Hall effect influence in evolution and equilibria of magnetic fields in Neutron stars crust

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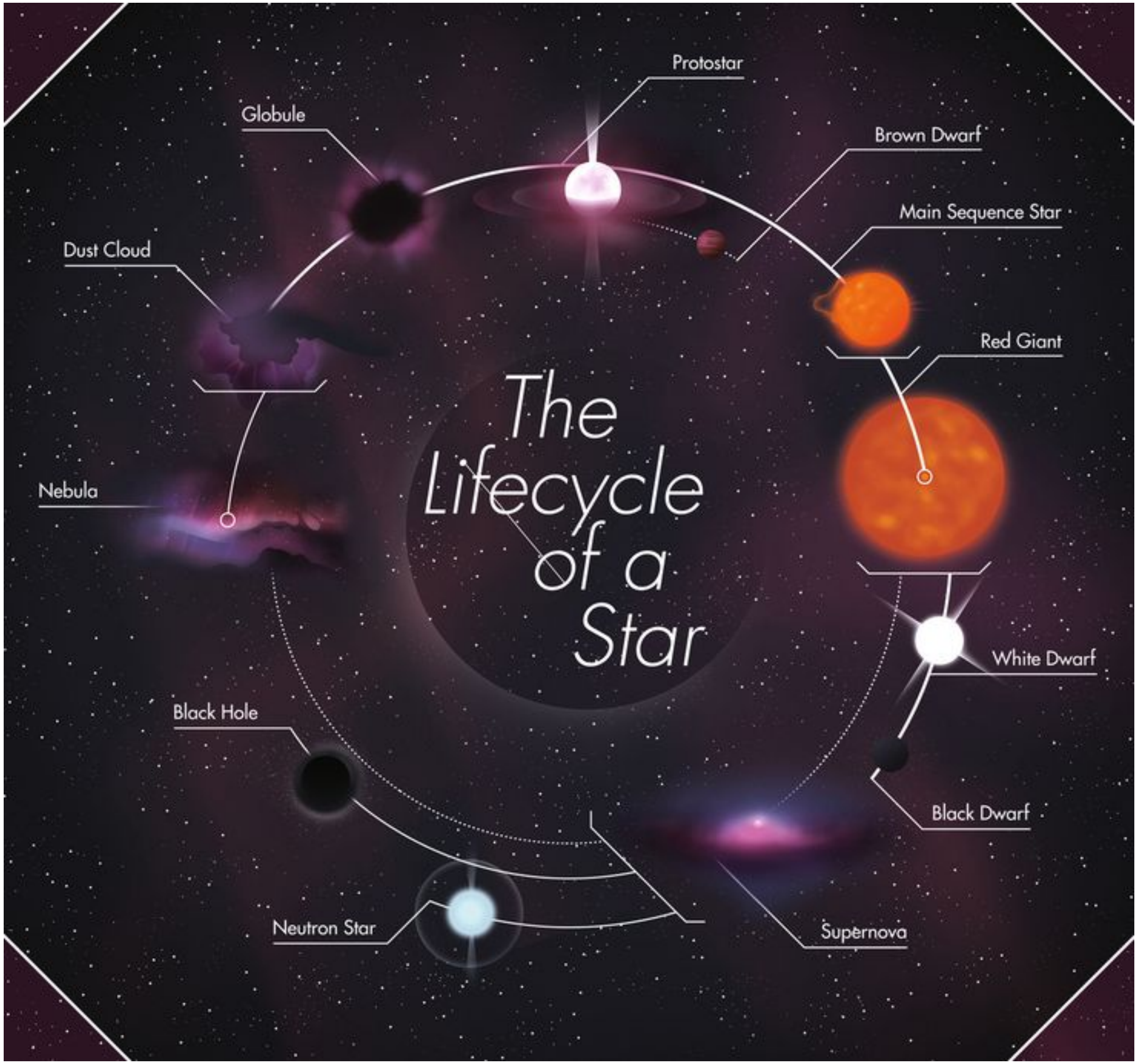
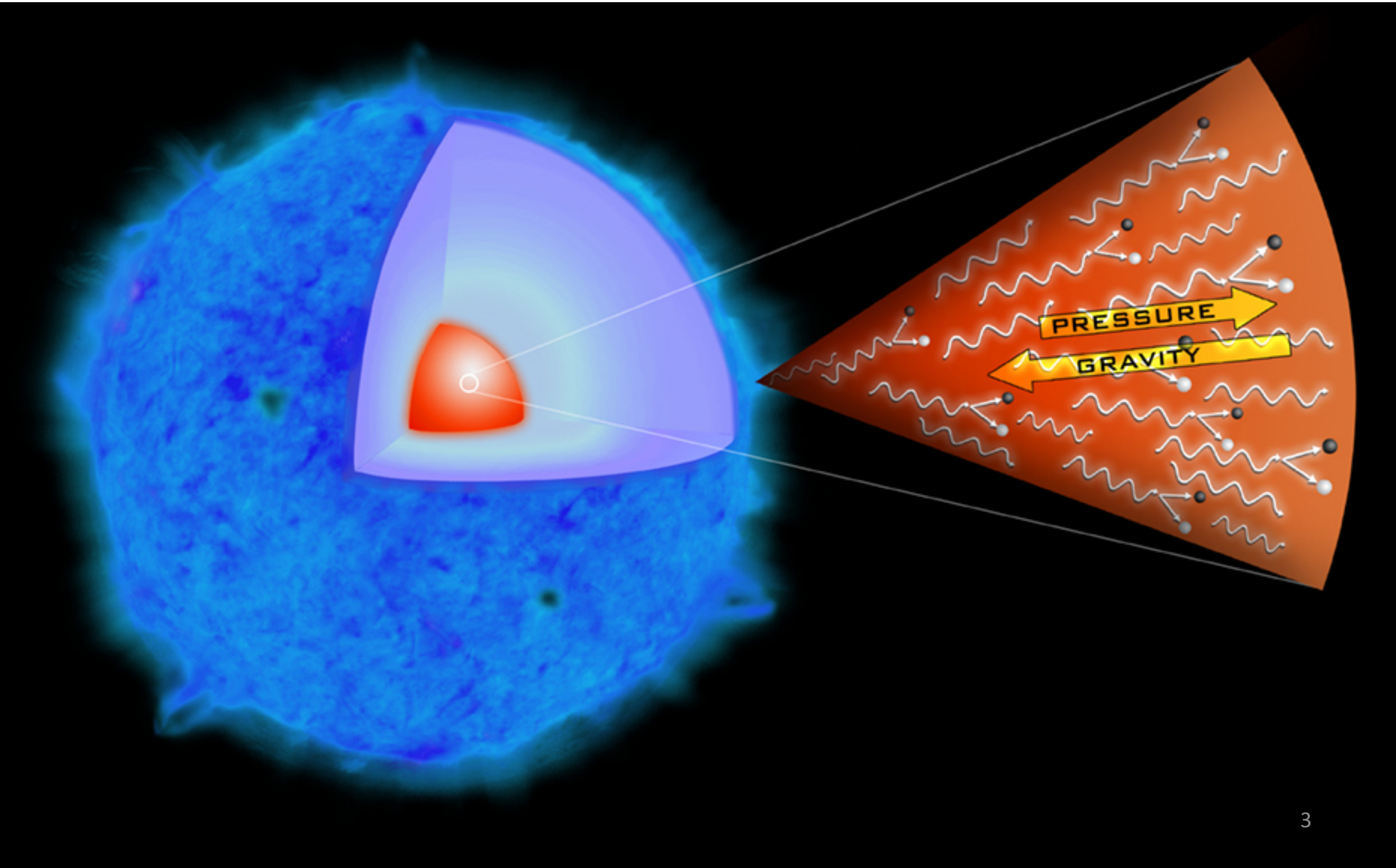
PCM
Computational Applications

Stellar evolution



How a NS is built?



Chandrasekhar Limit

$$M_{\mathrm{limit}} = \frac{\omega_3^0 \sqrt{3\pi}}{2} \left(\frac{\hbar c}{G} \right)^{3/2} \frac{1}{(\mu_e m_H)^2},$$

No equilibrium between electron degeneracy and gravitational force, then the star begin to collapse itself

Maximum mass value for a cold star

M>1.44 Solar mass

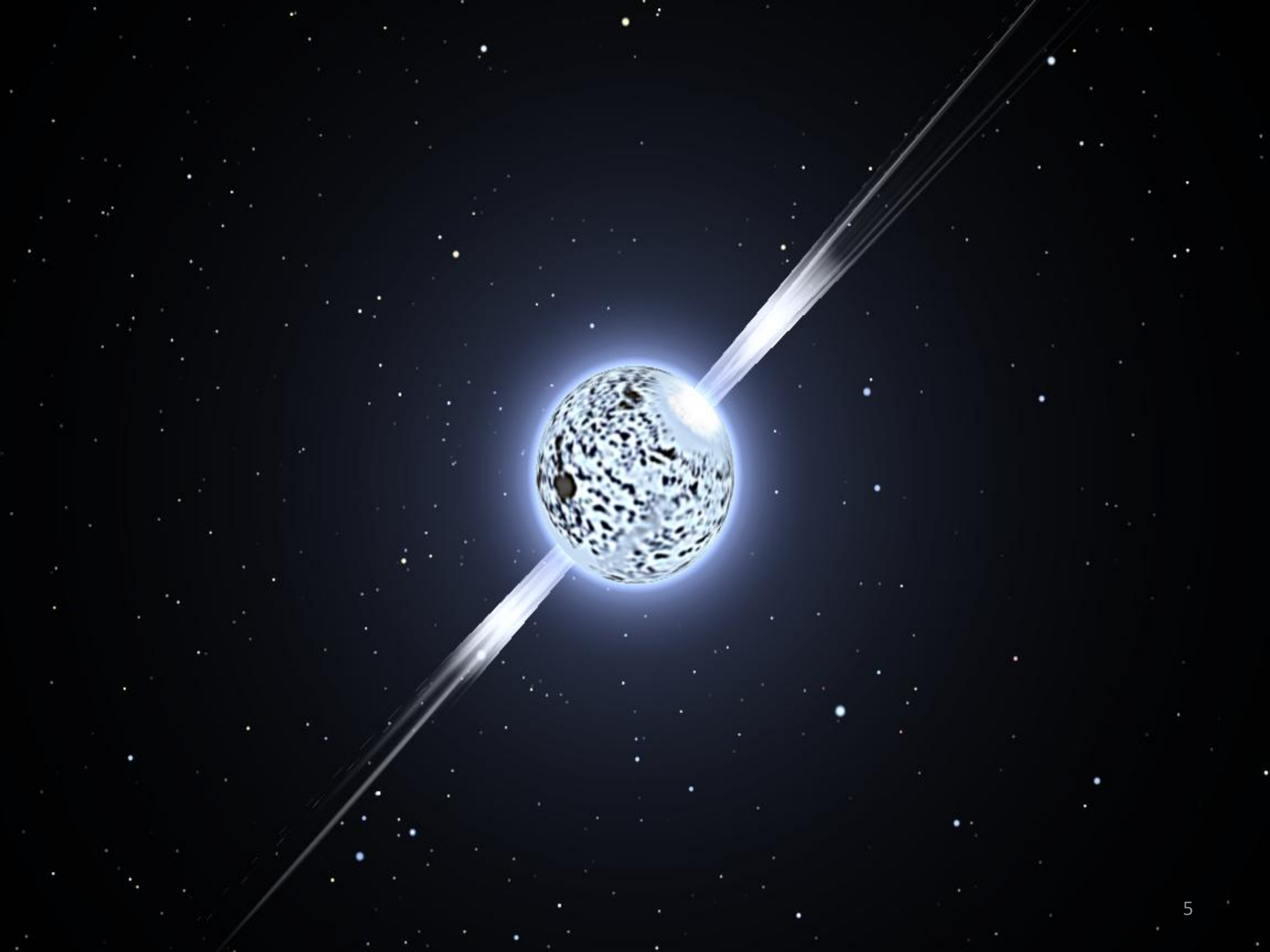
Tolman-Oppenheimer-Volkoff Limit

M>3 Solar mass → **Black hole**

$$\frac{dp}{dr} = -G(\rho(1+\epsilon/c^2)+p/c^2)\frac{m+4\pi r^3 p/c^2}{r(r-2Gm/c^2)}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho(1+\epsilon/c^2)\ ,$$

$$\frac{d\phi}{dr} = \frac{m+4\pi r^3 p/c^2}{r(r-2Gm/c^2)}\ .$$



A star made of Neutrons… principal idea

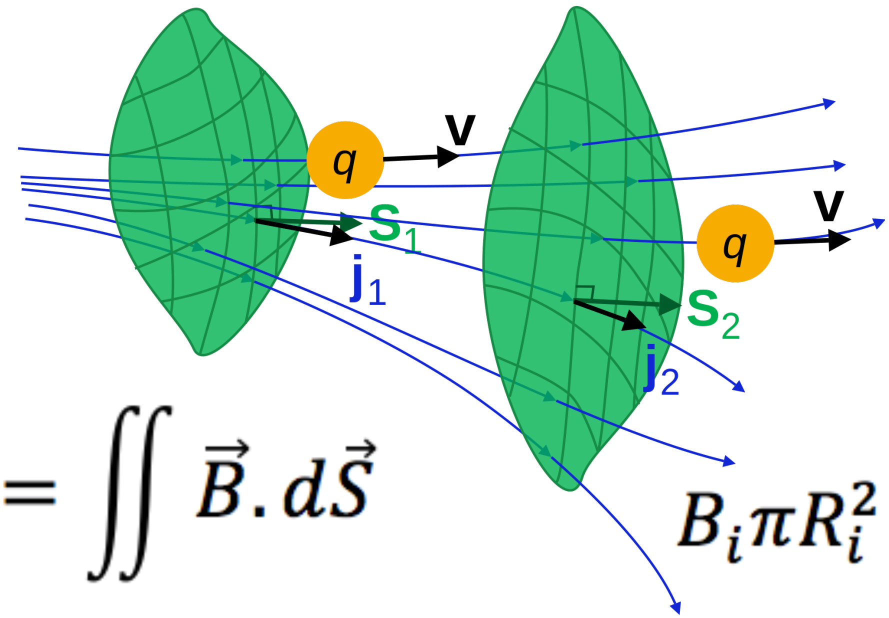
No really

It is not the only characteristic

- Strongest gravitational force. (canditates for gravitational waves sources)
- Strongest magnetic field known by human beings. (Gamma rays source?)
- Matter phase state transitions from their surface to interior.
- Much stuff of particles and nuclear interactions.

Sources of Magnetic Field

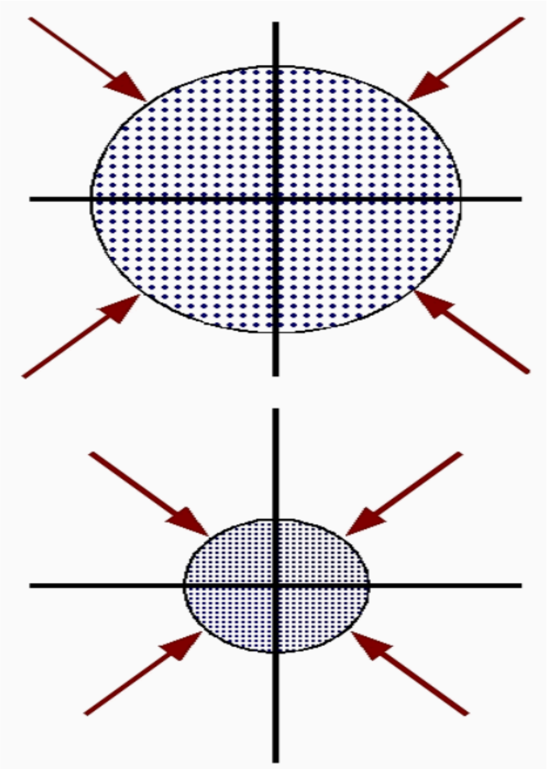
- **The fossil field hypothesis: Flux conservation**



$$\phi = \iint \vec{B}.d\vec{S}$$

$$B_i \pi R_i^2 = B_f \pi R_f^2$$

$$B_f = B_i \left(\frac{R_i}{R_f}\right)^2$$



Model Description

Induction Equation for Magnetic Field in Neutron Star

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} \rightarrow \frac{\partial \vec{B}}{\partial t} = -c\nabla \times \vec{E} \rightarrow \frac{\vec{j}}{\sigma} = \vec{E},$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \left(\frac{c^2}{4\pi\sigma}\nabla \times \vec{B}\right) \rightarrow \frac{\partial \vec{B}}{\partial t} = \vec{\nabla}^2\vec{B},$$

Ohmic Diffusion

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \left(\frac{c^2}{4\pi\sigma} \nabla \times \vec{B} \right)$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}^2 \vec{B}$$

$$\tau_{ohm} = \frac{4\pi\sigma L^2}{c^2} \sim 50 \ billion \ years$$

The Universe is only 14 billion years old

That model absolutely does not work.
But… From MHD model, Ohm's Law

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)$$

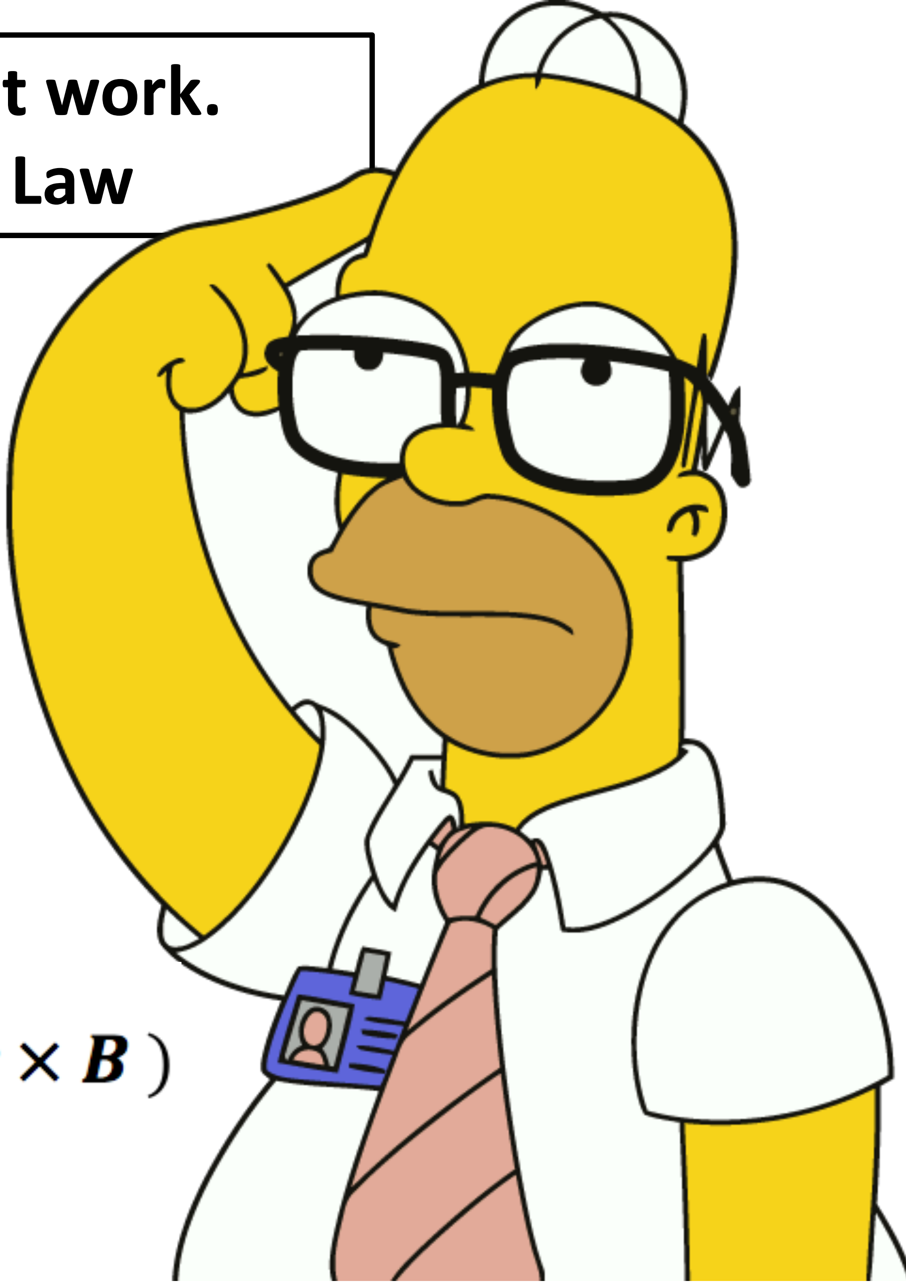
$$\downarrow$$

$$\boldsymbol{E} = \frac{c}{4\pi\sigma}(\nabla\times\boldsymbol{B}) - \boldsymbol{v}\times\boldsymbol{B}$$

$$\partial\boldsymbol{B}/\partial t = -c\nabla\times\boldsymbol{E}$$

$$\downarrow$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = -\frac{c^2}{4\pi\sigma}[\nabla\times(\nabla\times\boldsymbol{B})] + c\nabla\times(\boldsymbol{v}\times\boldsymbol{B})$$



$$\nabla\times(\nabla\times\boldsymbol{B}) = -\nabla^2\boldsymbol{B} + \nabla(\nabla.\boldsymbol{B})$$

$$\boxed{\frac{\partial \boldsymbol{B}}{\partial t} = \frac{c^2}{4\pi\sigma}\nabla^2 \boldsymbol{B} + c\nabla\times(\boldsymbol{v} \times \boldsymbol{B})}$$

$\nabla\times(\boldsymbol{v} \times \boldsymbol{B})$ Dominant term – Convection

This case corresponds if you take a value of conductivity infinite, or ideal MHD. The magnetic field and fluid are connected closer. Field lines are transported by convection motion and the highest conductivity prevents magnetic field diffusion in fluid and internal electric field. The magnetic flux is freezing into the fluid.

$\frac{c^2}{4\pi\sigma}\nabla^2 \boldsymbol{B}$ Dominant term – Diffusion

The induction equation is similar to diffusion equation, and then the magnetic force lines can diffuse through plasma fluid. There are not coupling between magnetic field and conducting fluid.

But → $J = nev$

Then...

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left[\frac{c}{4\pi ne} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \right] - \nabla \times \left[\frac{c^2}{4\pi\sigma} \nabla \times \boldsymbol{B} \right]$$

$$\downarrow$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \nabla^2 \boldsymbol{B} - \frac{c}{4\pi ne} \{\nabla \times [(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}]\}$$

$\eta = c^2/4\pi\sigma$ **Magnetic Diffusivity**

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla^2(\mathbf{B}) - R_{\mathcal{M}}\{\nabla\times[(\nabla\times(\mathbf{B}))\times(\mathbf{B})]\}$$

Ohmic Difussion

Hall Effect

$$R_{\mathcal{M}} = \frac{\sigma B_0}{cne}$$

Magnetic Reynolds number

The value is adimentional and its function is to give and estimate of magnetic field advective effects due to plasma movement in proportion to magnetic diffusion.

Introduction to Ohmic Modes

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla^2 \boldsymbol{B}$$

$$(\nabla^2 \boldsymbol{B})_r = \nabla^2 B_r - \frac{2B_r}{r^2} - \frac{2}{r^2}\frac{\partial B_\theta}{\partial \theta} - \frac{2\ \cot\theta B_\theta}{r^2} - \frac{2}{r^2 \sin\theta}\frac{\partial B_\varphi}{\partial \varphi}$$

$$(\nabla^2 \boldsymbol{B})_\theta = \nabla^2 B_\theta + \frac{2}{r^2}\frac{\partial B_r}{\partial \theta} - \frac{B_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial B_\varphi}{\partial \varphi}$$

$$(\nabla^2 \boldsymbol{B})_\varphi = \nabla^2 B_\varphi - \frac{B_\theta}{r^2 \sin^2\theta} + \frac{2}{r^2 \sin\theta}\frac{\partial B_r}{\partial \varphi} + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial B_\theta}{\partial \varphi}$$

Chandrasekhar Construction

$$\boldsymbol{B} = \nabla \times [g(r,\theta,\varphi,t)\hat{r}] + \nabla \times \nabla \times [h(r,\theta,\varphi,t)\hat{r}]$$

$$h(r,\theta,\varphi,t) = \sum_{l,m} h_{lm}(r,t)\frac{Y_{lm}(\theta,\varphi)}{N_{lm}}$$

$$g(r,\theta,\varphi,t) = \sum_{l,m} g_{lm}(r,t)\frac{Y_{lm}(\theta,\varphi)}{N_{lm}}$$

$$B_r = -\frac{1}{r^2}\left(\frac{\partial^2 h(r,\theta,\varphi,t)}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial h(r,\theta,\varphi,t)}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 h(r,\theta,\varphi,t)}{\partial\varphi^2}\right)$$

$$B_\theta = \frac{1}{r\sin\theta}\frac{\partial g(r,\theta,\varphi,t)}{\partial\varphi} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial h(r,\theta,\varphi,t)}{\partial\theta}\right)$$

$$B_\varphi = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{\partial h(r,\theta,\varphi,t)}{\partial\varphi}\right) - \frac{\partial g(r,\theta,\varphi,t)}{\partial\theta}\right]$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla\times(\nabla\times\mathbf{B})$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla\times\{\nabla\times\nabla\times[g(r,\theta,\varphi,t)\hat{r}] + \nabla\times\nabla\times\nabla\times[h(r,\theta,\varphi,t)\hat{r}]\}$$

Radial component

$$B_r = -\sum_{l,m}\frac{1}{r^2}\frac{h_{lm}(r,t)}{N_{lm}}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y_{lm}(\theta,\varphi)}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}(\theta,\varphi)}{\partial\varphi^2}\right]$$

$$\frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}(\theta,\varphi)}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y_{lm}(\theta,\varphi)}{\partial\theta}\right) = -l(l+1)Y_{lm}(\theta,\varphi)$$

$$B_r = \sum_{l,m}\frac{l(l+1)}{r^2}\frac{h_{lm}(r,t)}{N_{lm}}Y_{lm}(\theta,\varphi)$$

Solving the rotationals

$$\nabla\times\nabla\times\nabla\times[g(r,\theta,\varphi,t)\hat{r}] = 0$$

$$\nabla\times\nabla\times\nabla\times\nabla\times[h(r,\theta,\varphi,t)\hat{r}] =$$

$$\begin{aligned}
\frac{1}{r^4}\sum_{l,m}\Bigg\{&(3+\cos 2\theta)\csc^4\theta\,\frac{\partial^2 h_{lm}(r,t)}{\partial\varphi^2} + \csc^4\theta\,\frac{\partial^4 h_{lm}(r,t)}{\partial\varphi^4} - \frac{\partial^2 h_{lm}(r,t)}{\partial\theta^2} + \frac{\partial^4 h_{lm}(r,t)}{\partial\theta^4} \\
&+ \csc\theta\left[2\cos\theta\,\frac{\partial^3 h_{lm}(r,t)}{\partial\theta^3}\right. \\
&+ \csc\theta\left[\cot\theta\left(\frac{\partial h_{lm}(r,t)}{\partial\theta} - 2\,\frac{\partial h_{lm}(r,t)}{\partial\theta}\frac{\partial^2 h_{lm}(r,t)}{\partial\varphi^2}\right) - \frac{\partial^2 h_{lm}(r,t)}{\partial\theta^2}\right. \\
&\left.+ 2\,\frac{\partial^2 h_{lm}(r,t)}{\partial\theta^2}\frac{\partial^2 h_{lm}(r,t)}{\partial\varphi^2} + r^2\,\frac{\partial^2 h_{lm}(r,t)}{\partial r^2}\frac{\partial^2 h_{lm}(r,t)}{\partial\varphi^2}\right] \\
&\left.+ r^2\cos\theta\,\frac{\partial^2 h_{lm}(r,t)}{\partial r^2}\frac{\partial h_{lm}(r,t)}{\partial\theta}\right] + r^2\,\frac{\partial^2 h_{lm}(r,t)}{\partial r^2}\frac{\partial^2 h_{lm}(r,t)}{\partial\theta^2}\Bigg\}
\end{aligned}$$

Taking Br component and last result. You can replace in radial component of Chandrasekhar Construction

$$\frac{\partial}{\partial t}\left[\sum_{l,m}\frac{l(l+1)}{r^2}\frac{h_{lm}(r,t)}{N_{lm}}Y_{lm}(\theta,\varphi)\right]$$

$$= \frac{1}{r^4}\sum_{l,m}\left\{(3+\cos 2\theta)\csc^4\theta\,\frac{\partial^2 h_{lm}(r,t)}{\partial\varphi^2} + \csc^4\theta\,\frac{\partial^4 h_{lm}(r,t)}{\partial\varphi^4} - \frac{\partial^2 h_{lm}(r,t)}{\partial\theta^2}\right.$$

$$+ \frac{\partial^4 h_{lm}(r,t)}{\partial\theta^4}$$

$$+ \csc\theta\left[2\cos\theta\,\frac{\partial^3 h_{lm}(r,t)}{\partial\theta^3}\right.$$

$$+ \csc\theta\left[\cot\theta\left(\frac{\partial h_{lm}(r,t)}{\partial\theta} - 2\frac{\partial h_{lm}(r,t)}{\partial\theta}\frac{\partial^2 h_{lm}(r,t)}{\partial\varphi^2}\right) - \frac{\partial^2 h_{lm}(r,t)}{\partial\theta^2}\right.$$

$$\left. + 2\frac{\partial^2 h_{lm}(r,t)}{\partial\theta^2}\frac{\partial^2 h_{lm}(r,t)}{\partial\varphi^2} + r^2\frac{\partial^2 h_{lm}(r,t)}{\partial r^2}\frac{\partial^2 h_{lm}(r,t)}{\partial\varphi^2}\right]$$

$$\left.\left. + r^2\cos\theta\,\frac{\partial^2 h_{lm}(r,t)}{\partial r^2}\frac{\partial h_{lm}(r,t)}{\partial\theta}\right] + r^2\frac{\partial^2 h_{lm}(r,t)}{\partial r^2}\frac{\partial^2 h_{lm}(r,t)}{\partial\theta^2}\right\}$$

You obtain

$$\sum_{l,m} \frac{l(l+1)}{r^2}\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial r^2} + \frac{l(l+1)}{r^2}\right]\frac{h_{lm}(r,t)}{N_{lm}} Y_{lm}(\theta,\varphi) = 0$$

Returning to original Induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla^2(\mathbf{B}) - R_{\mathcal{M}}\{\nabla\times[(\nabla\times(\mathbf{B}))\times(\mathbf{B})]\}$$

$$\sum_{m,l} \frac{l(l+1)}{r^2}\left[\frac{\partial}{\partial t} - \mathcal{L}_l\right] h_{lm}(r,t)\,\frac{Y_{lm}(\theta,\varphi)}{N_{lm}} = \hat{r}.\nabla\times[(\nabla\times\mathbf{B})\times\mathbf{B}]$$

$$\sum_{m,l} \frac{l(l+1)}{r^2}\left[\frac{\partial}{\partial t} - \mathcal{L}_l\right] g_{lm}(r,t)\,\frac{Y_{lm}(\theta,\varphi)}{N_{lm}} = \hat{r}.\nabla\times\{\nabla\times[(\nabla\times\mathbf{B})\times\mathbf{B}]\}$$

Taking only 2D components

$$\sum_{m,l} \frac{l(l+1)}{r^2}\left[\frac{\partial}{\partial t} - \mathcal{L}_l\right] h_{lm}(r,t) P_l^{|m|}(\cos\theta) = 0$$

$$\mathcal{L}_l = \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}$$

$$\sum_{m,l} \frac{l(l+1)}{r^2}\left[\frac{\partial}{\partial t} - \mathcal{L}_l\right] g_{lm}(r,t) P_l^{|m|}(\cos\theta) = 0$$

You obtain a differential equations system

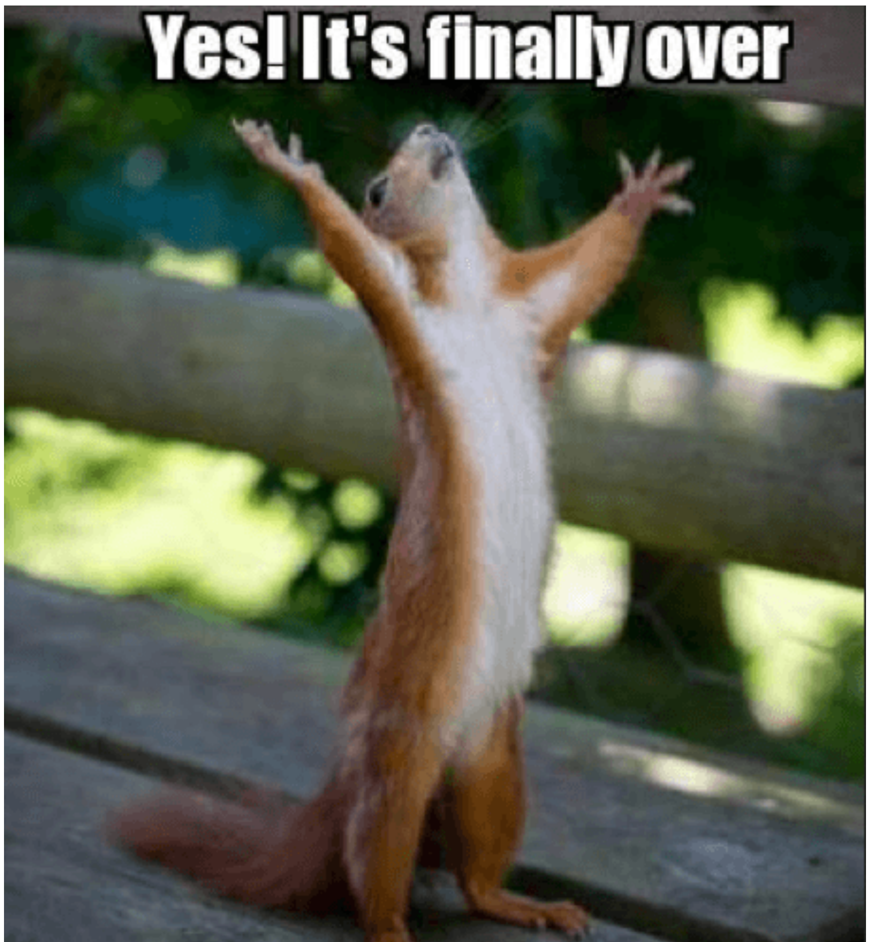
$$\frac{\partial}{\partial t} h_l(r,t) - \frac{\partial^2 h_l(r,t)}{\partial r^2} + \frac{l(l+1)}{r^2} h_l(r,t) = 0$$

$$\frac{\partial}{\partial t} g_l(r,t) - \frac{\partial^2 g_l(r,t)}{\partial r^2} + \frac{l(l+1)}{r^2} g_l(r,t) = 0$$

$$h_l(r,t) = R_l(t)T_l(t),\ g_l(r,t) = R_l(t)T_l(t)$$

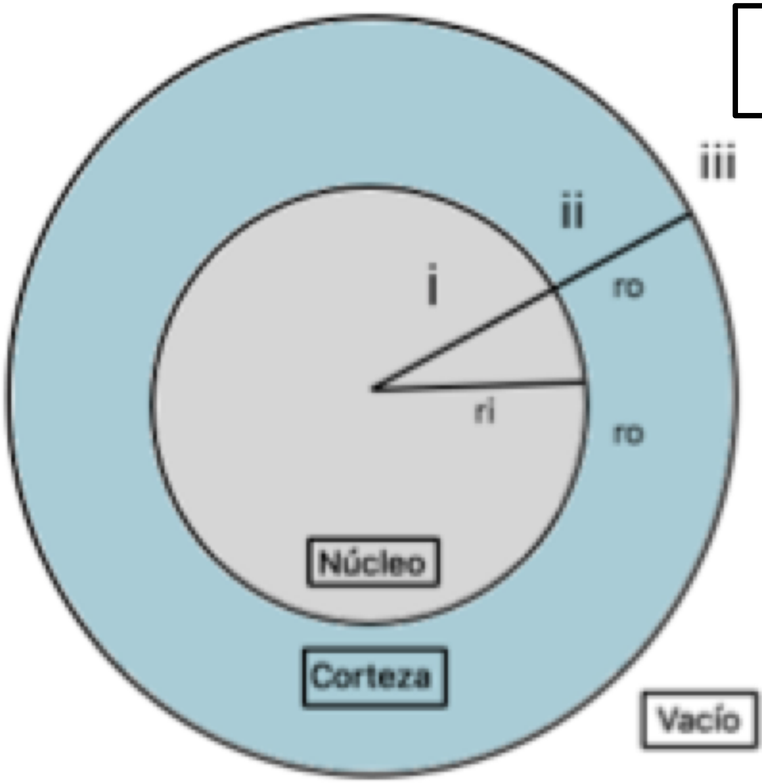
$$h(r,\theta,t) = -r[Aj_l(kr) + By_l(kr)]P_l^1(\cos\theta)e^{-\frac{t}{\tau}}$$

$$g(r,\theta,t) = -r[Cj_l(kr) + Dy_l(kr)]P_l^1(\cos\theta)e^{-\frac{t}{\tau}}$$



$$k \equiv (\tau R_B^{-1})^{-1/2}$$

Boundary conditions



Region I-II

$$B_r(r = r_i) = 0$$

$$h_l(r_i) = Aj_l(kr_i) + By_l(kr_i) = 0$$
$$g_l(r_i) = Cj_l(kr_i) + Dy_l(kr_i) = 0$$

Region II-III

$$\boldsymbol{B} = \nabla\psi$$

$$\psi(r,\theta,\varphi) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{a_{lm}}{r^{l+1}}Y_{lm}(\theta,\varphi)$$

$$\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{r^2}\frac{h_{lm}(r,t)}{N_{lm}}Y_{lm}(\theta,\varphi)=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{a_{lm}}{r^{l+1}}Y_{lm}(\theta,\varphi)$$

$$\frac{dh_l(r_0)}{dr}+\frac{l}{r}h_l(r_0)=0$$

Poloidal

$$y_l'(k)j_l(kr_i)-y_l(kr_i)j_l'(k)+\frac{(l+1)}{k}[y_l(k)j_l(kr_i)-y_l(kr_i)j_l(k)]=0$$

$$\frac{A}{B}=-\frac{y_l(kr_i)}{j_l(kr_i)}$$

$$g_l(r_o)=Cj_l(kr)+Dy_l(kr)=0$$

Toroidal

$$j_l(k)y_l(kr_i)-j_l(kr_i)y_l(k)=0$$

$$\frac{C}{D}=-\frac{y_l(k)}{j_l(k)}$$

Ohmic constants

n	l	k_P	A/B	k_T	C/D
1	1	7.03266	1.07456	12.67071	-0.18517
2	1	19.12793	3.49901	25.18557	-0.09278
3	1	31.584668	5.86928	37.73441	-0.06187
1	2	7.81795	10.93545	12.87682	1.65544
2	2	19.46616	-0.78889	25.29089	3.52035
3	2	31.79340	-0.43334	37.80492	5.33887
1	3	8.63565	-2.11402	13.17984	-1.82690
2	3	19.86013	0.43365	25.44808	-0.61489
3	3	32.04098	1.14319	37.91043	-0.38745

Objetives

$$h(r,\theta,\varphi,0) = h^{(0)}(r,\theta) + P\left[h^{(1)}(r,\theta,\varphi,0) + h^{(2)}(r,\theta,\varphi,0)\ \right]$$

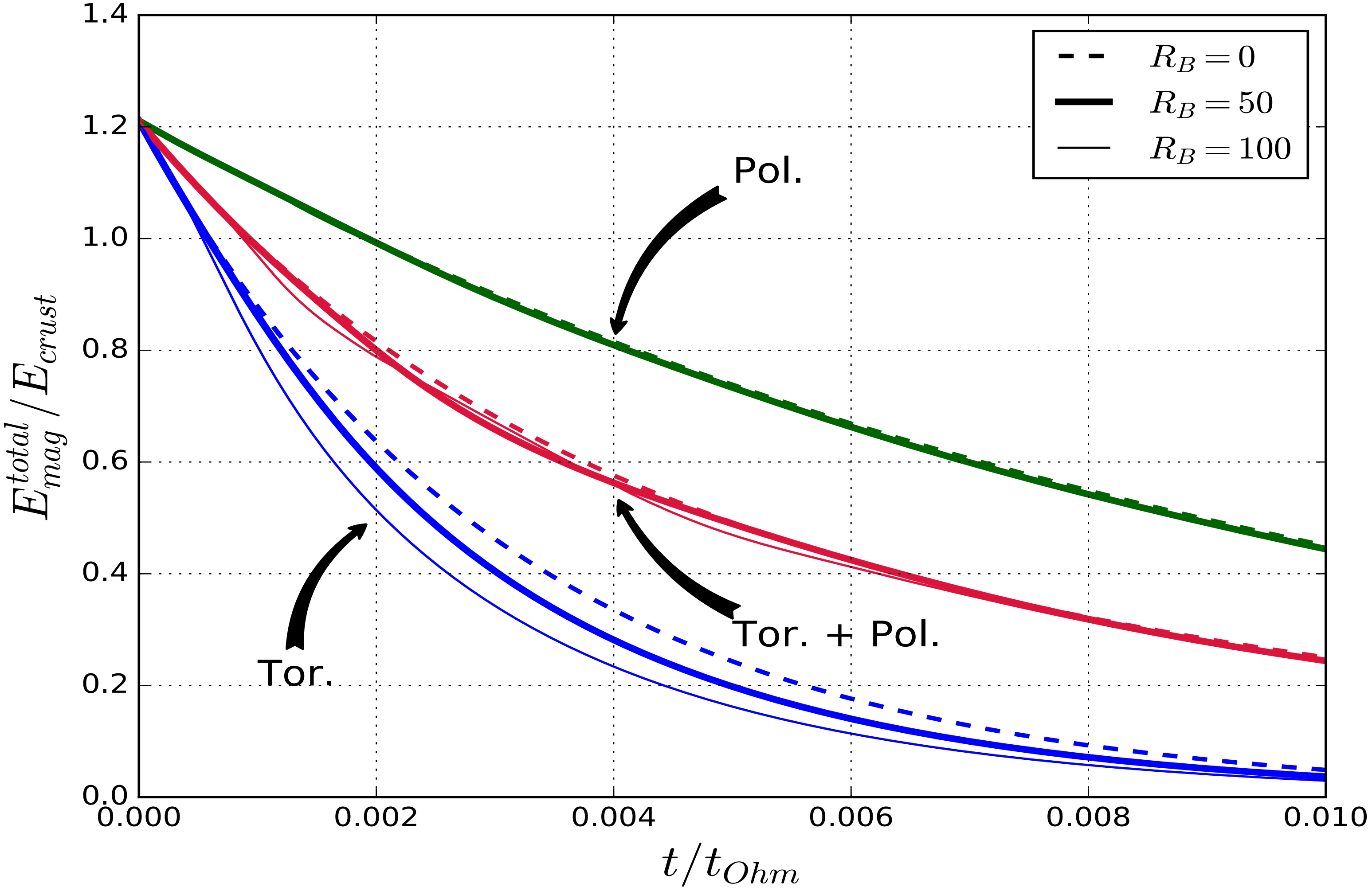
$$g(r,\theta,\varphi,0) = g^{(0)}(r,\theta) + P\left[g^{(1)}(r,\theta,\varphi,0) + g^{(2)}(r,\theta,\varphi,0)\right]$$

Forget for a while the perturbation

$$h(r,\theta,\varphi,0) = h^{(0)}(r,\theta)$$

$$g(r,\theta,\varphi,0) = g^{(0)}(r,\theta)$$

Total magnetic energy



Thanks for your time