Curve Repulsion - 1

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1 Theory behind Discretization

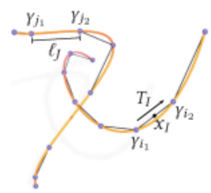

Figure 1: Discretization process

We now consider a discretization of a curve. Index the points on the curve by $\mathcal{I} = \{1, 2, \cdots, J\}$, such that points are given by $\{\gamma_1, \gamma_2, \cdots, \gamma_J\}$

Using the similar notation to the paper by Yu, Schumacher, and Crane, for edge $I = \{\gamma_i, \gamma_j\} \in E$ and for function $u : \mathbb{R}^3 \to \mathbb{R}$

- $l_I \coloneqq |\gamma_i - \gamma_j|$
- $T_I \coloneqq \frac{\gamma_j - \gamma_i}{l_I}$
- $\mathbf{x}_I \coloneqq \frac{\gamma_i + \gamma_j}{2}$
- $u_I \coloneqq \frac{u_i + u_j}{2}$
  - Syntactic sugar: $u_i \equiv u(\gamma_i)$
- $u[I] \coloneqq \begin{pmatrix} u_i \\ u_j \end{pmatrix}$

1.1 Discrete Energy

The naïve discretization of $\mathcal{E}_\beta^\alpha \coloneqq \iint_{M^2} k_\beta^\alpha(\gamma(x), \gamma(y), T(x))\, \mathrm{d}x_\gamma\, \mathrm{d}y_\gamma$ where $k_\beta^\alpha(p, q, T) \coloneqq \frac{|T \times (p-q)|^\alpha}{|p-q|^\beta}$ is given by

$$\sum_{I \in E} \sum_{J \in E} \int_{\bar{I}} \int_{\bar{J}} k_\beta^\alpha(\gamma(x), \gamma(y), T_I)\, \mathrm{d}x_\gamma\, \mathrm{d}y_\gamma \tag{1}$$

However, in a polygonal curve (hence the discretized curve), (1) is ill-defined.

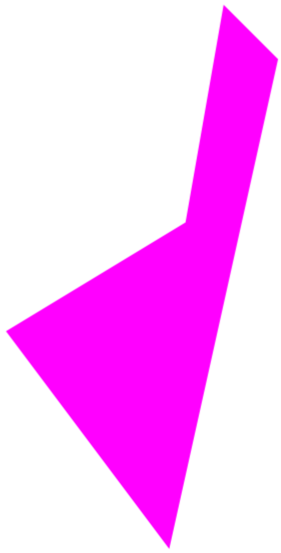

Figure 2: Near each vertex, the integrand is unbounded.

So resolve this by removing the two neighboring edges.[1] Also approximate the kernel by the average of the kernel evaluated at each pair of appropriate edges (total: 4)

$$\hat{\mathcal{E}}_\beta^\alpha := \sum_{I,J\in E, I\cap J=\emptyset} \left(\hat{k}_\beta^\alpha\right)_{I,J} l_I l_J \tag{2}$$

$$\left(\hat{k}_\beta^\alpha\right)_{I,J} := \frac{1}{4} \sum_{i\in J, j\in J} k_\beta^\alpha \left(\gamma_i, \gamma_j, T_I\right) \tag{3}$$

2 Discrete Gradient Flow in L^2 for Closed Loop

Suppose a curve is discretized as position vectors: $x_1, x_2, \cdots, x_J$ (and $x_{J+1} := x_1$).

Also denote the edge from x_i to x_{i+1} as I_i (as opposed to the previous section).

[1] In the limit, the contribution from this removed edge goes to zero.

The **discretized energy** E can be expressed as:

$$E = \sum_{i=1}^{J} \sum_{\substack{j=1 \\ |j-i|>1}} k_{i,j} ||x_{i+1} - x_i|| \, ||x_{j+1} - x_j|| \tag{4}$$

$$k_{i,j} = \frac{1}{4} \left(k_\beta^\alpha \left(x_i, x_j, T_i \right) + k_\beta^\alpha \left(x_i, x_{j+1}, T_i \right) + k_\beta^\alpha \left(x_{i+1}, x_j, T_i \right) + k_\beta^\alpha \left(x_{i+1}, x_{j+1}, T_i \right) \right) \tag{5}$$

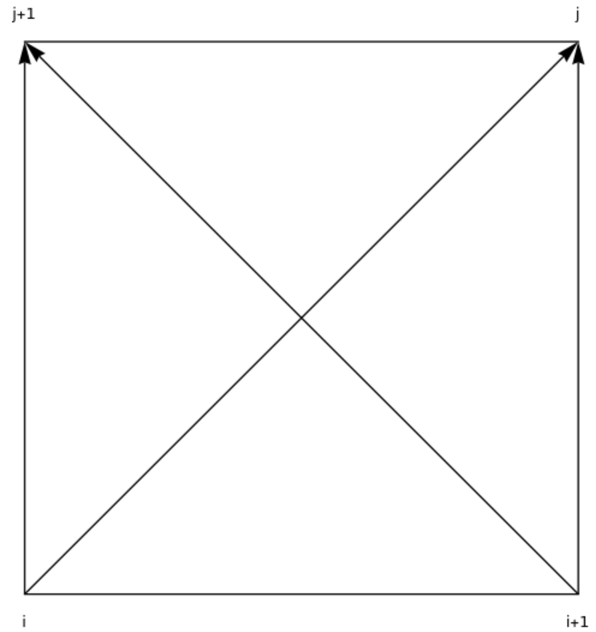

Figure 3: Kernel $k_{i,j}$ computation

Recall the definition of differential, gradient, and gradient flow.

Definition 1 (Differential)**.** Given functional $\mathcal{E}(\gamma)$, the **differential** is defined as:

$$\mathrm{d}\mathcal{E}|_\gamma(u) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\mathcal{E} \left(\gamma + \epsilon u \right) - \mathcal{E} \left(\gamma \right) \right) \tag{6}$$

Definition 2 (Gradient)**.** Given functional $\mathcal{E}\left(\gamma\right)$ and space V, the **gradient** $\operatorname{grad} \mathcal{E}$ is the unique function satisfying the following for any function u:

$$\langle\langle \operatorname{grad} \mathcal{E}, u \rangle\rangle_V = \mathrm{d}\mathcal{E}(u) \tag{7}$$

Note that the LHS is a inner product of two vector-valued functions. A natural inner product in L^2 to define is:

$$\langle\langle u, v \rangle\rangle_{L^2} := \int_\Omega u \cdot v \, \mathrm{d}x \tag{8}$$

Definition 3 (Gradient Flow)**.** Given functional $\mathcal{E}(\gamma)$, the **gradient flow** equation is defined as:

$$\frac{d}{dt}\gamma = -\operatorname{grad}\mathcal{E}(\gamma) \tag{9}$$

Note that $\gamma = (\gamma_x, \gamma_y, \gamma_z)^T \in \mathbb{R}^3$, so it might be clearer to write:

$$\frac{d}{dt}\begin{pmatrix}\gamma_x\\ \gamma_y\\ \gamma_z\end{pmatrix} = \operatorname{grad}\mathcal{E}\left(\begin{pmatrix}\gamma_x\\ \gamma_y\\ \gamma_z\end{pmatrix}\right) \tag{10}$$

Gradient flow equation in L^2 is given by[2]:

$$\frac{d\gamma}{dt} = -\underbrace{\frac{\partial\mathcal{E}}{\partial\gamma}}_{\text{Functional Derivative}} \tag{11}$$

or in our case with discrete energy,

$$\dot{x}_i = -\frac{\partial E}{\partial x_i} \tag{12}$$

For an explicit definition of functional derivatives, see link in the footnote.

Remark 1. Note that each x_i is a 3D vector, meaning in reality, (12) is

$$\dot{x}_{i,1} = -\frac{\partial E}{\partial x_{i,1}} \tag{13}$$

$$\dot{x}_{i,2} = -\frac{\partial E}{\partial x_{i,2}} \tag{14}$$

$$\dot{x}_{i,3} = -\frac{\partial E}{\partial x_{i,3}} \tag{15}$$

2.1 Explicit Euler Scheme

One could now write an explicit Euler scheme based on (13) $\sim$ (15)

$$\frac{X_{i,1}^{m+1} - X_{i,1}^{m}}{\Delta t} = -\frac{E\left(X_1^m, \cdots, X_i^m + \Delta x e_x, \cdots, X_J^m\right) - E\left(X_1^m, \cdots, X_i^m - \Delta x e_x, \cdots, X_J^m\right)}{2\Delta x} \tag{16}$$

where Δt and Δx are small parameters.

Note that in the case that we use (4), by computing the explicit form of the differential of the energy, the computation work for RHS can be greatly reduced.

[2] https://math.stackexchange.com/questions/1687804/what-is-the-l2-gradient-flow

3 Constraint

There is a risk that the curve might keep expanding in order to minimize the energy.

To mitigate that, we put an additional "penalty" for the length to the energy.

3.1 Length Constraint I

In the case of discrete energy, we modify E to F by:

$$F = E + \lambda \sum_i \frac{|x_{i+1} - x_i|^2}{2} \tag{17}$$

which the gradient flow equation turns into

$$\dot{x}_i = -\frac{\partial F}{\partial x_i} = -\frac{\partial E}{\partial x_i} - \lambda \left(2x_i - x_{i+1} - x_{i-1}\right) \tag{18}$$

where λ is a parameter which one could experiment with.

3.2 Length Constraint II

In the case that the points show "clustering behavior", one could attempt to penalize it by the following energy.

$$F = E + \sum_i \frac{\lambda_i \left(l_i - |x_i - x_{i+1}|\right)^2}{2} \tag{19}$$

where λ_i are positive "strengths" of the constraints and l_i are prescribed lengths.

Note that λ_i might have to be substantially large.

3.3 Length + Center Constraint

Alternatively one may attempt

$$F = E + \lambda \sum_i |x_i|^2 \tag{20}$$

which the gradient flow equation turns into

$$\dot{x}_i = -\frac{\partial F}{\partial x_i} = -\frac{\partial E}{\partial x_i} - \lambda x_i \tag{21}$$

Seems to parallel Lagrange constant in the method of Langrange multipliers.

4 Appendix

4.1 Explicit Form of Derivative of Discrete Energy

Given (4), instead of approximating the differential by central scheme, it is possible to compute the exact differential. Recall the definition.

$$E = \sum_{i=1}^{J} \sum_{\substack{j=1 \\ |j-i|>1}} k_{i,j} ||x_{i+1} - x_i|| \, ||x_{j+1} - x_j|| \tag{22}$$

$$k_{i,j} = \frac{1}{4} \left(k_\beta^\alpha (x_i, x_j, T_i) + k_\beta^\alpha (x_i, x_{j+1}, T_i) + k_\beta^\alpha (x_{i+1}, x_j, T_i) + k_\beta^\alpha (x_{i+1}, x_{j+1}, T_i) \right) \tag{23}$$

$$k_\beta^\alpha (p, q, T) \coloneqq \frac{|T \times (p - q)|^\alpha}{|p - q|^\beta} \tag{24}$$

Fix k. Indices[3] of terms in E involving x_k can be enumerated:

- $(k, 1)$, $\cdots$, $(k, k-2)$, $(k, k+2)$, $\cdots$, (k, J)
- $(k-1, 1)$, $\cdots$, $(k-1, k-3)$, $(k-1, k+1)$, $\cdots$, $(k-1, J)$
- $(1, k)$, $\cdots$, $(k-2, k)$, $(k+2, k)$, $\cdots$, (J, k)
- $(1, k-1)$, $\cdots$, $(k-3, k-1)$, $(k+1, k-1)$, $\cdots$, $(J, k-1)$

4.1.1 (k, j)

First note that

$$\frac{\partial}{\partial x_{k,n}} ||x_k - x_j|| = \frac{x_{k,n} - x_{j,n}}{||x_k - x_j||} \tag{25}$$

$$\frac{\partial}{\partial x_k} ||x_k - x_j|| = \frac{x_k - x_j}{||x_k - x_j||} \tag{26}$$

where (26) is simply a vector version of (25)

Also note:

$$k_\beta^\alpha (x_k, x_j, T_k) = k_\beta^\alpha \left(x_k, x_j, \frac{x_{k+1} - x_k}{||x_k - x_{k+1}||} \right) \tag{27}$$

$$= \frac{\sqrt{||x_{k+1} - x_k||^2 ||x_k - x_j||^2 - ((x_{k+1} - x_k) \cdot (x_k - x_j))^2}^{\,\alpha}}{||x_k - x_j||^\beta ||x_k - x_{k+1}||^\alpha} \tag{28}$$

$$= \frac{\xi_{k,j}^{\alpha/2}}{\eta_{k,j}} \tag{29}$$

[3] NOTE: CYCLIC

where $\xi_{k,j} := ||x_{k+1} - x_k||^2 ||x_k - x_j||^2 - ((x_{k+1} - x_k) \cdot (x_k - x_j))^2$ and $\eta_{k,j} = ||x_k - x_j||^\beta ||x_k - x_{k+1}||^\alpha$.

So,

$$\frac{\partial k_\beta^\alpha (x_k, x_j, T_k)}{\partial x_k} = \frac{1}{\eta_{k,j}^2} \left(\frac{\alpha}{2} \xi_{k,j}^{\alpha/2-1} \frac{\partial \xi_{k,j}}{\partial x_k} \eta_{k,j} - \xi_{k,j}^{\alpha/2} \frac{\partial \eta_{k,j}}{\partial x_k} \right) \tag{30}$$

Now, we note:

$$\begin{aligned} \frac{\partial \xi_{k,j}}{\partial x_k} &= 2 (x_k - x_{k+1}) ||x_k - x_j||^2 + 2||x_{k+1} - x_k||^2 (x_k - x_j) \\ &\quad - 2 ((x_{k+1} - x_k) \cdot (x_k - x_j)) (x_j + x_{k+1} - 2x_k) \qquad (31) \\ \frac{\partial \eta_{k,j}}{\partial x_k} &= \beta ||x_k - x_j||^{\beta-2} (x_k - x_j) ||x_k - x_{k+1}||^\alpha \qquad (32) \\ &\quad + ||x_k - x_j||^\beta \alpha ||x_k - x_{k+1}||^{\alpha-2} (x_k - x_{k+1}) \qquad (33) \end{aligned}$$

4.1.2 $(k-1, j)$

This time, the component of interest is:

$$\begin{aligned} k_\beta^\alpha (x_{k-1}, x_j, T_{k-1}) &= k_\beta^\alpha \left(x_{k-1}, x_j, \frac{x_k - x_{k-1}}{||x_k - x_{k-1}||} \right) \qquad (34) \\ &= \frac{\sqrt{||x_k - x_{k-1}||^2 ||x_{k-1} - x_j||^2 - ((x_k - x_{k-1}) \cdot (x_{k-1} - x_j))^2}^\alpha}{||x_{k-1} - x_j||^\beta ||x_k - x_{k-1}||^\alpha} \qquad (35) \\ &= \frac{\xi_{k-1,j}^{\alpha/2}}{\eta_{k-1,j}} \qquad (36) \end{aligned}$$

Refer to (30) for the derivative form.

Note the following derivatives.

$$\frac{\partial \xi_{k-1,j}}{\partial x_k} = 2||x_{k-1} - x_j||^2 (x_k - x_{k-1}) - 2 ((x_k - x_{k-1}) \cdot (x_{k-1} - x_j)) (x_{k-1} - x_j) \tag{37}$$

$$\frac{\partial \eta_{k-1,j}}{\partial x_k} = ||x_{k-1} - x_j||^\beta \alpha ||x_k - x_{k-1}||^{\alpha-2} (x_k - x_{k-1}) \tag{38}$$

4.1.3 (j,k)

This time,

$$k_\beta^\alpha\left(x_j,x_k,T_j\right)=k_\beta^\alpha\left(x_j,x_k,\frac{x_{j+1}-x_j}{||x_{j+1}-x_j||}\right) \tag{39}$$

$$=\frac{\sqrt{||x_{j+1}-x_j||^2||x_k-x_j||^2-\left(\left(x_{j+1}-x_j\right)\cdot\left(x_k-x_j\right)\right)^2}^{\alpha}}{||x_k-x_j||^\beta||x_j-x_{j+1}||^\alpha} \tag{40}$$

$$=\frac{\xi_{j,k}^{\alpha/2}}{\eta_{j,k}} \tag{41}$$

The relevant derivatives are:

$$\frac{\partial\xi_{j,k}}{\partial x_k}=2||x_{j+1}-x_j||^2(x_k-x_j)-2\left(\left(x_{j+1}-x_j\right)\cdot\left(x_k-x_j\right)\right)\left(x_{j+1}-x_j\right) \tag{42}$$

$$\frac{\partial\eta_{j,k}}{\partial x_k}=||x_j-x_{j+1}||^\alpha\beta||x_k-x_j||^{\beta-2}\left(x_k-x_j\right) \tag{43}$$

4.1.4 $(j,k-1)$

Finally,

$$k_\beta^\alpha\left(x_j,x_{k-1},T_j\right)=k_\beta^\alpha\left(x_j,x_{k-1},\frac{x_{j+1}-x_j}{||x_{j+1}-x_j||}\right) \tag{44}$$

$$=\frac{\sqrt{||x_{j+1}-x_j||^2||x_{k-1}-x_j||^2-\left(\left(x_{j+1}-x_j\right)\cdot\left(x_{k-1}-x_j\right)\right)^2}^{\alpha}}{||x_{k-1}-x_j||^\beta||x_j-x_{j+1}||^\alpha} \tag{45}$$

$$=\frac{\xi_{j,k-1}^{\alpha/2}}{\eta_{j,k-1}} \tag{46}$$

Thank God, both ξ and η derivative this time is zero.

4.1.5 Putting Everything Together

Now, the derivative of energy can be written as:

$$\frac{\partial E}{\partial x_k}=\sum\left(\frac{\partial k_{i,j}}{\partial x_k}||x_{i+1}-x_i||||x_{j+1}-x_j||+k_{i,j}\frac{\partial}{\partial x_k}\left(||x_{i+1}-x_i||||x_{j+1}-x_j||\right)\right) \tag{47}$$