Tackling $u_t = \Delta\left(u^2\right)$

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1 The Equation

The equation we are concerned with is:

$$u_t = \Delta\left(u^2\right) \tag{1}$$

2 Explicit Scheme

For convenience we consider explicit Euler schemes.

2.1 Scheme Construction After Simplification

Note that (1) can be written as:

$$\begin{aligned} u_t &= \Delta\left(u^2\right) && (2)\\ &= 2u\Delta u + 2||\nabla u||^2 && (3)\end{aligned}$$

In 1-dimensional case, we simplify it as:

$$\frac{\partial u}{\partial t} = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2 \tag{4}$$

Explicit Euler scheme can now be written

$$\begin{aligned} \frac{U_j^{m+1} - U_j^m}{\Delta t} &= 2U_j^m\left(\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{\Delta x^2}\right) + 2\left(\frac{U_{j+1}^m - U_{j-1}^m}{2\Delta x}\right)^2 && (5)\\ &= 2U_j^m\left(\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{\Delta x^2}\right) + \frac{\left(U_{j+1}^m - U_{j-1}^m\right)^2}{2\Delta x^2} && (6)\end{aligned}$$

which can be rewritten as:

$$U_j^{m+1} = U_j^m + 2\mu U_j^m\left(U_{j+1}^m - 2U_j^m + U_{j-1}^m\right) + \frac{1}{2}\mu\left(U_{j+1}^m - U_{j-1}^m\right)^2 \tag{7}$$

where $\mu := \frac{\Delta t}{\Delta x^2}$ is the “CFL number”. [1]

[1] Proving stability condition is quite involved apparently…

2.2 Scheme Construction Without Simplification

Define $v \coloneqq u^2$, then (1) can be written as

$$\frac{\partial u}{\partial t} = \Delta v \tag{8}$$

Explicit Euler scheme can now be written

$$\begin{cases} \frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{V_{j+1}^m - 2V_j^m + V_{j-1}^m}{(\Delta x)^2} \\ V_j^m = \left(U_j^m\right)^2 \end{cases} \tag{9}$$