Alternative Tracking

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1 Alternative Tracking Problem

$$
\begin{aligned}
\dot{x_i} &= (f_i(s)u_i(t) - D)\, x_i \quad \forall i \in G_1 \\
\dot{x_i} &= (f_i(s)u_i(t) - D)\, x_i \quad \forall i \in G_2 \\
\dot{s_1} &= (s_{in} - s_1)D - \sum_{i \in G_1} \frac{1}{y_i} f_i(s)u_i(t)x_i \\
\dot{s_2} &= -s_2 D + \sum_{i \in G_1} \frac{1}{y_i} f_i(s)u_i(t) - \sum_{i \in G_2} \frac{1}{y_i} f_i(s)u_i(t)x_i \\
\dot{s_3} &= -s_3 D + \sum_{i \in G_2} \frac{1}{y_i} f_i(s)u_i(t)x_i \\
\dot{u_i} &= v_i \\
y(t) &= g(x, s)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\min \quad & \int_0^T \|g(x,s) - z\|_Q + \|v\|_R dt \\
s.t. \quad & (x, s_1, s_2, s_3, u) \text{ solution of (1)} \\
& v_i(t) \in \mathbb{R}
\end{aligned} \tag{2}
$$

Following the work of Cimen one rewrites system (1) as

$$\dot{X} = A(X)X + B(X)v \tag{3}$$

$$y(x) = C(X)X \tag{4}$$

And the cost functional as

Where the state $X = (x, s_1, s_2, s_3, u)$, and $A(X)$ and $B(X)$ are represented below

$$A\left(X\right) = \begin{bmatrix} A_{11}(X) & A_{12}(X) & A_{13}(X) \\ A_{21}(X) & A_{22}(X) & A_{23}(X) \\ A_{31}(X) & A_{32}(X) & A_{33}(X) \end{bmatrix} \tag{5}$$

$$B\left(X\right) = \begin{bmatrix} B_1(X) \\ B_2(X) \end{bmatrix} \tag{6}$$

$$A_{11}(X) = \mathbf{diag}(\mathbf{diag}(u)f(s) - D_{n\times 1}) \tag{7}$$
$$A_{12}(X) = 0_{n\times 3} \tag{8}$$
$$A_{13}(X) = 0_{n\times n} \tag{9}$$
$$A_{21}(X) = \begin{pmatrix} (\mathbf{diag}(u)f(s))^\top \begin{bmatrix} -\mathbf{diag}(k_{G_1}) & 0_{n_1\times n_2} \end{bmatrix}^\top \\ (\mathbf{diag}(u)f(s))^\top \begin{bmatrix} \mathbf{diag}(k_{G_1}) & -\mathbf{diag}(k_{G_2}) \end{bmatrix}^\top \\ (\mathbf{diag}(u)f(s))^\top \begin{bmatrix} 0_{n_2\times n_1} & \mathbf{diag}(k_{G_2}) \end{bmatrix}^\top \end{pmatrix} \tag{10}$$
$$A_{22}(X) = \begin{pmatrix} \left(\frac{s_{in}}{s_1} - 1\right) D & 0 & 0 \\ 0 & -D & 0 \\ 0 & 0 & -D \end{pmatrix} \tag{11}$$
$$A_{31}(X) = 0_{n\times n} \tag{12}$$
$$A_{32}(X) = 0_{n\times 3} \tag{13}$$
$$A_{33}(X) = 0_{n\times n} \tag{14}$$

$$B(X) = \begin{bmatrix} 0_{(n+3)\times n} \\ I_n \end{bmatrix} \tag{15}$$

$$C(X) = \begin{bmatrix} I_n & 0_{n\times(n+3)} \end{bmatrix} \tag{16}$$

Define

$$\dot{X}^{[i]} = A(X^{[i]})X^{[i]} + B(X^{[i]})v^{[i]} \quad i \in \mathbb{N} \tag{17}$$
$$y^{[i]} = X^{[i]} \quad i \in \mathbb{N} \tag{18}$$
$$X^{[i]}(t_0) = X_0 \quad i \in \mathbb{N} \tag{19}$$

And for $i = -1$ define $X^{[-1]}(t) = X_0$.
The control law is given by

$$v^{[i]}(t)_j = \left(-R^{-1}B^\top \left(X^{[i-1]}(t)\right) \left(P^{[i]}(t)X^{[i]}(t) - s_f^{[i]}(t)\right)\right)_j \forall j \in [n] \tag{20}$$

Where $P^{[i]}(t) \in \mathcal{M}_{2n+3\times 2n+3}(\mathbb{R})$ and $s_f^{[i]}(t) \in \mathbb{R}^{2n+3}$ are the solution to the differential equations:

$$\dot{P}^{[i]} = -C^T \left(X^{[i-1]}(t)\right) QC \left(X^{[i-1]}(t)\right) - P^{[i]}A \left(X^{[i-1]}(t)\right) - A^\top \left(X^{[i-1]}(t)\right) P^{[i]} \tag{21}$$
$$+ P^{[i]}B \left(X^{[i-1]}(t)\right) R^{-1}B^\top \left(X^{[i-1]}(t)\right) P^{[i]} \tag{22}$$
$$P^{[i]}(t_f) = C^\top \left(X^{[i-1]}(t_f)\right) FC \left(X^{[i-1]}(t_f)\right) \tag{23}$$

$$\dot{s_f^{[i]}} = -C^\top \left(X^{[i-1]}(t)\right) Qz(t) - \left[A \left(X^{[i-1]}(t)\right) - B \left(X^{[i-1]}(t)\right) R^{-1}B^\top \left(X^{[i-1]}(t)\right) P^{[i]}(t)\right]^\top s_f^{[i]} \tag{24}$$
$$s_f^{[i]}(t_f) = C^\top \left(X^{[i-1]}(t_f)\right) Fz(t_f) \tag{25}$$

Note

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \tag{26}$$

$$P_{11} \in M_{n\times n} \tag{27}$$

$$P_{12} \in M_{n\times 3} \tag{28}$$

$$P_{13} \in M_{n\times n} \tag{29}$$

$$P_{21} \in M_{3\times n} \tag{30}$$

$$P_{22} \in M_{3\times 3} \tag{31}$$

$$P_{23} \in M_{3\times n} \tag{32}$$

$$P_{31} \in M_{n\times n} \tag{33}$$

$$P_{32} \in M_{n\times 3} \tag{34}$$

$$P_{33} \in M_{n\times n} \tag{35}$$

Replacing the matrices of the problem (and dropping the state dependence notation. Note that $A_{12}, A_{13}, A_{23}, A_{31}, A_{32}, A_{33}$ are all zero.

$$\dot{P}^{[i]}(t) = -\begin{bmatrix} Q & 0_{n\times(3+n)} \\ 0_{(3+n)\times n} & 0_{(3+n)\times(3+n)} \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{36}$$

$$- \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^\top \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \tag{37}$$

$$+ \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} 0_{(n+3)\times n} \\ I_n \end{bmatrix} R^{-1} \begin{bmatrix} 0_{(n+3)\times n} \\ I_n \end{bmatrix}^\top \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \tag{38}$$

$$P^{[i]}(t_f) = 0_{(2n+3)\times(2n+3)} \tag{39}$$

$$\dot{P}^{[i]}(t) = -\begin{bmatrix} Q & 0_{n\times(3+n)} \\ 0_{(3+n)\times n} & 0_{(3+n)\times(3+n)} \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} & 0_{(n+3)\times n} \\ 0_{n\times(n+3)} & 0_{n\times n} \end{bmatrix} \tag{40}$$

$$- \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^\top \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} & 0_{(n+3)\times n} \\ 0_{n\times(n+3)} & 0_{n\times n} \end{bmatrix} \tag{41}$$

$$+ \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} 0_{(n+3)\times(n+3)} & 0_{(n+3)\times n} \\ 0_{n\times n+3} & R^{-1} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \tag{42}$$

$$P^{[i]}(t_f) = 0_{(2n+3)\times(2n+3)} \tag{43}$$

$$
\begin{aligned}
\dot{P}^{[i]}(t) = &- \begin{bmatrix} \begin{bmatrix} Q & 0_{n\times 3} \\ 0_{3\times n} & 0_{3\times 3} \end{bmatrix} & 0_{n+3\times n} \\ 0_{n\times n+3} & 0_{n\times n} \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} & 0_{(n+3)\times n} \\ 0_{n\times(n+3)} & 0_{n\times n} \end{bmatrix} && (44) \\
&- \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^\top \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} & 0_{(n+3)\times n} \\ 0_{n\times(n+3)} & 0_{n\times n} \end{bmatrix} && (45) \\
&+ \begin{bmatrix} \begin{bmatrix} P_{13} \\ P_{23} \end{bmatrix} R^{-1} \begin{bmatrix} P_{31} & P_{32} \end{bmatrix} & \begin{bmatrix} P_{13} \\ P_{23} \end{bmatrix} R^{-1} P_{33} \\ P_{33} R^{-1} \begin{bmatrix} P_{31} & P_{32} \end{bmatrix} & P_{33} R^{-1} P_{33} \end{bmatrix} && (46) \\
P^{[i]}(t_f) = &\ 0_{(2n+3)\times(2n+3)} && (47)
\end{aligned}
$$

$$
\begin{aligned}
\dot{P}^{[i]}(t) = &- \begin{bmatrix} \begin{bmatrix} Q & 0_{n\times 3} \\ 0_{3\times n} & 0_{3\times 3} \end{bmatrix} & 0_{n+3\times n} \\ 0_{n\times n+3} & 0_{n\times n} \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} P_{11}A_{11} + P_{12}A_{21} & P_{12}A_{22} \\ P_{21}A_{11} + P_{22}A_{21} & P_{22}A_{22} \end{bmatrix} & 0_{(n+3)\times n} \\ 0_{n\times(n+3)} & 0_{n\times n} \end{bmatrix} && (48) \\
&- \begin{bmatrix} \begin{bmatrix} A_{11}^\top P_{11} + A_{21}^\top P_{21} & A_{11}^\top P_{12} + A_{21}^\top P_{22} \\ A_{22}^\top P_{21} & A_{22}^\top P_{22} \end{bmatrix} & 0_{(n+3)\times n} \\ 0_{n\times(n+3)} & 0_{n\times n} \end{bmatrix} && (49) \\
&+ \begin{bmatrix} \begin{bmatrix} P_{13} \\ P_{23} \end{bmatrix} R^{-1} \begin{bmatrix} P_{31} & P_{32} \end{bmatrix} & \begin{bmatrix} P_{13} \\ P_{23} \end{bmatrix} R^{-1} P_{33} \\ P_{33} R^{-1} \begin{bmatrix} P_{31} & P_{32} \end{bmatrix} & P_{33} R^{-1} P_{33} \end{bmatrix} && (50) \\
P^{[i]}(t_f) = &\ 0_{(2n+3)\times(2n+3)} && (51)
\end{aligned}
$$

One can see that P_{12}, P_{21} and P_{22} identically zero solve the equation, and therefore by existence and uniqueness, they are zero. Therefore the only entry to be calculated is P_{11}, but one can further reduce that system and noticing that only diagonal elements of P_{11} become zero as solution.

System reduction theorem:

$$
\begin{aligned}
P_{12} &= 0 && (52) \\
P_{13} &= 0 && (53) \\
P_{21} &= 0 && (54) \\
P_{22} &= 0 && (55) \\
P_{23} &= 0 && (56) \\
P_{31} &= 0 && (57) \\
P_{32} &= 0 && (58) \\
P_{33} &= 0 && (59)
\end{aligned}
$$

$$
\begin{aligned}
\dot{s_f}^{[i]}(t) &= - \begin{bmatrix} Qz(t) \\ 0_{(n+3)\times 1} \end{bmatrix} - \begin{bmatrix} A\left(X^{[i-1]}(t)\right) - \begin{bmatrix} 0_{(n+3)\times n} \\ I_n \end{bmatrix} R^{-1} \begin{bmatrix} 0_{(n+3)\times n} \\ I_n \end{bmatrix}^\top P^{[i]}(t) \end{bmatrix}^\top s_f^{[i]} && (60) \\
s_f^{[i]}(t_f) &= 0_{(2n+3)\times 1} && (61)
\end{aligned}
$$