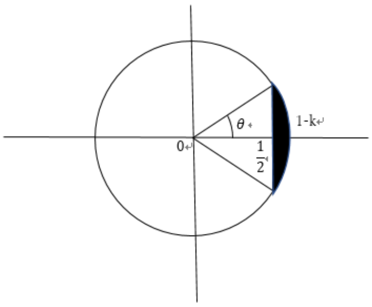
問題.1　(3) $z=k$ と置く. S の存在範囲から k が $0 \leq k \leq \frac{1}{2}$ の時の断面積を積分すればよい. 各 k の断面積を S_k と置き, また次の断面図のとおり θ を置く.



すると $\cos\theta = \frac{1}{2(1-k)}$ $(0 \leq \theta < \frac{\pi}{2}$) が成立する.

$$\begin{aligned}
-\sin\theta d\theta &= \frac{1}{2(1-k)^2}dk \\
S_k &= \frac{1}{2}(1-k)^2 2\theta - \frac{1}{2}(1-k)^2\sin 2\theta \\
&= (1-k)^2\theta - \frac{1}{2}(1-k)^2\sin 2\theta
\end{aligned}$$

であって k が 0 から $\frac{1}{2}$ まで動くとき θ は $\frac{\pi}{3}$ から 0 まで単調に動くので

$$\begin{aligned}
\int_0^{\frac{1}{2}} S_k dk &= \int_{\frac{\pi}{3}}^0 -\frac{\theta\sin\theta}{8\cos^4\theta} + \frac{\sin\theta\sin 2\theta}{16\cos^4\theta}d\theta \\
&= \int_{\frac{\pi}{3}}^0 -\frac{\theta\sin\theta}{8\cos^4\theta} + \frac{\sin^2\theta}{8\cos^3\theta}d\theta \\
&= \int_{\frac{\pi}{3}}^0 \frac{\theta}{24}(\cos^{-3}\theta)' + \frac{1}{8\cos^3\theta} - \frac{1}{8\cos\theta}d\theta \\
&= \frac{1}{24}\Big[\frac{\theta}{\cos^3\theta}\Big]_{\frac{\pi}{3}}^0 - \frac{1}{24}\int_{\frac{\pi}{3}}^0 \frac{1}{\cos^3\theta} + \frac{1}{8}\int_{\frac{\pi}{3}}^0 \frac{1}{\cos^3\theta} - \frac{1}{8}\int_{\frac{\pi}{3}}^0 \frac{1}{\cos\theta} \\
&= \frac{\pi}{9} - \frac{1}{12}I_3 - \frac{1}{8}I_1 \\
&= \frac{\pi}{9} - \frac{1}{12}\Big(\frac{\sqrt{3}}{2} + \frac{1}{2}I_1\Big) - \frac{1}{8}I_1
\end{aligned}$$