Principles of Abstract Interpretation

MIT press

Ch. **39**, Graphs

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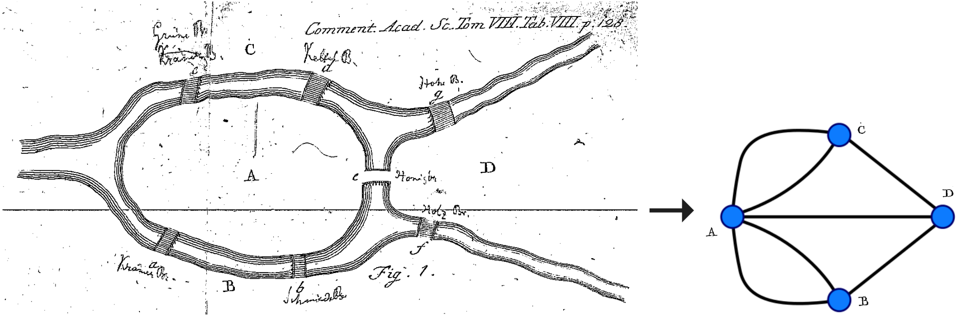
These slides are available at
http://github.com/PrAbsInt/slides/slides-39--graphs-PrAbsInt.pdf

Chapter 39

Ch. **39**, Graphs (1/4)

The origin of graph theory

Devise a walk through the city of Königsberg that would cross each of its seven bridges once and only once. Euler proved that the problem has no solution.



Leonhard Euler, "Solutio problematis ad geometriam situs pertinentis". Commentarii academiae scientiarum Petropolitanae 8, 1736, published 1741, pp. 128-140. (`http://eulerarchive.maa.org/docs/originals/E053.pdf`)

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We have split our review of Chapter **39** into four videos

This first video is about

- the definition of graphs, paths, and cycles

Graphs

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Definition of graphs

- A (directed) *graph* $G = \langle V, E \rangle$ is a pair of a set V of *vertices* (or *nodes* or *points*) and a set $E \in \wp(V \times V)$ of *edges* (or *arcs*).
- An edge $\langle x, y \rangle \in V$ has *origin* x and *end* y collectively called *extremities*.
- So E is a binary relation of Section **2.2.2** on V.
- Conversely, a binary relation can be understood as a graph on its field.
- A graph is *finite* when the set of V of vertices (hence E) is finite.

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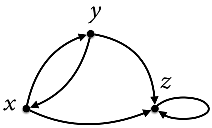
Example of graph, 39.1



$$\left[\begin{array}{lcl} V & = & \{x, y, z\} \\ E & = & \{\langle x,\ y\rangle, \langle x,\ z\rangle, \langle y,\ x\rangle, \\ & & \langle y,\ z\rangle, \langle z,\ z\rangle\} \end{array}\right.$$

$$\mathbf{G} = \left[\begin{array}{c|ccc} & x & y & z \\ \hline x & 0 & 1 & 1 \\ y & 1 & 0 & 1 \\ z & 0 & 0 & 1 \end{array}\right]$$

Paths and cycles in a graph

Paths and cycles in a graph

- A *path* from y to z in a graph $G = \langle V,\ E\rangle$ is a finite sequence of vertices

$$x_1, \ldots, x_n \in V^n, \quad n > 1$$

starting at *origin* $y = x_1$, finishing at *end* $z = x_n$, and linked by edges $\langle x_i,\ x_{i+1}\rangle$, $i \in [1, n[$.

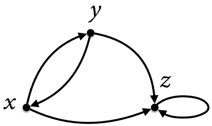
- Let $V^{>1} \triangleq \bigcup_{n>1} V^n$ be the sequences of vertices of length at least 2.
- Formally the set $\Pi(G) \in \wp(V^{>1})$ of all paths of a graph $G = \langle V,\ E\rangle$ is

$$\begin{aligned} \Pi(G) &\triangleq \bigcup_{n>1} \Pi^n(G) \\ \Pi^n(G) &\triangleq \{x_1 \ldots x_n \in V^n \mid \forall i \in [1, n[\ .\ \langle x_i,\ x_{i+1}\rangle \in E\} \quad (n > 1) \end{aligned} \tag{39.2}$$

Paths and cycles in a graph(cont'd)

- The *length* $|\pi|$ of the path $\pi = x_1 \ldots x_n \in V^n$ is the number of edges that is $n-1 > 0$.
- We do not consider the case $n = 1$ of paths of length 0 with only one vertex since paths must have at least one edge.
- A *subpath* is forming a strict part of another path (which, being strict, is not equal to that path).
- The *vertices of a path* $\pi = x_1 \ldots x_n \in \Pi^n(G)$ of a graph G is the set $\mathsf{V}(\pi) = \{x_1 \ldots x_n\}$ of vertices appearing in that path π.
- A *cycle* is a path $x_1 \ldots x_n \in \Pi^n(G)$ with $x_n = x_1$, $n > 1$.
- *Self-loops* *i.e.* $\langle x,\ x\rangle \in E$ yield a cycle $x\,x$ of length 1.

In Example 39.1,



$$\left[\begin{array}{rcl} V & = & \{x, y, z\} \\ E & = & \{\langle x,\ y\rangle, \langle x,\ z\rangle, \langle y,\ x\rangle, \\ & & \langle y,\ z\rangle, \langle z,\ z\rangle\} \end{array}\right] \qquad \mathbf{G} = \left[\begin{array}{c|ccc} & x & y & z \\ \hline x & 0 & 1 & 1 \\ y & 1 & 0 & 1 \\ z & 0 & 0 & 1 \end{array}\right]$$

- x, y, z and x, z are elementary paths (z, z is a path of length 1).
- The path x, y, x is an elementary cycle (z, z is a cycle of length 1).
- The paths x, y, z, z and x, y, x, y, x, z and the cycles y, x, y, x, y and z, z, z are not elementary.
- We do not consider infinite paths such as $x, y, x, y, x, \ldots$.

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Concatenation

- The *concatenation* of sets of finite paths is

$$P \circledcirc Q \quad \triangleq \quad \{x_1 \ldots x_n y_2 \ldots y_m \mid x_1 \ldots x_n \in P \wedge y_1 y_2 \ldots y_m \in Q \wedge x_n = y_1\}. \tag{39.3}$$

Fixpoint characterization of the paths of a graph

- Many possible recursive definitions:
  - $\text{path} = \text{arc} \mid \text{path} \odot \text{arc}$
  - $\text{path} = \text{arc} \mid \text{arc} \odot \text{path}$
  - $\text{path} = \text{arc} \mid \text{path} \mid \text{path} \odot \text{path}$
  - $\text{path} = \text{path} \mid \text{path} \odot \text{path} \quad \& \quad \text{arc} \subseteq \text{path}$
  - ...

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Fixpoint characterization of the paths of a graph

We have the following fixpoint characterization of the paths of a graph [P. Cousot and R. Cousot, 2004], which, by Tarski iterative fixpoint Theorem 15.21 and its variants yields an iterative algorithm (converging in finitely many iterations for finite graphs without infinite paths).

Theorem (39.4, Fixpoint characterization of the paths of a graph) The paths of a graph $G = \langle V,\ E\rangle$ are

$$
\begin{array}{rclrcll}
\Pi(G) & = & \mathsf{lfp}^{\subseteq}\, \overrightarrow{\mathcal{F}}_{\Pi}, & \overrightarrow{\mathcal{F}}_{\Pi}(X) & \triangleq & E \cup X \circledcirc E & (39.4.a)\\
 & = & \mathsf{lfp}^{\subseteq}\, \overleftarrow{\mathcal{F}}_{\Pi}, & \overleftarrow{\mathcal{F}}_{\Pi}(X) & \triangleq & E \cup E \circledcirc X & (39.4.b)\\
 & = & \mathsf{lfp}^{\subseteq}\, \overleftrightarrow{\mathcal{F}}_{\Pi}, & \overleftrightarrow{\mathcal{F}}_{\Pi}(X) & \triangleq & E \cup X \circledcirc X & (39.4.c)\\
 & = & \mathsf{lfp}^{\subseteq}_{E}\, \widehat{\mathcal{F}}_{\Pi}, & \widehat{\mathcal{F}}_{\Pi}(X) & \triangleq & X \cup X \circledcirc X & (39.4.d)
\end{array}
$$

□

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This concludes our definition of

- graphs, paths, and cycles

from Chapter **39** (Graphs)

The End

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Chapter 39

Ch. **39**, Graphs (2/4)

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In this second video, we study

- path problems

Path problems

Path problem

- Classical definition: *path problems* are solved by graph algorithms that have the same algebraic structure
- Abstract interpretation: A *path problem* in a graph $G = \langle V, E\rangle$ consists in specifying/computing an abstraction $\alpha(\Pi(G))$ of its paths $\Pi(G)$ defined by a Galois connection

$$\langle\wp(V^{>1}),\ \subseteq,\ \cup\rangle \xleftarrow[\alpha]{\gamma}\!\!\xrightarrow{}\ \langle A,\ \sqsubseteq,\ \sqcup\rangle.$$

Abstraction of the paths of a graph

- A path problem can be solved by a fixpoint definition/computation.

Theorem 39.6 (Fixpoint characterization of a path problem) Let $G = \langle V, E\rangle$ be a graph with paths $\Pi(G)$ and $\langle\wp(V^{>1}), \subseteq, \cup\rangle \xleftarrow[\alpha]{\gamma} \langle A, \sqsubseteq, \sqcup\rangle$.

$$
\begin{array}{rlllll}
\alpha(\Pi(G)) & = & \mathsf{lfp}^{\sqsubseteq}\, \overrightarrow{\mathscr{F}}^{\sharp}_{\Pi}, & \overrightarrow{\mathscr{F}}^{\sharp}_{\Pi}(X) & \triangleq & \alpha(E) \sqcup X \mathbin{\overline{\circledcirc}} \alpha(E) \quad (39.6.a)\\
& = & \mathsf{lfp}^{\sqsubseteq}\, \overleftarrow{\mathscr{F}}^{\sharp}_{\Pi}, & \overleftarrow{\mathscr{F}}^{\sharp}_{\Pi}(X) & \triangleq & \alpha(E) \sqcup \alpha(E) \mathbin{\overline{\circledcirc}} X \quad (39.6.b)\\
& = & \mathsf{lfp}^{\sqsubseteq}\, \overleftrightarrow{\mathscr{F}}^{\sharp}_{\Pi}, & \overleftrightarrow{\mathscr{F}}^{\sharp}_{\Pi}(X) & \triangleq & \alpha(E) \sqcup X \mathbin{\overline{\circledcirc}} X \quad (39.6.c)\\
& = & \mathsf{lfp}^{\sqsubseteq}_{\alpha(E)}\, \widehat{\mathscr{F}}^{\sharp}_{\Pi}, & \widehat{\mathscr{F}}^{\sharp}_{\Pi}(X) & \triangleq & X \sqcup X \mathbin{\overline{\circledcirc}} X \quad (39.6.d)
\end{array}
$$

where $\alpha(X) \mathbin{\overline{\circledcirc}} \alpha(Y) = \alpha(X \circledcirc Y)$.

- The proof is by calculational design using the classical exact least fixpoint abstraction with commutation of Theorem 18.21.

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Path problem 1: paths between any two vertices

- Projection abstraction

$$\alpha^{\multimap\!\circ}(X) \quad \triangleq \quad (y, z) \mapsto \{x_1 \ldots x_n \in X \mid y = x_1 \wedge x_n = z\}$$

such that

$$\langle \wp(V^{>1}),\ \subseteq,\ \cup \rangle \xleftrightarrow[\alpha^{\multimap\!\circ}]{\gamma^{\multimap\!\circ}} \langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq},\ \dot{\cup} \rangle \tag{39.7}$$

- Paths between any two vertices

$$\mathsf{p} \quad \triangleq \quad \alpha^{\multimap\!\circ}(\Pi(G))$$

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Fixpoint characterization of the paths of a graph between any two vertices

Theorem 39.10 Let $G = \langle V, E\rangle$ be a graph. The paths between any two vertices of G are $\mathsf{p} = \alpha^{\circ\!\!-\!\!\circ}(\Pi(G))$ such that

$$
\begin{array}{rcllcll}
\mathsf{p} & = & \mathsf{lfp}^{\dot{\subseteq}}\, \overrightarrow{\mathscr{F}}^{\circ\!-\!\circ}_{\Pi}, & \overrightarrow{\mathscr{F}}^{\circ\!-\!\circ}_{\Pi}(\mathsf{p}) & \triangleq & \dot{E} \mathbin{\dot{\cup}} \mathsf{p} \mathbin{\ddot{\circledcirc}} \dot{E} & \text{(Th.39.10.a)}\\
 & = & \ldots & & & & \text{(Th.39.10.b)}\\
 & = & \ldots & & & & \text{(Th.39.10.c)}\\
 & = & \mathsf{lfp}^{\dot{\subseteq}}_{\dot{E}}\, \widehat{\mathscr{F}}^{\circ\!-\!\circ}_{\Pi}, & \widehat{\mathscr{F}}^{\circ\!-\!\circ}_{\Pi}(\mathsf{p}) & \triangleq & \mathsf{p} \mathbin{\dot{\cup}} \mathsf{p} \mathbin{\ddot{\circledcirc}} \mathsf{p} & \text{(Th.39.10.d)}
\end{array}
$$

where $\dot{E} \triangleq x, y \mapsto (E \cap \{\langle x,\ y\rangle\})$ and $\mathsf{p}_1 \mathbin{\ddot{\circledcirc}} \mathsf{p}_2 \triangleq x, y \mapsto \bigcup_{z\in V} \mathsf{p}_1(x, z) \circledcirc \mathsf{p}_2(z, y)$. □

- The proof is by calculational design using the classical exact fixpoint abstraction with commutation of Theorem 18.21

Path problem 2: Elementary paths and cycles

- A cycle is *elementary* if and only if it contains no internal subcycle (*i.e.* subpath which is a cycle).
- A path is *elementary* if and only if it contains no subpath which is an internal cycle (so an elementary cycle is an elementary path).
- The only vertices that can occur twice in an elementary path are its extremities in which case it is an elementary cycle.
- Notation: $\text{elem?}(x_1 \ldots x_n)$
- Abstraction

$$\alpha^{\ni}(P) \triangleq \{\pi \in P \mid \text{elem?}(\pi)\}.$$

$$\langle \wp(V^{>1}),\ \subseteq \rangle \xrightleftharpoons[\alpha^{\ni}]{\gamma^{\ni}} \langle \wp(V^{>1}),\ \subseteq \rangle \qquad \langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq} \rangle \xrightleftharpoons[\dot{\alpha}^{\ni}]{\dot{\gamma}^{\ni}} \langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq} \rangle$$

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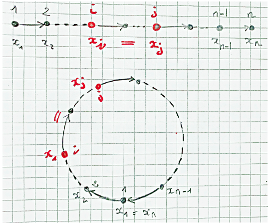
Elementary paths

Lemma 39.20 A path $x_1 \ldots x_n \in \Pi^n(G)$ is elementary if and only if

$$\text{elem?}(x_1 \ldots x_n) \triangleq (\forall i, j \in [1, n] \,.\, (i \neq j) \Rightarrow (x_i \neq x_j)) \vee \tag{39.20}$$
$$(x_1 = x_n \wedge \forall i, j \in [1, n[\,.\, (i \neq j) \Rightarrow (x_i \neq x_j)) \quad \text{(case of a cycle)}$$

is true.

Counter examples:



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Fixpoint characterization of the elementary paths of a graph

Theorem 39.23 Let $G = \langle V, E\rangle$ be a graph. The elementary paths between any two vertices of G are $\mathsf{p}^{\ni} \triangleq \alpha^{\circ\!\!-\!\!\circ} \circ \alpha^{\ni}(\Pi(G))$ such that

$$
\begin{array}{rcllcll}
\mathsf{p}^{\ni} & = & \mathsf{lfp}^{\dot\subseteq}\, \overrightarrow{\mathscr{F}}^{\ni}_{\Pi}, & \overrightarrow{\mathscr{F}}^{\ni}_{\Pi}(\mathsf{p}) & \triangleq & \dot{E} \mathbin{\dot\cup} \mathsf{p} \mathbin{\dot{\circledcirc}^{\ni}} \dot{E} & \text{(Th.39.23.a)}\\
& = & \ldots & & & & \text{(Th.39.23.b)}\\
& = & \ldots & & & & \text{(Th.39.23.c)}\\
& = & \mathsf{lfp}^{\dot\subseteq}_{\dot{E}}\, \widehat{\mathscr{F}}^{\ni}_{\Pi}, & \widehat{\mathscr{F}}^{\ni}_{\Pi}(\mathsf{p}) & \triangleq & \mathsf{p} \mathbin{\dot\cup} \mathsf{p} \mathbin{\dot{\circledcirc}^{\ni}} \mathsf{p} & \text{(Th.39.23.d)}
\end{array}
$$

where $\dot{E} \triangleq x, y \mapsto (E \cap \{\langle x, y\rangle\})$ and $\mathsf{p}_1 \mathbin{\dot{\circledcirc}^{\ni}} \mathsf{p}_2 \triangleq x, y \mapsto \bigcup_{z \in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge \mathsf{elem\text{-}conc?}(\pi_1, \pi_2)\}$. $\square$

- Proof by calculational design using the classical exact fixpoint abstraction
- (Th.39.23.d) is almost Floyd-Roy-Warshall but in n^4! (n number of vertices)

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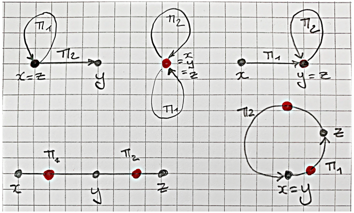
Concatenation of elementary paths

Lemma 39.21 $x\pi_1 z$ and $z\pi_2 y$ are elementary paths then their concatenation $x\pi_1 z \odot z\pi_2 y = x\pi_1 z\pi_2 y$ is elementary if and only if

$$\begin{aligned}\text{elem-conc?}(x\pi_1 z, z\pi_2 y) \quad \triangleq \quad & (x \neq z \wedge y \neq z \wedge \mathsf{V}(x\pi_1 z) \cap \mathsf{V}(\pi_2 y) = \varnothing) \\ & \vee (x = y \neq z \wedge \mathsf{V}(\pi_1 z) \cap \mathsf{V}(\pi_2) = \varnothing)\end{aligned} \tag{39.21}$$

is true.

Counter examples:



Iteration multiple abstraction

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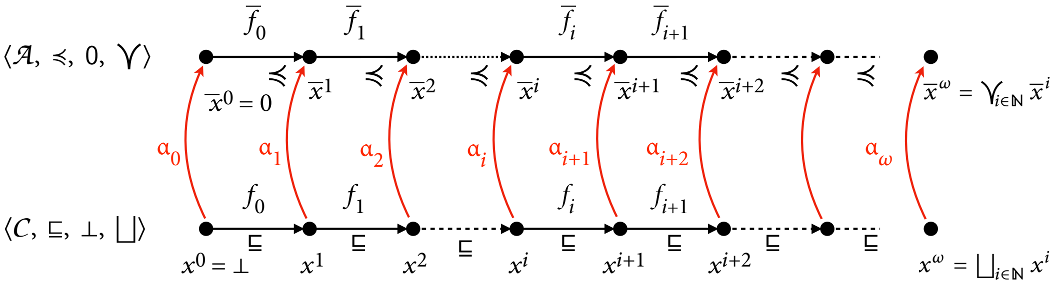
Exact abstraction of iterates (intuition)



If

- $\alpha_0(\bot) = 0$
- $\alpha_{i+1} \circ f_i = \overline{f}_i \circ \alpha_i$
- $\alpha_\omega(\bigsqcup_{i\in\mathbb{N}} x_i) = \curlyvee_{i\in\mathbb{N}} \alpha_i(x_i)$ for all increasing chains $\langle x_i \in \mathcal{C},\ i \in \mathbb{N}\rangle$.

then $\alpha_\omega(x^\omega) = \overline{x}^\omega$.

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Exact abstraction of iterates (formally)

Theorem 18.33 Let $\langle C, \sqsubseteq, \bot, \bigsqcup\rangle$ be a cpo, $\forall i \in \mathbb{N} \mathrel{.} f_i \in C \to C$ be such that $\forall x, y \in C \mathrel{.} x \sqsubseteq y \Rightarrow f_i(x) \sqsubseteq f_{i+1}(y)$ with iterates $\langle x^i,\ i \in \mathbb{N} \cup \{\omega\}\rangle$ defined by $x^0 = \bot$, $x^{i+1} = f_i(x^i)$, $x^\omega = \bigsqcup_{i\in\mathbb{N}} x^i$. Then these concrete iterates and $f \triangleq \dot{\bigsqcup}_{i\in\mathbb{N}} f_i$ are well-defined.

Let $\langle \mathcal{A}, \preccurlyeq, 0, \curlyvee\rangle$ be a cpo, $\forall i \in \mathbb{N} \mathrel{.} \overline{f}_i \in \mathcal{A} \to \mathcal{A}$ be such that $\forall \overline{x}, \overline{y} \in \mathcal{A} \mathrel{.} \overline{x} \preccurlyeq \overline{y} \Rightarrow \overline{f}_i(\overline{x}) \preccurlyeq \overline{f}_{i+1}(\overline{y})$ with iterates $\langle \overline{x}^i,\ i \in \mathbb{N} \cup \{\omega\}\rangle$ defined by $\overline{x}^0 = 0$, $\overline{x}^{i+1} = \overline{f}_i(\overline{x}^i)$, $\overline{x}^\omega = \curlyvee_{i\in\mathbb{N}} \overline{x}^i$. Then these abstract iterates and $\overline{f} \triangleq \dot{\curlyvee}_{i\in\mathbb{N}} \overline{f}_i$ are well-defined.

For all $i \in \mathbb{N} \cup \{\omega\}$, let $\alpha_i \in C \to \mathcal{A}$ be such that $\alpha_0(\bot) = 0$, $\alpha_{i+1} \circ f_i = \overline{f}_i \circ \alpha_i$, and $\alpha_\omega(\bigsqcup_{i\in\mathbb{N}} x_i) = \curlyvee_{i\in\mathbb{N}} \alpha_i(x_i)$ for all increasing chains $\langle x_i \in C,\ i \in \mathbb{N}\rangle$. It follows that $\alpha_\omega(x^\omega) = \overline{x}^\omega$.

If, moreover, $\forall i \in \mathbb{N} \mathrel{.} f_i \in C \xrightarrow{uc} C$ is upper-continuous then $x^\omega = \mathsf{lfp}^{\sqsubseteq} f$.

Similarly $\overline{x}^\omega = \mathsf{lfp}^{\preccurlyeq} \overline{f}$ when the $\overline{f}_i$ are upper-continuous.

If both the f_i and $\overline{f}_i$ are upper-continuous then $\alpha_\omega(\mathsf{lfp}^{\sqsubseteq} f) = \alpha_\omega(x^\omega) = \overline{x}^\omega = \mathsf{lfp}^{\preccurlyeq} \overline{f}$.

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Back to the elementary path problems

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Elementary paths of finite graphs $G = \langle V,\ E\rangle \qquad (|V| = n > 0)$

- Elementary paths in are of length at most $n+1$ so the fixpoint iterates in Theorem 39.23 converge in at most $n+2$ iterates.
- If $V = \{z_1 \ldots z_n\}$ is finite, then the elementary paths of the $k+2^{\text{nd}}$ iterate can be restricted to $\{z_1, \ldots, z_k\}$.
- Applying Theorem 18.33 with

$$
\begin{array}{lcll}
\alpha^{\ni}_0(\mathsf{p}) & \triangleq & \mathsf{p} & \\
\alpha^{\ni}_k(\mathsf{p}) & \triangleq & x, y \mapsto \{\pi \in \mathsf{p}(x, y) \mid \mathsf{V}(\pi) \subseteq \{z_1, \ldots, z_k\} \cup \{x, y\}\}, & k \in [1, n] \\
\alpha^{\ni}_k(\mathsf{p}) & \triangleq & \mathsf{p}, & k > n
\end{array}
\tag{39.24}
$$

$$
\langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq}\rangle \xrightleftharpoons[\alpha^{\ni}_k]{\gamma^{\ni}_k} \langle V \times V \to \bigcup_{k=2}^{n+1} V^k,\ \dot{\subseteq}\rangle. \tag{39.25}
$$

we get an iterative algorithm.

Iterative characterization of the elementary paths of a finite graph

Theorem 39.26 Let $G = \langle V, E\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$. Then

$$\mathsf{p}^{\partial} = \mathsf{lfp}^{\dot\subseteq}\, \overrightarrow{\mathscr{F}}^{\partial}_{\Pi} = \overrightarrow{\mathscr{F}}^{\partial n+2}_{\pi} \quad \text{where} \quad \overrightarrow{\mathscr{F}}^{\partial}_{\Pi}(\mathsf{p}) \triangleq \dot E \,\dot\cup\, \mathsf{p} \,\dot\circledcirc^{\partial}\, \dot E \text{ in (Th.39.23.a)} \qquad \text{(Th.39.26.a)}$$

$$\overrightarrow{\mathscr{F}}^{\partial 0}_{\pi} \triangleq \dot\varnothing, \qquad \overrightarrow{\mathscr{F}}^{\partial 1}_{\pi} \triangleq \dot E,$$

$$\overrightarrow{\mathscr{F}}^{\partial k+2}_{\pi} \triangleq \dot E \,\dot\cup\, \overrightarrow{\mathscr{F}}^{\partial k+1}_{\pi} \,\dot\circledcirc_{z_{k+1}}\, \dot E, \quad k \in [0, n], \quad \overrightarrow{\mathscr{F}}^{\partial k+1}_{\pi} = \overrightarrow{\mathscr{F}}^{\partial k}_{\pi}, \quad k \geqslant n+2$$

$$= \ldots \qquad \text{(Th.39.26.b)}$$

$$= \ldots \qquad \text{(Th.39.26.c)}$$

$$= \mathsf{lfp}^{\dot\subseteq}_{\dot E}\, \widehat{\mathscr{F}}^{\partial}_{\Pi} = \widehat{\mathscr{F}}^{\partial n+1}_{\pi} \quad \text{where} \quad \widehat{\mathscr{F}}^{\partial}_{\Pi}(\mathsf{p}) \triangleq \mathsf{p} \,\dot\cup\, \mathsf{p} \,\dot\circledcirc^{\partial}\, \mathsf{p} \text{ in (Th.39.23.d)} \qquad \text{(Th.39.26.d)}$$

$$\widehat{\mathscr{F}}^{\partial 0}_{\pi} \triangleq \dot E, \quad \widehat{\mathscr{F}}^{\partial k+1}_{\pi} \triangleq \widehat{\mathscr{F}}^{\partial k}_{\pi} \,\dot\cup\, \widehat{\mathscr{F}}^{\partial k}_{\pi} \,\dot\circledcirc^{\partial}_{z_{k+1}}\, \widehat{\mathscr{F}}^{\partial k}_{\pi}, \quad k \in [0, n],$$

$$\widehat{\mathscr{F}}^{\partial k+1}_{\pi} = \widehat{\mathscr{F}}^{\partial k}_{\pi}, \quad k \geqslant n+2$$

$$\mathsf{p}_1 \,\dot\circledcirc_{z}\, \mathsf{p}_2 \triangleq x, y \mapsto \{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge z \notin \{x, y\}\}$$

$$\mathsf{p}_1 \,\dot\circledcirc^{\partial}_{z}\, \mathsf{p}_2 \triangleq x, y \mapsto \{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge \text{elem-conc?}(\pi_1, \pi_2)\}. \qquad \square$$

Iterative characterization of an *over-approximation* of the elementary paths of a finite graph

Corollary 39.28 Let $G = \langle V, E\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$. Then

$$\mathsf{p}^{\partial} = \ldots \qquad \text{(Cor.39.28.c)}$$

$$= \mathsf{lfp}^{\dot\subseteq}_{\dot E}\, \widehat{\mathscr{F}}^{\partial}_{\Pi} \ \dot\subseteq\ \widehat{\mathscr{F}}^{n+1}_{\boldsymbol{\pi}} \qquad \text{(Cor.39.28.d)}$$

$$\text{where} \quad \widehat{\mathscr{F}}^{0}_{\boldsymbol{\pi}} \triangleq \dot E, \quad \widehat{\mathscr{F}}^{k+1}_{\boldsymbol{\pi}} \triangleq \widehat{\mathscr{F}}^{k}_{\boldsymbol{\pi}} \,\dot\cup\, \widehat{\mathscr{F}}^{k}_{\boldsymbol{\pi}} \,\dot{\circledcirc}_{z_k}\, \widehat{\mathscr{F}}^{k}_{\boldsymbol{\pi}}$$

□

replacing $\dot{\circledcirc}^{\partial}_{z}$ by $\dot{\circledcirc}_{z}$ (with no check that concatenated paths are elementary).

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Algorithm 39.30 **(Roy-Floyd-Warshall algorithm over-approximating the elementary paths of a finite graph)** Let $G = \langle V, E \rangle$ be a graph with $|V| = n > 0$ vertices. The Roy-Floyd-Warshall algorithm

$$
\begin{array}{l}
\text{for } x, y \in V \text{ do} \\
\qquad \mathsf{p}(x, y) := E \cap \{\langle x, y \rangle\} \\
\text{done;} \\
\text{for } z \in V \text{ do} \\
\quad \text{for } x, y \in V \setminus \{z\} \text{ do} \\
\qquad \mathsf{p}(x, y) := \mathsf{p}(x, y) \cup \mathsf{p}(x, z) \odot \mathsf{p}(z, y) \\
\quad \text{done} \\
\text{done}
\end{array}
$$

computes an over-approximation of all elementary paths p of G.

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This concludes our study of

- path problems

from Chapter **39** (Graphs)

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Ch. **39**, Graphs

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These slides are available at
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Chapter 39

Ch. **39**, Graphs (3/4)

In this third video, we study

- shortest paths in weighted graphs

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Weighted graphs, Section **39.7**

Groups

- A *group* $\langle \mathbb{G}, 0, + \rangle$[1] is a set $\mathbb{G}$ with $0 \in \mathbb{G}$ and $+ \in \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ such that
  - $\forall a, b, c \in \mathbb{G} \,.\, (a + b) + c = a + (b + c)$ (associativity)
  - $\forall a \in \mathbb{G} \,.\, 0 + a = a + 0 = a$ (identity denoted 0)
  - $\forall a \in \mathbb{G} \,.\, \exists b \in \mathbb{G} \,.\, a + b = 0$ (inverse, b is denoted $-a$ or a^{-1}.)
- For example the scalars $\langle \mathbb{F}, 0, + \rangle$ are a group where 0 is the null scalar and $+$ is scalar addition.
- Integers modulo n are a group under addition.

[1] We call group both the algebraic structure $\langle \mathbb{G}, 0, + \rangle$ and its support $\mathbb{G}$.

Weighted graphs

- Let $\langle \mathbb{G}, 0, + \rangle$ be a group.
- A (directed) graph $G = \langle V, E, \boldsymbol{\omega} \rangle$ weighted on the group $\mathbb{G}$ is a finite graph $\langle V, E \rangle$ equipped with a weight $\boldsymbol{\omega} \in E \rightarrow \mathbb{G}$ mapping arcs to their weight.

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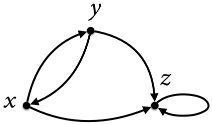
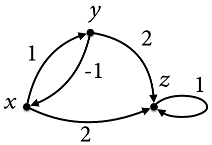
Example 39.11

Continuing Example 39.1,



$$\left[\begin{array}{lcl} V & = & \{x, y, z\} \\ E & = & \{\langle x,\ y\rangle, \langle x,\ z\rangle, \langle y,\ x\rangle, \\ & & \langle y,\ z\rangle, \langle z,\ z\rangle\} \end{array}\right.$$

$$\mathbf{G} = \left[\begin{array}{c|ccc} & x & y & z \\ \hline x & 0 & 1 & 1 \\ y & 1 & 0 & 1 \\ z & 0 & 0 & 1 \end{array}\right]$$



$$\left[\begin{array}{lclclcl} \boldsymbol{\omega}(\langle x,\ y\rangle) & = & 1 & \boldsymbol{\omega}(\langle x,\ z\rangle) & = & 2 \\ \boldsymbol{\omega}(\langle x,\ z\rangle) & = & 2 & \boldsymbol{\omega}(\langle y,\ x\rangle) & = & -1 \\ \boldsymbol{\omega}(\langle y,\ z\rangle) & = & 2 & \boldsymbol{\omega}(\langle z,\ z\rangle) & = & 1 \end{array}\right.$$

$$\mathbf{G} = \left[\begin{array}{c|ccc} & x & y & z \\ \hline x & \infty & 1 & 2 \\ y & -1 & \infty & 2 \\ z & \infty & \infty & 1 \end{array}\right]$$

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Shortest distances

Totally ordered groups

- A *totally (or linearly) ordered group* $\langle \mathbb{G}, \leqslant, 0, +\rangle$ is a group $\langle \mathbb{G}, 0, +\rangle$ with a total order $\leqslant$ on $\mathbb{G}$ such that
  - $\forall a, b, c \in \mathbb{G} \,.\, (a \leqslant b) \Rightarrow (a + c \leqslant b + c)$ (translation-invariance.)
- An element $x \in \mathbb{G}$ of a totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$ is said to be strictly negative if and only if $x \leqslant 0 \wedge x \neq 0$.
- Following Section **10.3**, if $S \subseteq \mathbb{G}$ then we define $\min S$ to be the greatest lower bound of S in $\mathbb{G}$ or $-\infty$:

$$\begin{array}{rcl@{\quad}c@{\quad}lr}
\min S & = & m & \Leftrightarrow & m \in \mathbb{G} \wedge (\forall x \in S \,.\, m \leqslant x) \wedge (\forall y \in S \,.\, (\forall x \in S \,.\, y \leqslant x) \Rightarrow y \leqslant m) & \\
 & = & -\infty & \Leftrightarrow & \forall x \in S \,.\, \exists y \in S \,.\, y < x & (\text{where } -\infty \notin \mathbb{G}) \\
 & = & \infty & \Leftrightarrow & S = \varnothing & (\text{where } \infty \notin \mathbb{G})
\end{array}$$

- So if $\mathbb{G}$ has no infimum $\min \mathbb{G} = \max \varnothing = -\infty \notin \mathbb{G}$.
- Similarly, $\max S$ is the least upper bound of S in $\mathbb{G}$, if any, ∞ otherwise, with $\max \mathbb{G} = \min \varnothing = \infty \notin \mathbb{G}$ when $\mathbb{G}$ has no supremum.

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Weight of paths

- The weight of a path is

$$\boldsymbol{\omega}(x_1, \ldots, x_n) \triangleq \sum_{i=1}^{n-1} \boldsymbol{\omega}(\langle x_i,\ x_{i+1}\rangle)$$

  which is 0 when $n \leqslant 1$ and $\boldsymbol{\omega}(\langle x_1,\ x_2\rangle) + \sum_{i=2}^{n-1} \boldsymbol{\omega}(\langle x_i,\ x_{i+1}\rangle)$ when $n > 1$.

- The (minimal) weight of a set of paths is

$$\boldsymbol{\omega}(P) \triangleq \min\{\boldsymbol{\omega}(\pi) \mid \pi \in P\} \tag{39.13}$$

- We have $\boldsymbol{\omega}(\bigcup_{i\in\Delta} P_i) = \min_{i\in\Delta} \boldsymbol{\omega}(P_i)$ so a Galois connection

$$\langle \wp(\bigcup_{n\in\mathbb{N}^+} V^n),\ \subseteq\rangle \xrightleftharpoons[\boldsymbol{\omega}]{} \langle \mathbb{G} \cup \{-\infty, \infty\},\ \geqslant\rangle$$

  and the complete lattice $\langle \mathbb{G} \cup \{-\infty, \infty\},\ \geqslant,\ \infty,\ -\infty,\ \min,\ \max\rangle$.

- Extending pointwise to $V \times V \to \wp(\bigcup_{n\in\mathbb{N}^+} V^{\,n})$ with $\dot{\boldsymbol{\omega}}(\mathsf{p})\langle x,\ y\rangle \triangleq \boldsymbol{\omega}(\mathsf{p}(x, y))$, $\mathsf{d} \mathrel{\dot{\leqslant}} \mathsf{d}' \triangleq \forall x, y \mathrel{.} \mathsf{d}\langle x,\ y\rangle \leqslant \mathsf{d}'\langle x,\ y\rangle$, and $\dot{\geqslant}$ is the inverse of $\dot{\leqslant}$, we have

$$\langle V \times V \to \wp(\bigcup_{n\in\mathbb{N}^+} V^{\,n}),\ \ddot{\subseteq}\rangle \xleftrightarrow[\dot{\boldsymbol{\omega}}]{} \langle V \times V \to \mathbb{G} \cup \{-\infty, \infty\},\ \dot{\geqslant}\rangle.$$

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Fixpoint characterization of the shortest distances of a graph

Theorem 39.17 Let $G = \langle V, E, \boldsymbol{\omega}\rangle$ be a graph weighted on the totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$. Then the distances between any two vertices are

$$
\begin{aligned}
\mathrm{d} &= \dot{\boldsymbol{\omega}}(\mathrm{p}) = \mathrm{gfp}^{\dot{\leqslant}}_{E^{\boldsymbol{\omega}}}\, \widehat{\mathscr{F}}^{\delta}_{G} \qquad \text{where} && (39.17)\\
E^{\boldsymbol{\omega}} &\triangleq (x, y) \mapsto [\![\langle x, y\rangle \in E \mathrel{?} \boldsymbol{\omega}(x, y) \mathrel{:} \infty]\!]\\
\widehat{\mathscr{F}}^{\delta}_{G}(X) &\triangleq (x, y) \mapsto \min\{X(x, y), \min_{z \in V}\{X(x, z) + X(z, y)\}\} && \square
\end{aligned}
$$

- by calculational design, applying the exact fixpoint abstraction of Theorem 18.21 to Theorem 39.6.d with abstraction $\dot{\boldsymbol{\omega}} \circ \alpha^{\circ\!\!-\!\!\circ}$

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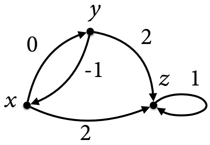
Example 39.16, Graph with cycle of strictly negative weight

- The following graph



has

  - a cycle x, y, x of weight -1,
  - a cycle x, y, x, y, x of weight -2,
  - a cycle x, y, x, y, x, y, x of weight -3, *etc*.

so that the minimum distance between x and y is $-\infty$.

- Theorem 39.17 requires an infinite iteration -1, -2, -3, … with limited $-\infty$

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This concludes our study of

- shortest paths in weighted graphs

from Chapter **39** (Graphs)

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Chapter 39

Ch. **39**, Graphs (4/4)

In this fourth video, we study the

- Roy, Floyd, Warshall shortest path algorithm

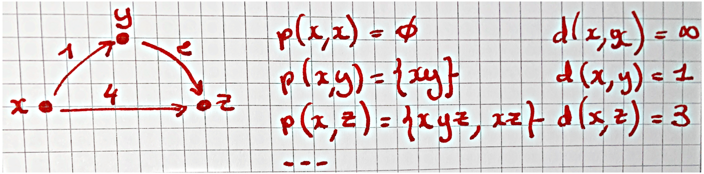
The Roy, Floyd, Warshall shortest path algorithm

Path problem 3: shortest distances between any two vertices of a weighted graph $G = \langle V, E, \boldsymbol{\omega}\rangle$ on a group $\langle \mathbb{G}, 0, +\rangle$

- The shortest distance $\mathsf{d}(x, y)$ between an origin $x \in V$ and an extremity $y \in V$ of a weighted finite graph $G = \langle V, E, \boldsymbol{\omega}\rangle$ on a totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$ is defined as the length $\boldsymbol{\omega}(\mathsf{p}(x, y))$ of the shortest path between these vertices

$$\mathsf{d} \quad \triangleq \quad \dot{\boldsymbol{\omega}}(\mathsf{p})$$

where p has a fixpoint characterization given by Theorem 39.10.



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Iterative characterization of the shortest path length of a graph

Theorem 39.33 Let $G = \langle V, E, \boldsymbol{\omega}\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$ weighted on the totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$ with no strictly negative weight.

Then the distances between any two vertices are

$$\mathrm{d} = \dot{\boldsymbol{\omega}}(\mathrm{p}) = \widehat{\boldsymbol{\mathscr{F}}}_{\delta}^{n+1} \quad \text{where} \tag{Th.39.33}$$

$$\widehat{\boldsymbol{\mathscr{F}}}_{\delta}^{0}(x, y) \triangleq [\![\langle x, y\rangle \in E \; ? \; \boldsymbol{\omega}(x, y) \; ⁏ \; \infty]\!],$$

$$\widehat{\boldsymbol{\mathscr{F}}}_{\delta}^{k+1}(x, y) \triangleq [\![z_k \in \{x, y\} \; ? \; \widehat{\boldsymbol{\mathscr{F}}}_{\delta}^{k}(x, y) \; ⁏ \; \min(\widehat{\boldsymbol{\mathscr{F}}}_{\delta}^{k}(x, y), \widehat{\boldsymbol{\mathscr{F}}}_{\delta}^{k}(x, z_k) + \widehat{\boldsymbol{\mathscr{F}}}_{\delta}^{k}(z_k, y))]\!] \quad \square$$

Proof by calculational design based on Theorem 18.33.

Roy-Floyd-Warshall shortest distances of a graph

Algorithm 39.34 The Roy-Floyd-Warshall algorithm computes the shortest distances $\dot{\boldsymbol{\omega}}(\mathrm{p}) \in V \times V \rightarrow \mathbb{G} \cup \{-\infty, \infty\}$ in a finite graph with no cycle with strictly negative weight:

$$
\begin{array}{l}
\text{for } x, y \in V \text{ do} \\
\qquad \mathrm{d}(x, y) := \text{if } \langle x,\ y\rangle \in E \text{ then } \boldsymbol{\omega}(x, y) \text{ else } \infty \\
\quad \text{done;} \\
\text{for } z \in V \text{ do} \\
\quad \text{for } x, y \in V \text{ do} \\
\qquad \mathrm{d}(x, y) := \min(\mathrm{d}(x, y), \mathrm{d}(x, z) + \mathrm{d}(z, y)) \\
\quad \text{done} \\
\text{done.}
\end{array}
$$

The graph has no cycle with strictly negative weight if and only if $\forall x \in V \,.\, \mathrm{d}(x, x) \geqslant 0$, in which case $\mathrm{d}(x, y)$ is the length of the shortest path from x to y.

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Adjacency matrix, Section **39.18**

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Adjacency matrix

- The boolean adjacency matrix of a finite graph $G = \langle [1, n], E \rangle$, $n \in \mathbb{N}^+$ is $\mathbf{G} = (\llbracket \langle i, j \rangle \in E \mathrel{?} 1 \mathrel{:} 0 \rrbracket)_{\substack{i=1,n \\ j=1,n}} \in \{0, 1\}^{n \times n}$, see Example 39.1.
- For a graph $G = \langle V, E, \boldsymbol{\omega} \rangle$ weighted on the field $\mathbb{F}$, the adjacency matrix is $\mathbf{G} = (\llbracket \langle i, j \rangle \in E \mathrel{?} \boldsymbol{\omega}(i, j) \mathrel{:} \infty \rrbracket)_{\substack{i=1,n \\ j=1,n}} \in (\mathbb{F} \cup \{\infty\})^{n \times n}$, see Example 39.11.
- The infimum is the empty graph $\langle \emptyset, \emptyset \rangle$ encoded with the empty matrix $[\emptyset]$.
- The supremum is $(\infty)_{\substack{i=1,n \\ j=1,n}}$.

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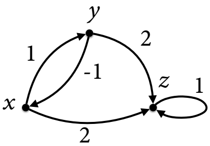
- There is a Galois isomorphism between $\mathbf{G}$ and $\langle i, j\rangle \mapsto \boldsymbol{\omega}(E \cap \{\langle i, j\rangle\})$ and similarly the distance d can be encoded, up to an isomorphism into $\mathbf{D} = (\mathrm{d}(i, j))_{\substack{i=1,n \\ j=1,n}} \in (\mathbb{F} \cup \{\infty\})^{n\times n}$, so that by an abstraction of (39.23.c) similar to Theorem 39.17, it follows that

Corollary 39.40 **(Shortest distances in a weighted graph with adjacency matrix)**
Let $G = \langle V, E, \boldsymbol{\omega}\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$ weighted on the totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$ with no strictly negative weight.

Then the matrix of distances between any two vertices is $\mathbf{D} = \mathscr{F}_{\mu}^{n+1}$ where $\mathscr{F}_{\mu}^{0} = \mathbf{G}$ and $\mathscr{F}_{\mu\,ij}^{\ell+1} = \min(\mathscr{F}_{\mu\,ij}^{\ell}, \min_{k\in[1,n]}(\mathscr{F}_{\mu\,ik}^{\ell} + \mathscr{F}_{\mu\,kj}^{\ell}))$.

Example

The iterates for Example 39.11



starting from $\mathbf{G} = \begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \hline \begin{array}{c} x \\ y \\ z \end{array} & \left[\begin{array}{ccc} \infty & 1 & 2 \\ -1 & \infty & 2 \\ \infty & \infty & 1 \end{array}\right] \end{array}$ are $\mathscr{F}^{\mu 1} = \begin{bmatrix} \infty & 1 & 2 \\ -1 & \infty & 2 \\ \infty & \infty & 1 \end{bmatrix}$,

$\mathscr{F}^{\mu 2} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ \infty & \infty & 1 \end{bmatrix} = \mathscr{F}^{\mu 3} = \mathbf{D}$.

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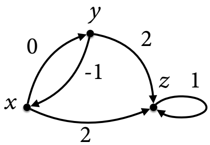
Graph with cycle of strictly negative weight

Continuing Example 39.16



the iterates starting from $\mathbf{G}$ are $\mathscr{F}^{\mu 0} = \begin{bmatrix} \infty & 0 & 2 \\ -1 & \infty & 2 \\ \infty & \infty & 1 \end{bmatrix}$, $\mathscr{F}^{\mu 1} = \begin{bmatrix} -1 & 0 & 2 \\ -1 & -1 & 1 \\ \infty & \infty & 1 \end{bmatrix}$,

$\mathscr{F}^{\mu 2} = \begin{bmatrix} -2 & -1 & 1 \\ -2 & -2 & 0 \\ \infty & \infty & 1 \end{bmatrix}$, $\mathscr{F}^{\mu 3} = \begin{bmatrix} -3 & -3 & -1 \\ -4 & -4 & -2 \\ \infty & \infty & 1 \end{bmatrix}$, $\mathscr{F}^{\mu 4} = \begin{bmatrix} -7 & -7 & -5 \\ -8 & -8 & -6 \\ \infty & \infty & 1 \end{bmatrix}$, ...,

$\mathscr{F}^{\mu \omega} = \begin{bmatrix} -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty \\ \infty & \infty & 1 \end{bmatrix}$.

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Conclusion

- The Roy-Floyd–Warshall algorithm finds [Floyd, 1962; Roy, 1959; Warshall, 1962] shortest paths in a weighted graph (with no cycle with strictly negative weight).
- It is an abstract interpretation of a fixpoint path finding algorithm.
- For comments on the Roy-Floyd-Warshall algorithm, see [Hansen and Werra, 2002, p. 26–29], [Naur, 1994] and [Schrijver, 2003, p. 129].
- The calculational design of the transitive closure by abstraction of a fixpoint path semantics is in [P. Cousot and R. Cousot, 2004].
- See [Sergey, Midtgaard, and Clarke, 2012] for the calculational design of other graph algorithms by abstract interpretation.
- Read the proofs in the book!

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Home work

Read Ch. **39** "Graphs" of

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