Asymptotic Properties of the Hill estimator

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Abstract

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Preface

I want to thank Professor Pirjo Professori and my instructor Dr Alan Advisor for their good and poor guidance.

Otaniemi, 24.4.2018

Eddie E. A. Engineer

Contents

Symbols and abbreviations

Symbols

$x^* = \sup\{x : F(x) < 1\}$	right endpoint of the distribution
γ	extreme value index
$F^{\leftarrow}(y) = \inf\{x : F(x) \geq y\}$	left-continuous inverse
U	left-continuous inverse of $\frac{1}{1-F}$
$\mathbb{1}(p) = \begin{cases} 1, \text{if p is true} \\ 0, \text{otherwise} \end{cases}$	indicator fuction
$X_{i,n}$	ith order statistic

Abbreviations

cdf	cumulative distribution function
i.d.d.	independent and identically distributed
a.s.	almost surely

1 Introduction

2 Backround

2.1 Fisher-Tippett-Gnedenko Theorem

First approach to study the behavior of extreme events could be to find limiting distribution of the sample maxima $M_n = \max(X_1, X_2, ..., X_n)$. Here $X_1, X_2, ..., X_n$ are i.d.d. random variables from cdf F_X. Function for the cdf of M_n can be easily derived, since $X_1, X_2, ..., X_3$ are i.d.d.

$$P(\max(X_1, X_2, ..., X_n) \leq x) = P(X_1 \leq x, X_2 \leq x, ..., X_n \leq x) = \\ P(X_1 \leq x)P(X_2 \leq x)...P(X_n \leq x) = F^n(x).$$

Now it can be shown that this approach is not very useful since

$$\lim_{n\to\infty} F^n(x) = \begin{cases} 0, x < x^* \\ 1, x \geq x^*. \end{cases}$$

To achieve a nondegerate distribution it is necessary to normalize the sample maxima M_n. After normalization a nondegenate distribution is gained as stated in the Fisher-Tippett-Gnedenko Theorem [1].

Theorem 2.1. *There exists real constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} F^n(a_n x + b_n) = G_\gamma(ax + b),$$

where

$$G_\gamma(x) = \begin{cases} \exp(-(1+\gamma x)^{-\frac{1}{\gamma}}), \gamma \neq 0 \\ \exp(-e^{-x}), \gamma = 0, \end{cases}$$

for all x with $1 + \gamma x > 0$ where $\gamma \in \mathbb{R}$.

2.2 Regularly Varying Functions

2.3 Domain of Attraction: Case $\gamma > 0$

3 Hill Estimator

3.1 Consistency

The following theorem states that Hill estimator is consistent i.e. estimator converges in probability to extreme value index. [1]

Theorem 3.1. *Let $X_1, X_2, ...$ be i.d.d. variables with cdf F_X. Suppose $F_X \in D(G_\gamma)$ with $\gamma > 0$. Then as $n \to \infty$, $k = k(n) \to \infty$, $\frac{k}{n} \to \infty$,*

$$\hat{\gamma}_H \xrightarrow{p} \gamma.$$

For the proof of the above theorem following lemmas are needed, firstly the Renyi's representation [2].

Lemma 3.2. *If $E_1, E_2, ...$ are i.d.d. random variables from the standard exponential distribution and $E_{1,n} \leq E_{2,n} \leq ... \leq E_{n,n}$ then for $k \leq n$ we have*

$$\Big(E_{1,n}, E_{2,n}, ..., E_{k,n}\Big) \stackrel{d}{=} \left(\frac{E_1^*}{n}, \frac{E_1^*}{n} + \frac{E_2^*}{n-1}, ..., \frac{E_1^*}{n} + \frac{E_2^*}{n-1} + ... + \frac{E_k^*}{n-k+1}\right),$$

where $E_1^*, E_2^*, ...$ are i.d.d. random variables from standard exponential distribution.

Secondly the lemma about the order statistics of Pareto distribution is necessary [1].

Lemma 3.3. *Let $Y_1, Y_2, ...$ be i.d.d. random variables from Pareto distribution $F_Y(y) = 1 - \frac{1}{y}$, $y \geq 0$ and let $Y_{1,n} \geq Y_{2,n} \geq ... \geq Y_{n,n}$ be the nth order statistics. Then with such $k = k(n)$ that $k \to \infty$, $\frac{k}{n} \to 0$ as $n \to \infty$,*

$$\lim_{n \to \infty} Y_{n-k,n} = \infty \quad a.s.$$

Last lemma that we need says that U(Y) is equal in distribution to X, where Y is random variable from pareto distribution and X is random variable from some distribution F_X.

Lemma 3.4. *Let Y be random variable from Pareto distribution $F_Y = 1 - \frac{1}{y}$, $y \geq 0$ Let X be random variable with cdf F_X then $U(Y) \stackrel{d}{=} X$.*

Next we prove the lemma 3.3. Proof of the lemma 3.2 is omitted here.

Proof. Let us assume that $Y_{n-k,n} < r$ for some $r > 0$ infinitely often. In other words

$$\frac{k}{n} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(Y_i > Y_{n-k,n}) > \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(Y_i > r).$$

Now the left side of the equation converges to zero, since

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(Y_i > Y_{n-k,n}) = \lim_{n\to\infty} \frac{k}{n} = 0.$$

But the right side converges to $1/r$ almost surely, since

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(Y_i > r) \xrightarrow{a.s} P(Y_i > r) = 1 - F_Y(r) = \frac{1}{r}$$

by the strong law of large numbers [3]. So the assumption cannot hold which implies that

$$P(\lim_{n\to\infty} Y_{n-k,n} = \infty) = 1.$$

$\square$

Now we prove the last lemma 3.4 that is needed for the proof of theorem 3.1

Proof. Let's study the condition $U(Y) \leq a, a \in \mathbb{R}$.

$$\begin{aligned} & U(Y) \leq a \\ \Leftrightarrow & \inf\Big\{x : \frac{1}{1 - F_X(x)} \geq Y\Big\} \leq a \\ \Leftrightarrow & \inf\Big\{x : 1 - \frac{1}{Y} \leq F_X(x)\Big\} \leq a \end{aligned} \tag{1}$$

Let $S = \Big\{x : 1 - \frac{1}{Y} \leq F_X(x)\Big\}$ and $b = \inf S$. Notice that F is increasing and right-continuous, since F is a cdf. So S is an interval of form $[b, \infty)$ or (b, ∞), since F is increasing. Let's define a sequence $x_n = b + \frac{1}{n}, n \in \mathbb{N}$. Notice that $x_n \to b$ and $x_n \in S$ for all n. Now right-continuity implies that $b \in S$ i.e S is an interval $[b, \infty)$. Additionally $a \in S$ since $a \geq b$ so a satisfies the condition $1 - \frac{1}{Y} \leq F(a)$. Therefore the equation 1 implies

$$U(Y) \leq a \Leftrightarrow 1 - \frac{1}{Y} \leq F_X(a) \Leftrightarrow Y \leq \frac{1}{1 - F(a)},$$

So now from the cdf of U(Y) we have

$$\begin{aligned} F_{U(Y)} = P(U(Y) \leq x) = P\Big(Y \leq \frac{1}{1 - F_X(x)}\Big) &= F_Y\Big(\frac{1}{1 - F_X(x)}\Big) \\ &= 1 - \Big(\frac{1}{1 - F_X(x)}\Big)^{-1} = F_X(x). \end{aligned}$$

$\square$

Now we are equipped to prove the theorem 3.1.

Proof. $F \in D(G_{\gamma>0})$ is equivalent to the fact that $U \in RV_\gamma$ i.e.

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\gamma.$$

From the uniform convergence of the regularly varying functions follows that for $x > 1$ and $t \geq t_0$,

$$(1-\varepsilon)x^{\gamma-\delta} < \frac{U(tx)}{U(t)} < (1+\varepsilon)x^{\gamma+\delta},$$

for all $\varepsilon > 0$ and $\delta > 0$. By taking natural logarithm from both sides of the equation above, it can be written as

$$\begin{aligned} \log(1-\varepsilon) + (\gamma-\delta)\log(x) &< \log(U(tx)) - \log(U(t)) \\ &< \log(1+\varepsilon) + (\gamma+\delta)\log(x). \end{aligned} \tag{2}$$

If $Y_1, Y_2, ...$ are i.d.d random variables from Pareto distribution with cdf $F_Y(y) = 1 - \frac{1}{y}$ then $U(Y_i) \stackrel{d}{=} X_i$ as stated in theorem 3.4. Hence it is sufficient to prove the result for $\hat{\gamma}_H = \frac{1}{k}\sum_{i=0}^{k-1} \log(U(Y_{n-i,n})) - \log(U(Y_{n-k,n}))$. For $t = Y_{n-k,n}$ and $x = \frac{Y_{n-i,n}}{Y_{n-k,n}}$ equation 2 has the form

$$\begin{aligned} \log(1-\varepsilon) + (\gamma-\delta)\log\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right) &< \log(U(Y_{n-i,n})) - \log(U(Y_{n-k,n})) \\ &< \log(1+\varepsilon) + (\gamma+\delta)\log(\frac{Y_{n-i,n}}{Y_{n-k,n}}). \end{aligned} \tag{3}$$

Notice that we can replace t with $Y_{n-k,n}$ because we can always find some n_0 such that $Y_{n_0-k,n_0} \geq t_0$ according to lemma 3.3. Furthermore, $Y_{n-i,n}$ is greater than $Y_{n-k,n}$ always when $i < k$. Therefore x can be replaced with $\frac{Y_{n-i,n}}{Y_{n-k,n}}$.

Equation 3 applies for every $i = 0, 1, 2, ..., k-1$. Thus we can write

$$\begin{aligned} \log(1-\varepsilon) + (\gamma-\delta)\frac{1}{k}\sum_{i=0}^{k-1}\log\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right) &< \frac{1}{k}\sum_{i=0}^{k-1}\log(U(Y_{n-i,n})) - \log(U(Y_{n-k,n})) \\ &< \log(1+\varepsilon) + (\gamma+\delta)\frac{1}{k}\sum_{i=0}^{k-1}\log\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right). \end{aligned}$$

The term in the middle is the hill estimator $\hat{\gamma}_H$, hence above becomes

$$\begin{aligned} \log(1-\varepsilon) + (\gamma-\delta)\frac{1}{k}\sum_{i=0}^{k-1}\log\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right) &< \hat{\gamma}_H \\ < \log(1+\varepsilon) + (\gamma+\delta)\frac{1}{k}\sum_{i=0}^{k-1}\log\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right)&. \end{aligned}$$

Now it is sufficient to only prove that

$$\frac{1}{k}\sum_{i=0}^{k-1}\log\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right)\xrightarrow{p}1.$$

$\log(Y_i)$ has a standard exponential distribution, since

$$F_{\log(Y_i)}(x)=P(\log(Y_i)<x)=P(e^{\log(Y_i)}<e^x)=P(Y_i<e^x)=F_Y(e^x)=1-e^{-x}.$$

Therefore we can write

$$\frac{1}{k}\sum_{i=0}^{k-1}\log\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right)=\frac{1}{k}\sum_{i=0}^{k-1}E_{n-i,n}-E_{n-k,n},$$

where $E_1, E_2, ...$ are i.d.d. random variables from standard exponential distribution. Now Renyi's representation 3.2 implies

$$\begin{aligned}
&\left\{E_{n-i,n}-E_{n-k,n}\right\}_{i=0}^{k-1}\\
\stackrel{d}{=}&\left\{\left(\frac{E_1^*}{n}+\frac{E_2^*}{n-1}+...+\frac{E_{n-(i+1)}^*}{n-(n-(i+1))+1}+\frac{E_{n-i}^*}{n-(n-i)+1}\right)\right.\\
&\left.-\left(\frac{E_1^*}{n}+\frac{E_2^*}{n-1}+...+\frac{E_{n-k}^*}{n-(n-k)+1}\right)\right\}_{i=0}^{k-1}\\
=&\left\{\frac{E_{n-i}^*}{i+1}+\frac{E_{n-(i+1)}^*}{i+2}+...+\frac{E_{n-(k-2)}^*}{k-1}+\frac{E_{n-(k-1)}^*}{k}\right\}_{i=0}^{k-1}\\
\stackrel{d}{=}&\left\{E_{k-i,k}\right\}_{i=0}^{k-1}.
\end{aligned}$$

Consequently we have

$$\frac{1}{k}\sum_{i=0}^{k-1}\log\left(\frac{Y_{n-i,n}}{Y_{n-k,n}}\right)\stackrel{d}{=}\frac{1}{k}\sum_{i=0}^{k-1}E_{k-i,k}=\frac{1}{k}\sum_{i=0}^{k-1}E_i\xrightarrow{p}E[E_i]=1$$

by the weak law of large numbers [3]. Notice that the expected value of a standard exponential is one.

□

3.2 Simulations

References

[1] L. D. Haan and A. Ferreira. *Extreme Value Theory: An Introduction.* Springer Series in Operations Research and Financial Engineering. Springer, New York, 2006.

[2] A. Rényi. On the theory of order statistics. *Acta Mathematica Academiae Scientiarum Hungarica*, 4(3):191–231, Sep 1953.

[3] J. S. Rosenthal. *A First Look at Rigorous Probability Theory.* World Scientific Publishing Co., Singapore, second edition edition, 2006.

Appendix