Implement basic complex numbers using upvalues. The point of this will be to implement Pauli matrices using complex numbers that are distinct from the Mathematica built-in Complex type.

```
ClearAll[ complex, real, imag, conjugate, complexQ,
 notComplexQ, fMatrix, conjugate, conjugateTranspose, complexI ]

complex /: complex[r1_, i1_] + complex[r2_, i2_] := complex[r1 + r2, i1 + i2 ] ;

complex /: - complex[re_, im_] := complex[-re, -im];

complex /: complex[re_] := complex[re, 0] ;

complex /: complex[re_] := complex[re, 0] ;

complex /: complex[r1_, i1_] complex[r2_, i2_] :=
  complex[r1 r2 - i1 i2, r1 i2 + r2 i1 ] ;
complexQ[z_complex] := True ;
complexQ[_] := False ;
notComplexQ[v_] := Not[complexQ[v]] ;

complex /: (v_?notComplexQ ) complex[re_, im_] := complex[ v re, v im ];

real[z_complex] := (z // First) ;
imag[z_complex] := (z // Last) ;
conjugate[z_complex] := complex[z // First, -z // Last] ;

complexI := complex[0, 1] ;

fMatrix[p_, f_] := (Function[a, f@a, Listable]@p)
real[m_List] := fMatrix[m, real];
imag[m_List] := fMatrix[m, imag];
conjugate[m_List] := fMatrix[m, conjugate];

ClearAll[pauliMatrix, conjugateTranspose]
pauliMatrix[1] := PauliMatrix[1] ;
pauliMatrix[2] :=
  (PauliMatrix[2] /. {Complex[0, 1] → complexI, Complex[0, -1] → -complexI}) ;
pauliMatrix[3] := PauliMatrix[3] ;
conjugateTranspose[m_List] := Transpose[conjugate[m]] ;

(*Unprotect[TraditionalForm, DisplayForm, StandardForm];
TraditionalForm[z_complex] := (((z // real) + I (z // imag)) // TraditionalForm)
   DisplayForm[z_complex] := (((z // real) + I (z // imag)) // DisplayForm)
    StandardForm[z_complex] := (((z // real) + I (z // imag)) // StandardForm)
    Protect[TraditionalForm, DisplayForm, StandardForm];*)
```

```
ClearAll[ a, b, c, d ]
$Assumptions = {a, b, c, d} > 0;
Column[ (# // DisplayForm) &/@{complex[1, 2] + complex[2, 3] + complex[4, 5],
   -complex[1, 2],
   complex[1, 2] - complex[2, 3],
   complex[1],
   complexI,
   complex[a, b] complex[c, d]} ]
(a + b I) (c + d I) // Expand
Column[ (# // DisplayForm) &/@{
   real[complex[2, 3]],
   imag[complex[2, 3]],
   conjugate[complex[2, 3]],
   3 complex[1, 2],
   complex[1, 2] 3}
]

complexQ[complex[1]]
notComplexQ[complex[1]]

complexQ[1]
notComplexQ[1]
```

```
7 + 10 i
-1 - 2 i
-1 - i
1
i
a c - b d + i (b c + a d)

a c + i b c + i a d - b d

2
3
2 - 3 i
3 + 6 i
3 + 6 i

True

False

False

True
```

```
pauliMatrix[2] // MatrixForm
pauliMatrix[1] // MatrixForm
pauliMatrix[3] // MatrixForm
```

```
{{{0, 1}, {1, 0}}, {{0, -i}, {i, 0}}, {{1, 0}, {0, -1}}}
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \text{complex}[0, -1] \\ \text{complex}[0, 1] & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \text{complex}[0, -1] \\ \text{complex}[0, 1] & 0 \end{pmatrix}$$

```
ClearAll[p]
p = {{complex[1, 2], complex[2, 3]}, {complex[3, 4], complex[4, 5]}} ;

(MatrixForm[#] & /@ {p, (p // real), (p // imag),
    (p // conjugate), (p // conjugateTranspose)}) // Column
```

$$\begin{pmatrix} \text{complex}[1, 2] & \text{complex}[2, 3] \\ \text{complex}[3, 4] & \text{complex}[4, 5] \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} \text{complex}[1, -2] & \text{complex}[2, -3] \\ \text{complex}[3, -4] & \text{complex}[4, -5] \end{pmatrix}$$

$$\begin{pmatrix} \text{complex}[1, -2] & \text{complex}[3, -4] \\ \text{complex}[2, -3] & \text{complex}[4, -5] \end{pmatrix}$$

```
complex[1, 2] // DisplayForm
complex[1, 2] // TraditionalForm
pauliMatrix[2] // DisplayForm
```

$1 + 2\,i$

$1 + 2\,i$

```
{{0, complex[0, -1]}, {complex[0, 1], 0}}
```