Dirichlet Energy in Different Spaces

December 8, 2022

1 Relevant Definitions

Definition 1 (Gradient Flow)**.** Given an energy (functional) $\mathcal{E}(f)$, *gradient flow* is:

$$\frac{d}{dt} f = -\operatorname{grad} \mathcal{E}(f) \tag{1}$$

Definition 2 (Differential)**.** *Differential* $d\mathcal{E}$ describes change in $\mathcal{E}$ due to[1] u:

$$d\mathcal{E}|_f(u) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\mathcal{E}(f + \epsilon u) - \mathcal{E}(f) \right) \tag{2}$$

Definition 3 (Gradient)**.** Given a space X, *gradient* of $\mathcal{E}$ is the unique function $\operatorname{grad}_X \mathcal{E}$ such that,

$$\langle \operatorname{grad}_X \mathcal{E}, u \rangle_V = d\mathcal{E}(u) \qquad \forall u \in X \tag{3}$$

2 Dirichlet Energy Example

Definition 4 (Dirichlet Energy)**.** Dirichlet Energy is defined as:

$$\mathcal{E}_D(f) := \frac{1}{2} \int_\Omega |\operatorname{grad} f(x)|^2 \, \mathrm{d}x = \frac{1}{2} \langle \nabla f, \nabla f \rangle_{L^2} = -\frac{1}{2} \langle \Delta f, f \rangle_{L^2} \tag{4}$$

Computing the differential $d\mathcal{E}_D|_f(u)$ of Dirichlet energy using (2),

$$\begin{aligned}
d\mathcal{E}_D|_f(u) &= -\frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\langle \Delta(f + \epsilon u), f + \epsilon u \rangle_{L^2} - \langle \Delta f, f \rangle_{L^2} \right) &(5)\\
&= -\frac{1}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\epsilon \langle \Delta f, u \rangle_{L^2} + \epsilon \langle \Delta u, f \rangle_{L^2} + \epsilon^2 \langle u, u \rangle_{L^2} \right) &(6)\\
&= -\frac{1}{2} \left(\langle \Delta f, u \rangle_{L^2} + \langle \Delta u, f \rangle_{L^2} \right) &(7)\\
&= -\frac{1}{2} \left(\langle \Delta f, u \rangle_{L^2} + \langle \Delta f, u \rangle_{L^2} \right) &(8)\\
&= -\langle \Delta f, u \rangle_{L^2} &(9)
\end{aligned}$$

where the last step is by integration by parts (twice)[2].

[1] analogous in traditional vector space would be "in the direction of" u

[2] This trick will be used all over the place when constructing gradients. See 4.1 in the Appendix

2.1 Gradient Flow in L^2

By (3), we may spot from (9) that the gradient can be written as

$$\operatorname{grad} \mathcal{E}_{L^2} = -\Delta f \tag{10}$$

Hence, the gradient flow equation (1) can be written as:

$$\frac{d}{dt} f = \Delta f \tag{11}$$

which is the heat equation.

2.2 Gradient Flow in H^1

We may express the differential (9) as

$$\begin{aligned} d\mathcal{E}_D|_f(u) &= -\langle \Delta f, u \rangle_{L^2} && (12)\\ &= \langle \nabla f, \nabla u \rangle_{L^2} && (13)\\ &= \langle f, u \rangle_{H^1} && (14) \end{aligned}$$

So by (3), the gradient in H^1 can be written as

$$\operatorname{grad} \mathcal{E}_{H^1} = f \tag{15}$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt} f = -f \tag{16}$$

which describes exponential decay.

2.3 Gradient Flow in H^{-1}

We may express the differential (9) as

$$\begin{aligned} d\mathcal{E}_D|_f(u) &= -\langle \Delta f, u \rangle_{L^2} && (17)\\ &= \langle \nabla f, \nabla u \rangle_{L^2} && (18)\\ &= \langle \nabla^3 f, \nabla^{-1} u \rangle_{L_2} && (19)\\ &= \langle \Delta f, \Delta^{-1} u \rangle_{H^1} && (20)\\ &= \langle \Delta^2 f, u \rangle_{H^{-1}} && (21) \end{aligned}$$

So by (3), the gradient in H^{-1} can be written as

$$\operatorname{grad} \mathcal{E}_{H^{-1}} = \Delta^2 f \tag{22}$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt} f = -\Delta^2 f \tag{23}$$

2.4 Gradient Flow in H^2

We may express the differential (9) as

$$d\mathcal{E}_D|_f(u) = -\langle \Delta f, u\rangle_{L^2} \tag{24}$$
$$= -\langle f, \Delta u\rangle_{L^2} \tag{25}$$
$$= -\langle \Delta^{-1} f, u\rangle_{H^2} \tag{26}$$

So by (3), the gradient in H^2 can be written as

$$\operatorname{grad} \mathcal{E}_{H^2} = -\Delta^{-1} f \tag{27}$$

The gradient flow equation (1) can be written as

$$\frac{d}{dt} f = \Delta^{-1} f \tag{28}$$

3 Numerically Solving Gradient Flow Equations

3.1 Gradient Flow in L^2

For L^2, given the gradient flow equation (11), we may write down explicit Euler scheme:

$$\frac{f_j^{m+1} - f_j^m}{\Delta t} = \frac{f_{j+1}^m - 2f_j^m + f_{j-1}^m}{(\Delta x)^2} \tag{29}$$

which can be rewritten as

$$f_j^{m+1} = f_j^m + \mu \left(f_{j+1}^m - 2f_j^m + f_{j-1}^m \right) \tag{30}$$

where $\mu \coloneqq \frac{\Delta t}{(\Delta x)^2}$ is called the *CFL number*.

This method is known to have consistency error

$$T_j^m = \mathcal{O}\left((\Delta x)^2 + \Delta t \right) \tag{31}$$

3.2 Gradient Flow in H^1

Given the gradient flow equation (16), we may write down explicit Euler scheme:

$$\frac{f_j^{m+1} - f_j^m}{\Delta t} = -f_j^m \tag{32}$$

which can be rewritten as

$$f_j^{m+1} = f_j^m - (\Delta t) f_j^m \tag{33}$$

3.3 Gradient Flow in H^{-1}

Given the gradient flow equation (23), we may write down explicit Euler scheme:

$$\frac{f_j^{m+1} - f_j^m}{\Delta t} = -\frac{f_{j+2}^m - 4f_{j+1}^m + 6f_j^m - 4f_{j-1}^m + f_{j-2}^m}{(\Delta x)^4} \tag{34}$$

3.4 Gradient Flow in H^2

Given the gradient flow equation (28), it can be rewritten as

$$\frac{d}{dt}\Delta f = f \tag{35}$$

we may write down explicit Euler scheme:

$$\frac{1}{\Delta t}\left(\frac{f_{j+1}^{m+1} - 2f_j^{m+1} + f_{j-1}^{m+1}}{(\Delta x)^2} - \frac{f_{j+1}^{m} - 2f_j^{m} + f_{j-1}^{m}}{(\Delta x)^2}\right) = f_j^m \tag{36}$$

which can be rewritten as

$$\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} f_{j-1}^{m+1} \\ f_j^{m+1} \\ f_{j+1}^{m+1} \end{pmatrix} = \begin{pmatrix} 1 & -2+\Delta t(\Delta x)^2 & 1 \end{pmatrix}\begin{pmatrix} f_{j-1}^{m} \\ f_j^{m} \\ f_{j+1}^{m} \end{pmatrix} \tag{37}$$

Assuming Dirichlet boundary condition $f_0^m = a$ and $f_J^m = b$, we may write this as a matrix equation:

$$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ \vdots & & & & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{J-4} \\ f_{J-3} \\ f_{J-2} \\ f_{J-1} \end{pmatrix}^{m+1}$$

$$= \begin{pmatrix} -2+\Delta t(\Delta x)^2 & 1 & & & & & \\ 1 & -2+\Delta t(\Delta x)^2 & 1 & & & & \\ & 1 & -2+\Delta t(\Delta x)^2 & 1 & & & \\ \vdots & & & & & & \\ & & & & 1 & -2+\Delta t(\Delta x)^2 & 1 \\ & & & & & 1 & -2 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{J-4} \\ f_{J-3} \\ f_{J-2} \\ f_{J-1} \end{pmatrix}^{m}$$

It might be worth noting that

$$A_n := \underbrace{\begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}^{-1}}_{n\times n} = -\frac{1}{n+1}\underbrace{\begin{pmatrix} n & (n-1) & (n-2) & \cdots & 1 \\ (n-1) & 2(n-1) & 2(n-2) & \cdots & 2 \\ (n-2) & 2(n-2) & 3(n-2) & \cdots & 3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}}_{\text{Symmetric Matrix}} \tag{38}$$

A (potentially) useful fact about this matrix is that the characteristic equation can be written recursively.

$$\begin{cases} |A_n - \lambda I| =: J_n = -\delta J_{n-1} - J_{n-2} \\ J_1 = -\delta \\ J_2 = \delta^2 - 1 \end{cases} \tag{39}$$

where $\delta \coloneqq \lambda + 2$.[3]

This method has consistency error

$$T_j^m = \mathcal{O}\left((\Delta x)^2 + (\Delta t)^2\right) \tag{40}$$

[3] The term *continuant* might be interesting to look at

4 Appendix

4.1 A (Somewhat) Careful Justification for IBP

Suppose we are concerned with the integral $I \coloneqq \langle \Delta u, f \rangle_{L^2} = \int_R (\Delta u) f \, \mathrm{d}\mathbf{x}$, where R is a bounded region with boundary ∂R.

4.1.1 IBP (on Gradient Operator)

One could show that

$$\langle \Delta u, f \rangle_{L^2} = -\langle \nabla u, \nabla f \rangle_{L^2} = -\int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} \tag{41}$$

using the identity[4]:

$$\nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi \tag{42}$$

Note that

$$\begin{aligned} \langle \Delta u, f \rangle_{L^2} &= -\int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} && (43) \\ &= \int_R \nabla \cdot (f \nabla u) \, \mathrm{d}\mathbf{x} - \int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} && (44) \\ &= \oint_{\partial R} f \frac{du}{dn} \, \mathrm{d}s - \int_R \nabla u \cdot \nabla f \, \mathrm{d}\mathbf{x} && (45) \end{aligned}$$

(Need the first term of last line to be zero. . .)

[4]This is also known as Green's Identity in some literature