Tangent-Point Energy of a Circle

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1 Circle

Given the tangent point energy from Yu, Crane, Schumacher with $\alpha = 2$, $\beta = 4$ (might be identical to the one to Buck, Orloff version)

$$\mathcal{E}_4^2(\gamma) \coloneqq \iint_{M^2} k_4^2(\gamma(x), \gamma(y), T(x))\, \mathrm{d}x_\gamma\, \mathrm{d}y_\gamma \tag{1}$$

where tangent-point kernel is defined as

$$k_4^2(p, q, T) \coloneqq \frac{|T \wedge (p-q)|^2}{|p-q|^\beta} \tag{2}$$

one could show that the tangent-point energy of a circle to be π^2.

Consider parameterizing a circle at the origin with radius a as:

$$\mathbf{r}_1 = a(\cos\theta, \sin\theta, 0)^T \tag{3}$$

$$\mathbf{r}_2 = a(\cos\varphi, \sin\varphi, 0)^T \tag{4}$$

Note we may express T as $T(\theta) = (-\sin\theta, \cos\theta, 0)^T$

Then, the tangent point energy is:

$$\iint_{S^1\times S^1} \frac{|T \wedge (\mathbf{r}_1 - \mathbf{r}_2)|^2}{|\mathbf{r}_1 - \mathbf{r}_2|^4}\, \mathrm{d}s_1\, \mathrm{d}s_2 = \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi} \frac{|T|^2|\mathbf{r}_1 - \mathbf{r}_2|^2 - (T\cdot(\mathbf{r}_1 - \mathbf{r}_2))^2}{|\mathbf{r}_1 - \mathbf{r}_2|^4} a\, \mathrm{d}\theta a\, \mathrm{d}\varphi \tag{5}$$

$$= \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi} \frac{4a^2 \sin^2\frac{\theta-\varphi}{2}}{4a^4(-1+\cos(\theta-\varphi))^2} a\, \mathrm{d}\theta a\, \mathrm{d}\varphi \tag{6}$$

$$= \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi} \frac{\sin^2\frac{\theta-\varphi}{2}}{(-1+\cos(\theta-\varphi))^2}\, \mathrm{d}\theta\, \mathrm{d}\varphi \tag{7}$$

$$= \pi^2 \tag{8}$$

Note that this is scale invariant.

2 Kernel Behavior at Singularity

For $\mathbf{r} = (x(s), y(s), z(s))$ parameterized by arc-length, we may write $\mathbf{T} = \mathbf{r}'$. Note that $||\mathbf{T}|| = 1$.

By Taylor expansion:

$$\mathbf{r}(s+\epsilon) = \mathbf{r}(s) + \epsilon\mathbf{r}'(s) + \frac{1}{2}\epsilon^2\mathbf{r}''(s) + O\left(\epsilon^2\right) \tag{9}$$

Then near $\epsilon = 0$,

$$\begin{aligned}
k_\beta^\alpha\left(\mathbf{r}(s), \mathbf{r}(s+\epsilon), \mathbf{r}'(s)\right) &= \frac{||\mathbf{r}'(s) \wedge (\mathbf{r}(s+\epsilon) - \mathbf{r}(s))||^\alpha}{||\mathbf{r}(s+\epsilon) - \mathbf{r}(s)||^\beta} &(10)\\
&= \frac{||\mathbf{r}'(s) \wedge \left(\epsilon\mathbf{r}'(s) + \frac{1}{2}\epsilon^2\mathbf{r}''(s) + O\left(\epsilon^3\right)\right)||^\alpha}{||\epsilon\mathbf{r}'(s) + O\left(\epsilon\right)||^\beta} &(11)\\
&= \epsilon^{2\alpha-\beta}\left(\frac{1}{2}\right)^\alpha \frac{||\mathbf{r}'(s) \wedge \mathbf{r}''(s) + O\left(\epsilon\right)||^\alpha}{||\mathbf{r}'(s) + O\left(\epsilon\right)||^\beta} &(12)\\
&\sim \frac{\epsilon^{2\alpha-\beta}}{2^\alpha} \frac{||\mathbf{r}'(s) \wedge \mathbf{r}''(s)||^\alpha}{||\mathbf{r}'(s)||^\beta} &(13)\\
&= \frac{\epsilon^{2\alpha-\beta}}{2^\alpha} \frac{||\mathbf{r}'(s)||^\alpha||\mathbf{r}''(s)||^\alpha}{||\mathbf{r}'(s)||^\beta} &(14)\\
&= \frac{\epsilon^{2\alpha-\beta}}{2^\alpha}||\mathbf{r}''(s)||^\alpha = \frac{\epsilon^{2\alpha-\beta}}{2^\alpha}\kappa^\alpha &(15)
\end{aligned}$$