Principles of Abstract Interpretation
MIT press

Ch. **37**, Basic linear algebra

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These slides are available at
http://github.com/PrAbsInt/slides/slides-37--linear-algebra-PrAbsInt.pdf

Chapter 37

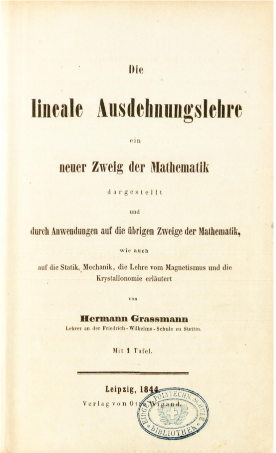
Ch. **37**, Basic linear algebra (1/4)

We have split our review of Chapter **37** into four videos

This first video is about

- fields and vector spaces

The origin of linear algebra



Die

lineale Ausdehnungslehre

ein

neuer Zweig der Mathematik

dargestellt

und

durch Anwendungen auf die übrigen Zweige der Mathematik,

wie auch

auf die Statik, Mechanik, die Lehre vom Magnetismus und die Krystallonomie erläutert

von

Hermann Grassmann

Lehrer an der Friedrich-Wilhelms-Schule zu Stettin.

Mit 1 Tafel.

Leipzig, 1844.

Verlag von Otto Wigand.

[1]

en.wikipedia.org/wiki/Hermann_Grassmann

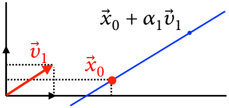
[1] The theory of linear extension, a new branch of mathematics, is presented and explained through applications to other branches of mathematics as well as to structural engineering, mechanics, magnetism, and crystallography.

Introduction

- Encoding values $\rho(\mathsf{x}_1), \ldots, \rho(\mathsf{x}_n)$ of the program variables $\mathsf{x}_1, \ldots, \mathsf{x}_n \in \mathbb{V}$ in environment ρ by mathematical variables $x_1, \ldots, x_n$, we look for linear equality invariants of the form

$$\{\langle x_1, \ \ldots, \ x_n\rangle \mid \bigwedge_{i=1}^{m} a_1^i x_1 + \ldots + a_n^i x_n = b^i\}$$

  where the coefficients $a_1^i, \ldots, a_n^i$ and b^i, $i \in [1, m]$ are discovered by the analysis
- This is a subset of the Euclidean space $\mathbb{R}^n$ (or $\mathbb{Q}^n$) with cartesian coordinates
- This subset can be encoded:
  - by an extended matrix $\left[a_1^i \ \ \ldots \ \ a_n^i \ \middle| \ b^i\right]_{i=1}^{m}$
  - by an affine space *i.e.* a point $\vec{x}_0$ and directions $\langle\vec{v}_1, \ \ldots, \ \vec{v}_k\rangle, 0 \leqslant k \leqslant n$ from a vector space $\overrightarrow{\mathbb{R}}^n$ defining the set of points reachable form $\vec{x}_0$ by following only the directions $\vec{v}_1 \ldots \vec{v}_k$: $\{\vec{x}_0 + \sum_{i=1}^{n} \alpha_i \vec{v}_i | \alpha_i \in \mathbb{R}\}$



Fields

A *Field* $\langle \mathbb{F}, +, -, \times, / \rangle$ is a set of *scalars* $a, b, c, \ldots \in \mathbb{F}$ with $+, -, \times, / \in \mathbb{F}^2 \rightarrow \mathbb{F}$ such that $\forall a, b, c, \ldots \in \mathbb{F}$,

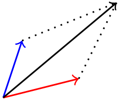
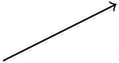
- $a + (b + c) = (a + b) + c$ associativity of addition
- $a \times (b \times c) = (a \times b) \times c$ associativity of multiplication
- $a + b = b + a$ commutativity of addition
- $a \times b = b \times a$ commutativity of multiplication
- $\exists\, 0 \in \mathbb{F} \,.\, a + 0 = a$ additive identity
- $\exists\, 1 \in \mathbb{F} \,.\, 1 \neq 0 \wedge a \times 1 = a$ multiplicative identity
- $\forall a \in \mathbb{F} \,.\, \exists -a \in \mathbb{F} \,.\, a + (-a) = 0$ additive inverse
- $\forall a \in \mathbb{F} \setminus \{0\} \,.\, \exists a^{-1} \in \mathbb{F} \,.\, a \times a^{-1} = 1$ multiplicative inverse (also denoted $1/a$ or $\frac{1}{a}$)
- $a \times (b + c) = (a \times b) + (a \times c)$ distributivity of multiplication over addition
- $a - b \triangleq a + (-b)$ definition of subtraction
- $a/b \triangleq a \times b^{-1}$ definition of division

`en.wikipedia.org/wiki/Field_(mathematics)`

Vector spaces

Vector spaces

Vector spaces are sets of vectors $\vec{v} =$  that can be

- added: $\vec{v}_1 =$  , $\vec{v}_2 =$  , $\vec{v}_1 + \vec{v}_2 =$ 
- scaled: $\vec{v}_1 =$  , $3\vec{v}_1 =$ 

Vectors represent *e.g.* a direction of displacement from one point to another or forces in physics.

The zero vector $\vec{0}$ plays the rôle of the origin, its position is irrelevant.

en.wikipedia.org/wiki/Vector_space

Definition of a vector space

A *vector space* $\langle \vec{\mathbb{V}}, \langle \mathbb{F}, +, -, \times, / \rangle, +, -, \times, / \rangle$ (or *linear space*) on a field $\langle \mathbb{F}, +, -, \times, / \rangle$ of scalars $a, b, c, \ldots \in \mathbb{F}$ is a set of *vectors* $\vec{u}, \vec{v}, \vec{w}, \ldots \in \vec{\mathbb{V}}$ equipped with a vector addition $+$ and substraction $-$ and a scalar multiplication $\times$ and division $/$ such that $+$, $- \in \vec{\mathbb{V}}^2 \rightarrow \vec{\mathbb{V}}$, $\times \in \mathbb{F} \times \vec{\mathbb{V}} \rightarrow \vec{\mathbb{V}}$, $/ \in \vec{\mathbb{V}} \times \mathbb{F} \rightarrow \vec{\mathbb{V}}$, and

- $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ associativity of vector addition
- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ commutativity of vector addition
- $\exists \vec{0} \in \vec{\mathbb{V}} \,.\, \vec{0} + \vec{u} = \vec{u}$ zero vector, identity element of addition
- $\forall \vec{u} \in \mathbb{F} \,.\, \exists -\vec{u} \in \vec{\mathbb{V}} \,.\, \vec{u} + (-\vec{u}) = \vec{0}$ additive inverse
- $a \times (b \times \vec{u}) = (a \times b) \times \vec{u}$ mixed associativity
- $1 \times \vec{u} = \vec{u}$ identity element of scalar multiplication
- $a \times (\vec{u} + \vec{v}) = (a \times \vec{u}) + (a \times \vec{v})$
  distributivity of scalar multiplication with respect to vector addition
- $(a + b) \times \vec{u} = (a \times \vec{u}) + (b \times \vec{u})$
  distributivity of scalar multiplication with respect to field addition
- $\vec{u} - \vec{v} \triangleq \vec{u} + (-\vec{v})$ definition of vector subtraction
- $\vec{u}/a \triangleq a^{-1} \times \vec{u}$ definition of vector division

Vector subspace

A *vector subspace* of a vector space $\vec{u}, \vec{v}, \vec{w}, \ldots \in \vec{\mathbb{V}}$ on a field $a, b, c, \ldots \in \mathbb{F}$ is a non-empty subset $\overrightarrow{W}$ of $\vec{\mathbb{V}}$ such that

- $\varnothing \neq \overrightarrow{W} \subseteq \vec{\mathbb{V}}$
- $\forall \vec{u}, \vec{v} \in \overrightarrow{W} \,.\, \vec{u} + \vec{v} \in \overrightarrow{W}$ closure by vector addition
- $\forall \vec{u} \in \overrightarrow{W} \,.\, \forall a \in \mathbb{F} \,.\, a \times \vec{u} \in \overrightarrow{W}$ closure by scalar multiplication

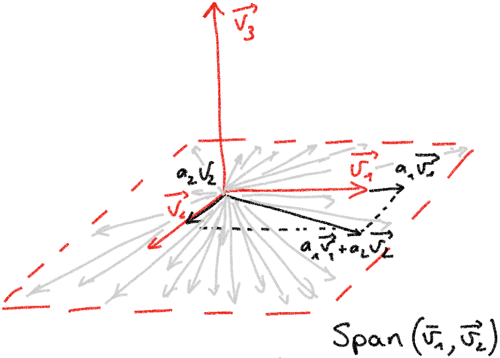
It follows that the zero vector belongs to the vector subspace $\vec{0} \in \overrightarrow{W}$ as well as the additive inverses $\forall \vec{u} \in \overrightarrow{W} \,.\, -\vec{u} \in \overrightarrow{W}$ so that the subspace $\overrightarrow{W}$ is itself a vector space $\langle \overrightarrow{W}, \langle \mathbb{F}, +, -, \times, / \rangle, +, -, \times, / \rangle$.

en.wikipedia.org/wiki/Linear_subspace

Span, Section **37.2.3**

Given a family $\overrightarrow{W} = \langle \vec{v}_i,\ i \in \Delta\rangle$ of vectors in the vector space $\mathbb{V}$, the *span* or *linear hull* or set of all *linear combinations* of these vectors $\overrightarrow{W}$ is

$$\begin{aligned}\mathsf{Span}(\varnothing) &\triangleq \{\vec{0}\}\\ \mathsf{Span}(\langle \vec{v}_i,\ i \in \Delta\rangle) &\triangleq \Big\{\sum_{i\in\Delta} a_i \times \vec{v}_i \;\Big|\; \forall i \in \Delta \,.\, a_i \in \mathbb{F}\Big\}\end{aligned}$$



Subspace generated by a family of vectors

The vector addition and scalar multiplication of linear combinations is a linear combination so the linear combinations are a subspace of the vector space.

$$\mathrm{Span}(\langle \vec{v}_i,\ i \in \Delta\rangle) \text{ is a subspace of the vector space } \vec{\mathbb{V}}.$$

en.wikipedia.org/wiki/Linear_subspace

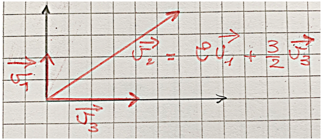
Linear independence

- A non-empty family $\langle \vec{v}_i,\ i \in \Delta\rangle$ of vectors is *free* (*i.e.* its vectors are *linearly independent*) if and only if the only linear combination equal to the zero vector has all scalar coefficients equal to zero *i.e.*

$$\forall \langle a_i,\ i \in \Delta\rangle \ .\ \Big(\sum_{i \in \Delta} a_i \times \vec{v}_i = \vec{0}\Big) \Rightarrow \big(\forall a_i \in \Delta \ .\ a_i = 0\big)$$

  Moreover $\varnothing$ is linearly independent.

- Otherwise, the vectors are *linearly dependent* *i.e.* one vector in the family can be defined as a linear combination of the others.

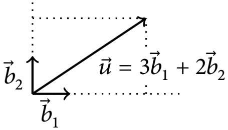


$\{\vec{v}_1, \vec{v}_2\}$, $\{\vec{v}_1, \vec{v}_3\}$, $\{\vec{v}_2, \vec{v}_3\}$ linearly independent, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly dependent

en.wikipedia.org/wiki/Linear_independence

Basis of a vector space

- A *basis* (or *linear basis*) of a vector space $\vec{\mathbb{V}}$ is a set $\langle \vec{b}_i,\ i \in \Delta \rangle$ of linearly independent vectors that generates this vector space $\mathrm{Span}(\langle \vec{b}_i,\ i \in \Delta \rangle) = \vec{\mathbb{V}}$.
- Any vector $\vec{u} \in \vec{\mathbb{V}}$ of the vector space $\vec{\mathbb{V}}$ is then a unique linear combination $\sum_{i \in \Delta} a_i \times \vec{b}_i$ of the base vectors $\langle \vec{b}_i,\ i \in \Delta \rangle$ for coefficients $\prod_{i \in \Delta} a_i$ called the *coordinates* of the vector in this basis.
- For example for the standard basis $\vec{b}_1$ and $\vec{b}_2$, we would have



with coodinates $(3, 2)$.

en.wikipedia.org/wiki/Basis_(linear_algebra)

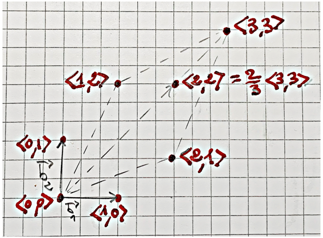
Dimension of a vector space

- All bases of the vector space have the same cardinality called the *dimension* $\dim(\vec{\mathbb{V}})$ of the vector space $\vec{\mathbb{V}}$.
- A vector line has dimension 1
- A vector plane has dimension 2.

`en.wikipedia.org/wiki/Dimension_(vector_space)`

Example of the coordinate vector space

- Vectors are given by their coordinates $\langle a,\ b\rangle$
- The basis is $\{\langle 1,\ 0\rangle, \langle 0,\ 1\rangle\}$



- The vector sum is the sum of the coordinates
- The scalar multiplication is the multiplication of the coordinates

en.wikipedia.org/wiki/Vector_space#Coordinate_space

Example of the coordinate vector space (cont'd)

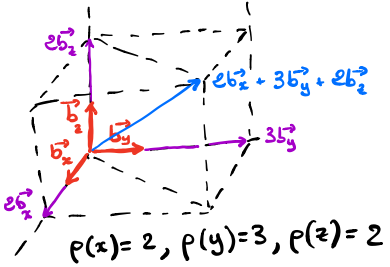
- For any strictly positive integer n, the space

$$\vec{\mathbb{F}}^n \triangleq \{\vec{x} \mid \vec{x} = \langle \vec{x}_1, \ldots, \vec{x}_n \rangle \wedge \forall i \in [0, n] \mathrel{.} \vec{x}_i \in \mathbb{F}\}$$

  of all n-tuples of elements of the field $\mathbb{F}$ is a vector space over $\mathbb{F}$ of dimension n with operations
  - $\vec{x} + \vec{y} \triangleq \langle \vec{x}_1 + \vec{y}_1, \ldots, \vec{x}_n + \vec{y}_n \rangle$ vector addition
  - $a \times \vec{x} \triangleq \langle a \times \vec{x}_1, \ldots, a \times \vec{x}_n \rangle$ scalar multiplication
  - $\vec{0} = \langle 0, \ldots, 0 \rangle$ zero vector
  - $-\vec{x} = \langle -\vec{x}_1, \ldots, -\vec{x}_n \rangle$ additive inverse
- The *standard (or canonical) basis* is $\{\vec{b}_i \mid i \in [1, n] \wedge \vec{b}_i(i) = 1 \wedge \forall j \in [1, n] \setminus \{i\} \mathrel{.} \vec{b}_i(j) = 0\}$ where 0 is the additive identity and 1 is the multiplicative identity in $\mathbb{F}$.
- $\dim(\vec{\mathbb{F}}^n) = n$.

Example of the coordinate vector space (cont'd)

- Typically in program analysis, $\mathbb{F}$ is $\mathbb{Q}$ or $\mathbb{R}$ and an element $\langle \vec{x}_1, \ldots, \vec{x}_n \rangle$ of $\vec{\mathbb{F}}^n$ is the vector of values $\rho(\mathsf{x})$ of the n variables $\mathsf{x} \in \mathbb{V}$ of the program.



- If a family of vectors $\langle \vec{v}_i,\ i \in \Delta \rangle$ of a vector space $\vec{\mathbb{V}}$ has a cardinality greater than the dimension $\dim(\vec{\mathbb{V}})$ of the vector space then these vectors are linearly dependent.

This concludes our study of

- vector spaces

from Chapter **37** (Basic linear algebra)

The End

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Chapter 37

Ch. **37**, Basic linear algebra (2/4)

In this second video, we study

- the resolution of systems of linear equations

Solving systems of linear equations

Systems of linear equations

- The affine or linear equality analysis of programs in next Chapter **38**, aims at automatically discovering a conjunction of linear equalities of the form

$$a_1 \times x_1 + \ \ldots + a_j \times x_j + \ \ldots + a_n \times x_n = b$$

  among the values $x_1, \ldots, x_n$ of the numerical variables $\mathtt{x}_1, \ldots, \mathtt{x}_n$ of the program where the coefficients $a_1, \ldots, a_j, \ldots, a_n, b$ are constants automatically inferred by the analysis.
- This motivates the study of system of linear equations, matrices, and affine spaces.

`en.wikipedia.org/wiki/System_of_linear_equations`

Definition of a system of linear equations

- A system of m linear equations in n *unknowns* x_1, …, x_n has the form

$$\left\{\begin{array}{ccccccc} a_{11}\times x_1 + & \ldots & +a_{1j}\times x_j + & \ldots & +a_{1n}\times x_n & = & b_1 \\ a_{21}\times x_1 + & \ldots & +a_{2j}\times x_j + & \ldots & +a_{2n}\times x_n & = & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \\ a_{m1}\times x_1 + & \ldots & +a_{mj}\times x_j + & \ldots & +a_{mn}\times x_n & = & b_n \end{array}\right.$$

  where the *coefficients* a_{ij}, $i \in [1, m]$, $j \in [1, n]$, b_j, $j \in [1, n]$, and the unknowns x_j, $j \in [1, n]$ all belong to a field $\mathbb{F}$.
- It is understood as a conjunction of the linear constraints on the unknown x_1, …, x_n.

Example of system of linear equations

- For example the system of linear equations

$$\left\{\begin{array}{rcrcr} 2x & + & 3y & = & 0 \\ -x & + & y & = & -1 \end{array}\right.$$

has solution $x_0 = 3/5$ and $y_0 = -2/5$

$$\left\{\begin{array}{rcrcrcrcrcr} 2x_0 & + & 3y_0 & = & 2 \times 3/5 & + & 3 \times -2/5 & = & 6/5 - 6/5 & = & 0 \\ -x_0 & + & y_0 & = & -3/5 & + & -2/5 & = & -5/5 & = & -1 \end{array}\right.$$

Matrix notation, Section **37.3.2**

- linear equations can be written as a single *matrix equation* $\mathbf{A} \times \vec{x} = \vec{b}$ or $\mathbf{A} \times \vec{x} - \vec{b} = \vec{0}$, by defining
- the *matrix* $\mathbf{A} = (a_{ij})_{\substack{i=1,m \\ j=1,n}} \in \mathbb{F}^{m \times n}$ over a field $\mathbb{F}$ (also denoted $(a_{ij})^{m \times n}$) is the array of numbers $a_{ij} \in \mathbb{F}$ with m rows and n columns

$$\mathbf{A} = \begin{bmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2j} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}$$

- The $n \times 1$ *column vectors* in $\mathbb{F}^n$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix} \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

- The *multiplication* of a matrix $\mathbf{A}$ by a vector $\vec{x}$, $\mathbf{A} \times \vec{x} =$

$$\begin{bmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2j} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + & \ldots & +a_{1j}x_j + & \ldots & +a_{1n}x_n \\ a_{21}x_1 + & \ldots & +a_{2j}x_j + & \ldots & +a_{2n}x_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1}x_1 + & \ldots & +a_{mj}x_j + & \ldots & +a_{mn}x_n \end{bmatrix}$$

(we omit the multiplication operator writing ab and $\mathbf{A}\vec{b}$ and $a \times b$ and $\mathbf{A} \times \vec{b}$).

- The difference of column vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \ldots \\ x_n - y_n \end{bmatrix}$$

- The system $\mathbf{A} \times \vec{x} = \vec{b}$ of equations is *homogeneous* if $\vec{b} = \vec{0}$, *inhomogeneous* otherwise.
- If $\mathbf{A} = (a_{ij})_{\substack{i=1,m \\ j=1,n}} \in \mathbb{F}^{m \times n}$, $\mathbf{B} = (a_{ij})_{\substack{i=1,m \\ j=1,n}} \in \mathbb{F}^{m \times n}$, and $c \in \mathbb{F}$, we also define
  - the matrix addition $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})_{\substack{i=1,m \\ j=1,n}}$,
  - the matrix difference $\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})_{\substack{i=1,m \\ j=1,n}}$, and
  - the matrix multiplication by the scalar $c \in \mathbb{F}$, $c\mathbf{A} = (c \times a_{ij})_{\substack{i=1,m \\ j=1,n}}$.

en.wikipedia.org/wiki/Matrix_(mathematics)

Solving a system of linear equations

- A system of linear equations $\mathbf{A}\vec{x} = \vec{b}$ may have 0, 1 or many solutions.
- The Gauss-Jordan elimination algorithm can be used to effectively compute them.

Basic row transformations

- The system $\mathbf{A}\vec{x} = \vec{b}$ is augmented to $(\mathbf{A}|\vec{b})$ by adding the extra column $\vec{b}$ and put in reduced row echelon form by Gauss–Jordan elimination, with three *basic row transformations*:
  (1) the swap of two rows,
  (2) the multiplication of a row by a non-zero scalar $a \in \mathbb{F} \setminus \{0\}$,
  (3) addition of a multiple of one row to another different row.
- These basic row transformations of $(\mathbf{A}|\vec{b})$ do not change the solutions to $\mathbf{A}\vec{x} = \vec{b}$.

Row echelon form

- A matrix is in row echelon form if and only if
  - all nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (so all zero rows, if any, belong at the bottom of the matrix), and
  - the leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always equal to 1 and strictly to the right of the leading coefficient of the row above it.
- These two conditions imply that all entries in a column below a pivot are zeros.
- For example $\begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{2}{5} \end{bmatrix}$ is in reduced row echelon form.
- Backward substitution yields the solution $y = -\frac{2}{5}$ and replacing y by its value in the first equation $x + \frac{3}{2}(-\frac{2}{5}) = 0$, we get $x = \frac{3}{5}$.

https://en.wikipedia.org/wiki/Row_echelon_form

- Carl Friedrich Gauss [Gavss, 1811, pp. 21–25] showed that by successive applications of the basic row transformations, the augmented matrix can be transformed in *row echelon form*[2].
- **Example**: Assume that $\mathbf{A}\vec{x} = \vec{b}$ is

$$\begin{cases} 2x & + & 3y & = & 0 \\ -x & + & y & = & -1 \end{cases}$$

and must be put in row echelon form. The augmented matrix is

$$\begin{bmatrix} 2 & 3 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

[2] See the history of this algorithm in Joseph F. Grcar: How ordinary elimination became Gaussian elimination. *Historia mathematica* 38 (2010) 163-218. `https://arxiv.org/abs/0907.2397`

$$\begin{bmatrix} 2 & 3 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

Dividing the first row by 2.

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

Adding the first row to the second.

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & \frac{5}{2} & -1 \end{bmatrix}$$

Dividing the second row by $\frac{5}{2}$.

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{2}{5} \end{bmatrix}$$

which yields $y = -\frac{2}{5}$ and $x = \frac{3}{5}$ as shown above.

Reduced row echelon form

- Wilhelm Jordan [Jordan, 1888, pp. 62–83] showed that by successive applications of the basic row transformations, the augmented matrix can be transformed in *reduced row echelon form*.
- A matrix is in reduced row echelon form if and only if it is in row echelon form and for each column vector containing a pivot, this pivot is the only non-zero element in the column.
- The above $\begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{2}{5} \end{bmatrix}$ is not in reduced form but $\begin{bmatrix} 1 & 0 & \frac{3}{5} \\ 0 & 1 & -\frac{2}{5} \end{bmatrix}$ is.
- The solution is obtained directly without needing a final backward substitution.
- More importantly (and unlike the row echelon form), the reduced row echelon form of a matrix is unique so provides a canonical form.

`en.wikipedia.org/wiki/Gaussian_elimination`

Example of Gauss-Jordan elimination

For the augmented matrix

$$\begin{bmatrix} 2 & 3 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

the first step locates the pivot at $A(0,0)$, divides the first row by 2 to set the pivot to 1

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

and then nullifies the column below $A(0,0)$ by adding the first to the second row.

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & \frac{5}{2} & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & \frac{5}{2} & -1 \end{bmatrix}$$

The second step locates the pivot at $A(1,1)$, divides the second row by $\frac{5}{2}$ to set the pivot to 1

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{2}{5} \end{bmatrix}$$

and then nullifies the column above $A(1,1)$ by subtracting the second row multiplied by $\frac{3}{2}$ from the first row.

$$\begin{bmatrix} 1 & 0 & \frac{3}{5} \\ 0 & 1 & -\frac{2}{5} \end{bmatrix}$$

□

Gauss-Jordan elimination algorithm, Section **37.4.4**

- Assume that $A(n \times m)$ is the augmented matrix with n rows and m columns.
- After the $i-1^{\text{th}}$-step $(i = 0, \ldots, n-1)$, the Gauss-Jordan elimination algorithm [Althoen and McLaughlin, 1987] transforms the matrix A in the following form

$$\begin{array}{cc}
 & \begin{array}{ccccccccccc} & & & & & & j & & & & \end{array} \\
\begin{array}{c} \\ \\ \\ \text{i-1} \\ \text{i} \\ \\ \\ \end{array} &
\left[\begin{array}{ccccccccccc}
1 & * & 0 & * & \ldots & 0 & * & * & * & * & * \\
 & & 1 & * & \ldots & 0 & * & * & * & * & * \\
 & & & & \ldots & 0 & & \ldots & \ldots & \ldots & \\
 & & & & & 1 & * & * & * & * & * \\
 & \mathbf{0} & & & & & * & * & * & * & * \\
 & & & & & & * & * & * & * & * \\
 & & & & & & * & * & * & * & *
\end{array}\right]
\end{array}$$

where the pivots are 1 and unique non-zero element in their column and $*$ represents an arbitrary value.

- The i^{th}-step is the following [Isaacson and Keller, 1994, p. 50].
  (1) Find the first column j containing a non-zero element at row k, at or below row i.
  If no such j exists, the bottom right of the matrix is 0 and we are done.

  (2) If $k \neq i$, exchange rows k and i so that the pivot at $A(i, j)$ is non-zero.

  (3) Set the pivot to 1 by dividing line i by $A(i, j)$.

  (4) Set the column j above and below the pivot to 0 by updating each row $A(\ell, k)$, $\ell \in [0, n-1] \setminus \{i\}$, $k \in [0, m-1]$ to $A(\ell, k) - A(i, k)\frac{A(\ell, j)}{A(i, j)}$.

en.wikipedia.org/wiki/Gaussian_elimination#Finding_the_inverse_of_a_matrix

Theorem (37.9) After elimination of the zero rows, a system of linear equations $\mathbf{A}\vec{x} = \vec{b}$ in reduced row echelon form has solutions if and only if (1) there is no row $\vec{0}\vec{b}_\ell$, and $\vec{b}_\ell \neq 0$ (2) the number of non-zero rows is less than or equal to the number of variables $\vec{x}_i$.

Proof of Theorem 37.9 Let $(\mathbf{A}|\vec{b})$ be the reduced row echelon form of a system of linear equations. Assume $\mathbf{A}$ has a zero row $\mathbf{A}_{i,j=0,\ldots,m}$. Then either $\vec{b}_i = 0$ and $\vec{0}\vec{x} = 0$ is satisfied by any $\vec{x}$ so his row can be eliminated without changing the solutions or $\vec{b}_i \neq 0$ and then there is no solution. Else $\mathbf{A}$ has no zero line. By def. of the reduced row echelon form, all rows start with a pivot 1, and these pivots are the only non-zero in a column and strictly to the right of the pivot of the row above. It follows that there are less lines than columns since otherwise a line would have two pivots, which is impossible. $\square$

This concludes our study of

- the resolution of systems of linear equations

from Chapter **37** (Basic linear algebra)

The End

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Ch. **37**, Basic linear algebra

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These slides are available at
http://github.com/PrAbsInt/slides/slides-37--linear-algebra-PrAbsInt.pdf

Chapter 37

Ch. **37**, Basic linear algebra (3/4)

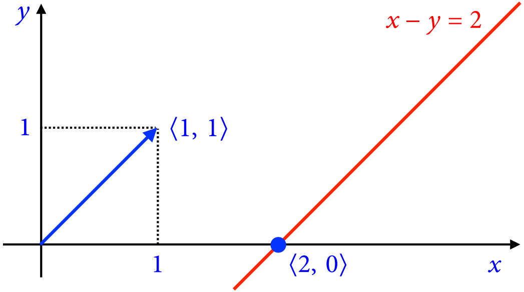
In this third video, we study

- the representation of the solutions of systems of linear equations

Representation of the solutions of a system of linear equations

Representation of the solution set of a system of linear equations

- The solutions $\gamma_{\equiv}(\mathbf{A}|\vec{b})$ of $\mathbf{A}\vec{x} = \vec{b}$ can be represented by a frame (or system of generators) consisting of a point $\vec{x}_0$ (which is any solution to $\mathbf{A}\vec{x} = \vec{b}$) and a vector space $\mathsf{Ker}(\mathbf{A})$ (called the kernel of $\mathbf{A}$).
- For example the solutions to $x - y = 2$ in $\mathbb{R}^2$ can be represented by the point $\langle 2, 0\rangle$ and the vector space generated by the vector $\langle 1, 1\rangle$.



Kernel of a matrix

- The *kernel* or *nullspace* $\mathsf{Ker}(\mathbf{A})$ of a matrix $\mathbf{A}$ is the set of solutions of the system of equations $\mathbf{A}\vec{x} = \vec{0}$.

$$\mathsf{Ker}(\mathbf{A}) \quad \triangleq \quad \{\vec{x} \in \mathbb{F}^m \mid \mathbf{A}\vec{x} = \vec{0}\}$$

- If $\vec{x}_1, \vec{x}_2 \in \mathsf{Ker}(\mathbf{A})$ and $a \in \mathbb{F}$ then $\vec{x}_1 + \vec{x}_2 \in \mathsf{Ker}(\mathbf{A})$ and $a\vec{x}_1 \in \mathsf{Ker}(\mathbf{A})$ and $\vec{0} \in \mathsf{Ker}(\mathbf{A})$

- Therefore $\mathsf{Ker}(\mathbf{A})$ is a non-empty vector subspace of the coordinate vector space $\langle \vec{\mathbb{F}}^n, \langle \mathbb{F}, +, -, \times, / \rangle, +, -, \times, / \rangle$

en.wikipedia.org/wiki/Kernel_(linear_algebra)

Basis of the kernel of a matrix

- To find a basis of $\text{Ker}(\mathbf{A})$, we establish the equation

$$a_1\mathbf{A}_{i=1,n,1} + a_2\mathbf{A}_{i=1,n,2} + \ldots + a_m\mathbf{A}_{i=1,n,m} = 0$$

  to find which vectors are independent.
- Then we put $(\mathbf{A}|\vec{0})$ in reduced echelon form $\mathbf{A}'$ by Gauss-Jordan algorithm.
- The pivot column vectors $\mathbf{A}'_{i=1,n,j}$ of $\mathbf{A}'$ are those with a 1 in the column with zeros below and to the left of this 1.
- The other non-pivot columns $\mathbf{A}'_{i=1,n,k}$ of $\mathbf{A}'$ are called *free*.
- The variables corresponding to the pivot columns depends upon the variables corresponding to the free columns.
- The variables corresponding to the free columns depends upon no other variable.
- So the free column vectors form a basis.

Example I

- Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & -4 & 0 \\ 1 & 0 & 1 & -3 & -1 \\ 1 & 0 & -1 & 1 & -1 \end{bmatrix}$.
- It's kernel is the solution to $\mathbf{A}\vec{x} = \vec{0}$.
- The reduced row echelon form is $\mathbf{A}' = \begin{bmatrix} \boxed{1} & 0 & 0 & -1 & -1 \\ 0 & \boxed{1} & 0 & -1 & 1 \\ 0 & 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

  (the 1s in the pivot columns have been framed).
- The kernel of $\mathbf{A}$ consists of the solutions of $\mathbf{A}'\vec{x} = \vec{0}$ that is
$$\left\{ \begin{array}{ccccccc} x_1 & & & -x_4 & -x_5 & = & 0 \\ & x_2 & & -x_4 & +x_5 & = & 0 \\ & & x_3 & -2x_4 & & = & 0 \end{array} \right. .$$

Example II

- This is the set of vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} a+b \\ a-b \\ 2a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

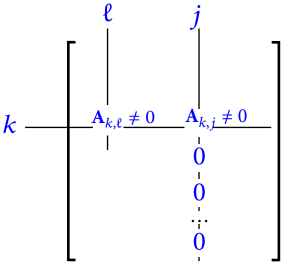
  where the values $a, b \in \mathbb{Q}$ of x_4 and x_5 are arbitrary (there is no corresponding pivot in the reduced row echelon form).

- So $\mathrm{Ker}(\mathbf{A}) = \mathrm{Span}(\langle \vec{b}_4, \vec{b}_5 \rangle)$ for the basis $\vec{b}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{b}_5 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Basis of the kernel of a matrix I

Let $\mathbf{A}_{i=1,n,j}$ be the j^{th} column of matrix $\mathbf{A}$ in reduced row echelon form. This column is *free* if and only if

- it is a zero column ($\forall i \in [1, n] \,.\, \mathbf{A}_{i,j} = 0$), or else
- let $k = max\{i \in [1, n] \mid \mathbf{A}_{i,j} \neq 0\}$ be the row of the last non-null element $\mathbf{A}_{k,j} \neq 0$ in this column j, then there is a non-null element in this row k strictly before column j *i.e.* $\exists \ell \in [0, j[\,.\, \mathbf{A}_{k,\ell} \neq 0$.



$\mathbf{A}_{i=1,n,j}$ is a free column

Basis of the kernel of a matrix II

Lemma (37.12) Let $\{\mathbf{A}'_{i=1,n,j} \mid j \in F\}$ be the free columns of the reduced row echelon form $\mathbf{A}'$ of the matrix $\mathbf{A}$.
Define the column vectors $\vec{b}^j$, $j \in F$ such that $\vec{b}^j_j = 1$, $\vec{b}^j_i = -\mathbf{A}'_{i,j}$, $i \in [1,n] \setminus \{j\}$, and $\vec{b}^j_i = 0$, $i \in]n,m] \setminus \{j\}$.
Then $\{\vec{b}^j \mid j \in F\}$ is a basis of $\mathsf{Ker}(\mathbf{A})$.
Each $\vec{b}^j$ is the solution of the linear equations when fixing $x_j = 1$ and $x_k = 0$ for $k \in F \setminus \{j\}$.

The variables corresponding to the free columns are called *free* while the others, corresponding to pivot columns, are called *principal*.

Solution set of a system of linear equations

The *concretization* or *solution set* $\gamma_{\equiv}(\mathbf{A}|\vec{b})$ of a system of linear equations $\mathbf{A}\vec{x} = \vec{b}$ where $\mathbf{A} \in \mathbb{F}^{m\times n}$ and $\vec{b} \in \mathbb{F}^{m}$ is the set of all its possible solutions (which may be empty, a singleton, ..., infinite).

$$\gamma_{\equiv}(\mathbf{A}|\vec{b}) \quad \triangleq \quad \{\vec{x} \in \mathbb{F}^{m} \mid \mathbf{A}\vec{x} = \vec{b}\}$$

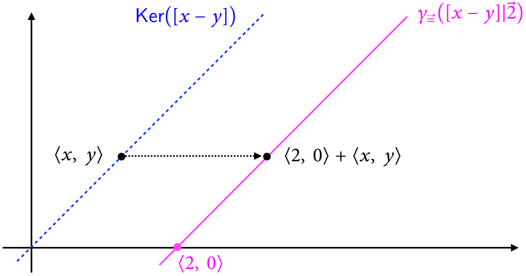
Lemma (37.14) If $\vec{x}_0 \in \mathbb{F}^{m}$ is any solution *i.e.* $\mathbf{A}\vec{x}_0 = \vec{b}$ then

$$\gamma_{\equiv}(\mathbf{A}|\vec{b}) \quad = \quad \{\vec{x}_0 + \vec{x} \mid \vec{x} \in \mathrm{Ker}(\mathbf{A})\}$$

□

Example

The following picture illustrates Lemma 37.14 on $x - y = 2$.



We now have two (equivalent) ways to represent the set $\gamma_{\equiv}(\mathbf{A}|\vec{b})$.

- by the augmented matrix $(\mathbf{A}|\vec{b})$, or
- by a so-called a *frame* (or *system of generators*) $\langle \vec{x}_0, \mathsf{Ker}(\mathbf{A})\rangle$.

This concludes our study of

- the representation of the solutions of systems of linear equations

from Chapter **37** (Basic linear algebra)

The End

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Chapter 37

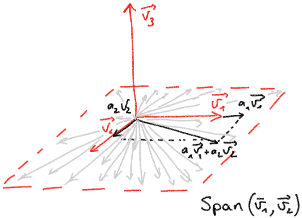
Ch. **37**, Basic linear algebra (4/4)

In this fourth video, we study

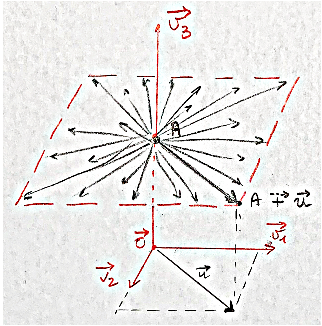
- affine spaces

Affine spaces

Affine space versus affine space



Vector space



Affine space

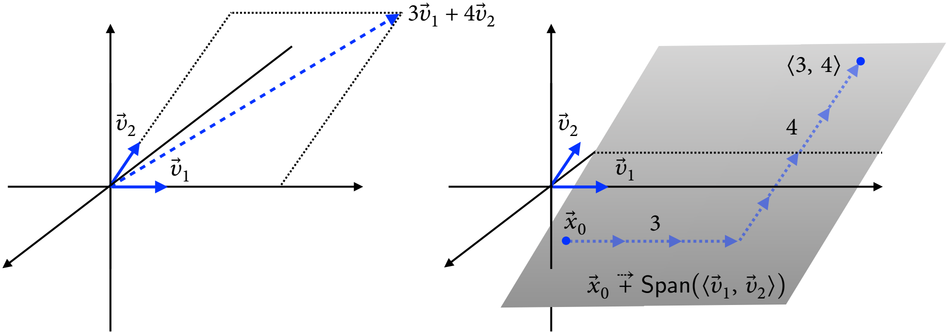
Affine spaces

- An affine space is a vector space which origin has been forgotten!
- An affine space $\mathbb{A}$ on a vector space $\vec{\mathbb{V}}$ with field $\mathbb{F}$ is a set of points equipped with a translation operation which given a point $A \in \mathbb{A}$ and a vector $\vec{u} \in \vec{\mathbb{V}}$ defines the point $A \overset{\dashrightarrow}{+} \vec{u}$ translated of point A by vector $\vec{u}$ $\quad A \bullet \xrightarrow{\vec{u}} \bullet\, A \overset{\dashrightarrow}{+} \vec{u}$.
- The translation aims at forgetting about the origin $\vec{0}$ of the vector space $\vec{\mathbb{V}}$ in that the translation of any point of $\mathbb{A}$ is a point of $\mathbb{A}$ so translating the origin does not matter, a way to forget about it!
- Without origin, the addition of vectors become meaningless in an affine space.

en.wikipedia.org/wiki/Affine_space

Example

- However, the position of any point in the affine space can be uniquely expressed by a linear combination of vectors $\langle \vec{v}_1, \vec{v}_2, \ldots \rangle$ (of a vector space) relative to a fixed point $\vec{x}_0$.
- The difference between a vector space and an affine space is illustrated below (the choice of the origin point O of coordinates $\vec{x}_0$ on the plane is arbitrary).



Definition of an affine space (Section **37.6.1**)

An *affine space* $\langle \mathbb{A}, \langle \vec{\mathbb{V}}, \langle \mathbb{F}, +, -, \times, / \rangle, +, -, \times, / \rangle, \vec{+} \rangle$ on a vector space $\langle \vec{\mathbb{V}}, \langle \mathbb{F}, +, -, \times, / \rangle, +, -, \times, / \rangle$ with field $\langle \mathbb{F}, +, -, \times, / \rangle$ is a set of *points* $A, B, C, \ldots \in \mathbb{A}$ equipped with a *translation* operation $A \vec{+} \vec{u}$ of a point A by vector $\vec{u}$ such that for all $\vec{u}, \vec{v}, \vec{w}, \ldots \in \vec{\mathbb{V}}$ and $a, b, c, \ldots \in \mathbb{F}$,

- $\vec{+} \in \mathbb{A} \times \vec{\mathbb{V}} \to \mathbb{A}$,
- $\forall A \in \mathbb{A} \,.\, \forall \vec{u}, \vec{v} \in \vec{\mathbb{V}} \,.\, (A \vec{+} \vec{u}) \vec{+} \vec{v} = A \vec{+} (\vec{u} + \vec{v})$
- $\forall A, B \in \mathbb{A} \,.\, \exists! \vec{u} \in \vec{\mathbb{V}} \,.\, A \vec{+} \vec{u} = B$

The zero vector $\vec{0}$ is the only vector such that $\forall A \in \mathbb{A} \,.\, A \vec{+} \vec{0} = A$.

Affine coordinates

- A *coordinate system* or *frame* or *system of generators* of an affine space $\mathbb{A}$ on a vector space $\vec{\mathbb{V}}$ with field $\mathbb{F}$ is a point $O \in \mathbb{A}$ (called the *origin*) and a basis $\langle \vec{b}_i, i \in \Delta \rangle$ of the vector space $\vec{\mathbb{V}}$.
- A point can be uniquely written in the form

$$A = O \vec{+} \sum_{i \in \Delta} a_i \times \vec{b}_i$$

where the $\langle \vec{a}_i, i \in \Delta \rangle$ are the *affine coordinates* of A.

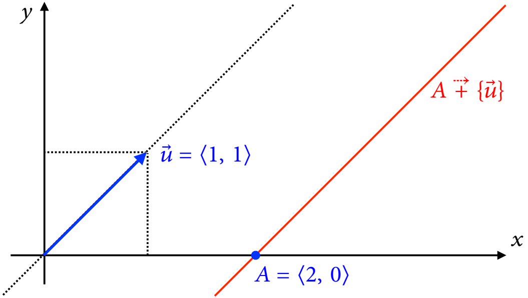
en.wikipedia.org/wiki/Affine_space

Affine subspace, Section **37.6.3**

- An affine subspace of a vector space is a translation of a linear subspace.
- An *affine subspace* of an affine space $A, B, C \ldots \in \mathbb{A}$ on a vector space $\vec{u}, \vec{v}, \vec{w}, \ldots \in \vec{\mathbb{V}}$ with field $a, b, c, \ldots \in \mathbb{F}$ is a subset $\mathbb{B}$ of $\mathbb{A}$ of the form

$$A \mathbin{\vec{+}} \vec{U} \triangleq \{A \mathbin{\vec{+}} \vec{u} \mid \vec{u} \in \vec{U}\}$$

where $A \in \mathbb{A}$ and $\vec{U}$ is a vector subspace of $\vec{\mathbb{V}}$.



- An affine subspace of an affine space is itself an affine space for the operations of this affine space.
- Moreover the intersection $\bigcap_{i\in\Delta} \mathbb{B}_i$ of affine subspaces $\langle\mathbb{B}_i,\ i \in \Delta\rangle$ of an affine space $\mathbb{A}$ is an affine subspace of $\mathbb{A}$.

Example of the coordinate affine space, Section **37.6.4**

- Continuing Section **37.2.8**, consider the affine space

$$\langle \mathbb{F}^n, \langle \vec{\mathbb{F}}^n, \langle \mathbb{F}, +, -, \times, / \rangle, +, -, \times, / \rangle, \vec{+} \rangle$$

of finite dimension $n > 0$ with

$$A \vec{+} \vec{u} \quad = \quad \langle A_1, \ldots, A_n \rangle \vec{+} \langle \vec{u}_1, \ldots, \vec{u}_n \rangle \quad \triangleq \quad \langle A_1 + \vec{u}_1, \ldots, A_n + \vec{u}_n \rangle$$

Lemma (37.15) If $\vec{x}_0 \in \mathbb{F}^m$ is any solution *i.e.* $\mathbf{A}\vec{x}_0 = \vec{b}$ then

$$\gamma_{\equiv}(\mathbf{A}|\vec{b}) \quad = \quad \vec{x}_0 \vec{+} \mathsf{Ker}(\mathbf{A})$$

- So an affine subspace is the solution set of an inhomogeneous linear system and inversely.

- Since $\mathsf{Ker}(\mathbf{A})$ is a vector space of dimension n, it has a basis, and so we have

> **Lemma (37.16)** If $\vec{x}_0 \in \mathbb{F}^m$ is any solution *i.e.* $\mathbf{A}\vec{x}_0 = \vec{b}$ and $\langle \vec{v}_i,\ i \in [1, n] \rangle$ is a basis of $\mathsf{Ker}(\mathbf{A})$ then
>
> $$\gamma_{\vec{\equiv}}(\mathbf{A}|\vec{b}) \quad = \quad \vec{x}_0 \ \vec{+}\ \mathsf{Span}(\langle \vec{v}_i,\ i \in [1, n] \rangle)$$

Machine representation of subspaces of the coordinate affine space, Section **37.6.5**

In the affine space $\langle \mathbb{F}^n, \langle \vec{\mathbb{F}}^n, \langle \mathbb{F}, +, -, \times, /\rangle, +, -, \times, /\rangle, \vec{+}\rangle$ of finite dimension $n > 0$, we can represent an affine subspace

- by a *system of equalities* represented by the pair $(\mathbf{A}|\vec{b})$ of a matrix $\mathbf{A} \in \mathbb{F}^{m\times n}$ and vector $\vec{b} \in \mathbb{F}^n$ encoding the affine subspace $\gamma_{\equiv}(\mathbf{A}|\vec{b})$.

  The matrix $\mathbf{A}$ is in reduced row-echelon form;

- by a *frame* or *system of generators* $\langle \vec{x}_0, \langle \vec{v}_i, i \in [1,n]\rangle\rangle$ consisting of the coordinates $\vec{x}_0$ of a point O and a basis $\langle \vec{v}_i, i \in [1,n]\rangle$ of a vector space (precisely $\mathsf{Ker}(\mathbf{A})$) encoding the affine space $\gamma_{\equiv}(\langle \vec{x}_0, \langle \vec{v}_i, i \in [1,n]\rangle\rangle) = \vec{x}_0 \vec{+} \mathsf{Span}(\langle \vec{v}_i, i \in [1,n]\rangle)$.

Matrix representation of an affine coordinate subspace

- Let W be an affine subspace of the coordinate affine space given by a system of generators $W = \vec{x}_0 \dot{\vec{+}} \mathrm{Span}(\mathbf{B})$ where $\mathbf{B} = \langle \mathbf{B}_i,\ i \in [1,n] \rangle$ is a basis.
- We look for a matrix $(\mathbf{A}|\vec{b})$ which solution set is W.
- We have $\vec{x} \in W$ if and only if $\exists \lambda_1, \ldots, \lambda_n \,.\, \vec{x} = \lambda_1 \mathbf{B}_1 + \ldots + \lambda_n \mathbf{B}_n + \vec{x}_0$ that is $\vec{x} = \lambda_1 \mathbf{B}_1 + \ldots + \lambda_n \mathbf{B}_n - \vec{x} + \vec{x}_0 = \vec{0}$ or in matrix form

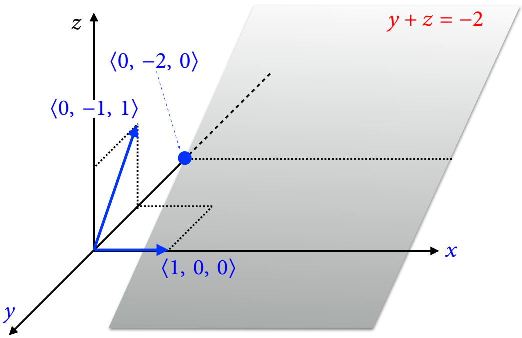
$$
(\mathbf{B}, \left\{ \begin{array}{cccc} -1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & -1 \end{array} \right\}, \vec{x}_0) \left\{ \begin{array}{c} \lambda_1 \\ \ldots \\ \lambda_n \\ \vec{x}_1 \\ \ldots \\ \vec{x}_n \\ 1 \end{array} \right\} = \vec{0}
$$

$$
\left(\mathbf{B}, \left\{\begin{array}{cccc} -1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & -1 \end{array}\right\}, \vec{x}_0\right) \left\{\begin{array}{c} \lambda_1 \\ \ldots \\ \lambda_n \\ \vec{x}_1 \\ \ldots \\ \vec{x}_n \\ 1 \end{array}\right\} = \vec{0}
$$

- Transforming this matrix in reduce row echelon form yields the λ_i in function of the free $\vec{x}$.
- The matrix $(\mathbf{A}|\vec{b})$ is given by the rows with zero coefficients for the λ_i in the first n columns.
- They put constraints on $\vec{x}$ for the solution to exist *i.e.* for $\vec{x} \in W$.

Example

- Consider the affine space



generated by $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$.

- $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$.
- The equation matrix is $(\mathbf{B}, \overrightarrow{-1}, \vec{x}_0) = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}$
- it is transformed in row echelon form as $\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}$.
- The matrix $(\mathbf{A}|\vec{b})$ is given by the rows with zero coefficients for the λs in the first 2 columns that is the last row $0x + 1y + 1z + 2 = 0$.

Elimination of a variable

- If $\vec{x} \in \mathbb{F}^n$ is a vector of scalars, $i \in [1,n]$, and $v \in \mathbb{F}$, we let $\vec{x}' = \vec{x}[i \leftarrow v]$ be the assignment of v to $\vec{x}$ at position i such that $\forall j \in [1,n] \setminus \{i\} \,.\, \vec{x}'_j = \vec{x}_j$ and $\vec{x}'_i = v$.
- Given an affine space $W = \{\vec{x} \mid \mathbf{A}\vec{x} = \vec{b}\}$, we look for a characterization of the set $W' = \{\vec{x} \mid \exists v \in \mathbb{F} \,.\, \vec{x}[i \leftarrow v] \in W\}$.
- For example $\exists z \,.\, x + z = 0 \wedge y = z + 1$ is $x + y = 1$ so if $W = \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \middle\| \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ then $W' = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle\| \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

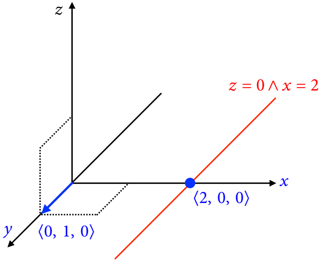
Elimination of a variable

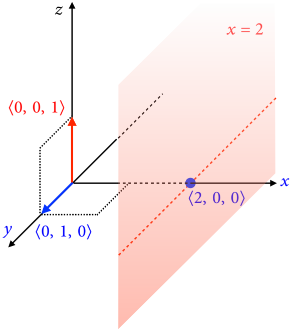
- When considering the system of generators, the elimination of a variable consists in adding a vector to the basis in the dimension of this variable.

Elimination of a variable

- In the example below, $\{\langle x,\ y,\ z\rangle \mid z = 0 \wedge x = 2\}$ is generated by $\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \rangle$

  while $\{\langle x,\ y,\ z\rangle \mid \exists z \,.\, z = 0 \wedge x = 2\} = \{\langle x,\ y,\ z\rangle \mid x = 2\}$ is generated by $\langle \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \rangle$.





Elimination of a variable

- More generally,

Lemma (37.19) If W is generated by $\langle \mathbf{B},\ \vec{x}_0\rangle$ then $\{\vec{x} \mid \exists v \in \mathbb{F} \mathrel{.} \vec{x}[i \leftarrow v] \in W\}$ is generated by $\langle (\mathbf{B}, \vec{0}[i \leftarrow 1]),\ \vec{x}_0\rangle$ (where $\vec{x}[i \leftarrow v]_j = \vec{x}_j$ when $j \neq i$ and $\vec{x}[i \leftarrow v]_i = v$).

Conclusion

Conclusion

- Linear algebra pervades mathematics and science.
- Considering variable properties *i.e.* sets of environments as sets of points, it is possible to abstract them to coordinate affine subspaces, which yields the linear equality static analysis of programs of Chapter **38**.
- We can represent a coordinate affine subspace by an extended matrix or a system of generators
- There are algorithms to convert between the two representations
- Operations on coordinate affine subspaces are often easier in one of the two representations (*e.g.* system of generators for variable elimination)

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Home work

Read Ch. **37** "Basic linear algebra" of

Principles of Abstract Interpretation
Patrick Cousot
MIT Press

The End, Thank you