

Tackling $u_t = \Delta (u^2)$

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1 The Equation

The equation we are concerned with is:

$$u_t = \Delta (u^2) \quad (1)$$

2 Explicit Scheme

For convenience we consider explicit Euler schemes.

2.1 Scheme Construction After Simplification

Note that (1) can be written as:

$$u_t = \Delta (u^2) \quad (2)$$

$$= 2u\Delta u + 2||\nabla u||^2 \quad (3)$$

In 1-dimensional case, we simplify it as:

$$\frac{\partial u}{\partial t} = 2u \frac{\partial^2 u}{\partial x^2} + 2 \left(\frac{\partial u}{\partial x} \right)^2 \quad (4)$$

Explicit Euler scheme can now be written

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = 2U_j^m \left(\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{\Delta x^2} \right) + 2 \left(\frac{U_{j+1}^m - U_{j-1}^m}{2\Delta x} \right)^2 \quad (5)$$

$$= 2U_j^m \left(\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{\Delta x^2} \right) + \frac{(U_{j+1}^m - U_{j-1}^m)^2}{2\Delta x^2} \quad (6)$$

which can be rewritten as:

$$U_j^{m+1} = 2\mu U_j^m (U_{j+1}^m - 2U_j^m + U_{j-1}^m) + \frac{1}{2}\mu (U_{j+1}^m - U_{j-1}^m)^2 \quad (7)$$

where $\mu := \frac{\Delta t}{\Delta x^2}$ is the CFL number.