Spectral Methods for Elliptic PDEs

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Outline

Our goal is to solve the general elliptic PDE

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0,$$

with

$$B^2 - AC < 0.$$

Subject to boundary conditions

$$(a + b\vec{n} \cdot \nabla)u|_{\partial\Omega} = g.$$

This equation arises in numerical relativity.

The Hamiltonian constraint

$$\bar{\nabla}^2\psi - \frac{1}{8}\bar{R}\psi - \frac{1}{12}K^2\psi^5 + \frac{1}{8}\bar{A}_{ij}\bar{A}^{ij}\psi^{-7} = -2\pi\psi^5\rho.$$

The lapse equation

$$\nabla^2\alpha = \alpha\left(K_{ij}K^{ij} + 4\pi\left(\rho + S\right)\right).$$

Where ∇^2 is the Laplace-Beltrami operator

$$\nabla^2 f = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right).$$

Spectral Methods can be used to represent the differential equation as an algebraic system of equations.

Consider the linear operators present in elliptic equations, they all have the form

$$\mathcal{L}u = \partial_y^e \partial_x^d \left(c(x,y) \partial_y^b \partial_x^a u \right),$$

where $a, b, d, e = 0, 1$. This can be represented with the matrix

$$L = (E \otimes D)\,\mathrm{diag}(C(:))(B \otimes A),$$

where $A, D = 1, \partial_x$; $B, E = 1, \partial_y$; $C_{ij} = c(x_i, y_j)$; $X \otimes Y$ is the Kronecker product, and $X(:)$ denotes vectorization. This allows the compact operation

$$LX(:) = D\left(C \odot \left(AXB^T\right)\right)E^T$$

So the linear system has the form

$$\sum_i (E_i \otimes D_i)\,\mathrm{diag}(C_i(:))(B_i \otimes A_i)X(:) = F(:)$$

To impose boundary conditions we must replace equations on our system.

Now let $X, F \in \mathbb{R}^{m \times n}$, $AX(:) = F(:)$ represents a $mn \times mn$ system of equations.
Let $K = K_2 \otimes K_1$ be the matrix that keeps the interior elements of a vector, i.e.

$$KX(:) = K_1 X {K_2}^T = X(2 : m-1, 2 : n-1).$$

We discard the outer equations by multiplying the system by K

$$KAX(:) = KF(:).$$

Now we have room at the boundary to impose the BCs

$$BX(:) = b.$$

The algebraic system becomes

$$\begin{bmatrix} KA \\ B \end{bmatrix} X(:) = \begin{bmatrix} KF(:) \\ b \end{bmatrix}.$$

Easy, just build the matrix and invert it. Problem solved.

Actually, no. That will take a lot of memory and a very long time. We will work around a more clever way of solving this linear system.

We assume that boundary operators $\mathcal{B}_i = \cos\theta_i + \sin\theta_i \partial_i$ have constant coefficients along each boundary. This will allow us to decouple the degrees of freedom associated with the enforcement of BCs from our solution. We are able to express our solution with a basis that does this

$$S_i = \begin{bmatrix} E_i & N_i \end{bmatrix}$$

$$B_i S_i = \begin{bmatrix} B_i E_i & B_i N_i \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$$

$$\begin{aligned} X &= (S_2 \otimes S_1)\tilde{X} \\ &= (E_2 \otimes E_1)\tilde{X}_{11} + (E_2 \otimes N_1)\tilde{X}_{\Gamma 1} \\ &+ (N_2 \otimes E_1)\tilde{X}_{1\Gamma} + (N_2 \otimes N_1)\tilde{X}_{\Gamma\Gamma} \\ &= E\tilde{X}_{11} + X_\Gamma \end{aligned}$$

We still have some freedom on the choice of the basis S_i.

For simplicity, let us impose the condition $KX(:) = \tilde{X}_{11}(:)$. This choice implies $K_i E_i = I$ and $K_i N_i = 0$, i.e.

$$\begin{bmatrix} K_i \\ B_i \end{bmatrix} \begin{bmatrix} E_i & N_i \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Hence,

$$S_i = \begin{bmatrix} E_i & N_i \end{bmatrix} = \begin{bmatrix} K_i \\ B_i \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ B_{i,1} & B_{i,\Gamma} \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ G_i & B_{i,\Gamma}^{-1} \end{bmatrix},$$

where

$$G_i = -B_{i,\Gamma}^{-1} B_{i,1}.$$

In this basis, the boundary operators do not combine interior dofs.

$$\begin{bmatrix} I \otimes B_1 \\ B_2 \otimes I \end{bmatrix} X = \begin{bmatrix} S_2 \otimes [0\ I] \\ [0\ I] \otimes S_1 \end{bmatrix} \tilde{X}$$

$$= \begin{bmatrix} 0 & E_2 \otimes I & 0 & N_2 \otimes I \\ 0 & 0 & I \otimes E_1 & I \otimes N_1 \end{bmatrix} \begin{bmatrix} \tilde{X}_{11} \\ \tilde{X}_{\Gamma 1} \\ \tilde{X}_{1\Gamma} \\ \tilde{X}_{\Gamma\Gamma} \end{bmatrix}$$

Note that $\tilde{X}_{11}$ does not appear on the equations. The solution to the system $BX(:) = b$ by least squares yields

$$X_\Gamma = X_0 + (N_1 b_1 - X_0)P_2 + P_1(b_2 {N_2}^T - X_0)$$

Boundary conditions have been decoupled!

At this point X_Γ has been determined, and can be moved to the RHS.

$$(KAE)\,\tilde{X}_{11}(:) = K\left(F(:) - AX_\Gamma(:)\right)$$

The matrix $\tilde{A} = KAE$ is nothing more than the Schur complement of the system.

And recall that the final solution can be obtained from

$$X = E\tilde{X}_{11} + X_\Gamma.$$

The final step seems pretty obvious: invert $\tilde{A}$.

Well, that's a huge problem. The matrix $\tilde{A}$ is very large and dense, but still, posses a structure that can be exploited.

Suppose that Σ can be diagonalized by separable eigenvectors.

$$\tilde{A} = (V_2 \otimes V_1)\Lambda(V_2^{-1} \otimes V_1^{-1}).$$

which implies that Σ may be expanded as a linear combination of terms which can be diagonalized by those eigenvectors

$$\Sigma = s_0 I \otimes I + I \otimes \Sigma_1 + \Sigma_2 \otimes I + \mathcal{O}(\Sigma_2 \otimes \Sigma_1).$$

We may find the eigenvectors form the first order terms

$$\Sigma_i = V_i \Lambda_i V_i^{-1}.$$

And the eigenvalues should appear in

$$\operatorname{diag} \Lambda = \operatorname{diag}\left((V_2^{-1} \otimes V_1^{-1})\Sigma(V_2 \otimes V_1)\right).$$

The inverse operation becomes trivial and fast to compute.

$$\Sigma^{-1} = (V_2 \otimes V_1)\Lambda^{-1}(V_2^{-1} \otimes V_1^{-1}).$$

The previous result was valid only under the assumption of separability of the eigenvectors. But in the general case, this is the exact first-order perturbation approximation. This can be used as a **preconditioner for an iterative method**.

Hence, we must find the terms Σ_1, Σ_2 that appear in a Kronecker product with the identity in the expansion of Σ.

References