Math 578B – Fall 2015 – Homework #1

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due 1 September

1. Let X be a Markov chain on $\mathcal{A} = \{0, 1\}$ with transition probabilities

$$\mathbb{P}\{X_{t+1} = 1 \mid X_t = 0\} = \alpha \tag{1}$$
$$\mathbb{P}\{X_{t+1} = 0 \mid X_t = 1\} = \beta. \tag{2}$$

a. Show that if $X_0 = 0$ then the distribution of

$$\tau := \min\{n > 0 : X_n = 1\} \tag{3}$$

is geometric.

b. Find the spectral decomposition of the chain and use it to find an expression for

$$\mathbb{P}\{X_n = 1 \mid X_0 = 0\}. \tag{4}$$

c. What happens to the spectral expansion if $\alpha + \beta = 1$? Explain.

2. For the Markov chain in **(1)**, suppose we choose a random starting state with $\mathbb{P}\{X_1 = 0\} = \beta/(\alpha + \beta)$, and run it for n steps, producing a string of n digits, $X = X_1, \ldots, X_n$. Let Y be the reversed string, i.e., $Y_k = X_{n+1-k}$ for $1 \leq k \leq n$.

a. For a given string of digits $a_1, \ldots, a_n \in \{0, 1\}^n$, what is $\mathbb{P}\{Y_1 = a_1, \ldots, Y_n = a_n\}$? (Don't use the spectral decomposition.)

b. Suppose we generate a string of length n as in *(a)* but it is reversed with probability 1/2: formally, let $\theta = H$ with probability 1/2 and $\theta = T$ otherwise, and define

$$Z = \begin{cases} X & \text{if } \theta = H \\ Y & \text{otherwise.} \end{cases} \tag{5}$$

A colleague has developed a procedure that guesses whether the string was reversed, i.e., a function $f : \{0,1\}^n \to \{H, T\}$ that is given Z and guesses the value of θ. Show that

$$\mathbb{P}\{f(Z) = \theta\} = 1/2. \tag{6}$$

3. Let X_n be the simplified "polymerase complex assembly" Markov chain defined in class, with transition matrix (where "†" means transcription):

$$P = \begin{array}{c} \\ \varnothing \\ \alpha \\ \beta \\ \alpha+\beta \\ \text{pol} \\ \dagger \end{array} \begin{array}{c} \begin{array}{cccccc} \varnothing & \alpha & \beta & \alpha+\beta & \text{pol} & \dagger \end{array} \\ \begin{bmatrix} * & k_\alpha & k_\beta & 0 & 0 & 0 \\ k_\alpha & * & 0 & k_\beta & 0 & 0 \\ k_\beta & 0 & * & k_\alpha & 0 & 0 \\ 0 & k_\beta & k_\alpha & * & k_{\text{pol}} & 0 \\ 0 & 0 & 0 & 0 & * & 1 \\ 0 & 0 & 0 & 1 & 0 & * \end{bmatrix} \end{array} \tag{7}$$

Here the "$*$"s on the diagonal are set so that rows sum to 1. Let

$$\tau = \min\{n \geq 0 : X_n = \dagger\}. \tag{8}$$

a. Set $k_\alpha = k_\beta = 0.2$ and $k_{\text{pol}} = 0.5$, and compute numerically $\mathbb{E}[\tau]$ for all starting states. (And, explain how you do it!)

```r
ka <- kb <- 0.2
kpol <- 0.5
M <- matrix( c(
    0, ka, kb, 0, 0, 0,
    ka, 0, 0, kb, 0, 0,
    kb, 0, 0, ka, 0, 0,
    0, kb, ka, 0, kpol, 0,
    0, 0, 0, 0, 0, 1,
    0, 0, 0, 1, 0, 0
    ), byrow=TRUE, nrow=6 )
diag(M) <- 1-rowSums(M)
```

Now, since $\mathbb{E}^a[\tau] = h$ solves $(I - Q)^{-1}1$ where Q is

```r
pander::pander(M[1:5,1:5])
```

0.6	0.2	0.2	0	0
0.2	0.6	0	0.2	0
0.2	0	0.6	0.2	0
0	0.2	0.2	0.1	0.5
0	0	0	0	0

the solution is

```r
h <- solve( diag(5) - M[1:5,1:5], rep(1,5) )
pander::pander(h)
```

19, *16.5*, *16.5*, *9* and *1*

b. Compute numerically the stationary distribution. What is the long-term average rate of transcription (i.e., mean number of visits to † per unit time)?

   The stationary distribution π is the first left eigenvector of P:

```r
pivec <- eigen(t(M))$vectors[,1]
pivec <- Re( zapsmall(pivec)/sum(pivec) )
```

   and the mean time between visits to † is 9.9999983.

c. Verify your computations in *(a)* and *(b)* using a simulation of the chain.