Tangent-Point Energy of a Circle

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1 Circle

Given the tangent point energy from Yu, Crane, Schumacher with $\alpha = 2$, $\beta = 4$ (might be identical to the one to Buck, Orloff version)

$$\mathcal{E}_4^2(\gamma) := \iint_{M^2} k_4^2(\gamma(x), \gamma(y), T(x)) \, \mathrm{d}x_\gamma \, \mathrm{d}y_\gamma \tag{1}$$

where tangent-point kernel is defined as

$$k_4^2(p, q, T) := \frac{|T \wedge (p - q)|^2}{|p - q|^\beta} \tag{2}$$

one could show that the tangent-point energy of a circle to be π^2.

Consider parameterizing a circle at the origin with radius a as:

$$\mathbf{r}_1 = a(\cos\theta, \sin\theta, 0)^T \tag{3}$$

$$\mathbf{r}_2 = a(\cos\varphi, \sin\varphi, 0)^T \tag{4}$$

Note we may express T as $T(\theta) = (-\sin\theta, \cos\theta, 0)^T$

Then, the tangent point energy is:

$$\iint_{S^1 \times S^1} \frac{|T \wedge (\mathbf{r}_1 - \mathbf{r}_2)|^2}{|\mathbf{r}_1 - \mathbf{r}_2|^4} \, \mathrm{d}s_1 \, \mathrm{d}s_2 = \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{2\pi} \frac{|T|^2 |\mathbf{r}_1 - \mathbf{r}_2|^2 - (T \cdot (\mathbf{r}_1 - \mathbf{r}_2))^2}{|\mathbf{r}_1 - \mathbf{r}_2|^4} a \, \mathrm{d}\theta a \, \mathrm{d}\varphi \tag{5}$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{2\pi} \frac{4a^2 \sin^2 \frac{\theta - \varphi}{2}}{4a^4 (-1 + \cos(\theta - \varphi))^2} a \, \mathrm{d}\theta a \, \mathrm{d}\varphi \tag{6}$$

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{2\pi} \frac{\sin^2 \frac{\theta - \varphi}{2}}{(-1 + \cos(\theta - \varphi))^2} \, \mathrm{d}\theta \, \mathrm{d}\varphi \tag{7}$$

$$= \pi^2 \tag{8}$$

Note that this is scale invariant.

2 Ellipse

For an ellipse parameterized by $\mathbf{r}(\theta) = (\alpha \cos\theta, \beta \sin\theta, 0)$, after a hefty computation,

$$\mathcal{E}_4^2(\gamma) = \frac{\pi^2 (\alpha^2 + \beta^2)}{2\alpha\beta} \tag{9}$$