

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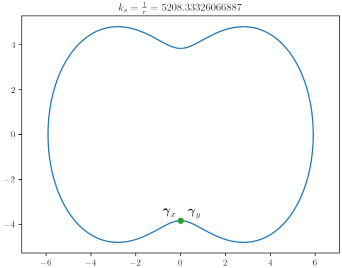

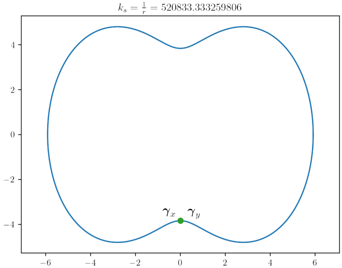

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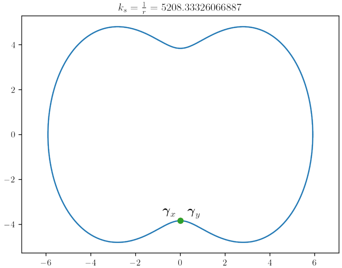

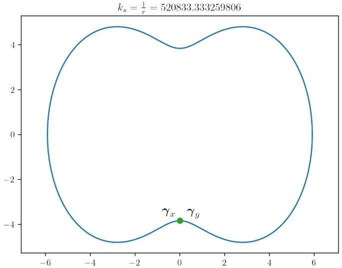

This energy is not well-defined for a lot of curves!

Buck-Orloff Tangent-Point Energy

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Definition (Buck-Orloff Tangent-Point Energy)

For a smooth curve γ, define

$$\mathcal{E}(\gamma) := \iint_{M^2} k_4^2\left(\gamma_x, \gamma_y, \mathbf{T}_x\right)\, \mathrm{d}\gamma_x\, \mathrm{d}\gamma_y$$

where $\mathbf{T}_x$ is the unit tangent vector at γ_x, with the kernel defined as

$$k_4^2(\mathbf{p}, \mathbf{q}, \mathbf{T}) := \frac{||\mathbf{T} \wedge (\mathbf{p} - \mathbf{q})||^2}{||\mathbf{p} - \mathbf{q}||^4}$$

as **Buck-Orloff Tangent-Point Energy**.[1]

Intuition

Bibliography

[1] Gregory Buck and Jeremey Orloff. "A simple energy function for knots". In: *Topology and its Applications* 61.3 (Feb. 1995), pp. 205–214. DOI: 10.1016/0166-8641(94)00024-w.

[2] Chris Yu, Henrik Schumacher, and Keenan Crane. "Repulsive Curves". In: *ACM Transactions on Graphics* 40.2 (Apr. 2021), pp. 1–21. DOI: 10.1145/3439429.