Consecutive Submodularity and Relaxation Techniques for Combinatorial Optimization

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Abstract

We introduce the notion of consecutive submodularity and a notion of relaxation by expectation that lead to a simplification and exact solutions of certain combinatorial problems.

1 Preliminaries

Let $n \in \mathbb{N}$ be positive and set $\mathcal{V} = \{1, \ldots, n\}$. We consider optimization problems of the type

$$F^* = \max_{P=\{S_1,\ldots,S_t\}} \sum_{j=1\ldots t} F\left(S_j\right) \tag{1}$$

where $F\colon 2^{\mathcal{V}} \to \mathbb{R}$ is a set function, and $P = \{S_1, \ldots, S_t\}$ is a partition of $\mathcal{V}$. Setting $t = 1$ yields the well-studied case

$$F^* = \max_{S\subseteq\mathcal{V}} F\left(S\right) \tag{2}$$

Regularity assumptions on F must be made, since it is only discrete, and (quasi)convexity, concavity are not suitable What has turned out to be relevant in a wide variety of applications is the notion of *submodularity*.

Definition 1. *A set function $F\colon 2^{\mathcal{V}} \to \mathbb{R}$ is* submodular *if, for any $S, T \subseteq \mathcal{V}$, we have*

$$F\left(S\right) + F\left(T\right) \geq F\left(S \cap T\right) + F\left(S \cup T\right) \tag{3}$$

It can be shown that the submodularity property is equivalent to a notion of *diminishing returns* [Bach] in the sense that, for $A \subseteq B \subseteq \mathcal{V}$, and $x \in \mathcal{V} \setminus B$, we have

$$F\left(A \cup \{x\}\right) - F\left(A\right) \geq F\left(B \cup \{x\}\right) - F\left(B\right) \tag{4}$$

Assuming submodularity in F allows for approximate maximization in polynomial time, and exact minimization routines . . . [elaborate... we will elaborate in all its entirety the approximation and relaxation techniques for unconstrained submodular maximization and minimization]

In [PehlNeill], the authors arrive at (1) from the point of view of Spatial Scan Statistics. The *spatial scan statistic* is a widely used methodological approach for spatial and space-time cluster detection, first proposed by Kulldorff Kull1,KN and building on prior work in scan statistics by Naus, Glaz, and others Naus1,Naus2,GN. Spatial scanning has been used for detection of high-risk clusters for cancer and other chronic diseases, emerging infections in human and animal populations, suspicious network activity, areas of increased brain activity from imaging data, and many other applications Kull1,Neill1. In the usual spatial scan setting, data elements s_i, for $i \in \{1, \ldots, n\}$, represent spatial locations with associated values x_i (representing counts or concurrent measurements) and y_i (representing baselines, expectations, or populations). The goal is to identify a subset of locations with unusual (typically, increased) values of x_i as compared to the expected y_i. This is done by maximizing a log-likelihood ratio statistic $F(S)$ over subsets $S \subseteq \{s_1, \ldots, s_n\}$, assuming a parametric model for the x_i, usually within problem-specific constraints, etc.

Formally, let $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ be finite real sequences with $y_i > 0$, for all i. Denote by $\mathcal{D} = \mathcal{D}_{X,Y}$ the set of tuples $\{(x_i, y_i)\} = \{s_i\}$ associated with X, Y. $\mathcal{D}$ is assumed to have an order induced by a *priority function* on the sequences X, Y, in our case, $\mathcal{D}$ is ordered by the priority $g(x, y) = \frac{x}{y}$. It is assumed in the following that X, Y are ordered to satisfy $\frac{x_1}{y_1} \leq \frac{x_2}{y_2} \leq \cdots \leq \frac{x_n}{y_n}$. The diversity measurement is given by a *score function* which measures the anomaly at a location i by $f(x_i, y_i)$ or in a region $S \subseteq \mathcal{V}$ by $f\left(\sum_{i \in S} x_i, \sum_{i \in S} y_i\right)$. The notion of score function from the spatial scan statistics literature formalizes this.

Definition 2. *A* score function *is a continuous function* $f(x,y)\colon \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ *with continuous extension to the origin in any wedge* $\mathcal{W}(\mu_1, \mu_2) = \left\{(x,y) : y > 0, \mu_1 \leq \frac{x}{y} \leq \mu_2\right\}$*, for* $-\infty < \mu_1 \leq \mu_2 < \infty$*, with* $\lim_{(x,y) \in \mathcal{W} \to (0,0)} f(x,y) = 0$.

The regularity condition on $\mathcal{W}$ in wedges simply guarantees a continuous extension to the origin on any positive cone in $\mathbf{R}^+$, for *rational score functions* of the form $f(x,y) = x^\alpha y^{-\beta}$, $\alpha, \beta > 0$ the constraint corresponds to the requirement $\alpha > \beta$. We do not assume smoothness beyond continuity, nor (quasi)convexity, etc., unless explicitly stated. As above, f can be associated with a real-valued set function on $2^{\mathcal{V}}$ by defining $F(S) = f(\sum_{i \in S} x_i, \sum_{i \in S} y_i)$.

For a given f, We are interested in maximal partitions for F, i.e., solutions to the program

$$P^* = \underset{P=\{S_1,\ldots,S_t\}}{\operatorname{argmax}} \sum_{j=1\ldots t} F(S_j) = \underset{P=\{S_1,\ldots,S_t\}}{\operatorname{argmax}} \sum_{j=1\ldots t} f\left(\sum_{i \in S_j} x_i, \sum_{i \in S_j} y_i\right). \tag{5}$$

According the results in [PehlNeill], under certain conditions, we can restrict our attention to consecutive partitions.

Definition 3. *A* consecutive subset *of* $\mathcal{V}$ *is a subset of the form* $U_j^k = \{j, j+1, \ldots, k\}$ *for some* $1 \leq j \leq k \leq n$*. A* consecutive partition $\mathcal{P} = \{S_1, \ldots, S_t\}$ *is a partition of* $\mathcal{V}$ *such that each* S_i *is a consecutive subset.*

Letting $\mathcal{P}_c$ be the set of consecutive partitions of $\mathcal{V}$, it is easy to see that $|\mathcal{P}_c| = \frac{n(n+1)}{2} + 1$.

We fix the notation $U_j^k = \{j, j+1, \ldots, k\}$, $U_j = \{1, 2, \ldots, j\}$, and $U_{-k} = \{k, k+1, \ldots, n\}$., for $1 \leq j, k \leq n$. Call U_j an *ascending* consecutive set, while U_{-k} is *descending*. In addition let $C_x^{j,k} = \sum_{i=j}^k x_i$, $C_x^j = \sum_{i=1}^j x_i$, $C_x^{-k} = \sum_{i=k}^n x_i$, the ascending and descending versions. Similarly define $C_y^{j,k}$, C_y^j, C_y^{-k}, and write $C^{j,k} = \frac{C_x^{j,k}}{C_y^{j,k}}$, $C^j = \frac{C_x^j}{C_y^j}$, $C^{-k} = \frac{C_x^{-k}}{C_y^{-k}}$.

Definition 4. *The set function* F *satisfies the Consecutive Partitions Property (CPP) if the solution*

$$\mathcal{P}^* = \underset{\substack{\mathcal{P} \\ |\mathcal{P}|=t}}{\operatorname{argmax}} \sum_{j=1}^{T} F(S_j) = \underset{\substack{\mathcal{P} \\ |\mathcal{P}|=t}}{\operatorname{argmax}} \sum_{j=1}^{T} f(\sum_{i \in P_j} x_i, \sum_{i \in P_j} y_i) \tag{6}$$

is a consecutive partition, for all X, Y*.* F *satisfies the Weak Consecutive Partitions Property (WCPP) if the solution*

$$\mathcal{P}^* = \underset{\substack{\mathcal{P} \\ |\mathcal{P}|\leq t}}{\operatorname{argmax}} \sum_{j=1}^{T} F(S_j) \tag{7}$$

is a consecutive partition, for any X, Y*.* F *satisfies CPP(*$\mathbf{R}^+$*), WCPP(*$\mathbf{R}^+$*) if it satisfies CPP, WCPP, respectively, for* $X \subseteq \mathbf{R}^+$, Y*.*

The following was shown in [PehlNeill].

Theorem 1. *Let* f *be a score function and let* F *be the restriction of* f*,* $F\colon 2^{\mathcal{V}} \to \mathbb{R}$*, defined by* $F(S) = f\left(\sum_{i \in S} x_i, \sum_{i \in S} y_i\right)$*. If* f *is convex, then* F *satisfies WCPP. If in addition* f *is subadditive, then* F *satisfies CPP. For functions* F *satisfying either CPP or WCPP, there is an worst-case* $\mathcal{O}\left(n^2 t\right)$ *algorithm which finds an exact solution to (6) or (7).*

1.1 Relaxation Methods

These results can be placed in the context of some well-known relaxation methods for solving (1) directly, for which F is not *a priori* defined as a restriction of an ambient f. Although the program in (2) is more common in the literature, our aim is to study it as the $t = 1$ case of the more more general (1). The standard approaches specify a relaxation f of F, defined on the unit hypercube $[0,1]^n \subseteq \mathbb{R}^n$. For unconstrained minimization problems, the Lovasz extension F_L [Bach] defines a continuous, convex function on $[0,1]^n$ for which $F_L(\mathbb{1}_S) = F(S)$, for each $S \subseteq \mathcal{V}$, in the case that F is submodular. An exact minimization can be found by using an ellipsoid-like method directly on F_L. In contrast, the maximization in (1) is NP-hard, even for F submodular. An approximate solution can be obtained by defining the multilinear extension

$$F_M(x) = \sum_{S \subseteq \mathcal{V}} F(S) \prod_{i \in S} x_i \prod_{i \in \mathcal{V} \setminus S} (1 - x_i). \tag{8}$$

For F submodular, F_M satisfies a *diminishing returns* [Bian] condition $\frac{\partial^2 F_M}{\partial x_i \partial x_j} \leq 0$, for all i, j (this condition is sometimes referred to as *continuous submodularity* (Bach2)), so it is convex along all lines $\mathbf{e_i} - \mathbf{e_j}$ in $\mathbf{R}^n$. F_M admits an $(1 - \frac{1}{e})$ approximation to (1) by applying the greedy algorithm. Note that the results in the previous section do not rely on submodularity assumptions.

Example: Define $f\colon \mathbf{R}^2 \to \mathbf{R}$ by $f(x,y) = \frac{x^2}{y}$, and let F be the restriction of F to $2^{\mathcal{V}}$ For $X = \{0,7,8,9\}$, $Y = \{4,7,1,1\}$, and $S = \{1,3\}$. $T = \{0,2,3\}$, we have $F(S) + F(T) \approx 80.1667$ while $F(S \cup T) + F(S \cap T) \approx 125.3077$, hence F is not submodular. f is convex, subadditive on the upper half-plane, so by (1) an exact solution to (1) be found in quadratic time in n. F does satisfy a weaker, related condition.

Definition 5. *A set function $F\colon 2^{\mathcal{V}} \to \mathbb{R}$ is* weakly consecutive submodular *if, for any $U, V \subseteq \mathcal{V}$, U, V consecutive subsets, with $U \cup V$ consecutive, we have*

$$F(U) + F(V) \geq F(U \cap V) + F(U \cup V) \tag{9}$$

F is (strongly) consecutive submodular *if (9) holds for any consecutive $U, V \subseteq \mathcal{V}$.*

The condition $U \cup V$ be consecutive, for U, V consecutive is equivalent to $U \cap V \neq \emptyset$. There is an equivalent characterization of weak consecutive submodularity in terms of diminishing returns.

Proposition 1. *Let F be as above, then F is weakly consecutive submodular if and only if for any consecutive $U \subseteq \mathcal{V}$, we have*

(i) if $\max(U) = k \leq n$, and $U \subseteq V \subseteq U_k$ for V consecutive with $\max(V) = k$ then

$$F(U) - F(U \setminus \{k\}) \geq F(V) - F(V \setminus \{k\}), \tag{10}$$

(ii) if $\min(U) = j \geq 1$, and $U \subseteq V \subseteq U_{-j}$ for V consecutive with $\min(V) = j$ then

$$F(U) - F(U \setminus \{j\}) \geq F(V) - F(V \setminus \{j\}). \tag{11}$$

Proof. (i) To show sufficiency, let $U, V \subseteq \mathcal{V}$ be consecutive with $U \cap V \neq \emptyset$. If $U \subseteq V$ then the inequality (9) trivially holds. So without loss of generality we assume $\max V > \max U$. Write $V \setminus U = \{v_1, \ldots, v_r\}$, with $v_i > \max(U)$, for all i. Assume $\{v_1, \ldots, v_r\}$ is ordered. For each $i = 1, \ldots, r$,

$$\begin{aligned} &F((U \cap V) \cup \{v_1, \ldots, v_i\}) - F((U \cap V) \cup \{v_1, \ldots, v_{i-1}\}) \geq \\ &\qquad F(U \cup \{v_1, \ldots, v_i\}) - F(U \cup \{v_1, \ldots, v_{i-1}\}) \end{aligned}$$

so

$$\sum_{i=1}^{r}[F((U\cap V)\cup\{v_1,\dots,v_i\})-F((U\cap V)\cup\{v_1,\dots,v_{i-1}\})]\geq$$
$$\sum_{i=1}^{r}[F(U\cup\{v_1,\dots,v_i\})-F(U\cup\{v_1,\dots,v_{i-1}\})].$$

Therefore $F(U)-F(U\cap V)\geq F(U\cup V)-F(V)$. So F is weakly consecutive submodular. This concludes the proof of sufficiency. To show necessity, take $U\subseteq\mathcal{V}$ consecutive, with $\min(U)=j\geq 1$, and let $V\subseteq\mathcal{V}$ satisfy $U\subseteq V\subseteq U_{-j}$. Set $S=U$, $T=V\cap U_{-(j+1)}$, then $S\cap T=U\setminus\{j\}$, and $S\cup T=V$, and since S, T are both consecutive, $F(S)-F(S\cap T)\geq F(S\cup T)-F(T)$, and it follows that $F(U)-F(U\setminus\{k\})\geq F(V)-F(V\setminus\{k\})$.

(ii) This case is handled analogously.

□

Note that weak consecutive submodularity does not allow us to conclude that $F\left(C^{1,j-1}\right)-F\left(C^{1,1}\right)\geq F\left(C^{1,j}\right)-F\left(C^{1,1}\right)$, for example. We will see that there are consecutive submodular functions which are also *ascending boundary expanding,* the above inequality implies that $F\left(C^{1,j-1}\right)\geq F\left(C^{1,j}\right)$, whic the expanding property would imply $F\left(C^{1,j-1}\right)\leq\frac{C_y^{1,j-1}}{C_y^{1,j}}F\left(C^{1,j}\right)<F\left(C^{1,j}\right)$, a contradiction. Also note that the strong consecutive submodularity condition doesn't lend itself to an obvious analogous notion of diminishing returns; the argument requires that $U\cup\{v\}$ be consecutive, for some $v\in V$, which may not hold in the general $U\cap V=\emptyset$ case. [Is there a stronger notion of diminishing returns that is equivalent to strong consecutive submodularity?]

Example: Consider $f(x,y)=\frac{x^2}{y}$ as before, for arbitrary subsets $X\subseteq\mathbf{R},Y\subseteq\mathbf{R}^+$. We saw that there are subsets X,Y for which F is not submodular. We will show that F is consecutive submodular. To this end choose consecutive A,B in $\mathcal{V}$. Note that the cases $A\subseteq B$ and $A\cap B=\emptyset$ are trivial or follow from subadditivity of f, repsectively. So assume $A=\{i,\dots,k\}$, $B=\{j,\dots,l\}$, with $i\leq j\leq k\leq l$. Set

$$x_{A\setminus B}=\sum_{i}^{j}x_i\quad x_{A\cap B}=\sum_{j}^{k}x_i\quad x_{B\setminus A}=\sum_{k}^{l}x_i$$
$$y_{A\setminus B}=\sum_{i}^{j}y_i\quad y_{A\cap B}=\sum_{j}^{k}y_i\quad x_{B\setminus A}=\sum_{k}^{l}y_i$$

Then

$$Q_{A\setminus B}\leq Q_{A\cap B}\leq Q_{B\setminus A}$$

where $Q_{A\setminus B}=\frac{x_{A\setminus B}}{y_{A\setminus B}}$, $Q_{A\cap B}=\frac{x_{A\cap B}}{y_{A\cap B}}$ and $Q_{B\setminus A}=\frac{x_{B\setminus A}}{y_{B\setminus A}}$. To show that

$$I:=\frac{(x_{A\setminus B}+x_{A\cap B})^2}{y_{A\setminus B}+y_{A\cap B}}+\frac{(x_{A\cap B}+x_{B\setminus A})^2}{y_{A\cap B}+y_{B\setminus A}}-\frac{(x_{A\setminus B}+x_{A\cap B}+x_{B\setminus A})^2}{y_{A\setminus B}+y_{A\cap B}+y_{B\setminus A}}-\frac{(x_{A\cap B})^2}{y_{A\cap B}}\geq 0 \tag{12}$$

write

$$\begin{aligned}\frac{\partial I}{\partial x_{B\setminus A}}&=\frac{2\left(y_{A\setminus B}\left(x_{A\cap B}+x_{B\setminus A}\right)-x_{A\setminus B}\left(y_{A\cap B}-y_{B\setminus A}\right)\right)}{(y_{A\cap B}+y_{B\setminus A})(y_{A\setminus B}+y_{A\cap B}+y_{B\setminus A})}\\&=\frac{2y_{A\setminus B}y_{A\cap B}(Q_{A\cap B}-Q_{A\setminus B})y_{A\cap B}+(Q_{B\setminus A}-Q_{A\setminus B})y_{B\setminus A})}{(y_{A\cap B}+y_{B\setminus A})(y_{A\setminus B}+y_{A\cap B}+y_{B\setminus A})}\end{aligned}$$

Since this is everywhere nonnegative, and since $\frac{\partial}{\partial Q_{B\setminus A}} = \frac{1}{y_{B\setminus A}}\frac{\partial}{\partial x_{B\setminus A}}$ it is sufficient to show that the substitution $Q_{B\setminus A} = Q_{A\cap B}$ in 12 makes the expression nonnegative. Writing $x_{B\setminus A} = Q_{b\setminus A}y_{B\setminus A}$, this gives

$$\frac{(Q_{A\setminus B} - Q_{A\cap B})^2 y_{A\setminus B}^2 y_{B\setminus A}}{(y_{A\cap B} + y_{B\setminus A})(y_{A\setminus B} + y_{A\cap B} + y_{B\setminus A})}$$

which is positive. So f is consecutive submodular. That the cases $\alpha, \beta = \{(4,3), (5,4), \dots\}$ are also consecutive submodular will be shown from a more general result below.

We will see that the notion of consecutive submodularity alone is not enough to guarantee even WCPP. That it can come from restrictions of concave functions, as in the case of $f(x,y) = \log\left((1+x)(1+y)\right)$ is easy to see.

Proposition 2. *Let f be concave and 2-homogenous, and let $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$ be finite real sequences with $y_i > 0$. Then $F \colon 2^{\mathcal{V}} \to \mathbf{R}$ defined by $F(S) = f\left(\sum_{i\in S} x_i, \sum_{i\in S} y_i\right)$ is consecutive submodular.*

Proof. Under the assumptions f is also subadditive, as $f(x+y) = f(2(\frac{1}{2}x + \frac{1}{2}y)) \leq f(x) + f(y)$. Now assume $S_1 = \{j, \dots, l\}$, $S_2 = \{k, \dots, m\}$, with $j \leq k \leq l \leq m$. Writing $f(\sum_j^k)$ for $f(\sum_j^k x_i, \sum_j^k y_i)$, etc., we have

$$\begin{aligned}
f(S_1) + f(S_2) &= f(\sum_j^l) + f(\sum_k^m) = f(\sum_j^{k-1} + \sum_k^l) + f(\sum_k^l + \sum_{l+1}^m) \\
&\geq \frac{1}{2}f(2\sum_j^{k-1}) + \frac{1}{2}f(2\sum_k^l) + \frac{1}{2}f(2\sum_k^l) + \frac{1}{2}f(2\sum_{l+1}^m) \\
&= f(\sum_j^{k-1}) + 2f(\sum_k^l) + f(\sum_{l+1}^m) \\
&\geq f(\sum_j^m) + f(\sum_k^l) = f(S_1 \cup S_2) + f(S_1 \cap S_2).
\end{aligned}$$

□

1.2 Relaxation results for $X \subseteq \mathbf{R}^+$

The general idea contained in the results in Theorem 1 relate a dataset $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^n$ and a score function f to a natural set function F, and obtains simplifications to the unconstrained maximization problem when f enjoys certain properties (convexity, subadditivity, or both). We would like to understand when the converse is true: given a set function $F \colon 2^{\mathcal{V}} \to \mathbb{R}$ and sequences X, Y of $\mathbb{R}$, under what conditions can we view the function F as acting on a subset of $\mathcal{C}$ by $f\left(\sum_{S\subseteq\mathcal{V}} x_i, \sum_{S\subseteq\mathcal{V}} y_i\right) = F(S)$, for some well-behaved f? In our case we are looking for an extension whose domain lies in $\mathbf{R}^2$, not $\mathbf{R}^{\mathcal{V}}$, in order to use the CPP and WCPP machinery. The natural domain is the *partition polytope* of X, Y.

Definition 6. *For $S \subseteq \mathcal{V}$, let $p_S = (\sum_{i\in S} x_i, \sum_{i\in S} y_i) \in \mathbf{R}^2$. The* partition polytope *$\mathcal{C}$ and the* constrained partition polytope *$\underline{\mathcal{C}}$ are defined by $\mathcal{C} = \hat{P}$ and $\underline{\mathcal{C}} = \hat{\underline{P}}$ respectively, where $P = \{p_S : S \subseteq \mathcal{V}\}$ and $\underline{P} = \{p_S : S \subseteq \mathcal{V}, S \neq \emptyset, S \neq \mathcal{V}\}$, where $\hat{S}$ denotes the convex hull of the set S.*

When is it the case that F extends to a possibly convex, subadditive f on $\mathcal{C}$ or $\underline{\mathcal{C}}$ which allows for exact $\mathcal{O}\left(n^2 t\right)$ WCPP- or CPP-based exact maximiation over partitions of size t?

Submodularity alone does not help in our case. Take $X,\ Y \subseteq \mathbf{R}^+$ and consider $F(S) = \log\left(1+\sum_{i\in S} x_i\right)\left(1+\sum_{i\in S} y_i\right)$, the restricution of $f(x,y) = \log\left((1+x)(1+y)\right)$ Then F is submodular, subadditive, $F(0) = 0$, but concave in the upper half-plane. In fact, it is easy to see that F does not satisfy WCPP($\mathbf{R}^+$) so that there can be no convex extension to $\mathcal{C}$. The Lovasz extension to the n-dimensional hypercube, however, is guaranteed convex. f is not decreasing in y and is not a natural candidate for a score function in the context of spatial scan statistics, for that we may use $F(S) = \sqrt{\sum_{i\in S} x_i}$, which is submodular, subadditive, and whose natural extension to the ambient partition polytope is nondecreasing in x, nonincreasing in y, $f(0) = 0$, yet doesn't satisfy WCPP($\mathbf{R}^+$). We first investigate what properties the restriction F of a smooth score function f, along with X, Y arbitrary, that satisfies WCPP or CPP.

Lemma 1. *Let $f\colon R^2_+ \to \mathbf{R}$ be convex, with $f(0) = 0$. Then*

(i) if f is nondecreasing in x, then $\frac{x_1}{y_1} \le \frac{x_2}{y_2} \implies \frac{f(x_1,y_1)}{y_1} \le \frac{f(x_1+x_2,y_1+y_2)}{y_1+y_2}$

(ii) if f is nonincreasing in x, then $\frac{x_1}{y_1} \le \frac{x_2}{y_2} \implies \frac{f(x_2,y_2)}{y_2} \le \frac{f(x_1+x_2,y_1+y_2)}{y_1+y_2}$

(iii) if f is nondecreasing in y, then $\frac{x_1}{y_1} \le \frac{x_2}{y_2} \implies \frac{f(x_2,y_2)}{x_2} \le \frac{f(x_1+x_2,y_1+y_2)}{x_1+x_2}$

(iv) if f is nonincreasing in y, then $\frac{x_1}{y_1} \le \frac{x_2}{y_2} \implies \frac{f(x_1,y_1)}{x_1} \le \frac{f(x_1+x_2,y_1+y_2)}{x_1+x_2}$

Proof. (i) Let $(x_1, y_1),\ (x_2, y_2) \in \mathbf{R}^+ \times \mathbf{R}^+$ with $\frac{x_1}{y_1} \le \frac{x_2}{y_2}$. It then follows that $\frac{x_1}{y_2} \le \frac{x_1+x_2}{y_1+y_2} \le \frac{x_2}{y_2}$, so for $\alpha = \frac{y_1}{y_1+y_2}$ we have $\alpha(x_1 + x_2, y_1 + y_2) = (\tilde{x}, \tilde{y})$ satisfies $\tilde{x} \ge x_1,\ \tilde{y} = y_1$. Then

$$\begin{aligned}\alpha f(x_1 + x_2, y_1 + y_2) &= \alpha f(x_1 + x_2, y_1 + y_2) + (1-\alpha) f(0, 0)\\ &\ge f(\tilde{x}, \tilde{y})\\ &\ge f(x_1, y_1)\end{aligned}$$

as f is nondecreasing in x, so that $\frac{f(x_1+x_2,y_1+y_2)}{y_1+y_2} \ge \frac{f(x_1,y_1)}{y_1}$.

(ii) For $\alpha = \frac{y_2}{y_1+y_2}$ we have $\alpha(x_1 + x_2, y_1 + y_2) = (\tilde{x}, \tilde{y})$ satisfies $\tilde{x} \le x_2,\ \tilde{y} = y_2$. Then as above $\alpha f(x_1 + x_2, y_1 + y_2) \ge f(x_1, y_1)$ and the result follows.

Cases (iii), (iv) follow similar lines of reasoning. □

Lemma 2. *Let $f\colon \mathbf{R}^2_+ \to \mathbf{R}$ be convex, subadditive, with $f(0) = 0$. Then*

(i) if f is nondecreasing in x, then $\frac{x_1}{y_1} \le \frac{x_2}{y_2} \implies \frac{f(x_1,y_1)}{y_1} \le \frac{f(x_1+x_2,y_1+y_2)}{y_1+y_2} \le \frac{f(x_2,y_2)}{y_2}$

(ii) if f is nonincreasing in x, then $\frac{x_1}{y_1} \le \frac{x_2}{y_2} \implies \frac{f(x_2,y_2)}{y_2} \le \frac{f(x_1+x_2,y_1+y_2)}{y_1+y_2} \le \frac{f(x_1,y_1)}{y_1}$

(iii) if f is nondecreasing in y, then $\frac{x_1}{y_1} \le \frac{x_2}{y_2} \implies \frac{f(x_2,y_2)}{x_2} \le \frac{f(x_1+x_2,y_1+y_2)}{y_1+y_2} \le \frac{f(x_1,y_1)}{x_1}$

(iv) if f is nonincreasing in y, then $\frac{x_1}{y_1} \le \frac{x_2}{y_2} \implies \frac{f(x_1,y_1)}{x_1} \le \frac{f(x_1+x_2,y_1+y_2)}{y_1+y_2} \le \frac{f(x_2,y_2)}{x_2}$

Proof. (i) The inequality $\frac{f(x_1,y_1)}{y_1} \le \frac{f(x_1+x_2,y_1+y_2)}{y_1+y_2}$ follows from convexity of f and the previous lemma. If f is convex, subadditive, with $f(0) = 0$, then $f(1) \le 1$, and

$$f(nx) \le nf(x) \Rightarrow f((n+1)x) \le f(nx) + f(s) \le (n+1)f(x),$$

so that $f(nx) \le nf(x)$ for all $n \in \mathbf{N}$ by induction. Now let $\lambda \in \mathbf{R}$ and set $l = \lfloor \lambda \rfloor$, $\rho = \lambda - l$. Then $\lambda = \rho(l+1) + (1-\rho)l$ so by convexity, $f(\lambda x) \le \rho f((l+1)x) + (1-\rho)f(lx)$. Finally, let $\lambda \in [1, \infty)$. Then

$$\begin{aligned}f(\lambda x) &\le \rho f((l+1)x) + (1-\rho) f(lx)\\ &\le \rho(l+1)f(x) + (1-\rho)lf(x)\\ &= \lambda f(x),\end{aligned}$$

so that f is $[1,\infty)$-subhomogeneous. With (x_1,y_1), (x_2,y_2), $\frac{x_1}{y_1} \leq \frac{x_2}{y_2}$, setting $\alpha = \frac{y_1+y_2}{y_2}$, and $\alpha(x_2,y_2) = (\tilde{x},\tilde{y})$, it follows that $\tilde{x} \geq x_1+x_2$, and $\tilde{y} = y_1+y_2$. By subhomogeneity,

$$\begin{aligned}\alpha f(x_2,y_2) &\geq f(\tilde{x},\tilde{y})\\ &\geq f(x_1+x_2,y_1+y_2),\end{aligned}$$

from which it follows that $\frac{f(x_2,y_2)}{y_2} \geq \frac{f(x_1+x_2,y_1+y_2)}{y_1+y_2}$.

(ii) The inequality $\frac{f(x_2,y_2)}{y_2} \leq \frac{f(x_1+x_2,y_1+y_2)}{y_1+y_2}$ follows from the previous lemma, now set $\alpha = \frac{y_1+y_2}{y_1}$, then $\alpha\left(x_1,y_1\right) = \left(\tilde{x},\tilde{y}\right)$ with $\tilde{x} \leq x_1+x_2$, $\tilde{y} = y_1+y_2$. Again by subhomogeneity, $\alpha f\left(x_1,y_1\right) \geq f\left(\tilde{x},\tilde{y}\right)$, implying $\frac{f(x_1+x_2,y_1+y_2)}{y_1+y_2} \leq \frac{f(x_1,y_1)}{y_1}$

Cases (iii), (iv) follow similar lines of reasoning. □

We can apply this to the sequence

$$0 \leq \frac{C_x^1}{C_y^1} \leq \cdots \leq \frac{C_x^n}{C_y^n}$$

obtaining, for a convex score function f, nondecreasing in x

$$0 \leq \frac{f(U_1)}{C_y^1} \leq \cdots \leq \frac{f(U_n)}{C_y^n}. \tag{13}$$

Similarly, for the sequence

$$0 \leq \frac{C_x^n}{C_y^n} \leq \frac{C_x^{-2}}{C_y^{-2}} \leq \cdots \leq \frac{C_x^{-n}}{C_y^{-n}}$$

for a convex, subadditive score function f, nondecreasing in x

$$0 \leq \frac{f(U_n)}{C_y^n} \leq \frac{f(U_{-2})}{C_y^{-2}} \leq \cdots \leq \frac{f(U_{-n})}{C_y^{-n}}. \tag{14}$$

It was shown in [PehlNeill] that the points

$$\left\{\left(C_x^0,C_y^0\right),\left(C_x^1,C_y^1\right),\ldots,\left(C_x^n,C_y^n\right),\left(C_x^{-2},C_y^{-2}\right),\left(C_x^{-3},C_y^{-3}\right),\ldots\left(C_x^{-n},C_y^{-n}\right)\right\}$$

are precisely the vertices of the partition polytope $\mathcal{C}$ for a given $X \subseteq \mathbf{R}$, $Y \subseteq \mathbf{R}$ traversed in clockwise order starting at the origin. The relations in 13 state that $f\left(x,y\right)/y$ or $f\left(x,y\right)/x$ is increasing on the boundary points as they are traversed in a clockwise direction, starting at 0.

Definition 7. *Order the elements in $\mathcal{U}$ by $S_1,\ldots,S_{2n-1}$, and the corresponding polytope points $p_{S_1},\ldots,p_{S_{2n-1}}$ by*

$$S_1 = C^1,\ldots,S_n = C^n, S_{n+1} = C^{-2},\ldots,S_{2n-1} = S^{-n}$$

and let $p_{S_j} = (p_{j,x},p_{j,y})$, for all $j = 1,\ldots,2n-1$. The representation (F,X,Y) is boundary expanding in yif

$$0 \leq \frac{F(S_1)}{p_{1,y}} \leq \cdots \leq \frac{F(S_{2n-1})}{p_{2n-1,y}},$$

and boundary expanding in x if

$$0 \leq \frac{F(S_1)}{p_{n,x}} \leq \cdots \leq \frac{F(S_n)}{p_{n,x}}.$$

We say (F, X, Y) is boundary expanding if it is boundary expanding in y. In addition, (F, X, Y) is ascending boundary expanding in y if

$$0 \leq \frac{F(S_1)}{p_{n,y}} \leq \cdots \leq \frac{F(S_n)}{p_{n,y}},$$

where $S_1, \ldots, S_n$ are the ascecending consecutive nonsplitting subsets of $\mathcal{V}$,m and descending boundary expanding in y if

$$0 \leq \frac{F(S_n)}{p_{n,y}} \leq \frac{F(S_{-2})}{p_{-2,y}} \leq \cdots \leq \frac{F(S_{-n})}{p_{in,y}}.$$

The ascending and descending boundary expanding properties in x are defined similarly.

Corollary 1. *If the score function f is convex and nondecreasing in x, then its restriction F is boundary ascending. If f is also subadditive, F is boundary descending.*

In fact f as above satisfies a slightly strong condition which will be important later, it is *boundary expanding convex.*

Definition 8. *A set function F is* boundary expanding convex *if for any consecutive splitting sets $U_{i-1}, U_i, Ui+1$ with $\frac{C_x^{i-1}}{C_{i-1y}} \leq \frac{C_x^i}{C_{iy}} \leq \frac{C_x^{i+1}}{C_{i+1y}}$, we have*

$$\frac{\frac{F(U_i)}{C_y^i} - \frac{F(U_{i-1})}{C_y^{i-1}}}{\frac{C_x^i}{C_y^i} - \frac{C_x^{i-1}}{C_y^{i-1}}} \leq \frac{\frac{F(U_{i+1})}{C_y^{i+1}} - \frac{F(U_i)}{C_y^i}}{\frac{C_x^{i+1}}{C_y^{i+1}} - \frac{C_x^i}{C_y^i}}$$

Corollary 2. *If the score function f is convex, subadditive, and nondecreasing in x then its restriction F is boundary expanding convex.*

Proof. Define the function $h \colon \mathcal{C}_x \to \mathbf{R}^+$ by $h(x) = f(x, 1)$. Assume that the three sets U_{i-1}, U_i, U_{i+1} are all ascending consecutive ascending subsets of $\mathcal{V}$, so that $\frac{C_x^{i-1}}{C_y^{i-1}} \leq \frac{C_x^i}{C_y^i} \leq \frac{C_x^{i+1}}{C_y^{i+1}}$. Then, since f is homogeneous,

$$\frac{\frac{f\left(C_x^i, C_y^i\right)}{C_y^i} - \frac{f\left(C_x^{i-1}, C_y^{i-1}\right)}{C_y^{i-1}}}{\frac{C_x^i}{C_y^i} - \frac{C_x^{i-1}}{C_y^{i-1}}} \leq \frac{\frac{f\left(C_x^{i+1}, C_y^{i+1}\right)}{C_y^{i+1}} - \frac{f\left(C_x^i, C_y^i\right)}{C_y^i}}{\frac{C_x^{i+1}}{C_y^{i+1}} - \frac{C_x^i}{C_y^i}}$$

$$\Leftrightarrow \frac{h\left(\frac{C_x^i}{C_y^i}\right) - h\left(\frac{C_x^{i-1}}{C_y^{i-1}}\right)}{\frac{C_x^i}{C_y^i} - \frac{C_x^{i-1}}{C_y^{i-1}}} \leq \frac{h\left(\frac{C_x^{i+1}}{C_y^{i+1}}\right) - h\left(\frac{C_x^i}{C_y^i}\right)}{\frac{C_x^{i+1}}{C_y^{i+1}} - \frac{C_x^i}{C_y^i}}$$

This inequality follows directly from convexity of the auxiliary function h defined by $x \mapsto f(x, 1)$. The same argument holds if the sets are mixed ascending and descending, or all descending, as long as the first inequality holds, as h is increasing convex. □

We now give conditions on f under which the restriction F is consecutive submodular.

Proposition 3. *If the score function f is C^2-smooth, convex and subadditive, and $X \subseteq \mathbf{R}_+$, $Y \subseteq \mathbf{R}$, then the restriction F is weakly consecutive submodular.*

Proof. Let $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in \mathbf{R}^2$ satsify $x_1 > x_2$, $y_1 < y_2$. Denote by $p_1 \wedge p_2, p_1 \vee p_2$ the coordinate-wise minimum, maximum. Then

$$\int \int_R f_{xy} dx dy = f(p_1 \wedge p_2) + f(p_1 \vee p_2) - f(p_1) - f(p_2)$$

where R is the rectangle with vertices $\{p_1, p_2, p_1 \wedge p_2, p_1 \vee p_2\}$. So if $f_{xy} \le 0$ on R, then the quantity $f(p_1) + f(p_2) - f(p_1 \wedge p_2) - f(p_1 \vee p_2)$ is nonnegative.

We will first show that f satisfies property (ii) of 1. To this end consider the parallelogram P with vertices

$$\{q_1, q_2, q_3, q_4\} = \left\{\left(C_x^{j,k}, C_y^{j,k}\right), \left(C_x^{j,k-1}, C_y^{j,k-1}\right), \left(C_x^{j-1,k-1}, C_y^{j-1,k-1}\right), \left(C_x^{j-1,k}, C_y^{j-1,k}\right)\right\}$$

in order when traversing the parallelogram counterclockwise, and define $I = f(q_1) + f(q_3) - f(q_2) - f(q_4)$. By the diminishing returns property (1) it suffices to show that $I \ge 0$.

Consider the affine mapping $\Theta(x, y) = (\theta_1(x, y), \theta_2(x, y))$, where

$$\begin{aligned} \theta_1(x, y) &= C_x^{k,k} x + C_x^{j-1,j-1} y + C_x^{j,k-1} \\ \theta_2(x, y) &= C_y^{k,k} x + C_y^{j-1,j-1} y + C_y^{j,k-1} \end{aligned}$$

Θ maps the unit square S to the parallelogram, retaining the order of the vertices $\{\mathbf{e_1}, 0, \mathbf{e_2}, \mathbf{e_1} + \mathbf{e_2}\}$ and q_1, q_2, q_3, q_4. By considering the lifting $g = (f \circ \Theta)\colon S \to \mathbf{R}$ it suffices to show that $g_{xy} \le 0$. We have

$$\begin{aligned} \frac{\partial^2 g}{\partial x \partial y} &= \left[\frac{\partial^2 f}{\partial \theta_1^2}\frac{\partial \theta_1}{\partial y} + \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2}\frac{\partial \theta_2}{\partial y}\right]\frac{\partial \theta_1}{\partial x} + \frac{\partial f}{\partial \theta_1}\frac{\partial^2 \theta_1}{\partial x \partial y} \\ &+ \left[\frac{\partial^2 f}{\partial \theta_2 \partial \theta_1}\frac{\partial \theta_1}{\partial y} + \frac{\partial^2 f}{\partial \theta_1^2}\frac{\partial \theta_2}{\partial y}\right]\frac{\partial \theta_2}{\partial x} + \frac{\partial f}{\partial \theta_2}\frac{\partial^2 \theta_2}{\partial x \partial y} \\ &= \left[C_x^{k,k} C_y^{j-1,j-1} + C_x^{j-1,j-1} C_y^{k,k}\right] f_{xy} + C_x^{k,k} C_x^{j-1,j-1} f_{xx} + C_y^{k,k} C_y^{j-1,j-1} f_{yy} \end{aligned} \tag{15}$$

It was shown in the proof of 2 that for f convex, subadditive, with $f(0,0) = 0$, f is also $[1,)$-subhomogeneous. It is easy to see that f is $(0,1]$-subhomogeneous: for $\alpha \in (0, \infty]$ we have $\alpha f(x, y) = \alpha f(x, y) + (1 - \alpha) f(0, 0) \ge f(\alpha x, \alpha y)$, so that f is positive subhomogeneous. Since, for any $\lambda \in [0, \infty)$, $f\left(\lambda^{-1}\lambda x, \lambda^{-1}\lambda y\right) \le \lambda^{-1} f(\lambda x, \lambda y)$, so that $f(\lambda x, \lambda y) \ge \lambda f(x, y)$, f is also positive superhomogeneous, and therefore homogeneous. By Euler's theorem we can write $f(x, y) = f_x(x, y)\, x + f_y(x, y)\, y$. Differentiating this expression with respect to x, y gives

$$\begin{aligned} f_{xx} &= -\frac{y}{x} f_{xy} \\ f_{yy} &= -\frac{x}{y} f_{xy} \end{aligned}$$

The expression in (15) can then be written

$$\frac{\partial^2 g}{\partial x \partial y} = \left(C_x^{k,k} C_y^{j-1,j-1} + C_x^{j-1,j-1} C_y^{k,k} - \frac{\theta_2(x, y)}{\theta_1(x, y)} C_x^{k,k} C_x^{j-1,j-1} - \frac{\theta_1(x, y)}{\theta_2(x, y)} C_y^{k,k} C_y^{j-1,j-1}\right) f_{xy}$$

To show this is positive over the rectangle R, substitute $z = \frac{\theta_1(x,y)}{\theta_2(x,y)}$, and multiply the term in parentheses by z, obtaining the quadratic

$$q(z) = -C_y^{k,k} C_y^{j-1,j-1} z^2 + \left(C_x^{k,k} C_y^{j-1,j-1} + C_x^{j-1,j-1} C_y^{k,k}\right) z - C_x^{k,k} C_x^{j-1,j-1}.$$

It is easy to see that $q(z_1) = q(z_2) = 0$, where $z_1 = \frac{C_x^{j-1,j-1}}{C_y^{j-1,j-1}}, z_2 = \frac{C_x^{k,k}}{C_y^{k,k}}$, and the leading coefficient is negative. So q is positive for those points whose image is in the wedge $\mathcal{W}(\mu_1, \mu_2) = \left\{(x, y) : y > 0, \frac{C_x^{j-1,j-1}}{C_y^{j-1,j-1}} \le \frac{x}{y} \le \frac{C_x^{k,k}}{C_y^{k,k}}\right\}$. But the parallelogram P clearly lies in the wedge, being bounded by $\frac{C_x^{j-1,k-1}}{C_y^{j-1,k-1}} \le \frac{x}{y} \le \frac{C_x^{j,k}}{C_y^{j,k}}$. So $\frac{\partial^2 g}{\partial x \partial y} \le 0$ and $I \ge 0$, from which consecutive submodularity follows.

To show that property (i) of 1 is satisfied, use similar reasoning on the parallelogram P defined by

$$\{q_1, q_2, q_3, q_4\} = \left\{\left(C_x^{j+1,k}, C_y^{j+1,k}\right), \left(C_x^{j+1,k+1}, C_y^{j+1,k+1}\right), \left(C_x^{j,k+1}, C_y^{j,k+1}\right), \left(C_x^{j,k}, C_y^{j,k}\right)\right\}$$

The affine mapping which takes the unit square R to P is $\Theta(x,y) = (\theta_1(x,y), \theta_2(x,y))$, where

$$\theta_1(x,y) = C_x^{k+1,k+1}x + C_x^{j,j}y + C_x^{j+1,k}$$
$$\theta_2(x,y) = C_y^{k+1,k+1}x + C_y^{j,j}y + C_y^{j+1,k}$$

We again calculate the mixed partial $\frac{\partial^2(\circ\Theta)}{\partial x \partial y}$ and after applying the homogeneity relations we obtain the quadratic

$$q(z) = -C_y^{j,j}C_y^{k+1,k+1}z^2 + \left(C_x^{k+1,k+1}C_y^{j,j} + C_x^{j,j}C_yk+1,k+1\right)z - C_x^{k+1,k+1}C_x^{j,j}$$

which has roots $z_1 = \frac{C_x^{j,j}}{C_y^{j,j}}, z_2 = \frac{C_x^{k+1,k+1}}{C_y^{k+1,k+1}}$ which form a cone which bounds P, from which the desired inequality follows. □

Corollary 3. *For f C^2-smooth, convex and subadditive, and $X \subseteq \mathbf{R}^+$, $Y \subseteq \mathbf{R}$, the restriction F is strongly consecutive submodular and boundary expanding convex.*

Proof. F is weakly consecutive submodular by the previous Proposition. For $U, V \subseteq \mathcal{V}$ consecutive with $\cap V = \emptyset$, the inequality (9) follows trivially by subadditivity. The boundary expanding convex property follows from Corrollary 2. □

The set of rational score functions $f \colon \mathbf{R}_+^2 \to \mathbf{R}$ of the form $f(x,y) = x^\alpha y^{-\beta}$, for $\alpha > \beta \geq 0$ can easily be classified. The principal minors of the Hessian are given by

$$M_1 = \alpha(\alpha-1)x^{\alpha-2}y^{-\beta},$$
$$M_2 = \alpha\beta(\alpha-\beta-1)x^{2(\alpha-1)}y^{-2(\beta+1)},$$

and f is subadditive if and only if $\alpha - \beta \leq 1$, hence, for F, the restriction of F given $X, Y \subseteq \mathbf{R}^+$

$$F \text{ is consecutive submodular} \Leftrightarrow \alpha - \beta = 1$$

None of the rational score functions are submodular [1].

$$f \text{ rational satisfies CPP} \Leftrightarrow \alpha - \beta = 1, \alpha \text{ even} \Leftrightarrow f \text{ consecutive submodular}$$
$$f \text{ rational satisfies CPP}(\mathbf{R}^+) \Leftrightarrow \alpha - \beta = 1, \Leftrightarrow f \text{ consecutive submodular on } \mathbf{R}^+$$

It is clear from the examples $F(S) = \sqrt{(\sum_{i\in S} x_i)^2 + (\sum_{i\in S} y_i)^2}$, $F(S) = \log\left((1 + \sum_{i\in S} x_i)(1 + \sum_{i\in S} y_i)\right)$ that consecutive submodularity is not sufficient to guarantee even that WCPP holds. While both are consecutive submodular, the first has a natural convex, subadditive extension to the ambient polytope while the second cannot. But that may not be clear for an arbitrary set function not obtained as a restriction of a continuous function. We seek properties of F, likely coupled with consecutive submodularirity, that guarantee CPP. CPP itself will be obtain as a result of Theorem 1, that is, for a given F we seek an extension to the ambient polytope which is convex, subadditive. The extension will not be used as the objective in any relaxation problem, but rather to indicate the desirability of optimizing F directly - by restricting attention to consecutive partitions. So the extension is not of practical use, although in many cases it can

[1] For $X = \{0,7,8,9\}$, $Y = \{4,7,1,1\}$, and $A = \{1,3\}$. $B = \{0,2,3\}$, we have $F(A) + F(B) \approx 80.1667$ while $F(A \cup B) + F(A \cap B) \approx 125.3077$

be readily calculated (see examples below). To this end, assume $X, Y \subseteq \mathbf{R}^+$, and assume that the set function $F\colon 2^{\mathcal{V}} \to \mathbf{R}$ is defined by $F(S) = f(\sum_{i\in S} x_i, \sum_{i\in S} y_i)$ for some f, not necessarily the restriction of any real-valued function on $\mathbf{R}^2$. We will call the triple (F, X, Y) a *planar representation*; it essentially defines a real-valued mapping on a subset of points of the partition polytope of (X, Y) for F. We will generally consider the set of all planar representations for F. The additional property on F relevant to CPP($\mathbf{R}^+$) beyond consecutive submodularity is also defined with respect to the consecutive subsets of $\mathcal{V}$.

Note that the function f is positive subhomogeneous, as it was shown that it is both $(0,1)-$ and $1,-$ subhomogeneous. If f is positive homogeneous and twice-differentiable, then it can be written $f(x) = \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y$, and differentiating both sides with respect to x gives $x \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial x y^2} = 0$, or $\frac{\partial^2}{\partial x y} = -\frac{x}{y} \frac{\partial^2 f}{\partial x^2}$. Since f is coordinatewise convex, the left side is always negative. The negative mixed partial implies that the f is *continuous submodular* [**?**], [**?**], i.e., that

$$f(x) + f(y) \geq f(x \wedge y) + f(x \vee y), \tag{16}$$

where$\wedge$, $\vee$ denote the coordinate-wise minimum and maximum functions on the vectors x, y. Integrating $\frac{\partial f^2}{\partial x y}$ over any rectangle with vertices $(x_1, y_1), (x_2, y_2), (x_1 \wedge x_2, y_1 \wedge y_2), (x_1 \vee x2, y_1 \vee y_2)$ immediately yields the inequality in (16). The assumptions above only guarantee that f is real-subhomogeneous.

Corollary 4. *If the score function f is convex, then F is ascending boundary expanding. If f is convex and $(1,\infty)$-subhomogeneous, then it is ascending and descending boundary expanding.*

1.3 Notes on obtaining f from F

Notes on obtaining desirable relaxation f from arbitrary set function F. One of the nice aspects of obtaining F as a "restriction" is that there is an attached notion of boundary and interior for the domain of F viewed as a polytope in $\mathbf{R}^2$. All F have the same domain $\mathcal{V}$, so this domain really depends on $X, Y \subseteq \mathbf{R}^2_+$. We would like to eliminate the role of f and determine when F can be relaxed to obtain an f which is convex, subadditive, or both. We'll have to specify X, Y and an ambient f. If we have, for $S \subseteq \mathcal{V}$, $F(S) = f\left(\sum_{i\in S} x_i, \sum_{i\in S} y_i\right)$ for some convex, additive f then it is clear we can optimize F by simply evaluating f on a discrete subset of the boundary of $\mathcal{C}$. Note that this is also true if $F(U) = f\left(\sum_{i\in U} x_i, \sum_{i\in U} y_i\right)$ holds for consecutive nonsplitting $U \subseteq \mathcal{V}$ (i.e. on the "boundary"), and $F(S) \geq f\left(\sum_{i\in S} x_i, \sum_{i\in S} y_i\right)$ for any other S. We can use this to define a relaxation for F. Denote by $\mathcal{U}$ the set of consecutive nonsplitting subsets in $2^{\mathcal{V}}$ and set

$$\underline{f}(x) = \min \left\{ \sum_{U} \lambda_U F(U) \colon U \in \mathcal{U},\, x = \sum_{U \in \mathcal{U}} \lambda_U \cdot p_U,\, \lambda_U \geq 0, \text{ for all } U \in \mathcal{U} \right\}$$

By knowledge of the geometry of $\mathcal{C}$ (vertices are subsets in $\mathcal{U}$), f^- is well-defined on $\mathcal{C}$. We have

$$\begin{aligned}
\underline{f}(x) &= \min\left\{\sum_U \lambda_U F(U) \colon U \in \mathcal{U},\, x = \sum_{U\in\mathcal{U}} \lambda_U \cdot p_U,\, \lambda_U \geq 0, \text{ for all } U \in \mathcal{U}\right\} \\
&= \min_{\substack{\lambda_U \geq 0 \\ U\in\mathcal{U}}} \left\{\sum_U \lambda_U F(U) \colon x = \sum_{U\in\mathcal{U}} \lambda_U \cdot p_U\right\} \\
&= \min_{\substack{\lambda_U \geq 0 \\ U\in\mathcal{U}}} \max_{s\in\mathbf{R}^2} \left\{\sum_{U in \mathcal{U}} \lambda_U F(U) + \sum_{k=1}^{2} s_k \left(x_k - \sum_{U\in\mathcal{U}} \lambda_U \cdot p_U\right)\right\} \\
&= \min_{\substack{\lambda_U \geq 0 \\ U\in\mathcal{U}}} \max_{s\in\mathbf{R}^2} \left\{s' \cdot x - \sum_{U\in\mathcal{U}} \lambda_U \left(s' \cdot p_U - F(U)\right)\right\} \\
&= \max_{s\in\mathbf{R}^2} \min_{\substack{\lambda_U \geq 0 \\ U\in\mathcal{U}}} \left\{s' \cdot x - \sum_{U\in\mathcal{U}} \lambda_U \left(s' \cdot p_U - F(U)\right)\right\} \\
&= \max_{s\in\mathcal{B}(F)} x' \cdot s = \overline{f}\,(x)\,,
\end{aligned}$$

where $\mathcal{B}\,(F) = \left\{s \in \mathbf{R}^2 \colon s' \cdot p_U - F(U) \leq 0, \text{ for all } U \in \mathcal{U}\right\}$. Call $\mathcal{B}(F)$ the *dual polytope* of F. We also define second function

$$\underline{\hat{f}}(x) = \min\left\{\sum_U \lambda_U F(U) \colon U \in \mathcal{U},\, x = \sum_{U\in\mathcal{U}} \lambda_U \cdot p_U,\, \sum_{U\in\mathcal{U}} \lambda_U = 1,\, \lambda_U \geq 0, \text{ for all } U \in \mathcal{U}\right\},$$

which by similar reasoning can be written as

$$\overline{\hat{f}}\,(x) = \max_{s\in\hat{\mathcal{B}}(F)} (x_0, x_1, 1)' \cdot s$$

, (17)

where $\hat{\mathcal{B}}\,(F) = \left\{s \in \mathbf{R}^3 \colon s' \cdot p_U + s_3 - F(U) \leq 0, \text{ for all } U \in \mathcal{U}\right\}$

There are two primals, two duals there, since these are linear programs we have strong duality so that $\underline{f} == \overline{f}$ and $\underline{\hat{f}} == \overline{\hat{f}}$, but it is not generally the case that $\underline{f} == \underline{\hat{f}}$, etc.

Some stylized facts

(i) Let F be boundary expanding. The inequalities $0 \leq \frac{C_x^1}{C_y^1} \leq \cdots \leq \frac{C_x^n}{C_y^n} \leq \frac{C_x^{-2}}{C_y^{-2}} \leq \cdots \leq \frac{C_x^{-n}}{C_y^{-n}}$, and the boundary expanding property imply that

$$\begin{aligned}
0 &\geq \frac{-C_x^1}{C_y^1} \geq \cdots \geq \frac{-C_x^n}{C_y^n} \geq \frac{-C_x^{-2}}{C_y^{-2}} \geq \cdots \geq \frac{-C_x^{-n}}{C_y^{-n}} \\
0 &\leq \frac{F(C_x^1, C_y^1)}{C_y^1} \leq \cdots \leq \frac{F(C_x^n, C_y^n)}{C_y^n} \leq \frac{F(C_x^{-2}, C_y^{-2})}{C_Y^{-2}} \leq \cdots \leq \frac{F(C_x^{-n}, C_y^{-n})}{C_y^{-n}}
\end{aligned}$$

Fix $U \in \mathcal{U}$. Then either $U = \{1, \ldots, j\}$, for $1 \leq j \leq n$, or $U = \{k, \ldots, n\}$, for $1 \leq k \leq n$. The defining inequality for the dual polytope can be written $s_2 \leq \frac{-C_x}{C_y} + \frac{F(C_x, C_y)}{C_y}$ consecutive nonsplitting set U with $p_U = (C_x, C_y)$. The inequalities above reveal the geometry of the dual polytope - each successive hyperplane boundary has a larger y-intercept and a larger negative

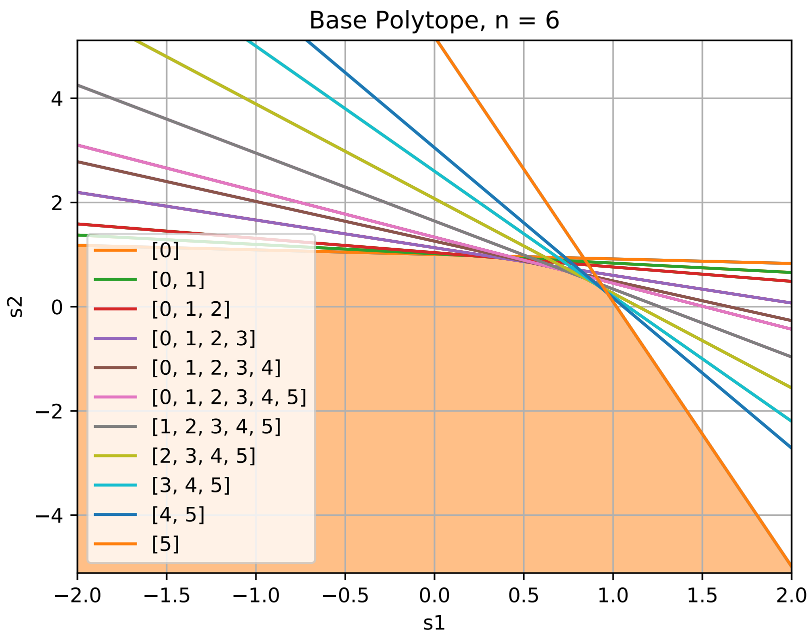


Figure 1: Base polytope for case $n = 6$, corresponding to $F(S) = \frac{(\sum_{i \in S} x_i)^2}{\sum_{i \in S} y_i}$, $x = \{0.85, 2.05, 2.67, 7.21, 8.2, 4.03\}$, $y = \{9.79, 6.35, 4.07, 3.86, 3.45, 0.79\}$

slope (see Figure ??) The resulting region is convex as it is the intersection of convex sets (it is always convex whether F is boundary expanding or not). It is nicely behaved when F is boundary expanding, it is even nicer when F satisfies a stronger condition on the boundary. Two consecutive (in the sense of the ordering above) consecutive nonsplitting sets U_{i-1}, U_i, determine a boundary point in $\mathcal{B}(F)$ defined by the intersection of the lines

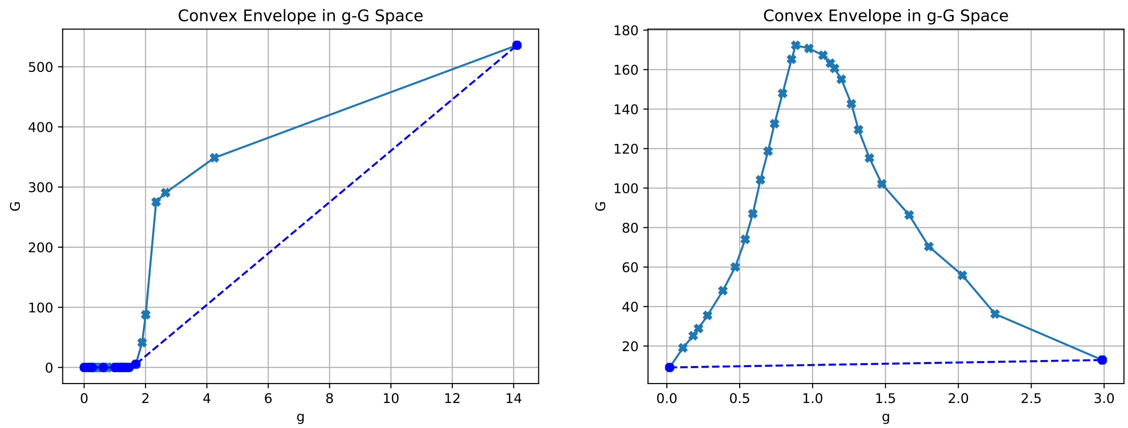
$$l_{i-1} = C_x^{U^{i-1}} s_1 + C_x^{U^{i-1}} s_2 = F(U_{i-1})$$
$$l_i = C_x^{U^i} s_1 + C_x^{U^i} s_2 = F(U_i)$$

A sufficient condition for the intersection of l_{i-1}, l_i to be a binding constraint in $\mathcal{B}(F)$ is that the x-intercept of the point in $l_{i-1} \cap l_i$ is smaller than the x-intercept of the point in $l_i \cap l_{i+1}$. A little algebra shows that this is equivalent to

$$\frac{\left[\frac{F(U_i)}{C_y^i} - \frac{F(U_{i-1})}{C_y^{i-1}}\right]}{\left[\frac{C_x^{U_i}}{C_y^{U_i}} - \frac{C_x^{U_{i-1}}}{C_y^{U_{i-1}}}\right]} \leq \frac{\left[\frac{F(U_{i+1})}{C_y^{i+1}} - \frac{F(U_i)}{C_y^i}\right]}{\left[\frac{C_x^{U_{i+1}}}{C_y^{U_{i+1}}} - \frac{C_x^{U_i}}{C_y^{U_i}}\right]} \tag{18}$$

Writing $g_i = \frac{C_x^{U_i}}{C_y^{U_i}}$ and $G_i = \frac{F(U_i)}{C_y^{U_i}}$ the above can be written as $\frac{G_i - G_{i-1}}{g_i - g_{i-1}} \leq \frac{G_{i+1} - G_i}{g_{i+1} - g_i}$. If this inequality holds then each successive consecutive $U \in \mathcal{U}$ with the ordering above defines a new vertex for the base polytope as in the figure.

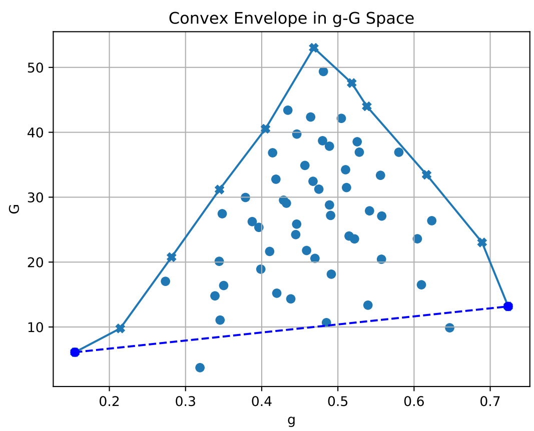
(ii) It is convenient to view the base polytope as a plot in $g - G$ space by the above equivalence (each $U \subseteq \mathcal{U}$ is associated with the point (g_i, G_i). That F is boundary expanding means that this graph is monotonic nondecreasing. The inequality in the previous item means that it is convex (and nondecreasing, since X, Y consist of positive reals). The binding constraints in the



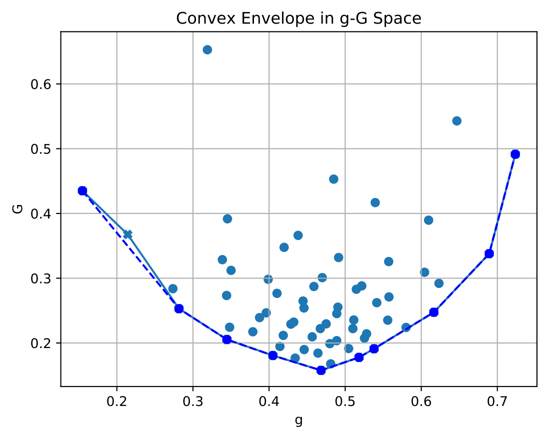
(a) Typical plot of set $\mathcal{U}$ with convex envelope as dotted line.

(b) Typical plot of set $\mathcal{U}$ with convex envelope as dotted line.

base polytope are associated with the convex envelope of the graph of the set $\mathcal{U}$ in $g - G$ space, the largest convex function that is dominated by the graph (Figure ??)



(a) Typical plot of set $\mathcal{U}$ with convex envelope as dotted line and image of interior points.



(b) Typical plot of set $\mathcal{U}$ with convex envelope as dotted line and image of interior points.

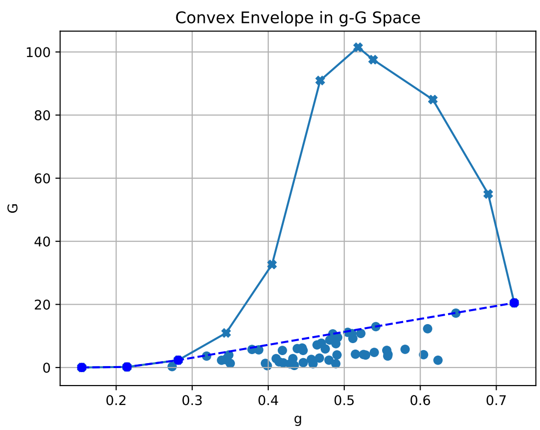
(iii) The function $\underline{f}$ is convex, subadditive, and homogeneous, with $f(0) = 0$. (any function on $\mathbf{R}^n$ which is convex and subadditive with $f(0) = 0$ is homogeneous, while for the converse, for f homogeneous and $f(0) = 0$, any one implies the other. The function $\overline{f}$ is convex with $f(0) = 0$. So formally defining our extension to be one or the other, we can obtain the properties we want.

(iv) The main remaining detail is to find conditions that ensure that one of our relaxations, or both, dominate F on the interior. We can see that $\underline{f} \le \overline{f}$, therefore $\hat{\underline{f}} \le \hat{\overline{f}}$. I found that it is easier to work with the base polytope and the "hat", or max versions of the relaxations. I then think of the base polytope as an image in $g - G$ space. In order for the $\underline{f}(p_S) \ge F(S)$ to be satisfied, the line $l_S = C_x^{U_S} s_1 + C_x^{U_S} s_2 = F(S)$ must intersect $\mathcal{B}(F)$, as we need some nontrivial portion of $\mathcal{B}(F)$ to be on the "negative" side of hyperplane l_S. For this we need the condition that the point(g_S, G_S) be below the convex envelope. If the previous inequality holds, we can find the adjacent consecutive splitting subsets U_{i-1}, U_i satisfying $g_{i-1} \le g_S \le g_i$, and check that

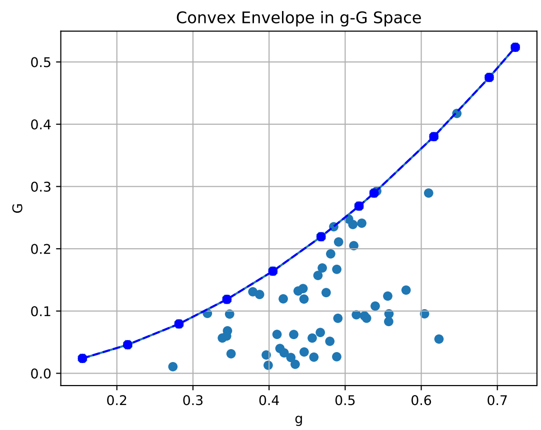
$$\frac{G_S - G_{i-1}}{g_S - g_{i-1}} \le \frac{G_i - G_{i-1}}{g_i - g_{i-1}}, \tag{19}$$

i.e. that the point is below the segment joining (g_{i-1}, G_{i-1}) and (g_i, G_i). Otherwise we have to check that the point is below the convex envelope. If we do a scatterplot overlay of the image interior points with the image of the consecutive nonsplitting points and their convex envelope, we can distinguish good cases from bad cases. In the first pair of graphs, the interior points will not in general satisfy $\underline{f}(p_S) \ge F(S)$ while in the second pair, they will.

(v) I think it is the case that if F allows for a convex relaxation, it will be defined by $\underline{f}$ and WCPP will be satisfied, while $\overline{f}$ will define a convex subadditive relaxation for which we can apply CPP. The conditions I've found are similar to the above inequality, they are analytical, not set-theoretical. As discussed, we have to furnish an X, Y, then define a relaxation as above. The boundary conditions only seem to stipulate conditions on Y, so something is needed in the interior. Consecutive submodularity doesn't specify any constraints on X - maybe there is a condition similar to it that is the one we're looking for. Also note that I now don't think that the boundary expanding condition is needed; if F is boundary expanding convex as described above, the base polytope is easy to handle and each consecutive nonsplitting subset gives rise to a binding constraint. Otherwise we can say this: f allows for a nice relaxation (convex, subadditive) if the image of all interior points is below the convex envelope. Is there a nicer characterization?



(a) Typical plot of set $\mathcal{U}$ with convex envelope as dotted line and image of interior points.



(b) Typical plot of set $\mathcal{U}$ with convex envelope as dotted line and image of interior points.