Two step nitrification model study

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$$s_1 \xrightarrow{\mu_1(s,x)} s_2 + y_1 x_1 \tag{R G1}$$

$$s_2 \xrightarrow{\mu_2(s,x)} s_3 + y_2 x_2 \tag{R G2}$$

$$\dot{x_1} = (\mu_1(s,x) - D)\, x_1 \tag{1}$$

$$\dot{x_2} = (\mu_2(s,x) - D)\, x_2 \tag{2}$$

$$\dot{s_1} = (s_{in} - s_1)D - k_1\mu_1(s,x)x_1 \tag{3}$$

$$\dot{s_2} = -s_2 D + k_1\mu_1(s,x)x_1 - k_2\mu_2(s,x)x_2 \tag{4}$$

$$\dot{s_3} = -s_3 D + k_2\mu_2(s,x)x_2 \tag{5}$$

Case 1:

$$\mu_1(s,x) = \bar{\mu}_1 \frac{s_1}{K_1 + s_1} \tag{6}$$

$$\mu_2(s,x) = \bar{\mu}_2 \frac{s_2}{K_1 + s_2} \tag{7}$$

Equilibrium points It is assumed that the points here presented are non-negative. Three cases can take place:

Coexistence: From (1) one gets:

$$s_1^{eq} = \frac{K_1 D}{\bar{\mu}_1 - D} \tag{8}$$

$$x_1^{eq} = \frac{s_{in} - s_1^{eq}}{k_1} \tag{9}$$

$$s_2^{eq} = \frac{K_2 D}{\bar{\mu}_2 - D} \tag{10}$$

$$x_2^{eq} = \frac{s_{in} - s_1^{eq} - s_2^{eq}}{k_2} \tag{11}$$

$$s_3^{eq} = s_{in} - s_1^{eq} - s_2^{eq} \tag{12}$$

Washout of x_2

$$s_1^{eq} = \frac{K_1 D}{\bar{\mu}_1 - D} \tag{13}$$

$$x_1^{eq} = \frac{s_{in} - \frac{K_1 D}{\bar{\mu}_1 - D}}{k_1} \tag{14}$$

$$s_2^{eq} = s_{in} - s_1^{eq} \tag{15}$$

$$x_2^{eq} = 0 \tag{16}$$

$$s_3^{eq} = 0 \tag{17}$$

Washout

$$s_1^{eq} = s_{in} \tag{18}$$
$$x_1^{eq} = 0 \tag{19}$$
$$s_2^{eq} = 0 \tag{20}$$
$$x_2^{eq} = 0 \tag{21}$$
$$s_3^{eq} = 0 \tag{22}$$

Jacobian of the system:

$$J_{11} = \bar{\mu}_1 \frac{s_1}{K_1 + s_1} - D \tag{23}$$
$$J_{12} = 0 \tag{24}$$
$$J_{13} = \bar{\mu}_1 \frac{K_1}{(K_1 + s_1)^2} x_1 \tag{25}$$
$$J_{14} = 0 \tag{26}$$
$$J_{15} = 0 \tag{27}$$
$$J_{21} = 0 \tag{28}$$
$$J_{22} = \bar{\mu}_1 \frac{s_2}{K_2 + s_2} - D \tag{29}$$
$$J_{23} = 0 \tag{30}$$
$$J_{24} = \bar{\mu}_2 \frac{K_1}{(K_1 + s_2)^2} x_2 \tag{31}$$
$$J_{25} = 0 \tag{32}$$
$$J_{31} = -k_1 \bar{\mu}_1 \frac{s_1}{K_1 + s_1} \tag{33}$$
$$J_{32} = 0 \tag{34}$$
$$J_{33} = -D - k_1 \bar{\mu}_1 \frac{K_1}{(K_1 + s_1)^2} x_1 \tag{35}$$
$$J_{34} = 0 \tag{36}$$
$$J_{35} = 0 \tag{37}$$
$$J_{41} = k_1 \bar{\mu}_1 \frac{s_1}{K_1 + s_1} \tag{38}$$
$$J_{42} = -k_2 \bar{\mu}_2 \frac{s_2}{K_2 + s_2} \tag{39}$$
$$J_{43} = k_1 \bar{\mu}_1 \frac{K_1}{(K_1 + s_1)^2} x_1 \tag{40}$$
$$J_{44} = -D - k_2 \bar{\mu}_2 \frac{K_2}{(K_2 + s_2)^2} x_2 \tag{41}$$
$$J_{45} = 0 \tag{42}$$
$$J_{51} = 0 \tag{43}$$
$$J_{52} = 0 \tag{44}$$
$$J_{53} = 0 \tag{45}$$
$$J_{54} = 0 \tag{46}$$
$$J_{55} = -D \tag{47}$$
$$\tag{48}$$

Evaluating the equilibrium point in the jacobian gives:

Case 1

$$J = \begin{bmatrix} 0 & 0 & \frac{D(\bar{\mu}_1 - D)(s_{in} - \frac{K_1 D}{\bar{\mu}_1 - D})}{\bar{\mu}_1 k_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{D(\bar{\mu}_2 - D)(s_{in} - \frac{K_1 D}{\bar{\mu}_1 - D} - \frac{K_2 D}{\bar{\mu}_2 - D})}{\bar{\mu}_2 k_2} & 0 \\ -k_1 D & 0 & -D - \frac{D(\bar{\mu}_1 - D)(s_{in} - \frac{K_1 D}{\bar{\mu}_1 - D})}{\bar{\mu}_1} & 0 & 0 \\ k_1 D & -k_2 D & \frac{D(\bar{\mu}_1 - D)(s_{in} - \frac{K_1 D}{\bar{\mu}_1 - D})}{\bar{\mu}_1} & -D - \frac{D(\bar{\mu}_2 - D)(s_{in} - \frac{K_1 D}{\bar{\mu}_1 - D} - \frac{K_2 D}{\bar{\mu}_2 - D})}{\bar{\mu}_2} & 0 \\ 0 & 0 & 0 & 0 & -D \end{bmatrix} \tag{49}$$

Case 2