Chebyshev Analogue of Fourier Series Curve Repulsion in a Finite Line Domain

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Recall the reduction of energy problem where the one attempts to reduce the energy functional of the form

$$\mathcal{E}(\boldsymbol{\gamma}) \coloneqq \int_{C_\gamma} \int_{C_\gamma} k\left(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2\right) \, \mathrm{d}\gamma_1 \, \mathrm{d}\gamma_2 \tag{1}$$

where $\boldsymbol{\gamma} : E \to \mathbb{R}^3$ is a parameterized function of a finite (continuous) curve with ends at $\boldsymbol{\gamma}(-1)$ and $\boldsymbol{\gamma}(1)$ (here we take $E = [-1, 1]$ without loss of generality).

1 Multidimensional Chebyshev Series

1.1 1D Chebyshev Series

Given a Lipschitz continuous 1D $f : [-1, 1] \to \mathbb{R}$, there exists a Chebyshev series representation:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n T_k(x) \tag{2}$$

where the coefficient $\{a_n\}$ are given by

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_n(x)}{\sqrt{1 - x^2}} \tag{3}$$

Chebyshev-analogous theorem of Fourier convergence theorem states that, for f that has absolutely continuous derivative upto order $\nu - 1$ and $f^{(\nu)}$ has bounded variation, then the coefficients decay as $O\left(\frac{1}{n^{\nu+1}}\right)$.

1.2 Multidimensional Extension

For a vector valued function of dimension N (which we will take $N = 3$ for our case), we have Chebyshev series representation in each of the coordinates.

For $\mathbf{f} : \mathbb{R} \to \mathbb{R}^N$, we write its Chebyshev series as:

$$\mathbf{f} = \frac{1}{2} \begin{pmatrix} a_{1,0} \\ a_{2,0} \\ \vdots \\ a_{N,0} \end{pmatrix} + \sum_{n=1}^{\infty} \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{N,n} \end{pmatrix} T_n(x) \tag{4}$$

where the coefficients $\{a_{i,n}\}$ are given by

$$a_{i,n} = \frac{2}{\pi} \int_{-1}^{1} \frac{f_i(x) T_n(x)}{\sqrt{1-x^2}} \tag{5}$$

for $i = 1, 2, \cdots, N$

2 Gradient Flow to Continuous Optmization

As it was done in Fourier Series, consider expressing $\boldsymbol{\gamma}(t)$ as a 3D Chebyshev series.

$$\boldsymbol{\gamma}(t) = \frac{\mathbf{a}_0}{2} + \sum_{n=1}^{\infty} \mathbf{a}_n T_n(x) \tag{6}$$

Truncate to order J term for approximation:

$$\boldsymbol{\gamma}_J(t) = \frac{\mathbf{a}_0}{2} + \sum_{n=1}^{\infty} \mathbf{a}_n T_n(x) \tag{7}$$

Then the reduction process of $\mathcal{E}(\boldsymbol{\gamma})$ can be approximated by reduction process of $\mathcal{E}(\boldsymbol{\gamma}_J)$. Notice that we can consider $\mathcal{E}(\boldsymbol{\gamma}_J) : \mathbb{R}^{3(J+1)} \to \mathbb{R}$ where the parameters are the Chebyshev coefficients in 3D. Now we may consider the standard optimization problem:

$$\min_{[\mathbf{a}_0, \cdots, \mathbf{a}_J]} \mathcal{E}(\boldsymbol{\gamma}_J) \tag{8}$$

Unlike Fourier series curve repulsion case, this time we have a lower-dimension optimization problem, which can be solved with higher performance. (This maybe due to less information needed to capture the notion of a line segment than a closed curve.)

3 Interpolation

Initial curve should be interpolated. The issue with a finite interval curve is that equispaced interpolation is considered a bad practice; it is known that the naïve interpolation using equispaced points is unstable. A natural choice here would be points interpolated at Chebyshev points.

One now needs a fast algorithm to get coefficients fast. A possibility is to use barycentric interpolation formula, then exploit the fact that Chebyshev polynomials are orthogonal polynomials.