Principles of Abstract Interpretation
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Ch. **22**, Chaotic iterations

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These slides are available at
http://github.com/PrAbsInt/slides/slides-22--chaotic-iterations-PrAbsInt.pdf

Chapter 22

Ch. **22**, Chaotic iterations

Chaotic Iterations

- Problem: solve systems of equations iteratively
- In general the result depends on the iteration strategy
- Chaotic iterations allow to choose at each iteration which components evolve while the others are unchanged
- Result: for continuous equations on complete partial orders, the limit of the chaotic iterates is always the least fixpoint.

`en.wikipedia.org/wiki/Iterative_method`

Systems of equations

System of equations

- Let $\vec{D} = \prod_{i=1}^{n} D_i$ be the cartesian product of $n \geqslant 1$ sets D_i.
- Let $\vec{F} \in \vec{D} \to \vec{D}$.
- When we write the vectorial equation

$$\vec{X} = \vec{F}(\vec{X})$$

we mean the system of equations

$$\begin{cases} X_i = F_i(X_1, \ldots, X_n) \\ i = 1, \ldots, n \end{cases}$$

where

- $\vec{X} = \langle X_1, \ldots, X_n \rangle \in \vec{D}$ is a vector of variables
- the i^{th} component of $\vec{F}(\vec{X})$ is

$$F_i(X_1, \ldots, X_n) \triangleq \text{let } \langle X_1', \ldots, X_n' \rangle = \vec{F}(\langle X_1, \ldots, X_n \rangle) \text{ in } X_i'$$

Solutions to systems of equations — I

- The variables $X_i \in \mathbb{V}$, $i = 1, \ldots, n$ are identifiers with values in D_i
- $\vec{F} \in \vec{D} \to \vec{D}$ so $\vec{F}(\vec{X})$ is an abuse of notation since $\vec{D} \neq \prod_{i=1}^{n} \mathbb{V}$.
- It is meant that a solution to

$$\begin{cases} X_i & = & F_i(X_1, \ldots, X_n) \\ i = 1, \ldots, n \end{cases}$$

is a map $\rho \in \{X_i \in \mathbb{V} \mid i = 1, \ldots, n\} \mapsto \rho(X_i) \in D_i$ such

$$\begin{cases} \rho(X_i) & = & F_i(\rho(X_1), \ldots, \rho(X_n)) \\ i = 1, \ldots, n \end{cases}$$

where now $F_i \in \vec{D} \to D_i$.

Solutions to systems of equations — II

- $\vec{X} = \vec{F}(\vec{X})$ confuses the function $\vec{F}$ with its denotation
- To be fully rigorous, the denotation $\vec{\mathsf{F}}$ of $\vec{F}$ is written in some formally defined language
- This language has a semantics $\mathcal{S}[\![\vec{\mathsf{F}}]\!] = \vec{F}$
- A solution to $\vec{X} = \vec{\mathsf{F}}(\vec{X})$ is $\rho(\vec{X}) = \mathcal{S}[\![\vec{\mathsf{F}}]\!](\rho(\vec{X}))$
- Outside of mathematical logic, function notations $\vec{\mathsf{F}}$ are identified with the function $\vec{F} = \mathcal{S}[\![\vec{\mathsf{F}}]\!]$ that they denote
- Variables X_i are identified with their value $\rho(X_i)$
- Ignoring the incoherence, write $\vec{X} = \vec{F}(\vec{X})$ for brevity!

en.wikipedia.org/wiki/System_of_equations

Historical iterative methods

Jacobi iterations

- All components evolve simultaneously at all iterations.

$$\begin{cases} X_i^{k+1} = F_i(X_1^k, \ldots, X_n^k) \\ i = 1, \ldots, n, k = 1, \ldots, +\infty \end{cases}$$

- Two arrays are needed to record both $\vec{X}^k$ and $\vec{X}^{k+1}$.
- This is the iteration method considered in Chapter **15** (Fixpoints), $\vec{X}^0 = 0$, $\vec{X}^{k+1} = \vec{F}(\vec{X}^k)$, and pass to the limit $\bigsqcup_{i\in\mathbb{N}} \vec{X}^{k+1}$.

en.wikipedia.org/wiki/Jacobi_method
en.wikipedia.org/wiki/Carl_Gustav_Jacob_Jacobi

Gauss-Seidel (or successive) iterations

One array is enough to program Gauss-Seidel (or successive) iteration(s) method [Isaacson and Keller, 1994, Sect. 4.2]

$$\begin{cases} X_i^{k+1} \quad = \quad F_i(X_1^{k+1}, \ldots, X_{i-1}^{k+1}, X_i^k, \ldots, X_n^k) \\ i = 1, \ldots, n, k = 1, \ldots, +\infty \end{cases}$$

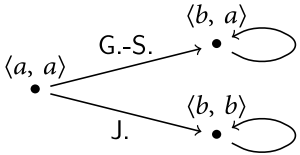
where the components evolve one after another.

en.wikipedia.org/wiki/Gauss-Seidel_method
en.wikipedia.org/wiki/Carl_Friedrich_Gauss
en.wikipedia.org/wiki/Philipp_Ludwig_von_Seidel

The result depends on the iteration strategy (Example 22.1)

In general the two iteration strategies yield different results.

$\langle x,\ y\rangle$	$F_1(x,y)$	$F_2(x,y)$
$\langle a,\ a\rangle$	b	b
$\langle a,\ b\rangle$	b	b
$\langle b,\ a\rangle$	b	a
$\langle b,\ b\rangle$	b	b



Jacobi iterations

$$
\begin{array}{lcccc}
\vec{X}^0 & = & \langle a,\ a\rangle & & \\
\vec{X}^1 & = & \langle F_1(a,a),\ F_2(a,a)\rangle & = & \langle b,\ b\rangle \\
\vec{X}^2, \vec{X}^3, \ldots & = & \langle F_1(b,b),\ F_2(b,b)\rangle & = & \langle b,\ b\rangle
\end{array}
$$

Gauss-Seidel iterations

$$
\begin{array}{lcccc}
\vec{X}^0 & = & \langle a,\ a\rangle & & \\
\vec{X}^1 & = & \langle F_1(a,a),\ a\rangle & = & \langle b,\ a\rangle \\
\vec{X}^2, \vec{X}^4, \ldots & = & \langle b,\ F_2(b,a)\rangle & = & \langle b,\ a\rangle \\
\vec{X}^3, \vec{X}^5, \ldots & = & \langle F_1(b,a),\ a\rangle & = & \langle b,\ a\rangle
\end{array}
$$

□

Chaotic iterations

Chaotic iterations

- At each step one can arbitrarily choose which components do evolve
- no component is omitted for ever (fairness condition)
- If all components evolve at each iteration, this is Jacobi iterations.
- If the components evolve one after another, this is Gauss-Seidel.
- A chaotic iteration is defined by an infinite sequence $\mathfrak{J}$ of subsets of $\{1, \ldots, n\}$ specifying that all components in $\mathfrak{J}(k)$ should evolved at iterate k (while those in $\{1, \ldots, n\} \setminus \mathfrak{J}(k)$ remain unchanged).

Definition 22.2

- Let $\vec{D} = \prod_{i=1}^{n} D_i$ be the cartesian product of $n \geqslant 1$ sets D_i.
- Let $\vec{F} \in \vec{D} \to \vec{D}$.
- Let $\mathfrak{J} \in \mathbb{N}^+ \to \wp(\{1, \ldots, n\}) \setminus \{\varnothing\}$ satisfying the fairness condition

$$\forall i \in \{1, \ldots, n\} \, . \, \forall k \in \mathbb{N} \, . \, \exists k' > k \, . \, i \in \mathfrak{J}(k') \ .$$

- The chaotic iterations $\vec{X}^k$, $k \in \mathbb{N}$ from $\vec{X}_0$ defined by $\mathfrak{J}$ for the system of equations $\vec{X} = \vec{F}(\vec{X})$ is

$$\begin{cases} \vec{X}_i^{k+1} \triangleq F_i(\vec{X}^k) & \text{when } i \in \mathfrak{J}(k) \\ \vec{X}_i^{k+1} \triangleq \vec{X}_i^k & \text{when } i \notin \mathfrak{J}(k) \ . \end{cases}$$

□

Convergence of chaotic iterations of continuous operators on complete partial orders

Convergence of chaotic iterations [P. Cousot and R. Cousot, 1977]

Theorem 22.4 The chaotic iterations of componentwise continuous operator $\vec{F} \in L^n \xrightarrow{uc} L^n$ on a CPO $\langle L^n, \dot{\sqsubseteq}, \dot{\bot}, \dot{\sqcup}\rangle$ from $\dot{\bot}$ converge to the $\mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F}$.

Proof of Theorem 22.4 — (1) We first prove that the iterates form an increasing chain less than $\mathsf{lfp}^{\dot\sqsubseteq}\,\vec{F}$ (which exists by Kleene/Scott iterative fixpoint Theorem 15.26).

- By recurrence.
- For the basis, $\vec{X}^0 = \dot\bot \mathrel{\dot\sqsubseteq} \mathsf{lfp}^{\dot\sqsubseteq}\,\vec{F}$ by def. infimum $\dot\bot$.
- Assume, for the induction step, that for $k \in \mathbb{N}$, $\forall k' \leqslant k\,.\,\vec{X}^{k'} \mathrel{\dot\sqsubseteq} \vec{X}^{k} \mathrel{\dot\sqsubseteq} \mathsf{lfp}^{\dot\sqsubseteq}\,\vec{F}$ so that $\forall i \in \{1, \ldots, n\}\,.\,\forall k' \leqslant k\,.\,\vec{X}_i^{k'} \sqsubseteq \vec{X}_i^{k} \sqsubseteq (\mathsf{lfp}^{\dot\sqsubseteq}\,\vec{F})_i$ componentwise.
- Let $i \in \{1, \ldots, n\}$ be any component
  - If $i \notin \mathfrak{J}(k+1)$ then
    - $\vec{X}_i^{k+1} = \vec{X}_i^{k}$ by def. of the iterates
    - so $\vec{X}_i^{k} \sqsubseteq \vec{X}_i^{k+1} \sqsubseteq (\mathsf{lfp}^{\dot\sqsubseteq}\,\vec{F})_i$ by reflexivity and induction hypothesis
    - so $\forall k' \leqslant k\,.\,\vec{X}_i^{k'} \sqsubseteq \vec{X}_i^{k+1} \sqsubseteq (\mathsf{lfp}^{\dot\sqsubseteq}\,\vec{F})_i$ by transitivity.

- Else $i \in \mathfrak{I}(k+1)$.
  - If there is no $k' \leqslant k$ such that $i \in \mathfrak{I}(k')$ then
    - $\bot = \vec{X}_i^0 = \ldots = \vec{X}_i^k$
    - so $\vec{X}_i^k \sqsubseteq \vec{X}_i^{k+1} = F_i(\vec{X}^k) \sqsubseteq (\mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F})_i$
      - by def. infimum $\bot$, and
      - for all $\vec{X} \mathrel{\dot{\sqsubseteq}} \mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F}$, we have $\vec{F}(\vec{X}) \mathrel{\dot{\sqsubseteq}} \vec{F}(\mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F}) = \mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F}$ since, by

    Exercise 15.24, $\vec{F}$ is continuous hence increasing and by def. fixpoints so $F_i(\vec{X}) \triangleq \vec{F}(\vec{X})_i \sqsubseteq (\mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F})_i$ by componentwise def. of $\dot{\sqsubseteq}$.
  - Otherwise let $k' \leqslant k$ be the largest such that $i \in \mathfrak{I}(k')$ so that
    - $\vec{X}_i^{k'-1} \sqsubseteq \vec{X}_i^{k'} = \vec{X}_i^{k'+1} = \ldots = \vec{X}_i^k \sqsubseteq (\mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F})_i$ by def. iterates and ind. hyp.
    - By def. of the iterates, it follows that
    $F_i(\vec{X}^{k'-1}) = \vec{X}_i^{k'} = \vec{X}_i^{k'+1} = \ldots = \vec{X}_i^k \sqsubseteq F_i(\vec{X}^k) = \vec{X}_i^{k+1} \sqsubseteq (\mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F})_i$
    since F_i is continuous hence increasing.
    - By transitivity, $\forall k' \leqslant k \;.\; \vec{X}_i^{k'} \sqsubseteq \vec{X}_i^{k+1} \sqsubseteq (\mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F})_i$.

- By componentwise definition of $\dot{\sqsubseteq}$, we conclude that $\forall k' \leqslant k \;.\; \vec{X}^{k'} \mathrel{\dot{\sqsubseteq}} \vec{X}^{k} \mathrel{\dot{\sqsubseteq}} \mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F}$
- The iterates form an increasing chain bounded by $\mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F}$.
- By def. of a complete partial order in Section **10.9** and that of a lub, they have a limit $\dot{\bigsqcup\limits_{k\in\mathbb{N}}} \vec{X}^{k} \mathrel{\dot{\sqsubseteq}} \mathsf{lfp}^{\dot{\sqsubseteq}}\, \vec{F}$.

— (2)

- Let $\vec{J}^0 = \dot{\perp}$, $\vec{J}^{k+1} = \vec{F}(\vec{J}^k)$ be the Jacobi iterates
- by Kleene/Scott iterative fixpoint Theorem 15.26, $\vec{J}^\omega = \dot{\bigsqcup}_{k\in\mathbb{N}} \vec{J}^k = \mathsf{lfp}^{\dot{\sqsubseteq}} \vec{F}$
- We prove that any Jacobi iterate is ultimately bounded by a chaotic iterate *i.e.* $\forall k \in \mathbb{N} \,.\, \exists k' \geqslant k \,.\, \vec{J}^k \mathrel{\dot{\sqsubseteq}} \vec{X}^{k'}$.
- By the fairness hypothesis of Def. 22.2, $\eta(k) = \max\{\min\{k' \geqslant k \mid i \in \mathfrak{I}(k')\} \mid i \in \{1, \ldots, n\}\}$ is well defined.
- In the chaotic iterations, all components have evolved at least once between k and $\eta(k)$.
- Let us extract the subsequence $\lambda(0) = 0$ and $\lambda(k+1) = \eta(\lambda(k))$.
- We have $\forall k \in \mathbb{N} \,.\, \vec{J}^k \mathrel{\dot{\sqsubseteq}} \vec{X}^{\lambda(k)}$ *i.e.* by waiting long enough, any Jacobi iterate will be overapproximated by some chaotic iterate.
- The proof is by recurrence.

- For $k = 0$, $\vec{J}^0 = \vec{X}^{\lambda(0)} = \vec{X}^0 = \dot{\perp}$ by def. λ so we conclude by reflexivity.
- Assume by induction hypothesis that $\forall k' \leqslant k \in \mathbb{N} \;.\; \vec{J}^{k'} \mathrel{\dot{\sqsubseteq}} \vec{X}^{\lambda(k')}$ where the $\vec{J}^\ell$ and $\vec{X}^\ell$, $\ell \in \mathbb{N}$, are $\dot{\sqsubseteq}$-increasing.
- $\vec{F}$ is continuous hence increasing so $\vec{J}^{k+1} = \vec{F}(\vec{J}^k) \mathrel{\dot{\sqsubseteq}} \vec{F}(\vec{X}^{\lambda(k)})$.
- For all $i \in \{1, \ldots, n\}$, the i^{th} component has evolved at least once at k_i' between $\lambda(k)$ and $\eta(\lambda(k)) = \lambda(k+1)$
- So the increasing chain has the form $\vec{X}^{\lambda(k)} \sqsubseteq \vec{X}^{k_i'-1} \sqsubseteq \vec{X}^{k_i'} \sqsubseteq \vec{X}^{\eta(\lambda(k))} = \vec{X}^{\lambda(k+1)}$ where $\vec{X}_i^{k_i'} = \vec{F}_i(\vec{X}^{k_i'-1})$.
- $\vec{F}$ hence $\vec{F}_i$ is continuous hence increasing so by def. of the iterates, $\vec{F}_i(\vec{X}^{\lambda(k)}) \sqsubseteq \vec{F}_i(\vec{X}^{k_i'-1}) = \vec{X}^{k_i'} \sqsubseteq \vec{X}_i^{\lambda(k+1)}$.
- By componentwise def. of $\vec{F}$ and $\dot{\sqsubseteq}$, we have $\vec{F}(\vec{X}^{\lambda(k)}) \mathrel{\dot{\sqsubseteq}} \vec{X}^{\lambda(k+1)}$ hence, by transitivity $\vec{J}^{k+1} \mathrel{\dot{\sqsubseteq}} \vec{X}^{\lambda(k+1)}$.
- By recurrence $\forall k \in \mathbb{N} \;.\; \vec{J}^k \mathrel{\dot{\sqsubseteq}} \vec{X}^{\lambda(k)}$. $\square$

— (3) In conclusion,

- We have shown $\{\vec{X}^{\lambda(k)} \mid k \in \mathbb{N}\} \subseteq \{\vec{X}^{k} \mid k \in \mathbb{N}\}$
- So $\mathsf{lfp}^{\dot{\sqsubseteq}}\,\vec{F} = \bigsqcup_{k\in\mathbb{N}} \vec{J}^{k} \mathrel{\dot{\sqsubseteq}} \bigsqcup_{k\in\mathbb{N}} \vec{X}^{\lambda(k)} \mathrel{\dot{\sqsubseteq}} \bigsqcup_{k\in\mathbb{N}} \vec{X}^{k} \mathrel{\dot{\sqsubseteq}} \mathsf{lfp}^{\dot{\sqsubseteq}}\,\vec{F}$ (by def. of lubs,)
- So $\bigsqcup_{k\in\mathbb{N}} \vec{X}^{k} = \mathsf{lfp}^{\dot{\sqsubseteq}}\,\vec{F}$ by antisymmetry. $\square$

Conclusion

Conclusion

- Chaotic iterations cover all iteration algorithms used to solve systems of equations in static analysis such as the *work list* [Kildall, 1973].
- The chaotic iterations generalize from continuous to increasing operators on CPOs using transfinite iterations and to asynchronous iterations where the components evolve in parallel [P. Cousot, 1977, 1978][1]

`en.wikipedia.org/wiki/Data-flow_analysis`

[1] see also [Wei, 1993] (with stronger hypotheses).

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Home work

Read Ch. **22** "Chaotic iterations" of

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The End, Thank you