Principles of Abstract Interpretation
MIT press
Ch. **23**, Abstract equational semantics

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These slides are available at
http://github.com/PrAbsInt/slides/slides-23--equational-forward-abstract-semantics-PrAbsInt.pdf

Chapter 23

Ch. **23**, Abstract equational semantics

Traditional data flow analysis

- Traditional data flow analyses are expressed as solutions to systems of boolean equations attached to program graphs/flowcharts [Cocke and Schwartz, 1969, Second revised version, April 1970, pp. 429–461] [Hecht, 1977], not as an abstract structural semantics.
- These equations are abstractions of the structural semantics essentially loosing the structural information on programs.
- For example, circular dependencies appearing in equations because of iterations must be rebuilt [Cocke and Schwartz, 1969, Second revised version, April 1970, pp. 435–437] (nowadays using Tarjan's strongly connected components algorithm [Tarjan, 1972] and Bourdoncle's iteration algorithm [Bourdoncle, 1993]) whereas iterations are straightforward to locate knowing the program syntax.

Reachability proof methods

The traditional methods to prove reachability properties of programs are:

- Hoare logic
  - It is based on the structural reachability semantics of Chapter **19** (and so generalizes to the abstract semantics of Chapter **21**, see [P. Cousot, R. Cousot, Logozzo, and Barnett, 2012])
  - It will be developed in Chapter **26**
- Turing/Naur/Floyd invariance proof method
  - It is based on the equational semantics developed in this Chapter **23** (using fixpoint induction introduced in next Chapter **24**)
  - It will be developed in Chapter **25**

So although the two abstract structural and equational semantics are equivalent, they are both needed to explain verification and static analysis methods and prove them equivalent.

Principle of the equational semantics

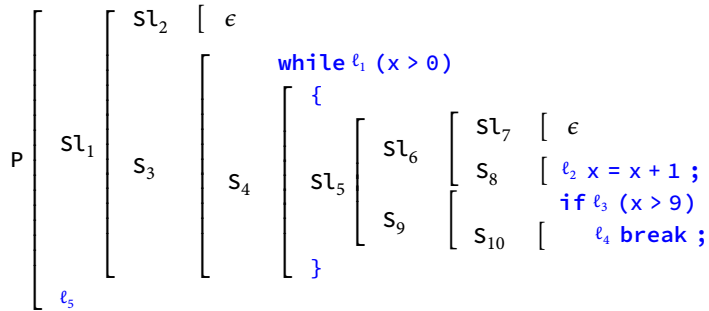
- A variable $\mathcal{X}_\ell$ is attached to each label ℓ of the program $\mathtt{P}$.
- The equations must be designed so that their solutions satisfy $\mathcal{X}_\ell = \widehat{\mathcal{S}}^{\,\natural}[\![\mathtt{P}]\!]\,\mathcal{R}_0\,\ell$ for all program labels ℓ.
- So the relations between the variables $\mathcal{X}_\ell$ of the equations directly follow from the structural definition of the semantics $\widehat{\mathcal{S}}^{\,\natural}[\![\mathtt{P}]\!]\,\mathcal{R}_0\,\ell$ by expressing the relationships between the abstract properties of the subcomponents of each program component.
- Then $\widehat{\mathcal{S}}^{\,\natural}[\![\mathtt{P}]\!]\,\mathcal{R}_0\,\ell$ corresponds to a particular iteration strategy for solving the equations by structural induction, all such strategies being equivalent by the chaotic iteration Theorem 22.4.

Example of equational semantics of a program

The program

```
while ℓ1 (x > 0) {
    ℓ2 x = x + 1 ;
    if ℓ3 (x > 9)
        ℓ4 break ;
    } ℓ5
```
(19.7) □

Grammatical structure:



(23.1)

Labelling

$$
\mathtt{P}\left[\begin{array}{l}
\mathtt{Sl}_1\left[\begin{array}{ll}
\mathtt{Sl}_2 & [\ \epsilon \\
\mathtt{S}_3 & \\
\end{array}\right. \\
\mathtt{S}_4\left[\begin{array}{l}
\texttt{while}\ \ell_1\ \texttt{(x > 0)} \\
\texttt{\{} \\
\mathtt{Sl}_5\left[\begin{array}{ll}
\mathtt{Sl}_6 & \left[\begin{array}{ll} \mathtt{Sl}_7 & [\ \epsilon \\ \mathtt{S}_8 & [\ \ell_2\ \texttt{x = x + 1 ;} \end{array}\right. \\
\mathtt{S}_9 & \left[\begin{array}{ll} \mathtt{S}_{10} & [\ \ell_4\ \texttt{break ;} \end{array}\right.
\end{array}\right. \\
\texttt{\}}
\end{array}\right. \\
\ell_5
\end{array}\right.
\qquad (23.1)
$$

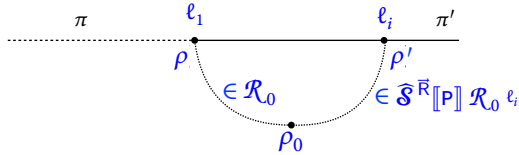
The labelling of program (23.1) is as follows.

$\mathtt{S}$	$\mathtt{P}, \mathtt{Sl}_1$	$\mathtt{Sl}_2$	$\mathtt{S}_3$	$\mathtt{S}_4, \mathtt{Sl}_5$	$\mathtt{Sl}_6, \mathtt{S}_8$	$\mathtt{Sl}_7$	$\mathtt{S}_9$	$\mathtt{S}_{10}$
at$[\![\mathtt{S}]\!]$	ℓ_1	ℓ_1	ℓ_1	ℓ_2	ℓ_2	ℓ_2	ℓ_3	ℓ_4
after$[\![\mathtt{S}]\!]$	ℓ_5	ℓ_1	ℓ_5	ℓ_1	ℓ_3	ℓ_2	ℓ_1	ℓ_1

and breaks-of$[\![\mathtt{S}_4]\!] = \{\ell_4\}$, breaks-of$[\![\mathtt{Sl}_1]\!] = \varnothing$.

The relational invariants

- The structural relational forward reachability semantics $\widehat{\mathcal{S}}^{\vec{\mathsf{R}}}[\![\mathsf{P}]\!]\,\mathcal{R}_0\,\ell_i$ is a relation between environments that can be encountered during an execution of program P starting from any initial environment statisfying $\mathcal{R}_0$ and reaching program label ℓ_i (see Chapter **19**).



- We denote by
  - x_0 the initial value of the variable x and by
  - x the value of this variable x during execution when reaching program label ℓ_i.

Relational invariants

P	ℓ_i	$\widehat{\mathcal{S}}^{\vec{\mathsf{R}}}[\![\mathsf{P}]\!]\mathcal{R}_0\ \ell_i, \quad i = 1, \ldots, 5$
/* $x = x_0$ */		$\mathcal{R}_0 = \{\langle \rho_0, \rho\rangle \mid \rho = \rho_0\}$
/* $x = x_0 \leqslant 0 \vee x_0 > 0$ */		
while ℓ_1 (x > 0)	ℓ_1	$\{\langle \rho_0, \rho\rangle \mid \rho(\mathsf{x}) = \rho_0(\mathsf{x}) \leqslant 0 \vee \rho_0(\mathsf{x}) > 0\}$
{		
ℓ_2 /* $x_0 > 0 \wedge x > 0$ */	ℓ_2	$\{\langle \rho_0, \rho\rangle \mid \rho_0(\mathsf{x}) > 0 \wedge \rho(\mathsf{x}) > 0\}$
x = x + 1 ;		
/* $x_0 > 0 \wedge x > 1$ */		
if ℓ_3 (x > 9)	ℓ_3	$\{\langle \rho_0, \rho\rangle \mid \rho_0(\mathsf{x}) > 0 \wedge \rho(\mathsf{x}) > 1\}$
ℓ_4 /* $x_0 > 0 \wedge x > 9$ */	ℓ_4	$\{\langle \rho_0, \rho\rangle \mid \rho_0(\mathsf{x}) > 0 \wedge \rho(\mathsf{x}) > 9\}$
break ;		
}		
ℓ_5 /* $(x = x_0 \leqslant 0) \vee (x_0 > 0 \wedge x > 9)$ */	ℓ_5	$\{(\langle \rho_0, \rho\rangle \mid \rho(\mathsf{x}) = \rho_0(\mathsf{x}) \leqslant 0) \vee (\rho_0(\mathsf{x}) > 0 \wedge \rho(\mathsf{x}) > 9)\}$

S_4

The system of equations

The system of equations $\vec{\mathscr{E}}^{\mathsf{R}}[\![\mathtt{P}]\!]\,\mathcal{R}_0$ for the program `P` (23.1) is

```
while ℓ1 (x > 0) {        (19.7)
    ℓ2 x = x + 1 ;
    if ℓ3 (x > 9)
        ℓ4 break ;
    } ℓ5
```

$$\begin{cases} \mathcal{X}_{\ell_1} &= \mathcal{R}_0 \cup \overleftarrow{\mathtt{test}}^{\mathsf{R}}[\![\mathtt{x > 9}]\!]\mathcal{X}_{\ell_3} \\ \mathcal{X}_{\ell_2} &= \mathtt{test}^{\mathsf{R}}[\![\mathtt{x > 0}]\!]\mathcal{X}_{\ell_1} \\ \mathcal{X}_{\ell_3} &= \mathtt{assign}_{\mathsf{R}}[\![\mathtt{x, x + 1}]\!]\,\mathcal{X}_{\ell_2} \\ \mathcal{X}_{\ell_4} &= \mathtt{test}^{\mathsf{R}}[\![\mathtt{x > 9}]\!]\mathcal{X}_{\ell_3} \\ \mathcal{X}_{\ell_5} &= \overleftarrow{\mathtt{test}}^{\mathsf{R}}[\![\mathtt{x > 0}]\!]\mathcal{X}_{\ell_1} \cup \mathcal{X}_{\ell_4} \end{cases} \qquad (23.2)$$

The equations express the fact that

- The values of x at ℓ_1 are the initial values of $\mathcal{R}_0$ or those after the loop body that is those at ℓ_3 that do not pass the test `(x > 9)`;
- The values of x at ℓ_2 are those at ℓ_1 that pass the test `(x > 0)`;
- The values of x at ℓ_3 are those at ℓ_2 after incrementation;
- The values of x at ℓ_4 are those at ℓ_3 that pass the test `(x > 9)`;
- The values of x at ℓ_5 are are those at ℓ_1 that do not pass the test `(x > 0)` or those at ℓ_4 when the **`break ;`** statement exists the loop.

Solving the system of equations iteratively

- In the example (23.2), $\mathcal{X}_{\ell_1}$ recursively depends upon itself while all other $\mathcal{X}_{\ell_i}$, $i = 2, \ldots, 5$ depend only of this $\mathcal{X}_{\ell_1}$.
- So we first have to solve the fixpoint equation

$$\mathcal{X}_{\ell_1} \quad = \quad \mathcal{R}_0 \cup \overrightarrow{\overline{\mathsf{test}}^{\mathsf{R}}}[\![\mathtt{x} > 9]\!](\mathsf{assign}_{\mathsf{R}}[\![\mathtt{x}, \mathtt{x} + 1]\!]\ (\overrightarrow{\mathsf{test}^{\mathsf{R}}}[\![\mathtt{x} > 0]\!]\mathcal{X}_{\ell_1}))$$

that is (by definitions (19.12) and (19.16))

$$\mathcal{X}_{\ell_1} \quad = \quad \mathcal{R}_0 \cup \{\langle \rho_0, \rho[\mathtt{x} \leftarrow \rho(\mathtt{x}) + 1]\rangle \mid \langle \rho_0, \rho\rangle \in \mathcal{X}_{\ell_1} \wedge 0 < \rho(\mathtt{x}) \leqslant 9\} \tag{23.3}$$

- and then carry forward the result to all other $\mathcal{X}_{\ell_i}$, $i = 2, \ldots, 5$. The equation is solved iteratively using Tarski iterative fixpoint Theorem 15.21.

Iterates

The iterates $\mathcal{X}^n_{\ell_1}$, $n \geqslant 0$ are as follows.

$$\mathcal{X}^0_{\ell_1} = \varnothing$$

$$\begin{aligned}\mathcal{X}^1_{\ell_1} &= \mathcal{R}_0 \cup \{\langle \rho_0,\ \rho[\mathrm{x} \leftarrow \rho(\mathrm{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in \mathcal{X}^0_{\ell_1} \wedge 0 < \rho(\mathrm{x}) \leqslant 9\} \\ &= \mathcal{R}_0 \\ &= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\}\end{aligned}$$

$$\begin{aligned}\mathcal{X}^2_{\ell_1} &= \mathcal{R}_0 \cup \{\langle \rho_0,\ \rho[\mathrm{x} \leftarrow \rho(\mathrm{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in \mathcal{X}^1_{\ell_1} \wedge 0 < \rho(\mathrm{x}) \leqslant 9\} \\ &= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathrm{x}) + 1]\rangle \mid 0 < \rho_0(\mathrm{x}) \leqslant 9\}\end{aligned}$$

.../...

$$
\begin{aligned}
\mathcal{X}^3_{\ell_1} &= \mathcal{R}_0 \cup \{\langle \rho_0,\ \rho[\mathtt{x} \leftarrow \rho(\mathtt{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in \mathcal{X}^2_{\ell_1} \wedge 0 < \rho(\mathtt{x}) \leqslant 9\} \\
&= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho[\mathtt{x} \leftarrow \rho(\mathtt{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in (\{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0, \\
&\quad \rho_0[x \leftarrow \rho_0(\mathtt{x}) + 1]\rangle \mid 0 < \rho_0(\mathtt{x}) \leqslant 9\}) \wedge 0 < \rho(\mathtt{x}) \leqslant 9\} \\
&= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho[\mathtt{x} \leftarrow \rho(\mathtt{x}) + 1]\rangle \mid \rho = \rho_0 \wedge 0 < \rho_0(\mathtt{x}) \leqslant 9\} \cup \{\langle \rho_0, \\
&\quad \rho[\mathtt{x} \leftarrow \rho(\mathtt{x}) + 1]\rangle \mid \rho = \rho_0[x \leftarrow \rho_0(\mathtt{x}) + 1] \wedge 0 < \rho_0(\mathtt{x}) \leqslant 9 \wedge 0 < \rho(\mathtt{x}) \leqslant 9\} \\
&= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathtt{x}) + 1]\rangle \mid 0 < \rho_0(\mathtt{x}) \leqslant 9\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \\
&\quad \rho_0(\mathtt{x}) + 2]\rangle \mid 0 < \rho_0(\mathtt{x}) \leqslant 8\}
\end{aligned}
$$

So we make the following induction hypothesis (which holds for $1 \leqslant n \leqslant 3$ where $\bigcup \varnothing = \varnothing$).

$$
\mathcal{X}^n_{\ell_1} \quad = \quad \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \bigcup_{i=1}^{n-1} \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathtt{x}) + i]\rangle \mid 0 < \rho_0(\mathtt{x}) \leqslant (9 - i)\}
$$

.../...

Induction

We check that the hypothesis is inductive.

$$\mathcal{X}^{n+1}_{\ell_1} = \mathcal{R}_0 \cup \{\langle \rho_0,\ \rho[\mathtt{x} \leftarrow \rho(\mathtt{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in \mathcal{X}^n_{\ell_1} \wedge 0 < \rho(\mathtt{x}) \leqslant 9\}$$

$$= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho[\mathtt{x} \leftarrow \rho(\mathtt{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \bigcup_{i=1}^{n-1}\{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathtt{x}) + i]\rangle \mid 0 < \rho_0(\mathtt{x}) \leqslant (9 - i)\}) \wedge 0 < \rho(\mathtt{x}) \leqslant 9\}$$

$$= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathtt{x}) + 1]\rangle \mid \rho_0(\mathtt{x}) \leqslant 9\} \cup \{\langle \rho_0,\ \rho[\mathtt{x} \leftarrow \rho(\mathtt{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in \bigcup_{i=1}^{n-1}\{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathtt{x}) + i]\rangle \mid 0 < \rho_0(\mathtt{x}) \leqslant (9 - i)\}) \wedge 0 < \rho(\mathtt{x}) \leqslant 9\}$$

$$= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathtt{x}) + 1]\rangle \mid \rho_0(\mathtt{x}) \leqslant 9\} \cup \bigcup_{i=1}^{n-1}\{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathtt{x}) + i][x \leftarrow \rho_0[x \leftarrow \rho_0(\mathtt{x}) + i](\mathtt{x}) + 1]\rangle \mid 0 < \rho_0(\mathtt{x}) \leqslant (9 - i) \wedge 0 < \rho_0[x \leftarrow \rho_0(\mathtt{x}) + i](\mathtt{x}) \leqslant 9\}$$

.../...

$$= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathsf{x}) + 1]\rangle \mid \rho_0(\mathsf{x}) \leqslant 9\} \cup \bigcup_{i=1}^{n-1} \{\langle \rho_0,$$

$$\rho_0[x \leftarrow \rho_0(\mathsf{x}) + i + 1]\rangle \mid i + 1 < \rho_0(\mathsf{x}) + i + 1 \leqslant 10 \wedge 1 < \rho_0(\mathsf{x}) + i + 1 \leqslant 9\}$$

$$= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \bigcup_{j=1}^{(n+1)-1} \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathsf{x}) + j]\rangle \mid 0 < \rho_0(\mathsf{x}) \leqslant (9 - j)\}$$

(letting $j = i + 1$)

.../...

Limit

Passing to the limit, the solution to the fixpoint equation (23.3) is

$$\begin{aligned}\mathcal{X}_{\ell_1} &= \bigcup_{n\geqslant 0} \mathcal{X}^n_{\ell_1}\\ &= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathrm{x}) + i]\rangle \mid i \in [1,9] \wedge 0 < \rho_0(\mathrm{x}) \leqslant (9-i)\}\end{aligned}$$

(since the join is empty when $9 - i \leqslant 0$)

.../...

Propagation

We derive the other components of the equations (23.2).

$$\mathcal{X}_{\ell_2} = \{\langle \rho_0,\ \rho\rangle \in \mathcal{X}_{\ell_1} \mid \rho(\mathrm{x}) > 0\}$$
$$= \{\langle \rho_0,\ \rho_0\rangle \mid \rho_0(\mathrm{x}) > 0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathrm{x}) + i]\rangle \mid i \in [1,9] \wedge 0 < \rho_0(\mathrm{x}) \leqslant (9-i)\}$$

$$\mathcal{X}_{\ell_3} = \{\langle \rho_0,\ \rho[\mathrm{x} \leftarrow \rho(\mathrm{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in \mathcal{X}_{\ell_1} \wedge \rho(\mathrm{x}) > 0\}$$
$$= \{\langle \rho_0,\ \rho[\mathrm{x} \leftarrow \rho(\mathrm{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in (\{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathrm{x}) + i]\rangle \mid i \in [1,9] \wedge 0 < \rho_0(\mathrm{x}) \leqslant (9-i)\}) \wedge \rho(\mathrm{x}) > 0\}$$
$$= \{\langle \rho_0,\ \rho[\mathrm{x} \leftarrow \rho(\mathrm{x}) + 1]\rangle \mid \rho = \rho_0 \wedge \rho(\mathrm{x}) > 0\} \cup \{\langle \rho_0,\ \rho[\mathrm{x} \leftarrow \rho(\mathrm{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathrm{x}) + i]\rangle \mid i \in [1,9] \wedge 0 < \rho_0(\mathrm{x}) \leqslant (9-i)\}) \wedge \rho(\mathrm{x}) > 0\}$$
$$= \{\langle \rho_0,\ \rho[\mathrm{x} \leftarrow \rho(\mathrm{x}) + 1]\rangle \mid \rho = \rho_0 \wedge \rho(\mathrm{x}) > 0\} \cup \{\langle \rho_0,\ \rho[\mathrm{x} \leftarrow \rho(\mathrm{x}) + 1]\rangle \mid \exists i \in [1,9]\ .\ \rho = \rho_0[x \leftarrow \rho_0(\mathrm{x}) + i] \wedge 0 < \rho_0(\mathrm{x}) \leqslant (9-i) \wedge \rho(\mathrm{x}) > 0\}$$
$$= \{\langle \rho_0,\ \rho[\mathrm{x} \leftarrow \rho(\mathrm{x}) + 1]\rangle \mid \rho = \rho_0 \wedge \rho(\mathrm{x}) > 0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathrm{x}) + i][x \leftarrow \rho_0[x \leftarrow \rho_0(\mathrm{x}) + i](\mathrm{x}) + 1]\rangle \mid i \in [1,9] \wedge 0 < \rho_0(\mathrm{x}) \leqslant (9-i) \wedge \rho_0[x \leftarrow \rho_0(\mathrm{x}) + i](\mathrm{x}) > 0\}$$
$$= \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathrm{x}) + 1]\rangle \mid \rho_0(\mathrm{x}) > 0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathrm{x}) + i]\rangle \mid i \in [1,9] \wedge 0 < \rho_0(\mathrm{x}) \leqslant (9-i)\}$$

$$\begin{aligned}
\mathcal{X}_{\ell_4} &= \{\langle \rho_0,\ \rho[\mathsf{x} \leftarrow \rho(\mathsf{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in \mathcal{X}_{\ell_1} \wedge \rho(\mathsf{x}) > 9\} \\
&= \{\langle \rho_0,\ \rho[\mathsf{x} \leftarrow \rho(\mathsf{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in (\{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathsf{x}) + i]\rangle \mid i \in [1, 9] \wedge 0 < \rho_0(\mathsf{x}) \leqslant (9 - i)\}) \wedge \rho(\mathsf{x}) > 9\} \\
&= \{\langle \rho_0, \rho_0[x \leftarrow \rho_0(\mathsf{x})+1]\rangle \mid \rho_0(\mathsf{x}) > 9\} \cup \{\langle \rho_0, \rho_0[x \leftarrow \rho_0(\mathsf{x})+i+1]\rangle \mid i \in [1, 9] \wedge \rho_0(\mathsf{x}) = (9 - i)\} \\
&= \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathsf{x}) + 1]\rangle \mid \rho_0(\mathsf{x}) > 9\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow 10]\rangle \mid 1 \leqslant \rho_0(\mathsf{x}) \leqslant 9\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{X}_{\ell_5} &= \{\langle \rho_0,\ \rho[\mathsf{x} \leftarrow \rho(\mathsf{x}) + 1]\rangle \mid \langle \rho_0,\ \rho\rangle \in \mathcal{X}_{\ell_1} \wedge \rho(\mathsf{x}) > 9\} \cup \{\langle \rho_0,\ \rho\rangle \in \mathcal{R}_0 \mid \rho(\mathsf{x}) \leqslant 0\} \\
&= \{\langle \rho_0,\ \rho_0\rangle \mid \rho_0(\mathsf{x}) \leqslant 0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(\mathsf{x}) + 1]\rangle \mid \rho_0(\mathsf{x}) > 9\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow 10]\rangle \mid 1 \leqslant \rho_0(\mathsf{x}) \leqslant 9\}
\end{aligned}$$

.../...

Least solution to the fixpoint equations

So the least solution $\mathcal{S}$ to the system of equations (23.2) is the following.

$$\begin{cases}
\mathcal{S}_{\ell_1} &= \{\langle \rho_0,\ \rho\rangle \mid \rho = \rho_0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(x) + i]\rangle \mid i \in [1,9] \wedge 0 < \rho_0(x) \leqslant (9-i)\} \\
\mathcal{S}_{\ell_2} &= \{\langle \rho_0,\ \rho_0\rangle \mid \rho_0(x) > 0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(x) + i]\rangle \mid i \in [1,9] \wedge 0 < \rho_0(x) \leqslant (9-i)\} \\
\mathcal{S}_{\ell_3} &= \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(x) + 1]\rangle \mid \rho_0(x) > 0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(x) + i]\rangle \mid i \in [1,9] \wedge 0 < \rho_0(x) \leqslant (9-i)\} \\
\mathcal{S}_{\ell_4} &= \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(x) + 1]\rangle \mid \rho_0(x) > 9\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(x) + i + 1]\rangle \mid i \in [1,9] \wedge \rho_0(x) = (9-i)\} \\
\mathcal{S}_{\ell_5} &= \{\langle \rho_0,\ \rho_0\rangle \mid \rho_0(x) \leqslant 0\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow \rho_0(x) + 1]\rangle \mid \rho_0(x) > 9\} \cup \{\langle \rho_0,\ \rho_0[x \leftarrow 10]\rangle \mid 1 \leqslant \rho_0(x) \leqslant 9\}
\end{cases}$$

```
while ℓ1 (x > 0) {
    ℓ2 x = x + 1 ;
    if ℓ3 (x > 9)
        ℓ4 break ;
    } ℓ5
```
(19.7)

On program exit at ℓ_5, $\mathcal{S}_{\ell_5}$ environments that the value x of x is

- the initial value x_0 of x when x_0 is negative or zero (since the loop is never entered);
- or, the initial value x_0 of x plus 1 when x_0 is greater than 9;
- or 10, when the initial value x_0 of x is between 1 and 9 since in that case x is incremented by successive loop iterations until reaching 10 in which case the loop exits by the break statement.

Structural equational semantics

- The equational semantics of a program P is designed so that its least solution is the abstract interpreter of Chapter **21**.
- We define the equational semantics of a program P by structural induction on the syntax of program P.
- So the equational semantics (specifying how the equations are derived for a program) is structural but the equations are not (*i.e.* the program structure is not apparent in the equations).

- The equations $\widehat{\mathscr{E}}^{\natural}[\![\mathsf{s}]\!]$ for a program component s have the form

$$\mathcal{X} \quad = \quad \mathsf{E}[\![\mathsf{s}]\!]\, \mathcal{R}_0\, \mathcal{X} \tag{23.4}$$

  which can be detailled as

$$\begin{cases} \mathcal{X}_\ell \quad = \quad \mathsf{E}[\![\mathsf{s}]\!]\, \mathcal{R}_0\, (\prod_{\ell' \in \mathsf{labs}[\![\mathsf{s}]\!]} \mathcal{X}_{\ell'})\, \ell \\ \ell \in \mathsf{labs}[\![\mathsf{s}]\!] \end{cases}$$

  so that the equational semantics is the set of equations

$$\widehat{\mathscr{E}}^{\natural}[\![\mathsf{s}]\!] = \{\mathcal{X}_\ell \quad = \quad \mathsf{E}[\![\mathsf{s}]\!]\, \mathcal{R}_0\, (\textstyle\prod_{\ell' \in \mathsf{labs}[\![\mathsf{s}]\!]} \mathcal{X}_{\ell'})\, \ell \mid \ell \in \mathsf{labs}[\![\mathsf{s}]\!]\}.$$

- By (19.30), the set of reachable environments of a statement s not at a label of that statement is empty, abstracted by $\bot^{\natural}$. So $\mathcal{X}^\ell = \bot^{\natural}$ for all $\ell \notin \mathsf{labs}[\![\mathsf{s}]\!]$ and to determine this no specific equation need to be considered.

Structural equational semantics

Equational semantics of a program $\mathtt{P} ::= \mathtt{Sl}\ \ell'$

$$\widehat{\mathscr{E}}^{\natural}[\![\mathtt{P}]\!] \triangleq \widehat{\mathscr{E}}^{\natural}[\![\mathtt{Sl}]\!] \tag{23.5}$$

- The equational semantics of a program $\mathtt{P} ::= \mathtt{Sl}\ \ell'$ with variables $\mathcal{X}^{\ell}$, $\ell \in \mathit{labs}[\![\mathtt{S}]\!]$ is that of its statement list $\mathtt{Sl}$.

Equational semantics of a skip statement $\mathtt{S} ::= \mathtt{;}$

$$\widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0 = \{\mathcal{X}_{\mathit{at}[\![\mathtt{S}]\!]} = \mathcal{R}_0, \mathcal{X}_{\mathit{after}[\![\mathtt{S}]\!]} = \mathcal{X}_{\mathit{at}[\![\mathtt{S}]\!]}\} \tag{23.6}$$

- The equational semantics of a skip statement $\mathtt{S} ::= \mathtt{;}$ maps the (abstraction of the) initial environments to themselves since they are not modified.

Equational semantics of an assignment statement S ::= x = E ;

$$\mathscr{E}^{\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0 = \{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathcal{R}_0, \quad \mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]} = \mathsf{assign}_{\natural}[\![\mathtt{x}, \mathtt{E}]\!]\,\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]}\} \tag{23.7}$$

- The equational semantics of an assignment statement S ::= x = E ; maps the (abstraction of the) initial environments to themselves on entry and to themselves as modified by the assignment on exit.

Equational semantics of a conditional statement $\mathtt{S} ::= \mathtt{if\ (B)\ S}_t$

$$\begin{aligned}\mathscr{E}^{\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0 \quad = \quad & \mathsf{let}\,\{\mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]} = E\} \uplus \mathcal{E} = \mathscr{E}^{\natural}[\![\mathtt{S}_t]\!](\mathsf{test}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]})\ \mathsf{in} \\ & \{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathcal{R}_0\} \cup \mathcal{E} \cup \{\mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]} = E \sqcup^{\natural} \overline{\mathsf{test}}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]}\}\end{aligned} \tag{23.8}$$

- We write $X \uplus Y = Z$ to mean that X, Y is a partition of Z that is $X \cap Y = \varnothing$ and $X \cup Y = Z$. This is essential to ensure that there is only one equation per variable $\mathcal{X}_\ell$, $\ell \in \mathsf{labs}[\![\mathtt{S}]\!]$
- The equational semantics of a conditional statement $\mathtt{S} ::= \mathtt{if\ (B)\ S}_t$ is the (abstraction of the) set of reachable environments in $\mathtt{S}_t$ under the hypothesis that the test B is true and, after the conditional statement, the union of the (abstraction of the) environments reaching the end of the true alternative $\mathtt{S}_t$ (as given by the equation for $\mathcal{X}_{\mathsf{after}[\![\mathtt{S}_t]\!]} = \mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]}$ in the alternative $\mathtt{S}_t$) and those for which the conditional statement is skipped when the test B is false.

Equational semantics of a conditional statement $\mathtt{S} ::= \mathtt{if\ (B)\ S}_t\ \mathtt{else\ S}_f$

$$\begin{aligned}\widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0 \quad = \quad & \mathsf{let}\,(\{\mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]} = E_t\} \cup \mathcal{E}_t) = \widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}_t]\!](\mathsf{test}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]}) \\ & \text{and}\ \ (\{\mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]} = E_f\} \cup \mathcal{E}_f) = \widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}_f]\!](\overline{\mathsf{test}}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]})\ \mathsf{in} \\ & \quad \{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathcal{R}_0\} \cup \\ & \quad \mathcal{E}_t \cup \mathcal{E}_f \cup \\ & \quad \{\mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]} = E_t \sqcup^{\natural} E_f\}\end{aligned} \tag{23.9}$$

- The equational semantics of a conditional statement $\mathtt{S} ::= \mathtt{if\ (B)\ S}_t\ \mathtt{else\ S}_f$ is the (abstraction of the) set of reachable environments in $\mathtt{S}_t$ under the hypothesis that the test $\mathtt{B}$ is true, the set of reachable environments in $\mathtt{S}_f$ under the hypothesis that the test $\mathtt{B}$ is false, and, after the conditional statement, the union of the sets of environments reaching the end of the true and false alternatives $\mathtt{S}_t$ and $\mathtt{S}_f$.

Equational semantics of a statement list $\mathtt{sl} ::= \mathtt{sl}'\ \mathtt{s}$

$$\begin{array}{lcll} \widehat{\mathscr{E}}^{\natural}[\![\mathtt{sl}]\!]\mathcal{R}_0 & = & \widehat{\mathscr{E}}^{\natural}[\![\mathtt{s}]\!]\mathcal{R}_0 & \text{when} \quad \mathtt{sl}' ::= \epsilon \\ & = & \text{let } (\{\mathcal{X}_{\mathsf{after}[\![\mathtt{sl}']\!]} = E'\} \cup \mathcal{E}') = \widehat{\mathscr{E}}^{\natural}[\![\mathtt{sl}']\!]\mathcal{R}_0 & \text{otherwise} \\ & & \quad \text{and } (\{\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]} = E\} \cup \mathcal{E}) = \widehat{\mathscr{E}}^{\natural}[\![\mathtt{s}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]} \text{ in} & \\ & & \quad\quad \{\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]} = E' \sqcup^{\natural} E\} \cup \mathcal{E}' \cup \mathcal{E} & \end{array} \tag{23.10}$$

- The equational semantics of a statement list $\mathtt{sl} ::= \mathtt{sl}'\ \mathtt{s}$ is the set of equations of $\mathtt{sl}'$ and those of $\mathtt{s}$ after executing $\mathtt{sl}'$ where the equation for $\mathcal{X}_{\mathsf{after}[\![\mathtt{sl}']\!]} = \mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]}$ is shared.

Equational semantics of an empty statement list $\mathtt{sl} ::= \epsilon$

$$\widehat{\mathscr{E}}^{\natural}[\![\mathtt{sl}]\!]\mathcal{R}_0 = \{\mathcal{X}_{\mathsf{at}[\![\mathtt{sl}]\!]} = \mathcal{R}_0\} \tag{23.11}$$

- The empty statement list $\mathtt{sl} ::= \epsilon$ does not involve any computation so that the (abstraction of the) reachable environments are the initial environments.

Equational semantics of an iteration statement $\mathtt{S} ::= \mathtt{while}^{\ell}\ \mathtt{(B)}\ \mathtt{S}_b$

$$
\begin{aligned}
\widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0 \quad = \quad & \mathsf{let}\,(\{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = E\} \cup \mathcal{E}) = \widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}_b]\!](\mathsf{test}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]})\ \ \mathsf{in} \\
& \{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathcal{R}_0 \sqcup^{\natural} E\} \cup \\
& \mathcal{E} \cup \\
& \{\mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]} = \overline{\mathsf{test}}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} \sqcup^{\natural} \bigsqcup_{\ell \in \mathsf{breaks\text{-}of}[\![\mathtt{S}_b]\!]}^{\natural} \mathcal{X}_{\ell}\}
\end{aligned}
\tag{23.12}
$$

- The equational semantics of an iteration statement $\texttt{S} ::= \texttt{while}^{\ell}\ \texttt{(B)}\ \texttt{S}_b$, is a fixpoint which successive iterates provide the (abstraction of the) reachable environments after at most 0, 1, ...or more iterations in the loop.
- If the loop terminates, the fixpoint iterates beyond the maximal number of iterations add no new reachable environment.
- If the loop does not terminate the (abstraction of the) reachable environments after any number of iterations (is) are obtained by passing to the limit in the fixpoint iterates.
- The reachable environments $\mathcal{X}_{\mathsf{at}[\![\texttt{S}]\!]}$ are the entry environments or the reachable environments after one more iteration from reachable states with test true.
- The reachable environments $\mathcal{X}_{\mathsf{after}[\![\texttt{S}]\!]}$ on loop exit, are those $\mathsf{at}[\![\texttt{S}]\!]$ for which the test is false or the breaking environments reachable from reachable environments in an iteration with test true.

Equational semantics of a break statement $\mathtt{S} ::= \ell\ \mathtt{break}\ ;$

$$\mathscr{E}^{\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0 \quad = \quad \{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathcal{R}_0,\ \mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]} = \bot^{\natural}\} \tag{23.13}$$

- The equational semantics of a break statement $\mathtt{S} ::= \mathtt{break}\ ;$ states that execution never goes after the break at $\mathsf{after}[\![\mathtt{S}]\!]$.

Equational semantics of a compound statement $\mathtt{S} ::= \{\ \mathtt{Sl}\ \}$

$$\mathscr{E}^{\natural}[\![\mathtt{S}]\!] \quad = \quad \mathscr{E}^{\natural}[\![\mathtt{Sl}]\!] \tag{23.14}$$

- The equational semantics of a compound statement $\mathtt{S} ::= \{\ \mathtt{Sl}\ \}$ is that of the statement list $\mathtt{Sl}$.

Theorem (23.20) The equations $\widehat{\mathscr{E}}^{\natural}[\![\mathsf{P}]\!]\,\mathcal{R}_0$ have the form

$$\begin{cases} \mathcal{X}_\ell \;=\; \mathsf{E}[\![\mathsf{P}]\!]\,\mathcal{R}_0\Big(\prod_{\ell' \in \mathsf{labs}[\![\mathsf{P}]\!]} \mathcal{X}_{\ell'}\Big)\,\ell \\ \ell \in \mathsf{labs}[\![\mathsf{P}]\!] \end{cases}$$

such that

- there is exactly one equation $\mathcal{X}_\ell = \mathsf{E}[\![\mathsf{P}]\!]\,\mathcal{R}_0\,(\prod_{\ell' \in \mathsf{labs}[\![\mathsf{P}]\!]} \mathcal{X}_{\ell'})\,\ell$ for each program point $\ell \in \mathsf{labs}[\![\mathsf{P}]\!]$;
- $\mathsf{E}[\![\mathsf{P}]\!]\,\mathcal{R}_0 \in \prod_{\ell \in \mathsf{labs}[\![\mathsf{P}]\!]} \mathbb{P}^{\natural} \xrightarrow{uc} \prod_{\ell \in \mathsf{labs}[\![\mathsf{P}]\!]} \mathbb{P}^{\natural}$ is well-defined and continuous;
- The equations have a least pointwise fixpoint $\mathsf{lfp}^{\dot{\sqsubseteq}^{\natural}}\,\mathsf{E}[\![\mathsf{P}]\!]\,\mathcal{R}_0 = \widehat{\mathscr{S}}^{\natural}[\![\mathsf{P}]\!]\,\mathcal{R}_0$, as defined in Section **21.2**.

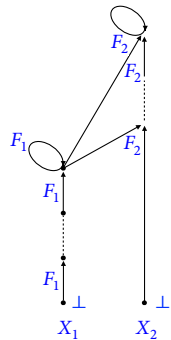
Theorem 23.20 shows that the equational definition of the semantics is exactly the same as the structural functional one of Chapter **21**. The idea is that the abstract interpreter is a function f which can be abstracted by an equation $\mathcal{X} = F(\mathcal{X})$ which least solution is $f = \mathsf{lfp}\,F$, and conversely.

Transfinite chaotic iterations

Equations:

$$\begin{cases} X_1 &= F_1(X_1) \\ X_2 &= F_2(X_1, X_2) \end{cases}$$

Example transfinite chaotic iterations [P. Cousot, 1977, 1978]:



Properties of the system of equations

(1) The variables of equations $\widehat{\mathscr{E}}^{\,\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0$ for a program component $\mathtt{S} \in \mathbb{P}\mathtt{c}$ are in $\{\mathcal{X}_\ell \mid \ell \in \mathsf{labs}[\![\mathtt{S}]\!]\}$

(2) No system of equations for a program component $\mathtt{S} \in \mathbb{P}\mathtt{c}$ has any equations depending on $\mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]}$.

(3) For all program components $\mathtt{S}$, $\underline{\underline{NOT}}$ starting with an iteration, the equation for $\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]}$ is

$$\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathcal{R}_0$$

The proofs are by structural induction

- Base cases, *e.g.* assignment (23.7)
- Induction cases, *e.g.* iteration (23.12) or statement list (23.10)

Proof of Theorem 23.20 — I

We prove the theorem for all program components $\mathsf{s} \in \mathbb{P}c$, by structural induction of s.

The equations $\widehat{\mathscr{E}}^{\,\natural}[\![\mathsf{s}]\!]\,\mathcal{R}_0$ have the form $\mathcal{X} = E(\mathcal{X})$, that is componenwise

$$\begin{cases} \mathcal{X}_\ell = \mathsf{E}[\![\mathsf{s}]\!]\,\mathcal{R}_0\,(\prod_{\ell' \in \mathsf{labs}[\![\mathsf{s}]\!]} \mathcal{X}_{\ell'})\,\ell \\ \\ \ell \in \mathsf{labs}[\![\mathsf{s}]\!] \end{cases}$$

— The upper continuity of these equations follows pointwise from the continuity of each equation $\mathcal{X} \mapsto \mathsf{E}[\![\mathsf{P}]\!]\,\mathcal{R}_0\,\mathcal{X}\,\ell$, $\ell \in \mathsf{labs}[\![\mathsf{P}]\!]$ which itself follows from

- the continuity of the identity and constant function,
- of the primitives of the abstract domain (including the join) in Definition 21.1,
- of the conditional with continuous alternatives, and
- the fact that the composition of continuous functions is continuous.

Proof of Theorem 23.20 — II

— The fact that there is one equation per program label is proved by structural induction.

- For the basis, this is true for the assignment, empty statement list, skip, and break statement.
- For the induction step,
  - for a program component made of several subcomponents, this remains true, by structural induction, for disjoint labels;
  - moreover, when a label is shared between subcomponents their equations are regrouped into one single equation.

Proof of Theorem 23.20 — III

For example, for the statement list $\mathtt{Sl} ::= \mathtt{Sl}'\ \mathtt{S}$, $\mathtt{Sl}' \neq \epsilon$,

- the equations $\widehat{\mathscr{E}}^{\,\natural}[\![\mathtt{Sl}']\!]\mathcal{R}_0$ and $\widehat{\mathscr{E}}^{\,\natural}[\![\mathtt{S}]\!]\mathcal{R}_0$ share, by definition of labs in Section **4.2.7**, only one common variable $\mathcal{X}_{\mathsf{after}[\![\mathtt{Sl}']\!]} = \mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]}$
- the equation $\mathcal{X}_{\mathsf{after}[\![\mathtt{Sl}']\!]} = \mathrm{RHS}'$ for $\mathtt{Sl}'$ and the equation $\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathrm{RHS}$ for $\mathtt{S}$ are grouped into a single equation $\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathrm{RHS}' \sqcup^{\natural} \mathrm{RHS}$ for $\mathtt{Sl}$.
- so, by (23.10), the system of equation for $\mathtt{Sl} ::= \mathtt{Sl}'\ \mathtt{S}$ has exactly one equation for each program point of $\mathtt{Sl}$

The reasoning is the same for the conditional, iteration, *etc*.

Proof of Theorem 23.20 — IV

— It remains to prove that the equations $\widehat{\mathscr{E}}^{\,\natural}[\![\mathsf{S}]\!]\,\mathcal{R}_0$ for a statement S have a least pointwise fixpoint equal to the structural abstract semantics defined in Section **21.2**

$$\mathrm{lfp}^{\dot{\subseteq}^{\natural}}\, \mathsf{E}[\![\mathsf{P}]\!]\,\mathcal{R}_0 \quad = \quad \widehat{\mathcal{S}}^{\,\natural}[\![\mathsf{P}]\!]\,\mathcal{R}_0$$

— The proof is by structural induction

Proof of Theorem 23.20 — V

— For the basis, consider the assignment, (23.7) the equations are

$$\{\mathcal{X}_{\mathsf{at}[\![\mathsf{s}]\!]} = \mathcal{R}_0, \mathcal{X}_{\mathsf{after}[\![\mathsf{s}]\!]} = \mathsf{assign}^{\natural}[\![\mathsf{x}, \mathsf{E}]\!]\, \mathcal{X}_{\mathsf{at}[\![\mathsf{s}]\!]}\}$$

The unique solution (hence $\dot{\sqsubseteq}^{\natural}$-solution) is

$$\mathcal{S}_{\mathsf{at}[\![\mathsf{s}]\!]} = \mathcal{R}_0, \quad \mathcal{S}_{\mathsf{after}[\![\mathsf{s}]\!]} = \mathsf{assign}^{\natural}[\![\mathsf{x}, \mathsf{E}]\!]\, \mathcal{R}_0$$

and, by convention, $\mathcal{S}_\ell = \bot^{\natural}$ when $\ell \notin \{\mathsf{at}[\![\mathsf{S}]\!], \mathsf{after}[\![\mathsf{S}]\!]\}$.
By (21.7), this solution is

$$\mathcal{S} = \widehat{\mathcal{S}}^{\natural}[\![\mathsf{S}]\!]\, \mathcal{R}_0$$

The proof is similar for an empty statement list (23.11), a skip statement (23.6), a break statement (23.13), and it is trivial, by structural induction, for a compound statement (23.14).

Proof of Theorem 23.20 — VI

For the induction step of the proof.

— The equations for a program $\mathtt{P} ::= \mathtt{Sl}\ \ell'$ are the same as those for the statement list $\mathtt{Sl}$ since $\mathsf{labs}[\![\mathtt{P}]\!] = \mathsf{in}[\![\mathtt{Sl}]\!] \cup \{\mathsf{after}[\![\mathtt{Sl}]\!]\}$.

- By structural induction the equations $\widehat{\mathscr{E}}^{\,\natural}[\![\mathtt{Sl}]\!]$ have solution $\mathscr{S} = \widehat{\mathscr{S}}^{\,\natural}[\![\mathtt{Sl}]\!]\,\mathcal{R}_0$
- So the solution is the same for the program in accordance with (21.4).

— For the statement list $\mathtt{Sl} ::= \epsilon\ \mathtt{S}$, reduced to a statement $\mathtt{S}$, the result directly follows from $\mathsf{in}[\![\mathtt{Sl}]\!] = \mathsf{in}[\![\mathtt{S}]\!]$ and $\mathsf{after}[\![\mathtt{Sl}]\!] = \mathsf{after}[\![\mathtt{S}]\!]$, (21.5), (23.10), and the induction hypothesis.

Proof of Theorem 23.20 — VII

— For the statement list $\mathtt{sl} ::= \mathtt{sl}'\ \mathtt{s}$

$$\widehat{\mathscr{E}}^{\,\natural}[\![\mathtt{sl}]\!]\mathcal{R}_0 \quad = \quad \begin{array}{l}\text{let } (\{\mathcal{X}_{\mathsf{after}[\![\mathtt{sl}']\!]} = \mathrm{RHS}'\} \cup \mathcal{E}') = \widehat{\mathscr{E}}^{\,\natural}[\![\mathtt{sl}']\!]\mathcal{R}_0 \\ \quad \text{and } (\{\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]} = \mathrm{RHS}\} \cup \mathcal{E}) = \widehat{\mathscr{E}}^{\,\natural}[\![\mathtt{s}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]} \text{ in} \\ \quad\quad \{\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]} = \mathrm{RHS}' \sqcup^{\natural} \mathrm{RHS}\} \cup \mathcal{E}' \cup \mathcal{E}\end{array} \tag{23.10}$$

- By induction hypothesis, the equations $\{\mathcal{X}_{\mathsf{after}[\![\mathtt{sl}']\!]} = \mathrm{RHS}'\} \cup \mathcal{E}' = \widehat{\mathscr{E}}^{\,\natural}[\![\mathtt{sl}']\!]\mathcal{R}_0$ have solution $\mathcal{S}'_{\ell'} = \widehat{\mathcal{S}}^{\,\natural}[\![\mathtt{sl}']\!]\,\mathcal{R}_0\,\ell'$ for $\ell' \in \mathsf{labs}[\![\mathtt{sl}']\!]$
- So the value of $\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]} = \mathcal{X}_{\mathsf{after}[\![\mathtt{sl}']\!]}$ is $\mathcal{S}'_{\mathsf{at}[\![\mathtt{s}]\!]} = \widehat{\mathcal{S}}^{\,\natural}[\![\mathtt{sl}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathtt{s}]\!]$.
- By induction hypothesis, the equations

$$\{\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]} = \mathrm{RHS}\} \cup \mathcal{E} = \widehat{\mathscr{E}}^{\,\natural}[\![\mathtt{s}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]}$$

have therefore solution

$$\widehat{\mathcal{S}}^{\,\natural}[\![\mathtt{s}]\!]\,\mathcal{S}'_{\mathsf{at}[\![\mathtt{s}]\!]}\,\ell \quad = \quad \widehat{\mathcal{S}}^{\,\natural}[\![\mathtt{s}]\!]\,(\widehat{\mathcal{S}}^{\,\natural}[\![\mathtt{sl}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathtt{s}]\!])\,\ell \quad = \quad \widehat{\mathcal{S}}^{\,\natural}[\![\mathtt{sl}]\!]\,\mathcal{R}_0\,\ell \quad \text{by (21.5)}$$

for $\ell \in \mathsf{labs}[\![\mathtt{s}]\!]$

Proof of Theorem 23.20 — VIII

It remains to show that the equations $\{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathrm{RHS}' \sqcup^{\natural} \mathrm{RHS}\} \cup \mathcal{E}' \cup \mathcal{E} = \widehat{\mathscr{E}}^{\natural}[\![\mathtt{Sl}]\!]\mathcal{R}_0$ for $\mathtt{Sl}$ have the same solution.

- Remember that no system of equations for a program component $\mathtt{S} \in \mathbb{P}\mathrm{c}$ has any equations depending on $\mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]}$.
- This means that a chaotic iteration strategy can solve $\mathcal{E}'$ first to get the solution $\mathcal{S}'_{\ell'} = \widehat{\mathscr{E}}^{\natural}[\![\mathtt{Sl}]\!]\mathcal{R}_0\ \ell'$ for $\mathcal{X}_{\ell'}$ when $\ell' \in \mathsf{labs}[\![\mathtt{Sl}']\!] \setminus \{\mathsf{after}[\![\mathtt{Sl}']\!]\}$.
- Then, we can calculate the value of RHS' which depends only on $\mathcal{S}'_{\ell'}$, $\ell' \in \mathsf{labs}[\![\mathtt{Sl}']\!] \setminus \{\mathsf{after}[\![\mathtt{Sl}']\!]\}$ and is

$$\widehat{\mathcal{S}}^{\natural}[\![\mathtt{Sl}']\!]\ \mathcal{R}_0\ \mathsf{after}[\![\mathtt{Sl}']\!] = \widehat{\mathcal{S}}^{\natural}[\![\mathtt{Sl}']\!]\ \mathcal{R}_0\ \mathsf{at}[\![\mathtt{S}]\!]$$

- So the chaotic iteration strategy is left with solving

$$\{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \widehat{\mathcal{S}}^{\natural}[\![\mathtt{Sl}']\!]\ \mathcal{R}_0\ \mathsf{at}[\![\mathtt{S}]\!] \sqcup^{\natural} \mathrm{RHS}\} \cup \mathcal{E}$$

- There are two subcases.

Proof of Theorem 23.20 — IX

– (First subcase) For all program components $\mathsf{s} \in \mathcal{PC}$ but the iteration,

- The equations $\widehat{\mathscr{E}}^{\,\natural}[\![\mathsf{s}]\!]\,\mathcal{R}_0$ have an entry equation of the form $\mathcal{X}_{\mathsf{at}[\![\mathsf{s}]\!]} = \mathcal{R}_0$.
- Therefore, $\mathcal{X}_{\mathsf{at}[\![\mathsf{s}]\!]} = \mathrm{RHS}$ is $\mathcal{X}_{\mathsf{at}[\![\mathsf{s}]\!]} = \mathcal{X}_{\mathsf{after}[\![\mathsf{s}\mathfrak{l}']\!]}$.
- Then the equations $\{\mathcal{X}_{\mathsf{at}[\![\mathsf{s}]\!]} = \widehat{\mathcal{S}}^{\,\natural}[\![\mathsf{s}\mathfrak{l}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathsf{s}]\!] \sqcup^{\natural} \mathrm{RHS}\} \cup \mathcal{E}$ are

  $$\{\mathcal{X}_{\mathsf{at}[\![\mathsf{s}]\!]} = \widehat{\mathcal{S}}^{\,\natural}[\![\mathsf{s}\mathfrak{l}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathsf{s}]\!] \sqcup^{\natural} \mathcal{X}_{\mathsf{after}[\![\mathsf{s}\mathfrak{l}']\!]}\} \cup \mathcal{E}$$

  that is $\{\mathcal{X}_{\mathsf{at}[\![\mathsf{s}]\!]} = \widehat{\mathcal{S}}^{\,\natural}[\![\mathsf{s}\mathfrak{l}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathsf{s}]\!] \sqcup^{\natural} \mathcal{X}_{\mathsf{at}[\![\mathsf{s}]\!]}\} \cup \mathcal{E}$,
- Their least solution is, by def. of the lub, the same as

  $$\{\mathcal{X}_{\mathsf{at}[\![\mathsf{s}]\!]} = \widehat{\mathcal{S}}^{\,\natural}[\![\mathsf{s}\mathfrak{l}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathsf{s}]\!]\} \cup \mathcal{E}$$

  which we have shown to have solution

  $$\widehat{\mathcal{S}}^{\,\natural}[\![\mathsf{s}]\!]\,(\widehat{\mathcal{S}}^{\,\natural}[\![\mathsf{s}\mathfrak{l}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathsf{s}]\!])\,\ell \quad = \quad \widehat{\mathcal{S}}^{\,\natural}[\![\mathsf{s}\mathfrak{l}]\!]\,\mathcal{R}_0\,\ell, \qquad \ell \in \mathsf{labs}[\![\mathsf{s}]\!]$$

Proof of Theorem 23.20 — X

– (Second subcase) Otherwise, $\mathtt{S} ::= \mathtt{while}^{\ell}\ \mathtt{(B)}\ \mathtt{S}_b$ is an iteration.

- By (23.12), the entry equation is $\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathcal{X}_{\mathsf{after}[\![\mathtt{Sl}']\!]} \sqcup^{\natural} \mathrm{RHS}''$ where
$$\{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathrm{RHS}''\} \cup \mathcal{E}'' = \widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}_b]\!](\mathsf{test}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]})$$
(where $\mathsf{after}[\![\mathtt{Sl}']\!] = \mathsf{at}[\![\mathtt{S}]\!] = \mathsf{after}[\![\mathtt{S}_b]\!]$ is the iteration label).
- So $\mathrm{RHS} = \mathcal{X}_{\mathsf{after}[\![\mathtt{Sl}']\!]} \sqcup^{\natural} \mathrm{RHS}''$ and the chaotic iterations are left to solve
$$\{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \widehat{\mathscr{S}}^{\natural}[\![\mathtt{Sl}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathtt{S}]\!] \sqcup^{\natural} \mathrm{RHS}\} \cup \mathcal{E}$$
that is
$$\{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \widehat{\mathscr{S}}^{\natural}[\![\mathtt{Sl}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathtt{S}]\!] \sqcup^{\natural} \mathcal{X}_{\mathsf{after}[\![\mathtt{Sl}']\!]} \sqcup^{\natural} \mathrm{RHS}''\} \cup \mathcal{E}$$
which simplifies to
$$\{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \widehat{\mathscr{S}}^{\natural}[\![\mathtt{Sl}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathtt{S}]\!] \sqcup^{\natural} \mathrm{RHS}''\} \cup \mathcal{E}$$
since the solution to $\mathcal{X}_{\mathsf{after}[\![\mathtt{Sl}']\!]}$ is $\widehat{\mathscr{S}}^{\natural}[\![\mathtt{Sl}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathtt{S}]\!]$ and $\sqcup^{\natural}$ is idempotent.
- This is exactly the equations $\widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]}$ which we have shown to have solution $\widehat{\mathscr{S}}^{\natural}[\![\mathtt{S}]\!]\,(\widehat{\mathscr{S}}^{\natural}[\![\mathtt{Sl}']\!]\,\mathcal{R}_0\,\mathsf{at}[\![\mathtt{S}]\!])\,\ell = \widehat{\mathscr{S}}^{\natural}[\![\mathtt{Sl}]\!]\,\mathcal{R}_0\,\ell$ for $\ell \in \mathsf{labs}[\![\mathtt{S}]\!]$.
- By Theorem 22.4, all chaotic iterations are equivalent and yields the least fixpoint proving that, in case of a statement list, $\mathsf{lfp}^{\dot{\sqsubseteq}^{\natural}}\,\widehat{\mathscr{E}}^{\natural}[\![\mathtt{Sl}]\!]\,\mathcal{R}_0 = \widehat{\mathscr{S}}^{\natural}[\![\mathtt{Sl}]\!]\,\mathcal{R}_0$.

Proof of Theorem 23.20 — XI

— The proof for the conditionals (23.8) and (23.9) is very similar.

Proof of Theorem 23.20 — XII

— For the iteration $\mathtt{S} ::= \mathtt{while}^{\ell}\ \mathtt{(B)}\ \mathtt{S}_b$ (23.12), the Jacobi calculation of the least fixpoint $\widehat{\mathscr{S}}^{\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0 \;=\; \mathrm{lfp}^{\sqsubseteq}(\mathscr{F}^{\natural}[\![\mathtt{while}^{\ell}\ \mathtt{(B)}\ \mathtt{S}_b]\!]\,\mathcal{R}_0)$ corresponds to a specific chaotic iteration strategy for the loop equations

$$\begin{array}{rcl}
\widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0 & = & \mathsf{let}\,(\{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = E\} \cup \mathcal{E}) = \widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}_b]\!](\mathsf{test}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]})\ \mathsf{in} \\
& & \quad \{\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} = \mathcal{R}_0 \sqcup^{\natural} E\} \cup \\
& & \quad \mathcal{E} \cup \\
& & \quad \{\mathcal{X}_{\mathsf{after}[\![\mathtt{S}]\!]} = \overline{\mathsf{test}}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]} \sqcup^{\natural} \bigsqcup_{\ell \in \mathsf{breaks\text{-}of}[\![\mathtt{S}_b]\!]}^{\natural} \mathcal{X}_{\ell}\}
\end{array} \qquad (23.12)$$

where the equations $\widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}_b]\!](\mathsf{test}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{S}]\!]})$ for the loop body are solved by iteration before considering another iteration of the loop fixpoint $\widehat{\mathscr{E}}^{\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0$.

Proof of Theorem 23.20 — XIII

More precisely, by (21.11), $\widehat{\mathcal{S}}^{\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0$ is the least solution to the system of equations

$$\begin{cases} X(\ell') & = & \mathcal{F}^{\natural}[\![\mathtt{while}\ \ell\ \mathtt{(B)}\ \mathtt{S}_b]\!]\,\mathcal{R}_0\,X\,\ell' \\ \ell' \in \mathsf{labs}[\![\mathtt{S}]\!] \end{cases}$$

(where $\mathsf{labs}[\![\mathtt{S}]\!] = \mathsf{labs}[\![\mathtt{S}_b]\!] \cup \{\mathsf{after}[\![\mathtt{S}]\!]\}$ for iteration) that is

$$\begin{cases} X(\ell') & = & (\!|\ \ell' = \ell\ ?\ \mathcal{R}_0 \sqcup^{\natural} \widehat{\mathcal{S}}^{\natural}[\![\mathtt{S}_b]\!]\,(\mathsf{test}^{\natural}[\![\mathtt{B}]\!]X(\ell))\,\ell \\ & & \ |\ \ell' \in \mathsf{in}[\![\mathtt{S}_b]\!] \setminus \{\ell\}\ ?\ \widehat{\mathcal{S}}^{\natural}[\![\mathtt{S}_b]\!]\,(\mathsf{test}^{\natural}[\![\mathtt{B}]\!]X(\ell))\,\ell' \\ & & \ |\ \ell' = \mathsf{after}[\![\mathtt{S}]\!]\ ?\ \overline{\mathsf{test}}^{\natural}[\![\mathtt{B}]\!]X(\ell) \sqcup^{\natural} \bigsqcup_{\ell'' \in \mathsf{breaks\text{-}of}[\![\mathtt{S}_b]\!]}^{\natural} \widehat{\mathcal{S}}^{\natural}[\![\mathtt{S}_b]\!]\,(\mathsf{test}^{\natural}[\![\mathtt{B}]\!]X(\ell))\,\ell'' \\ & & \ :\ \bot^{\natural}\ |\!) \\ \ell' \in \mathsf{labs}[\![\mathtt{S}]\!] \end{cases} \qquad \text{(a)}$$

We let X^k, $k \in \mathbb{N}$ be the Jacobi iterations for this system of equations (a).

Proof of Theorem 23.20 — XIV

- Let
$$\{\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]} = \mathrm{RHS}\} \cup \mathcal{E} = \widehat{\mathscr{E}}^{\,\natural}[\![\mathtt{s}_b]\!](\mathsf{test}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]})$$
be the equations of the loop body $\mathtt{s}_b$.
- By (23.12), the equations $\widehat{\mathscr{E}}^{\,\natural}[\![\mathtt{S}]\!]\,\mathcal{R}_0$ for the iteration $\mathtt{S} ::= \mathtt{while}^{\,\ell}\ \mathtt{(B)}\ \mathtt{s}_b$ are
$$\{\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]} = \mathcal{R}_0 \sqcup^{\natural} \mathrm{RHS}\} \cup \mathcal{E} \cup \{\mathcal{X}_{\mathsf{after}[\![\mathtt{s}]\!]} = \overline{\mathsf{test}}^{\natural}[\![\mathtt{B}]\!]\mathcal{X}_{\mathsf{at}[\![\mathtt{s}]\!]} \sqcup^{\natural} \bigsqcup_{\ell \in \mathsf{breaks\text{-}of}[\![\mathtt{s}_b]\!]}^{\natural} \mathcal{X}_\ell\} \qquad \text{(b)}$$
- We consider a chaotic iteration (generalized to transfinite iteration [P. Cousot, 1977, 1978]) for these equations (b) with a subsequence $\mathcal{X}^k$, $k \in \mathbb{N}$ defined as follows.

Proof of Theorem 23.20 — XV

- The initialization is the infimum $\mathcal{X}^0 = X^0 = \bot$;
- Assume, by recurrence hypothesis, that $\mathcal{X}^k = X^k$;
- We next iterate the subequations

$$\{\mathcal{X}_{\mathsf{at}[\![\mathsf{S}]\!]} = \mathcal{R}_0 \sqcup^\natural \mathrm{RHS}\} \cup \mathcal{E}$$

  in (b) and pass to the limit.
  - By $\mathcal{R}_0 \sqsubseteq^\natural \mathcal{X}^k_{\mathsf{at}[\![\mathsf{S}]\!]}$ the limit is the least solution to $\widehat{\mathscr{E}}^\natural[\![\mathsf{S}_b]\!](\mathsf{test}^\natural[\![\mathsf{B}]\!]\mathcal{X}^k_{\mathsf{at}[\![\mathsf{S}]\!]})$
  - By structural induction, it is $\widehat{\mathscr{S}}^\natural[\![\mathsf{S}_b]\!]\,(\mathsf{test}^\natural[\![\mathsf{B}]\!]\mathcal{X}^k(\mathsf{at}[\![\mathsf{S}]\!])) = \widehat{\mathscr{S}}^\natural[\![\mathsf{S}_b]\!]\,(\mathsf{test}^\natural[\![\mathsf{B}]\!]X^k(\mathsf{at}[\![\mathsf{S}]\!]))$ since $\mathcal{X}^k = X^k$ by recurrence hypothesis.
- Then we do a Jacobi iteration on the remaining equations in (b) which is then exactly the same as the Jacobi iteration in (a) so we get $\mathcal{X}^{k+1} = X^{k+1}$.

Proof of Theorem 23.20 — XVI

- By recurrence, $\forall k \in \mathbb{N} \,.\, \mathcal{X}^k = X^k$
- So the infinite increasing chaotic iterations for (b) contains as a subsequence the infinite increasing Jacobi iteration for (a)
- So the limits are the same (see Exercise 10.11) that is, by Theorem 22.4 and (21.11), $\widehat{\mathcal{S}}^{\,\natural}[\![\mathsf{s}]\!]\,\mathcal{R}_0$. □

Constraints

Constraints versus equations

- Many static program analysis methods solve inequations instead of equations (*e.g.* [P. Cousot and R. Cousot, 1995; Heintze and Jaffar, 1994]).
- This follows from the fact that the least solution $\mathrm{lfp}^{\dot{\sqsubseteq}}\, \vec{F}$ to the system of equations $\vec{X} = \vec{F}(\vec{X})$ that is

$$\left\{ \begin{array}{l} X_i \quad = \quad F_i(X_1, \ldots, X_n) \\ \mathrm{i}{=}1, \ldots, \mathrm{n} \end{array} \right.$$

on the cartesian product $\vec{D} = \prod_{i=1}^{n} D_i$ of complete lattices $\langle D_i, \sqsubseteq_i \rangle, i = 1, \ldots, n$, $n \geqslant 1$ ordered pointwise is also, by Tarski's fixpoint Theorem 15.6, the least solution to the inequations $\vec{F}(\vec{X}) \mathrel{\dot{\sqsubseteq}} \vec{X}$ that is

$$\left\{ \begin{array}{l} F_i(X_1, \ldots, X_n) \quad \sqsubseteq_i \quad X_i. \\ \mathrm{i}{=}1, \ldots, \mathrm{n} \end{array} \right.$$

Constraints versus equations

- It is trivial to modify Section **22** to generate such inequations from the program text.
- When the complete lattices $\langle D_i, \sqsubseteq_i \rangle, i = 1, \ldots, n$ are powersets,
  - $\{x\} \subseteq X$ can be rewritten $x \in X$,
  - $X \cup Y \subseteq Z$ as $X \subseteq Z \wedge Y \subseteq Z$,
  - $X \subseteq Y$ as $\forall x \in X \,.\, x \in Y$, *etc*.
- Solving such inequations is a problem that looks like quite different from solving equations iteratively, but it is not!
- Static analysis by constraint resolution fits perfectly in the abstract interpretation framework.

Conclusion

Conclusion on the structural equational semantics

- The understanding of static program analysis as solving fixpoint equations associated to the program (graph) dates back from the origin of dataflow analysis [Allen, 1971; Cocke and Schwartz, 1969, Second revised version, April 1970].
- This point of view is much broader and applies equally well to semantics, verification, and static analysis.
- We give more importance to the functional semantics $\widehat{\mathcal{S}}^{\natural}[\![\mathsf{s}]\!]$ over the equational semantics $\widehat{\mathscr{E}}^{\natural}[\![\mathsf{s}]\!]$ because we prefer to reason on the solution $\widehat{\mathcal{S}}^{\natural}[\![\mathsf{s}]\!]$ of the equations rather than on the equations $\widehat{\mathscr{E}}^{\natural}[\![\mathsf{s}]\!]$.
- The abstract interpreter $\widehat{\mathcal{S}}^{\natural}[\![\mathsf{s}]\!]$ is simpler, can be reasoned upon by structural induction, and does not introduce an intermediate step of building equations before solving them.
- However, $\mathcal{X} = \mathscr{F}^{\natural}(\mathcal{X})$, $\mathscr{F}^{\natural}(\mathcal{X}) \sqsubseteq^{\natural} \mathcal{X}$ and $\mathsf{lfp}^{\sqsubseteq^{\natural}} \mathscr{F}^{\natural}$ are perfectly interchangeable since they only depend on the choice of the abstract domain of Definition 21.1 and its soundness (and completeness) of Definition 27.1.

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Home work

Read Ch. **23** "Abstract equational semantics" of

Principles of Abstract Interpretation
Patrick Cousot
MIT Press

The End, Thank you