Curve Repulsion - 1

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1 Theory behind Discretization

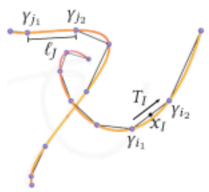

Figure 1: Discretization process

We now consider a discretization of a curve. Index the points on the curve by $\mathcal{I} = \{1, 2, \cdots, J\}$, such that points are given by $\{\gamma_1, \gamma_2, \cdots, \gamma_J\}$

Using the similar notation to the paper by Yu, Schumacher, and Crane, for edge $I = \{\gamma_i, \gamma_j\} \in E$ and for function $u : \mathbb{R}^3 \to \mathbb{R}$

- $l_I \coloneqq |\gamma_i - \gamma_j|$
- $T_I \coloneqq \frac{\gamma_j - \gamma_i}{l_I}$
- $\mathbf{x}_I \coloneqq \frac{\gamma_i + \gamma_j}{2}$
- $u_I \coloneqq \frac{u_i + u_j}{2}$
  - Syntactic sugar: $u_i \equiv u(\gamma_i)$
- $u[I] \coloneqq \begin{pmatrix} u_i \\ u_j \end{pmatrix}$

1.1 Discrete Energy

The naïve discretization of $\mathcal{E}_\beta^\alpha \coloneqq \iint_{M^2} k_\beta^\alpha\left(\gamma(x), \gamma(y), T(x)\right) \mathrm{d}x_\gamma \, \mathrm{d}y_\gamma$ where $k_\beta^\alpha(p, q, T) \coloneqq \frac{|T \times (p-q)|^\alpha}{|p-q|^\beta}$ is given by

$$\sum_{I \in E} \sum_{J \in E} \int_{\bar{I}} \int_{\bar{J}} k_\beta^\alpha\left(\gamma(x), \gamma(y), T_I\right) \mathrm{d}x_\gamma \, \mathrm{d}y_\gamma \tag{1}$$

However, in a polygonal curve (hence the discretized curve), (1) is ill-defined.

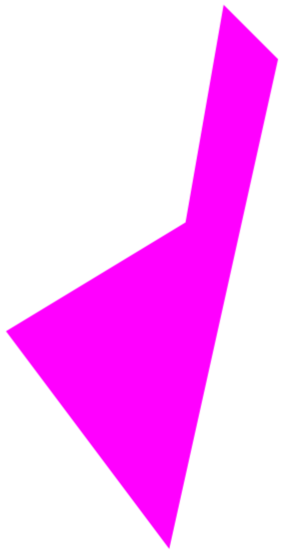

Figure 2: Near each vertex, the integrand is unbounded.

So resolve this by removing the two neighboring edges.[1] Also approximate the kernel by the average of the kernel evaluated at each pair of appropriate edges (total: 4)

$$\hat{\mathcal{E}}_\beta^\alpha \coloneqq \sum_{I,J\in E, I\cap J=\emptyset} \left(\hat{k}_\beta^\alpha\right)_{I,J} l_I l_J \tag{2}$$

$$\left(\hat{k}_\beta^\alpha\right)_{I,J} \coloneqq \frac{1}{4} \sum_{i\in J, j\in J} k_\beta^\alpha\left(\gamma_i, \gamma_j, T_I\right) \tag{3}$$

2 Discrete Gradient Flow in L^2 for Closed Loop

Suppose a curve is discretized as position vectors: $x_1, x_2, \cdots, x_J$ (and $x_{J+1} \coloneqq x_1$).

Also denote the edge from x_i to x_{i+1} as I_i (as opposed to the previous section).

[1]In the limit, the contribution from this removed edge goes to zero.

The **discretized energy** E can be expressed as:

$$E = \sum_{i=1}^{J} \sum_{\substack{j=1 \\ |j-i|>1}} k_{i,j} ||x_{i+1} - x_i|| \, ||x_{j+1} - x_j|| \tag{4}$$

$$k_{i,j} = \frac{1}{4} \left(k_\beta^\alpha \left(x_i, x_j, T_i\right) + k_\beta^\alpha \left(x_i, x_{j+1}, T_i\right) + k_\beta^\alpha \left(x_{i+1}, x_j, T_i\right) + k_\beta^\alpha \left(x_{i+1}, x_{j+1}, T_i\right) \right) \tag{5}$$

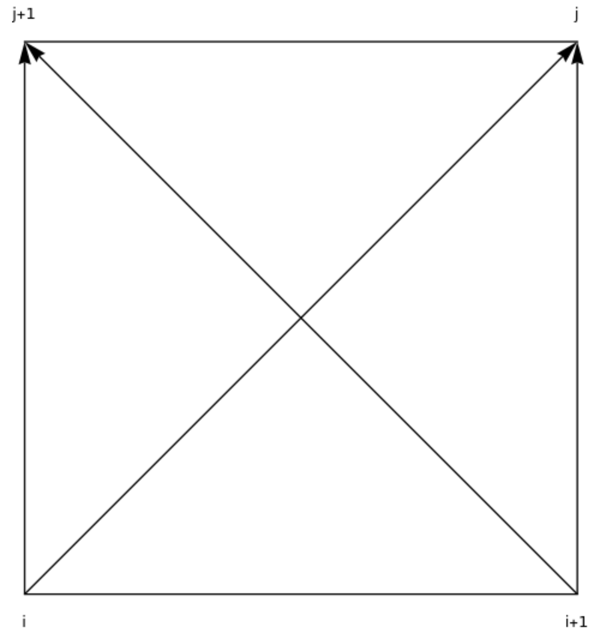

Figure 3: Kernel $k_{i,j}$ computation

Recall the definition of differential, gradient, and gradient flow.

Definition 1 (Differential)**.** Given functional $\mathcal{E}(\gamma)$, the **differential** is defined as:

$$\mathrm{d}\mathcal{E}|_\gamma(u) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\mathcal{E}\left(\gamma + \epsilon u\right) - \mathcal{E}\left(\gamma\right) \right) \tag{6}$$

Definition 2 (Gradient)**.** Given functional $\mathcal{E}\left(\gamma\right)$ and space V, the **gradient** $\operatorname{grad} \mathcal{E}$ is the unique function satisfying the following for any function u:

$$\langle\langle \operatorname{grad} \mathcal{E}, u \rangle\rangle_V = \mathrm{d}\mathcal{E}(u) \tag{7}$$

Note that the LHS is a inner product of two vector-valued functions. A natural inner product in L^2 to define is:

$$\langle\langle u, v \rangle\rangle_{L^2} := \int_\Omega u \cdot v \, \mathrm{d}x \tag{8}$$

Definition 3 (Gradient Flow)**.** Given functional $\mathcal{E}(\gamma)$, the **gradient flow** equation is defined as:

$$\frac{d}{dt}\gamma = -\operatorname{grad}\mathcal{E}(\gamma) \tag{9}$$

Note that $\gamma = (\gamma_x, \gamma_y, \gamma_z)^T \in \mathbb{R}^3$, so it might be clearer to write:

$$\frac{d}{dt}\begin{pmatrix}\gamma_x\\ \gamma_y\\ \gamma_z\end{pmatrix} = \operatorname{grad}\mathcal{E}\left(\begin{pmatrix}\gamma_x\\ \gamma_y\\ \gamma_z\end{pmatrix}\right) \tag{10}$$

Gradient flow equation in L^2 is given by[2]:

$$\frac{d\gamma}{dt} = -\underbrace{\frac{\partial\mathcal{E}}{\partial\gamma}}_{\text{Functional Derivative}} \tag{11}$$

or in our case with discrete energy,

$$\dot{x}_i = -\frac{\partial E}{\partial x_i} \tag{12}$$

For an explicit definition of functional derivatives, see link in the footnote.

Remark 1. Note that each x_i is a 3D vector, meaning in reality, (12) is

$$\dot{x}_{i,1} = -\frac{\partial E}{\partial x_{i,1}} \tag{13}$$

$$\dot{x}_{i,2} = -\frac{\partial E}{\partial x_{i,2}} \tag{14}$$

$$\dot{x}_{i,3} = -\frac{\partial E}{\partial x_{i,3}} \tag{15}$$

2.1 Explicit Euler Scheme

One could now write an explicit Euler scheme based on (13) $\sim$ (15)

$$\frac{X_{i,1}^{m+1} - X_{i,1}^{m}}{\Delta t} = -\frac{E\left(X_1^m, \cdots, X_i^m + \Delta x e_x, \cdots, X_J^m\right) - E\left(X_1^m, \cdots, X_J^m\right)}{\Delta x} \tag{16}$$

3 Constraint

There is a risk that the curve might keep expanding in order to minimize the energy.

[2] https://math.stackexchange.com/questions/1687804/what-is-the-l2-gradient-flow

To mitigate that, we put an additional "penalty" for the length to the energy. In the case of discrete energy, we modify E to F by:

$$F = E + \lambda \sum_i \frac{|x_i|^2}{2} \tag{17}$$

which the gradient flow equation turns into

$$\dot{x}_i = -\frac{\partial E}{\partial x_i} - \lambda x_i \tag{18}$$

where λ is a parameter which one could experiment with.

Seems to parallel Lagrange constant in the method of Langrange multipliers.

4 Appendix

4.1 Index of Regular-Polygonness

If a closed curve is topologically equivalent to a hoop, we expect the untangling process to approach a perfect circle. In a discrete scheme, we expect the curve to approach a regular polygon.

To measure how "regular-polygonlike" a curve is, one may define the following function:

Definition 4. Regularity $\mathcal{R}$ of a polygon P with verticies at $(x_1, x_2, \cdots, x_J)$:

$$\mathcal{R} \coloneqq \mathrm{Var}\,(I) + \mathrm{Var}\,(\Theta) \tag{19}$$

where

- I is a tuple of edge lengths.
- Θ is a tuple of inner angles.
- $\mathrm{Var}(A)$ computes the variance of tuple A by

$$\mathrm{Var}(A) \coloneqq \frac{1}{|A|} \sum_{j=1}^{|A|} \left(\frac{1}{|A|} \sum_{i=1}^{|A|} A_i - A_j \right)^2 \tag{20}$$

This quantity penalizes a polygon which has high "variance" in its edge lengths and its angles; minimized when all the edge lengths are equal and all the angles are equal.