Principles of Abstract Interpretation

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Ch. **13**, Topology

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These slides are available at
http://github.com/PrAbsInt/slides/slides/slides-13--topology-PrAbsInt.pdf

Chapter 13

Ch. **13**, Topology

- A topology can be thought of as abstracting away all properties irrelevant to the notion of limit (of functions, sequences, *etc*.)
- We will use topology to discuss safety and liveness program properties

`en.wikipedia.org/wiki/General_topology`

Topology (Section **13.1**)

Topology

- Let $\mathcal{X}$ be a non empty set.
- A *topology* $\mathcal{T}$ on $\mathcal{X}$ is a family $\mathcal{T} \in \wp(\mathcal{X})$ of subsets of $\mathcal{X}$, called the *open sets* such that

  (a) The union of open sets is an open set

  $$\forall P \in \wp(\mathcal{T}) \,.\, \bigcup P \in \mathcal{T}$$

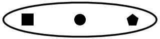
  (b) The *finite* intersection of open sets is an open set

  $$\forall n \in \mathbb{N} \,.\, \forall P_1, \ldots, P_n \in \mathcal{T} \,.\, \bigcap_{i=1}^{n} P_i \in \mathcal{T}$$

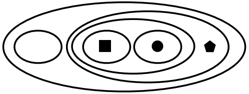
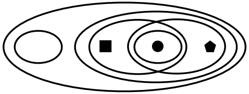
  (so in particular $\bigcap \varnothing = \mathcal{X} \in \mathcal{T}$ and $\bigcup \varnothing = \varnothing \in \mathcal{T}$).
- The pair $\langle \mathcal{X}, \mathcal{T} \rangle$ is called a *topological space*.

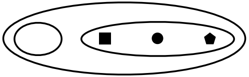
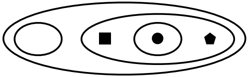
Example 13.1

set $\mathcal{X}$:

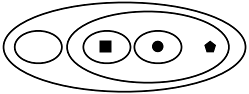
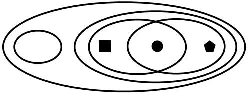


topologies on $\mathcal{X}$:

$\mathcal{T}_A =$  $\mathcal{T}_B =$ 

$\mathcal{T}_E =$  $\mathcal{T}_F =$ 

non-topologies on $\mathcal{X}$:

$\mathcal{S}_C =$  $\mathcal{S}_D =$ 

Closed sets

- The *closed sets* $\overline{\mathcal{T}} \in \wp(\mathcal{X})$ of a topology $\mathcal{T}$ are the *complements* of the open sets.
- For $P \in \wp(\mathcal{X})$, define $\neg P \triangleq \mathcal{X} \setminus P$.

$$\overline{\mathcal{T}} \quad \triangleq \quad \{\neg U \mid U \in \mathcal{T}\}$$

- $\mathcal{T}$ is closed by union so $\overline{\mathcal{T}}$ is closed by intersection (by De Morgan laws)
- Define the closure $\rho \in \wp(\mathcal{X}) \rightarrow \overline{\mathcal{T}}$

$$\rho(P) \quad \triangleq \quad \bigcap\{C \in \overline{\mathcal{T}} \mid P \subseteq C\} \qquad (13.4)$$

en.wikipedia.org/wiki/Open_set
en.wikipedia.org/wiki/Closed_set

Topological closure (Section 13.2)

Topological closure

Lemma 13.5 $\rho \in \wp(\mathcal{X}) \mapsto \overline{\mathcal{T}}$ in (13.4) is a *topological closure* satisfying the Kazimierz Kuratowski closure axioms [Kuratowski, 1958, 1961]

(a) ρ is expansive i.e. $P \subseteq \rho(P)$

(b) ρ is idempotent i.e. $\rho \circ \rho = \rho$

(c) ρ is strict i.e. $\rho(\varnothing) = \varnothing$

(d) ρ preserves finite joins i.e. $\rho(P \cup Q) = \rho(P) \cup \rho(Q)$[a] $\square$

[a] by recurrence this holds for any finite numbers of joins.

Notice that a topological closure ρ is an upper closure (since 13.5-(d) implies that ρ is increasing:
if $P \subseteq Q$ then $P \cup Q = Q$ so $\rho(Q) = \rho(P \cup Q) = \rho(P) \cup \rho(Q)$ proving that $\rho(P) \subseteq \rho(Q)$).

en.wikipedia.org/wiki/Kuratowski_closure_axioms

Proof of Lemma 13.5 $\rho(P) \triangleq \bigcap\{C \in \overline{\mathcal{T}} \mid P \subseteq C\}$

- $\rho \in \wp(\mathcal{X}) \mapsto \overline{\mathcal{T}}$ since the intersection of closed sets is closed
- It $P \in \overline{\mathcal{T}}$ is closed then $\rho(P) = P$
- ρ is expansive, idempotent and increasing
- Therefore $\langle \wp(\mathcal{X}), \subseteq\rangle \xleftarrow[\rho]{\mathbb{1}} \mathrel{\mkern-14mu}\twoheadrightarrow \langle \rho(\wp(\mathcal{X})), \subseteq\rangle$ is a Galois connection
- $\varnothing \subseteq \rho(\varnothing)$ and $\varnothing \subseteq \mathbb{1}(\varnothing)$ implies $\rho(\varnothing) \subseteq \varnothing$, so $\rho(\varnothing) = \varnothing$ by antisymmetry.
- Since ρ is $\subseteq$-increasing and $\cup$ is a lub, we have $\rho(P) \cup \rho(Q) \subseteq \rho(P \cup Q)$.
- By expansivity, $P \subseteq \rho(P)$ and $Q \subseteq \rho(Q)$ so $P \cup Q \subseteq \rho(P) \cup \rho(Q)$
- Hence $\rho(P \cup Q) \subseteq \rho(\rho(P) \cup \rho(Q))$ since ρ is increasing.
- Because the finite intersection of open sets is an open set, the finite union of closed sets is a closed set so $\rho(\rho(P) \cup \rho(Q)) = \rho(P) \cup \rho(Q)$
- Hence $\rho(P \cup Q) \subseteq \rho(P) \cup \rho(Q)$
- Proving $\rho(P \cup Q) = \rho(P) \cup \rho(Q)$ by antisymmetry. $\square$

Topology defined by a topological closure

Topology defined by a topological closure

Lemma 13.6 Any topological closure ρ on $\wp(\mathcal{X})$ defines a topology

$$\mathcal{T} \triangleq \{\neg\rho(P) \mid P \in \wp(\mathcal{X})\}$$ $\square$

The sets $\{\rho(P) \mid P \in \wp(\mathcal{X})\}$ are the *closed sets*.

Lemma 13.6 A topological closure ρ on $\wp(\mathcal{X})$ defines a topology

$$\mathcal{T} \triangleq \{\neg\rho(P) \mid P \in \wp(\mathcal{X})\}.$$

$\square$

Proof of Lemma 13.6

- Let ρ be topological closure operator (hence an upper closure) on $\wp(\mathcal{X})$.
- Let $\mathcal{T} \triangleq \{\neg\rho(P) \mid P \in \wp(\mathcal{X})\}$ be the complements of the closed sets $\{\rho(P) \mid P \in \wp(\mathcal{X})\}$.
- $\langle\wp(\mathcal{X}), \subseteq\rangle$ is a complete lattice
- So $\langle\rho(\wp(\mathcal{X})), \subseteq\rangle$ is a complete lattice (Theorem 11.86)
- So $\langle\rho(\wp(\mathcal{X})), \subseteq\rangle$ is closed by intersection.
- By De Morgan laws, the complement $\mathcal{T}$ is therefore closed by union.
- Moreover by 13.5-(d), the finite union of closed sets is closed
- So, by De Morgan laws, the complement $\mathcal{T}$ is therefore closed by finite intersection.
- This proves that $\mathcal{T}$ is a topology on $\mathcal{X}$. $\square$

Dense sets

Dense sets

Definition 1 The *dense sets* P of the topological space $\langle\mathcal{X}, \mathcal{T}\rangle$ are such that $\rho(P) = \mathcal{X}$ that is belong to $\rho^{-1}(\mathcal{X})$. $\square$

Example A classical example is the rationals $\mathbb{Q}$ are dense in the reals $\mathbb{R}$.

- Let $\rho(X)$, $X \in \wp(\mathbb{Q})$ be the operation that adds the limits of the sequences of elements of X.
- For example the limit of the sequence of rationals 3, 3.1, 3.14, 3.141, 3.1415, 3.14159, ... is π.
- Then $\rho(\mathbb{Q}) = \mathbb{R}$.

en.wikipedia.org/wiki/Rational_number
en.wikipedia.org/wiki/Construction_of_the_real_numbers (Construction from Cauchy sequences)
hen.wikipedia.org/wiki/Dense_set

Lemma 13.11 $P \in \wp(\mathcal{X})$ is dense if and only if $P = \neg\rho(P) \cup P$.

Proof of Lemma 13.11

- We first prove that for all $P \in \wp(\mathcal{X})$, $\rho(\neg\rho(P)) = \neg\rho(P)$.
  - Indeed $\neg\rho(P) \subseteq \rho(\neg\rho(P))$ since ρ is expansive.
  - Moreover $P \subseteq \rho(P)$ so $\neg\rho(P) \subseteq \neg P$
  - Hence $\rho(\neg\rho(P)) \subseteq \rho(\neg P)$
  - Proving $\neg\rho(P) = \rho(\neg\rho(P))$ by antisymmetry.
- If P is dense then $\rho(P) = \mathcal{X}$ so $\neg\rho(P) = \varnothing$ proving $P = \neg\rho(P) \cup P$.
- Reciprocally, if $P = \neg\rho(P) \cup P$ then

$$
\begin{aligned}
& \rho(P) \\
= {} & \rho(\neg\rho(P) \cup P) \\
= {} & \rho(\neg\rho(P)) \cup \rho(P) \\
= {} & \neg\rho(P) \cup \rho(P) \qquad \text{(as proved above)} \\
= {} & \mathcal{X}
\end{aligned}
$$

  proving, by def., that P is dense. □

Lemma 13.12 Any subset $P \in \wp(\mathcal{X})$ is the intersection $P = \rho(P) \cap (P \cup \neg\rho(P))$ of a closed set $\rho(P)$ and a dense set $(P \cup \neg\rho(P))$.

Lemma 13.12 Any subset $P \in \wp(\mathcal{X})$ is the intersection $P = \rho(P) \cap (P \cup \neg\rho(P))$ of a closed set $\rho(P)$ and a dense set $(P \cup \neg\rho(P))$.

Proof of Lemma 13.12

Let $P \in \wp(\mathcal{X})$. Since $P \subseteq \rho(P)$, we have

- $$\begin{aligned} & P \\ = \ & \rho(P) \cap P \\ = \ & (\rho(P) \cap P) \cup (\rho(P) \cap \neg\rho(P)) \\ = \ & \rho(P) \cap (P \cup \neg\rho(P)). \end{aligned}$$
- By def. $\rho(P)$ is a closed set.
- $P \cup \neg\rho(P)$ is dense by previous Lemma 13.11 $\square$

Conclusion

- Lemma 13.12 will be used in next Chapter **14** to characterize safety and liveness trace properties.
- To learn more about topology: [Armstrong, 1983; Bourbaki, 1966; Simmons, 1963].

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Home work

- Read Ch. **13** “Topology” of

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The End, Thank you