Chapter 7 Summary

NEW SYMBOLS

$\sin^{-1} x$ (arcsin x)	inverse sine of x
$\cos^{-1} x$ (arccos x)	inverse cosine of x
$\tan^{-1} x$ (arctan x)	inverse tangent of x
$\cot^{-1} x$ (arccot x)	inverse cotangent of x
$\sec^{-1} x$ (arcsec x)	inverse secant of x
$\csc^{-1} x$ (arccsc x)	inverse cosecant of x

QUICK REVIEW

CONCEPTS / **EXAMPLES**

7.1 Fundamental Identities

Reciprocal Identities

$$\cot\theta = \frac{1}{\tan\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Negative-Angle Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

If θ is in quadrant IV and $\sin\theta = -\frac{3}{5}$, find $\csc\theta$, $\cos\theta$, and $\sin(-\theta)$.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{-\frac{3}{5}} = -\frac{5}{3}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(-\frac{3}{5}\right)^2 + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$$
$$\cos\theta = +\sqrt{\frac{16}{25}} = \frac{4}{5}$$
$\cos\theta$ is positive in quadrant IV.

$$\sin(-\theta) = -\sin\theta = \frac{3}{5}$$

7.2 Verifying Trigonometric Identities

See the box titled Hints for Verifying Identities on page 613.

7.3 Sum and Difference Identities

Cofunction Identities

$$\cos(90° - \theta) = \sin\theta \qquad \cot(90° - \theta) = \tan\theta$$
$$\sin(90° - \theta) = \cos\theta \qquad \sec(90° - \theta) = \csc\theta$$
$$\tan(90° - \theta) = \cot\theta \qquad \csc(90° - \theta) = \sec\theta$$

Find a value of θ such that $\tan\theta = \cot 78°$.

$$\begin{aligned} \tan\theta &= \cot 78° \\ \cot(90° - \theta) &= \cot 78° \\ 90° - \theta &= 78° \\ \theta &= 12° \end{aligned}$$

Sum and Difference Identities

$$\begin{aligned} \cos(A - B) &= \cos A\cos B + \sin A\sin B \\ \cos(A + B) &= \cos A\cos B - \sin A\sin B \\ \sin(A + B) &= \sin A\cos B + \cos A\sin B \\ \sin(A - B) &= \sin A\cos B - \cos A\sin B \end{aligned}$$

Find the exact value of $\cos(-15°)$.

$$\begin{aligned} \cos(-15°) &= \cos(30° - 45°) \\ &= \cos 30°\cos 45° + \sin 30°\sin 45° \\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4} \end{aligned}$$

CONCEPTS	EXAMPLES

Sum and Difference Identities

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Write $\tan\left(\frac{\pi}{4} + \theta\right)$ in terms of $\tan \theta$.

$$\tan\left(\frac{\pi}{4} + \theta\right) = \frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \tan \theta} = \frac{1 + \tan \theta}{1 - \tan \theta} \qquad \tan \frac{\pi}{4} = 1$$

7.4 Double-Angle Identities and Half-Angle Identities

Double-Angle Identities

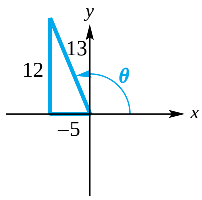
$$\cos 2A = \cos^2 A - \sin^2 A \qquad \cos 2A = 1 - 2\sin^2 A$$

$$\cos 2A = 2\cos^2 A - 1 \qquad \sin 2A = 2 \sin A \cos A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Given $\cos \theta = -\frac{5}{13}$ and $\sin \theta > 0$, find $\sin 2\theta$.

Sketch a triangle in quadrant II and use it to find $\sin \theta$: $\sin \theta = \frac{12}{13}$.



$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$= 2\left(\frac{12}{13}\right)\left(-\frac{5}{13}\right) = -\frac{120}{169}$$

Product-to-Sum Identities

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$$

Write $\sin(-\theta) \sin 2\theta$ as the difference of two functions.

$$\sin(-\theta) \sin 2\theta = \frac{1}{2}[\cos(-\theta - 2\theta) - \cos(-\theta + 2\theta)]$$

$$= \frac{1}{2}[\cos(-3\theta) - \cos \theta]$$

$$= \frac{1}{2}\cos(-3\theta) - \frac{1}{2}\cos \theta$$

$$= \frac{1}{2}\cos 3\theta - \frac{1}{2}\cos \theta$$

Sum-to-Product Identities

$$\sin A + \sin B = 2 \sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\sin A - \sin B = 2 \cos\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

$$\cos A + \cos B = 2 \cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

Write $\cos \theta + \cos 3\theta$ as a product of two functions.

$$\cos \theta + \cos 3\theta = 2 \cos\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right)$$

$$= 2 \cos\left(\frac{4\theta}{2}\right)\cos\left(\frac{-2\theta}{2}\right)$$

$$= 2 \cos 2\theta \cos(-\theta)$$

$$= 2 \cos 2\theta \cos \theta$$

Half-Angle Identities

$$\cos \frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}} \qquad \sin \frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$$

$$\tan \frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}} \qquad \tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}$$

$$\tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}$$

(The sign is chosen based on the quadrant of $\frac{A}{2}$.)

Find the exact value of $\tan 67.5°$.

We choose the last form with $A = 135°$.

$$\tan 67.5° = \tan \frac{135°}{2} = \frac{1 - \cos 135°}{\sin 135°} = \frac{1 - \left(-\frac{\sqrt{2}}{2}\right)}{\frac{\sqrt{2}}{2}}$$

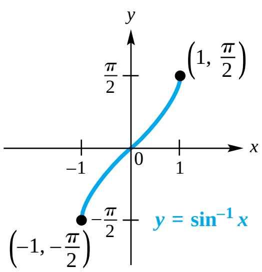
$$= \frac{1 + \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \cdot \frac{2}{2} = \frac{2 + \sqrt{2}}{\sqrt{2}} \quad \text{or} \quad \sqrt{2} + 1$$

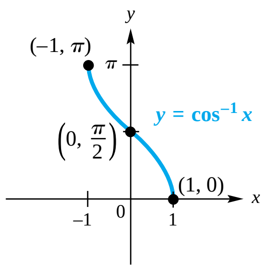
Rationalize the denominator; simplify.

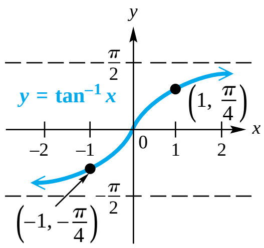
CONCEPTS | **EXAMPLES**

7.5 Inverse Circular Functions

Inverse Function	Domain	Range: Interval	Range: Quadrants of the Unit Circle
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	I and IV
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	I and II
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	I and IV
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$	I and II
$y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi], y \neq \frac{\pi}{2}$	I and II
$y = \csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \neq 0$	I and IV







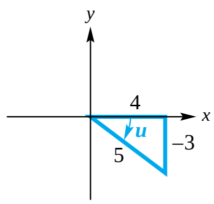
See page 649 for graphs of the other inverse circular (trigonometric) functions.

Evaluate $y = \cos^{-1} 0$.

Write $y = \cos^{-1} 0$ as $\cos y = 0$. Then $y = \frac{\pi}{2}$, because $\cos \frac{\pi}{2} = 0$ and $\frac{\pi}{2}$ is in the range of $\cos^{-1} x$.

Evaluate $\sin\left[\tan^{-1}\left(-\frac{3}{4}\right)\right]$.

Let $u = \tan^{-1}\left(-\frac{3}{4}\right)$. Then $\tan u = -\frac{3}{4}$. Since $\tan^{-1} x$ is negative in quadrant IV, sketch a triangle as shown.



We want $\sin\left[\tan^{-1}\left(-\frac{3}{4}\right)\right] = \sin u$. From the triangle, $\sin u = -\frac{3}{5}$.

7.6 Trigonometric Equations

Solving a Trigonometric Equation

1. Decide whether the equation is linear or quadratic in form, so you can determine the solution method.
2. If only one trigonometric function is present, first solve the equation for that function.
3. If more than one trigonometric function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to 0 to solve.
4. If the equation is quadratic in form, but not factorable, use the quadratic formula. Check that solutions are in the desired interval.
5. Try using identities to change the form of the equation. It may be helpful to square both sides of the equation first. If this is done, check for extraneous solutions.

Solve $\tan \theta + \sqrt{3} = 2\sqrt{3}$ over the interval $[0°, 360°)$. Use a linear method.

$$\begin{aligned} \tan \theta + \sqrt{3} &= 2\sqrt{3} \\ \tan \theta &= \sqrt{3} \\ \theta &= 60° \end{aligned}$$

Another solution over $[0°, 360°)$ is

$$\theta = 60° + 180° = 240°.$$

The solution set is $\{60°, 240°\}$.