

# Gradient Flow to Continuous Optimization via Fourier Series

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We've been concerned about minimizing the energy functional of the form:

$$\mathcal{E}(\gamma) := \int_{C_\gamma} \int_{C_\gamma} k(\gamma_1, \gamma_2) \, d\gamma_1 \, d\gamma_2 \quad (1)$$

where  $\gamma : E \rightarrow \mathbb{R}^3$  is a parameterization function of a closed curve on the interval  $E = [0, 2\pi)$  (without loss of generality).

Note that we assume  $\gamma$  to be a periodic function of period  $2\pi$ .

## 1 Multidimensional Fourier Series

### 1.1 1D Fourier Series

Given a continuous 1D  $2\pi$ -periodic function  $f : \mathbb{R} \rightarrow \mathbb{R}$  (where we only need to define  $f$  on  $[0, 2\pi)$ ), there exists a Fourier series representation:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \quad (2)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad (3)$$

where the coefficients  $\{a_n\}, \{b_n\}, \{c_n\}$  are given by

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) \, dx \quad \in \mathbb{R} \quad (4)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) \, dx \quad \in \mathbb{R} \quad (5)$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, dx \quad \in \mathbb{C} \quad (6)$$

Fourier convergence theorem states that the rate of convergence is  $O\left(\frac{1}{n^{p+1}}\right)$  where  $f$  has the first jump discontinuity in the  $p^{\text{th}}$  derivative.<sup>1</sup>

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<sup>1</sup>Lecture note

## 1.2 Multidimensional Extension

For a vector valued function of dimension  $N$  (which we will take  $N = 3$  for our case), we have Fourier series representation in each of the coordinates.

For  $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}^N$ , we write its Fourier series as:

$$\mathbf{f}(x) = \frac{1}{2} \begin{pmatrix} a_{1,0} \\ a_{2,0} \\ \vdots \\ a_{N,0} \end{pmatrix} + \sum_{n=1}^{\infty} \begin{pmatrix} a_{1,n} & b_{1,n} \\ a_{2,n} & b_{2,n} \\ \vdots & \vdots \\ a_{N,n} & b_{N,n} \end{pmatrix} \begin{pmatrix} \cos(nx) \\ \sin(nx) \end{pmatrix} \quad (7)$$

$$= \sum_{n=-\infty}^{\infty} \begin{pmatrix} c_{1,n} \\ c_{2,n} \\ \vdots \\ c_{N,n} \end{pmatrix} e^{-inx} \quad (8)$$

where the coefficients  $\{a_{i,n}\}, \{b_{i,n}\}, \{c_{i,n}\}$  are given by

$$a_{i,n} = \frac{1}{\pi} \int_0^{2\pi} f_i(x) \cos(nx) \, dx \quad (9)$$

$$b_{i,n} = \frac{1}{\pi} \int_0^{2\pi} f_i(x) \sin(nx) \, dx \quad (10)$$

$$c_{i,n} = \frac{1}{2\pi} \int_0^{2\pi} f_i(x) e^{-inx} \, dx \quad (11)$$

for  $i = 1, 2, \dots, N$