Untangling Knots Through Curve Repulsion

Joo-Hyun Paul Kim

March 2, 2023

What the curious folks ponder about

Introduction

A Cool Knot

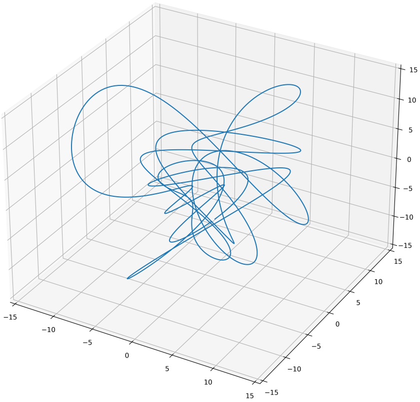

Figure: Imagine your earphones getting tangled like this...

Aim

- Finding a “homotopy” from a knot to an unknot.

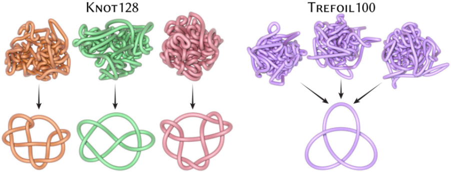

Figure: Unknots of test knots.[3]

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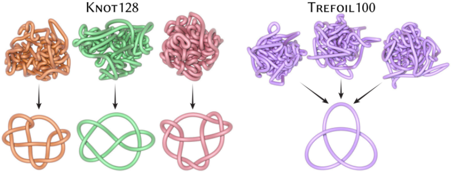

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- “Avoiding self-intersection”

General Strategy

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2. Attempt to decrease the curve energy by continuously deforming the curve.
   - We evolve the curve according to the gradient flow equation.
   - There is a freedom in choosing the "gradient" here.
3. We expect the stationary state to be the "unknot"
   - Or at least a simpler state...

Tangent-Point Energy

Defining Curve Energy

Given an (arc-length parameterised) curve $\gamma : M \to \mathbb{R}^3$, we wish to assign energy of the form:

$$\mathcal{E}(\gamma) := \iint_{M^2} k\left(\gamma_x, \gamma_y\right)\, \mathrm{d}\gamma_x\, \mathrm{d}\gamma_y \tag{1}$$

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A naïve choice is $k\left(\gamma_x, \gamma_y\right) := \frac{1}{||\gamma_x - \gamma_y||}$

Pitfall of the "Simple Energy"

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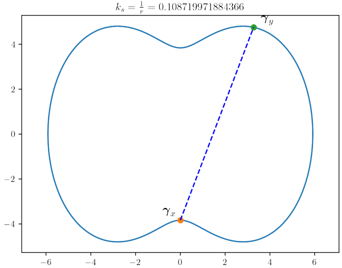

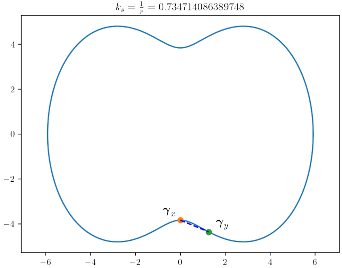

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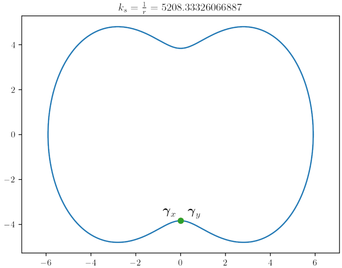

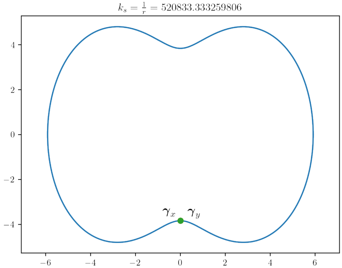

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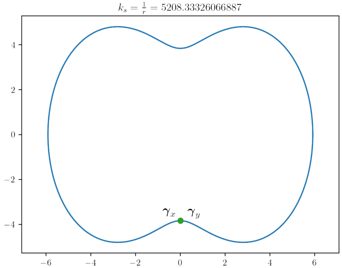

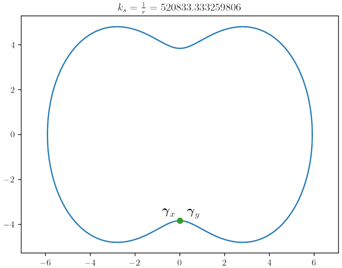

This energy is not well-defined for a lot of curves!

Buck-Orloff Tangent-Point Energy

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Definition (Buck-Orloff Tangent-Point Energy)

For a smooth curve γ, define

$$\mathcal{E}(\gamma) := \iint_{M^2} k_4^2\left(\gamma_x, \gamma_y, \mathbf{T}_x\right)\, \mathrm{d}\gamma_x\, \mathrm{d}\gamma_y$$

where $\mathbf{T}_x$ is the unit tangent vector at γ_x, with the kernel defined as

$$k_4^2(\mathbf{p}, \mathbf{q}, \mathbf{T}) := \frac{||\mathbf{T} \wedge (\mathbf{p} - \mathbf{q})||^2}{||\mathbf{p} - \mathbf{q}||^4}$$

as **Buck-Orloff Tangent-Point Energy**.[1]

Intuition

What is the intuition behind the kernel $k_4^2(\mathbf{p}, \mathbf{q}, \mathbf{T}) := \frac{||\mathbf{T} \wedge (\mathbf{p}-\mathbf{q})||^2}{||\mathbf{p}-\mathbf{q}||^4}$?

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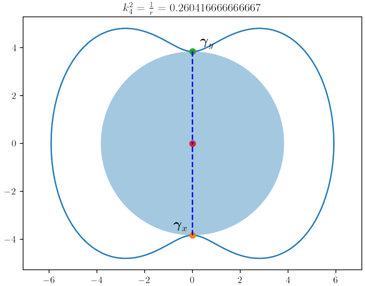

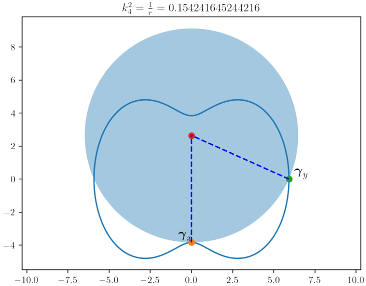

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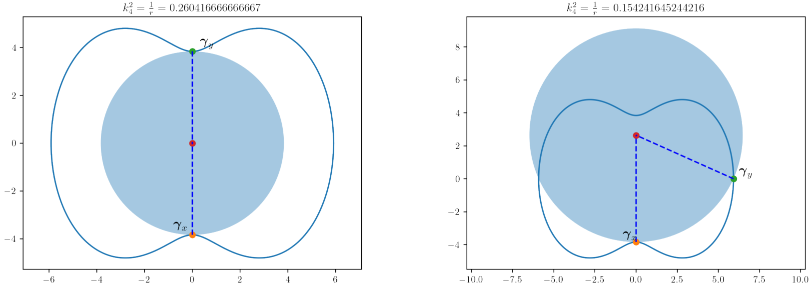

Remark

Note that closer does not necessarily mean the kernel is larger.

Intuition

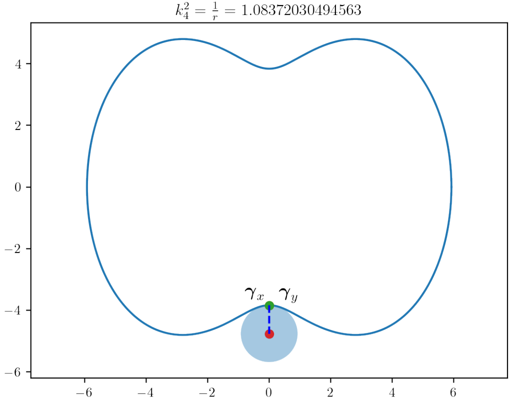

Figure: When two points are very close, the kernel converges to the curvature of the curve.

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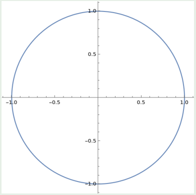

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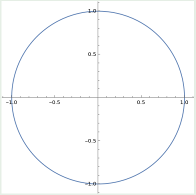

Then write:

$$\begin{cases} \gamma_x(\theta) = (\cos\theta, \sin\theta, 0) \\ \gamma_y(\phi) = (\cos\phi, \sin\phi, 0) \\ \mathbf{T}_x(\theta) = (-\sin\theta, \cos\theta, 0) \end{cases}$$

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Substituting to Buck-Orloff Tangent-Point energy formula:

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Using a few identities:

$$\mathcal{E}(\gamma) = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi} \frac{||\mathbf{T}_x||^2||\gamma_x - \gamma_y||^2 - \left(\mathbf{T}_x \cdot (\gamma_x - \gamma_y)\right)^2}{||\gamma_x - \gamma_y||^4}\, d\theta\, d\phi$$

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$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi} \frac{\sin^2\frac{\theta-\phi}{2}}{(-1+\cos(\theta-\phi))^2}\, \mathrm{d}\theta\, \mathrm{d}\phi \tag{3}$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi} \frac{\sin^2\frac{\theta-\phi}{2}}{\sin^4\frac{\theta-\phi}{2}}\, \mathrm{d}\theta\, \mathrm{d}\phi \tag{4}$$

$$= \pi^2 \tag{5}$$

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Remark

Note that (4) suggests the order at “singularity” is inverse-square.

A more general form of tangent-point energy comes from Yu, Schumacher, and Crane [3]:

Definition (Generalised Tangent-Point Energy)

$$\mathcal{E}_{\beta}^{\alpha}(\gamma) := \iint_{M^2} \frac{||\mathbf{T}_x \wedge (\gamma_x - \gamma_y)||^{\alpha}}{||\gamma_x - \gamma_y||^{\beta}} \, d\gamma_x \, d\gamma_y$$

where $\alpha > 1$ and $\beta \in [\alpha + 2, 2\alpha + 1)$

Remark

When $\alpha = 2$ and $\beta = 4$, we are back to Buck-Orloff.

Gradient Flow

Motivating Gradient Flow

A simple method of minimising a (differentiable) function $f : \mathbb{R}^n \to \mathbb{R}$ is the steepest descent [2]

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k \nabla f \left(\mathbf{x}^k \right) \tag{6}$$

where $\alpha^k > 0$.

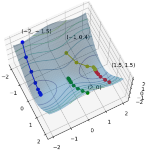

Figure: Steepest Descent

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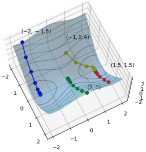

Figure: Steepest Descent

For motivation, take $\alpha^k = \alpha \equiv$ const.

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \alpha_k \nabla f\left(\mathbf{x}^k\right)$$

In the case of functional $\mathcal{E} : X \to \mathbb{R}$, analogously write steepest descent step:

$$f^{k+1} = f^k - \alpha \operatorname{grad}_X f$$

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Take limit as $\Delta t \to 0$, then scale time variable to nondimensionalise to get the gradient flow equation.

Definition (Gradient Flow Equation)

$$\frac{\partial f}{\partial t} = -\,\mathrm{grad}_X f$$

But what is $\mathrm{grad}_X f$?

Gradient in Space of Function?

(Gradient of Function)
$\nabla f(\mathbf{x})$ is such that for all $\mathbf{y} \in \mathbb{R}^n$

$$\nabla f(\mathbf{x}) \cdot \mathbf{y} = \frac{\partial}{\partial \epsilon} f(\mathbf{x} + \epsilon \mathbf{y}) \Big|_{\epsilon=0}$$

(Gradient of Functional)
$\text{grad}_X E(f)$ is such that for all $g \in X$,

$$\langle\langle \text{grad}_X E, g \rangle\rangle_X = \frac{\partial}{\partial \epsilon} E(f + \epsilon g) \Big|_{\epsilon=0}$$

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[1] Gregory Buck and Jeremey Orloff. "A simple energy function for knots". In: *Topology and its Applications* 61.3 (Feb. 1995), pp. 205–214. DOI: 10.1016/0166-8641(94)00024-w.

[2] "Chapter 8: Primal and Dual Projected Subgradient Methods". In: *First-Order Methods in Optimization*, pp. 195–245. DOI: 10.1137/1.9781611974997.ch8. eprint: https://epubs.siam.org/doi/pdf/10.1137/1.9781611974997.ch8. URL: https://epubs.siam.org/doi/abs/10.1137/1.9781611974997.ch8.

[3] Chris Yu, Henrik Schumacher, and Keenan Crane. "Repulsive Curves". In: *ACM Transactions on Graphics* 40.2 (Apr. 2021), pp. 1–21. DOI: 10.1145/3439429.